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A Survey of Ekeland's Variational Principle and Related Theorems and Applications

Jessica Robinson
University of Nevada, Las Vegas

A STUDY OF EKELAND'S VARIATIONAL PRINCIPLE AND RELATED THEOREMS AND APPLICATIONS

by

Jessica Robinson

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Department of Mathematical Sciences

David Costa, Ph.D., Committee Chair

Xin Li, Ph.D., Committee Member

Hossein Tehrani, Ph.D., Committee Member

Paul Schulte, Ph.D., Graduate College Representative

Kathryn Hausbeck Korgan, Ph.D., Interim Dean of the Graduate College

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ABSTRACT

A Study of Ekeland's Variational Principle and Related Theorems and Applications

by

Jessica Robinson

⟨David Costa ⟩, Examination Committee Chair
Professor of Mathematical Sciences
University of Nevada, Las Vegas

Ekeland's Variational Principle has been a key result used in various areas of analysis such as fixed point analysis, optimization, and optimal control theory. In this paper, the application of Ekeland's Variational Principle to Caristi's Fixed Point Theorem, Clarke's Fixed Point Theorem, and Takahashi's Minimization theorem is the focus. In addition, Ekeland produced a version of the classical Pontryagin Minimum Principle where his variational principle can be applied. A further look at this proof and discussion of his approach will be contrasted with the classical method of Pontryagin. With an understanding of how Ekeland's Variational Princple is used in these settings, I am motivated to explore a multi-valued version of the principle and investigate its equivalence with a multi-valued version of Caristi's Fixed Point Theorem and Takahashi's Minimization theorem.

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CHAPTER 1

INTRODUCTION

In 1972, Ekeland proved an important result in nonlinear analysis. Since then, Ekeland's Variational Principle (EVP for short) has been proven equivalent with other theorems such as Caristi's Fixed Point Theorem and Takahashi's Minimization Theorem. This fundamental theorem is seamlessly connected to other results that are used in a myriad of mathematical areas.

Theorem 1.1 (Ekeland's Variational Principle-weak form)**.** *Let X be a complete metric space. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function, which is bounded below. Then for any $\epsilon > 0$, there exists $v \in X$ such that*

$$F(v) \leq inf_X F + \epsilon,$$

$$\forall w \in X, \ F(v) - \epsilon d(v, w) \leq F(w).$$

Ekeland also stated a stronger form and gave a visualization in order to geometrically interpret the theorem.

Theorem 1.2 (Ekeland's Variational Principle-strong form)**.** *Let X be a complete metric space. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function which is* *bounded below. Let $\epsilon > 0$ and a point $u \in X$ be given such that*

$$F(u) \leq inf_X F + \epsilon.$$

Then $\forall \lambda > 0$ there exists some point $v \in X$ such that

$$F(v) \leq F(u), \tag{1.1}$$

$$d(u, v) \leq \lambda, \tag{1.2}$$

$$F(v) - \frac{\epsilon}{\lambda} d(v, w) < F(w) \; \forall w \neq v. \tag{1.3}$$

A visualization of EVP (strong form) is shown in Figure 1.1.

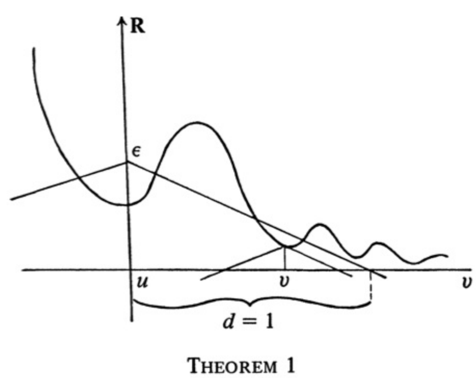

Figure 1.1: EVP

In the above diagram, taking $\lambda = 1$, we design a line with slope equal to $-\epsilon = -\tan\theta$. Then, the theorem guarantees that for any given ϵ there is a point $(v, F(v))$

such that if we create an open downwards cone with that point as its vertex and having angle 2θ, the function values for all other inputs will stay above the cone. Essentially, this theorem says that when a function isn't guaranteed to have a minimum there is a "good" approximate substitute. Under these conditions of lower semicontinuity and bounded below the best you can get is an approximate minimum. By requiring X to be a Banach space and F differentiable, EVP can be thought of as obtaining an approximate minimum with also a small derivative. Coupling this with the "Palais-Smale condition" a function must achieve its minimum and therefore have a critical point. These results are essential in areas of nonlinear analysis and optimization theory.

Outside of these applications, EVP provides equivalences with other important theorems, which we present in Chapter 2. While these results have been known for some time, they illustrate how useful EVP is in making connections with different areas of mathematics. By using EVP the proofs of these results are quite short and simple. The method of arriving at these equivalences are found also in Chapter 4 when considering set valued versions.

One area that will be discussed as an application of EVP is Control Theory. Beginning with a review of variational methods we find motivation for the formulation of Pontryagin's Maximization Principle. Then, we shall give an example of how the principle is applied. Ekeland's Variational Principle also yields an "approximate" Pontryagin Minimum Principle. We will discuss how Ekeland's version compares to Pontryagin's.

With the centralization that EVP provides, it has been a goal of many to generalize it. Various versions of Ekeland's Variational Principle have been presented since its initial presentation in 1972. There are three ways in which to alter Ekeland's Principle: change the space, the metric, or the function type. Recently, various people have proven the equivalence of these different Ekeland-type theorems with different corresponding versions of Caristi's Fixed Point Theorem and Takahashi Minimization Theorem. In this paper we wish to only look at changing the function from a single-valued function to a multi-valued one, which initially began with work by Nemath[14], Tammer [18], and Isac [12]. In 1997 Chen and Huang [5] sought to unify the work of these collegues. Here we will examine an approximate multivalued version of EVP presented by Chen and Huang in [3]. Finally, by considering a generalized EVP by Chen and Huang, we investigate the similarities in proving this EVP's equivalence with set-valued Caristi Fixed Point Theorem and set-valued Takahashi minimization result. In conclusion, we will again highlight the significance of the original EVP and that of current research being done on this topic.

CHAPTER 2

FIXED POINT RELATIONSHIPS

In the following chapter we review the equivalences between the original Ekeland's Variational Principle and other influential theorems. The proof techniques used here appear to also be effective in proving set-valued versions as we will see in Chapter 4.

EVP equivalence with Caristi's Fixed Point Theorem

Theorem 2.1 (Caristi's Fixed Point Theorem)**.** *Let X be a complete metric space.*

Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function which is bounded below. Let $T : X \to 2^X$ be a multivalued mapping such that

$$F(w) \leq F(v) - d(v, w) \; \forall v \in X, \forall w \in Tx.$$

Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Theorem 2.1, hereby referred to as CFPT, has many applications in Ordinary and Partial Differential Equations (see e.g. [6]). Theorem 1.2 and Theorem 2.1 are proven equivalent in [10]. We present the proof here of its equivalence with EVP as we will want to compare with the set valued proofs in Chapter 4.

EVP⇒CFPT

Proof. Suppose $\epsilon = 1$. Then we know by EVP that there exists $x_0 \in X$ such that the following holds for all $y \in X$, $y \neq x_0$

$$F(x_0) < F(y) + d(x_0, y),$$

i.e.,

$$F(x_0) - F(y) < d(x_0, y) \tag{1}$$

Claim: $x_0 \in Tx_0$. Indeed if $x_0 \notin Tx_0$, then $y \neq x_0$ $\forall y \in Tx_0$ by CFPT we get

$$F(y) \leq F(x_0) - d(x_0, y),$$

i.e.,

$$d(x_0, y) \leq F(x_0) - F(y). \tag{2}$$

Inequalities (1) and (2) give

$$F(x_0) - F(y) < d(x_0, y) \leq F(x_0) - F(y),$$

a contradiction. Therefore $x_0 \in Tx_0$.

□

EVP⇐CFPT

In order to prove the reverse implication (CFPT⟹EVP) we also use a proof by contradiction.

Proof. Let us define $d_1(x, y) = \epsilon d(x, y)$ which is an equivalent distance in X. Define $T(x) = \{y \in X : F(x) \geq F(y) + d_1(x, y); y \neq x\}$ for each $x \in X$ and $T(x) \neq \emptyset$ $\forall x \in X$. $T(x)$ is a multivalued map, which satisfies

$$F(y) \leq F(x) - d_1(x, y),$$

so that according to CFPT, there is a fixed point $x_0 \in T(x_0)$. However if x_0 is in $T(x_0)$, we have violated the definition of $T(x)$. Thus a contradiction. □

EVP equivalence with Takahashi

Definition 2.1 (Takahashi's Condition)**.** Let $Z := \{z \in X : F(z) = inf\, F\}$ be the set of possible minimizers of the function F.

The Takahashi condition says the following:

$\exists\, \alpha_0 > 0$, such that $\forall x \in X \setminus Z$, $\exists\, y \neq x$ where the following inequality holds:

$$F(y) + \alpha_0 d(y, x) \leq F(x).$$

Theorem 2.2 (Takahashi's Minimization Theorem [11])**.** *Let X be a complete metric space.*

Let $F : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function which is bounded below. Assume that F satisfies the Takahashi condition. Then there exists some $x_0 \in X$ such that

$$F(x_0) = \inf F.$$

Initially, Takahashi demonstrated his theorem implied EVP. Then in [11] the reverse implication was also presented.

EVP$\Rightarrow$ Takahashi

Proof. Let $0 < \alpha < \alpha_0$. Then by EVP, there exists $x \in X$ such that

$$F(x) < F(y) + \alpha d(x, y) \; \forall x \neq y.$$

But by Takahashi Condition, $\forall z \in X \setminus Z$, $\exists\, y \neq z$ such that

$$F(y) + \alpha d(y, z) \leq F(z).$$

The above two inequalities can only hold together if $x \in Z$. Thus $Z \neq \emptyset$. So we are guaranteed that there exists an $x_0 \in X$ such that $F(x_0) = inf\ F$.

$\square$

Clarke's Fixed Point Theorem

It turns out that Ekeland's Variational Principle also implies a different fixed point theorem due to Clarke in [**?**].

Theorem 2.3 (Clarke's Fixed Point Theorem)**.** *Let X be a closed convex subset of a Banach Space.*

Suppose $f : X \to X$ is continuous, and there exists $\sigma \in (0,1)$ such that for any $x \in X$, $f(x) \neq x$, there exists $y \in [x, f(x)] \setminus \{x\}$, such that

$$d(f(x), f(y)) \leq \sigma d(x, y).$$

Then $f(x)$ has a fixed point.

It should be noted the above condition is referred to as a directional contraction where $[x, f(x)] := \{z \in X \mid d(x,z) + d(z, f(x)) = d(x, f(x))\}$ represents the line segment connecting x and $f(x)$. It is easy to check that every contraction is a directional contraction. Thus, Clarke's Fixed Point Theorem generalizes Banach Fixed Point Theorem although uniqueness is not guaranteed. An example is presented in [**?**] where Clarke's Theorem applies and Banach does not.

Proof. Apply EVP to $F(a) = d(a, f(a))$ since it is clearly bounded below and continuous with $0 < \epsilon < 1 - \sigma$. We know $\exists b \in X$ such that $\forall z \in X$

$$F(b) - \epsilon d(z, b) < F(z), \quad z \neq b$$

i.e.

$$d\left(b, f(b)\right) - \epsilon d(z, b)) < d(z, f(z)) \tag{a}$$

Assume $f(b) \neq b$. By $f(x)$ being a directional contraction there exists $z \neq b$, $z \in [b, f(b)]$, i.e.

$$d(b, z) + d(z, f(b)) = d(b, f(b)) = F(b)$$

satisfying

$$d(f(z), f(b)) \leq \sigma d(z, b). \tag{b}$$

Now apply the reverse triangle inequality to get

$$d(z, f(z)) - d(z, f(b) \leq d(f(b), f(z)) \leq \sigma d(z, b). \tag{c}$$

Substitute $d(b, z) + d(z, f(b)) = d(b, f(b))$ into (a) to get

$$d(b, z) + d(z, f(b)) - d(z, f(z)) < \epsilon d(z, b). \tag{d}$$

Combining inequalities (c) and (d) results in

$$d(b, z) \leq (\epsilon + \sigma) d(z, b).$$

This is a contradiction since $\epsilon + \sigma < 1$.

$\square$

Summary

While many of the results presented in this chapter have been known for some time, it is still interesting to note the far reaching effects of Ekeland's Variational Principle. Currently we have the following relationships CFPT$\Longleftrightarrow$ EVP $\Longleftrightarrow$ Takahashi. This shows there is an equivalence between a fixed point theorem and an minimization theorem through Ekeland's Variational Principle, and the proofs are straightforward.

CHAPTER 3

APPLICATIONS

Ekeland's Variational Principle has been a useful tool in Control Theory. Here we will consider a classic result in Control Theory known as Pontryagin's Principle. Pontryagin's Principle can be stated either for finding a minimum or maximum. Some special cases include free end-point vs fixed end-point. In the original proof, the author states the principle for the most general setting. This will not be necessary for what is to be explored here. We will briefly consider the setup from Euler-Lagrange equations to motivate Pontryagin's Principle. Then we give the statement of Pontryagin's Maximum Principle (PMP) specifically for the free end-point problem without running cost. This version is the most closely associated with Ekeland's approximate PMP. We will discuss the relationship between the original version and Ekeland's version. In addition we mention work by Tammer [17] involving an approximate PMP for set-valued functions.

Preliminaries on Calculus of Variations

First we must investigate the motivation for PMP by reviewing original techniques in the Calculus of Variations.

The basic problem can be presented as the following:

Let a smooth function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L = L(x, v)$ be given (L is called the "Lagrangian"), and let $T > 0$, x^0, $x^1 \in \mathbb{R}^n$ be given.

Find a curve $x^*(\cdot) : [0, T] \to \mathbb{R}^n$ that minimizes the functional

$$I\left[x(\cdot)\right] := \int_0^T L(x(t), \dot{x}(t))dt \tag{I}$$

among all functions $x(\cdot)$ satisfying $x(0) = x^0$ and $x(T) = x^1$

Now suppose we have found $x^*(\cdot)$ that does minimize the function I. How is $x^*(\cdot)$ characterized? To discover the answer we must look at the Euler-Lagrange Equations.

NOTATION: We often write $L = L(x, v)$, where x denotes the position and v denotes the velocity. The partial derivatives of L are denoted by

$$\frac{\partial \mathrm{L}}{\partial x_i} = L_{x_i}, \frac{\partial L}{\partial v_i} = L_{v_i} \qquad (1 \leq i \leq n)$$

Theorem 3.1 (Euler-Lagrange Equations)**.** *Let $x^*(\cdot)$ solve the calculus of variations problem (I). Then $x^*(\cdot)$ solves the Euler-Lagrange differential equations below*

$$\frac{d}{dt}\left[\nabla_v L\left(x^*(t), \dot{x}^*(t)\right)\right] = L_{x_i}\left(x^*(t), \dot{x}^*(t)\right). \tag{3.1}$$

The implication of Theorem 3.1 is that the solution of the original calculus of variations problem (if it exists) will be among the solutions of the Euler-Lagrange differential equation. The proof of Theorem 3.1 is omitted here (see e.g. [8]).

Now we want to see how can one convert the Euler-Lagrange equations into a system of first order ODE.

Definition 3.1 (generalized momentum)**.** For the given curve $x(\cdot)$, define

$$p(t) := \nabla_v L\left(x(t), \dot{x}(t)\right) \qquad (0 \leq t \leq T)$$

We call $p(t)$ the generalized momentum. Using this definition and the following hypothesis we can rewrite Euler-Lagrange into a system of ODE's.

Hypothesis: Assume that for all $x, p \in \mathbb{R}^n$ we can solve the equation

$$p = \nabla_v L(x, v) \tag{3.2}$$

for v in terms of x and p (i.e $v = \mathbf{v}(x, p)$).

Definition 3.2 (Hamilitonian)**.** Define the dynamical system's Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by the formula

$$H(x, p) = p \cdot \mathbf{v}(x, p) - L\left(x, \mathbf{v}\left(x, p\right)\right)$$

where $\mathbf{v}$ is defined above.

NOTATION: The partial derivatives of H are denoted by

$$\frac{\partial H}{\partial x_i} = H_{x_i}, \, \frac{\partial H}{\partial p_i} = H_{p_i} \qquad (1 \leq i \leq n)$$

Theorem 3.2 (Hamiltonian dynamics)**.** *Let $x(\cdot)$ solve the Euler-Lagrange equations and define $p(\cdot)$ as above. Then the pair $(x(\cdot), p(\cdot))$ solves the Hamilitonian equations:*

$$\begin{cases} \dot{x}(t) = \nabla_p H\left(x(t), p(t)\right) \\ \\ \dot{p}(t) = -\nabla_x H\left(x(t), p(t)\right) \end{cases} \tag{3.3}$$

Furthermore, the mapping $t \mapsto H\left(x(t), p(t)\right)$ is constant.

The proof of this theorem is also omitted here (See [8]).

Now given this background information, we construct the setting for Pontryagin's Principle.

PROBLEM: Given $A \subseteq \mathbb{R}^m$ and also $f : \mathbb{R}^n \times A \to \mathbb{R}^n$. We denote the set of admissible controls by

$$\mathcal{A} = \left\{\alpha\left(\cdot\right) : [0, \infty) \to A \,|\, \alpha\left(\cdot\right) \textit{ measurable}\right\}.$$

Then given $\alpha\left(\cdot\right) \in \mathcal{A}$, we solve the following evolution system

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)) \quad (0 \leq t \leq T) \\ \\ x(0) = x^0 \end{cases} \tag{ODE}$$

and introduce the "terminal payoff functional"

$$P\left[\alpha\left(\cdot\right)\right] = g(x(T)) \tag{P}$$

for a given $g : \mathbb{R}^n \to \mathbb{R}$. Then P is to be maximized. With the above terminal payoff functional, the Hamiltonian is defined as

$$H(x, p, \alpha) = f(x, \alpha) \cdot p.$$

Theorem 3.3 (Pontryagin's Maximization Principle)**.** *Assume that $\alpha^*(t)$ is an optimal control for the above problem, with a corresponding optimal trajectory $x^*(t)$. Then there exists a function $p^* : [0, T] \to \mathbb{R}^n$ such that*

$$\dot{x}^*(t) = \nabla_p H\left(x^*(t), p^*(t), \alpha^*(t)\right)$$

and

$$\dot{p}^*(t) = -\nabla_x H(x^*(t), p^*(t), \alpha^*(t)) \qquad (0 \leq t \leq T) \tag{ADJ}$$

and

$$H(x^*(t), p^*(t), \alpha^*(t)) = \max_{\alpha \in A} H(x^*(t), p^*(t), \alpha) \qquad (0 \leq t \leq T). \tag{M}$$

Futhermore, the mapping $t \mapsto H\left(x^*(t), p^*(t), \alpha^*(t)\right)$ is constant.

Finally we also have the terminal condition

$$p^*(T) = \nabla g\left(x^*(T)\right). \tag{T}$$

The proof of PMP can be found in [8].

REMARKS: We refer to the identities (ADJ) as the adjoint equations and (M) as the maximazation principle. Notice that (ODE) and (ADJ) resemble the structure of Hamilton's equations above. Also note another way of writing (ADJ) is the following

$$\dot{p}^{i*}(t) = -H_{x_i}\left(x^*(t), p^*(t), \alpha^*(t)\right)$$

$$= -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i}(x^*(t), \alpha^*(t)) p_j^*(t).$$

This corresponds closely to Ekeland's version.

Example of Pontryagin's Maximum Principle

Now we present an example of this version of Pontryagin without any running cost from [8].

Example 3.1 (Commodity Trading)**.** Here is a simple model for trading a commodity, say wheat. We let T be the fixed length of trading period, and introduce the following variables:

$y(t)$ = money on hand at time t

$w(t)$ = amount of wheat owned at time t

$\alpha(t)$ = rate of buying or selling wheat

$q(t)$ = price of wheat at time t (known)

λ = cost of storing a unit amount of wheat for a unit of time

We require that the price of wheat $q(t)$ is known for the entire trading period. We also assume that the rate of selling and buying is constrained:

$$|\alpha(t)| \leq M,$$

where $\alpha(t) > 0$ means wheat is being bought and $\alpha(t) < 0$ means wheat is being sold. The purpose of the problem is to maximize the holdings at the end time $t = T$, meaning we want the sum of cash and value of wheat to be at a premium. This is modeled with the following payoff functional

$$P\left[\alpha(\cdot)\right] = y(T) + q(T)w(T). \tag{P}$$

Therefore y(T)+q(T)w(T) represents the g(x(T)) in the Pontryagin setup. The ODE is as follows

$$\begin{cases} \dot{y}(t) = -\lambda w(t) - q(t)\alpha(t) \\ \dot{w}(t) = \alpha(t). \end{cases} \tag{ODE}$$

Note the above pair $(y(t), w(t))$ represents the $x(t)$ in the Pontryagin set up. Now we will apply PMP to find the optimal buying and selling strategy. First compute the Hamilitonian

$$H(x, p, \alpha) = f(x, \alpha) \cdot p$$

i.e.

$$H(y, w, p, t, \alpha) = p_1\left(-\lambda w - q(t)\alpha\right) + p_2\alpha.$$

Then the adjoint equation $\dot{p}(t) = -\nabla_x H(x(t), p(t), \alpha(t))$ now reads as

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = \lambda p_1 \end{cases} \tag{ADJ}$$

with terminal condition

$$p(T) = \nabla g\left(x(T)\right). \tag{3.4}$$

In our case $g(y, w) = x + q(T)w$. Hence the above becomes

$$\begin{cases} p_1(T) = 1 \\ p_2(T) = q(T) \end{cases} \tag{T}$$

We can thensolve for the costate by integrating and using the terminal conditions i.e.

$$\begin{cases} p_1(t) \equiv 1 \\ p_2(t) = \lambda(t - T) + q(t). \end{cases}$$

By maximation principle (M), we know

$$H(y, w, p, t, \alpha) = \underset{|\alpha| \leq M}{max} \left\{ p_1(t)\left(-\lambda w(t) - q(t)\alpha\right) + p_2(t)\alpha \right\}$$

$$= -\lambda p_1(t) w(t) + \underset{|\alpha| \leq M}{max} \left\{ \alpha\left(-q(t) + p_2(t)\right) \right\}.$$

So the optimal control or rate of buying or selling wheat can be described as

$$\alpha(t) = \begin{cases} M & q(t) < p_2(t) \\ \\ -M & q(t) > p_2(t). \end{cases}$$

Note. This optimal control says that one should buy as much wheat as possible when the price of wheat is below a certain threshold, but then one should sell as much wheat as possible when the price has now gone above that same threshold. Remember that $p_2(T) = q(T)$ when $t = T$. This is an example of a so-called bang-bang control.

From the statement of PMP it is fundamental that there exist an optimal control and corresponding trajectory that solves the (ODE). The conditions of PMP guarantee to maximize the Payoff functional. Note that above we formulated the Maximum principle for the purposes of seeing an example. However all the prior work can be done for the minimum principle with a slightly different definition of the Hamilitonian. This is sufficient background to now consider a result on Ekeland's approximate PMP.

Ekeland's Approximate PMP

Here is Ekeland's approximate version on the Pontryagin Minimal Principle which is an application of the weak form of Ekeland Variational Principle.

Theorem 3.4 (Approximate Pontryagin)**.** *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. For every $\epsilon > 0$ there exists a measurable control $\alpha^*(t) \in A$ with a corresponding trajectory*

$x^*(t)$ that solves the differential equation (ODE) and is such that

$$\langle p^*, f(x^*, \alpha^*)\rangle \leq \min_{\alpha \in A} \langle p^*, f(x^*, \alpha)\rangle + \epsilon, \tag{H}$$

$$g(x^*(T)) \leq \inf g(x(T)) + \epsilon,$$

for almost every $t \in [0, T]$ and where $p^*(t)$ solves the following:

$$\frac{dp^*}{dt} = -\sum_{j=1}^{n} \frac{\partial f_j}{\partial x}(x^*(t), \alpha^*) p_j^*(t), \tag{ADJ}$$

$$p^*(T) = g'\left(x^*\left(T\right)\right). \tag{T}$$

Proof. Let A be the set of all measurable controls $\alpha : [0, T] \to K$, where K is a compact metrizable space with the following metric

$$d(a, b) = meas\left\{t \in [0, T] : a \neq b\right\}$$

By a Lemma in [7] this space of measurable controls is a complete metric space. Consider the function

$$F : \alpha \to g(x(T))$$

where $\alpha \in A$, x is a corresponding solution of the (ODE). This F function is shown continuous and bounded below in [7]. Therefore we will apply weak form of Ekeland's

Variational Principle. There exists $\alpha^* \in A$ such that

$$F(\alpha^*) < inf\, F + \epsilon$$

$$F(\alpha) \geq F(\alpha^*) - \epsilon d(\alpha, \alpha^*) \;\; \forall \alpha \in A$$

The second equation can be further analyzed by taking into account the differential equation $\dot{x}(t) = f(x(t), \alpha(t))$ which holds almost everywhere on $[0, T]$. Take any t_0 where the differential equation holds, and any $k_0 \in K$, and define $a_\tau \in A$, $\tau \geq 0$ by:

$$\begin{cases} a_\tau(t) = k_0 & t \in [0, T] \cap (t_0 - \tau, t_0) \\ a_\tau(t) = \alpha^*(t) & t \notin [0, T] \cap (t_0 - \tau, t_0) \end{cases}$$

Notice that $d(a_\tau, \alpha^*) = \tau$ when τ is small enough. Denote the trajectory associated with a_τ as x_τ. Therefore $F(x_\tau) = g(x_\tau(T))$. Using this in the EVP inequality we arrive at

$$g(x_\tau(T)) \geq g(x^*(T)) - \epsilon\tau \quad \forall \tau$$

Hence we can see that the derivative of g with respect to τ is bounded below:

$$\frac{d}{d\tau} g(x_\tau(T)) \mid_{\tau=0} \geq -\epsilon.$$

However through a linearization argument the left hand side can be calculated as the following (see [7])

$$\frac{d}{d\tau} g(x_\tau(T)) = \langle f(t_0, x^*, k_0) - f(t_0, x^*, \alpha^*(t_0), p(t_0)\rangle \geq -\epsilon. \tag{*}$$

This completes the proof since k_0 was taken to be any element in the target space of the control functions and t_0 is any point in $[0, T]$ where the (ODE) holds.

$\square$

In Ekeland's approximate version whenever $\epsilon = 0$, one can see essentially Pontryagin's Principle. However Ekeland's approximate version always holds even if an optimal control does not exist. Ekeland admits in [6] that this approximate Pontryagin Principle may be less than desirable given a particular problem. Even if ϵ is small, the solution may become highly irregular such that the result is inadequate for any practical application. The approximate Pontryagin displays another prominent usage of EVP. In [17], Tammer develops a similar approximate Pontryagin Principle only in the setting of multi-objective optimal control problems. Tammer mirrors Ekeland's approach starting with a vector-valued function F and complete metric space X. The proof uses a set-valued EVP given by Tammer. Then, following along the lines of Ekeland, Tammer eventually shows a corresponding statement to (*) with respect to set-valued functions. Tammer's application of the set-valued EVP once again demonstrates the significance of EVP to mathematical problems.

CHAPTER 4

MULTI-VALUED EVP

As stated in the introduction recent work has been done to formulate EVP type statements in other areas of mathematics. This has been accomplished by changing one of the follow: the space, the metric or the function type. By generalizing EVP, it can be applied to more mathematical situations. Changing the function from single-valued to multi-valued was first explored by Nemeth (1986), Tammer (1992), and Isac (1996). However each of these multi-valued versions had varying conditions on the function. Then in 1997, Chen and Huang [5] unified these results in "A unifed approach to the existing three types of variational principles for vector valued functions." The set-valued EVP they prove in that paper requires very generalized conditions for set valued function and the space. From that version Chen and Huang derive a version, which they prove equivalent to set valued Caristi Fixed Point Theorem and set valued Takahashi. At the end of this chapter we will consider these proofs for comparision to their originl counterparts. In addition, Chen and Huang explored $\epsilon - solutions$ of a set valued function which led to a strong EVP that is similar to original. This result is the most closely associated with our goal of keeping as much as possible of the original EVP intact and only changing to a multi-valued function.

Set Valued Function Background

For this section we will let X be a set and Y be a locally convex topological space. $K \subseteq Y$ is a nonempty, nontrivial, pointed, closed, convex cone with nonempty interior $int\, K$. A pointed cone is one such that $K \cap -K = \{0\}$. A convex cone is such that $K + K \subset K$ and $\forall \lambda \geq 0,\ \lambda K \subset K$. K induces the following partial order in Y: $\forall\, y_1, y_2 \in Y, y_1 \leq_k y_2$ iff $y_2 - y_1 \in K$.

In Ekeland's original variational principle the function needed to be proper, lower semi-continuous, and bounded below. In comparison we will need the set valued mapping to be proper, upper semicontinuous, bounded below, and compact valued. We will define these terms for set-valued mappings.

Definition 4.1. A set valued mapping $F : X \to 2^Y$ is called proper, if dom(F)$\neq \emptyset$.

Definition 4.2 (upper semicontinuity)**.** Let X be a topological space. A set-valued mapping $F : X \to 2^Y$ is said to be upper semicontinuous at $x_0 \in X$ if for any neighborhood U of $F(x_0)$, there exists a neighborhood V of x_0 such that

$$F(x) \subset U, \ \ \forall x \in V.$$

If F is upper semicontinuous at every $x \in X$, then we say F is upper semicontinuous on X.

Definition 4.3 (lower semicontinuity)**.** Let X be a topological space. A set-valued mapping $F : X \to 2^Y$ is said to be lower semicontinuous at $x_0 \in X$ if for any $y_0 \in F(x_0)$ and any neighborhood U of y_0, there exists a neighborhood V of x_0 such that

$$F(x) \cap U \neq \emptyset, \quad \forall x \in V.$$

If F is lower semicontinuous at every $x \in X$, then we say F is lower semicontinuous on X.

Note. The definition of upper semicontinuity can be thought of as the regular definition of continuity when the function is single-valued. However in the setting of set-valued mappings there are examples of functions that are upper semicontinuous but not lower semicontinuous and vice versa.

Example 4.1. Note in the following example that the function is not lower semicontinuous at $x = 0$, but it is upper semicontinuous.

$$f(x) = \begin{cases} 0 & x < 0 \\ [-1, 1] & x = 0 \\ 0 & x > 0 \end{cases}$$

Meanwhile the next function is lower semicontinuous at $x = 0$, but not upper semicontinuous.

$$f(x) = \begin{cases} [-1, 1] & x < 0 \\ 0 & x = 0 \\ [-1, 1] & x > 0 \end{cases}$$

Definition 4.4. A set valued mapping $F : X \to 2^Y$ is said to be bounded below on

X, if $\exists\, y \in Y$ such that

$$F(x) - y \subset K, \;\; \forall x \in X.$$

Definition 4.5. A set valued mapping $F : X \to 2^Y$ is said to be compact valued if $\forall x \in X$, $F(x)$ is a compact subset of Y.

Also, in order to prove the Chen-Huang version of EVP we will need to introduce a few other definitions.

Definition 4.6. Let $\alpha \in int\, K$. Given $F : X \to 2^Y$ and $\epsilon > 0$, $x^\star \in X$ is called an $\epsilon - solution$ of F if $\exists y^\star \in F(x^\star)$ such that

$$[F(x^\star) - y^\star] \cap [-K \setminus \{0\}] = \emptyset \tag{4.1}$$

and

$$[F(x) - y^\star + \epsilon\alpha] \cap [-K \setminus \{0\}] = \emptyset, \; \forall x \in X \setminus \{x^\star\} \tag{4.2}$$

Note. When $\epsilon = 0$ in the above definition, then the y^* is referred to as an "efficient point" of the set $F(X) = \bigcup\limits_{x \in domF} F(x)$ as defined below.

Definition 4.7 (efficient points)**.** The set of efficient points for $A \subset Y$ is defined by

$$E(A) =: \{a \in A \mid [A - a] \cap [-K \setminus \{0\}] = \emptyset\}.$$

The following lemma is straightforward.

Lemma 4.1 (domination property [3]). *If $B \subseteq Y$ is compact and nonempty, then for any $y_0 \in B$, there exists $y_1 \in E(B)$ such that $y_1 \leq_k y_0$.*

Theorem 4.2 ($\epsilon - $ *solutions* of F). *Given a set valued mapping $F : X \to 2^Y$ that is proper, compact valued, and bounded below, and $\epsilon > 0$, there exists an $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$ such that*

$$[F(\bar{x}) - \bar{y}] \cap [-K \setminus \{0\}] = \emptyset \tag{a}$$

$$[F(x) - \bar{y} + \epsilon\alpha] \cap [-K \setminus \{0\}] = \emptyset \ \forall x \in X \setminus \{\bar{x}\} \tag{b}$$

Remark. *Notice this theorem gives sufficient conditions for $\bar{x}$ to be an $\epsilon - $ solution of a function.*

Proof. Suppose by contradiction that there exists a real number $\epsilon_0 > 0$ such that the conclusions of Theorem 4.2 do not hold. Then choose an arbitrary $x_1 \in domF$ and $y_1 \in F(x_1)$. Since $F(x_1)$ is compact we can apply Lemma 4.1, to get $y_1' \in E\left(F(x_1)\right)$ such that $y_1' - y_1 \in -K$. At this time conclusion (b) is assumed to not hold. Therefore $\exists x_2 \in X$ and $y_2 \in F(x_2)$ such that

$$(y_2 - y_1' + \epsilon_0\alpha) \in -K.$$

Since $F(x_2)$ is compact, we can again apply Lemma 4.1 to deduce $\exists y_2' \in E\left(F(x_2)\right)$ such that $(y_2' - y_2) \in -K$. This combined with the above statement would yield

$$(y_2' - y_1' + \epsilon_0\alpha) \in -K.$$

Once again conclusion (b) does not hold for y_2'. Therefore $\exists x_3 \in X$ and $y_3' \in E\left(F(x_3)\right)$ such that

$$(y_3' - y_2' + \epsilon_0\alpha) \in -K.$$

Continuing this way, one obtains sequences $\{x_n\} \subset X$ and $\{y_n'\} \subset E\left(F(x_n)\right)$ such that

$$\left(y_i' - y_{i-1}' + \epsilon_0\alpha\right) \in -K, \quad i = 2\ldots.$$

Then, summing the above for $i = 2, ..., n$, gives

$$\sum_{i=2}^{n} \left(y_i' - y_{i-1}' + \epsilon_0\alpha\right) = y_n' - y_1' + (n-1)\epsilon_0\alpha \quad \forall n \geq 2$$

It follows that

$$\left(\frac{y_n' - y_1'}{n-1} + \epsilon_0\alpha\right) \in -K \quad \forall n \geq 2.$$

Since F is bounded below, $\exists y \in Y$ such that $y_n' - y \in K$ for all $n \in \mathbb{N}$. Then we get

$$\frac{y - y_n'}{n-1} + \frac{y_n' - y_1'}{n-1} + \epsilon_0\alpha = \left(\frac{y - y_1'}{n-1} + \epsilon_0\alpha\right) \in -K.$$

Now, letting $n \to \infty$, we have that $\epsilon_0\alpha \in -K$. However this is impossible by the assumptions $\epsilon > 0$ and $\alpha \in intK$. This completes the proof of $\epsilon - solution$ theorem.

$\square$

Visualizing the $\epsilon - solution$ Theorem

Recall that the original EVP gave a visual interpretation of existence of a cone such that the function would always stay above the said cone. We want to check whether we can visualize the set valued theorem in a similar manner.

Example 4.2. Suppose we have the following single valued function $f : \mathbb{R} \to \mathbb{R}$, $K = \mathbb{R}^+$, $\alpha \in IntK$ where $f(x) = \sin(x)$.

Notice this function trivially meets the criteria of being bounded below, compact valued, and proper.(See Figure 4.1)

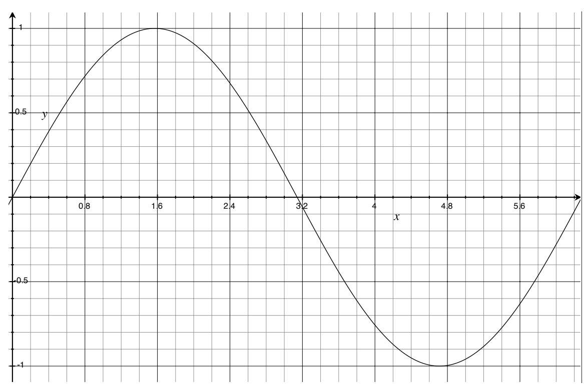

FIGURE 4.1: F(X)=SIN(X)

Now let's evaluate if we have met the conclusions (a) and (b). We can see in the case of a single valued function that $[F(\bar{x}) - \bar{y}] \cap [-K \setminus \{0\}] = \emptyset$ is true because $\bar{y} = F(\bar{x})$. Clearly $\{0\} \cap [-K \setminus \{0\}] = \emptyset$. Also, $[F(x) - \bar{y} + \epsilon\alpha] \cap [-K \setminus \{0\}] = \emptyset$ in the case of the single valued function, can be read as the following:

$$f(x) - \bar{y} + \epsilon\alpha \geq 0,$$

i.e.

$$f(x) \geq \bar{y} - \epsilon\alpha.$$

We realize that finding the $\bar{y}$ can always be accomplished because the function is bounded below. So in the single-valued case if $f(x) \geq M \; \forall x \in X$ holds, we can find $\bar{y}$ satisfying

$$M \geq \bar{y} - \epsilon\alpha,$$

which renders (b) true. In fact, in this example $-1 \in E\,(F(x))$.

We infer that the above example is typical and Theorem 4.2 holds true for any single valued function bounded below. However, $M = inf\,F$ may not be an efficient point of $F(X)$ as the next example shows. Let's consider a different singled valued function that does not attain its minimum.

Example 4.3. Suppose we have the following single valued function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \exp(-x)$.

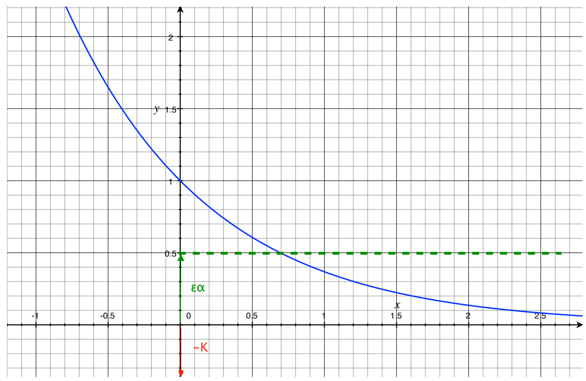

Figure 4.2: f(x)=exp(-x)

In the above example as long as $\bar{y} \leq \epsilon\alpha$ then the conclusions of $\epsilon - \mathit{solution}$ theorem will be satisfied. However, in this example, there are no efficient points. Indeed any such $\bar{y} = F(\bar{x})$ will not satisfy

$$F(x) - \bar{y} \geq 0 \quad \forall x \in X.$$

As we see, in the single valued case, boundedness from below with existence of a minimum is necessary and sufficient for existence of efficient points. Note also that, in the single-valued case, with $Y = \mathbb{R}$, the requirement of $\alpha > 0$ is superfluous ($\epsilon > 0$ suffices). The next example shows the need of $\alpha \in intK$ when $Y = \mathbb{R}^2$.

Example 4.4. *Suppose* $g : [0, 2\pi) \to \mathbb{R}^2$*,* $K = [0, \infty) \times [0, \infty)$*, where* $g(t) = (\cos(t), \sin(t))$*.*

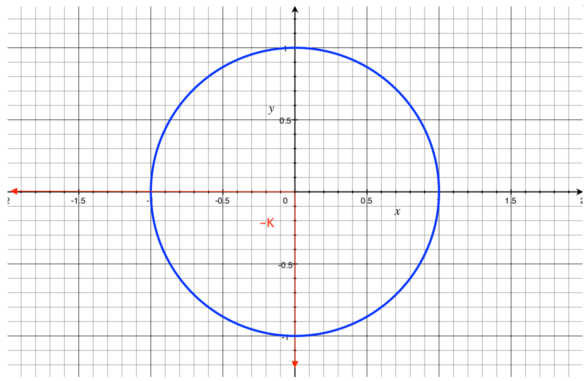

Figure 4.3: circle

In this example, $g(t)$ is bounded below because $y = (-1, -1)$ will suffice to move the circle to inside the cone. Also, $g(t)$ is compact because the image is just a single point. Let's check if the conclusions of the $\epsilon - solution$ theorem are met. Once again the first statement is true vacuously because $g(\bar{t}) = \bar{y}$, hence $\{0\} \cap -K \setminus \{0\} = \emptyset$. The second statement in this example shows the necessity of $\alpha \in intK$. One can think of $g(t) + \epsilon\alpha$ as shifting the function by a vector towards the cone. Then the $\epsilon - solution$ theorem says that there exists some $\bar{y} \in g(\bar{t})$, which shifts the function out of the negative cone. This can be illustrated in the following way.

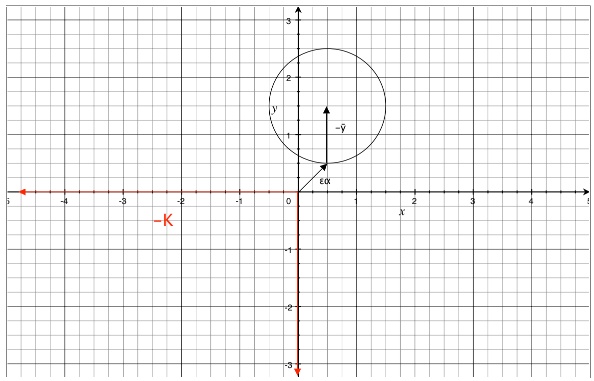

Figure 4.4: circle shifted

In the above diagram we picked $t = \frac{3\pi}{2}$, $\bar{y} = (0, -1)$. It turns out that $(0, -1) \in E\,(F(x))$. This example has several points that are efficient points. Earlier we established that the attainment of a minimum produced efficient points. Are there examples when $Y = \mathbb{R}^2$ that do not have efficient points, but do have $\epsilon - solutions$? The next example answers the question in the affirmative.

Example 4.5. *Suppose* $g : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^2$ *where* $g(t) = (t, \tan(t))$.

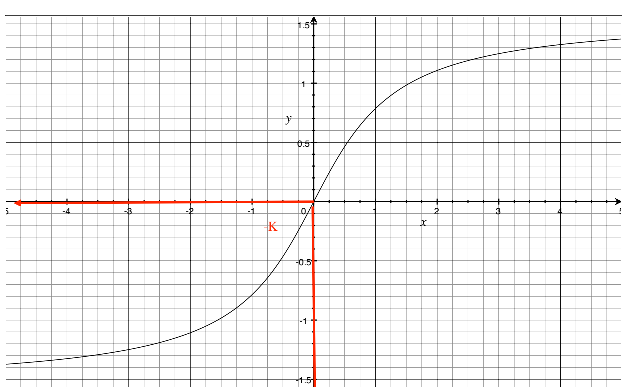

Figure 4.5: tan inverse

This function is trivially compact valued, bounded below, and proper, thus meeting the conditions of the $\epsilon - solution$ theorem, and given any $\epsilon\alpha$, one can find the $\bar{y}$ by solving the following inequality:

$$(0, -\frac{\pi}{2}) \geq_K \bar{y} - \epsilon\alpha.$$

However, this example has no efficient points. Indeed any such $\bar{y} = g(\bar{t})$ will not satisfy

$$[g(t) - \bar{y}] \cap [-K \setminus \{0\}] = \emptyset \quad \forall t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In summary, these examples demonstrate that $\epsilon - solutions$ of single-valued functions are connected to the infimum inequality of the original EVP. Let's restate the infimum inequality using the ideas of set-valued functions.

$$y_0 = F(x_0) \leq infF + \epsilon$$

$$0 \leq infF - y_0 + \epsilon$$

To convert this inequality into a statement using a cone, we must add the α as follows

$$[infF - y_0 + \epsilon\alpha] \cap [-K \setminus \{0\}] = \emptyset.$$

If the above statement is true for the infimum of the function, which is bounded below, then it should be true for any other value of the function. Thus we arrive at

$$[F(x) - y_0 + \epsilon\alpha] \cap [-K \setminus \{0\}] = \emptyset$$

When the infimum is not attained but the function is bounded below, the y_0 can be specifically calculated based on the $\epsilon\alpha$ given. When the infimum is attained it does not matter the $\epsilon\alpha$ that is given, one can always choose an efficient point to be y_0 .

Strong-set valued EVP

Using the idea of $\epsilon - solutions$ from the last section Chen and Huang [4], give a strong form of EVP for set valued functions, which we will refer to as SVEVP.

Theorem 4.3 (SVEVP)**.** *Let (X, d) be a complete metric space, Y be a locally convex Hausdorff space, and K be a nontrivial, pointed, closed and convex cone with nonempty interior $int\, K$ and $\alpha \in int\, K$. Let $F : X \to 2^Y$ satisfy the following:*

(i) F is proper on X;

(ii) F is compact valued;

(iii) F is upper semicontinuous on X;

(iv) F is bounded below on X.

Also, one is given a real number $\epsilon > 0$ and $x_1 \in dom(F)$ and $y_1 \in F(x_1)$ such that

(v) x_1 is an $\epsilon - solution$ of F.

Then, for any real number $\lambda > 0$, there exists $x_2 \in dom(F)$, $y_2 \in F(x_2)$ such that

(vi) $y_2 \leq_K y_1$;

(vii) $d(x_1, x_2) \leq \lambda$;

(viii) x_2 *is an* $\epsilon -$ *solution of* F;

(ix) $\left[F(x) - y_2 + \frac{\epsilon}{\lambda} d(x, x_2)\alpha\right] \cap [-K] = \emptyset \; \forall x \in X \setminus \{x_2\}$.

It is interesting to see that in the proof of this theorem Chen and Huang use the original Ekeland's Variational Principle to achieve most of the conclusions. However in order to understand the proof, we will need some information regarding a scalarization function.

Definition 4.8 (nonlinear scalarization function)**.** Given (X, d) a complete metric space, and Y, K, and $\alpha \in IntK$ as stated earlier. A scalarization function $\phi : Y \to \mathbb{R}$ is defined as

$$\phi(y) = min \left\{ t \, : \, y \in t\alpha - K \right\}.$$

Some properties of this scalarization function are given below.

Lemma 4.4 ([3])**.** *The above function* ϕ *is a monotone, subadditive, and convex continuous function.*

Property. The following properties can be observed from the definition of the non-linear scalarization function.

$$\phi(y) \leq t \Leftrightarrow y \in \alpha t - K, \tag{4.3}$$

i.e.

$$\phi(y) > t \Leftrightarrow y \notin \alpha t - K. \tag{4.4}$$

Now we show the proof of the SVEVP by breaking it down into four parts below.

Proof of SVEVP. **Part I**: Let's consider a set valued mapping $F_1 = X \to 2^Y$ defined as

$$F_1(x) = \{y \in F(x) : y \leq_K y_1\} = F(x) \cap [y_1 - K].$$

The assumption that F is proper leads us to find $y_1 \in F_1(x_1)$. Thus $F(x) \cap [y_1 - K] \neq \emptyset$ proving $F_1(x)$ is proper. Also the assumption that $F(x)$ is compact valued and bounded below easily shows that $F_1(x)$ is also compact valued and bounded below.

Part II. Now define a real function

$$f(x) = min\{\phi(y - y_1) : y \in F_1(x)\} \quad if\ x \in dom(F_1)$$

$$f(x) = \infty \qquad\qquad\qquad if\ x \notin dom(F_1).$$

This is the function we wish to show that meets the conditions of the original Ekeland's Variational Principle. Therefore let's show that f is bounded below, lower semicontinuous, and proper.

PROPER: By the assumption that x_1 is an $\epsilon - solution$, there exists $y_1 \in F(x_1)$ such that $[F(x_1) - y_1] \cap [-K \setminus \{0\}] = \emptyset$ which yields $x_1 \in dom(F_1)$ and $f(x_1) = min\{\phi(y - y_1) : y \in F_1(x_1)\} \geq 0$. On the other hand if we choose $y = y_1$ in the above then we get that $f(x_1) \leq \phi(y_1 - y_1) = 0$. Putting these two inequalities together gives $f(x_1) = 0$.

BOUNDED BELOW: Since F_1 is bounded below, $\exists y \in Y$ such that $F_1(x) \subset y + K$ and $\phi(x)$ is monotone, then $f(x)$ must be bounded below.

LOWER SEMICONTINUOUS: In order to show f is lower semicontinuous, we only need to show that $\forall t \in \mathbb{R}$, $A = \{x \in X : f(x) \leq t\}$ is closed. Suppose $x_n \in A$ and $x_n \to x^\star$. We want to show $x^\star \in A$. By the definition of f, $\exists y_n \in F_1(x_n)$ where $y_n \leq_K y_1$ such that

$$\phi(y_n - y_1) \leq t \tag{4.5}$$

and

$$(y_n - y_1) \in -K. \tag{4.6}$$

Since F is upper semicontinuous at $x^\star$, for any neighborhood U around $F(x^\star)$, there exists a neighborhood V around $x^\star$ such that $F(x) \subset U$ $\forall x \in V$. Given $\delta > 0$, take a covering of $F(x^\star)$ with open balls having diameter equal to $\frac{\delta}{2}$. Then we denote the union of these open balls U. Take $\{x_{k_1}\} \in V$ with $F(x_{k_1}) \subset U$. Hence there is y_{k_1} in one of the balls such that if you pick another element z_{k_1} in that ball one will have $d(y_{k_1}, z_{k_1}) \leq \frac{\delta}{2}$. We can repeat this argument to get y_{k_n} and z_{k_n} for $n \in \mathbb{N}$ such that $d(y_{k_n}, z_{k_n}) \leq \frac{\delta}{2^n}$. Thus $y_{k_n} - z_{k_n} \to 0$. However since $F(x^\star)$ is compact, z_{k_n} has a convergent subsequence such that $z_{k_{n_l}} \to y^\star$ where $y^\star \in F(x^\star)$. From this, we can deduce that $y_{k_{n_l}} \to y^\star$. Finally, we know that $y^\star \leq_K y_1$ and $y^\star \in F_1(x^\star)$, hence $y^\star \in [F(x^\star) \cap (y_1 - K)]$. This implies by definition of f that $f(x^\star) = min\phi(y^\star - y_1) \leq t$. Therefore $x^\star \in A$ which demonstrates that A is closed.

Part III. Now apply the original EVP to f. Therefore $\forall \lambda > 0$, there exists $x_2 \in X$ such that

$$f(x_2) \leq f(x_1) = 0, \tag{a}$$

$$d(x_1, x_2) \leq \lambda, \tag{b}$$

$$f(x) + \frac{\epsilon}{\lambda} d(x, x_2) > f(x_2) \quad \forall x \in X \setminus \{x_2\}\,. \tag{c}$$

Part IV. Finally each of the conclusions (vi)-(ix) of this theorem 4.3 are reached. We omit the details here (see [3]).

□

SVEVP is most closely associated with the original EVP. Both versions require a complete metric space, the function to be bounded below, and the conclusions are similar. However the single-valued function is required to be lower semicontinuous and the multi-valued function is required to be upper semicontinuous. The set-valued version also has this extra requirement of the cone. This is necessary when dealing with a set-valued function. The compact valued condition on the function is required because of the set-valued nature of the functions. If we interpret the compact valued definition for single-valued functions, then every single-valued function is compact valued. Therefore in the single-valued setting, the requirement that the function be compact valued is redundant.

In the next section, for comparision reasons with the single-valued situation, we state corresponding set-valued versions of Caristi and Takahashi Theorems. The main idea is to make evident the similarities in the corresponding proofs of equivalence. For that, rather than going into the details of all definitions we will only state the corresponding theorems and the assumptions needed for these theorems to hold.

Generalized EVP

Assumptions (A1)**.** Let Y be a locally convex space, $K \subset Y$ is a nonempty, non-trivial, convex cone. Y is ordered by K. Let $K_0 \subset K$ be a K bound regular complete convex cone, $K_0 \cap -K \subset -K_0$. Let (X, r) be a complete K_0 metric space. Let $F : X \to 2^Y$ be a strict set-valued map such that $\forall x \in X$ $F(x)$ has the domination property and F is bounded below on $X_1 = \{x \in X : [y_0 - K] \cap F(x) \neq \emptyset\}$. Also one of the following conditions must also hold.

(I) K is closed and F is submonotone with respect to K, $\forall a \in X, r(a, .)$is continuous with respect to the topology of X induced by r.

(II) $\forall x_0 \in X, y_0 \in F(x_0)$, and a net $\{x_x\} \subset X$, $x_x \to \bar{x} \in X$ and $y_x \in F(x_x)$ such that $y_x - y_0 + \epsilon r(x_x, x_0) \in -K$, it follows $\exists \bar{y} \in F(\bar{x})$ such that $\bar{y} - y_0 + \epsilon r(\bar{x}, x_0) \in -K$ where $\epsilon > 0$.

Theorem 4.5 (GSVEVP)**.** *Let (A1) hold. Then* $\exists x^\star \in X$ *and* $y^\star \in E\left(F(x^\star)\right)$ *such that*

$$y^\star \leq_K y_0 \tag{4.7}$$

$$[F(x) - y^\star + r(x, x^\star)] \cap [-K] = \emptyset \ \forall x \in X \setminus \{x^\star\} \tag{4.8}$$

This theorem can be shown equivalent with the following set-valued Caristi Fixed Point Theorem.

Theorem 4.6 (SVCFPT)**.** *Let (A1) hold. Let* $T : X \to 2^Y$ *be a set valued map such that*

(b) $\forall \bar{x} \in X_1, \forall \bar{y} \in F(\bar{x})$ *and* $\bar{y} \leq_K y_0$, $\exists x \in T(\bar{x})$ *and* $y \in F(x)$ *such that*

$[y - \bar{y} + r(x, \bar{x})] \in -K$.

Then $\exists x^\star \in X_1$ *and* $y^\star \in F(x^\star)$ *such that*

$$y^* \leq_K y_0 \tag{4.9}$$

$$x^\star \in T(x^\star) \tag{4.10}$$

Next we provide the proofs of equivalence between theorems 4.5 and 4.6. Again we emphasize the similar nature of the proofs with the single-valued ones.

SVEVP⇒SVCFPT

Proof. From Theorem 4.5 we know $\exists x^\star \in X$ and $y^\star \in E\left(F(x^\star)\right)$ such that

$$y^\star \leq_K y_0, \tag{4.11}$$

$$[F(x) - y^\star + r(x, x^\star)] \cap [-K] = \emptyset \; \forall x \in X \setminus \{x^*\}. \tag{4.12}$$

We want to show $x^* \in T(x^*)$. By contradiction assume $x \in T(x^*) \setminus \{x^\star\}$ and $y \in F(x)$. Then by Theorem 4.6 $y - y^\star + r(x, x^\star) \in -K$, but this contradicts the above line. □

SVEVP⇐SVCFPT

Proof. Define $T(x) := \{w \in X \setminus \{x\} : \exists y_1 \in F(w) \, and \, y_2 \in F(x) \; y_1 - y_2 + r(x, w) \in -K\}$.

By contradiction suppose that 4.5 does not hold, meaning $\forall \bar{x} \in X_1, \bar{y} \in F(\bar{x})$ with

$\bar{y} \leq_K y_0, \exists w \in X \setminus \{\bar{x}\}$ and $y_1 \in F(w)$ such that

$$y_1 - \bar{y} + r(\bar{x}, w) \in -K$$

Since this satisfies 4.6 there should exist $x^\star \in T(x^\star)$. However by definition of $T(x)$, $x^\star$ cannot be in the set. Thus contradiction.

$\square$

Finally, Chen and Huang also give a set-valued Takahashi Theorem under the same (A1) assumptions and show its equivalence to SVEVP.

Theorem 4.7 (SVT)**.** *Let (A1) hold. In addition assume*

$\forall \bar{x} \in X, \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_K y_0$, there exists $x_1 \in X_1$ such that $[F(x_1) - \bar{y}] \cap [-K \setminus \{0\}] \neq \emptyset$, it follows that $\exists x_2 \in X_1 \setminus \{\bar{x}\}$ and $y_2 \in F(x_2)$ such that $y_2 - \bar{y} + r(\bar{x}, x_2) \leq_K 0$.

Then $\exists x^\star \in X_1$ and $y^\star \in E\left(F(x^\star)\right)$ with $y^\star \leq_K y_0$ such that

$$[F(x) - y^\star] \cap [-K \setminus \{0\}] = \emptyset,\ \forall x \in X \setminus \{x^\star\}.$$

SVEVP⇒SVT

Proof. From Theorem 4.5 $\exists x^\star \in X_1$ and $y^\star \in E\left(F(x^\star)\right)$ such that

$$y^\star \leq_K y_0$$

$$[F(x) - y^\star + r(x, x^\star)] \cap [-K] = \emptyset \quad \forall x \in X \setminus \{x^\star\}\,.$$

We now want to show that

$$[F(x) - y^\star] \cap [-K \setminus \{0\}] = \emptyset \quad \forall x \in X \setminus \{x^\star\}\,.$$

Otherwise $\exists x_1 \in X \setminus \{x^\star\}$ and $y_1 \in F(x_1)$ such that $y_1 - y^\star \in -K \setminus \{0\}$. Then $x_1 \in X_1$ and $y_1 \leq_K y_0$. Using the added assumption of Theorem 4.7, there exist $x_2 \in X_1 \setminus \{x^\star\}$ and $y_2 \in F(x_2)$ such that

$$y_2 - y^\star + r(x^\star, x_2) \leq_K 0.$$

However this contradicts $[F(x) - y^\star + r(x, x^\star)] \cap [-K] = \emptyset \quad \forall x \in X \setminus \{x^\star\}\,.$

□

As in the single-valued case we only show the forward implication. Again note the similarity in the proof here and the proof for the single-valued version. As already mentioned earlier, the (A1) assumptions are very general and require more background in set-valued functions and cone metric spaces. Chen and Huang have found the conditions that make the proofs of the set-valued versions follow a similar path to the proofs of the single-valued versions. Perhaps such a technical and general setting is indeed necessary to create connections to Caristi and Takahashi in the set-valued case.

CHAPTER 5

CONCLUSION

Even after the publication of EVP in 1972, it still has wide applications across various areas of mathematics. One of the most important ideas of EVP is that in the absence of a known minimum, one can use EVP to get close to a minimum. An interpretation of this idea when a function is bounded below and differentiable in Banach space provides that the derivative must also be small. This result is used extensively in optimization and control theory. In the exposition of EVP, we first conveyed the importance of EVP's connection with other known results: Caristi's and Clarke's Fixed Point Theorems and Takahashi's Minimization Theorem. Then in Chapter 3, we explored the background that led to Pontryagin's minimum principle. In addition, we gave Ekeland's approximate version of Pontryagin Principle.

The initial applications of EVP pose the question, "Can the theorem be generalized by changing from a single-valued function to a multi-valued function?" Unfortunately, a direct transformation using the same proof techniques of Ekeland are insufficient. In Chapter 4, we discovered that many authors were successful in giving various set-valued EVP's. The differences presented in these versions are the assumptions on the space, metric and function. Chen and Huang formulated $\epsilon - solutions$ of a set-valued function in which they use for a closely associated strong form of EVP for set-valued functions. They also unite other authors work to give a set-valued weak form of EVP. By comparing the original EVP wth Chen and Huang's generalized version we recognize that despite different assumptions, the proofs of equivalence with corresponding Caristi's and Takahashi's results are remarkably similar.

Recent work with set-valued EVP versions have been used in applications to vector optimization problems and vector equilibrium problems (see [1] and references therein). Results related to existence and well posedness in regards to vector optimization are current areas of research (e.g.[4]). With the multitude of EVP set-valued versions presented over the past two decades, the necessity has arisen to sort out the relationships amongst them. Some EVP versions have be proven as merely more specific cases through various lemmas. However others stand alone in sharp contrast from one another. For example in [9], the authors note how their EVP version differs from that of Chen and Huang. This has created a splintering of generalized EVPs. However each version has particular uses. In summary, EVP continues to be a necessary tool for various mathematical areas of research.

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VITA

Graduate College
University of Nevada, Las Vegas

Jessica Robinson

Home Address:
P.O. Box 19280
Jean, Nevada 89019

Degrees:
Bachelor of Science, Mathematics, 2010
Westminster College, Salt Lake City, Utah

Thesis Title: A Summary of Ekeland's Variational Principle and Related Theorems and Applications

Thesis Examination Committee:
Chairperson, Dr. David Costa, Ph.D.
Committee Member, Dr. Xin Li, Ph.D.
Committee Member, Dr. Hossein Tehrani, Ph.D.
Graduate Faculty Representative, Dr. Paul Schulte, Ph.D.