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Subcube embeddability and fault tolerance of augmented hypercubes

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SUBCUBE EMBEDDABILITY AND FAULT TOLERANCE OF
AUGMENTED HYPERCUBES

by

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Bachelor of Science in Engineering
University of Peradeniya, Sri Lanka
2002

A thesis submitted in partial fulfillment
of the requirement for the

**Masters of Science Degree in Electrical Engineering
Department of Electrical and Computer Engineering
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**Graduate College
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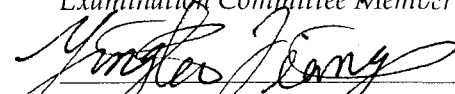
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ABSTRACT

Subcube Embeddability and Fault Tolerance of Augmented Hypercubes

by

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Hypercube networks have received much attention from both parallel processing and communications areas over the years since they offer a rich interconnection structure with high bandwidth, logarithmic diameter, and high degree of fault tolerance. They are easily partitionable and exhibit a high degree of fault tolerance. Fault-tolerance in hypercube and hypercube-based networks received the attention of several researchers in recent years.

The primary idea of this study is to address and analyze the reliability issues in hypercube networks. It is well known that the hypercube can be augmented with one dimension to replace any of the existing dimensions should any dimension fail. In this research, it is shown that it is possible to add i dimensions to the standard hypercube, Q_n to tolerate $(i - 1)$ dimension failures, where $0 < i \leq n$. An augmented hypercube, $Q_n^{+(n)}$ with n additional dimensions is introduced and compared with two other hypercube networks with the same amount of redundancy. Reliability analysis for the three hypercube networks is done using the combinatorial and Markov modeling. The MTTF

values are calculated and compared for all three networks. Comparison between similar size hypercube networks show that the augmented hypercube is more robust than the standard hypercube.

As a related problem, we also look at the subcube embeddability. Subcube embeddability of the hypercube can be enhanced by introducing an additional dimension. A set of new dimensions, characterized by the Hamming distance between the pairs of nodes it connects, is introduced using a measure defined as the magnitude of a dimension. An enumeration of subcubes of various sizes is presented for a dimension parameterized by its magnitude. It is shown that the maximum number of subcubes for a Q_n can only be attained when the magnitude of dimension is $n-1$ or n . It is further shown that the latter two dimensions can optimally increase the number of subcubes among all possible choices.

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CHAPTER 1

INTRODUCTION

Computer interconnection networks came into existence due to a large demand for transferring information. The rapid advancement in distributed databases and distributed computations ensures that interconnection networks will remain a key component in the successful performance of any computer system. Many interconnection network topologies have been proposed in the literature. Among them, the hypercube is one of the most popular and has been studied extensively in different aspects [1-7]. It has been long known as an attractive topology in distributed parallel systems, since it offers a rich connection structure.

The hypercube possesses several attractive attributes such as a logarithmic node degree and diameter, a large bandwidth, a homogeneous symmetric structure, and a high degree of fault tolerance. Many interconnection network topologies, such as meshes, rings, trees, can be embedded in the hypercube [2-8]. A number of hypercube machines have been designed and implemented since the 1980's: Ncube's 10, 2, and 3 [9]; Intel's iPSC series (iPSC/2 and iPSC/860) [10]; Ametek S-14/16 and 64 [11]; and JPL Mark III [12] are among those that have from 128 to 4096 node processors.

The hierarchical nature of the hypercube allows its partitioning into subnetworks (subcubes) of smaller size, but of the same topology. Therefore, jobs of different sizes

requiring the hypercube topology can be easily hosted on this subnetwork simultaneously, provided that there are a sufficient number of free subcubes to accommodate the arriving jobs. The techniques followed to host the jobs on a given network are referred to as *Job Scheduling*. Since 1985 several papers have addressed the problem of *Job Scheduling* (see [13-15] for example).

No matter how efficient and adaptive the job scheduling technique is, there will still be fragmentation as jobs with different execution times and deadlines arrive and leave the network. In many cases, a job needs to be queued due to the absence of a free subcube of the required size. Furthermore, if a higher priority job with a short deadline arrives but cannot be allocated, other job(s) need(s) to be pre-empted to free up a subcube of the desired size for the high priority job. Pre-emption of jobs requires interruption and migration of the jobs out of the system and queuing them in a buffer, so they can be later on moved back into the system after the high priority jobs ran to completion and left the system.

It is clear that if the fragmented processors (or subcubes) could be combined to makeup a larger subcube in more ways than what is offered in the standard hypercube, many benefits would result. The attained benefits include reduced overall running time for the jobs, number of pre-emptions, and communication overhead, and increased processor utilization. A natural way to achieve this is to provide more interconnections among the processors connected based on the hypercube than what exists in the standard topology.

The addition of links to the hypercube to achieve higher subcube embeddability can be done in many ways. To preserve node symmetry and regularity of the network, all

new links must connect pairs of nodes whose binary labels are different in a consistent way (ex. they all complement the second and third bits of the node label). These links are said to be *parallel* and the set of new links (there are 2^{n-1} of them) belong to the same *dimension*. In [16], the extra links connect processors whose labels are 1's complement of each other (e.g. the farthest nodes from each other). The author enumerated the additional subcubes resulting from this augmentation and showed how additional links can improve the overall network performance. In this thesis, we generalize the idea by adding a new dimension of arbitrary magnitude (to be defined later) to the network and, for each case, enumerate the additional subcubes that will result after dimension enhancement.

Fault-tolerance in hypercube and hypercube-based networks received the attention of several researchers in recent years [16-21]. The fault tolerance of networks is generally measured by how much of the network structure is preserved in the presence of a given number of node and/or link failures. In this study, we analyze the reliability issues in a fault tolerant hypercube network. It is assumed that only links fail and a dimension is declared faulty if one or more of its links fail. This assumption relies on the fact that in most applications, the communication takes place between one set of the nodes and the other set along parallel links that collectively belong to the same dimension. We show that it is possible to add n dimensions to the hypercube to replace $(n-1)$ consecutive dimension failures. We assess the reliability of the hypercube which is subject to dimension failures using the combinatorial and Markov modeling. We determine the reliability for three networks with the same amount of dimension redundancy, (i) a

network enhanced with additional dimensions in a straightforward manner; (ii) our proposed network; and (iii) an optimal network.

CHAPTER 2

AUGMENTATION OF HYPERCUBES

In this chapter, we consider an augmented hypercube network obtained by introducing extra dimensions to achieve higher subcube embeddability. By adding extra dimension we maintain a constant number of parallel links in each dimension to preserve node symmetry and regularity of the hypercube network structure. We begin the chapter with the necessary preliminaries to this study.

2.1 Notation and Terminology

An n -dimensional hypercube, denoted by Q_n , can be modeled as a graph $G_n(V, E)$ with $|V| = N = 2^n$ nodes and $|E| = n \times 2^{n-1}$ edges. Each node represents a processor and each edge represents a link between a pair of processors. Let u and v be two nodes in $G_n(V, E)$. If an edge $e = (u, v) \in E$, then the nodes u and v are said to be *adjacent* and the edge e is said to be *incident* on these nodes.

The nodes of Q_n can be uniquely labeled by binary numbers from 0 to $2^n - 1$. An edge (link) is labeled i if it connects two nodes whose labels differ in the i -th bit, $0 \leq i < n$. The set of links labeled i collectively form the dimension i . Consequently, there are 2^{n-1} edges in (or along) dimension i . Figure 2.1 shows the hypercube of dimensions 1, 2, 3, and 4.

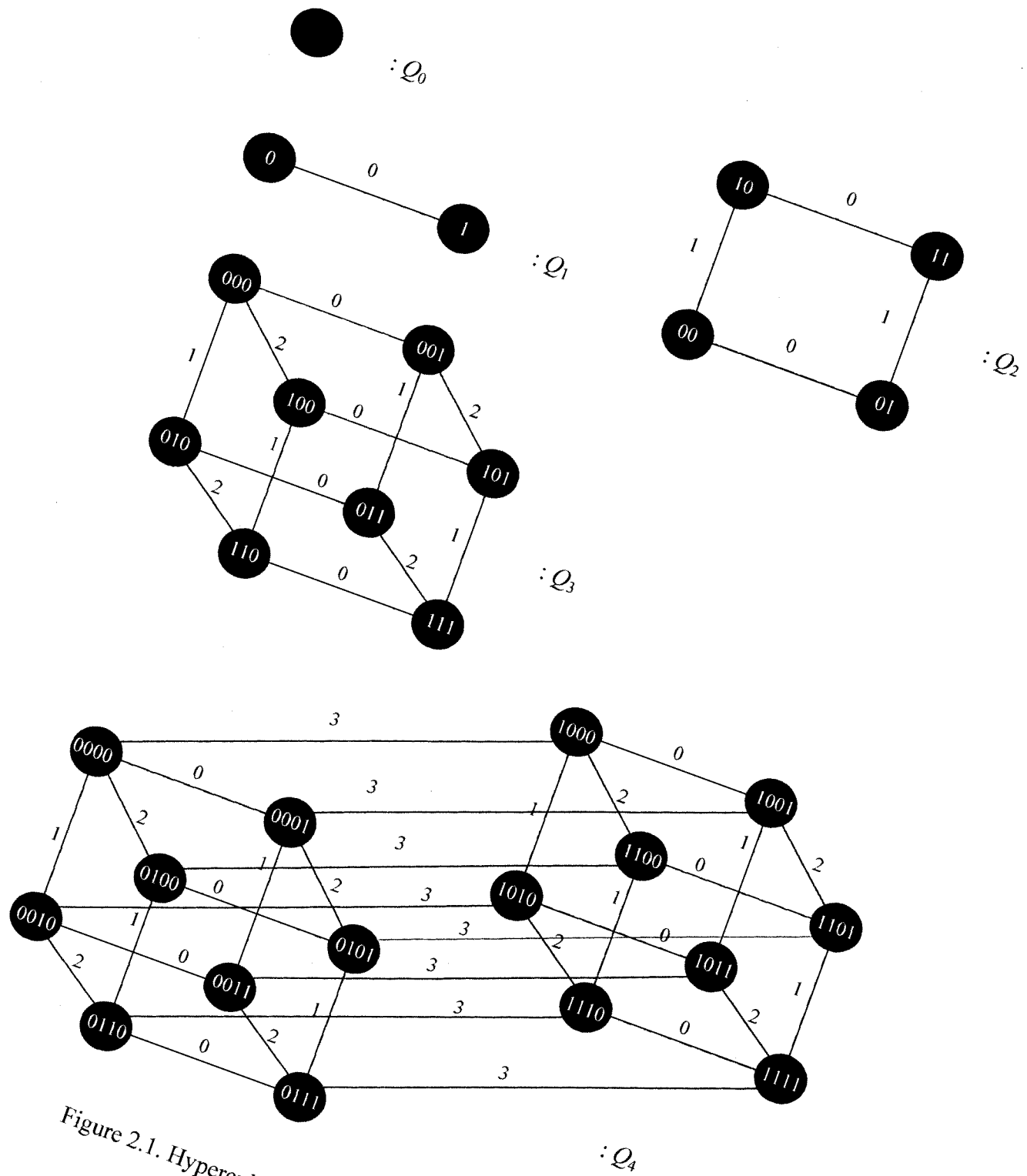


Figure 2.1. Hypercube of dimensions 1, 2, 3 and 4.

An n -dimensional *folded hypercube* ($FHC(n)$) is an n -dimensional hypercube to which extra links are added connecting every pair of nodes that are bit-wise complements of each other [16]. A 3-dimensional folded hypercube is illustrated in Figure 2.2. Dashed lines represent the complementary links.

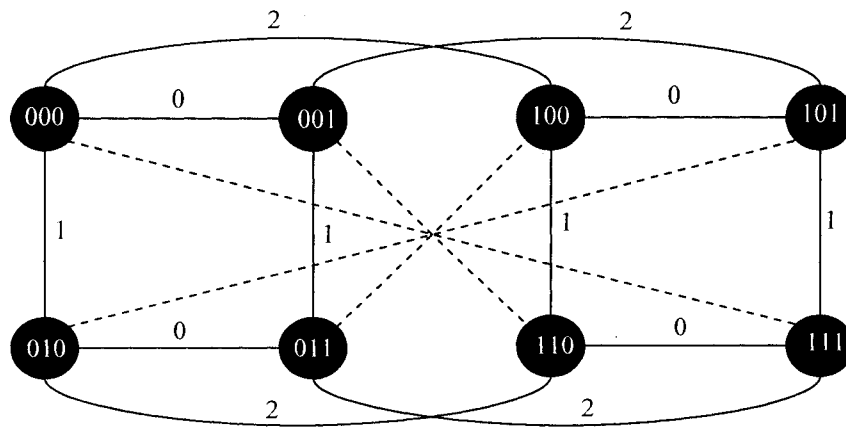


Figure 2.2. Folded hypercube of dimension 3.

Let Σ be the ternary set $\{0,1,X\}$, where X is a *don't care* symbol. Then, every *subcube* of a Q_n can be uniquely represented by a string of symbols in Σ . Such a string of symbols is called the *label* of the corresponding subcube. Each symbol in the label of a subcube corresponds to a certain dimension, starting from the right-most symbol in the label as dimension 0. If dimension i in the label of a subcube corresponds to symbol X , then the subcube is said to *span* dimension i ; otherwise dimension i is fixed to its corresponding symbol (i.e. 0 or 1). A Q_k contained in a Q_n spans k dimensions (i.e. has

k X 's in its label) and has $n-k$ fixed dimensions, where $0 \leq k < n$. Note that the number of X 's in the label of each subcube determines its dimensionality. For instance, a subcube of a Q_5 labeled as $0X00X$ is of dimension 2, spans dimensions 0 and 3 and has its 1st, 2nd, and 4th dimensions set to 0. A subcube contains a node if its label matches the label of the node. For example, the above subcube contains nodes with labels 00000, 01000, 00001, and 01001.

2.2 Dimension Enhancement

The 2^n nodes can be connected in many ways to form a hypercube topology, subject to the constraint that the set of links are comprised of n dimensions, with each dimension containing 2^{n-1} parallel links. Conventionally, each set of n dimensions affects exactly one of the n bits so that each link (labeled i) is incident on a pair of nodes whose labels differ in the i -th bit, where $0 \leq i < n$. The set of dimensions defined as such is denoted as $\{0, 1, \dots, n-1\}$. Nonetheless, a dimension may be defined to affect more than one bit of the node.

Lemma 1. There are $(2^n - 1)$ ways to define a set of dimensions in Q_n .

Proof: Denote a dimension as an n -bit vector $b_{n-1}b_{n-2} \dots b_1b_0$ wherein $b_i = 1$ only if the pair of nodes, incident on each of the parallel links in the dimension, differ in the i -th bit. There are 2^n such vectors; and excluding the all-zero vector, the number of different dimensions can be easily obtained. Denote the set of $(2^n - 1)$ dimensions by Δ_n .

Corollary 1. There are a total of $2^n - n - 1$ new dimensions that can be added to the standard hypercube Q_n .

Denote an n -dimensional hypercube with i additional dimensions by $Q_n^{+(i)}$. The *augmented hypercube* $Q_n^{+(i)}$ contains 2^n nodes and $(n+i)$ dimensions.

For convenience, we denote a dimension as $i_1i_2\cdots i_m$ whose corresponding vector has non-zero bits in positions i_1, i_2, \dots, i_m , where $i_1 < i_2 < \dots < i_m$ and $m < n$. For example, in a Q_4 , $\Delta_4 = \{0,1,2,3,01,02,03,12,13,23,012,013,023,123,0123\}$; the link labeled 012 (belonging to dimension 012) is incident on node (0000) and is also incident on node (0111).

Definition 1. The *magnitude* of a dimension is the number of differing bits in the labels of the nodes that parallel links in that dimension join.

For example in a Q_5 , the dimension 013 is of magnitude 3 and its links connect nodes of the form $(c_4c_3c_2c_1c_0)$ to nodes of the form $(c_4\overline{c_3}\overline{c_2}c_1\overline{c_0})$.

When specifying the magnitude, we use the terms links and dimensions interchangeably. It is understood that a dimension consists of 2^{n-1} links, with each link having the same magnitude as the dimension itself. For example, all dimensions in the standard hypercube have a magnitude of 1, the complementary dimension in $FHC(n)$ has a magnitude of n , and a dimension labeled $i_1i_2\cdots i_m$ has a magnitude of m .

Definition 2. A *k-covering* is a set of k distinct dimensions, where each dimension corresponds to one element in the set Δ_n .

Note that for a k -covering to produce a Q_n , it is necessary (but not sufficient) that $k \geq n$.

Definition 3. The *bit-wise union* (denoted by \cup) of two or more dimensions is another dimension, whose set of digits is the union of the digits of all dimensions composing the union.

For example, let $d_1 = 014$, $d_2 = 013$, and $d_3 = 2$. Then $d_1 \cup d_2 \cup d_3 = 01234$.

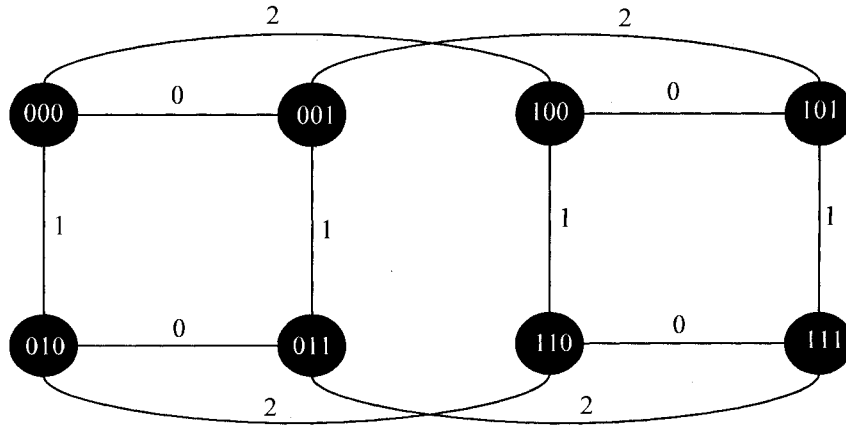
Observe that the bit-wise union applies only to bit positions and not to the actual links. In other words, the number of links in a dimension is constant irrespective of whether the dimension being the union of other dimensions or not. In the preceding example, the number of links in d_1, d_2, d_3 , and d_4 is the same and equal to 16.

A *minimal k -covering* is a set $\{d_1, d_2, \dots, d_k\}$ consisting of k dimensions such that $d_{i_1} \cup d_{i_2} \dots \cup d_{i_l} \neq d_p$ for every l and p , where $1 \leq i_1 < i_2 < \dots < i_l \leq k$, $p \in \{1, 2, \dots, k\}$ and $l \neq p$. In other words, the union of two or more dimensions will not generate an existing dimension in a minimal covering set. For example, the set $\{0, 1, 2, 013\}$ is a minimal 4-covering, whereas the set $\{0, 1, 2, 012\}$ is not.

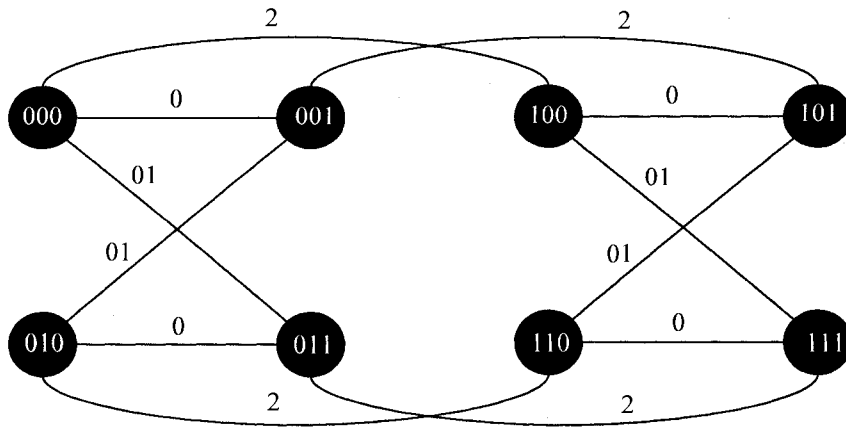
Corollary 2. A k -covering can generate a Q_k only if it is minimal.

For example, the set $\{0, 1, 2, 012\}$ can only generate a Q_3 even though it consists of 4 dimensions.

Definition 4. Two subcubes are said to be *node-identical* if they consist of exactly the same nodes, but a different covering (e.g. different set of dimensions). For example, the six subcubes containing the all zero node, (000), and spanning dimensions $\{0, 1, 2\}$, $\{0, 01, 2\}$, $\{0, 1, 02\}$, $\{0, 1, 12\}$, $\{0, 1, 012\}$, and $\{0, 01, 012\}$ are node-identical. Figure 2.3 shows two node-identical Q_3 's with two different coverings of $\{0, 1, 2\}$ and $\{0, 01, 2\}$.



{0, 1, 2}



{0, 01, 2}

Figure 2.3. Node-identical Q_3 's with different coverings

The significance of node-identical subcubes becomes clear when the number of new subcubes generated as the result of adding a new dimension is determined. From the subcube allocation viewpoint, node-identical subcubes are equivalent and should be counted once. Note that node-identical subcubes may differ in one or more different dimensions.

Lemma 2. Two minimal coverings will generate node-identical subcubes only if they contain the same number of dimensions and cover the same digits.

Proof. If coverings with the same number of dimensions are minimal, they generate the same-size subcube. Furthermore, since each covering includes the same digits, the same bit positions will be affected as a result of traversing the dimensions in the covering. Thus, the subcubes will be node-identical.

2.3 Subcubes in $Q_n^{+(1)}$

We wish to add a new dimension to Q_n to increase the number of available subcubes, since we are naturally seeking the dimension that produces the maximum number of new subcubes.

Due to the link symmetry of the hypercube, all dimensions of the same magnitude behave identically and would generate the same number of new subcubes, once added to the existing n dimensions in Q_n . For instance, in the same way the two dimensions 0123 and 2457 will contribute to the generation of new subcubes in Q_8 . This claim can be trivially shown true by re-labeling the dimensions.

In the following we enumerate the number of subcubes attained as a result of introducing a dimension of magnitude m . We will then prove that the two dimensions whose addition will produce the maximum number of subcubes have magnitudes of $(n-1)$ or n .

Consider the standard hypercube Q_n with the covering $\{0,1,\dots,n-1\}$. We wish to augment the covering with an additional dimension selected from the set of $(2^n - n - 1)$ new dimensions. Due to the edge transitivity of the hypercube, with no loss of generality,

we assume the dimension to be added is $\{01\cdots m-1\}$ with $1 < m \leq n$. We wish to enumerate the additional subcubes Q_i , $0 < i < n$ that emerge as a result of this enrichment. The new covering will be $\{0,1,\cdots,n-1,01\cdots m-1\}$ (note that this covering is not minimal; e.g. it produces a Q_n , even though it has $(n+1)$ dimensions).

Denote by $N_m(Q_i)$ the number of Q_i 's that exist in a Q_n augmented by a dimension of magnitude m . For $m < i < n-1$, there are i dimensions to be selected from the available $(n+1)$ dimensions. The remaining $(n-i)$ fixed positions can take on 0 or 1. Of all the choices, if the covering includes the set of dimensions $\{0,1,\cdots,m-1,01\cdots m-1\}$, it will produce smaller subcubes and may not count. The remaining $(i-m-1)$ dimensions can be taken from the set $\{m,m+1,\cdots,n-1\}$. There are C_{i-m-1}^{n-m} such redundant coverings which should be excluded from the total count. Another mutually exclusive set of redundant coverings occurs when they include the set $\{0,1,2,\cdots,p-1,p+1,\cdots,m-1,01\cdots m-1\}$, where $0 \leq p \leq m-2$. All these coverings are equivalent to $\{0,1,2,\cdots,m-1\}$ and should be excluded from the count, i.e. mC_{i-m}^{n-m} . For $i = m$, the redundant coverings include the set $\{0,1,2,\cdots,p-1,p+1,\cdots,m-1,01\cdots m-1\}$, where $0 \leq p \leq m-2$ and there are m such coverings. For $i < m$, there is no possibility of generating a redundant covering and $N_m(Q_i) = (C_i^{n+1})2^{n-i}$. For $i = n-1$, distinct subcubes can be obtained by picking $(n-1)$ dimensions out of the available $(n+1)$ dimensions and excluding any possible set that cannot make Q_{n-1} 's. It follows:

If $n < m+1$,

$$N_m(Q_{n-1}) = (C_{n-1}^{n+1})2$$

If $n = m + 1$,

$$N_m(Q_{n-1}) = (C_{n-1}^n + C_{n-2}^n - m)2 = (C_{n-1}^{n+1} - m)2$$

If $n > m + 1$,

$$\begin{aligned} N_m(Q_{n-1}) &= (C_{n-1}^n + C_{n-2}^n - C_{n-m-2}^{n-m} - mC_{n-m-1}^{n-m})2 \\ &= [C_{n-1}^{n+1} - C_{n-m-2}^{n-m} - m(n-m)]2 \end{aligned}$$

Summarizing the above results we have the following equations:

For $0 < i < n - 1$:

$$\begin{aligned} N_m(Q_{n-2}) &= (C_{n-2}^n + C_{n-3}^n - C_{n-m-3}^{n-m} - mC_{n-m-2}^{n-m})2^2 \\ &= [C_{n-2}^{n+1} - C_{n-m-3}^{n-m} - mC_{n-m-2}^{n-m}]2^2 \end{aligned}$$

$$\begin{aligned} N_m(Q_{n-3}) &= (C_{n-3}^n + C_{n-2}^n - C_{n-m-4}^{n-m} - mC_{n-m-3}^{n-m})2^3 \\ &= [C_{n-2}^{n+1} - C_{n-m-3}^{n-m} - mC_{n-m-2}^{n-m}]2^3 \end{aligned}$$

.....

$$\begin{aligned} N_m(Q_i) &= (C_i^n + C_{i-1}^n - C_{i-m-1}^{n-m} - mC_{i-m}^{n-m})2^{n-i} \\ &= (C_i^{n+1} - C_{i-m-1}^{n-m} - mC_{i-m}^{n-m})2^{n-i} \end{aligned}$$

.....

$$\begin{aligned} N_m(Q_{m+1}) &= [C_{m+1}^n + C_m^n - 1 - m(n-m)]2^{n-m-1} \\ &= [C_{m+1}^{n+1} - 1 - m(n-m)]2^{n-m-1} \end{aligned}$$

$$N_m(Q_m) = (C_m^n + C_{m-1}^n - C_{m-1}^m)2^{n-m} = (C_m^{n+1} - m)2^{n-m}$$

$$N_m(Q_{m-1}) = (C_{m-1}^n + C_{m-2}^n)2^{n-m+1} = (C_{m-1}^{n+1})2^{n-m+1}$$

.....

$$N_m(Q_i) = (C_i^n + C_{i-1}^n)2^{n-i} = (C_i^{n+1})2^{n-i}$$

.....

$$N_m(Q_2) = (C_2^n + C_1^n)2^{n-2} = (C_2^{n+1})2^{n-2}$$

$$N_m(Q_1) = (C_1^n + 1)2^{n-1} = (C_1^{n+1})2^{n-1}$$

For example, consider a Q_4 augmented with the dimension 012. The covering is $\{0,1,2,3,012\}$. The seven non-redundant dimensions for subcube Q_3 are $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,3\}$, $\{0,3,012\}$, $\{1,2,3\}$, $\{1,3,012\}$ and $\{2,3,012\}$.

2.3.1 Data Analysis

From the equations obtained in the previous section, it is clear that the number of subcubes attained from $Q_n^{+(1)}$ is dependent on the parameters n , i , and m . Hence a data analysis is necessary.

Table 2.1 lists the subcubes (Q_i 's) attained from $Q_4^{+(1)}$ for different values of i and m .

Table 2.1: Subcubes (Q_i 's) attained from $Q_4^{+(1)}$

Magnitude of the added dimension	$m = 2$	$m = 3$	$m = 4$
Covering	$\{0, 1, 2, 3, 01\}$	$\{0, 1, 2, 3, 012\}$	$\{0, 1, 2, 3, 0123\}$
Subcubes	Q_1	$\{0\}, \{1\}, \{2\}, \{3\}, \{01\}$	$\{0\}, \{1\}, \{2\}, \{3\}, \{0123\}$
	Q_2	$\{0, 1\}, (0, 2), \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 01\}, \{3, 01\}$	$\{0, 1\}, (0, 2), \{0, 3\}, \{0, 012\}, \{1, 2\}, \{1, 3\}, \{1, 012\}, \{2, 3\}, \{2, 012\}, \{3, 012\}$
	Q_3	$\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}, \{2, 3, 01\}$	$\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 0123\}, \{0, 2, 3\}, \{0, 2, 0123\}, \{0, 3, 0123\}, \{1, 2, 3\}, \{1, 2, 0123\}, \{1, 3, 0123\}, \{2, 3, 0123\}$

Table 2.2 gives the number of subcubes (Q_i 's) (all the values should be multiplied by 2^{8-i}) obtained from $Q_8^{+(1)}$ for different values of i and m . Figure 2.4 illustrates the number of subcubes obtained from $Q_8^{+(1)}$.

Table 2.2. Number of subcubes (Q_i 's) obtained from $Q_8^{+(1)}$

		Dimension of the subcube (i)						
		1	2	3	4	5	6	7
Magnitude of the added dimension (m)	2	9	34	71	90	71	34	9
	3	9	36	81	110	91	44	11
	4	9	36	84	122	109	56	14
	5	9	36	84	126	121	68	18
	6	9	36	84	126	126	78	23
	7	9	36	84	126	126	84	29
	8	9	36	84	126	126	84	36

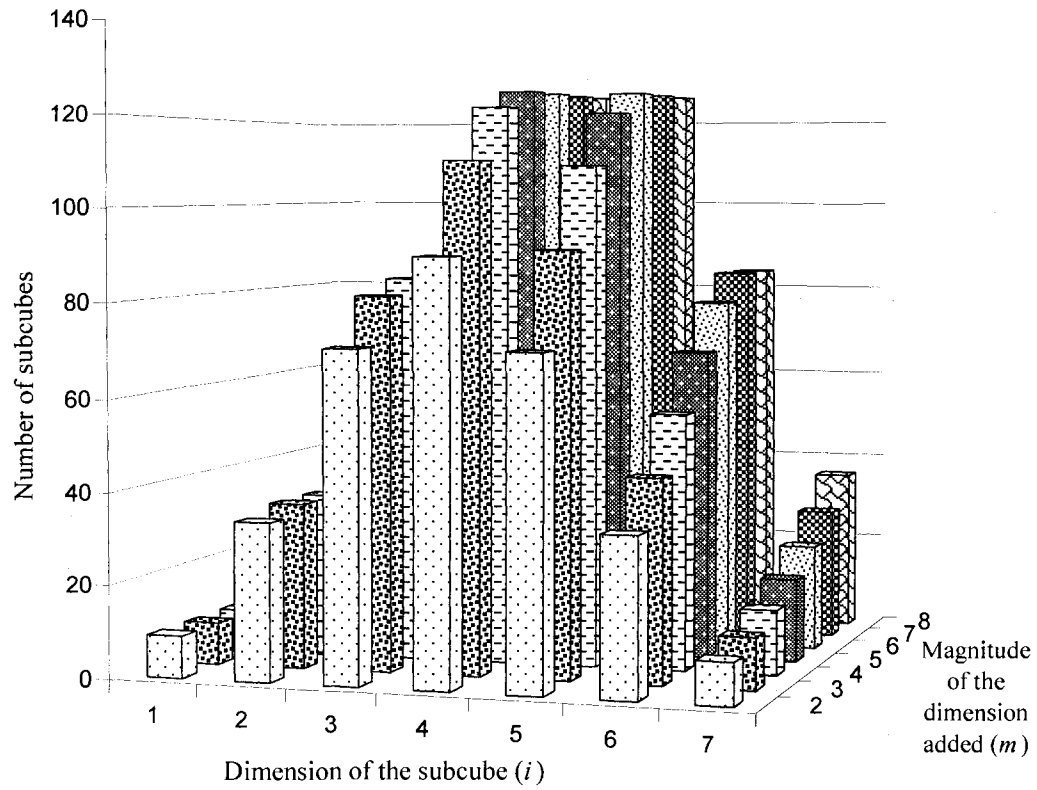


Figure 2.4. Number of subcubes obtained from $Q_8^{+(1)}$

Observe that the number of subcubes of dimension i increases as the magnitude of the added dimension increases. For $i < m$ The number of subcubes of dimension i are the same regardless of the value of m .

2.3.2 Special Cases

[i] $m = 2, \{0, 1, \dots, n-1, 0, 1\}$

$$N_2(Q_{n-1}) = (C_1^n + 1)2^1 = (C_1^{n+1})2$$

$$N_2(Q_{n-2}) = (C_{n-2}^n + C_{n-3}^{n-2})2^2$$

$$N_2(Q_{n-3}) = (C_{n-3}^n + C_{n-4}^{n-2})2^3$$

.....

$$N_2(Q_2) = (C_2^n + C_1^{n-2})2^{n-2}$$

$$N_2(Q_1) = (C_1^n + 1)2^{n-1}$$

[ii] $m = n-1, \{0,1,\dots,n-1,01\dots n-2\}$

$$N_{n-1}(Q_i) = (C_i^{n+1})2^{n-i}, 0 < i < n-1.$$

[iii] $m = n, \{0,1,\dots,n-1,01\dots n-1\}$

$$N_n(Q_i) = (C_i^{n+1})2^{n-i}, 0 < i < n. \quad (\text{in agreement with results in [9]})$$

Summarizing the above results, we will have the following claims:

Theorem 1. Consider a $Q_n^{+(1)}$ with the covering $\{0,1,\dots,n-1,012\dots m-1\}$. The optimal number of extra subcubes, Q_i 's (e.g. $N_m(Q_i) = C_i^{n+1} \times 2^{n-i}$) are obtained for $0 < i < m$.

proof. To have node-identical and non-minimal coverings, the covering of Q_i should have at least m number of dimensions. Since $i < m$ it is not possible to have redundant coverings.

Lemma 3. Consider a Q_n with the covering $\{0,1,\dots,n-1,012\dots m-1\}$. The optimal number of extra subcubes Q_i 's (e.g. $N_m(Q_i) = C_i^{n+1} \times 2^{n-i}$) are obtained when $m = n-1$ or $m = n$, for $0 < i < n-1$.

Proof. It is impossible to have a covering of i dimensions from the sets $\{0,1,\dots,n-1,01\dots n-2\}$ and $\{0,1,\dots,n-1,01\dots n-1\}$ which is not minimal.

2.4. Subcubes in $Q_n^{+(2)}$

In this section, we add the two dimensions $012\dots n-1$ and $0,1,\dots,n-2$ to Q_n . The reason for this selection is clear. The two dimensions should considerably improve the

subcube embeddability of the hypercube as they both generate the maximum number of subcubes when individually added to the network.

The new covering for the hypercube after augmentation is $\{0,1,\dots,n-1,01\dots n-2,01\dots n-1\}$. Clearly, when enumerating the subcubes, redundant coverings arise when both $012\dots n-1$ and $0,1,\dots,n-2$ are included in the covering of the subcubes (e.g. each of these dimensions cannot individually produce a redundant covering together with the standard dimensions). It is not difficult to see that the subcubes, whose covering includes the set $\{n-1,01\dots n-2,01\dots n-1\}$, are redundant and may not be counted. Denote the number of Q_i 's produced as a result of adding the said two dimensions by $N_{n-1,n}(Q_i)$. To construct a Q_i , $2 < i < n-1$, i dimensions would have to be selected out of the available $n+2$ dimensions. To enumerate redundant cubes, the 3 dimensions would have to be $n-1,01\dots n-2,01\dots n-1$, and the remaining $(i-3)$ dimensions can be selected from the remaining $(n-1)$ standard dimensions (i.e. $0,1,\dots,n-2$). It follows that:

$$N_{n-1,n}(Q_{n-1}) = (C_1^{n+2}) \times 2$$

.....

$$N_{n-1,n}(Q_i) = (C_i^{n+2} - C_{i-3}^{n-1}) 2^{n-i}$$

.....

$$N_{n-1,n}(Q_2) = (C_2^{n+2} - n - 1) 2^{n-2}$$

Next, we show that if we were to add two dimensions to Q_n to increase its subcube embeddability, the two said dimensions will render the maximum subcubes among all possibilities.

Suppose the added dimensions are $01\cdots m-1$ and $01\cdots m'-1$, where $1 < m < m' \leq n-1$. The subcubes existing in $Q_n^{+(2)}$ can be classified into four categories:

- (i) subcubes spanning standard dimensions only
- (ii) subcubes spanning dimension $01..m-1$ but not $01\cdots m'-1$
- (iii) subcubes spanning dimension $01\cdots m'-1$ but not $01..m-1$
- (iv) subcubes spanning both dimensions $01\cdots m-1$ and $01\cdots m'-1$

For the first case, the number of subcubes is the same regardless of the choice of two additional dimensions. For the next two cases, clearly the maximum number of subcubes is obtained when $m = n-1$ and $m' = n-2$.

For case (iv), the presence of standard dimensions of $0,1,\dots,m-1$ or $0,1,\dots,m'-1$ in the covering will make it redundant (or non minimal). Also, the presence of standard dimensions $m',m'+1,\dots,m-1$ will result in a redundant covering. To minimize the redundant covering, m and m' must be as close to each other and as large as possible.

Therefore, we establish the following;

Theorem 2. The $Q_n^{+(2)}$ contains the maximum number of subcubes when it is generated by the covering $\{0,1,\dots,n-1,01\cdots n-2,01\cdots n-1\}$.

CHAPTER 3

SPARE DIMENSIONS IN HYPERCUBE

A hypercube network is said to be fault tolerant, if its structure is preserved in the presence of failures, which could be in the form of node, link, or combined node/link failures. In this chapter we investigate the robustness of the hypercube under link failures by adding redundant dimensions called *spare dimensions*. The spare dimensions are considered to be *hot spares*, which are connected as part of the hypercube network and active, but not become operable until a dimension fails. It is assumed that a dimension fails, if one or more of its links fail.

3.1 One Dimension Fault Tolerant Covering

In this section we consider adding one spare dimension to Q_n in order to investigate the fault tolerance capability of Q_n under dimension failures. Naturally, we are seeking how many dimension failures can be tolerated by adding a spare dimension.

Let C_n denote the set of dimension in Q_n , i.e. $C_n = \{0, 1, \dots, n-1\}$.

Let $C_n^{+(i)} = \{0, 1, \dots, n-1, S_i\}$ be the augmented covering of $Q_n^{+(i)}$, where S_i is itself a set with i spare dimensions.

An immediate consequence of Lemma 2 in Chapter 2 is that a spare dimension can replace any existing dimension, as long as the same digits are covered in both cases discussed in Chapter 2. More specifically, in the non-minimal dimension set of

$\{0, 1, \dots, n-1, 01\dots m-1\}$ in Q_n , the spare dimension $01\dots m-1$ can replace any dimension i , $0 \leq i \leq m-1$ to recover the Q_n . It follows:

Theorem 3. Consider a $Q_n^{+(1)}$ with the non-minimal dimension set,

$$\{0, 1, \dots, n-1, 01\dots m-1\}.$$

The minimal dimension set $\{0, 1, \dots, j-1, j+1, \dots, m-1, m, \dots, n-1, 012\dots m-1\}$ can be used to recover Q_n , where $0 \leq j < m$ and $1 \leq m < n$.

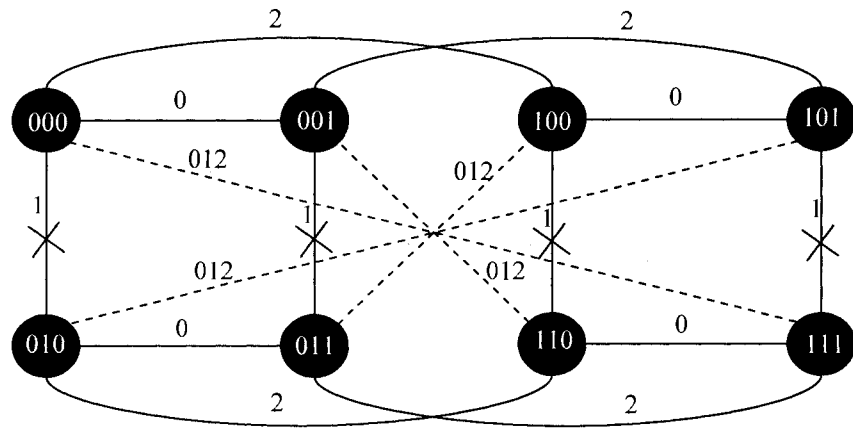
In other words, the extra dimension $01\dots m-1$ can replace any dimension $0, 1, \dots, m-1$. For example, in a Q_5 with the dimension set $\{0, 1, 2, 3, 4, 0123\}$, the dimension 0123 can replace any of the dimensions $0, 1, 2$, or 3 to form a non-redundant Q_5 with the dimension sets of $\{1, 2, 3, 4, 0123\}$, $\{0, 2, 3, 4, 0123\}$, $\{0, 1, 3, 4, 0123\}$, and $\{0, 1, 2, 4, 0123\}$, respectively. Nonetheless, the dimension 0123 cannot replace dimension 4 as this replacement results in a non-minimal dimension set of $\{0, 1, 2, 3, 0123\}$ which can only generate a Q_4 .

As a special case when $m = n$, Theorem 3 reduces to the following lemma:

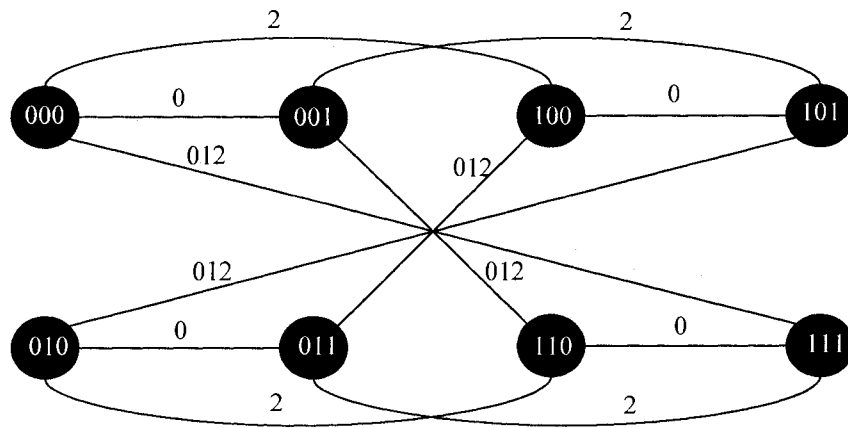
Lemma 4. The dimension set $C_n^{+(1)} = \{0, 1, \dots, n-1, 01\dots n-1\}$ in $Q_n^{+(1)}$ contains one spare dimension and any n dimensions in $C_n^{+(1)}$ can be used to form a Q_n .

In other words, the failure of any one of the dimensions in the set $C_n^{+(1)}$ can be tolerated to recover Q_n . Lemma 4 is proven in Chapter 2.

For example, from the covering set $C_3^{+(1)} = \{0,1,2,012\}$ in $Q_3^{+(1)}$, any 3 dimensions can be used to form Q_3 . Figure 3.1 illustrates the recovery of Q_3 upon the failure of dimension 1. The links in hot spare dimension 012 are indicated by dashed lines.



$Q_3^{+(1)}$ with failed dimension 1



Q_3 recovered

Figure 3.1. Spare dimension 012 replacing dimension 1

3.2 Two Dimension Fault Tolerant Covering

Even though one spare dimension is sufficient to achieve one fault tolerance capability, we consider two spare dimensions so that we can generalize to cases where more dimension failures can be tolerated. Let the two spare dimensions be $12\cdots n-1$ and $0.2\cdots n-2$. We obtain these spares by dropping digits 0 and 1 from the dimension $01\cdots n-1$, one at a time. Hence, we get the augmented covering set $C_n^{+(2)} = \{0,1,\dots,n-1,12\cdots n-1, 0.2\cdots n-2\}$.

We can see any dimension failure in $C_n^{+(2)}$ can be tolerated to form Q_n . For example, consider a 4-dimensional hypercube Q_4 with the covering set $C_4 = \{0,1,2,3\}$. The covering of the augmented hypercube $Q_4^{+(2)}$ is $C_4^{+(2)} = \{0,1,2,3,123, 023\}$. When dimension 1 fails, the covering will reduce to $\{0,123,2,3,023\}$, which will recover the 4-dimensional hypercube Q_4 .

3.3 Three Dimension Fault Tolerant Covering

In this section we consider adding three spare dimensions to Q_n . We obtain these distinct spares by rotating the digits of dimension $01\cdots n-1$ and dropping digits 0, 1, and 2 one at a time. It follows:

$$C_n^{+(3)} = \{0,1,\dots,n-1,12\cdots n-1, 23\cdots n-1.0,.34\cdots n-1.01\}.$$

Observe that at least two out of the three spares cover the digits $0,1,\dots,n-1$. Therefore, we can tolerate any two dimension failures. In other words, the spare dimensions can replace any two arbitrary dimension failures. Similarly, we can prove Lemma 5 by induction.

Lemma 5. In Q_n , adding $(n-i)$ extra dimensions will tolerate any arbitrary $(n-i-1)$ dimension failures, where $0 \leq i \leq n-1$.

3.4 Generalization of n Dimension Fault Tolerant Covering

By rotating and dropping one digit at a time from dimension $01 \cdots n-1$, we can generate n different dimensions shown below.

$$\begin{array}{cccccc}
 1 & 2 & \cdots & n-3 & n-2 & n-1 \\
 2 & 3 & \cdots & n-2 & n-1 & 0 \\
 3 & 4 & \cdots & n-1 & 0 & 1 \\
 & & & \vdots & & \\
 n-1 & 0 & 1 & \cdots & n-4 & n-3 \\
 0 & 1 & 2 & \cdots & n-3 & n-2
 \end{array}$$

An example for $n = 8$ is shown in Table 3.1.

Table 3.1. Spare Dimensions for Q_8

Spare Dimensions	Dropped Digit from Dimension 01234567
1234567	0
2345670	1
3456701	2
4567012	3
5670123	4
6701234	5
7012345	6
0123456	7

In a special case $i = 0$, we establish the following theorem from Lemma 5:

Theorem 4. Let Q_n be a standard cube with dimension set $\{0,1,\dots,n-1\}$. Consider adding the following n extra dimensions to this network:

$$\{1.2\dots n-2.n-1, 2.3\dots n-1.0, \dots, n-1.0.1\dots n-3, 0.1\dots n-2\}.$$

The resultant network (denoted by $Q_n^{+(n)}$) will tolerate any $(n-1)$ dimension failures.

Proof. The proof follows by way of considering all possibilities. Say a spare replaces the failed dimension i in such a way that the union of all new dimensions covers all digits $0,1,\dots,n-1$, yet the minimality of the new dimension set is preserved. This replacement recovers the original cube by Theorem 3. In the worst case, a dimension, say dimension i , fails followed by subsequent failures of its replacing spares (e.g. dimensions containing the same digit are depleted first). There are $(n-1)$ spares which include i in their labels, and therefore any $(n-1)$ consecutive failures can be tolerated. Q.E.D.

As an example, the $Q_5^{+(5)}$ contains the dimension set $\{0,1,2,3,4, 1234, 2340, 3401, 4012, 0123\}$. Any 4 consecutive dimension failures can be tolerated to recover Q_5 .

A natural question arises as to why we can't simply duplicate the dimension $01\dots n-1$ n times to form the n spare dimensions. Intuitively, this makes sense as dimension $01\dots n-1$ can replace any of the existing dimensions in a standard cube. However, this approach quickly results in the original cube's failure if the first two failed dimensions are among the standard dimensions (e.g. $0,1,\dots,n-1$), no matter how many times we replicate the $01\dots n-1$ dimension. For example, consider an augmented Q_5 with the dimension set $\{0,1,2,3,4, 01234, 01234\}$. If dimension 4 fails, the dimension set will reduce to $\{0,1,2,3, 01234, 01234\}$, which is still sufficient to recover the Q_5 . However, if

the second failed dimension is anything but 01234, say 3, the new dimension set will be $\{0,1,2,01234,01234\}$ which can only generate a Q_4 .

In our approach, by rotating the digits and dropping one digit at a time from the newly added dimensions, we ensure distinctiveness and, at the same time, maximum coverage for the spares.

3.5 Reconfiguration Algorithm

The reconfiguration algorithm specifies which spare should replace a failed dimension. It is simply described as follows.

The set of n dimensions in $Q_n - C_n = \{0,1,\dots,n-1\}$.

The set of n spare dimensions-

$$S_n = \{s_1, s_2, \dots, s_{n-1}, s_n\} = \{1.2 \dots n-2.n-1, 2.3 \dots n-1.0, \dots, n-1.0.1 \dots n-3, 0.1 \dots n-2\}$$

Let C_n' denote the covering obtained after replacing the failed dimension with a spare dimension.

Algorithm:

.....

input: C_n, S_n ;

begin

$C_n' = C_n$; $i = 1$; counter = 0.

if dimension in C_n fails

while $n >$ counter

choose spare dimension s_i from S_n .

update the covering C_n' by replacing s_i with the failed dimension.

counter = counter + 1

```

    If (union of all dimensions in  $C_n' = \text{union of all dimensions in } C_n$ )
     $C_n = C_n'$ 
    output: Hypercube  $Q_n$  recovered
    exit
  else
     $i = i+1$ 
  endif
endwhile
output: Hypercube  $Q_n$  cannot be recovered
exit
else
output: No dimension failure
endif
exit

```

.....

For example consider a Q_4 with the covering $\{0,1,2,3\}$ and $S_4 = \{123,230,301,012\}$. If dimension 3 fails, choose a spare dimension (say 012). The covering will reduce to $\{0,1,2,012\}$. In this case the union of all dimensions with the spare dimension is 012 is not equal the union of the original hypercube dimensions which is 0123. So the hypercube Q_4 cannot be recovered. Choose another spare dimension (say 013). The covering will then become $\{0,1,2,013\}$. In this case the union of all dimensions is equal the union of the original hypercube dimensions. So the hypercube Q_4 is recovered.

CHAPTER 4

RELIABILITY ANALYSIS IN HYPERCUBE NETWORKS

The fault tolerance of networks is generally measured by how much of the network structure is preserved in the presence of a given number of node and/or link failures. The most important requirements to achieve a fault tolerant network are reliability, availability, safety, performability, dependability, maintainability, and testability. As the size of a network grows, the probability of a fault occurring in the network increases. It is important to quantify the effects of the faults, so the fault-tolerant network can be pursued. In this chapter we analyze the reliability issues in fault tolerant hypercube networks.

4.1 Reliability and Evaluation

The *reliability* function of a system component at time t , $R(t)$, is defined as the conditional probability that the component will perform its intended function correctly throughout the time interval $[t_0, t]$, given that the system is operational at time t_0 [22]. The *unreliability* function $Q(t)$ of a component is defined as the conditional probability that the component will perform incorrectly throughout $[t_0, t]$, given that the system is operational at time t_0 . From the probability point of view $R(t) + Q(t) = 1.0$. The unreliability is often referred to as the probability of failure. Fault tolerance is a technique

that can improve reliability, but a fault-tolerant system does not necessarily have a high reliability.

The method for evaluating fault-tolerant networks can be divided into two major categories namely, *quantitative* and *qualitative*. Qualitative measures are typically subjective in nature and describe the benefits of one network over another. On the other hand, quantitative evaluation techniques produce numbers that can be used to compare two or more systems. We introduce two important quantitative evaluation parameters, *failure rate* and *mean time to failure (MTTF)*, which we use in the reliability assessment of hypercube networks.

- *Failure Rate*

Intuitively, the *failure rate* (λ) is defined as the expected number of failures of a device or system for a given time period. For example, if a component in a system fails once every 200 hours, the component has a failure rate of 0.005 failures/ hour.

Suppose we test N identical components. Let $N_f(t)$ be the number of failed components at time t and $N_g(t)$ be the number of good components at time t . The reliability of the component at time t is given by,

$$R(t) = \frac{N_g(t)}{N} = \frac{N_g(t)}{N_g(t) + N_f(t)}.$$

If we differentiate and rearrange $R(t)$ with respect to time, we obtain

$\frac{dN_f(t)}{dt} = -N \frac{dR(t)}{dt}$. Here the derivative of $N_f(t)$ with respect to time is the instantaneous rate of which component are failing. By dividing the derivative of $N_f(t)$ by $N_g(t)$ we obtain,

$$Z(t) = \frac{1}{N_g(t)} \frac{dN_f(t)}{dt}$$

$Z(t)$ is called the *hazard rate* or *failure rate function*. By mathematical manipulations we can write $Z(t)$ strictly in terms of the reliability function $R(t)$ as

$$Z(t) = -\frac{\frac{dR(t)}{dt}}{R(t)}$$

If we assume the system has a constant failure rate λ , the solution to the above differential equation is an exponential function given by $R(t) = e^{-\lambda t}$. This exponential relationship between the reliability and time is known as the *exponential failure law*.

- *Mean time to failure - MTTF*

MTTF is the expected time that a system will operate before the first failure occurs. MTTF can be calculated by finding the expected value of the time of failure. From the probability theory, we know that the expected value of a random variable X ,

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx, \text{ where } f(x) \text{ is the probability density function. In reliability analysis,}$$

we are interested in the expected value of MTTF. So, $MTTF = \int_0^{\infty} tf(t)dt$, where $f(t)$ is the

failure density function. Using integration by parts, and the fact that

$$f(t) = \frac{dQ(t)}{dt} = -\frac{dR(t)}{dt}, \text{ we can show that } MTTF = \int_0^{\infty} R(t)dt.$$

Reliability Modeling Methods

Reliability modelling is the process of predicting or understanding the reliability of a component or system. The reliability of a system can be determined analytically and experimentally. One problem with the experimental approach is the *number of systems*

that would be required to achieve a level of confidence in the experimental results. This is particularly a problem when costs limit the number of systems that can be built. The second problem is the *time* required to run such experiments. Many systems today are being designed to achieve a reliability of 0.97 or higher after ten hours of operation. Using the exponential failure law, a reliability of 0.97 corresponds to a failure rate of 10^{-8} failures per hour. Therefore, on average we would have to wait approximately 100 million hours for the first failure to occur. Clearly we need alternatives to the experimental approach.

The most popular reliability analysis techniques are the analytical approaches. Of the analytical techniques, *combinatorial modeling* and *Markov modeling* are the two most commonly used approaches. We use these two modeling methods to evaluate the reliability of the hypercube networks in this study.

4.2 Combinatorial Model

Combinatorial models use probabilistic techniques that enumerate the different ways in which a system can remain operational. The probabilities of the events that lead to a system being operational are calculated to form an estimate of the system reliability. The reliability of a system is generally derived in terms of reliabilities of the individual component of the system. Two models that are most common in practice are the series and the parallel. In a series system, each element is required to operate correctly for the system to operate correctly. In a parallel system, on the other hand, only one of several elements must be operational for the system to perform its function correctly. In practice, systems are typically combinations of series and parallel subsystems. These systems can be represented by *reliability block diagrams*. The reliability block diagram maps the

operational dependency of a system on its components and not on the on the actual physical structure.

- *Series System*

The series system can be considered as a system with no redundancy. Reliability block diagram of a series system with N components is shown in Figure 4.1. If any one of the component fails the system fails.

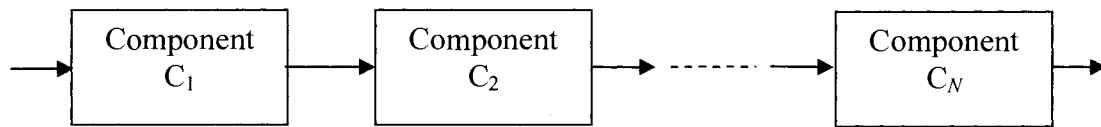


Figure 4.1. Series system

Let the reliability of component C_i at time t be $R_i(t)$. The reliability of the series system $R_{series}(t)$ is given by,

$$R_{series}(t) = \prod_{i=1}^N R_i(t).$$

- *Parallel System*

In basic parallel system, only one of the N identical components required for the system to function. Figure 4.2 illustrates a basic parallel system that contains N components.

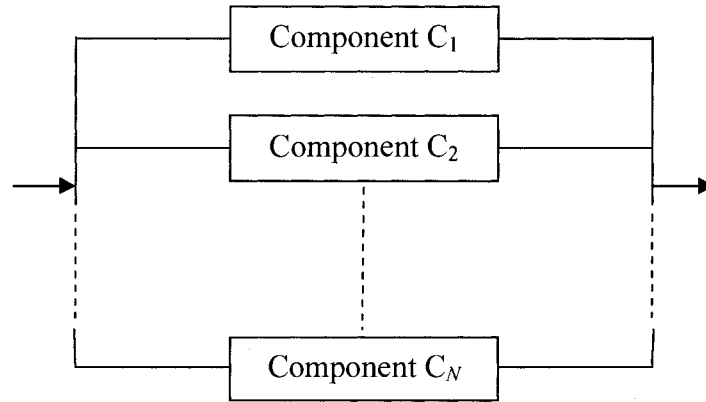


Figure 4.2. Parallel system

The reliability of the parallel system $R_{parallel}(t)$ is given by

$$R_{parallel}(t) = 1 - \prod_{i=1}^N [1 - R_i(t)].$$

m -out-of- N Systems are a generalization of the ideal parallel system. In the ideal parallel system only one of N modules is required to work for the system to work. However, in the m -out-of- N System, at least m of the total on N identical modules are required to function for the system to function properly, and the system can tolerate at most $N-m$ modules failures. The expression for the reliability of an m -out- N system can be written as (assuming each module has the same reliability R)

$$R_{m-out-of-N}(t) = \sum_{i=m}^N \binom{N}{i} R^i (1-R)^{N-i}, \text{ where } \binom{N}{i} = \frac{N!}{(N-i)!i!}.$$

4.2.1 Reliability analysis on hypercube networks

In this section we derive the reliability expression for three different hypercube networks with same number of redundant dimensions.

- *Augmented Hypercube Network*

The augmented cube $Q_n^{+(n)}$ as introduced in chapter 3 contains n standard dimensions and n spare dimensions. We proved that by adding n spare dimension to the standard hypercube Q_n , $(n-1)$ dimension failures can be tolerated. Hence, the augmented cube can be modeled as an m -out-of- N system with $2n$ components and at least $(n+1)$ component required functioning. We assume that all the dimension have the same reliability probability denoted by p . The reliability of the augmented cube (denoted by $R(Q_n^{+(n)})$) is equal to the probability of having at least $(n+1)$ reliable dimensions in the network. Therefore:

$$R(Q_n^{+(n)}) = \sum_{k=n+1}^{2n} \binom{2n}{k} p^k (1-p)^{2n-k}$$

- *Optimal Hypercube Network*

We introduce an optimal hypercube network (denoted by $Q_n^{Op(n)}$) as a hypothetical network with n redundant dimensions. We assume a pool of spares is used to replace failed dimensions, there is no restriction for replacement and n dimension failures can be tolerated. This network is not feasible and we introduce it to provide a frame of reference for the merit of the augmented hypercube. Let the Reliability of the optimal hypercube be $R(Q_n^{Op(n)})$. It follows:

$$R(Q_n^{Op(n)}) = \sum_{k=n}^{2n} \binom{2n}{k} p^k (1-p)^{2n-k}$$

- *Hypercube Network With Duplicate Dimensions*

To provide a realistic reference for comparison, in the following, we present the reliability for a hypercube in which each dimension is duplicated, e.g. the dimension set is $\{0,0,1,1,2,2,\dots,n-1,n-1\}$. Denote the hypercube network with duplicate dimension by $Q_n^{D(n)}$. This network can be modeled as parallel-series system illustrated in Figure 4.3. Hence, we can obtain the reliability of the hypercube with duplicated dimension as

$$R(Q_n^{D(n)}) = (1 - (1 - p)^2)^n$$

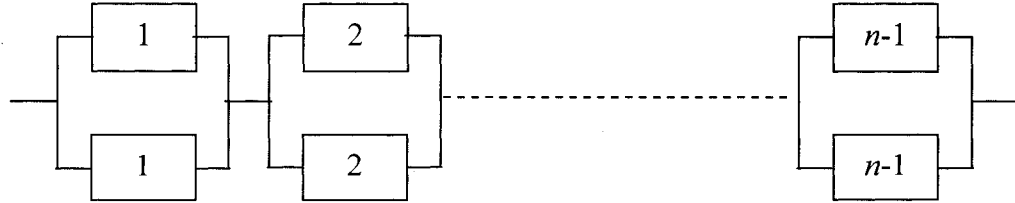


Figure 4.3. Modeling of $Q_n^{D(n)}$

The comparison of $Q_n^{+(n)}$ with $Q_n^{D(n)}$ is a fair comparison as both networks use the same number of spare dimensions.

4.2.2 Results and comparison

In this section, we compare the reliability and MTTF of the hypercube networks $Q_n^{+(n)}$, $Q_n^{Op(n)}$, and $Q_n^{D(n)}$ of different sizes.

The dimension reliability probability p is assumed to be homogeneous and follows an exponential distribution with a constant failure rate λ (failures/hour), i.e. $p = e^{-\lambda t}$.

The MTTF values for the three networks are given in Table 4.1.

Table 4.1. MTTF values.

	MTTF for $n=10$	MTTF for $n=20$
Hypercube with duplicate dimensions, $Q_n^{D(n)}$	$\frac{0.334}{\lambda}$	$\frac{0.224}{\lambda}$
Augmented hypercube, $Q_n^{+(n)}$	$\frac{0.669}{\lambda}$	$\frac{0.681}{\lambda}$
Optimal hypercube, $Q_n^{Op(n)}$	$\frac{0.769}{\lambda}$	$\frac{0.731}{\lambda}$

Observe that for $n=10$ ($n=20$), the MTTF for the Augmented cube is better than the duplicate hypercube by 100% (204%) while being worse than the optimal cube by only 15% (7%).

Figures 4.4, 4.5, and 4.6 show how the reliability of the three networks compare for different values of n . Here the failure rate is 0.05 per 10000 seconds.

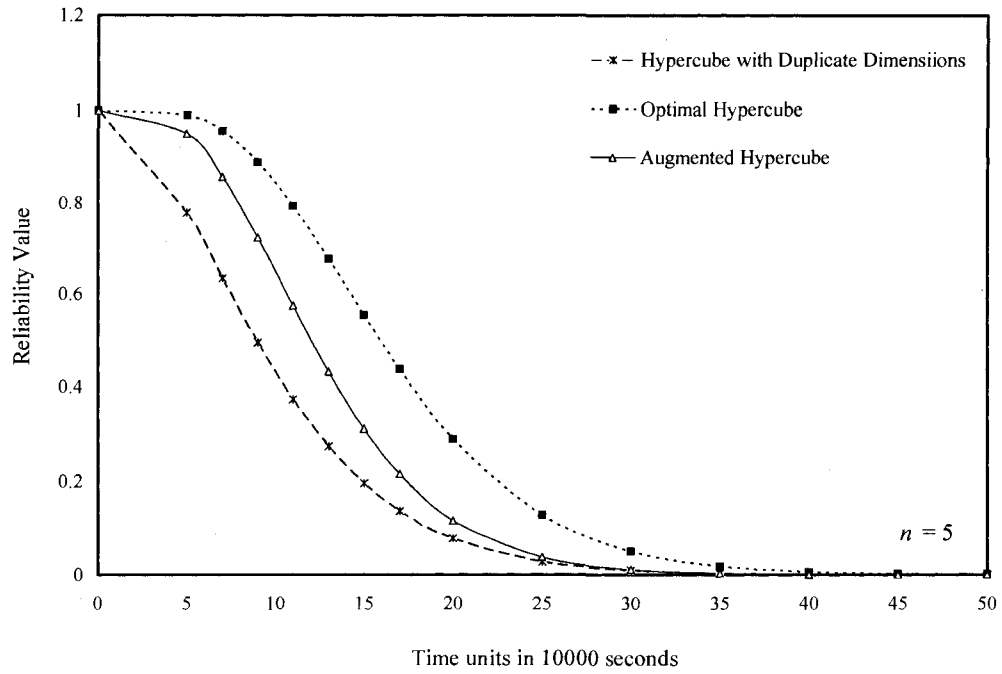


Figure 4.4. Reliability comparison between $Q_5^{+(5)}$, $Q_5^{D(5)}$ and $Q_5^{Op(5)}$

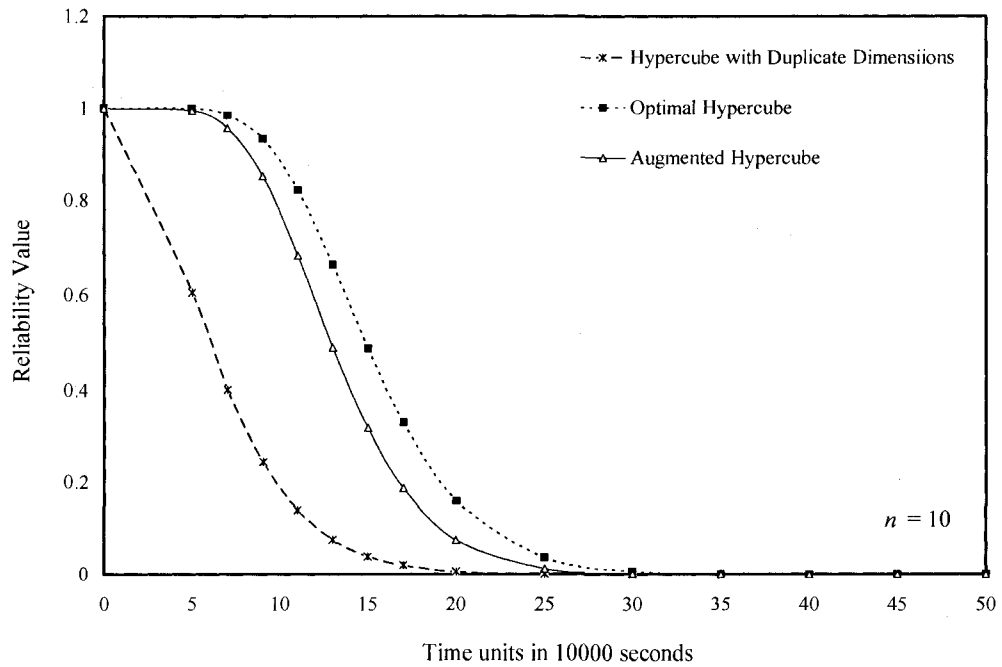


Figure 4.5. Reliability comparison between $Q_{10}^{+(10)}$, $Q_{10}^{D(10)}$ and $Q_{10}^{Op(10)}$

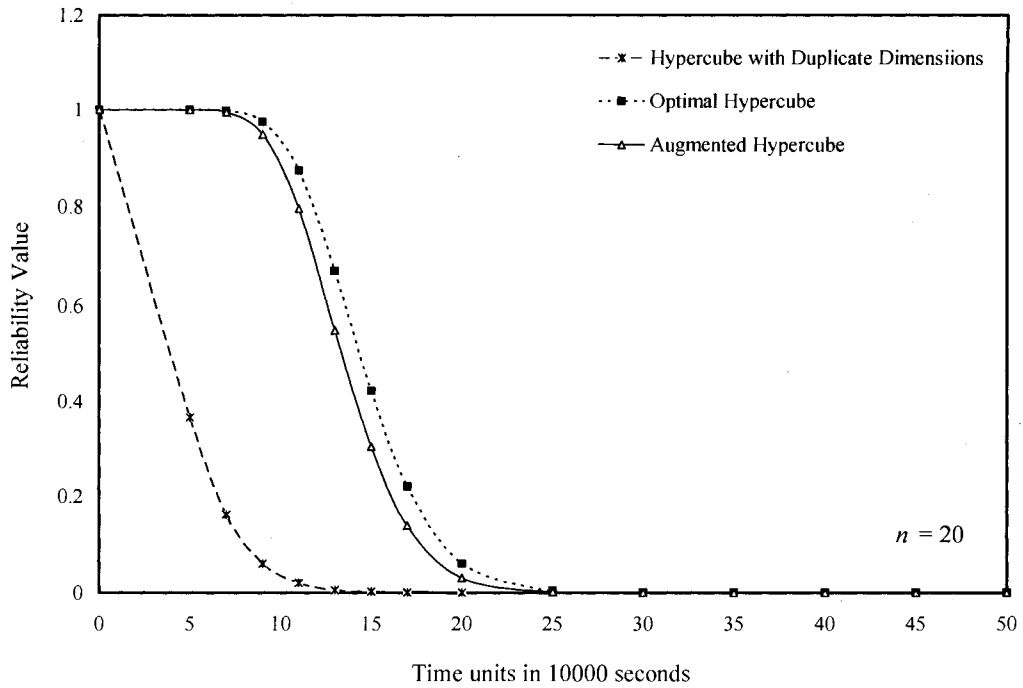


Figure 4.6. Reliability comparison between $Q_{20}^{+(20)}$, $Q_{20}^{D(20)}$ and $Q_{20}^{Op(20)}$

Observe that the larger the network, the more appreciable will be the superiority of the augmented hypercube network over the duplicate dimension network.

4.3 Markov Model

The primary difficulty with the combinatorial models is that many complex systems cannot be modeled easily in a combinatorial fashion. The reliability block diagrams can be extremely difficult to construct, and the resulting reliability expressions are often very complex. In addition, combinatorial models cannot accurately model dynamic system behavior. Because of its unique ability to handle dynamic cases, Markov analysis can be a powerful tool in the reliability analyses of dynamic systems [23].

The Markov model for reliability of a system is based on two concepts: the possible states of the system, and the transitions between states. The state of a system represents all that must be known to describe the system at any given instant of time. For reliability models, each state of the Markov model represents a distinct combination of faulty and fault-free modules.

The failed state is annotated as F. The reliability of the system is defined to be as the probability of the system to be in any of those states except F; it is the probability of being in any state other than F (which is the sum of the probabilities of each state), or (1-probability of the system to be in the F state).

In this section we use the Markov model to derive and compare the reliability of the hypercube networks, namely augmented hypercube ($Q_n^{+(n)}$), hypercube with duplicate dimensions ($Q_n^{D(n)}$) and optimal hypercube ($Q_n^{Op(n)}$). For this we make the following assumptions.

- The nodes in the network do not fail
- Spare dimensions replace the failed dimensions are hot spares, which means that they replace immediately the failed dimension
- Spare dimensions do not fail before replacing a failed dimension
- The reliability is the same for all dimensions.
- Dimension failures occur independently
- There is no repair capability in the system

4.3.1 Modeling of Augmented Hypercube for $n = 3$

For clarity, we first consider a special case of $n = 3$. The system fails when a standard cube of size 3 cannot be recovered. In a reliability state diagram the system fail

is indicated by circles containing F. For example, consider a standard Q_3 with coverings $\{0,1,2\}$. If dimension 2 fails the system fails as shown in Figure 4.7.

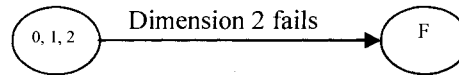


Figure 4.7. An example of system Failure

The reliability state diagram for $Q_3^{+(3)}$ is illustrated in Figure 4.8 in which the available standard dimensions and spare dimensions are indicated in the upper and lower half of the circles, respectively. Each arrow shows a transition from one state to the other, and is labeled by the dimension whose failure caused the transition.

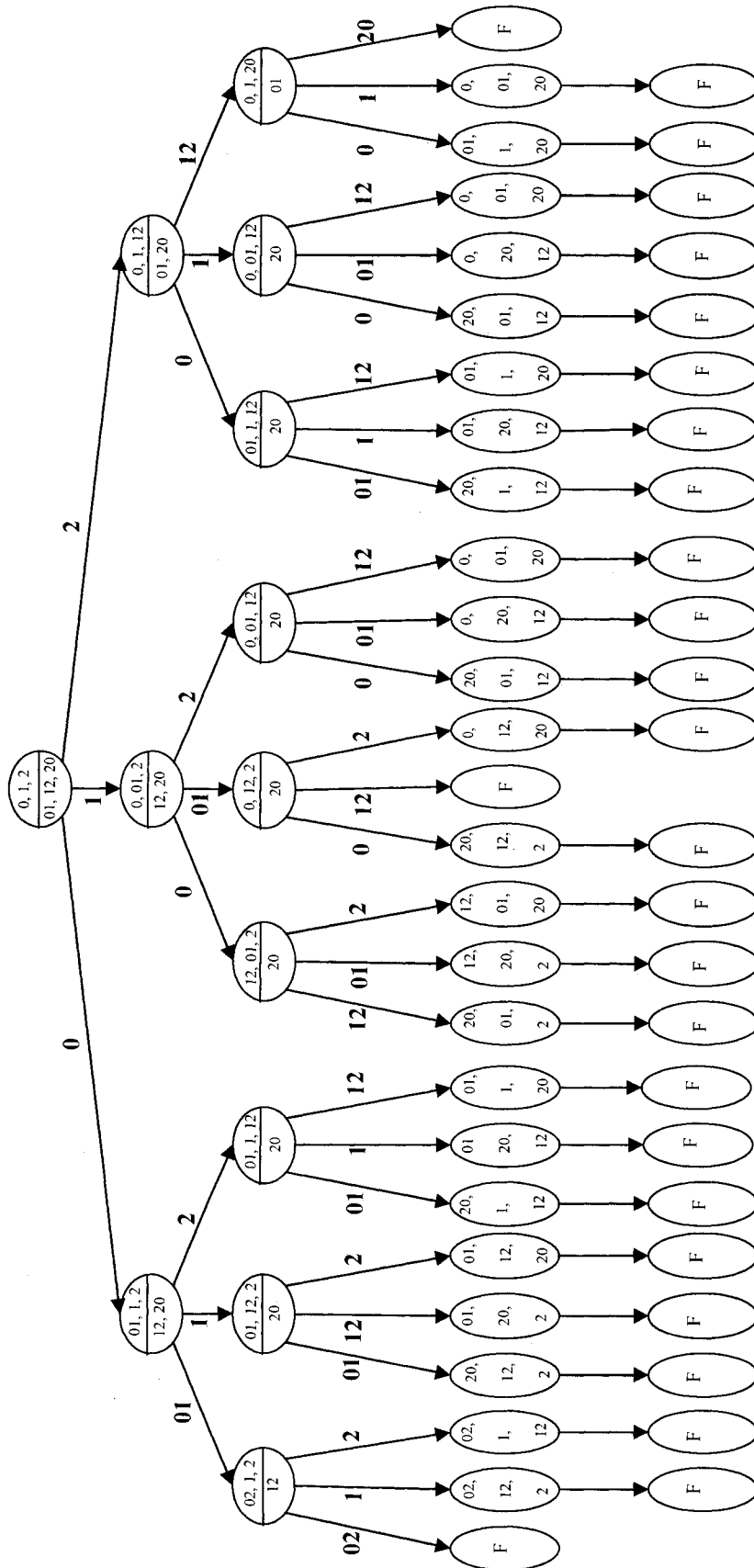


Figure 4.8. State diagram for $Q_3^{+(3)}$

Due to symmetry of states, the state diagram for $Q_3^{+(3)}$ can be reduced. We reduce the state diagram to five states: states containing three, two, one and no spare dimensions and the failed state. The reduced state diagram or Markov chain is given in Figure 4.9.

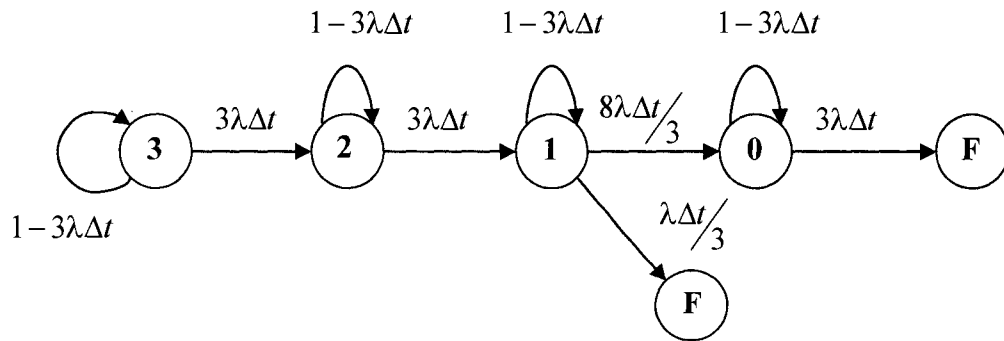


Figure 4.9. Reduced state diagram for $Q_3^{+(3)}$

Let $p_i(t)$ denote the probability that the system is in state i at time t . Initially, $p_3(0)=1, p_2(0)=p_1(0)=p_0(0)=p_F(0)=0$. By writing the expressions for probabilities associated with each state and letting $\Delta t \rightarrow 0$, one can derive a set of differential equations for each state probability. The method for solving the Markov chain with known transition probabilities is straightforward and can be found in any classical reliability books [22-23]. Therefore, details are omitted here. The transition matrix is given by:

$$\begin{bmatrix} \dot{p}_3(t) \\ \dot{p}_2(t) \\ \dot{p}_1(t) \\ \dot{p}_0(t) \\ \dot{p}_F(t) \end{bmatrix} = \begin{bmatrix} -3\lambda & 0 & 0 & 0 & 0 \\ 3\lambda & -3\lambda & 0 & 0 & 0 \\ 0 & 3\lambda & -\frac{3\lambda}{2} & 0 & 0 \\ 0 & 0 & \frac{8\lambda}{3} & -3\lambda & 0 \\ 0 & 0 & \frac{\lambda}{3} & 3\lambda & 0 \end{bmatrix} \begin{bmatrix} p_3(t) \\ p_2(t) \\ p_1(t) \\ p_0(t) \\ p_F(t) \end{bmatrix},$$

where $\dot{p}_i(t)$ denote is derivative of $p_i(t)$ with respect to t . It follows:

$$p_3(t) = e^{-3\lambda t}$$

$$p_2(t) = 3\lambda t e^{-3\lambda t}$$

$$p_1(t) = 9\lambda^2 \frac{t^2}{2!} e^{-3\lambda t}$$

$$p_0(t) = 4\lambda^3 t^3 e^{-3\lambda t}$$

$$p_F(t) = 1 - e^{-3\lambda t} - 3\lambda t e^{-3\lambda t} - \frac{9}{2} \lambda^2 t^2 e^{-3\lambda t} - 4\lambda^3 t^3 e^{-3\lambda t}$$

Observe that $p_3(t) + p_2(t) + p_1(t) + p_0(t) + p_F(t) = 1$.

The system is reliable as long as it is not in the failed state. Therefore the reliability of

$Q_3^{+(3)}$ is given by,

$$R(Q_3^{+(3)}) = p_3(t) + p_2(t) + p_1(t) + p_0(t) = \left(1 + 3\lambda t + \frac{9}{2} \lambda^2 t^2 + 4\lambda^3 t^3\right) e^{-3\lambda t}.$$

We can obtain the MTTF for $Q_3^{+(3)}$ as a function of failure rate λ using the equation,

$$\text{MTTF}_{Q_3^{+(3)}} = \int_0^{\infty} R(Q_3^{+(3)}) dt$$

It follows:

$$\text{MTTF}_{Q_3^{+(3)}} = \frac{35}{27\lambda} = \frac{1.296}{\lambda}$$

4.3.2 Modeling of Hypercube with Duplicate Dimensions for $n = 3$

Now, we proceed to derive the reliability expression for $Q_3^{D(3)}$ using the same method in the previous section. The state diagram and the reduced Markov chain are given in Figures 4.10 and 4.11, respectively.

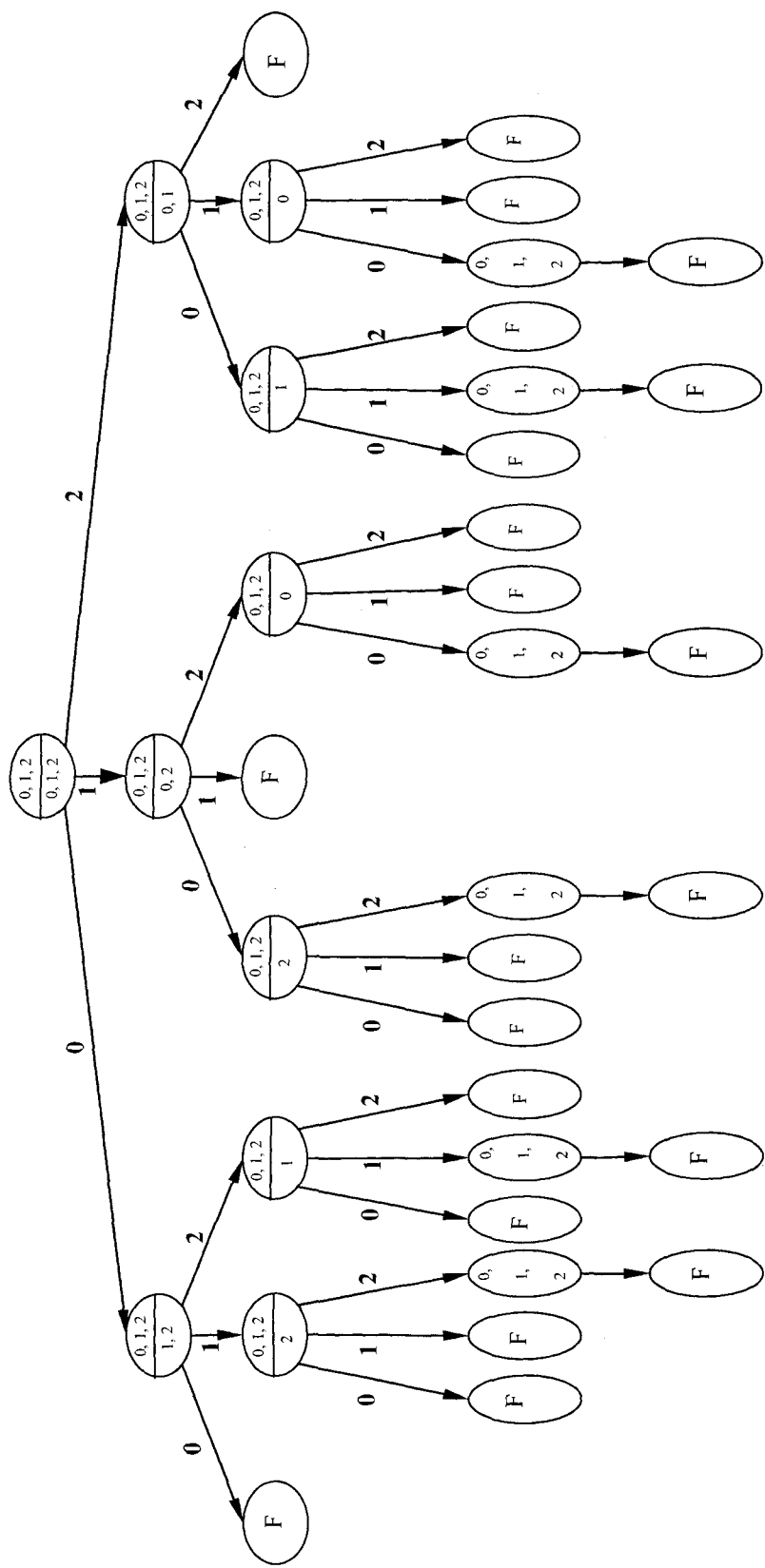


Figure 4.10. State diagram for $Q_3^{D(3)}$

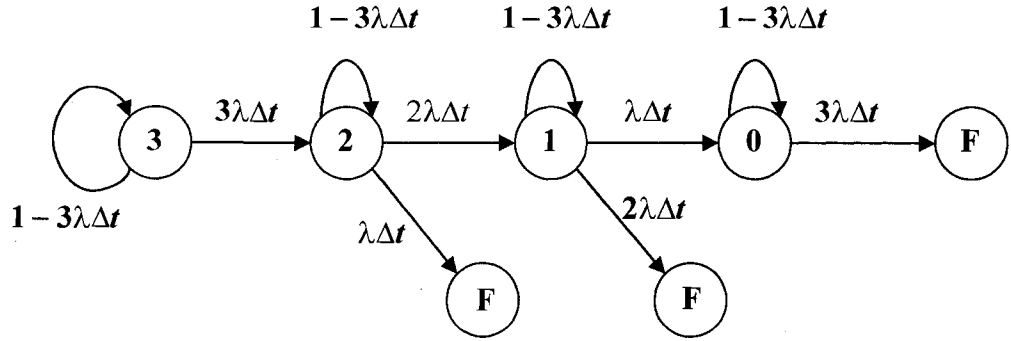


Figure 4.11. Reduced state diagram for $Q_3^{D(3)}$

The procedure for deriving the state probabilities is the same as in the previous case. For brevity, only the results are presented below.

Initially, $p_3(0) = 1, p_2(0) = p_1(0) = p_0(0) = p_F(0) = 0$.

$$p_3(t) = e^{-3\lambda t}$$

$$p_2(t) = 3\lambda t e^{-3\lambda t}$$

$$p_1(t) = 3\lambda^2 t^2 e^{-3\lambda t}$$

$$p_0(t) = \lambda^3 t^3 e^{-3\lambda t}$$

$$p_F(t) = 1 - e^{-3\lambda t} - 3\lambda t e^{-3\lambda t} - 3\lambda^2 t^2 e^{-3\lambda t} - \lambda^3 t^3 e^{-3\lambda t}$$

It follows that,

$$R(Q_3^{D(3)}) = \left(1 + 3\lambda t + 3\lambda^2 t^2 + 2\lambda^3 t^3\right) e^{-3\lambda t} \text{ and } \text{MTTF}_{Q_3^{D(3)}} = \frac{28}{27\lambda} = \frac{1.037}{\lambda}.$$

Comparing these two networks $Q_3^{+(3)}$ and $Q_3^{D(3)}$ in terms of MTTF, the augmented cube has the highest value independent of the value of λ .

4.3.3 Generalization to n dimensional networks

In this section we generalize the results for $n = 3$ to n for networks introduced in section 4.2.1 namely $Q_n^{+(n)}$, $Q_n^{Op(n)}$, and $Q_n^{D(n)}$. We follow the same procedure used in section 4.3.1 to derive the reliability equations for the three networks. Hence, for brevity only the results are given.

Modeling of $Q_n^{D(n)}$

For the hypercube network with duplicate dimension we can obtain the reduced state diagram as illustrated in Figure 4.12.

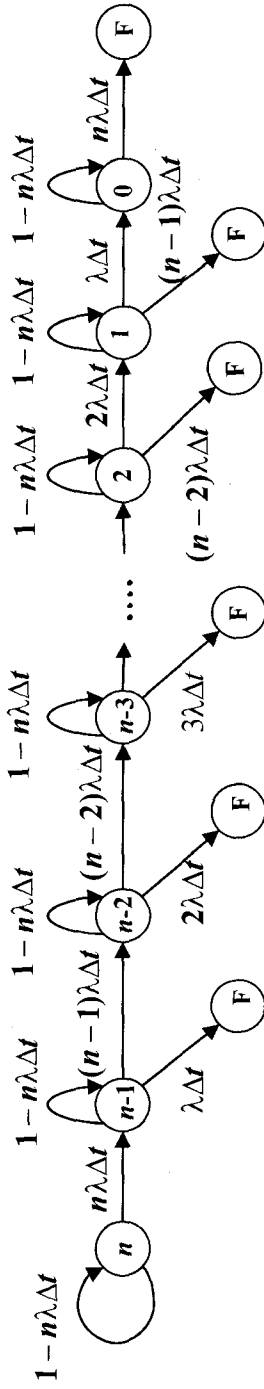


Figure 4.12. Reduced state diagram for $Q_n^{D(n)}$

It follows that:

$$R(Q_n^{D(n)}) = \left(1 + n\lambda t + \frac{n(n-1)}{2!} \lambda^2 t^2 + \dots + \frac{n!}{(n-i)!(i)!} \lambda^i t^i + \dots + \frac{n(n-1)}{2!} \lambda^{n-2} t^{n-2} + n\lambda^{n-1} t^{n-1} + \lambda^n t^n \right) e^{-n\lambda t}$$

or $R(Q_n^{D(n)}) = \left(\sum_{i=0}^n \binom{n}{i} \lambda^i t^i \right) e^{-n\lambda t}$, where $R(Q_n^{D(n)})$ is the reliability of $Q_n^{D(n)}$.

Modeling of Augmented Hypercube

Figure 4.13 shows the reduced state diagram for $Q_n^{(+n)}$.

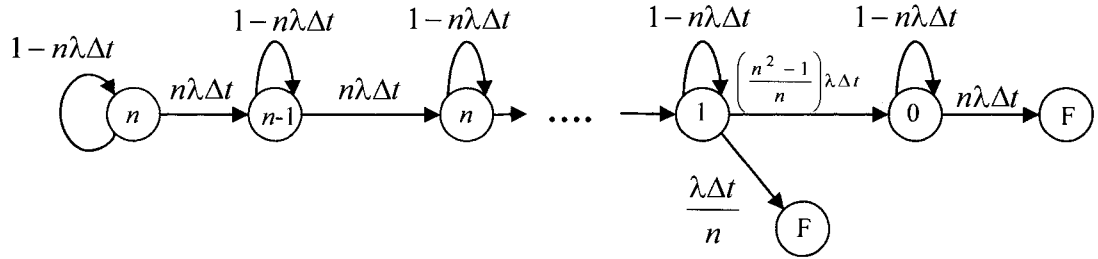


Figure 4.13. Reduced state diagram for $Q_n^{(+n)}$

It follows that the expression for reliability is given by:

$$R(Q_n^{(+n)}) = \left(1 + n\lambda t + n^2 \lambda^2 \frac{t^2}{2!} + \dots + n^i \lambda^i \frac{t^i}{i!} + \dots + n^{n-1} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} + n^{n-2} (n^2 - 1) \lambda^n \frac{t^n}{n!} \right) e^{-n\lambda t}$$

or $R(Q_n^{(+n)}) = \left(\sum_{i=0}^{n-1} n^i \lambda^i \frac{t^i}{i!} + n^{n-2} (n^2 - 1) \lambda^n \frac{t^n}{n!} \right) e^{-n\lambda t}$.

Modeling of Optimal Hypercube

In this section we derive the reliability function for the optimal hypercube ($Q_n^{Op(n)}$) in which n dimension failures can be handled. Reduced state diagram for an optimal hypercube network is illustrated in Figure 4.14.

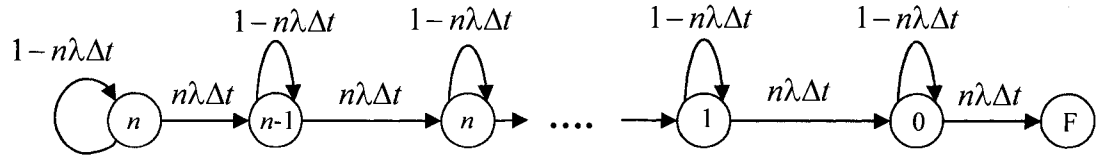


Figure 4.14. Reduced state diagram for $Q_n^{Op(n)}$ with n spares

It follows that the expression for reliability is given by:

$$R(Q_n^{Op(n)}) = \left(1 + n\lambda t + n^2 \lambda^2 \frac{t^2}{2!} + \dots + n^i \lambda^i \frac{t^i}{i!} + \dots + n^{n-1} \lambda^{n-1} \frac{t^{n-1}}{(n-1)!} + n^n \lambda^n \frac{t^n}{n!} \right) e^{-n\lambda t}$$

$$\text{or } R(Q_n^{Op(n)}) = \left(\sum_{i=0}^n n^i \lambda^i \frac{t^i}{i!} \right) e^{-n\lambda t}.$$

4.3.4 Reliability Comparison

In Figure 4.15, 4.16, and 4.17 the reliability of $Q_n^{+(n)}$, $Q_n^{Op(n)}$, and $Q_n^{D(n)}$ are compared for different values of n . Here the failure rate is 0.05 per 10000 seconds.

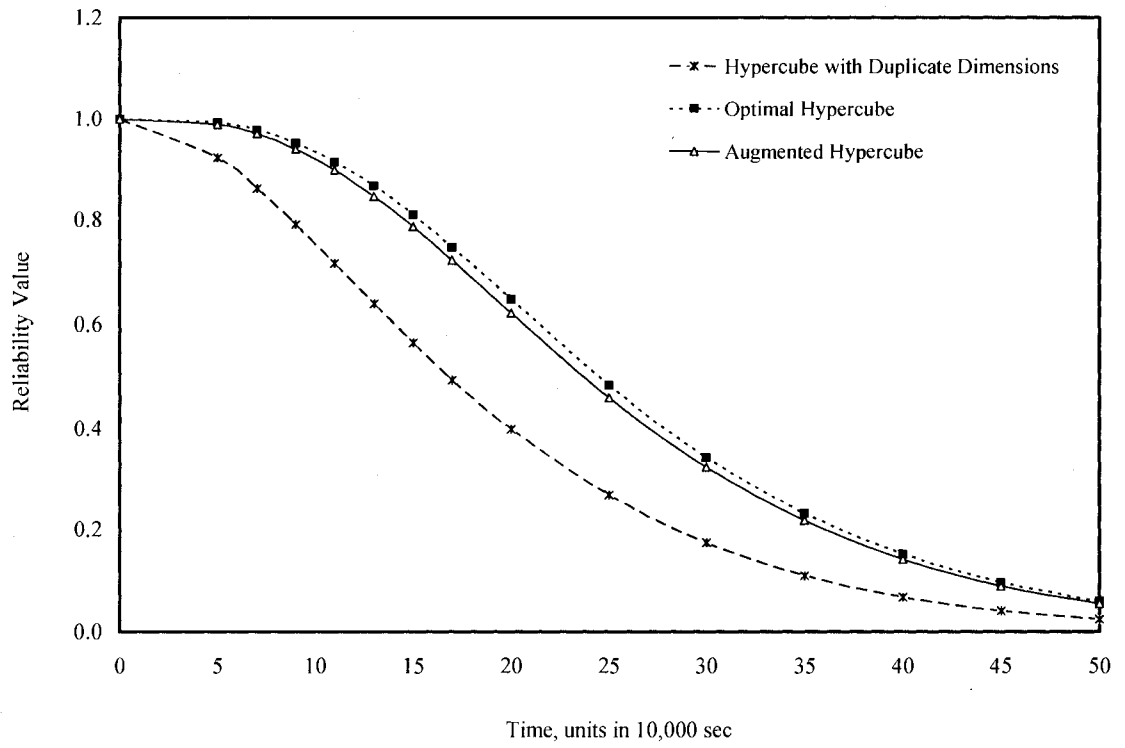


Figure 4.15. Reliability values for $n = 3$

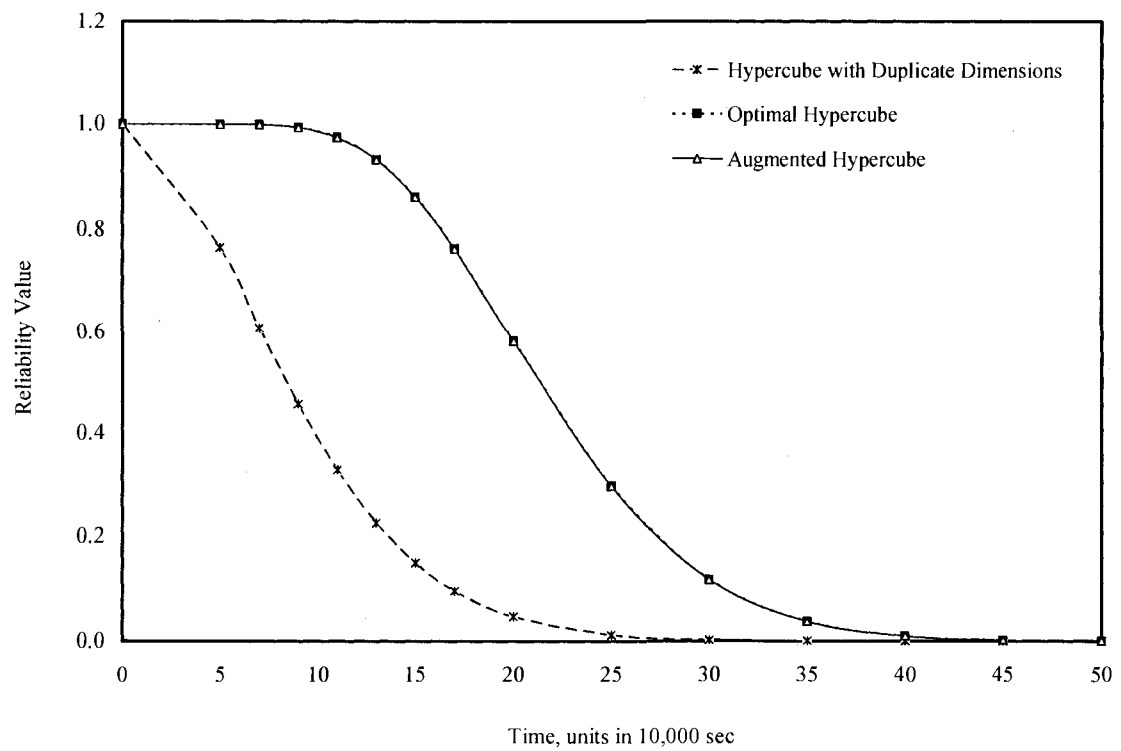


Figure 4.16. Reliability values for $n = 10$

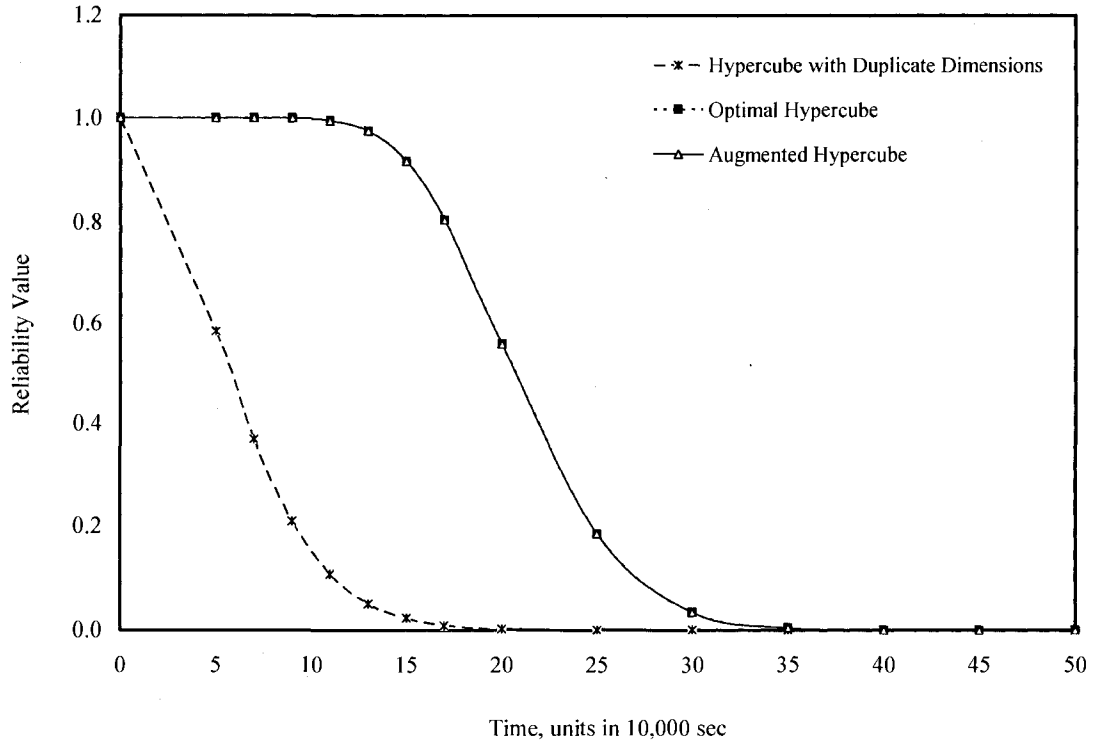


Figure 4.17. Reliability values for $n = 20$

Observe that the larger the network, the more appreciable will be the superiority of the augmented hypercube network over the duplicate dimension network.

CHAPTER 5

CONCLUSION AND RECOMMENDATION

In the first part of the research, we enumerated the newly generated subcubes in a standard hypercube when it is augmented by an extra dimension which would connect pairs of nodes whose Hamming distance is greater than one. Depending on this distance, dimensions were characterized by the notion of the magnitude. It was shown a total of $(2^n - n - 1)$ new dimensions can be added to the standard hypercube network. This approach could be regarded as a generalization of the concept of folded hypercube wherein the Hamming distance of nodes connected through the extra dimension is n . It was shown that the optimal number of subcubes is attainable when the magnitude of the new dimension is $n-1$ or n . As the magnitude of the new dimension decreases, so does the number of subcubes produced due to the added dimension. Having a maximum number of subcubes is important as it results in a higher utilization of nodes, lower completion times for jobs, less communication overhead due to job migration, and fewer jobs preemption to accommodate higher priority jobs.

In the second part, we investigated the robustness of the hypercube under dimension failures. It was shown that up to $(n-1)$ dimension failures can be tolerated by adding a near-optimal number of spare dimensions to standard hypercube. We proposed a new hypercube network with n spare dimensions called the augmented hypercube. To show

the superiority of our new network we introduced two other alternate hypercube networks with the same redundant dimensions. We presented the reliability of the three networks using the combinatorial and Markov modeling methods. The network reliability functions were obtained and plotted for different value of n . The reliability and the Mean-Time-To-Failure (MTTF) of the new network were compared to the alternative networks with the same redundancy. The comparison results indicate the superiority of the new network.

Future Research work includes:

- Simulation Program for identifying the subcubes in an augmented cube with extra dimensions when applications such as job scheduling are considered.
- Enumerating subcubes in an augmented cube after link failures
- The reliability analysis of augmented hypercube when considering link failure rather than dimension failures. Simulation results are necessary to further verify the numerical results.

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2. Sithy Shameema M. Yasim and Shahram Latifi, "Spare Dimensions in Hypercubes", in *Proc. of 2008 International Conference on Parallel and Distributed Processing Techniques and Applications (PDPTA)*, vol. 1, pp. 207-210, July 2008

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