Teacher exploration of instructional strategies to promote algebraic thinking

Cynthia Ann Hernon

University of Nevada, Las Vegas

Follow this and additional works at: https://digitalscholarship.unlv.edu/rtds

Repository Citation
https://digitalscholarship.unlv.edu/rtds/2586

This Dissertation is brought to you for free and open access by Digital Scholarship@UNLV. It has been accepted for inclusion in UNLV Retrospective Theses & Dissertations by an authorized administrator of Digital Scholarship@UNLV. For more information, please contact digitalscholarship@unlv.edu.
TEACHER EXPLORATION OF INSTRUCTIONAL STRATEGIES TO PROMOTE ALGEBRAIC THINKING

by

Cynthia Ann Hernon

Bachelor of Arts
University of Nevada, Las Vegas
1970

Master of Education
University of Nevada, Las Vegas
1975

A dissertation submitted in partial fulfillment of the requirements for the

Doctor of Philosophy Degree in Curriculum and Instruction
Department of Curriculum and Instruction
College of Education

Graduate College
University of Nevada, Las Vegas
August 2004
The Dissertation prepared by

Cynthia Ann Hernon

Entitled

Teacher Exploration of Instructional Strategies to Promote Algebraic Thinking

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Curriculum and Instruction

Examination Committee Co-Chair

Examination Committee Chair

Dean of the Graduate College

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
ABSTRACT

Teacher Exploration of Instructional Strategies to Promote Algebraic Thinking

by

Cynthia Ann Hernon

Dr. Jeffrey Shih, Examination Committee Co-Chair
Assistant Professor of Mathematics Education
University of Nevada, Las Vegas

Dr. William Speer, Examination Committee Co-Chair
Professor of Mathematics Education
University of Nevada, Las Vegas

The research study investigates the influence of teacher participation in a graduate course fostering the development of algebraic thinking for K-8 students on teacher understanding of the nature of algebraic thinking and on the incorporation of the teaching of algebraic thinking, guided by student discourse, into practice. The traditional high school algebra courses, with high failure rates, are not meeting the challenge of teaching algebra to every student. Thus, mathematics educators and researchers have proposed that problems in the teaching and learning of algebra be addressed before the middle school years by integrating the development of algebraic thinking and reasoning into the elementary school mathematics curriculum.

This study explores how three elementary teachers introduce the algebraic concepts of equivalence, relational thinking, and the development and justification of conjectures to first and third grade students. The research is framed against the examination of
teacher change in practice within the context of a professional development experience. The three aspects of change that constitute the conceptual framework are change as part of the learning to teach process, change in teacher understanding of the nature of mathematics, and change in teacher understanding of the nature of algebraic thinking. The qualitative case study of these three elementary teachers is focused on the personal, situational, and institutional factors that are conducive to effecting this change in practice.

The constant comparative analysis of the data collected from interviews, classroom observations, journal reflections, and survey responses revealed six common themes across the cases. All three teachers possess a high level of interest in teaching mathematics, believe that traditional teaching strategies are not working for their students, demonstrate ambiguity about the definition of algebraic thinking, cite a lack of curriculum resources to support the teaching of algebraic thinking, desire collaboration with like-minded teachers, and are committed to continuing the teaching of algebraic thinking. These teachers successfully added to their mathematics content knowledge and either incorporated new pedagogy into their teaching or refined an existing constructivist approach to teaching and learning as they integrated the teaching of algebraic thinking into the classroom.
ACKNOWLEDGEMENTS

I wish to thank the members of my committee for their support and encouragement as I progressed through the doctoral program. Each committee member made a unique and valuable contribution to my learning experiences as a prospective mathematics educator. Dr. Jeffrey Shih provided the inspiration for the dissertation, kept the research process alive, and guided me to the finish line. Dr. William Speer was the ultimate mentor and role model as a secondary mathematics educator. Dr. Marilyn Sue Ford enhanced my understanding of teaching and learning in the primary grades. Dr. Kent Crippen pushed the research and writing to a higher level with his relentless questioning of the process and attention to detail. Dr. Peter Shiue kept the focus on the subject matter content with his constant reminder of the beauty of the mathematics.

I would also like to extend a personal note of appreciation to Dr. Aimee Govett for facilitating my transition into higher education, collaborating in teaching and writing, and being a friend.
# TABLE OF CONTENTS

**ABSTRACT** ..................................................................................................................... iii

**ACKNOWLEDGMENTS** ................................................................................................. v

**CHAPTER 1  INTRODUCTION** ................................................................................ 1
  Teacher Development in the Context of Professional Development .......... 4
  Teacher Development in the Mathematics Classroom ............................... 7
  Teacher Development of the Understanding of the Nature of Algebraic Thinking . 9
  Significance of the Study ........................................................................................ 11

**CHAPTER 2  REVIEW OF RELATED LITERATURE** .......................................... 13
  Professional Development and Teacher Change ............................................ 14
  Mathematics Teaching and Teacher Change ................................................... 19
  Algebraic Thinking as a Basis for Change ....................................................... 25
  History of the Development of Algebraic Thought ......................................... 26
  Traditional Algebra ............................................................................................... 31
  Algebraic Thinking or Arithmetic Thinking ..................................................... 33
  Cognitive Obstacles in Understanding Algebra ............................................... 38
  Mathematics Curriculum Based on Algebraic Thinking ............................... 48
  Teacher Understanding of Algebraic Thinking and Practice .......................... 52

**CHAPTER 3  METHODOLOGY AND DATA DESCRIPTION** ............................ 56
  Structure of the Professional Development ....................................................... 59
  Content of the Professional Development ......................................................... 60
  Participants ............................................................................................................. 63
  Data Collection ...................................................................................................... 63
  Teachers in the Case Study .................................................................................. 67
  Case One: Sonya Henderson ........................................................................... 70
  Case Two: Josh Abernathy .................................................................................. 91
  Case Three: Paula Whitford ............................................................................... 112
  Summary ................................................................................................................ 132

**CHAPTER 4  DATA ANALYSIS** ........................................................................ 134
  Development of Themes ...................................................................................... 136
  Nature of Algebraic Thinking ............................................................................ 137
  Professional Development and Practice ............................................................ 140
  Student Discourse as the Basis for Instruction .................................................. 146
  Summary ................................................................................................................ 156

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER 1

INTRODUCTION

Access to instruction in algebra for every student in the schools is a significant issue in the overall movement for reform in mathematics teaching (Choike, 2000; Kaput 1995, 2000, 2000a, 2000b; Picciotto & Wah, 1993; Pugalee, 2001; Silver 1995, 1997). The National Council of Teachers of Mathematics proposed educational goals to support a reformed vision of the nature of mathematics teaching and learning in the 1989 *Curriculum and Evaluation Standards for School Mathematics*. These goals include providing opportunities for all students to become mathematically literate workers, lifelong learners of mathematics, and informed members of the electorate with the technical knowledge and understanding to read and interpret complex information. The *Principles and Standards for School Mathematics* (NCTM, 2000) expand on these ideas to promote teaching standards that demand a common foundation of mathematical knowledge, including algebra, to be learned by all students with emphasis on student discourse, engagement in worthwhile mathematical tasks, and learning through problem solving. Elementary and secondary teachers must be able to assist students of all ability levels to attain computational fluency, justify these computations, and understand the more complex concepts in algebra (Kaput, 2000, 2000b; Pegg & Redden, 1990).

State departments of education have implemented content and process standards for all grade levels, assessment measures, school accountability policies, and mandated the
use of specific teaching materials or strategies (California, 1997; Nevada, 2001; Texas, 1998; Virginia, 1995). Mathematics education has been described as a “state of ferment” (Schifter & Fosnot, 1993) that challenges the professional expertise and traditional roles or identities of new and experienced teachers. It is the individual teacher who implements the policies and makes the daily decisions about what and how the students are taught, thus the success of reform in mathematics teaching depends on the individual teacher showing and discussing a new vision of mathematics instruction (Schifter, 1996a, 1996b). Therefore, this study examines how, through case study, individual teachers implement the content of a graduate course focused on introducing inservice teachers to techniques of arithmetic instruction that are a foundation for algebraic thinking.

Algebra, as in the successful completion of an algebra course, has been called the “gatekeeper” (Silver, 1997; Choike, 2000) or “constricted gateway” (Kaput, 2000b) to the further study of mathematics and science, advanced vocational training, and quality employment opportunities. Education in algebra has even been proclaimed a civil right based on the fact that today’s technology-based society “has put advanced mathematics-math with the symbolic, abstract representations of algebra-on the table as an educational necessity for anyone who strives to enjoy the full rights of citizenship” (Moses in Successful Equation, 2002, p. 22). The NCTM Principles and Standards for School Mathematics state “All students should learn algebra” (2000, p. 37). The algebra standard for students from prekindergarten to grade 12 affirms that all students should understand patterns, relations, and functions, represent and analyze mathematical situations and structures using algebraic symbols, use mathematical models to represent
and understand quantitative relationships, and analyze change in various contexts. The Nevada State Content Standard 2.0: Patterns, Functions, and Algebra requires all students to solve problems, communicate, reason, and make connections within and beyond the field of mathematics. Students should also use various algebraic methods to analyze, illustrate, extend, and create numerous representations (words, numbers, tables, and graphs) of patterns, functions, and algebraic relations as modeled in practical situations.

Effective algebra instruction for all students requires a rethinking and restructuring of a mathematics curriculum that is currently based on the “late, abrupt, isolated, and superficial approach to school algebra” (Kaput & Blanton, 2000, p. 2). Algebra should not be viewed as a single course in the school curriculum, “but rather a collection of knowledge, skills, and dispositions prerequisite for understanding algebraic concepts” (Lodholz, 1990, p. 25). Problems in the teaching and learning of algebra need to be addressed before the students’ middle school years by integrating the development of algebraic thinking and reasoning into the elementary school mathematics curriculum (Carpenter, Franke, & Levi, 2003; Kaput, 2000a, 2000b, Kaput & Blanton, 2000).

A mathematics curriculum based on algebraic thinking “is a way of teaching and learning mathematics that enables younger students to be successful as they advance through increasingly sophisticated concepts and ideas” (Arens & Meyer, 2000, p. 6). Student assignments should be engaging situations that require exploration and discovery in the search for patterns that express the generalizations of algebra. Students should talk about the activity in simple language. The reflection on the experience should eventually lead to writing about the concepts, procedures, and results in appropriate symbolic
notation. The teacher starts the learning experience by asking a question or presenting a problem, not by telling the students what to do. The teacher listens to and documents student thinking in order to understand how students conceptualize ideas, execute procedures, or experience difficulties in learning a concept or a procedure (Bell in Arcavi, 1995; Booth & Watson, 1990).

The process of integrating algebraic thinking into elementary school arithmetic should motivate teachers to rethink the definition of algebra, explore the nature of algebraic thinking, investigate how students think about algebra, and use this student understanding of the nature of algebra to plan instruction (Arens & Meyer, 2000; Carpenter, Franke, & Levi, 2003; Kaput, 2000, 2000a). This research study investigates how teacher participation in a graduate course fostering the development of algebraic thinking for K-8 students influences teacher understanding of the nature of algebraic thinking, and how the teaching of algebraic thinking based on the use of student thinking and discourse is incorporated into practice. The conceptual framework of the study is based on the examination of teacher beliefs and change in practice within the context of a professional development experience. The three aspects of change that constitute this framework include the general notion of teacher change or reform of practice, mathematics teacher change, and teacher understanding of the nature of algebraic thinking as a basis for changing practice.

Teacher Development in the Context of Professional Development

The literature on change in instructional practice represents several different research perspectives. Richardson (1990) delineates these perspectives as teacher change, learning
to teach, and practical inquiry. The practical inquiry perspective is described as the relationship of the content of teachers’ reflection to classroom practice and how change may affect that content. The teacher change literature examines the reasons teachers do or do not adopt new teaching strategies or innovations (Richardson, 1990, 1994b). In the teacher change literature teachers are typically viewed as resistant to change and unwilling to adopt the instructional strategies proposed by educational scholars or experts when the new strategies do not match what the teacher intuitively believes is good teaching. Organizational factors such as administrative support for change, environment of faculty collegiality, and interest in coordination of instruction influence teacher willingness to adopt new programs. Personal factors such as beliefs, attitudes, goals, and practical knowledge also impact implementation of new teaching practices. The defining characteristic of the teacher change literature is the fact that the change is something that is mandated or suggested by someone other than the teacher (Richardson, 1990, 1994b).

The learning to teach literature addresses the issues of what teachers do and why and how they do it, but with the emphasis “more on individual teacher’s cognitions, beliefs, and other mental processes than on behaviors” (Richardson, 1990, p. 12). The teacher as person and the role of individual experience are significant in the development of the practical knowledge of teaching. The teacher change literature and the learning to teach literature can be integrated to focus on how the nature of teachers’ reflections and their relationship to classroom practice can facilitate change in teaching practice. In this study the emphasis is on the learning to teach aspect of change in practice, which is compatible with the term “teacher development” as described by Lieberman and Miller (1990) in characterizing professional growth in a professional practice school.
By teacher development, we mean continuous inquiry into practice. In this construction of professional development, we see the teacher as a "reflective practitioner," someone who has a tacit knowledge base and who then builds on that knowledge base through ongoing inquiry and analysis, continually rethinking and reevaluating values and practices (p. 107, 1990).

Quality professional development experiences are sustained over time, focused on subject-matter content and how children learn it, require the teachers to be actively engaged in the learning, strongly connected to other professional development activities, and encourage communication and collaboration with other teachers working to reform their teaching (Garet, Porter, Desimone, Birman, & Yoon, 2001).

The professional development experience in this study was a weeklong summer graduate course in which the content and pedagogy are derived from the Cognitively Guided Instruction professional development workshops with first grade teachers. The CGI workshops are designed from current research on: the development of student mathematical thinking, instruction that promotes student mathematical thinking, the influence of teachers' beliefs on their practice, and the effect of the teacher's understanding of student mathematical thinking on beliefs, knowledge, and practice (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema, et al., 1996; Franke, Fennema, & Carpenter, 1997; Knapp & Peterson, 1995). The CGI research study on the teaching of addition and subtraction of whole numbers demonstrated that teacher knowledge of student thinking has a positive relationship to student achievement. Students in CGI classes show higher levels of expertise in problem solving than students in control groups and no difference in performance on a number skills test (Fennema, et al., 1996). The teachers in the CGI professional development program emphasized problem solving over computational skills, encouraged students to explore multiple
strategies to solve problems, listened more to their students' ideas, and used what they learned from listening to students to inform their instruction (Fennema, et al., 1996). The success of this approach for the teaching of arithmetic is the basis for “studying how to help children build on their emerging knowledge of arithmetic to provide a foundation for learning algebra” (Carpenter, et al., 2003, p. xi). Thus, the summer graduate professional development in this study focused on the teaching of algebraic thinking as outlined in Thinking Mathematically Integrating Arithmetic and Algebra in Elementary School (Carpenter, Franke, & Levi, 2003).

The CGI model of professional development does not present procedures or “ways to teach” that teachers can implement, but instead shares the way problems can be categorized into particular types and describes the strategies students use to solve these problems (Knapp & Peterson, 1995). The teacher participants discuss and debate the findings that are outlined in readings and presented in videotapes of children working the problems. The teachers and the instructor work together to decide how this knowledge of student thinking and solution strategies can be used to improve student learning in the classroom. Teachers who successfully adapt the philosophy of basing instruction on student thinking to their classroom practice cite the opportunity to interact with other teachers as significant in the change process (Franke, Carpenter, Levi, & Fennema, 2001).

Teacher Development in the Mathematics Classroom

Promoting the development of algebraic thinking in the elementary classroom requires a reform vision of the teaching and learning of mathematics. In this vision
mathematics is not a static body of knowledge that is learned by applying set procedures to given problems, but is instead a changing and expanding body of patterns whose study requires reasoning, debating, and experimentation. Students learn mathematics with understanding by constructing their own knowledge with the teacher as facilitator, not transmitter of procedures. Students are actively engaged in the creation and solution of interesting mathematical problems, which reflect the ideas of arithmetic and the concepts of algebra. The content emphasis is on understanding the mathematical concepts and the constructivist pedagogy promotes the exploration of student thinking based on the introduction of mathematics that encourages student discourse and reflection.

Implementing this type of teaching requires teachers to examine their conceptions of the nature of mathematics and their understanding of how mathematics is learned. An integral part of this analysis is the examination of the role of the teacher. Engaging in activities that focus on the teacher as a learner of mathematics allows the teacher to consider teaching as a process of facilitation or negotiation as opposed to teaching as telling or transmitting knowledge (Franke, et al., 2001; Simon & Schifter, 1991; Wood, Cobb, & Yackel, 1991). An effective professional development effort to encourage teachers to rethink beliefs about teaching and learning mathematics should include engaging the teachers as learners of mathematics in activities that model the pedagogy and content of the prescribed program, supporting field experiences that allow teachers to practice the new pedagogy, creating multiple opportunities for reflection, encouraging teachers to share their experiences, and fostering collaboration (Borasi, Fonzi, Smith, & Rose, 1999; Grouws & Schultz, 1996).
Restructuring beliefs and integrating change in classroom practice will not likely be the outcome of a one-time workshop or course. Sustained change requires at least a 2-year commitment of intellectual, emotional, and material support of the classroom teacher (Brown, Cooney, & Jones, 1990). This research study documents the impact of the weeklong course with follow-up support on the teaching practice of the participants. Critical aspects of the study include the examination of teacher beliefs about the nature of mathematics, teacher beliefs about the teaching and learning of mathematics, and the relationship of these beliefs to classroom practice.

Teacher Development of the Understanding of the Nature of Algebraic Thinking

The effective teaching of algebra based on understanding the nature of algebraic thinking is difficult at the secondary level as demonstrated by the 40 to 50 percent student failure rate (Silver, 1995) and the fact that even successful high school students require additional instruction in algebra when entering college (Harvey, Waits, & Demana, 1995). Secondary teachers have the advantage of extensive mathematics coursework and an interest in teaching mathematics content, as opposed to the typical elementary teacher. Elementary teachers, who are usually only required to take two or three college courses in mathematics content, may perceive themselves as weak mathematically, and may state a lack of interest in mathematics (Ball in Schifter, 1996a). However, these same teachers want their students to be successful at learning and understanding the required elementary school mathematics. Suggesting that elementary school arithmetic be the basis for the teaching of algebraic thinking can be the impetus for teacher examination of the nature of algebra.
The traditional algebra course that most of these teachers took in high school or college was characterized by the study of the properties of different number systems and the rules and algorithms for solving equations, inequalities, and systems of equations. The course typically represented an exercise in memorization of rules and procedures, manipulation of symbols, and applications to word problems unrelated to situations outside the classroom (Kieran, 1992; Kaput, 1995; Picciotto & Wah, 1993; Schifter, 1999). As a result most elementary teachers have a weak conceptual understanding of algebra and limited ability to perform the procedures to solve basic linear equations (Harvey, Waits, & Demana, 1995). In the graduate course, “The Development of K-8 Algebraic Thinking”, the teacher participants are introduced to the ideas of equivalence, relational thinking, conjecture development, and justification and proof in the context of arithmetic operations and properties. All of these ideas form the foundation for algebraic thinking.

In this graduate course teachers are engaged as mathematics learners when they design number sentences that represent the idea of equivalence and demonstrate the relational thinking that is the basis for the “collection of knowledge, skills, and dispositions prerequisite for understanding algebraic concepts” described by Lodholz (1990, p. 25). The teachers use mathematical knowledge and skills that permeate the elementary curriculum to enhance their understanding of the nature of algebraic thinking. By focusing on arithmetic operations and procedures the teachers foster the development of algebraic thinking in all grades as an alternative to the traditional experience of algebra as a single course. A key component in the effort to explore the nature of algebraic thinking is using teacher thinking and teacher discourse to guide instruction. The teacher
participants, like their students, build on their arithmetic knowledge as a foundation for understanding the nature of algebraic thinking.

Significance of the Study

The reform vision of the teaching of mathematics calls for teachers to explore their understanding of the nature of mathematics as they teach this reform mathematics. Successful implementation of this vision in the classroom depends on the desire of individual teachers to grow and develop effective practices (Franke, et al., 2001). The investigation of teacher development in this study is based on the conception of teacher change as a process the teacher undertakes voluntarily (Richardson, 1990, 1994a). The literature on teacher change and learning to teach documents changes in the thinking and practice of teachers, but the description of the process by which the change occurs has been limited (Wood, et al., 1991). The focus in this study is on the documentation and description of the learning experiences of teachers as they work to facilitate student development of algebraic thinking in the elementary mathematics curriculum after completion of a graduate seminar on this topic.

This qualitative investigation of the process of teacher learning/development will add to the existing literature in several ways. First, it will build on the existing research by investigating how teachers translate the idea of fostering the development of algebraic thinking into practice. Previous CGI (Carpenter, et al., 1989) and SummerMath (Little, 1986; Simon & Schifter, 1991) research has documented sustained change in teacher practice following a workshop, institute, or course with a minimum of six months of extended support. The researcher focused on the teaching of elementary school
arithmetic concepts. This research study builds on prior work, but focuses on the teaching of algebraic thinking. The teaching of algebraic thinking in the elementary classroom is a relatively new area of mathematics study with limited available research. Examining teacher practice in this area has implications for the design of elementary teacher preparation courses in mathematics, for the content of secondary mathematics teacher education courses, and for reforming secondary mathematics instruction to insure that algebra is part of the curriculum for every student in school. Secondly, it will contribute to the learning to teach literature by examining the process of change, not just the occurrence of change. This study will combine ideas from the teacher change literature and the research on the teaching and learning of algebra in documenting and analyzing the process of teacher efforts to facilitate the development of students’ algebraic thinking.
CHAPTER 2

LITERATURE REVIEW

The following review of the literature contains a detailed description of the main components of the research that inform the conceptual framework of the study. The study is structured around the idea of supporting teacher incorporation of algebraic thinking into the existing elementary school arithmetic curriculum. There are three aspects to this examination of teacher change in mathematics teaching. They include: the general notion of teacher change within the context of professional development, change in the teaching of mathematics, and change in teacher understanding of the nature of algebraic thinking.

Teacher change is viewed from the perspective of the learning to teach literature and within the context of a professional development experience. The particular professional development experience is a graduate course titled “The Development of K-8 Algebraic Thinking”. The content and structure of the course and the planned follow-up support are examined against the components of effective professional development as outlined in the research literature. The general discussion of effective professional development as a vehicle for teacher reform is narrowed to focus on the teaching and learning of mathematics teachers. The research on mathematics teacher change is integrated with the research on the nature of algebraic thinking to establish the research rationale for the study.

13
Professional Development and Teacher Change

In general, change requires that an individual or an organization become different and that initially means operating in an uncertain restructured environment that may cause fear and possibly antagonism (Sprinthall, Reiman, & Thies-Sprinthall, 1996). When the vision of teacher change is “teachers doing something that others are suggesting they do” (Richardson, 1990, p. 11), then the level of fear and antagonism is exacerbated and so is the resistance to change. In the learning to teach view of change, the perception is that teachers are continually making changes in their classrooms, and that they voluntarily seek to incorporate effective new techniques into their practice. In the process of transforming the traditional classroom environment, the teacher must have the flexibility and confidence to explore different patterns of authority and control over student learning and behavior (Pradl, 1993). Richardson (1994a) asserts that genuine change will occur only when teachers begin to think differently about the teaching and learning in their classroom and are provided with teaching practices that align with this different way of thinking.

Teachers may be introduced to these practices with professional development in the form of college courses or seminars, research projects, school district in-service experiences, or federal programs for professional development. The source of the professional development experience is not as critical as the design of the professional development experience. The training model of professional development which focuses on “expanding an individual repertoire of well-defined and skillful classroom practice” (Little, 1993, p. 129) is not congruent with an educational emphasis on reform in subject matter teaching, assessment, social organization of the schools, equity, and
professionalization of teaching. Teachers need the time and opportunity to investigate, experiment, consult, and evaluate how broad principles translate into the daily practice in their particular classroom environment (Little, 1993). Teachers require access to effective and sustaining professional development experiences.

Guskey (1995) asserts that there is no precise statement of effective professional development, but there is a framework of guidelines that support quality professional development. Professional development should be an individual and organizational process. Change must be gradual and incremental, teams of individuals should work together, regular feedback on results is important, along with continued follow-up, support, and pressure which must be provided. Educational innovations must be integrated into the overall program. Guskey (1995) also stresses that successful professional development is a process not a single event. The Richardson (1994b, 1990) research on practical inquiry in literacy instruction, which was based on a staff development project, incorporated these guidelines into a specific content area. The teachers in the project adopted changes in instructional practice when the changes did not violate beliefs about the nature of teaching and learning, engaged students, allowed the teachers the level of control they felt was necessary, fit teachers' personal sense of what works, and helped them to respond to organizational demands such as the mandate for high test scores.

Examining teacher development in professional practice schools, Lieberman and Miller (1990) refine the idea of continual voluntary change into the concept of continuous inquiry into practice. They identify five essential elements of a culture that supports this continuous inquiry in teaching and learning. These elements include norms of
collegiality, trust and openness, time and opportunity for inquiry, teaching and learning in context of the actual classroom, restructuring of the teacher's leadership role, and networks of collaboration. In this new academic culture teachers learn how to work together and are involved in continuous learning about student motivation, engagement, connection, and prior knowledge. Dana, Campbell, and Lunetta (1997) analyze science teacher education and describe meaningful and worthwhile professional development based on the ideas of constructivism, reflection, and professional community. In their study the nature of constructivism is a vision which implies that the teachers function as learners and continuously construct knowledge about the teaching and learning of science connected to classroom practice. In addition, reflective inquiry requires that teachers examine, discuss, and evaluate their teaching in order to take action to change teaching practice. Professional community requires the individual teacher, peers, and the entire school community to collaboratively work to examine and improve teaching practice.

Three specific professional development programs for mathematics and science teachers demonstrate that the framework of guidelines and the five cultural elements that support teacher inquiry are integral to quality professional development experiences. These programs are the Educational Leaders in Mathematics Project in the SummerMath for Teachers Program, the Eisenhower Professional Development Program, and Cognitively Guided Instruction. The ELM (Educational Leaders in Mathematics) intervention consisted of two summer institutes, a year of classroom follow-up, and opportunities for the participants to conduct workshops for colleagues. The ELM professional development was based on encouraging teachers to examine the nature of mathematics and the process of learning mathematics in determining how to teach
mathematics, allowing teachers to function as mathematics learners at their level of mathematical understanding, and providing follow-up support and supervision (Simon & Schifter, 1991).

The results of a Teacher Activity Survey (Garet, et al., 2001) returned by 1,027 teachers (72% response rate), who participated in Eisenhower Professional Development activities, confirmed the research literature assumptions about the characteristics of professional development design that can positively impact teacher adoption of new practices and enhance content knowledge and skills. According to the survey response, quality professional development is sustained over time. That means the professional development is composed of longer activities with opportunities for in-depth discussion of content, pedagogy, and student difficulties and it extends over time to permit teachers to use the new practices in the classroom and receive feedback on their efforts. The focus is on subject matter content and how students learn. Teachers are engaged as active learners of the content. The active learning can include observing other teachers and being observed, implementing a new practice in the classroom, reviewing student work, giving presentations, or leading discussions. Finally, the professional development activities must be integrated with other aspects of school life. The Eisenhower Professional Development activities must connect with other professional development experiences, align with state and district standards and accountability assessments, and encourage communication among other teachers and administrators (Garet, et al., 2001).

The Cognitively Guided Instruction professional development focuses on using student thinking to teach first grade addition and subtraction. The CGI approach to professional development is not the training model where preconceived sets of
procedures are disseminated to the teachers. Instead, the instructor shared findings from a study on how first grade students solved complicated addition and subtraction problems. The sharing consisted of readings, presentations, and videotapes of children solving these problems. The teacher participants then discussed and analyzed the findings, interviewed children to see if they could actually solve the problems this way, and worked collaboratively with the researchers/instructors to determine how to use this knowledge about children's thinking in their teaching practice. Twenty-two elementary teachers who participated in the workshops were contacted three to four years later about their use of student thinking in their teaching. Those teachers who developed a new teaching practice based on the understanding of student thinking also demonstrated the ability to learn with understanding. Learning with understanding is characterized by continued or generative additions to understanding, acquiring knowledge rich in structure and connections, and learning motivated by one's own inquiry efforts. The teachers stated that support from other teachers and the long-term support and commitment of the researchers was important in their continuing efforts to change their teaching practice (Franke, et al., 2001).

Teacher engagement in the process to produce change in practice and enhanced understanding requires that teachers examine their beliefs about teaching and learning. A significant feature of professional development programs that produce sustained change in teachers' thinking and practice is the focus on teacher beliefs about the nature of teaching and learning (Dana, et al., 1997; Richardson, 1994a; Schifter, 1996a, 1996b; Schifter & Fosnot, 1993; Simon & Schifter, 1991; Sprinthall, et al., 1996). Professional development designed to encourage teachers to rethink beliefs about teaching and
learning mathematics should include engaging teachers as learners of mathematics in activities that model the pedagogy and content of the new program, supporting teachers in the field as they practice the new pedagogy, creating multiple opportunities for reflection, and encouraging collaboration among teachers at the same school or different sites (Borasi, et al., 1999; Grouws & Schultz, 1996). Restructuring beliefs and integrating sustained generative change in classroom practice requires at least a six month to 2-year commitment of intellectual, emotional and material support of the classroom teacher (Brown, Cooney, & Jones, 1990; Little, 1986). The graduate seminar on fostering the development of algebraic thinking and the follow-up supervision and support documented in this research study incorporate these elements of a quality professional development experience.

Mathematics Teaching and Teacher Change

The way teachers think about mathematics is a major determining factor in how they teach mathematics (Cooney, 1999; Simon & Schifter, 1991). Considering this finding, one of the critical aspects of this study is the examination of teacher beliefs about the nature of mathematics, beliefs about the teaching and learning of mathematics, and the relationship of these beliefs to classroom practice. The relationship between beliefs and teacher practice has been conceptualized in three distinct patterns (Cooney, 2001; Richardson, 1994a). One view is that teachers change their practice first and the success of the new teaching approach results in a change in beliefs. The counterpoint to this view is that changes in the teacher’s beliefs result in a change in practice. Usually the teacher becomes dissatisfied with student participation and learning that is occurring with the
current teaching methods and believes that an alternative approach is necessary before proceeding to initiate change in teaching practice. The third view is an interactive relationship between beliefs and practice. With this view the process of change may begin with either changes in beliefs or changes in practice based on the context of the teaching environment, individual beliefs and knowledge, and individual experience (Richardson, 1990, 1994a).

The general concept of a belief system appears to be a dynamic structure that changes and reorganizes as teachers constantly evaluate beliefs in the context of their experiences in the classroom. Typically a belief does not require factual empirical evidence to exist. Green (1971) identifies three components of a belief system and how they relate to each other within the system. The first component is the quasi-logical structure of primary and derivative beliefs. An example of this would be a teacher’s primary belief that mathematics must be presented in a “clear” manner to students. The derivative belief is the importance of thorough lesson planning and preparation for this “clear” presentation. The second component concerns the conviction with which beliefs are held. Beliefs can be either central or peripheral in the individual’s belief system. Central beliefs are the most strongly held beliefs and peripheral beliefs are more vulnerable to change. The derivative belief that a teacher must be prepared may be central to the teacher’s need to maintain control and authority. The third component of Green’s belief theory is that beliefs are held in isolated clusters and each cluster may be protected from a relationship with other belief clusters. The idea of belief clusters could explain why teachers may hold beliefs that appear to be inconsistent with their actual teaching practice (Brown, et al., 1990; Thompson, 1992).
Examining a teacher's view of the nature of mathematics and the translation of that view into classroom practice makes more sense when grounded in Green's belief theory which allows for primary and derivative beliefs, central and peripheral beliefs, and belief clusters. The importance of understanding teachers' conceptions of the nature of mathematics was significant in a study of 12 elementary teachers working to reform their teaching. Researchers discovered that teachers with limited visions of mathematics were not likely to incorporate genuine reform into their teaching (Cooney, 2001). Skemp (1978) delineates between instrumental and relational mathematics based on the type of knowledge each requires. Instrumental mathematics is a set of plans or step-by-step procedures for completing a particular type of problem or mathematical task. Relational mathematics allows for a conceptual framework that permits the construction of many plans for completing a problem or mathematical task. Teachers with an instrumental understanding of the nature of mathematics represent teachers with those limited visions of the nature of mathematics. Similarly, Ernest (1989) has described three conceptions of mathematics identified as the instrumentalist view, Platonist view, and the problem-solving view. For the instrumentalist mathematics is a set of useful facts, rules, and procedures to be used to pursue the solution to a problem. In the Platonist view mathematics is a unified static body of knowledge connected by logic and meaning. The mathematician can discover, but not create, the content of this body of knowledge. The problem-solver views mathematics as a growing field of human endeavor in which mathematical patterns are explored and formed into knowledge. The problem-solver approaches mathematics as a process of inquiry with the results always open to revision and expansion (Cooney & Shealy, 1997; Thompson, 1992).
Examining teacher beliefs within the context of mathematics, Cooney, Shealy, and Arvold (1998) characterize teachers as isolationists, naïve idealists, naïve connectionists, or reflective connectionists (Cooney, 1999, 2001). The positions echo the degree to which teachers accommodate or reject reform in their teaching methods and demonstrate reflective practice. The isolationist structures beliefs in clusters and resists any new ideas that are not consistent with his or her existing beliefs. The isolationist sees little worth in any aspect of a teacher education program that does not support an existing view of the right way to teach. A naïve idealist accepts without question whatever the existing authority recommends. The authority may be professors, classmates, supervising teachers, administrators or any significant other who is relied upon to provide the essence of knowledge. The naïve connectionist identifies connections between mathematics content and pedagogy and sees tensions or contradictions in the teaching of mathematics, but cannot resolve the differences between belief and practice. A reflective connectionist can identify the tensions and works to resolve the differences by restructuring beliefs and accommodating new pedagogy in reflective practice.

A fifteen-month study of the meanings about mathematics and teaching mathematics demonstrated by prospective secondary math teachers related the four characterizations to what and how the prospective teachers ultimately taught. Examining two preservice teachers with identical mathematics content background and similar grades, the researchers (Cooney, 1999) found that the preservice candidate with the isolationist stance performed with an uncritical acceptance of her teacher centered style emphasizing mathematics as a set of procedures to be learned by the students. The second preservice student exhibited a more reflective connectionist stance and was critical of his teaching
performance in the teacher-centered classroom, which did not allow him to teach the type of mathematics that he valued. A teacher's individual belief structures are important in deciding what gets taught and how it gets taught (Cooney, 1999; Wilson & Goldenberg, 1998). As a result, mathematics teacher education would benefit if focus were placed on the belief structures of preservice and in-service teachers to foster the development of a "more reflective and adaptive teacher."

In the CGI research on teacher engagement with children's mathematical thinking a continuum of teachers' instructional and belief levels was designed to document teacher understanding of student thinking and implementation in instruction. The original classification scheme separated beliefs and practice (Fennema, et al., 1996), but subsequently the researchers have integrated beliefs and practice into a single table of benchmarks. The researchers did not assume each teacher moves through the stages in the same time frame, nor is the movement unidirectional and linear. The stages represent skills and understandings acquired by the teachers and reflect how teachers regard the teaching and learning of mathematics. The levels of engagement with children's mathematical thinking range from Level 1 where the teacher does not believe students can solve problems unless they are taught how and does not ask children how they solved problems to Level 4B where knowledge of individual student thinking drives the teacher's curriculum (Appendix E).

In the original CGI study of 21 primary teachers, there were 17 teachers who achieved final ratings that were higher than initial ratings on beliefs and instruction. In this group of teachers 6 changed beliefs before they changed instruction, 5 changed instruction before changing beliefs, and 6 simultaneously changed beliefs and instruction.
(Fennema, et al., 1996). More significant than the pattern of change was the relationship between the level of change and the pattern of change. When teachers changed classroom practice before changing beliefs, the change transpired at Levels 1 to 3. All the teachers who moved beyond Level 3 experienced a change in beliefs. Level 4 beliefs were necessary for any teacher to change instruction to move classroom practice to Level 4. A change in beliefs must precede or accompany change in practice at Level 4 (Franke, et al., 1997). A different CGI research study (Fennema, et al., 2001) of 22 teachers who participated in professional development focused on teacher understanding of the development of students' mathematical thinking concluded that attending to student thinking facilitated sustained and generative change in teaching practice (Franke, et al., 2001). Both studies concluded that consideration of student thinking is related to instructional change. An integral part of the professional development experience in this study is the emphasis on understanding and using student thinking as the basis for instruction.

The goal of this research study is to document teacher integration of the ideas and activities presented in the graduate course into their teaching practice. That is, how the professional development activities and experiences translate into changed or reformed teaching practice. The described research states that teachers who work to genuinely reform their teaching have a broad view of the nature of mathematics and their beliefs about teaching and learning are consistent with this view (Cooney & Shealy, 1997). These beliefs will be regarded "as dispositions to act, which include both utterances and actions" (Cooney, 2001, p. 21) and the best way to access these "belief systems about mathematics and the teaching of mathematics is through the study of school
mathematics” (p. 27). The school mathematics studied is teacher understanding of the
nature of algebraic thinking.

Algebraic Thinking as a Basis for Change

The current promotion of “Algebra for All!” needs to be framed against the impact of
computers on how we work, communicate, live, and play and the resulting change in the
types of technical, and intellectual skills demanded in the workplace (House, 1988).
Factual knowledge and algorithmic competency will be required less than the
understanding of concepts and the capacity to apply mathematical ability to solve
problems in new and creative ways. The NCTM Standards and the Nevada Standards
support this reformed view of the type of mathematics required in a competitive
technological society. Teaching algebra to an increased number of students and having
more of these students acquire and demonstrate real competence in algebra is
problematic. The response to this problem has been to mandate an algebra course for all
students, require completion of an Algebra I course for high school graduation, and
include questions that necessitate knowledge of algebra on high school proficiency
examinations. The mandated algebra courses, particularly at grade 8, are often instituted
without comprehensive curriculum development or professional development for the
teachers (Kaput & Blanton, 2000). The actual content taught may bear little resemblance
to the ideas of algebra. Failure rates may be high and grade inflation may be common.
(Silver, 1995; 1997). Providing greater access to algebra instruction may also be
interpreted as placing more students in a traditional algebra course or instituting a two-
year Algebra I course that generally focuses on basic algorithms for solving equations.
But, traditional Algebra I courses typically have a failure rate of 40 to 50 percent (Silver, 1995), which means that enrolling more students with marginal skills in the course can only increase the failure rate and overall student frustration with traditional algebra.

It has been suggested (Lodholz, 1990) that efforts to provide algebra success for all students should focus on that lower one fourth of the student population which does not usually consider studying algebra. The algebra to be studied should not be a single course based on the teaching of rules and procedures, but a collection or web of knowledge and skills necessary for understanding algebraic concepts (Kaput, 1995, 2000a, 2000b; Lodholz, 1990). The lofty goal of providing access to a quality learning experience in algebra for all students needs to be examined in the context of the history of the development of algebra. Reviewing the format of a traditional algebra experience in contrast to the nature of algebraic thinking, delineating the difference between arithmetic and algebra, illustrating cognitive obstacles to the learning of algebra, and investigating teaching strategies and curricular changes that promote the development of algebraic thinking, each has implications for achieving a quality algebra experience for all students.

History of the Development of Algebraic Thought

An examination of the history of the development of algebraic thought provides a knowledge base of necessary agreed-upon facts, insight about the nature of learning algebra, an understanding of how difficult it is to comprehend and formalize particular concepts, and a context for valuing the thinking of students (Arcavi, 1995). Sfard’s generation of a three stage process (Kieran, 1992) explaining conceptual development
from operational to structural mathematical thinking parallels the historical stages in the creation of algebraic symbolism. The operational/procedural/structural duality or the process-object interpretation of mathematical concepts will be explored in the framework of the historical evolution of algebra.

Establishing the historical origins of algebra requires consensus on a definition or interpretation of the nature of algebra. Sfard (1995) asserts that most historians of mathematics agree that algebra represents generalized computations, but disagree on whether modern symbolic notation is the only means to represent these computations. Accepting the use of “the term algebra with respect to any kind of mathematical endeavor concerned with generalized computational processes, whatever the tools used to convey this generality” (Sfard, 1995, p. 18) allows an examination of algebra prior to Diophantus (c. 250 AD). The three stages of algebraic thought are rhetorical algebra, syncopated algebra, and symbolic algebra (Kieran, 1992).

Rhetorical algebra, which predates Diophantus (c. 250 AD), was purely verbal and relied on ordinary language to describe problems which allowed the computation of unknown values from available numerical data. No special symbols or notation were used to represent the unknown values. The syncopated algebra of Diophantus included some use of special symbols to represent fixed unknown values, but no general methods of expressing solutions appeared. The syncopated algebra was not accepted on any widespread scale until the beginning of the 17th century. Rhetorical and syncopated algebra relied purely on operational techniques of solving problems, that is by working backwards or reversing the calculations. Rhetorical algebra was the norm until Vieta (1540-1603) introduced the use of a letter to stand for a “given” quantity as well as the
unknown quantity. Vieta’s use of the variable heralded the beginning of symbolic algebra, which allowed mathematicians to state general solutions and to use algebra to prove rules for numerical operations and relations. Vieta’s notation moved algebra beyond the realm of computations/operations and “permitted algebra to be more than merely a procedural tool; it allowed the symbolic forms to be used structurally as objects” (Kieran, 1992, p. 391). British mathematician George Peacock is credited with introducing algebra as a formal science and as an abstract method of reasoning. Based on the principle of permanence of form, he distinguishes between symbolic and arithmetic algebra. Formal properties of a numerical structure do not depend on the numbers on which they are performed, but in the way in which the operation is defined or presented (Menghini, 1994). The introduction of a variable, as a given and subsequent creation of abstract algebra, moved the development of algebraic thinking to the structural level.

In the historical development of algebraic thought, the operational process preceded the structural concept, and a similar process characterizes the learning of algebraic concepts. Sfard (1991) states that the operational construct is the initial step in the acquisition of new mathematical ideas, and that the transition from computational operations to the structure of abstract algebraic objects is a long demanding process for the learner via the stages of interiorization, condensation, and reification.

Interiorization is the stage at which the learner becomes familiar with processes that will be the basis for new concepts. As the learner gradually becomes more skillful at performing these processes he can interiorize them, and no longer finds it necessary to physically perform the process, such as counting objects to understand natural numbers. The learner can then mentally represent the process. Condensation is a stage in which
long sequences of operations are compressed into smaller units. In other words, the process is regarded as a whole and may be combined with other processes for purposes of comparison and generalization. The learner can now easily alternate among different representations of a concept. The learner remains in the condensation phase of algebraic thinking as long as the new concept or entity is regarded as a process. When the operation or process is perceived as an object in its own right, then the concept is reified. "Reification, therefore, is defined as an ontological shift - a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure" (Sfard, 1991, pp.21-22). The process of concept development, which culminates in reification, requires that the learner sequentially proceed through all three stages.

The basis of Sfard’s (1991) model of concept development is the construct that specific mathematical ideas are only fully developed when they “are conceived both operationally and structurally” (p. 23). Operational competence and understanding must precede structural comprehension in Sfard’s model of the development of algebraic thought. As a result, moving from operational processes to abstract objects enhances understanding of mathematics, and at certain phases of acquisition of mathematical knowledge the lack of a structural conception may impede further growth (Sfard, 1991; Sfard & Linchevski, 1994). Just as the historical development of algebra from an operational process to a structural object was a lengthy and demanding process, the “difficulties experienced by an individual learner at different stages of knowledge
formation may be quite close to those that once challenged generations of mathematicians" (Sfard, 1995, pp. 15-16).

Kieran (1992) maintains that the vision of algebra represented as a development from an operational process to a structural object is not reflected in the traditional algebra course. Sfard and Linchevski (1994) assert that the structural approach is initially evident in instruction before the student is conceptually ready to deal with the duality of the process-object idea. Algebra teachers and algebra textbooks introduce the idea of variables as given values to be manipulated simultaneously with the presentation of a variable as an unknown, although historically mathematicians have demonstrated that accepting the use of variable as a given value is a more difficult concept. Students are exposed to stand-alone algebraic formulas and are expected to insert the formulas as objects into equations or inequalities before they have experience with the formulas in arithmetic operations. This example of the structural level of teaching does not “capitalize on the students’ natural propensity for an operational approach by beginning with processes rather than with ready-made algebraic objects” (Sfard & Linchevski, 1994, p. 224). Arithmetic instruction in prior courses can be modified to reflect the algebra model of thinking forward (Linchevski, 1995), thus providing the operational basis for future work in algebra. Until students conceptually understand the abstract ideas or underlying structure of algebraic manipulations, they view their experiences with algebra as meaningless symbol pushing (Arcavi, 1995; Bell, 1995; Booth, 1989; Picciotto & Wah, 1993; Sfard & Linchevski, 1994; Stacey & MacGregor, 1999).
Traditional Algebra

The structural view of algebra, as representing general numerical relationships and operations in special notation and symbols, is prevalent in traditional algebra textbooks. The general topics listed in the textbooks include properties of real and complex numbers, creating and solving first- and second-degree equations in one unknown, simplifying polynomial and rational expressions, symbolic and graphical representation of linear, quadratic, exponential, logarithmic, and trigonometric functions, and sequences and series (Kieran, 1992). Introduction of probability and statistics, increased work with inequalities, more emphasis on functions and functional notation, and problems requiring the use of graphing calculator technology have been incorporated into algebra textbooks in the past ten years.

Picciotto and Wah (1993) indicate that the weaknesses of the traditional algebra experience are a one-dimensional emphasis on symbol manipulation, the authoritarian role of the teacher as the sole dispenser of knowledge, the perceived pointlessness of the work, development of narrow skills based on repetitive drill, and organization of the topics as self-contained units of knowledge. For purposes of this study, the traditional algebra course is characterized by this emphasis on the manipulation of symbols, the memorization of rules and procedures as the essence of algebraic study, class work unrelated to other branches of math and science or situations outside the class, the teaching of topics in self-contained units or chapters, dissemination of all knowledge by the teacher, and inadequate time allowed for understanding and internalizing a new idea before proceeding to the next idea. Problem solving is not perceived as an integral part of the course, but is used as enrichment or an opportunity for the student to earn
additional grade points. The word problems presented require the use of a single skill or application of a step-by-step procedure modeled by the teacher (Kieran, 1992; Kaput, 1995; Picciotto & Wah, 1993). Hence, the main activity in this traditional algebra class is “to present rules for computation, which students are expected to diligently memorize, and then to give word problems as an exercise in application” (Schifter, 1999, p. 68). In the traditional version of school algebra, students in the lower one fourth of mathematical ability do not attempt the course and many marginal ability students fail. Even students who succeed in completing two years of high school algebra often “have poor conceptual and procedural knowledge of it and will take an additional algebra course upon entering a college or university” (Harvey, et al., p. 75).

The topical content of the school algebra curriculum is not the only limiting factor in the teaching and learning of algebra. The problem seems to be the view of the nature of algebra by the individual teacher or institution and how that view translates into instructional strategies and activities. Usiskin (1988) formulated four conceptions of algebra, which apply to an examination of the nature of school algebra. Algebra may be conceived as generalized arithmetic, procedures for solving specific types of problems, examination of relationships among quantities, and the study of structures. Algebra as generalized arithmetic is the study of properties of various number systems. Algebra as procedures constitutes the rules and algorithms for solving equations, inequalities, and systems of equations. School algebra is characterized by these two conceptions of the nature of algebra (Harvey, et al., 1995). Algebra as the examination of relationships among quantities is the study of numerical, symbolic, and graphical representations of functions and relations. The study of algebraic structures includes abstract algebra, linear
algebra, and the structure of comparative number systems beginning with the counting
numbers in elementary school. The traditional algebra course gives limited emphasis to
these last two conceptions of algebra although the study of functions, relations, and
algebraic structures may appear at a superficial level in advanced high school math
courses. The fact that these four conceptions of algebra usually appear sequentially in the
school curriculum does not mean that they have to be taught in that order to be learned
successfully (Harvey, et al. 1995). Analyzing the structure and operations of arithmetic
can be an effective basis for promoting the development of algebraic thinking in the
school curriculum (Demana & Leitzel, 1988; Herscovics & Linchevski, 1994; Lee &

Algebraic Thinking or Arithmetic Thinking

The implication in the previous description of a traditional algebra course is that the
course does not truly teach algebraic thinking nor does it allow the student to reach the
structural level of algebraic understanding. Exploring that premise requires an
explanation or definition of the nature of algebraic thinking. Researchers in mathematics
education describe a demarcation or difference between arithmetic thinking and algebraic
thinking. In a study of secondary students in Mexico City, Filloy and Rojano (1989)
established a “didactic cut” between arithmetic and algebra when operating on an
unknown value in an equation. In the evolution of student thinking from arithmetic to
algebra the cut appears in the transition between equations of the form $Ax \pm B = C$ and
$Ax \pm B = Cx$. Equations such as $3x + 5 = 11$ where the unknown value appears only on
one side can be solved by undoing or reversing the operations. The left side of the equation represents a sequence of arithmetic operations performed on known or unknown numbers, and the right side represents the results of performing these operations. The arithmetical nature of the equation does not require the student to operate on or with the unknown value.

Equations of the type $3x + 5 = x + 11$ cannot be solved simply by reversing or inverting the arithmetic operations. Solving this type of equation requires the student to operate, not on the familiar numbers of arithmetic, but on what is represented by the algebra symbols. The student has to modify existing ideas of arithmetic and at the same time preserve the arithmetic knowledge base. The secondary students participating in the Mexico City study could not immediately alter acquired arithmetic skills and concepts to solve the new equations. An area model and a balance model were presented to the students to help in solving these problems, but the concrete models exposed and created other problems in the development of the ability to solve the equations. Appropriate teacher intervention was deemed necessary to guide students in connecting these operations on a concrete model with operations at an abstract level. Achieving the ultimate goal of using the solution of a known situation to solve a more general abstract situation required expansion of arithmetic understanding and the acquisition of algebraic symbols and structure. The researchers state that the cut between arithmetic and algebra “corresponds to the major changes that took place in the history of symbolic algebra in connection with the conception of the ‘unknown’ and the possibility of ‘operating on the unknown’” (Filloy and Rojano, 1989, p. 20).
Herscovics and Linchevski (1994) describe a "cognitive gap" between the natures of arithmetical thinking and algebraic thinking based on the inability of students to operate spontaneously with or on the unknown. Twenty-two Montreal parochial school students, who had received no prior instruction in algebra, were interviewed and observed as they attempted to solve a set of 50 equations. The problems to be solved included equations with single quantities on the left or right side of the equal sign, the completion of single or multiple operations in the solution, and the double occurrence of unknowns on one or both sides of the equal sign. The students were encouraged to use calculators for the numerical calculations as they verbalized their thinking and subsequent solution strategies to the interviewers. Students demonstrated the ability to solve equations of the type $Ax \pm B = Cx \pm D$ with unknown quantities on both sides of the equals sign, but their solution strategies were based on the arithmetic procedures of inverse operations, substitution of numerical values, and approximations of techniques used in equations where the unknown appeared once. The study confirmed the inability of the students to operate spontaneously, which is without instruction, with or on unknown values.

The students worked around the unknown values in constructing solutions to the equations. Examination of the student solutions revealed problems such as lack of acceptance of different meanings for the equal sign, performing arithmetic operations sequentially from left to right without regard to the particular operation, misinterpreting the concatenation of a number and a letter in an expression such as $3x$, and detaching a number from a preceding minus sign. These problems, which can be significant obstacles in the learning of algebra, originate in the arithmetic background of the students. These researchers assert that "the traditional curriculum in arithmetic may not
be sufficient and a bridge between arithmetic and algebra has to be constructed” (Herscovics & Linchevski, 1994, p. 60). Both studies emphasize the importance of developing facility in arithmetic and expanding arithmetic knowledge in order to acquire the ideas of algebra.

The concept of a didactic cut between arithmetic and algebra and the idea of a cognitive gap in student ability to spontaneously operate on or with an unknown value suggest a need to facilitate the transition from arithmetic to algebra. The transitional level of pre-algebraic knowledge has been described as an operational level of mathematical knowledge between arithmetic and algebra (Filloy & Rojano, 1989) or as a field of inquiry based on intuitive algebraic ideas about solving for unknowns in first-degree equations and student generated solutions (Herscovics & Linchevshi, 1994).

Pillay, Wilss, and Boulton-Lewis (1998) propose that the transition from arithmetical to algebraic thinking is a cumulative sequence of knowledge development and that each stage of the sequence is a prerequisite for the next stage. The developmental model was the result of data collected from a three-year longitudinal study of students in four Brisbane, Australia state schools. Based on interviews and observations of student problem-solving procedures both before and after instruction in algebra the following transitional phases were identified:

<table>
<thead>
<tr>
<th>ARITHMETIC</th>
<th>PRE-ALGEBRA</th>
<th>ALGEBRA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 + 3 = 5</td>
<td>( x + 7 = 16 )</td>
<td>( x + 3 = 2x - 1 )</td>
</tr>
<tr>
<td>35/7 + 8 = 13</td>
<td>( 3(x + 7) = 24 )</td>
<td>( x + 3y + 4x - 2y = 15 )</td>
</tr>
<tr>
<td>Operational laws, numerical answers and</td>
<td>Recognition of unknown then variable in equations then</td>
<td>More than one unknown or variable, operating on</td>
</tr>
<tr>
<td>equals as meaning each side of equals is</td>
<td>expressions, concatenation, equals as meaning each side of the</td>
<td>with the unknown/variable, equals as equivalence.</td>
</tr>
<tr>
<td>the same value.</td>
<td>equation is the same value.</td>
<td>Lowest level solution:</td>
</tr>
<tr>
<td>Lowest level solution:</td>
<td>Lowest level solution: inverse procedures</td>
<td>balance procedures</td>
</tr>
<tr>
<td>numerical procedures</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Pillay, et al., 1998, p. 91)
Analysis of the data from the 33 students who remained in the study provided specific information about areas of cognitive difficulty in each phase. As outlined above, examples of the type of problem characteristic of each level accompanied with required concepts and solution techniques necessary to solve the problem enforced the view that "for students to understand algebraic concepts they must have a firm understanding of arithmetic laws and operations followed by pre-algebraic principles" (Pillay, et al., 1998, p. 99).

Stacey and McGregor (1999) assert that the essence of algebra, the necessity of operating on and with unknowns, is not present in sections of the algebra curriculum as outlined in *A National Statement for Australian Schools* and the *Victoria Curriculum and Standards Framework*. The Victorian Standards list four methods for solving equations. The methods include doing the same to both sides of the equals sign, using a graph, guess-and-check, and backtracking, which is working backwards or reversing the operations (Lovitt & Clarke, 1987). Algebraic thinking is required for doing the same to both sides of the equation and graphing. In guess-and-check the student must be able to read some algebra notation, but the substitution of the guessed values is an arithmetic operation. Backtracking, the dominant approach in classroom instruction and in the textbooks, is essentially an application of arithmetic. Examining 7 Year 10 textbooks that were supposed to require algebraic thinking documented the appearance of an average of 14 equations with the unknown on both sides of the equation and an average of 2.7 problems that required creation and solution of an equation with the unknown on both sides. There were a limited number of problems that required the algebraic approach of operating on and with unknowns; therefore, when "the answer to a problem
is obvious to students without algebra, by and large the algebraic ideas will not get across" (Stacey & McGregor, 1999, p. 31). The difference between arithmetic and algebra is again defined by the techniques and thinking required to solve an equation of the form \( Ax + B = C \) as opposed to solving \( Ax + B = Cx \).

In all the perspectives on the boundaries between arithmetic and algebra, the researchers confirmed the existence of specific learning difficulties or cognitive obstacles to the development of algebraic thinking. These areas or obstacles include the meaning of equals as equivalence, knowledge of arithmetic laws and operations, meaning of unknown and variable including concatenation in representation, understanding algebraic expressions, and solving linear equations with unknown on both sides (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; MacGregor & Stacey, 1997; Pillay et al., 1998; Stacey & MacGregor, 1997, 1999). Examining the character of these obstacles as related to the learning of algebra enhances the understanding of the nature of algebraic thinking.

Cognitive Obstacles in Understanding Algebra

A cognitive obstacle (Tall, 1989) is a specific piece of knowledge, embedded in the mind of the learner, which has been successfully used to solve particular problems over time. Eventually this specific knowledge is not sufficient to solve new problems nor is it readily adaptable to the acquisition of new ways of thinking. Thus, "the learner's existing cognitive structures are difficult to change significantly, their very existence becoming cognitive obstacles in the construction of new structures" (Herscovics, 1989, p. 62). Specific cognitive obstacles in the learning of algebra include the idea of the equals
sign representing equivalence, the notion of algebraic expressions, prior arithmetic content knowledge and facility with operations, the concept of variable, the ability to solve equations with the unknown on both sides of the equals sign, and translation of word problems into equations.

Algebraic thinking and the successful solution of equations require an understanding of the concept of equality. The equal sign, which is the symbol used to indicate equivalence, is not always understood in terms of an equivalence relationship. Preschool and elementary students typically interpret the equal sign as a "do something" signal or as an instruction to perform a computation or state the answer (Kieran, 1981, 1992; Linchevski, 1995; Pillay et al., 1998). Elementary students regard the equals sign in $2 + 3 = □$ as indicating what the numbers should add up to. If the problem is presented as $□ = 2 + 3$, then the typical response is to state that the problem is backwards and rewrite it as $2 + 3 = □$ to find the answer. Presented with an equality such as $4 + 3 = 2 + 5$ students respond that the answer should be after the "=" and compare the results of writing the two equalities $4 + 3 = 7$ and $2 + 5 = 7$. The students do not regard the "=" as denoting an equivalence relationship. High school algebra students continue with this unidirectional mode of reading an equivalence statement (Linchevski, 1995) and persist in viewing the left-hand side of the equation as a series of operations to be performed, and the right-hand side of the equation as the result of these operations. In the Herscovics and Linchevski (1994) study equations presented as $23 = n - 37$ were rewritten as $n - 37 = 23$ by students in order to successfully solve them.

Significant work needs to be done in order for students to develop the concept of an algebraic equation as an expression of equivalence. Kieran (1981) suggests teaching the
use of the equal sign as equivalence by having students construct arithmetic equalities. Starting with one operation on each side and moving on to two operations on each side these renamed “arithmetic identities” provide the groundwork for more meaningful experiences transforming expressions and equations to equivalent forms (Greenes & Findell, 1999). Examples of these arithmetic identities include:

\[
\begin{align*}
3 \times 6 &= 2 \times 9 \quad \text{(same operation)} \\
3 \times 6 &= 10 + 8 \quad \text{(different operations)} \\
5 \times 3 + 3 - 1 &= 6 \times 3 - 3 + 2 \quad \text{(two different operations)}
\end{align*}
\]

The right side of the equation does not have to be the answer, but can be an expression equivalent to the left side of the equation. The notion of equals as meaning equivalence is essential to any meaningful understanding of the nature of algebra.

Another essential element of thinking and working within the context of algebra is a good arithmetic knowledge base. Milton emphasizes “the need for children to be thoroughly prepared in arithmetic structure, and experience generalisation situations as a necessary part of being introduced to formal algebra” (1989, p. 14). The arithmetic curriculum should have children examine structure in numbers, emphasize the methods used in arithmetic investigations, and require children to explore patterns and record the resulting generalizations. Instructional methods must include the use of concrete materials or manipulatives that allow the learners to do, discuss, record, and reflect as they work on the tasks (Milton, 1988, 1989).

In this approach the concepts of algebra are a “by-product of making arithmetic sensible” (Peck & Jencks, 1988, p. 85). Utilizing numerical examples to deal with order of operations, negative numbers, commutativity, associativity, the distributive law,
bracketing, and detachment of terms from operations presents the concepts of algebra in a format accessible to students and addresses some of the cognitive difficulties in learning algebra (Demana & Leitzel, 1988; Linchevski, 1995; Linchevski & Herscovics, 1996; Pillay, et al., 1998).

Solutions to arithmetic word problems can be presented so that the students are encouraged to develop the algebra mode of thinking forward. A distance problem with two cars traveling toward each other on the same road, but moving at different speeds and leaving at different times from opposite locations could be solved by a chart detailing the distance for each car after 1, 2, 3, and . . . hours. The pattern revealed on the chart could then be generalized for an unknown number of hours. General principles such as the fact that the sum of two consecutive numbers is always an odd number could be verified by numerical examples or substitutions. Justification of results should be an integral part of the arithmetic curriculum. The traditional arithmetic curriculum and associated pedagogy does not support the transition from arithmetic to algebra (Lee & Wheeler, 1989).

The concept of variable is troublesome for students, and this "difficulty can be basic to a lack of success in algebra" (Demana & Leitzel, 1988, p. 64). MacGregor and Stacey (1997) categorized student work with variables in two groups based on Küchemann’s (Kieran, 1989; Küchemann, 1978) six levels of student interpretation of the use of algebraic letters. In the first group, students would ignore the letter, assign the letter a numerical value, or use the letter as the name for an object or as representing the object. In the second grouping, students would treat the letter as a specific unknown number which could be operated on directly, a generalized number which could take on several values, or a variable in the sense that the letter represented a range of specific values with
a systematic relationship existing between two sets of such values. These different uses of variable suggest increasing levels of difficulty.

A research study of 3,000 English secondary school children revealed that the majority of the 13 to 15-year old students participating in the study could not conceive of a variable as a generalized number nor could they consider a variable as representing unspecified values with a relationship between the sets of values (Küchemann, 1978). Also, some students appear not to understand that letters selected to represent function values are arbitrary, and mistakenly link the change in the name of a variable with a change in the value of a function (Sutherland, 1991). Concatenation or the linking of a variable with a number such as in the expression $8m$ exemplifies student misconceptions about the variable in arithmetic operations. When instructed to replace the letter $m$ with the number 2, the student who has an insufficient understanding of the implied multiplication process will write 82 as the result. The correct interpretation of $8m$ as $8 \times m$ will produce 16 as the result.

The importance assigned to different uses of variables relates to the four different conceptions of school algebra (Usiskin, 1988). In algebra as generalized arithmetic, the variable is used to generalize patterns such as translating the equality $4 + 5 = 5 + 4$ into the general notion of $m + n = n + m$. There are no unknowns in the generalization of patterns. Algebra, as the study of procedures for solving specific types of problems, requires that variables are unknowns or constants, as represented by $6x + 5 = 29$. The student is instructed to simplify and solve in this conception of algebra. Algebra as the study of relationships among quantities requires that the variables actually vary. The variable is an argument that signifies the domain value of the function or a parameter that
represents a number upon which other numbers depend. In finding the equation of a line through (5, 1) with slope 2, using the formula \( y = mx + b \), the \( m \) is constant and the unknown \( b \) can be found. The resulting \( y = 2x - 9 \) contains \( x \) and \( y \) as arguments or dummy variables which can be replaced with many values. The \( x \) and \( y \) are not unknown values. Algebra as the study of structures requires treating the variable as an arbitrary symbol to be manipulated. The variables are marks on paper, without numerical referents. An example of this structural algebra would be asking the student to factor an expression such as \( 5x^2 + 88mx - 36m^2 \) with the resulting factors \((5x - 2m)(x + 18m)\). In this algebra as a structure, there is no pattern to generalize, no equation to solve, and no function or relation with the variable as argument. The Küchemann hierarchy and the Usiskin uses of variable demonstrate the difficulty inherent in the concept of variable and the importance of introducing variables in arithmetic as pattern generalizers.

The traditional high school algebra course focuses on equations with one unknown. The sequence of instruction usually begins with the concept of variable, followed by algebraic expressions, and then the solution of equations (Kieran, 1989). Algebraic expressions are typically the vehicle for instruction on order of operations, grouping, combining like terms, and the implicit uses of variables. Researchers question this sequence of instruction because "simplification of algebraic expression creates serious difficulties for many students" (Linchevski & Herscovics, 1996, p. 42). From the perspective of the learner, the combining of terms and grouping in algebraic expressions is a significant problem. The operational or computational steps required to simplify the expression cannot be separated from the obtained result, which is an object. The algebraic expression must be viewed from both an operational and structural perspective.
In an expression such as $5x + y$ the operations cannot be performed, but the expression is the result of the simplification process (Linchevski & Herscovics, 1996). The algebra student must be capable of reifying the operations into the object which $5x + y$ represents. Sfard (1991) reminds us that moving from the operational conception of algebra to structural understanding is a demanding and lengthy process. Student difficulty with algebraic expressions exemplifies this.

Davis refers to the problem of operating with algebraic expressions as the "name-process" dilemma (Herscovics and Lincheski, 1994) because an expression such as $5x$ indicates the process of multiplication and at the same time is the name of the answer. Another theory pertinent to the examination of algebraic expressions as a cognitive obstacle to the learning of algebra is Collis' theory of Acceptance of the Lack of Closure. Collis states that algebra students regard expressions such as $(x + 9)$ as incomplete because they cannot "hold unevaluated operations in suspension" (Sutherland, 1991, p. 40). Students need to see $(x + 9)$ replaced by a third value before the expression is meaningful. The inability to accept this lack of closure in expressions is particularly evident in the thinking of 6 to 10 year old students. Equivalence statements such as $3 + 4 = 2 + 5$ need to be rewritten as $3 + 4 = 7$ or $2 + 5 = 7$ before students accept the identity. Collis (Herscovics, 1989) maintains that it is not until the age of 15 that students can hold unevaluated operations in suspension and operate with them as objects. Sequencing instruction in algebra so that equations, not algebraic expressions, are the vehicles for teaching order of operations, grouping, combining like terms, and introducing variables would be a more effective way to organize the algebra curriculum (Linchevski and Herscovics, 1996).
Using the ability to solve an equation with the unknown value on both sides of the equal sign as the benchmark that separates arithmetical thinking from algebraic thinking, Linchevski and Herscovics (1996) reported success teaching the following procedure for solving equations with the unknown on each side of the equal sign to students who had no prior formal instruction in algebra.

**Summary of solution procedure.**

8n + 11 = 5n + 50

8n + 11 = 5n + 11 + 39

8n = 5n + 39

5n + 3n = 5n + 39

3n = 39

n = 13

**Explanation of solution procedure.**

Original equation.

Decompose numbers and maintain equivalence.

Cancel identical numerical terms.

Decompose unknown.

Cancel identical terms.

Solve equation in with one unknown.

Connections to arithmetic via observation of students' intuitive thinking and knowledge of numerical relationships provided the basis for the interactions between student and interviewer in the instruction. The transition from arithmetic thinking to algebraic thinking, as evidenced by the ability to solve the equations with unknowns on both sides of the equal sign, was the result of teaching this solution technique.

The student-professor problem posed to 150 freshman engineering students is a classic example of the difficulties students experience translating word problems to equations. Researchers asked the students to write an equation representing the problem:

There are six times as many students as professors at this university. Use S for the number of students and P for the number of professors. Only 63% of the students
responded correctly and 68% of the errors were reversals of the variables. Students wrote
6S = P instead of S = 6P (Herscovics, 1989). The students' difficulties in stating the
correct equation were not the result of the inability to read and understand the nature of
the problem nor from the lack of algebraic fluency (Lochhead & Mestre, 1988). Based
on videotaped interviews the types of thinking that led to the mistake were identified and
labeled as syntactic or semantic thinking. Students who demonstrated syntactic thinking
literally mapped the sequence of words in the problem into a sequence of symbols or
performed a left-to-right match of words to symbols. Six times as many students as
professors became 6S = P. Semantic thinkers associated that same incorrect equation (6S
= P) with the written language meaning of the problem. Their equation was a description
of relative size or comparison of the two groups in which the equals sign did not
represent equivalence and the S became a label for the group of students, not a variable to
represent the number of students. The unsuccessful translation of written language to
mathematical language reflected the interference of natural language in the appropriate
algebraic application of variable and equivalence. Algebra instruction needs to provide
experience reading and writing in mathematics with an emphasis on the interpretation of
mathematical symbol strings (Lochhead & Mestre, 1988).

Fostering the development of algebraic thinking so that students can bridge the
didactic cut, cross the cognitive gap, explore the essence of algebra, or follow the
sequential development of knowledge for understanding algebraic concepts appears to
rest on teaching equivalence, number operations and properties, the concept of variable,
and number identities in the arithmetic curriculum. Instruction should also include
appropriate justification of the reasonableness of the performed operations and number
properties and opportunities to translate written language into mathematical language. A true understanding of equals as equivalence, facility with arithmetic operations, comprehension of number properties such as identities, inverses, multiplication by zero, commutativity, associativity, the distributive law, and the grasp of the concept of variable as a specific unknown or generalized number would be demonstrated by the student in such a curriculum.

Mathematics Curriculum Based on Algebraic Thinking

The essence of the algebra and the pedagogy a curriculum based on algebraic thinking has been described in a variety of ways by different researchers, but there are common elements. Bell (in Arcavi, 1995) states that school algebra courses should be based on the three strands of generalizing, forming and solving equations, and working with formulas and functions. Student assignments should be engaging situations based on inquiry and the search for patterns that express the structure of algebra in appropriate symbolic language. A task representative of this type of engaging activity would be to examine general patterns of the properties of numbers in a 2X2 matrix. The student should discover patterns such as the fact that the sum of the numbers on one diagonal equals the sum of the numbers on the other diagonal. Students would be required to justify their observations verbally and in mathematical language. The teacher should listen to student explanations of solution strategies and document student thinking to understand student conceptualization of ideas, execution of procedures, and difficulties in learning a concept or a procedure.
Seventeen years ago, participants at The Fifth International Congress on Mathematics Education (Davis, 1985) recommended the “thoughtful exploration of algebraic ideas” in the elementary schools. The exploration of algebra would allow the students to build mental representations of key concepts such as variable, open sentence, solutions of open sentences, graphs of the solutions, negative numbers, and functions based on learning experiences or activities designed by the teacher. The elementary school math learning experiences should be worthwhile, not just an exercise in fun and games. Students should be actively engaged in the learning experience and should talk about the activity in simple language. Student reflections should lead to writing about the concepts, procedures, and results in appropriate symbolic notation. The teacher would start the learning experience by presenting a problem situation such as the “guess the function” task. In this task one student makes up a rule, others tell what number to use in the rule, the number is used in the rule, and then other students attempt to guess the result. The elementary students would be looking for patterns, relationships, and appropriate mathematical notation to express the generalizations of algebra (Booth & Watson, 1990).

Picciotto and Wah (1993) recommend the same type of approach in the teaching of a formal ninth grade algebra course. Their vision of school algebra is an interface between themes and tools situated in the context of real world or invented problems that introduce algebra concepts for the student to explore, develop, and review. The spiral organization of the topic content of this algebra course requires the student to revisit concepts in multiple representations (Friedlander & Tabach, 2001) and permits extended exposure to the key ideas. A group of 24 investigations on the topic of area allowed students to explore the “area” concept using math tools such as graph paper, geoboards, Lab Gear
manipulatives, algebra tiles, or graphing calculators. The tools transform the traditional algebra class into an inquiry format that encourages discovery learning and cooperative problem solving. Symbol manipulation is regarded as another tool to promote understanding of numbers, variables, operations, equations, functions, and the general structure of algebra.

Harvey, Waits, and Demana (1995) pursue the nature of algebra and a new vision of the high school mathematics curriculum beyond the formal ninth grade course. Using Usiskin's four conceptions of algebra: algebra as generalized arithmetic, algebra as the study of procedures, algebra as the study of relationships among quantities, and algebra as the study of structures, these researchers assert that the first two conceptions dominate the teaching of school algebra through high school and college precalculus. The study of algebra beyond the introductory course should focus on algebra as the study of relationships among quantities, i.e. functions, with some attention to the formal study of abstract algebra structures. The teaching and learning of these conceptions of algebra are possible with the technology of the graphing calculator and computer systems utilizing programs such as Derive, Mathematica, Maple, Matlab, Cayley, and IBM Math Exploration Toolkit. The technology facilitates numerical, graphical, and symbolic representations of relationships, permits exploration of local and global function properties, supports a more complete study of classes of functions, allows the study of functions not in a recognized class, and generally promotes the inclusion of topics that are typically left to calculus or college level precalculus. The algebra curriculum can now encompass all four conceptions of algebra. Integrating the graphing technology into the teaching and learning of algebra will permit topics to be treated more
comprehensively in longer continuous time periods with less repetition. The graphing calculator is the tool for the students to explore, discover, and resolve problems as they work individually, in cooperative groups, or as a whole class. The teacher’s role in implementing the use of this technology is to create a learning environment that promotes student interaction, communication, and construction of meaning about math (Pugalee, 2001).

Although the emphasis in this view of algebra is on the use of the graphing technology, it also contains elements common to the elementary and ninth grade curricular models of algebraic thinking. All of the models feature the common aspects of generalizing, forming and solving equations, and working with functions that are implemented in a constructivist classroom environment characterized by student exploration and discovery, justification of procedures and results, and communication of these results in verbal language and appropriate symbolic notation.

A reform version of a mathematics curriculum founded on algebraic thinking has been proposed by James Kaput (1995, 2000a, 2000b) who regards algebra as a “web of knowledge and skill” and not as an institution founded on the traditional two year algebra courses distinguished by static teaching, perpetual remediation, and review based on textbook dictated topics and sequence. The web of algebra knowledge and skill is composed of five forms of reasoning: generalization and formalization of patterns, manipulation of formalisms, abstracted structures, functions and variables, and modeling and phenomena controlling languages.

Generalization and manipulation of formalisms are the kernel strands of algebra that serve as the foundation for the system. Generalization is described as identifying
common patterns, procedures, structures, and their relations in a situation and extending
the reasoning and communication about the common elements beyond a specific
situation. Formalizing demands that the student look beyond the manipulation of
symbols to see what the symbols and the manipulation represent, or in other words, to
learn with understanding. The abstracted structures of algebra develop from the
generalization and formalization experiences in mathematics, such as examining
symmetry of figures, working with modular arithmetic, or manipulating letters in words.
The concept of function or relation can be developed even in the elementary grades by
examining quantities that change over time such as temperature, heights of plants over
time, or cost of an item in relation to the number of items purchased. The initial
representation of a function would be expressed in the natural language of the students
followed by the acquisition of the formal symbolic language of algebra. Abstracted
structures and functions are the topic strands in the web of algebra.

The primary goal of the study of algebra is to use functions, relations, and their
accompanying algebraic structure to describe and to reason about phenomena. Modeling
can be done with computer simulations or graphing calculators allowing students to build
generalizations in the appropriate symbolic language from specific examples. The
teaching and learning of this new algebra must begin in the early grades by: building on
the informal knowledge of students, being integrated into the learning of other subjects,
including all the forms of algebraic thinking, building on the natural language and
cognitive abilities of the students, encouraging reflection and communication about what
they have learned, and promoting active student engagement in the learning process.
Kaput (2000a) includes the common elements of generalizing, forming and solving
equations, and working with functions as the basis of an algebra learned in an environment characterized by student exploration and discovery, justification of procedures and results, use of technology, and communication of results and reasoning in natural and mathematical language.

Teacher Understanding of Algebraic Thinking and Practice

The current organization and content of the algebra curriculum in the schools does not advance the agenda of providing a quality learning experience for all students nor does it foster the genuine development of algebraic thinking in the more capable students. A curriculum based on algebraic thinking would promote the teaching and learning of algebra as the knowledge, skills, and dispositions required for understanding the operational and structural nature of algebra. The traditional algebra course based on the structural level of teaching should be modified to allow a more operational approach based on arithmetic processes that are already familiar to the students. These arithmetic processes should include experience with the idea of equals as an equivalence relationship, justification of number operations and properties, writing and exploring patterns in number identities, working with variables, and translating word problems into the appropriate numerical or symbolic notation.

The numerical procedures of arithmetic would support inverse procedures (calculations) to solve for an unknown on one side of the equals sign. Both types of procedures would then support the concept of balance or equivalence in operating on and with unknowns on both sides of the equals sign as learners move from the realm of arithmetic into algebraic thinking. Significant changes in the mathematics curriculum of
the elementary school and the middle school would not be necessary to support this new vision of arithmetic. Significant changes in how this curriculum is taught would be necessary. Teaching algebraic thinking throughout school mathematics and implementing the formal teaching of algebra to an increased number of students demands authentic algebra reform.

Kaput (1995) proposes reform in the teaching and learning of algebra within the dimensions of breadth, integration, and pedagogy. Breadth of reform refers to his vision of algebra as a web of knowledge and skill, not exclusively as the content of the traditional two years of high school algebra. Integration implies the learning of algebra while studying other subjects such as science, business, computer science, or engineering or within mathematics connections as demonstrated by the different representations of algebra concepts in algebra, geometry, calculus, or statistics. Pedagogical reform requires an active exploratory learning environment where students investigate engaging problem situations, discover patterns and generalizations, utilize technology in the explorations, communicate their reasoning and solutions, reflect on the results, and understand that the teacher values student discourse and thinking. Implementing the pedagogical reform requires that teachers possess the mathematical knowledge to create this learning environment for students.

This study will examine the teacher's efforts to expand their understanding of the nature of algebra, integrate the learning of algebra with the teaching of arithmetic, and explore reform pedagogy that supports this expanded view of the nature of algebraic thinking. As part of this process the teachers have to examine the roles they assume in a classroom environment that focuses instruction on student explanations and
understanding of mathematics. Schifter (1996a, 1996b) calls this reconstructing the professional identities of teachers as they develop as mathematical thinkers, enlarge their understanding of content, manage the classroom environment, redefine responsibilities for student learning, listen to student thinking, collaborate about instructional issues, and contribute to the conversation about effective reform of mathematics teaching. This study of teacher development or continuous inquiry into practice will investigate the following questions:

1. How does participation in a professional development experience influence teacher understanding of the nature of algebraic thinking?
   - What aspects of the teacher vision of the nature of mathematics support interest in teaching to develop student algebraic thinking?
   - What changes can be documented in teacher understanding of algebraic thinking?

2. What is the effect of a professional development experience exploring the development of algebraic thinking on the practice of the teachers?
   - What changes can be documented in teacher use of arithmetic-based activities that promote algebraic thinking?
   - How do the teachers demonstrate they value the teaching of algebraic thinking?
   - What factors encourage/discourage teachers to engage in instructional practices that foster the development of algebraic thinking?
3. How do the teachers incorporate student discourse and examination of student thinking into mathematics teaching focused on the development of algebraic thinking?

- What changes can be documented in teacher use of student discourse/thinking in instruction?

- What changes can be documented in teacher comprehension of the nature of student learning that is occurring
CHAPTER 3

METHODOLOGY AND DATA DESCRIPTION

This study of the effect of a professional development experience on teacher practice uses qualitative research methods. Qualitative methodology aligns with the research goal of examining how teachers go about changing their mathematics instruction to use arithmetic operations and properties as a basis for developing student algebraic thinking. The aim of this research study is to focus on the process of change, in particular what personal, situational, and institutional factors are conducive to effecting this change in practice (Patton, 1987). Qualitative research is characterized by naturalistic inquiry, inductive analysis, fieldwork, understanding of the process from the participant’s perspective, and rich description of the findings (Merriam, 2001; Patton 1987). The design of the qualitative study is flexible and emergent in response to changing conditions in the work. The sample selection is small, nonrandom, and purposeful, and the researcher is the primary collector of data in the natural setting of the participants (Merriam, 2001).

Primary data collection occurs through observation of the teachers as they work in their own classrooms and through interviews, conducted at their respective schools. Assignments and journal reflections completed as part of the professional development experience are also examined. The research sample consists of three elementary teachers treated as individual case studies. The qualitative case study design is utilized to acquire
in-depth understanding of the teaching situation and meaning for those individuals involved. The research interest is the discovery of the process, not analysis of specific outcomes (Merriam, 2001). The advantage of a case study for this research is that it permits the intensive examination of the interaction of many variables in order to provide a complete understanding of the situation (Merriam, 1985). Case study data collection techniques of observation, interview, and document analysis are well documented in the literature on qualitative research methodology (Merriam, 1985; Patton, 1987). However, the guidelines for categorizing, coding, analyzing, and interpreting the findings in case study research are not as clearly delineated. The uniquely individual nature of the cases selected for study and the necessity of balancing detailed description with analysis precludes a precise framework for writing a case study (Merriam, 1985). Although Patton (1987) has proposed a process/outcomes matrix to analyze case study data, the task in this study is to create a systematic way to assess the meaning of the findings and present the findings in a form that the reader can understand. A significant aspect of this task is the description of the research process employed in the examination of the teachers' exploration to effect change.

The appropriateness of the case study method for this study of three teachers working to implement and sustain change in practice is derived from the fact that teaching is a very individual endeavor enacted in unique classroom environments, but the role of schoolteacher and school teaching contains some elements common to the classroom experiences of every teacher (Lortie, 1975). The uniqueness of the individual teacher's experience will be brought out in the detailed description of each case. The questions raised in the interviews and the content of the journal reflections guided the organization
of the written descriptions into a sequence of categories. Background information about teaching experience, mathematical expertise, beliefs about the teaching and learning process, and motivation for teaching algebraic thinking provided the context for observing each teacher in the classroom. The observation of an illustrative mathematics lesson, which is defined as a representative sample of how each of the three teachers goes about the practice of teaching mathematics, was followed by the observation of a lesson designed to develop algebraic thinking. Each case study teacher selected the particular teaching episodes, focused on algebraic thinking, for the researcher to observe. Informal discussions and scheduled interviews during the course of the study revealed each teacher’s definition of algebraic thinking, provided opportunities for feedback and reflection, and generally supported the teacher efforts to implement the algebraic thinking content and pedagogy.

The analyses within case and across-cases will reveal themes or patterns common to teacher exploration of change in practice. Examples of case study methodology are prevalent in mathematics education research (Franke, et al., 1997), particularly in studies that focus on teachers working to effect change in practice (Fennema, Franke, Carpenter, & Carey, 1993; Schifter & Fosnot, 1993; Wasley, 1994; Wilson & Goldenberg, 1998; Wood, et al., 1991). Schifter (1996a) declares that successful reform of teaching requires that teachers meet and communicate regularly to discuss the issues and pedagogical problems that arise in the change process. Furthermore, cases or stories are powerful vehicles for teachers to communicate what they have come to understand about their students, schools, subject matter, and the teaching and learning process as they work to construct new ways of being teachers (Schifter, 1996a, 1996b). The goal of this
 qualitative study is to add to this dialogue by documenting the efforts of three teachers to conceptualize and enact change in their teaching.

Structure of the Professional Development

Teachers in this study attended a summer graduate course titled “Topics in Elementary/Secondary School Mathematics – The Development of K-8 Algebraic Thinking.” This class met for 2½ hours each day on 5 consecutive days during the summer of 2003. Two separate sections of the class were taught, one in June and one in July. The instructor is an elementary mathematics education professor at the university. The professor had attended a Cognitively Guided Instruction seminar on this topic in the summer of 2002. In addition, he had taught this course once in Summer 2002. The professor focused on student explanations of problem solving strategies as the pedagogical bases for his teaching of algebraic thinking. This instructor believes the mathematics content is important but that it is the process of reasoning and problem solving which provides the vehicle for learning the content. The purpose of the course was to assist teachers in using the operations and properties of arithmetic as the basis for helping students develop algebraic thinking. This way of learning arithmetic is what Arens & Meyer (2000) previously described as teaching and learning based on algebraic thinking, allowing younger students to be successful as they encounter more complex mathematical ideas. The pedagogical emphasis is the focus on student thinking and student discourse.

Adhering to the Cognitively Guided Instruction philosophy of professional development, the professor did not present techniques or ways to teach, but instead
shared the arithmetic-based components of algebraic thinking that appear in the CGI
textbook (Carpenter, et al., 2003), which is based on current algebraic thinking research.
The teacher participants had the opportunity to function as mathematics learners as they
worked on the same type of problems given to the children in the study. Then the
teachers were asked to create problems appropriate for the grade or content level of their
respective students. The problems were shared and discussed in class. Videotapes of
children successfully working these types of problems and videotapes of teachers
demonstrating how to teach concepts based on student discourse and student solution
strategies were an integral part of each day's class. Teachers were required to reflect
daily on the class activities and videos. Their daily reflections were electronically
submitted to the instructor at the end of the course. In addition to the daily reflections,
teacher participants were required to write a response to a specific mathematics education
issue or question.

Content of the Professional Development

The topical components of algebraic thinking examined in the course were
equivalence, relational thinking, conjectures, justification, and proof. The topics,
examples, and videos are all part of the curriculum in Thinking Mathematically
Integrating Arithmetic and Algebra in Elementary School (Carpenter, et al., 2003). A
discussion of the types of arithmetic-based problems that are utilized to explore these
topics is appropriate in order to have a full understanding of the mathematical content
examined by the workshop participants. The focus of the mathematical content was on
the types of problems that represent cognitive obstacles to the learning of algebra. The
idea of equivalence meaning "the same as" instead of "equal to the answer" is the main goal of student work with equality. Students are asked to solve number sentences such as $8 + 4 = \square + 5$, determine whether number sentences such as $3 + 2 = 5 + 1$ are true or false, and write their own number sentences. The teacher's role was not to immediately reinforce correct responses or identify incorrect responses, but to facilitate student discussion of the proposed solutions.

Relational thinking facilitates the learning of arithmetic facts and concepts and is the basis for work in algebra. Examples of number sentences that relate number operations included:

$$3 \times 9 = 9 + 9 + 9$$
$$3 \times 9 = 18 + 9$$
$$4 \times 8 = 16 + 16$$

Examples of number sentences that represent relations among number facts included:

$$3 \times 6 = 2 \times 6 + 6$$
$$5 \times 7 = 4 \times 7 + 7$$
$$6 \times 9 = 5 \times 9 + 9$$
$$9 \times 5 = 10 \times 5 - 5$$

For older children or secondary students, the numbers were larger or number sentences using decimals, fractions, or exponents were created. Number sentences were used to suggest conjectures such as the fact that adding zero to any number results in that number, or that one times any number is that number. Videos of first grade students postulating these conjectures were viewed in the class.
Justification, or the proof of conjectures, was presented as a “range of arguments that children use to show a conjecture is true” (Carpenter, et al., 2003, p. 85). Children initially justified concepts and procedures to themselves, but when they shared or discussed these ideas they had to use convincing arguments to justify their results to others. The levels of justification were an appeal to authority, justification by example, or general arguments such as restating the original conjecture, building on concrete examples, building on previously justified conjectures, and using counterexamples (Carpenter, et al., 2003). The level of justification became more sophisticated as students grew in their ability to generalize arguments. As such, justification was presented as an important part of learning with understanding.

Teaching arithmetic based on algebraic thinking is a new area of teaching and learning, so it was important to delineate the topics and problems that are to serve as the foundation for this instruction. The pedagogical content requires the active engagement of students, discussion and writing about the mathematics in simple language, and structuring the curriculum and instruction on student thinking, while the mathematical content rests on the examination of the structure in numbers, exploration of patterns, and introduction of variables (Carpenter, et al., 2003; Milton, 1988, 1989; Peck & Jencks, 1988). The teacher participants were all products of a traditional algebra learning experience. This introduction to a vision of algebra as being a web of knowledge (Kaput, 2000a, 2000b) or a collection of dispositions, knowledge, and skills (Lodholz, 1990) served to challenge their understanding of the nature of algebra.
Participants

The participants in this study were drawn from the teachers enrolled in two sections of the Summer 2003 professional development focusing on the fostering of algebraic thinking. The teachers in the graduate course were either currently teaching or had completed a field experience in a large urban school district in the southwestern United States. There were 35 total participants in two separate workshops. Enrollment in the June session totaled 25, and the July session enrollment was 10 teachers. Gender distribution of the teachers was 29 females and 6 males. Teaching experience of the participants ranged from a single semester of student teaching to 16 years in the classroom. Grade level assignments of the teacher participants included 12 primary teachers, 10 upper elementary teachers, 9 middle school teachers, 1 high school teacher, and 3 teachers not currently teaching. Of those teachers who were not currently teaching, one had high school teaching experience, one completed a middle school field experience, and one had recently completed elementary student teaching and was beginning coursework for a graduate degree. Among the 35 participants, 24 individuals indicated interest in getting support from the university instructor to implement the ideas from the course in their teaching practice.

Data Collection

The three types of data collected in qualitative research studies include in-depth open-ended interviews, observations, and written documents. The written documents can include open-ended questionnaires, personal reflections, journal entries, or program records (Merriam, 2001; Patton, 1987). Merriam (2001) also asserts that qualitative
studies in education commonly use only one of the three data collection techniques and occasionally two of these techniques, but in qualitative case study research all three types of data collection are prevalent. The data collection techniques used in this qualitative case study are observation, interview, and analysis of written documents. The units of analysis, or the cases studied, are the three elementary teachers. Each teacher represents a bounded system or entity of intrinsic interest, not a sample from a population and as such is treated as a distinct case (Stake, 1978, 1985).

Initial data collection consisted of the distribution of a survey of teacher opinions regarding the nature of algebra and the teaching and learning of algebra. The survey required the completion of demographic information about years of teaching, educational background, and teaching assignment. The survey content consisted of 11 statements requiring a response on a six-part Likert Scale, two open-ended problem-solving questions, and one response to a general comment about the teaching of algebra (Appendix A). All the course participants completed the survey prior to any class instruction. Teacher participants were allowed to provide their names and school location as optional information.

The teachers completed the survey during the first class meeting. Question four addressed two different issues. The question asked the teachers to determine the appropriateness of a required algebra course in the middle school and at the same time to state their position on when algebraic thinking should be taught. One teacher commented that the question required two separate responses. The researcher and course instructor questioned the high rate of agreement with question eleven, which states that student mistakes can be effectively resolved by student explanation of the work and input from
other class members. The wording of the question and the fact that many of the teachers knew the university professor was an advocate of this type of teaching might have skewed the responses toward agreement with the statement.

The tallied responses represent a general picture of the overall attitudes of the 35 teachers at the beginning of the course (Table 1). Sonya, Josh, and Paula identified themselves on the survey. Their initials are written in the response category that each selected. It should be noted that the responses of the three case study teachers aligned with the responses of a majority of the teachers. The survey data was not a factor in the selection of teachers to participate in the case study.

Additional data collection during the graduate course included daily email reflections about the content and structure of the class, and email responses to questions posed by the instructor and researcher (Appendix B). The instructor required that the daily reflection and question response be submitted electronically. Each day the email from the previous day was to be included. On the Tuesday following the end of the class, all five daily responses were to be submitted as a final document. Number sentences and conjectures created by the teachers during the group work in class were also collected for analysis. The June class was videotaped for documentation of the topic content and pedagogy. Detailed field notes were taken during the July class. The researcher was present as an observer/participant during both classes. At the end of the week, the teachers were asked to provide contact information if they were interested in follow-up support in the fall to assist them in implementing the teaching of algebraic thinking.

During October 2003 an initial interview was conducted with each of the three individual teachers selected from the pool of teachers interested in follow-up support.
The interview took place in the teacher’s classroom at the end of a school day. Each interview was recorded on audiotape and extensive notes were taken. Four to six classroom observations were scheduled over a five-month time period from November 2003 to March 2004. Detailed field notes were taken during each classroom observation. The school district human subjects review board did not permit video-taping or audiotaping of the observations and requested that pseudonyms be used for any student names. Principals at all three sites provided written permission for the teachers to participate in the research study. The observations were followed by informal discussions about the observation and private email correspondence between the case study subject and the researcher. At the end of the five-month period of classroom observations, each teacher met with the researcher to respond to the exit interview questions (Appendix B). Again the interview was conducted in the teacher’s classroom. A copy of the description of each case was sent electronically to the individual teachers. The teachers were asked to comment on the accuracy of the description and to make any suggestions regarding any additional content that they thought should be included. The final descriptions are the result of this collaboration between subject and researcher. Although the formal observations were completed in March, all three teachers agreed to invite the researcher and university instructor to visit their classrooms before the end of the school year. The researcher also continued to provide support for the teachers through email communication for the duration of the school year.
Teachers in the Case Study

The selection of the individuals for the case studies was based on indicated interest in follow-up support, the content of their daily reflections, comments made in class, survey results, and their specific teaching assignment. The selection of three individuals, for further study represents a purposeful selection of information-rich cases for study in depth. These information-rich cases are the sources of information about significant issues central to the purpose of the evaluation, thus the term 'purposeful sampling' (Patton, 1987). The selection of the individual cases represents typical case sampling and maximum variation sampling. The pseudonyms Sonya Henderson, Josh Abernathy, and Paula Whitford were used to identify the individuals in the three cases. Pseudonyms were also used for all student names.

Teachers in the study are employed in a large metropolitan school district in the southwestern United States. During the 2003-2004 school year the district operated 289 schools with a total enrollment of 268,357 students. The student population of the district represents very diverse ethnic backgrounds (Table 2) with that diversity reflected in many of the 179 elementary schools. A concern in design of the study was to reflect this diverse student population in the choice of teachers and their corresponding school environments. Initially two first grade teachers, working in an "at risk" school, had volunteered to participate in the study. Their school was classified as "at risk" based on the number of students eligible for free or reduced lunches, school percentile scores on standardized tests, and the percentage of Second Language students. A new principal was appointed at the school over the summer. Both teachers expressed concern about exploring a nontraditional approach to the teaching of mathematics under the supervision
of a new administrator. As a result of the administrative change, the teachers decided not to participate in the research study.

The final list of candidates in the case study does reflect different levels of diversity in the student population and different achievement levels on standardized assessment instruments. Sonya Henderson teaches in a school with a student population that represents ethnic backgrounds almost identical to the district overall. The percentile scores on the Iowa Test of Basic Skills (Table 4) for her school are close to the average district scores. Josh Abernathy teaches in an environment with a different mixture of ethnic backgrounds than in the district and the standardized test scores of the students are slightly above the district average. Paula Whitford teaches in a suburban school with a predominantly White student population that achieves well above the district average on standardized tests.

Each teaching situation reflects a different level of collaboration or peer support for reform of practice. Ms. Whitford acknowledges that she is left alone in her portable to teach the way she wants. Mr. Abernathy is required to participate in 130 hours of mathematics and science teacher training because of school participation in a federal grant. He has the support of the grant project director and the teacher assigned to his school to assist classroom teachers with the teaching of mathematics and science. Ms. Henderson works closely with the teacher in the adjacent classroom, but has no formal collaborative network in her building.

Patton (1987) reminds us that "a qualitative profile of one or more typical cases is presented in order to describe and illustrate to those unfamiliar with the program what is typical-not to make generalized statements about the experiences of all participants" (p.
Sonya Henderson is that typical case in this study. Maximum variation sampling is purposeful sampling that aims to describe the central themes or major outcomes that appear across program or participant variation (Patton, 1987), and that is sometimes described as identifying and looking for individuals who represent a wide range of characteristics of interest in a study (Merriam, 2001). Although an at-risk or disadvantaged school population and environment is not represented in this study, there is a range of student ability and ethnicity represented in the three school environments.

In this study the typical teaching situation is one teacher working alone in a classroom. The professional development and teacher change literature have emphasized that follow-up support and the collaboration and interaction with other teachers is significant in sustaining reform (Borasi, et al., 1999; Franke, et al., 2001; Grouws & Schultz, 1996). In view of this literature it is important to include teachers working alone in their individual classrooms and teachers with a support network or the opportunity for collaboration in this study. Maximum variation sampling is represented in the selection of one teacher with on-site opportunities for collaboration and interaction and the selection of two teachers working on their own in their respective schools.

Qualitative research methodology is characterized by the efforts of the researcher to understand situations or programs as a whole and to design research that is responsive to emerging issues. Using the qualitative research framework, there is a search in this study for evidence that the teachers make changes in their practice. The analysis of their efforts includes exploring how teachers incorporated algebraic thinking into the teaching of arithmetic operations and properties, how this influenced their understanding of the nature of mathematics, and how instruction focused on student thinking. It is this process
of making change in mathematics teaching and learning that represents the situation to be understood in this case research study. The process begins with a detailed description of the background and experiences of each teacher.

Case One: Sonya Henderson

Sonya Henderson is a first grade teacher at Hilltop Elementary School. She is in her fifth year of teaching at the same elementary school. After a successful student teaching experience, Sonya was hired midyear to teach first grade at Hilltop Elementary School. During her second year of teaching she taught an all male fourth/fifth grade combination class because of low enrollment at the first grade level. In her third year of teaching she was reassigned to first grade and has since taught at that grade level. Ms. Henderson earned an undergraduate degree in Business and worked for the county Parks and Recreation Department in the Safe Key Office for a year and a half before returning to the university to earn a graduate degree in education. Sonya stated that she realized working in an office was not the appropriate environment for her. She found that she really loved working with kids, so she decided to become a teacher.

Mathematics is a subject that always came easy to her in school. It was not until her graduate work in education, particularly the elementary math methods course, that she began to “understand the concepts beneath the math that I had always known how to do.” The fact that math comes easy to her makes it a struggle to explain it to the students. She fears that her students will lack conceptual understanding of the mathematics even though they may perform well on standardized tests. Sonya asserts that she didn’t want to be the type of teacher that “just produced these robots that did things because that’s how they were told to do them and they didn’t understand them.” The decision to enroll in the
summer course, on fostering the development of algebraic thinking, was based on interest in taking a class that contained mathematics content. The majority of the university courses or the school district training classes for elementary teachers focus on the topic of literacy. All of her content courses taken after the completion of her Master's degree were in the area of literacy. While recognizing the importance of literacy instruction in the primary grades she states that effective teaching of mathematics is also important.

Professional development opportunities for the teachers are an integral part of the program at Hilltop Elementary School. The principal encourages teachers to keep up on current instructional strategies in literacy and mathematics. The school is in the second year of participation in a $500,000 National Reading Education Association (NREA) grant. As a direct result of the grant literacy specialists were hired, books and other resources were purchased, and a university professor meets with the teachers once a month to facilitate the Reading and Writing Workshop approach to literacy instruction. For mathematics, one teacher at the school is a designated Mathematics Site Trainer. The Mathematics Site teacher attends regional meetings on topics such as the components of an effective mathematics lesson, adoption of new textbooks, and aligning instructional materials to the state and district standards. Substitutes are provided so that the Mathematics Site teacher can attend the all day training sessions. The Mathematics Site teachers return to their schools and share the ideas from the training sessions with the other teachers at their school. There were three scheduled training days during this academic year. After seven months into the research study Ms. Henderson reluctantly acknowledged that she was the Hilltop Mathematics Site Trainer. She downplayed the significance of her contribution to the mathematics program at her school by explaining.
to the researcher, she had told the principal she would do it only if no other teacher wanted to.

The principal at Hilltop Elementary regards Ms. Henderson as an outstanding member of her staff and thoroughly supports her participation in the research study. The principal requested that I schedule a meeting with her to explain the purpose of the study and to discuss the instructional strategies that promote the development of algebraic thinking. At our meeting I demonstrated the types of number sentences that Sonya used to teach equivalence, relational thinking, and conjecture formulation. The principal demonstrated understanding of the concepts and genuine interest in how the students reacted to the ideas. She had approved participation in the study based on Sonya's interest in teaching algebraic thinking. The principal trusts Ms. Henderson to make good decisions regarding instruction and management of students. The whole atmosphere of the school is welcoming and nurturing with an emphasis on all the positive aspects of working with children in a public school.

The Hilltop Elementary School student population represents an ethnic distribution of students with approximately 5% more Hispanic students and 5% fewer Black students (not of Hispanic origin) than is represented in the overall student population of the school district (Table 2). The percentage of students eligible for free or reduced lunch is 5% less than the percentage for the entire school district. The special education population, the number of English Language Learner students, and the student transiency rate are within one point of the percentage rates for the district as a whole (Table 3). On the fourth grade 2002-2003 Iowa Test of Basic Skills the percentile rankings at Hilltop were 4% above the district average in Reading, 4% above in Language, 7% above in Mathematics, and 4%
above in Science (Table 4). The first grade class size is approximately 17 students. Ms. Henderson's class enrollment fluctuated from 17 to 20 during the research study observations. Using these measures the students at Hilltop Elementary represent a fairly typical student population in this large southwestern school district. Therefore, Ms. Henderson's class at Hilltop Elementary is where I begin the examination of teacher exploration of instructional strategies to promote algebraic thinking.

Illustrative Mathematics Lesson

The school district requires 70 minutes of daily mathematics instruction in the first grade. Ms. Henderson allocates approximately 20 minutes in the morning for calendar math and a short review of mathematics topics such as place value, counting to a specific number, or reading numbers aloud. The remaining 50 minutes of daily mathematics instruction is in the afternoon from about 1:15 P.M. to 2:05 P.M. Ms. Henderson prefers to break the mathematics lessons up a bit between the morning and the afternoon. For first grade students, she believes 70 minutes of mathematics at a time is too long. In addition, some students are better able to focus at different times of the day so they have the opportunity to work on mathematics throughout the day instead of just at one time. Her class is the designated first grade special education class. Special needs students whose Individual Education Plan (IEP) prescribes inclusion in a regular class are placed with her. In her first grade class this year are one autistic student, two students identified with speech problems, and one student who is in the process of being formally placed in special education. A special education teacher is assigned to work with Ms. Henderson and this teacher comes in daily during the afternoon math lessons to monitor the progress.
of the special needs students, especially the autistic student. The special education teacher assists Ms. Henderson by working with any of the students during this time. Ms. Henderson also has a sizeable group of parent volunteers, student volunteers from a nearby high school, and a university preservice student that assist her in working with the first graders.

A mathematics lesson typically starts with a whole group presentation. The presentation may be done with Ms. Henderson standing at the white board at the front of the room or sitting in the large leather chair in the corner. The students are sitting on the floor around her or at their tables. One observed lesson was on the relationship between three-dimensional objects and their two-dimensional representation. Ms. Henderson held up models of a cube, cylinder, rectangular prism, cone, pyramid, and sphere. She then traced around one side of each of the three-dimensional models at the overhead projector and called on a student to name the flat shape that had been traced. The object traced and the responses were as follows:

<table>
<thead>
<tr>
<th>Traced Object</th>
<th>Student Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>Cube, then square</td>
</tr>
<tr>
<td>Cylinder</td>
<td>Circle</td>
</tr>
<tr>
<td>Cone</td>
<td>Circle, ice cream</td>
</tr>
<tr>
<td>Pyramid</td>
<td>Triangle side</td>
</tr>
<tr>
<td>Rectangular Solid</td>
<td>Flat side, rectangle</td>
</tr>
</tbody>
</table>

Drawings of a square, rectangle, triangle, small circle, and a large circle are on the white board. Ms. Henderson identifies the shape and calls on a student for a definition. For the
square a student responded that means “all same sizes” and for a rectangle a student responded “two little sides, two big sides.”

Next the students were directed to go back to their seats and take out all the shapes from a plastic bag containing colored shapes of circles, squares, rectangles, and triangles. Ms. Henderson tells the students to put all the same shapes together. When finished the students raise their hands and Sonya inspects their work. As she examines their work she inquires, “How do you know it does not go there? How are they not the same? What’s different?” and waits for the students to recognize any problems with their individual sorted piles. The shapes are returned to the plastic bags. The students get out their math books and remove two specific pages. They work together with her to complete one page. That page requires the students to again respond with the name of the shape when one side of a three-dimensional object is traced. On the back they sign their name to a statement about what they learned about plane shapes.

They complete the second page working independently. Here they draw the traced side of a three-dimensional object and state the name of the plane figure. On the back they color shapes that match the first shape on a row and draw the largest possible square on an array of dots that is 6 by 17. When they are done, the students line up for Ms. Henderson to go over their work individually. If the work is correct, then she marks it with a star, tells the student to put it in their mailbox to take home, and directs the student to select a math game or puzzle to complete. If the work is not correct, she asks questions and guides the individual student toward a correct response. During this time the special education teacher enters the room to assist the students. The general format for a typical mathematics lesson is direct instruction or discussion with the entire class,
use of a manipulative or hands-on activity to explore the concept, guided practice in the
math book, individual work in the math book, inspection of individual student work by
the teacher, and use of a math activity from the selection of math games and activities in
the classroom. Ms. Henderson’s questioning strategies include fact recall, such as “What
is the name of that shape?”, process justification, such as “How did you do that?”, or
reference to defining criteria, such as “Is 8 dots by 6 dots a square?”. Student
explanations are either recall of vocabulary or definitions, stating the arithmetic
operation, or describing the criteria for selecting a shape or answer. Ms. Henderson is
working on her questioning technique and looking for ways to assist students to discover
their own mistakes and find appropriate solutions on their own.

Algebraic Thinking and Number Sentences

The teacher acknowledges her reliance on traditional methods to determine if students
understand a math concept and her awareness that teaching just algorithms and
procedures shuts down understanding. Ms. Henderson stated that her goal of teaching for
conceptual understanding is based on her belief that students forget what they have
memorized, but do not forget what they understand. With this belief she begins her
exploration of teaching to develop algebraic thinking. The initial lessons focus on
equivalence and Sonya uses the examples from the summer workshop and the Thinking
Mathematically Integrating Arithmetic and Algebra in Elementary School (Carpenter,
Franke, & Levi, 2003) textbook to get started.

The students are given number sentences to read, state whether they are true or false,
and then explain why they are true or false. The idea of balance was discussed prior to
working with the number sentences and part of the first lesson was to work on student understanding of the "=" sign as representing the "same as". The students had been working with number sentences for only a few days when they responded to the following problems.

\[
\begin{align*}
3 + 2 &= 5 \\
7 + 2 &= 9 \\
4 + 2 &= 8 \\
8 + 3 &= 10
\end{align*}
\]

All of the problems were read correctly. For example, the first problem was read as "three plus two is the same as five." The students correctly identified the first two problems as true and the second two problems as false. All of the students justified their work by counting on. That is seven, eight, nine is the same as nine. Or eight, nine, ten, eleven is not the same as ten.

Ms. Henderson introduced the next set of problems by stating that "I am going to give you something new to try." The directions were the same. Read the problem, state whether it is true or false, and explain why it is true or false.

Problem 1: \[8 + 2 = 5 + 5\]

Students initially decided that the first number sentence was false. One student stated that it is supposed to be right there. The plus should be there. He demonstrated what he meant by writing \[8 + 2 + 5 = 5\] and asserted that it is still false. It should be \[8 + 2 + 5 = 15\]. A different student responded it is false because 8 and 2 is not 5. Mrs. Henderson stated that this is very interesting and asked if the students had any other ideas. Finally the first student stated that backwards or frontwards it doesn’t matter. He stated \[8 + 2 =\]
10 and $5 + 5 = 10$. Two other students responded that it was true. One said that it was kind of backwards because $8 + 2 = 10$ and $5 + 5 = 10$. Ms. Henderson then wrote:

\[8 + 2 = 5 + 5\]
\[10 = 10\]

and asked if ten is equal to ten. The first response was no, but that was followed by yes, it’s a double. A student wants to show something and he writes: $8 + 2 = 10 = 5 + 5$. They both can be ten.

Ms. Henderson has the students look at another number sentence.

**Problem 2:** $3 + 3 = 5 + 1$

The first response is that it is false because $3 + 3$ does not equal 5. A second student responds that it is true because $3 + 3 = 6$ and $5 + 1 = 6$ so:

\[3 + 3 = 5 + 1\]
\[6 = 6\]

Ms. Henderson shows one more number sentence. The class reads the numbers and symbols out loud as she writes them.

**Problem 3:** $7 + 2 = 5 + 4$

Again the first response is false. The student explains it is false because $7 + 2$ does not equal 5. A different student responds that it is true. Both a double. That one ($7 + 2$) equals 9 and that one ($5 + 4$) equals 9. Ms. Henderson closes the session with the comment that “You guys are really thinking”. Students return to their tables to work on a sheet from the math book that requires the students to work on adding zero to the numbers one through ten, then adding one, adding two, adding three, adding four, and adding five.
The lesson described was the first time the students had worked on number sentences with more than one number on both sides of the equal sign. The material was new to the students. Ms. Henderson commented after the lesson that she knew she was not supposed to give the correct answers, but just to affirm the student response. She is uncomfortable with her new role and worried about students leaving the class uncertain about the procedure or the correct answer. The work on the number sentences used the math facts that the students were practicing in the mathematics textbook, but it was designed as a learning experience separate from the textbook work.

The students continued their work with number sentences the next day by examining these problems:

\[
\begin{align*}
3 + 1 &= 5 \\
7 + 2 &= 9 \\
3 + 1 &= 2 + 2 \\
3 + 4 &= 6 + 1 \\
4 + 1 &= 3 + 3
\end{align*}
\]

Students quickly and successfully responded to the first two problems. The first student to examine the problem \(3 + 1 = 2 + 2\) stated it was false and then exclaimed “Wait! It’s true like I did yesterday both equal four.” Ms. Henderson used his response to ask the students if they knew other ways to get \(4 = 4\). Student suggestions included: \(0 + 4\) and \(5 - 1\). She commented on the good thinking and how tricky it was to put the subtraction there.

Responding to the problem \(3 + 4 = 6 + 1\), a student read it correctly, stated it was true, and then it is true because I don’t know. Ms. Henderson had the student look at the
problem and asked her what is $3 + 4$? The student said seven and went on to say I think it is true and $6 + 1 = 7$. A different student volunteered that she had another way to make 7. She stated $8 - 1 = 7$ and $5 + 2 = 7$. A third student stated $9 - 2 = 7$. The last number sentence $4 + 1 = 3 + 3$ was read correctly and the student said it was false because $4 + 1 = 5$ and $3 + 3 = 6$ and then she said “No.” Ms. Henderson again guided her with the question “Is $5 = 6$? Are they the same?” The student said no. The number sentence discussion was followed with some written work on number sentences. The students were directed to look at the following number sentences, write true and why, or write false and why. They were not to worry about right or wrong. Ms. Henderson says she wants to know about their thinking. The four number sentences on the paper were:

- $3 + 5 = 8$
- $4 + 2 = 5$
- $2 + 1 = 3 + 2$
- $5 + 4 = 6 + 3$

Ms. Henderson reported that nine students got all the problems correct and eight students had some wrong. She did not provide a detailed description of the incorrect responses.

In subsequent lessons the students continue their work on equivalence and are also introduced to number sentences with a missing value. The missing value is represented with a □ or a ___. In the descriptions of the classroom observations the □ will be used to represent the missing value. They examined the following number sentences. The directions were to put a number in the square that makes the sentence true.

- $3 + 3 = □ + 2$
- $3 + 1 = □ + 2$
The solution to the first number sentence \(3 + 3 = \square + 2\) was quickly identified by a student, who thought the answer was “4 because then both make 6.” Ms. Henderson asked the class what would the left side of \(3 + 1 = \square + 2\) need to equal? The class responded 4 and told her to put in 2 for the square to make the number sentence true. A third student read “five plus four equals eight plus cube” and stated the answer is nine equals nine. Ms. Henderson had to ask him eight plus what equals nine to obtain the response that the number one goes in the cube. The students were quick to contribute appropriate solutions to the new problems with a missing number. This number sentence discussion was followed with more written work on number sentences. For the first two problems students are to state whether the sentences are true or false and explain why. For the last two problems the students are to fill in the blank to make a true number sentence. The four number sentences on the paper were:

\[
\begin{align*}
5 + 4 &= 8 + \square \\
3 + 3 &= 4 + 2 \\
4 + 1 &= 3 + 4 \\
2 + 3 &= 4 + \square \\
5 + 2 &= 4 + \square
\end{align*}
\]

Ms. Henderson states that she thinks the students are getting the concept of balance or equivalence. I observe that she seems to have developed a plan for teaching these ideas. Sonya laments that the problem is, “There is what I have to do and what I want to do in my teaching.” She states that she knows the content and how she needs to teach it so that the students learn. Furthermore, she explains that as teachers we are directed to design our instruction so that we can reach all the different ability levels in the class and yet we
are also working to standardize how we mark or grade all the students in first grade. It is
difficult. There is also the emphasis on accountability and the mandated testing that
limits what we do in the classroom. Within this context she outlines her plan for teaching
equivalence and relational thinking. During the morning mathematics time she will work
with number sentences that use addition and subtraction facts. Once a week she will
devote an afternoon math session to working on number sentences that use addition and
subtraction facts and principles that are a required part of the first grade mathematics
curriculum.

The next classroom observation is five weeks later. Ms Henderson now has her
student helper distribute a paper penny to students who participate in the number
sentence discussions. The student participant may receive a penny for a correct response
or an explanation of how they are solving or thinking about the problem. She already had
the money-based reward system as part of her classroom management procedures. She
extended it to include student participation in the number sentence discussions. The
student helper does not give a student the penny until the appropriate solution is stated or
a detailed explanation of the student thinking is articulated. Ms. Henderson states that
this system promotes student participation and allows her to acknowledge correct
solutions and different thinking strategies. She believes that the reward system helps to
keep the students focused and reinforces appropriate problem solving strategies and
solutions.

The students are working with missing values, adding zero, and rewriting number
sentences with the single number to the left of the equal sign. Sample problems from this
lesson include:
$3 + 1 = 2 + \Box$

$2 + 1 = 3 + \Box$

$3 + 1 = 3 + 1$

$5 + 3 = 5 + \Box$

$5 = 4 + 3$

$5 = 4 + \Box$

Tom’s response to $3 + 1 = 2 + \Box$ was to state that the number 2 would go in the box to make a true statement. His first explanation was that $3 + 1 = 4$ and $2 + 2 = 4$ so the answer was 2. When prompted to supply another way to make both sides 4, he replied that he could take 1 away from 3 so $2 + 1 + 1$ was on the left side. Put 2 on the right side so $2 + 2$. So it is $2 + 2 = 2 + 2$.

Ms. Henderson stated that Tom has really been shining in the number sentence discussions. He is the first student to bring up subtraction. He has caught on to the fact that you do not have to complete the operations to see if the equations are balanced. For example, if she showed $4 + 3 = 5 + \Box$ then he would say “five is four plus one more so the box has to be one less than three.” Her plan is to build off his responses to encourage the other students to look at these relationships. She acknowledges that during the number sentence discussions she first calls on the students that she thinks will give a correct response. The idea is to use these correct responses to encourage the students who are less confident of their mathematics skills to participate. During the number sentence lessons there is always a sea of hands waving to give a response, sometimes before she even writes the problem down.
Less than a week later the students are examining number sentences with two-digit numbers. Building on the truth of the number sentence $3 + 2 = 3 + 2$, the students decide that $55 + 30 = 55 + 30$ is a true statement. Sample problems from this lesson include:

$$35 + 28 = 35 + 29$$
$$52 + 10 = 53 + 9$$
$$65 + 32 = 64 + 33$$
$$37 + 18 = 36 + 17$$
$$75 + 5 = 73 + □$$

Karl asserts that $35 + 28 = 35 + 29$ is false because you would have to add one more to 28 to make it the same. Paula states that $52 + 10 = 53 + 9$ is true because 52 is less than or one smaller (than 53) and 10 is older, bigger than 9. If you add one more then it is true. Theodore examines $65 + 32 = 64 + 33$ and states that it is true because 65 is one more, one higher than 64 and 32 is one less than 33, so balances out.

The problem $37 + 18 = 36 + 17$ provoked more discussion. Tom initially responded by stating it was true, and then said I think it is different. Alice countered with it is false because two over there (on the left) and both are one more (than the numbers on the right). Ms. Henderson repeated “36 is one less than 37, 17 is one less than 18”. She wrote $37 + 18 = 36 + □$ on the board and asked what different number could be put here to make this true. She asks Charles to help with the problem. He first says 15, then says 19, then no wait 20. He settles on 19. She writes $37 + 18 = 36 + 19$ and says “Good job!”. The interesting aspect of this problem is that Ms. Henderson repeats the student response and builds on that response to ask a new question. She does not just accept the student response and move on to a different problem. Instead she uses the student
response to create a new problem and move the discussion to finding a missing value that satisfies the equivalence relationship.

The last problem $75 + 5 = 73 + □$ posed more difficulty because the 75 was 2 more than the 73. The students had become accustomed to the numbers being one more or one less than each other. Melanie responded 74. Ms. Henderson asked Melanie if she should add one more to 73? That question was followed by “How much is it from 75 to 73?” The whole class responded a lot, 5 bigger. Ms. Henderson counted with them 73, 74, 75 until they came up with the idea that 75 is two bigger than 73. Tom suggested 75 is two more than 73 and 3 is two more, smaller than 5. Hands are waving as Ms Henderson writes on the board:

$$\begin{align*}
2 \text{ bigger} \\
75 + 5 &= 73 + 6 \text{ (one bigger than 5)} \\
75 + 5 &= 73 + 7 \text{ (two bigger than 5)}
\end{align*}$$

Ms. Henderson ends the lesson by leaving this on the board for the students to think about for the next day.

Students are still struggling with problems written as $4 = 3 + 1$, $5 = 3 + 2$, or $5 = 4 + 3$. On a number sentence assignment the students missed this type of problem most often. About one third of the students stated that $4 = 3 + 1$ and $5 = 3 + 2$ were both false. Ms. Henderson did not provide details of the student explanations for the answers of “false”. In the class discussion a student responded that $5 = 4 + 3$ is false because 5 does not equal 4. The student correctly identified the statement as false, but for an inappropriate reason. Ms. Henderson and her students continue to work on placing the
single numerical value to the left of the equal sign and evaluating these number sentences.

Collaboration and Support

Ms. Henderson states that she has not given any number sentences as homework assignments, however one of the parent volunteers said that her son came home and taught her that "=" means "same as". She declared that she never knew that before. Other parent volunteers, who have been in the classroom when she is working on number sentences, stated that they never knew first grade students could do that kind of mathematics. A different volunteer shared her observations of the class work on number sentences with her mother. The volunteer's mother, who has been a first grade teacher for thirty years, was amazed at what the students were doing. Ms. Henderson has shared the number sentence work that she and her students are doing with the first grade teacher on the other side of the partition. This teacher had already been thinking about some work on equality and has used some things in his class that he got from her. She can hear him next door using some of the things they have talked about. It does make her feel empowered.

Ms. Henderson has talked with her principal about including some training focused on algebraic thinking during the next staff development day. She clearly articulates, "I would like to have a group of teachers involved. It is better if there is someone to work with or bounce ideas off, than just working alone." She further suggests that having someone to contact, either personally or via email, with questions or concerns would be an appropriate support mechanism or follow up for teachers taking the ideas from the
graduate course into their classrooms. "If it does not seem to work, then it is easy to just
give up and go back to teaching the way you were before. Especially with no one to ask
questions about what is going on. Important to be able to ask a question."

In response to a question about what she would change in the algebraic thinking class,
Ms. Henderson states that the teachers in the class should be required to purchase the
textbook, Thinking Mathematically Integrating Arithmetic and Algebra in Elementary
School (Carpenter, Franke, & Levi, 2003). The book should not be available to the
teachers until the end of the class. She thinks that having the book at the beginning of the
class would be a bad approach. Speaking as a student, if she had the book the first day
then she would read it and probably not be as involved in the class work and activities.
Ms. Henderson declares that during the course we teachers were placed in the same
situation as the students. It was all new to us and we had to figure out what was
happening and what to do on the assignments. It was a valuable experience to be in the
same situation as the students. For that reason it would be better to go back and read the
book after the workshop.

Sonya asserts that she would have implemented the ideas from the class on fostering
algebraic thinking without support, but might have given up without this same support.
Ms. Henderson acknowledged that participating in the case study meant there was
pressure to implement ideas from the summer workshop in her classroom. The
researcher coming in, observing, interviewing, and questioning helped her to make a
solid commitment. She is also participating in the literacy workshops. When the literacy
professor first shared his ideas with the teachers, she thought first graders could not do
this. Now she says that I know they can do it and what to expect. I hope the same will be
true in mathematics. I am trying to improve and change the way I approach the teaching of mathematics. The biggest change I have made is that I do not tell them when they are wrong. They find their own mistakes. When a student says the problem is true and that is not the correct answer then my response now is to say “Why?” Often the student will reply “Oh no, it is false” and then explain why. I try to guide them to find their own mistakes.

Personal Reflections

For the final journal reflection in the graduate class, Sonya Henderson identified five aspects of the course content that she planned to implement in her classroom. The main focus would be on the teaching of equivalence. The “=” sign would be approached as meaning “the same as” not as indicating the answer to a problem. Instead of 5 + 3 = 8, the students would be exposed to different ways of representing equivalence such as 8 = 5 + 3, or 5 + 3 = 6 + 2. The implementation of relational thinking based on equivalence was also a new topic to be included in her curriculum. She expressed some concern about the difficulty of teaching this because she had not been taught to think in this manner, but hoped to develop her own relational thinking skills as she worked with her first grade students. The writing of conjectures formulated by the students, group discussions about mathematics, and assessment of student understanding based on verbal or written explanations of solution strategies were the other three strategies that she wished to integrate into her teaching.

During the exit interview she concluded that equivalence, relational thinking, and the mathematics discussions were successfully integrated into her teaching practice. The
students do answer “same as” most of the time when they see the “=” sign and have demonstrated genuine understanding of the concept of equivalence. Some students come up with more on their own than others, and she tries to build off their responses to encourage and teach the other students. The students had just started to work with the larger two-digit numbers for two days before the observed lesson on relational thinking. When she first put a large two-digit number on the white board the student response was a chorus of “Oh’s and Ah’s”. They quickly became used to working with the large numbers and could assess relationships among numbers such as 65 + 32 and 64 + 33 before they acquired the skills to add these numbers. Ms. Henderson regarded this as pretty incredible. She had never had math discussions before in her class. The only thing she had done before, that was related to algebraic thinking, was to put a □ in a math problem.

The first day I was in her classroom Ms. Henderson pointed out the two conjectures that were on the wall. The first conjecture was a statement that to skip count by tens, you count the numbers that end in zero. The second conjecture refers to the fact that to skip count by fives, you count the numbers that end in five and zero. She acknowledged that she has not devoted much time to that. The students come up with conjectures, such the fact that adding zero to any number does not change the number. They are also starting to recognize that if a number is taken away from one number then they need to put that number on another to maintain equivalence. There are many other arithmetic properties that the students recognize and that they use in the number sentences, but they have not formally written them down. Nor have they actually called them “conjectures”.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Next year Sonya wants to use math discussions and number sentences to work on learning basic number facts. Right now the students just memorize number facts. They have one-minute number fact tests. They start with the addition of zero to the digits one through nine. The sheet looks like this:

\[
\begin{align*}
0 + 1 &= \\
0 + 2 &= \\
0 + 3 &= \\
0 + 4 &= \\
0 + 5 &= \\
0 + 6 &= \\
0 + 7 &= \\
0 + 8 &= \\
0 + 9 &= \\
\end{align*}
\]

After a student successfully completes the number fact test on zero then she goes to +1, +2, \ldots, +9. The next set of number facts is -0, -1, -2, \ldots, -9. She used to do the number facts test everyday, but now just has the students do them twice a week. She does not want to stop administering the number fact tests this year because the students have bought into the importance of learning their number facts based on the tests. It is a change she will implement in the coming school year. Sonya asserts that she wants to slowly implement the type of teaching talked about in the summer so that she will be comfortable and the students will be successful. She stated at the beginning of the research study, "I sometimes find changes that I want to implement, and I dive in full force trying to be the absolute perfect teacher. Often what happens is that I change too much too fast, I get in over my head, I end up failing, and I toss out the new idea." She tempered that comment with the remark that the first time you do anything it is a bumpy experience. She reiterated that in the textbook (Carpenter, et al., 2003) the authors stated that it is important to actually experience the scenarios they describe in your own
classroom. Her experiences with the teaching of algebraic thinking have changed her view of mathematics teaching and learning. The use of number sentences to teach math facts and the corresponding discussions can help the students to get the correct answer and understand the reasoning behind the correct response. In her words, “Good teaching does not involve pointing out and correcting student misconceptions. Instead, good teaching leads students to finding their own mistakes and to developing solutions. Good teaching is particularly important in math so that students do not just learn to apply algorithms. Good math instruction strives for students to reach conceptual understanding”.

Case Two: Josh Abernathy

Josh Abernathy is a first grade teacher at Grandview Elementary School. He is in his sixth year of teaching first grade at the same elementary school. Mr. Abernathy made the decision to major in elementary education after completing two years at the community college. The college counselor informed him that the three basic areas of work or academic study are technology, money, or people. Josh believed he would fit best in a career area that involved working with people. He enjoyed sports and originally thought he would study to become a physical education teacher. The instructor of his first university education class was a male professor with a background in elementary education. The professor was a positive role model for a young man interested in teaching young students and directly influenced his decision to become an elementary teacher.
Four years of high school mathematics, two mathematics content courses for elementary teachers, college accounting, and a Business math class constitute Mr. Abernathy's mathematics content preparation for teaching elementary school mathematics. Josh pointed out that every teacher has their own content area that they particularly like to teach. Some teachers specialize in reading, but "I am so interested in the math area, especially with the young ones." Josh stated that two years ago in one of the graduate mathematics education courses, he was required to devise an activity to assess student achievement on a part of the mathematics curriculum that he wanted to focus on more. He was intrigued by the idea of examining what teachers do algebraically in first grade. In the search for information he found that the Curriculum Framework for Elementary does not list a lot of topics that specifically relate to the teaching of algebra or algebraic thinking. There is reference to recognizing, describing, extending, and creating simple repeating patterns using symbols and creating, comparing, and describing sets of objects as having more, less, or equal amounts. He commented that whenever he thinks of algebra he thinks of symbols and missing values, such as the missing addend in an addition problem. Based these ideas and the Elementary Curriculum Framework he created his own algebra assessment activity (Table 5).

The content and pedagogy in the summer class reinforced his developing ideas about the teaching and learning of algebra. He declared that the summer workshop, "that class fires me up." Josh explains that the emphasis on making connections, reasoning, and communicating in mathematics never crossed his mind the first three or four years he taught first grade. He just drilled the facts and the students learned what he, the teacher, told them. His algebra assessment activity demonstrated that the students are already
programmed to interpret the number sentence $8 = 5 + 3$ as eight plus five equals three. Even though the students might later comment that it is not right, they are not prepared to explain why it is not right. He states that he wants his students to really be able to explain their thinking. "I'd write a student's explanation and put it up just to make it like a personal connection. Instead of always just what the teacher told them. Let them explain it in their words, I guess."

Grandview Elementary School is one of 32 elementary schools in the school district participating in the MASE K-5 project. The five-year Mathematics and Science Enhancement (MASE) K-5 project is a federally funded professional training and development program that mandates each teacher at a participating school complete 130 hours of training before the end of June 2005. The professional development activities include training to develop pedagogical knowledge related to designated instructional materials, content workshops in mathematics and science, and workshops for teacher leaders that focus on assessment and using data to inform instruction. Mr. Abernathy has had the benefit of the MASE training for four years. The designated mathematics instructional materials are Kathy Richardson's Developing Number Concepts Using Unifix Cubes and the Scott Foresman/Addison Wesley Investigations series. He feels very confident about his ability to help the students build number sense using the problem solving approach of the Investigations, but would also like to know what is out there with regard to fostering student algebraic thinking. Professional development is an integral part of the educational program at Grandview Elementary School. A new principal was appointed midyear during the previous academic year and she has continued to support faculty participation in ongoing professional development. Mr. Abernathy is regarded as
an outstanding teacher by the school principal and the MASE project facilitator for his school. He is also one of several teacher leaders at his campus.

The Grandview Elementary School student population represents an ethnic distribution of students with approximately 9% more Asian students, 4% less Hispanic students, and 4% less white students (not of Hispanic origin) than is represented in the overall student population of the school district (Table 2). The percentage of students enrolled in special education programs is identical to the percentage for the district as a whole. The percentage of students in English Language Learner programs, the number of students eligible for free or reduced lunches, and the student transiency rate were all below the district averages (Table 3). On the fourth grade 2002-2003 Iowa Test of Basic Skills the percentile rankings of the students at Grandview were 11% above the district average in Reading, 16% above in Language, 18% above in Mathematics, and 11% above in Science (Table 4). The first grade class size is approximately 19 students. Mr. Abernathy’s class attendance varied from 17 to 19 students during the research study observations. Using these measures the students at Grandview Elementary represent a population with above average scores on a standardized assessment instrument and with a larger percentage of Asian students than the average elementary school in this school district.

Illustrative Mathematics Lesson

The required 70 minutes of daily mathematics instruction is structured as approximately 60 - 70 minutes each afternoon from about 1:00 – 2:00 P.M. and 5 - 10 minutes of calendar math each morning. Mr. Abernathy introduces the concept on the
white board or easel to the entire class. The introduction is followed with some type of hands-on activity for the students to explore the concept. A few written problems might follow the exploration. Either after the hands-on activity or after the written exercise he brings the class back together and revisits the concept or discusses some of the problems that he saw the students struggle to understand. He emphasizes that the best way for students to learn mathematics is to work with representations of the concept at all three levels. First grade students need to manipulate concrete materials, draw or see pictures, and then try to do it abstractly with numerals on paper. He asserts that he can assess them concretely and demonstrate it abstractly or do the reverse; depending on the mathematical concept they are learning.

The same basic framework is apparent in each of Mr. Abernathy’s mathematics lessons. Using a balance beam with the numbers one through ten on each side of the supporting column he introduces the concept of balance or equivalence as it pertains to numbers. The balance looks something like this:

```
  1 2 3 4 5 6 7 8 9 10
```

The introduction begins with Mr. Abernathy placing a weight on the number 4. He asks the students how he can make this balance. Jerald responds by telling him to put a weight on the other 4. Next Mr. Abernathy puts a weight on the number 4 and on the other side puts a weight on the number 9. He asks are they the same and the students respond with a chorus of “No’s.” He removes the weight from the 9 and asks “three and what make four?” Alice responds you have 3. If you have more with the 3 then both sides will balance. He places a weight on the 1 next to the weight on the 3. Now the sides balance.
In the next problem Mr. Abernathy places a weight on the number 10 and asks the students how two weights could be used to make 10. Carlton suggests that a weight be put on 1 and another weight on 5 to make 6. Mr. Abernathy asks the students are 10 and 6 the same. The response is another chorus of “No’s.” Carlton states that he still does not get why the balance does not go down. Mr. Abernathy suggests that Carlton think about numbers, not about the weights. He further encourages the student to think about 5 and 1. That both are needed to get 10. Carlton now suggests the number 9 because it will make 10 (with the 1). Mr. Abernathy asks if it is now balanced. The students agree that it is. He asks is there another way to make 10? Barbara suggests 5 and 5. The lesson continues with students finding different ways to balance or make 7. Student suggestions include 4 and 3, 1 and 6, or 5 and 2.

Students then go to their seats and get out their dot cards. The dot cards only have the numbers one, two, three, four, or five on them. Mr. Abernathy writes $5 = 3 + 2$ on the board. Students use their dot cards to represent $3 + 2$.

\[
5 = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{array} + \begin{array}{c}
\cdot \\
\cdot \\
\end{array}
\]

Mr. Abernathy then writes $5 = 5$. Kassandra thinks 5 and 5 make 10. Mr. Abernathy then writes $5 + 5 = 10$ next to $5 = 5$. Kassandra now says they are different. There is a ten on the other one. He continues with $5 = 2 + 3$. Bailey says we already did that one. Mr. Abernathy asks how he knows that so fast. Bailey responds because it is the same numbers as the top one, just backwards. Howard adds that they are the same numbers just upside down. The next problem that the class works on as a group is to find different ways to make 9. Mr. Abernathy starts with $9 = 5 + 2 + 1 + 1$. Nicholas suggests $1 + 2 + \ldots$
5, but is encouraged by other students to add 1. Other responses include $1 + 3 + 5$, $1 + 5 + 1 + 1$, $4 + 5$, and nine individual 1's. Students verify all the responses by counting with the dot cards. The final problem is to find different ways to represent the number 15. Mr. Abernathy puts $15 = 5 + 5 + 5$ on the board. Students work with the dot cards for a few minutes and then they share their solutions. Different solutions include $2 + 3 + 5 + 5$, $3 + 3 + 4 + 5$, $2 + 2 + 2 + 1 + 1 + 1$, and $2 + 2 + 4 + 2 + 2 + 1 + 1$. As students check the results with their dot cards they declare that the last solution is wrong and that another 1 needs to be added.

Mr. Abernathy gives each student a long paper with 4 numbers on the paper. The students are to select dot cards that represent each of the numbers and use the glue stick to attach the dot cards to the paper. They may work alone or work with a partner. The paper looks like:

| 4 | 7 | 10 | 12 |

When most of the students are finished, then Mr. Abernathy goes to the front of the room to discuss the problems with the students. He reminds them that they built dots before. Next to the dots or on the back the students are to write the math problem that they did. He asks how would you write $2 + 2 = 4$ a different way? The students can select the way to write the problem that is most comfortable for them. In the remaining time the students complete the glue stick process and write their number sentences. Mr. Abernathy confides that he is just interested in seeing how the students choose to write
their problems. He collects their papers. Mr. Abernathy structures all his lessons around the basic format of an introductory discussion, a hands-on activity, some writing of the mathematics, and closure in the form of revisiting the concept, making suggestions for working on a problem, or illustrating a difficult problem or concept.

Algebraic Thinking and Number Sentences

All of the mathematics lessons observed in Mr. Abernathy’s classroom focused on some aspect of algebraic thinking. His initial work on an algebra assessment activity to measure student understanding of equivalence played a significant role in his decision to explore ways to foster the development of algebraic thinking in his teaching. The student responses to the assessment revealed misconceptions similar to the ones described in the workshop and the Thinking Mathematically text (Carpenter, et al., 2003). Josh showed the students the “+” sign on the list of symbols. He asked them, “What does this sign mean to you when we are working with numbers and working with math?” Some students could not tell him the name of the symbol. Others might say that means you add or that is the plus sign. One student said, “That’s the button you push on the remote control.” Josh believes a teacher could build off that comment. He stated that it made him think of environmental reading. A student might not know what a word is, but if for instance they see the word “McDonald’s” they know what a McDonald’s restaurant is and what the word represents to them. The remote control button is like that. The student knew that when you push the “+” there is more sound, the volume increases. At the beginning of the year in first grade the students do not really know what the other signs mean (−, ÷, □). They might call the “−” times or the reverse. He tells the students
that "□" is the box. He might follow up with a question about what number is hiding behind the box, but in the fall is too early to talk about missing values.

Mr. Abernathy remarks that most of the students read $4 + 3 = 7$ and $6 - 2 = 4$ correctly. The most interesting response is to $8 = 5 + 3$. Typically a student will read "eight plus five equals three" and then state that is not right. He is amazed that coming into first grade they are already programmed to do the problem in that one way. Josh adds that here is nothing in the textbooks to support work on equality or changing the signs around. The student response to seeing $10 = 10$ is that's not right. One student said because ten plus ten is twenty that problem is not right. When asked to read "$5 + 5 = □$", the students usually read it as "five plus five equals ten" even though the ten is not written there. In the next section, Mr. Abernathy reads the problem and asks the student to tell him if the problem sounds right or wrong. For $2 + 2 = 4$ they all declare it is correct. They justify the answer by reciting, "two plus two equals four". With $4 - 1 = 3$, some students may get it but at this point in first grade they do not necessarily know about subtraction. When he reads "$10 = 5 + 5$", the usual response is that it is wrong. It is backwards. That is not right. The same types of student comments accompany the reading of "six equals six". Mr. Abernathy remarks that when he

Took that class and talked about equality and just switching things around and how it blows these kids away. And how they are already telling you it's backwards. It's wrong. How do they know that already? I mean, it's just amazing. They don't, they don't write equations in first grade, but they just see them all the time like that.

Searching for resource materials to present number sentences in the "$c = a + b$" format instead of the traditional "$a + b = c$" format, Josh made some interesting discoveries. He found nothing in the elementary mathematics textbooks about equality or
writing the number sentences with the single value to the left of the equal sign. He stated
that it is not possible to enter $10 = 5 + 5$ in a calculator that would be used in the
elementary grades. There is a website called “A+ Math” that teachers can use to type up
worksheets for addition and subtraction. The only format that the website supports is the
traditional “$a + b = c$” format. In some of the primary level mathematics workbooks a
teacher might find problems such as $5 + □ = 10$. Part of the reason he got excited about
the content in the algebraic thinking class is because there are so few resource materials
to support this kind of teaching.

The emphasis on student thinking as the basis for designing instruction was the other
reason Mr. Abernathy was motivated to explore student development of algebraic
thinking in his teaching practice. Students respond to problems such as $5 + 5 = □ + 9$ by
stating 10 goes in the box. There is confusion when faced with a problem such as $□ = 3
+ 1$. The students who finally decide the answer is four usually state that is because $3 + 1
= 4$. They need to think about the problem is the traditional format of “$a + b = c$” to work
out a solution. He says that if he works on these ideas in the first grade he hopes the
students will be comfortable moving the numbers around and “trying to figure things out
instead of just - they’re like robots”. The experience of designing and implementing the
algebra assessment, exposure to the pedagogical and mathematical content of the summer
course, and the evolving concern about teaching for understanding, not just drilling the
basic facts, are reflected in the lessons Josh has developed for his students.

Mr. Abernathy introduces new concepts during what he describes as “number talks”.
Using the white board or the easel he writes a number sentence, asks a student to read the
number sentence, state whether it is true or false, and then explain why the student thinks
the problem is true or false. During one of the classroom observations Mr. Abernathy wrote $2 + 8 = 10$. Jasmine read “eight plus two equals ten”, stated it was true, and explained that if you have eight and add two more you have ten. Josh wrote a new problem, $10 = 2 + 8$. Georgeanne said it was just backwards because the eight (in the original problem) is before the two, the two is on the other side. Mr. Abernathy inquires if any student thinks it is false. Martha states that she thinks it is false if you make it backwards and wants to show how to write it. Martha says “two plus eight same as ten backwards”. Mr. Abernathy points out that she said it different, that she said “same as”. Martha responds, “Yes, like equals.” She goes on to say that equal tells us that it is the same. Another student suggests that when we do math today to see if it is true we could show it with pictures. Martha shows what she means by drawing stars:

$$\begin{array}{c}
\begin{array}{c}
\star \\
\star
\end{array} +
\begin{array}{c}
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star
\end{array} = 10
\end{array}$$

and

$$10 = \begin{array}{c}
\star \\
\star
\end{array} + 8$$

Mr. Abernathy asks Martha if two plus eight is the same as ten and if ten is the same as two plus eight. Is it okay to say? He asks for another way to make ten. Other students suggest $5 + 5$ and $9 + 1$. Bruce asks for a marker to demonstrate the long way to make ten. Under the drawings of the stars Bruce writes:

$$\begin{array}{c}
\begin{array}{c}
+ \ \ \ \ 10 \\
\ \ \ \ 8
\end{array} &
\begin{array}{c}
+ \ 2 \\
10 \\
8
\end{array}
\end{array}$$

Mr. Abernathy asks the students if they see the “equal” sign in the long way. He comments that something interesting is happening instead of equal. He further states it is
hard to read the tall or long way. Because we read left to right it is easier to read when written the horizontal way. To see the “equal” in the tall way we have to look at the line (___). Students also suggested replacing the ten with a □. The problems looked like this:

\[
\begin{array}{c}
2 \\
+8 \\
\hline
10 \\
\end{array}
\quad
\begin{array}{c}
\square \\
\hline
2 \\
+8 \\
\end{array}
\]

Mr. Abernathy told the students that there is no special way in the tall form. After the lesson he commented to the researcher that he really was unsure about what to tell the students. One of the difficulties of teaching these new concepts is that he is not totally sure what is the right way to answer when students come up with ideas such as writing a traditional presentation of addition backwards.

The next problem was 8 = 8. Bruce thought it was false because it did not make anything. It did not make a new number. It should make a math problem like 2 + 8 = 10. Mary said 8 + 8 = 16 makes a new number. Mr. Abernathy asked if we have to make a new number all the time and wrote 5 + 5 = 10. The students decided this was true. He then asked if this math problem is the same as 10 = 10. The students said this was true. He asked “Does it make a new number?” There was a loud chorus of “No’s.” He went on to a new type of problem. He wrote 8 + 4 = □ + 7. The students read the problem together and then he asked what number to put in the box. Karl responded 12 and explained he chose 12 because of 8 + 4. Mr. Abernathy asked if 8 + 4 is the same as 12 + 7. Karl said “No”. Mr. Abernathy then asked if we do the math problem and 12 goes in
the box, then why is plus seven on the other side? He drew lines to represent the numbers to illustrate the problem. It looked like:

\[
8 + 4 = \frac{12}{1111111111111111111} + 7
\]

He asked what is 8 lines and 4 lines? How many lines to have the same as 12?

Georgeanne said one more. Mr. Abernathy said to make 8 or to make 12?

Joseph said 5 lines. Mr. Abernathy drew the following:

\[
8 + 4 = \frac{5}{1111111111111111111} + 7
\]

Both sides are the same now. The number talk ended and the students went to work at the math stations.

Mr. Abernathy calls four students to work with him at the front table. He continues using number sentences to demonstrate equivalence, but has used paper plates and colored paper squares for hands-on number sentences. The paper plate activity is from the algebraic thinking book. He writes the problem \(3 + 5 = 8\). He puts 3 blue (solid) squares on the first plate, 5 green (checkerboard) squares on the second plate, and 4 red (diagonal lines) squares on the third plate. He asks what to put in the last plate.
A student says four. Mr. Abernathy places four red squares in the last plate and asks if it is okay to say that $3 + 5 = 4 + 4$?

Bruce responds by moving a green square to the plate with the blue squares so he has:

Mr. Abernathy asks if $4 + 4$ is the same as $4 + 4$. The students agree and go to work at the math stations. Toward the end of the mathematics lesson the teacher goes to the white board and talks about a problem that he noticed several students working with that day that he thought was interesting. He writes $0 + 5 = \square$ and asks for a response. The response is 5. He repeats the process with $0 + 4 = \square$ and $7 + 0 = \square$. The students point out that all the problems have a zero. He asks what happens when you add zero to any number? Martha asserts that $0 + 5 = 5$ and when you write it backwards $5 + 0 = 5$. He asks what happens with $22 + 0$, $0 + 349$, and $0 + 17,963$? The students correctly respond with 22, 349, and 17,963 respectively. Mr. Abernathy asks what is the rule? Karl says “Adding zero to any number, the number doesn’t change.” The students try two more
problems and correctly state that $39 + 0 = 39$ and $211 + 0 = 211$. After the lesson Mr. Abernathy states that he is planning number talks twice a week for about fifteen minutes. The number talks will incorporate arithmetic activities that promote algebraic thinking.

He expresses concern that there is much more in the curriculum to be completed although there are 90 days left of school. Grandview Elementary School is on a modified nine-month schedule. The students have a three-week break at Christmas and a two-week break in the spring. That means the school year ends two weeks later than the school year for a traditional nine-month school.

Another activity that Mr. Abernathy modified for working on equivalence was using cutouts of dogs with different number of spots on their backs. For example, there are two dogs in the neighbor’s backyard, one has three spots and the other has three spots. There are two dogs in your backyard, and one has four spots. How many spots on the other dog? Mr. Abernathy also used cubes placed in a square and a triangle for students to represent the number of different ways to make the numbers 5, 6, 7, 8. The students used the notion of equivalence ($1 + 4 = 5$, $4 + 1 = 5$, $2 + 3 = 5$, $3 + 2 = 5$, $5 + 0 = 5$, and $0 + 5 = 5$) and the concrete materials to find six ways to make the number 5. They repeated the process to make 6, 7, and 8. They left class thinking that the number of different ways to make a number is one more than the number itself. Mr. Abernathy told them to think about that at the end of class. They would work more later on this idea. He adapted both activities from the NCTM Navigating through Algebra K-2 text.

A different activity Josh uses is to hold up a card with small squares or dots for the students to view for several seconds. Then Mr. Abernathy asks the students how many
dots did they see and how were they grouped. The responses are written as number sentences as in the following examples.

\[
\begin{align*}
6 &= 3 + 3 \\
6 &= 1 + 1 + 1 + 1 + 1 + 1 \\
6 &= 2 + 2 + 2 \\
10 &= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \\
10 &= 2 + 2 + 2 + 2 + 2
\end{align*}
\]

The activity was to prepare the students to think about grouping by tens. Reading riddles from The Grapes of Math written by Greg Tang and illustrated by Harry Briggs, Mr. Abernathy used number sentences to group by tens. The students generated number sentences such as:

\[
\begin{align*}
6 + 1 + 3 &= 10 \\
3 + 6 + 1 &= 10 \\
3 + 6 + 1 &= 10
\end{align*}
\]

Josh Abernathy remarked that now the process of planning and teaching is different. “Before I just followed the curriculum that emphasized number sense and time. Now I use the number talks and emphasize the equal sign or the idea of equality. I try to integrate problems like:

\[
\begin{align*}
10 &= 3 + 2 + 3 + 2 \\
10 &= 5 + 5
\end{align*}
\]
into the curriculum. In first grade students are expected to master the basic facts up to the number ten. I would also like them to master equality, but perhaps I am expecting too much.”

Collaboration and Support

Mr. Abernathy declares that he likes the MASE program. Six years ago when he was hired at this school, the principal required the teachers to take the Kathy Richardson and Investigations training. Taking those classes made him feel confident and competent in teaching with a hands on approach. Enrolling in the required hours of MASE training and the recent completion of a Master’s degree in elementary mathematics education has given him a solid foundation of content knowledge and pedagogy for teaching first grade. Josh continues to search for resources that support the teaching of algebra in the elementary school. He joined the National Council of Teachers of Mathematics to be part of the professional mathematics education community, receive the elementary level periodical, and to qualify for the member discount on the purchase of books and materials. In a recent issue of Teaching Children Mathematics he read an article about algebra in the primary grades. The published article was a description of Investigations-based teaching that utilized math stations, number talks, and hands on activities. After reading the article he realized that “I do this in my class. I could get published.” Additional motivation to write about his teaching comes from the fact that one of the teachers in his graduate program recently had an article published. The teacher on special assignment who coordinates the MASE training at Grandview Elementary is working with Josh to draft an article describing how the teaching of algebraic thinking...
plays out in his classroom. The initial focus of the article is on the student misconceptions about equality that appear in the thinking and work of his first graders.

One of the other teachers at Grandview saw Mr. Abernathy's algebra assessment sheet (Table 5) and heard Josh discussing the students' responses. This teacher thought it was a cool activity and suggested it be put in the first grade portfolios. Some of the other teachers were uncomfortable with the idea. They became very defensive because of concern that their students do not know how to do this or that they do not teach this in class. Josh countered with "I don't either, but just interesting to find out what they (the students) say and why, so maybe we can do something about it. I don't know what to do. That's why I guess I get excited too." Josh further confided that the teachers got intimidated. They did not seem to think this should be part of the first grade curriculum. "But, a lot of the stuff you teach isn't necessarily always part of just what the curriculum says and I think there's more. You know maybe it comes along in the communicating part and in the reasoning aspect."

Building on his enthusiasm for fostering algebraic thinking in young students and the interest of other first grade teachers, Mr. Abernathy approached his principal about doing a grade level Book Talk using the Children's Mathematics (Carpenter, Fennema, Franke, Levi, & Empson, 1999) or Thinking Mathematically (Carpenter, et al., 2003) materials. The Grandview teachers already participate in a school-wide phonics book talk, so the principal did not want a second mandatory book study, but suggested it could be an optional activity. The teachers would read the chapters, view the videos of students at work, discuss the readings and videos, and try some of the ideas in their own classrooms. The book talk would be a forum for the teachers to discuss their successes, failures, and
overall reactions to the cognitively guided instruction philosophy of teaching based on student-generated solution strategies.

The book study group consisted of six first grade teachers and four second grade teachers. The book study began in February and continued for ten weeks with a two-hour session every Wednesday after school. The teachers could earn hours toward the 130 hours of required MASE training, receive a $10 an hour stipend, or be awarded professional development credits that count toward advancement on the district pay scale. The professor, who taught the summer course on algebraic thinking, facilitated the first book study session. Mr. Abernathy assumed the role of facilitator for the remaining sessions. After the sixth book talk meeting, he stated, “it stressed me out. All the responsibility was on me.” He believes an expert should guide the book study. He has reiterated to the members of the book study group that he is learning also and is just there to guide the discussion. He pointed out that another discouraging aspect of the book talk was that a few of the teachers did not actively participate in the sessions. They wanted to talk on other subjects or laugh and giggle like their students. Josh had to require that the teachers write in their journals and bring specific examples of teaching strategies or problems that were used in their classroom to share at each session. The teachers did respond positively to the chapter that described specific strategies for working on addition and subtraction problems. He suggested that the book talk might work better if it is started in the fall when teachers and students are more enthusiastic about the school year. Working with the teachers has revealed that teachers have misconceptions about the nature of equivalence and the appropriate construction of number sentences. The teachers thought the extended number sentence 12 = 8 + 4 + 2 = 14 was a correct way to
represent equality. Mr. Abernathy had to provide the teachers with verification that this was not an appropriate equivalence statement.

Commenting on what he would change about the structure of the summer course on algebraic thinking, he states that he would have preferred to have the book, Thinking Mathematically Integrating Arithmetic and Algebra in Elementary School (Carpenter, et al., 2003) available during the class. As a teacher he would have liked to have access to the book. He would then read it during the course, interpret the material for himself, and then listen to the instructor’s interpretation. He believes graduate students need the book because, unlike undergraduates, they would not regard the book as the ultimate authority. Mr. Abernathy would also like to see more university courses on algebraic thinking, perhaps even a three-credit class.

Josh wishes there were more collaboration in the teaching and learning of mathematics, particularly with regard to the development of algebraic thinking. As part of the ongoing experience it would be helpful to share questions or concerns with teachers at the same grade level. Being able to go online and email questions to someone with more expertise in this area would also support his teaching of algebra in the primary grades. Being part of the case study made it an ongoing experience. “It forced me to continue, to carry on from day one.” At the beginning of the research study Josh had indicated an interest in teaching third grade the next school year. He has been offered the opportunity to teach third grade next year at his present school. He is excited about the prospect of working with a new grade level and enthusiastic about learning to become an effective teacher leader in the development of student algebraic thinking.
Personal Reflections

In the journal for the summer course, Josh Abernathy identified three major mathematical ideas that he wanted his first grade students to know and understand at the end of the school year. He wrote that the students should “see numbers as patterns and relationships, not just equations for computations”, communicate mathematically by explaining their thinking process, and have a sense of number. Algebraic thinking was an integral part of his description of these big mathematical ideas. He stated that the first graders are capable of communicating algebraically about equality and his role would be to allow his students to “talk” about ideas and explanations. Putting student-generated conjectures on the walls of the classroom would support the student “talk”. He wants to pull more from the students. A student might say that a problem was easy, but that could mean different things. It could be that

It was actually too easy, the student didn’t really understand but is going to say that was easy to cope/slip through the cracks, etc. Which one is it? You can only find out through communication. Assessing student thinking and explanations of problems solved is key to guiding your curriculum.

Josh described himself as being on “algebraic thinking training wheels” and itching for the opportunity to take the training wheels off.

In the exit interview Mr. Abernathy asserted that he believed he had successfully integrated equality, understanding of the commutative property, and mathematical communication about student reasoning and student conjectures into his teaching practice. He is not completely sure what the students understand about the nature of algebraic thinking. He wants them to be comfortable with both \( a + b = c \) and \( c = a + b \) as number sentence formats, to be able to prove their theories, and to support their thinking.
with examples, conjectures, or number properties formulated in class. He is still working on making his class more student-centered and less teacher-centered. This is the first year he has tried talking less. Concentrating on changing the flow of the classroom dialogue has been, in his words, “like going back to high school, knowing what I know now.”

He thinks now that he has an idea of how to facilitate more student-to-student discussion. Josh Abernathy affirms that he will continue to explore strategies to promote the development of algebraic thinking in his students. Equivalence will be still be a major focus of his curriculum by using number sentences to teach arithmetic properties such as the addition of zero, the commutative property with respect to addition, grouping by tens, and addition/subtraction facts. Students have made conjectures during the number talks and hands on activities, but Mr. Abernathy did not formulize the writing of student conjectures. The five conjectures on the wall were worded and written by him. The conjectures include the strategy for counting on, the result of adding or subtracting zero, and the result of adding and subtracting one. He stated that he approaches his work on the teaching of algebraic thinking with the knowledge of a first year teacher, but with the determination of a veteran. Josh Abernathy’s perspective on his exploration of teaching strategies to promote algebraic thinking is in accord with his philosophy that “Good teaching is teaching that is open to change”.

Case Three: Paula Whitford

Paula Whitford is a third grade teacher at Evergreen Elementary School. She is in her eighth year of teaching at the same elementary school. Prior to moving to the
southwestern region of the country, she worked for five years as a substitute teacher in Pennsylvania. The depressed economic situation in that part of the country limited her prospects of obtaining a fulltime teaching position and influenced the family decision to move out West. After two years of teaching first grade in this school district, she transferred to third grade and has since taught at that grade level. Ms. Whitford describes her experience with learning algebra as very negative. She states that beginning in seventh grade that she “was lost and never caught up.” As a product of the type of teaching that relied on demonstration of procedures and memorization of rules she was never taught to “think, question, and explore ideas.” The same attitude affected her work as a substitute teacher. When she was a substitute she hated to teach the math lessons because in her words, “This is so boring and I can’t stand this and it’s awful.”

The mathematics content of her undergraduate program in education consisted of one algebra course and one elementary mathematics methods course. There was no separate mathematics content course for elementary teachers required or offered for undergraduates. She recalls that the undergraduate mathematics methods instructor brought out attribute blocks during one class, but never had the students develop lessons using these manipulatives. It was not until she began her graduate program at the local university that she became aware of the variety of materials that are available for teaching mathematics. The initial change in her teaching practice was the incorporation of more hands-on activities. Paula states that she started with candy hearts to do graphing and make arrays and has since built up a sizable personal collection of hands-on activities to teach mathematics.
The evolution from a traditional teaching style to her current style of teaching occurred during the past few years as she worked her Master's degree in mathematics education. Ms. Whitford declares that she has learned to "let go of control, allow students to find their own strategies for solving problems, and push them to do their own thinking." She laments that she robbed her first grade students because she did everything the exact opposite of how she teaches now. Then she did everything by the textbook, but now starts with the concrete materials, moves to the pictorial level, and then to the abstract representation of the concept. Paula Whitford's participation in the summer workshop on algebraic thinking was toward the end of her graduate program. The content and pedagogy of this course supported what she already does in the classroom and gave her "ideas and knowledge to strengthen my discussions through better thought-out questioning."

The principal of Evergreen Elementary School has high regard for Ms. Whitford's teaching, particularly in mathematics. One of the girls in Ms. Whitford's class started the school year in a 2nd grade class, but the student was moved into 3rd grade. The principal told Paula that she placed the girl in her class because of the way she teaches math. Mathematics is the girl's weakest area and you can really help her to learn with understanding. The principal told the researcher that Ms. Whitford is an outstanding teacher and is the designated Mathematics Site Trainer for mathematics. The school has adopted the Saxon series for a mathematics textbook. Paula does not advocate the use of a particular textbook for teaching mathematics. The principal supports Paula's decision to supplement the content in the adopted textbook with instructional materials of her own design. The administration of Evergreen Elementary School has established an
environment where good teachers can teach mathematics in the ways they believe are most effective. The standardized test scores for the students in Ms. Whitford's classes have been comparable to the scores for other students in the same grade level with one exception. Paula's student scores in the area of problem solving are very high. She remarked, "What I am doing is working."

The Evergreen Elementary School student population represents an ethnic distribution of students with approximately 14% fewer Hispanic students, 7% fewer Black students (not of Hispanic origin), and 21% more White students (not of Hispanic origin) than is represented in the overall student population of the school district (Table 2). The percentage of students enrolled in special education programs, percentage of students in English Language Learner programs, the number of students eligible for free or reduced lunches, and the student transiency rate are all below the district averages (Table 3). On the fourth grade 2002-2003 Iowa Test of Basic Skills the percentile rankings of the students at Evergreen were 17% above the district average in Reading, 25% above in Language, 17% above in Mathematics, and 16% above in Science (Table 4). The third grade class size is approximately 22 students. Ms. Whitford's class attendance varied from 20 to 24 students during the research study observations. Using these measures the students at Evergreen Elementary School represent a population with scores well above the district average on a standardized assessment instrument with a larger percentage of White students than is found in the average elementary school in this school district.
Illustrative Mathematics Lesson

The required 70 minutes of daily mathematics instruction is scheduled differently each day. On Monday and Tuesday mathematics is taught from in the morning from 11:00 to 11:50 A.M. and in the afternoon from 12:45 to 1:00 P.M. The schedule on Wednesday, Thursday, and Friday features fifteen minutes of mathematics instruction from 12:45 to 1:00 P.M. After a short break the third grade students return to working on mathematics for 75 to 55 minutes. The longer mathematics lesson may start at any time from 2:00 to 2:20 P.M., but the lesson always ends at 3:15 P.M.

Ms. Whitford emphasizes that she structures every mathematics lesson in a similar way. She starts with background information or knowledge and just sort of puts the big idea out there. The goal is to see what the students can give back, what they know about it, and what they think. When the students present ideas her response is “How do you know? Prove it”. She describes the sharing of strategies as unbelievable. During this process she learns things from them that even she did not know and can find misconceptions in their understanding. When she started teaching 3rd grade six years ago she taught directly out of the Addison-Wesley textbook and dabbled in Investigations, but now does not have one particular textbook that she uses.

Observing students work on a problem of the day verified her description of a typical mathematics lesson. Ms. Whitford distributed a small slip of paper with the following word problem typed on it.

Mike and Jane baked 18 snowman cookies. Eight fell on the floor and had to be thrown away so Mike and Jane baked 10 more. How many cookies did they have for the party?
Students used glue sticks to paste the problem in their math journals. Student comments of “That’s easy!” accompanied their work on solving the problem. Ms. Whitford asked students to share their solutions. Clifford says 28 is wrong. She asks if the students believe him or not? One student says he does not believe Clifford. She asks someone to share an answer, not a strategy. There are responses of 20, 18, 28, and 24. Now Ms. Whitford asks Louise to share her strategy by writing it on the white board. Louise writes:

This was her representation of the original 18 cookies. Then she crossed off the eight that fell on the floor.

She then added 10 more tally marks for the additional baked cookies. Counted up and got 20.

Ms. Whitford says that Laura counted up and got 20 and she is convinced this is correct. Is there anyone who did not get this and can prove it is wrong? Tena challenges. She uses the black marker to do the following.

\[
\begin{array}{ccc}
18 & 8 \\
-10 & +10 \\
08 & 18 \\
\end{array}
\]
Ms. Whitford reminds the student that 8 cookies fell on the floor and requests that Tena show us where the 8 cookies went. Tena goes up to the board and makes a change.

\[
\begin{array}{r|c}
18 & 10 \\
-8 & +10 \\
\hline
10 & 20
\end{array}
\]

Paula Whitford tells Tena that it is important that she caught herself and asks Tena if she is convinced that 20 is correct. Tena responds, “Kind of.” Ms. Whitford still thinks there might be more than one answer. Millie questions, “What answer? Twenty because eight fell.” Millie demonstrates by writing the number sentence \(18 - 8 = 10 + 10 = 20\). Ms. Whitford exclaims, “Great! Anyone know what I am going to say?” Annabelle replies that \(18 - 8 = 10\) is not equal. Ms. Whitford continues to wonder about Millie’s solution. Eli says it does not work. Tena responds that it did work. Eli persists by stating that you cannot put them together. Cassie says one is minus and one is plus. Tim states that you cannot have both in the same problem. Millie asserts they go in a different direction and that they are separate. Tim points to another problem (the problem above) that Tena did which is okay.

Ms. Whitford insists that she is going to be very picky about Millie’s number sentence, \(18 - 8 = 10 + 10 = 20\). She asserts that the problem is equivalent to \(10 = 20 = 20\) and asks the students if ten equals twenty? Cassie states that “=” means “is the same as” and ten is not the same as twenty. Ms. Whitford says “Perfect, beautiful!” She adds that Tim used different words to say the same thing when he said that you cannot put two math problems in the same problem. Perhaps you could change the problem to \(18 + 2 = 10 + 10 = 20\) to persuade us that it is true. Don says it is proved true because all are 20. After asking if \(20 = 20\), Ms. Whitford inquires if anyone has a different strategy. She
would like to see at least two more. A different student presents the same numbers but in one long vertical arrangement.

\[
\begin{array}{c}
18 \\
-8 \\
10 \\
+8 \\
18 \\
\end{array}
\]

Tena asserts that it is weird because two equal things in the same problem are not usually written up and down. Ms. Whitford asks if it is a problem to have two equal signs in one problem. Cassie declares that she is confused. Chris states there are 8 on the floor. Ms. Whitford offers that it does not fit the problem because there are 10 more cookies, but she has 8 more cookies. Annabelle changes the problem.

\[
\begin{array}{c}
18 \\
-8 \\
10 \\
+10 \\
20 \\
\end{array}
\]

Lila states that she does not think the problem is okay. Ms. Whitford again announces that she feels terrible being this picky but she is so confused. Louise also insists that you cannot do it. Ms. Whitford counters that Annabelle did it. Louise still insists that she should not do it. Chris declares that 10 does not equal 20. Ms. Whitford reminds the students that they were told that already. She thanks Annabelle for confusing us and tells the students to ponder the problem overnight. Meanwhile a two-student team presents one last solution strategy. The team uses circles to represent the cookies.

\[
\begin{array}{c}
\circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \\
\circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \\
\circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \\
\ = \ 20
\end{array}
\]
The students draw 18 circles and cross out 8 circles. Then they draw 10 more circles.

Ms. Whitford notices that one of the students has written more than one way to complete the problem. She points out that Marilyn has put a “letter” in the math problem. Cassie tells the class that it is algebra. You (Ms. Whitford) taught us. Ms. Whitford acts surprised and states that she did not know 3rd graders do algebra. Here is Marilyn’s problem:

\[18 - 8 = y\]
\[y = 10\]
\[10 + 10 = 20\]

Can the letter be any number? Cassie states that a letter can replace a number. Chris asserts that it is a variable. Ms. Whitford ends the discussion about the problem of the day, tells the students to put away their math journals, and begins a new activity. The mathematics lesson illustrated the teacher’s commitment to encouraging multiple solution strategies and different levels of representation of the solution.

Algebraic Thinking and Number Sentences

The introduction to equations and the examples of student misconceptions about the nature of equivalence were familiar to Ms. Whitford from work in a previous university course. Prior to the course on algebraic thinking she had been working with her students on this type of activity. The students examine a list of equations, determine whether each is true or false, and justify their respective solutions. Paula states that “we debate between those who believe it is true and those who believe a statement is false. It is a great assessment when listening to the arguments. I enjoy it and I believe they do too.”
The one thing that she added to the activity was to replace the term “equals” with “is the same as”. She admits that she is guilty of saying that both sides of a number sentence need to balance. Now she believes that this could impede some students from developing higher level thinking by taking the focus away from finding relationships. Early in the school year students typically use cubes and drawings to prove that they have a correct solution. The first problems that the students examined were:

\[
14 = 7 + 7 \\
6 + 2 = 8 + 1 \\
6 + 2 + 1 = 2 + 1 + 6 \\
1 + 2 = 3 + 2 = 5
\]

The typical student reaction to the problem \(14 = 7 + 7\) is to state that the problem is false because it is backwards. Students insist on changing the problem to \(7 + 7 = 14\) to make it true. Students also tend to state \(6 + 2 = 8 + 1\) is true because \(6 + 2 = 8\). One student responded that \(6 + 2 + 1 = 6 + 2 + 1\) is true because he could draw lines to connect each number on the left of the equal sign to a matching number to the right of the equal sign. He drew it like this:

\[
6 + 2 + 1 = 2 + 1 + 6
\]

The extended number sentence, \(1 + 2 = 3 + 2 = 5\), provokes the first class discussion about the meaning of equivalence when there is more than one equal symbol. Ms. Whitford maintains a checklist of which of these problems each student has answered correctly. It provides her with information about which students are not understanding the idea, which students have understanding, and which students are developing
understanding. It is assessment information that helps her to plan instruction and guide individual work for the students.

The initial work with number sentences, introduction of conjecture and proof, and pattern activities are the demarcation points for formal instruction in algebraic thinking. The pattern activity begins with a drawing of three squares. The students add squares to each new drawing to maintain the same general configuration. They use the drawings to determine the number of squares in the first eight to ten drawings. Looking for a pattern, they are asked to figure out the number of squares in the 20th drawing. The first few drawings looked like the following:

![Pattern Drawings](image)

<table>
<thead>
<tr>
<th>Figure 1</th>
<th>Figure 2</th>
<th>Figure 3</th>
<th>Figure 4</th>
<th>Figure 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 squares</td>
<td>5 squares</td>
<td>7 squares</td>
<td>9 squares</td>
<td>11 squares</td>
</tr>
</tbody>
</table>

Two students in the class of about 25 individuals successfully figured out the pattern to compute the number of squares in the 20th figure. All of the observed mathematics lessons in Ms. Whitford’s third grade class contain some element of algebraic thinking and are driven by student explanations and student understanding. There is no clear differentiation between an illustrative mathematics lesson and a lesson based on algebraic thinking in her classroom.

Ms. Whitford agrees that the pedagogical model advocated in the summer course in algebraic thinking is congruent with her methods of teaching mathematics. She adds that the course provided ideas for her to expand her questioning strategies, to continue the use
of number sentences to reinforce the concept of equivalence, and to increase opportunities for students to work with variables. Evidence of this was observed in a lesson on multiplication. Paula writes the following problem on the board.

\[ \_ \_ \_ = 4 \times 7 \]

Students have two minutes to solve the problem. The directions are to solve it more than one way. Students may work in small groups. They are to write the answer on the paper and then they will tell the class how they got the answer. The different strategies that the students proposed are listed below.

28 = 4 \times 7
2 \times 14 = 4 \times 7
4 + 4 + 4 + 4 + 4 + 4 = 4 \times 7
30 - 2 = 4 \times 7
20 + 8 = 4 \times 7
7 + 7 + 7 + 7 = 4 \times 7
7 + 4 + 5 + 8 = 4 \times 7

Each strategy was discussed in detail and the entire class had to agree that it was correct before it could be accepted as an appropriate solution. Some of the students justified their solutions based on the knowledge of multiplication facts. Ms. Whitford later told the researcher that she does not drill the multiplication tables, nor does she assign students to memorize multiplication facts. All instruction on multiplication is facilitated in lessons like the one just described.
Multiplication of two and three-digit numbers is taught in a similar manner. Ms. Whitford writes the problem, \( y = 3 \times 433 \). Students are again instructed to find as many different ways as they can to solve the problem. She says that two or three different solutions from each student would be wonderful. Students may work alone or in a group. Presentation of the different solutions is preceded by class discussion about a possible estimate or number that the answer is close to. Cassie suggests the number 1,000, but is unsure why she selected this value. Ms. Whitford guides the class to thinking about \( 3 \times 400 \) because 433 rounded to the nearest hundred is 400. Annabelle suggests twelve hundred as a reasonable estimate. The list of strategies follows:

Annabelle stated that the problem is times three so she wrote 433 down three times.

\[
\begin{array}{c}
433 \\
433 \\
433 \\
1299
\end{array}
\]

Marsha, Annabelle, and Susan developed this strategy working together.

\[
\begin{array}{ccc}
4 & 3 & 3 \\
\times3 & \times3 & \times3 \\
12 & 9 & 9
\end{array}
\]

Joseph started counting by 400's, then by 30's, and finally by 3's.

\[
\begin{array}{ccc}
/ & 400 & \text{ } & / & 800 & \text{ } & \rightarrow & 1200 \\
400 & \text{ } & 400 & \text{ } & 400 & \text{ } & \rightarrow & 90 \\
/ & 30 & \text{ } & / & 30 & \text{ } & / & 6 & \text{ } & \rightarrow & 9 \\
3 & \text{ } & 3 & \text{ } & 3 & \text{ } & 3 & \rightarrow & 1299
\end{array}
\]

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Marilyn multiplied across because she thinks well, $3 \times 3 = 9$ across.

\[
\begin{array}{c}
433 \\
\times 3 \\
\hline
1299
\end{array}
\]

Matilda suggested this strategy, which Ms. Whitford remarked really is high level thinking.

\[
3 \times 400 = 1200 \\
33 \\
33 \\
+ 33 \\
1299
\]

Norris used pictures of place value blocks and then counted them.

Jerald states that he knows how to add, so he solved the problem by adding three times.

\[
8 \ 6 \ 6 \\
433 + 433 = 866 \\
\]

\[
12 \ 9 \ 9 \\
866 + 433 = 1299
\]

Kurt offers the last strategy.

\[
\begin{array}{c}
4 \ 43 \ 433 \\
\times 3 \times 3 \times 3 \\
12 \ 129 \ 1299 \\
y = 1,299
\end{array}
\]

Ms. Whitford asks the students how many learned a different way to do the problem and if they learned it well enough to do it themselves. Paula explains that she does not teach
the algorithm for multiplying multi-digit numbers until after the students have proposed different strategies for the multiplication process.

Ms. Whitford is placing more emphasis on the concept of variable in her current teaching. The previous problem was introduced in the format $y = 3 \times 433$. After students work on a problem, she often rewrites the problem as an equation containing a letter. Students worked in groups to create concrete representations of three word problems by drawing tomato plants and using glue sticks to paste sample tomatoes on the plants. Next to the model they wrote their answers and corresponding solution strategies. After the students presented their group result, she inserted the letter “$L$” in the problems and solutions to encourage students to link the arithmetic problem to algebra equations with variables.

<table>
<thead>
<tr>
<th>$L \times 6 = 24$</th>
<th>$4 \times 6 = L$</th>
<th>$4 \times L = 24$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 4$</td>
<td>$L = 24$</td>
<td>$L = 6$</td>
</tr>
</tbody>
</table>

Gene has some tomato plants. There are 6 tomatoes on each plant. 24 tomatoes all together. How many tomato plants does he have?

Gene has 4 tomato plants. Each plant has 6 tomatoes. How many tomatoes all together?

Gene has 4 tomato plants. Each plant has the same number of tomatoes. There are 24 total tomatoes. How many tomatoes on each plant?

Paula comments that, “lessons should enable students to discover new ideas, build upon old ideas, and make connections.” I listen for students to say things such as “This is like what we did before” in order to tell if they understand enough to connections among math topics, between school and the real world, or across the curriculum.
Collaboration and Support

Ms. Whitford affirms that she has not had administrators or parents question the way she teaches mathematics. School staff and the parents in the community know that if they are looking to place their children in a traditional third grade class, then she is not the teacher for them. During the annual open house at Evergreen Elementary School Paula Whitford explains to the parents how she teaches math. In addition, Ms. Whitford and one of the other third grade teachers schedule a parent night in the fall to demonstrate how they teach and review addition and subtraction facts. The parents who attend participate in the instructional activities and leave with ideas for mathematics games and exercises that they can do at home with their children. In the spring a second parent night focuses on instructional activities for teaching multiplication and division. Paula says that most parents tell her that their experiences learning mathematics were terrible and that they are also terrible at doing mathematics. She asserts, “We say well that’s why we teach math the way we do. You know because it’s more interesting and they’re (the students) more apt to practice.” Ms Whitford and her parent night teaching partner have the same philosophy about the teaching of mathematics. They share their ideas and teaching materials with the other two third grade teachers. Of these other two teachers, one is a new teacher that Paula and her partner are guiding and mentoring. The new teacher is very open to adopting this non-traditional teaching style. The second teacher is an older woman who prefers teaching out of the Saxon textbook because it covers all the topics in the curriculum and is an easy book to use. However, even the older teacher is open to suggestions for some lessons not outlined in the textbook.
In her position as Mathematics Site Trainer for Evergreen Elementary School, Paula has the opportunity to attend all day training sessions on topics such as developing effective mathematics lessons, integrating new topics into the elementary mathematics curriculum, and facilitating the adoption of new textbooks with accompanying resource materials. She states that what saddens her about the training in her school’s region is that there is no real focus on mathematics that she can see. There are only three days of mathematics training scheduled this year, the rest of the allocated training days are devoted to literacy, i.e. reading. Literacy is important at the elementary level, but so is the development of mathematical skills. One of the mathematics site training sessions focused on examples of what algebra looks like in grades one through five. The instructor prepared a packet with sample algebra problems for each grade and he aligned these sample problems with the algebra portions of the state standards. These concrete examples of algebra helped Paula to understand the district’s vision of algebra in the elementary curriculum. Ms. Whitford would like to build on this information to teach the development of algebraic thinking. She needs support to “make use of our elementary curriculum to influence math later on.” Paula knows that she has to work on patterns that represent growth or increasing values, introducing the concept of variable, and writing number sentences in the form “c = a + b” so that students realize the answer can appear before the problem. She is “not sure if there’s anything else algebra entails that I could introduce at the 3rd grade level.”

Ms. Whitford declared that she would have preferred more emphasis on content in the summer course on the development of algebraic thinking. She realizes that is exactly what the course instructor does not do. He wants us to figure it out, just as I want my
students to figure out their solution strategies. Nor is the teaching of specific content part of the philosophy of the researchers and authors of *Thinking Mathematically Integrating Arithmetic and Algebra in Elementary School* (Carpenter, Franke, & Levi, 2003). She did not purchase this textbook, but mentioned borrowing the book from one of the other teachers she knows from the summer class. Paula would like some “direct teaching” to expand her background knowledge of algebra in order to more effectively put the algebra in the elementary context. She noted that equality was the one content area where seeing the videos of the elementary students working with it in a classroom was a huge help in her teaching. More examples of the difference between equality as represented by balance in number sentences such as $17 = 17$ and equality as number sentences that promote relational thinking were another request.

Support, in the form of having someone to email or talk with personally about what seems to work or doesn’t work in the teaching of algebraic thinking, would also be helpful in her efforts to teach the development of algebraic thinking. The contact person could be another teacher, course instructor, or the elementary mathematics specialist. It would be good “if we could try our activities with kids and come back and talk about them.” Ms. Whitford suggests a one-credit workshop or Professional Development Experience during the school year scheduled over four weeks or some similar timeframe. Teachers could meet once a week for a couple of hours, implement the content or strategies during the week, and then return to discuss their experiences at the next session. She thinks it would be helpful to view one of the algebraic thinking course videos as part of the inservice activities during a staff development days. Then other teachers would at least have “a view of one tiny piece of the course.” Paula Whitford
does acknowledge that she thinks her region of the school district is only region with
Math Site Trainers. She does not believe the other four regions in the school district have
site trainers or regional mathematics training. There is some district-wide training for
mathematics teachers.

Personal Reflections

The journal reflection for the summer course contained four big ideas in mathematics
that Ms. Whitford wanted her third grade students to know and understand before they
left her class. She wrote that the students should know “there are multiple ways to solve
problems and one way is not necessarily better than another.” Also, they should
understand that concepts in math overlap, look for connections among fractions, percents,
decimals, and division, and realize “math is more than memorization, timed tests, and
copying problems from a book. It is investigating, hypothesizing, games, exploring, etc.”
The algebra aspect of these big mathematical ideas included interpreting the equal sign to
mean “the same as”, using number sentences to think about relations among numbers and
between numbers and number properties, and playing around with numbers as an
acceptable learning strategy. Paula believes that she successfully integrated the idea of
variable into her teaching and that students are not misled by problems with the answer
placed before the equal sign. Her view of mathematics as investigating, hypothesizing,
and exploring is evident in the design and implementation of the observed mathematics
lessons.

Paula initially stated that she had been waiting since seventh grade for a definition of
the nature of algebraic thinking. She had “always thought algebraic thinking was
something you were either born with or not. It never occurred to me that it could be
developed from a young age.” Her current understanding of algebraic thinking is as a
way to represent mathematics using symbols, numbers, and patterns. Ms. Whitford
asserts that she plans to continue working on equality with her students. The focus will
be more on the relational aspect of equivalence than on the idea of keeping the number
sentences balanced. She will continue to push for students to understand the idea of a
variable, to explore expanded patterns, and to justify or prove the validity of their
solutions. Paula wants to be more consistent in her efforts to have students articulate
their conjectures. The students are making conjectures on their own, but probably do not
really know the word “conjecture”. She acknowledges the student-generated conjectures
and uses these conjectures in her teaching, but does not think it is really important to
formally write them for display in the classroom.

The roles of both teacher and student have not changed in Paula’s classroom as a
result of the algebraic thinking course. The evolution of her classroom from a teacher-
centered to a student-centered learning environment has been a gradual process. The
process started during the coursework for her Master’s degree in education. The
algebraic thinking course reinforced her belief that she is creating positive learning
experiences for her students with this teaching style. Her view of the teaching and
learning of mathematics as boring and awful has been replaced with the attitude that “if I
could teach math all day I’d be in business.” Ms. Whitford has considered transferring to
a middle school so she could teach mathematics all day, but has concerns about the
quality of her mathematics content knowledge. She states that she could never teach
middle school because she does not understand that level of mathematics well enough.
Paula is interested in tracking the mathematical progress of her students over the next couple of years. She thinks it would be informative to see if they continue to do as well with the traditional mathematics teaching that is the norm in the 4th and 5th grade classes at her school. Tracking student progress in subsequent math classes would be an excellent test of her definition of good teaching. Paula plans her lessons and designs instructional activities based on her belief that "good teaching in math means your students are chatty and excited while engaged in a task. Different solutions are explained by students and supported by the teacher."

Summary

Sonya Henderson, Josh Abernathy, and Paula Whitford have opened their classroom doors and shared their experiences implementing instructional strategies to foster the development of algebraic thinking. The analysis of their efforts focuses on investigating these research questions:

1. How does participation in a professional development experience influence teacher understanding of the nature of algebraic thinking?
   - What aspects of the teacher vision of the nature of mathematics support interest in teaching to develop student algebraic thinking?
   - What changes can be documented in teacher understanding of algebraic thinking?

2. What is the effect of a professional development experience exploring the development of algebraic thinking on the practice of the teacher?
• What changes can be documented in teacher use of arithmetic-based activities that promote algebraic thinking?

• How do the teachers demonstrate they value the teaching of algebraic thinking?

• What factors encourage/discourage teachers to engage in instructional practices that foster the development of algebraic thinking?

3. How do the teachers incorporate student discourse and examination of student thinking into mathematics teaching focused on the development of algebraic thinking?

• What changes can be documented in teacher use of student discourse/thinking in instruction?

• What changes can be documented in the teachers’ understanding of the student learning?
CHAPTER 4

DATA ANALYSIS

Participation in the professional development experience and the subsequent efforts of Sonya Henderson, Josh Abernathy, and Paula Whitford to implement the algebraic thinking content and pedagogy in their teaching practice is analyzed with regard to the three research questions. Each case description is examined for evidence that participation in the graduate course influenced teacher understanding of the nature of algebraic thinking. The individual perspectives on algebraic thinking are placed within the context of the teacher's vision of the nature of mathematics.

The translation this personal vision of the nature of algebraic thinking into practice can be documented with examples of arithmetic-based number sentences that the teachers and students examine together in mathematics lessons. The extent to which the teachers value instruction based on algebraic thinking is reflected in the ways that the mathematics lessons are restructured to incorporate student discourse and student thinking as the basis for instruction. The teacher beliefs about the teaching and learning of mathematics influence the process of exploring instructional strategies that foster the development of student algebraic thinking.

Analysis begins with the examination of all data sources using the interviews and observations as primary sources. The observations include evaluation of levels of teacher listening to student discourse (Appendix D) and evaluation of levels of engagement with
children's mathematical thinking (Appendix E). Journal reflections and survey responses are the secondary data sources. The collected descriptive data is examined using the constant comparative method of data analysis (Merriam, 2001). The constant comparative approach to data analysis involves comparing one portion of data with another to distinguish similarities and differences and organizing the data in categories or classifications to search for patterns in the data. In other words, "the researcher begins with a particular incident from an interview, field notes, or document and compares it with another incident in the same set of data or in another set. These comparisons lead to tentative categories that are then compared to each other and to other instances" (Merriam, 2001, p. 159). The initial level of data analysis in this study is a detailed description of each case to provide a comprehensive narrative of the experiences of the individual teachers (Merriam, 2001; Patton, 1987; Stake, 1985). The next level of analysis is cross-case analysis, which attempts to delineate outcomes and processes occurring in multiple cases and to understand the interaction of outcomes and processes with the teaching situations of the three teachers (Merriam, 2001; Patton, 1987).

The subjective nature of qualitative methods, in which the researcher is the primary instrument of data collection and analysis, makes it important for the researcher to employ some basic strategies to improve internal validity and credibility of the research. In this study, data analysis triangulation is accomplished by using multiple sources of data (Merriam, 1985, 2001; Patton, 1987; Stake, 1985). Comparing the written documents and the interview data to the observational data is done to confirm the analysis. Other strategies are the development of a thick description of each case, continuous member checks or corroboration of the data with those who provided the data,
consultation with the course instructor on the research findings as they emerge, and participatory research collaboration by involving case study subjects in the research. Examining whether the findings of the study are consistent with the data collected relies on the use of multiple methods of data collection or triangulation, detailed descriptions of data collection, comprehensive discussion of the development of themes or categories, and documentation of decision-making in the research process (Merriam, 2001).

Development of Themes

The organizational structure of the written descriptions of each case in the previous chapter evolved from the interview questions and journal responses. Each case description began with details of teaching experience, mathematical content knowledge, and experiences as learner of mathematics. Observations of an illustrative mathematics lessons and lessons based on algebraic thinking were followed by opportunities for the teacher and researcher to collaborate and reflect on the classroom learning experience. The researcher continued the collaboration and reflection by discussing the observations and follow up conversations with the university instructor of the summer course on algebraic thinking. The themes revealed in the research include a high level of interest in teaching mathematics, the belief that traditional teaching strategies are not working for their students, ambiguity about the definition of algebraic thinking, a lack of curriculum resources to support the teaching of algebraic thinking, a desire for collaboration with other teachers, and the commitment to continuing the teaching of algebraic thinking.
Nature of Algebraic Thinking

What aspects of the teacher vision of the nature of mathematics supported interest in teaching that develops student algebraic thinking? Many elementary teachers perceive themselves as weak mathematically, state a general lack of interest in the study of mathematics, or a lack of confidence to teach mathematics effectively (Schifter, 1996a). This was not the case for these three teachers. Sonya describes herself as the type of student that always found mathematics easy and that is the reason she believes it is sometimes difficult to adequately explain this same mathematics to her students.

Enrollment in the algebraic thinking course was based on interest in taking a mathematics content course. Sonya believes effective instruction in mathematics is important. Josh is fascinated with the idea of teaching algebraic thinking to young students. He has never made any comments that indicate he perceives a lack of ability to teach elementary mathematics or a lack of preparation in mathematics content. He is intrigued with seeing what his students can learn about algebraic reasoning. Paula did indicate that she had negative experiences as a student learning algebra and does not perceive herself as having the mathematics content knowledge to teach mathematics beyond the third grade level. She stated that she could teach fourth or fifth grade mathematics, but does not believe she understands the concepts well enough to teach it the way she wants to teach mathematics. However, her previous view of mathematics teaching and learning as boring has been replaced with the attitude that she would be happy to teach mathematics all day. All three teachers demonstrate a high level of interest in teaching and learning mathematics.

What changes were documented in teacher understanding of algebraic thinking? Teacher understanding of the nature of algebraic thinking and their respective perceptions

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
of student understanding of algebraic thinking are varied. Sonya Henderson describes algebraic thinking as based on the ability to recognize relationships between numbers and the use of these relationships to solve problems. The problems may be solved in a variety of ways. There is not one single way to solve a problem. Furthermore algebraic thinking examines patterns, requires students to look at problems from more than one viewpoint, inspires students to discover their own solution strategies, and makes arithmetic just a small part of mathematics instruction. Algebraic thinking pushes students to move beyond number identification and basic operations to understand the structure of number systems and to justify the use of particular operations to solve problems. Based on her vision of the nature of algebraic thinking, Sonya states that the number sentence work on equivalence and relational thinking has supported the idea of examining patterns and looking for relationships to solve problems. The math discussions about the number sentences have inspired sharing of multiple solution strategies and pushed students to justify solutions and problem solving strategies.

Josh Abernathy states that his understanding of the nature of algebraic thinking is limited, but it is growing. During the workshop he described algebraic thinking as mathematical thinking that allows students to see patterns and relationships both with and within numbers. It is not just computation. He further described this view of algebraic thinking as fresh and new or more specifically he is on “algebraic thinking training wheels.” After the last observation Josh stated that he believes his current vision of algebraic thinking is very broad. It includes the concept of variable and work with functions. He acknowledges that he and his students may be doing some of this and not be totally aware that they are. He is not completely sure what his students understand.
about the nature of algebraic thinking. He wants the students to be comfortable working with number sentences in the format \( a + b = c \) and the format \( c = b + a \), to be able to prove their theories, and to support their thinking with examples, conjectures, or number properties formulated in class.

Paula Whitford initially commented that she had been waiting for the definition of algebraic thinking since her seventh grade mathematics class. She tempered that comment with a description of algebraic thinking as a way to represent mathematics using symbols, numbers, and patterns. She had believed that the ability to think algebraically was something that an individual was born with. She had never previously considered the idea that algebraic thinking could be fostered in a child from a young age. Paula’s conception of algebraic thinking shifted from hating algebra as a mysterious mix of numbers and letters to thinking of algebra as patterns. Using number sentences as vehicles to teach equivalence, introduce the concept of variable, and place the single number before the equal sign has enhanced her understanding of algebra. One of her favorite examples is the student exclamation “I did 7 times a number equals 56. Oh! That’s algebra.”

The basic elements of the Algebra Standard in *Principles and Standards of School Mathematics* appear in all three perceptions of the nature of algebraic thinking. Each teacher recited a focus on patterns, relations, and functions. They use algebraic symbols to represent and examine mathematical situations, use models to represent quantitative relationships, and analyze change in various situations. Despite the teacher assertions that the definition of algebraic thinking is general and somewhat elusive, each of them has a vision of algebraic thinking that is compatible with the standards. Paula Whitford

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
offers an explanation for this sense that a precise definition of algebraic thinking is elusive. She described a packet of sample problems that aligned with the algebra strands of the state standards and how these concrete examples of algebra enhanced her understanding of the school district's vision of algebra in the elementary school. The teachers possess a theoretical understanding of algebraic thinking, but lack curriculum materials or resources to support their efforts to implement instruction focused on algebraic thinking.

Professional Development and Practice

Evidence of the value that the teachers place on the teaching of algebraic thinking is found in the mathematical content and pedagogy of the classroom observations. In all three cases the teacher description of a representative mathematics lesson was validated by the classroom observation. The lessons focused on developing algebraic thinking incorporate the general framework or structure of each teacher's illustrative mathematics lesson. Sonya Henderson is moving away from direct instruction as the major strategy for the whole group presentation. She has initiated the use of student discussion and sharing of strategies for all the morning math sessions and several times a week in the afternoon math sessions. A sample mathematics discussion begins with Ms. Henderson writing the problem $35 + 20 = 34 + \square$ on the board. She instructs the students to read the number sentence, find a value that makes a true statement, and explain why the statement is true.

Michael: Thirty-five plus twenty is the same as thirty-four plus box. Twenty. Thirty-four is lower, thirty-five is higher, it is the same for thirty-four and thirty-five.

Ms. Henderson: Okay to be the same?
Sam: Have to add one more.
Pamela: Thirty-five will be the same.
Sam: Change the thirty-four
Ms. Henderson: Is it okay to add to this one (points to the $34 + \square$)?
What number?
Sam: Twenty-one.
The class now agrees that it is true.

The concentration has been on the general notion of equivalence as representing "same as" and the use of relational thinking to introduce topics such as two-digit addition and subtraction. The students are provided with opportunities to examine number relationships such as $76 + 3 = 74 + 5$ prior to any instruction on the algorithm for adding two-digit numbers.

Josh Abernathy is teaching arithmetic procedures and properties via his "number talks" which also rely on student explanations and sharing of solutions as the method of instruction. Josh is talking less in his teaching and encouraging the students to talk more about their understanding of the problems. This is demonstrated in a lesson based on counting by groups. For a few seconds Josh holds up a card with squares on it and asks the students to quickly determine how many squares are on the card.

![Card with squares]

Helen: I think I saw ten.
Mr. Abernathy: In your head, how did you remember ten?
Helen: Three top, three middle. No. Three top, two middle, three again, two again.
Several students mumble: Pattern.
Helen: Pattern. After three and two, three and two.
Marsha: It's five and five. One, two, three, four, five.
One, two, three, four, five.
Mr. Abernathy: Where is five and five?
Justine: Five and five from the dots.
Mr. Abernathy: What is three and two?
Class response: Five.
Mr. Abernathy draws what Marsha saw as

\[
10 = 3 + 2 + 3 + 2
\]

\[
\begin{array}{c}
5 \\
/ \\
6
\end{array}
\begin{array}{c}
5 \\
/ \\
4
\end{array}
\]

Martha: I was looking at the dots and then I counted, almost by three’s. First three, then six. Start by two’s. I did three, six, eight, ten.
Mr. Abernathy draws what Martha saw as

\[
10 = 3 + 3 + 2 + 2
\]

\[
\begin{array}{c}
6 \\
/ \\
4
\end{array}
\]

Sharlene: Different. Same numbers, but backward.
Mr. Abernathy: Okay?
Students: Yea.
Mr. Abernathy: Nice job Martha. Thank you for sharing your brains.
Joseph: First I thought nine, then ten.
Mr. Abernathy: It takes about five seconds to count 1, 2, 3, . . . , 10.
It takes a long time. How long does it take to count three and two and three and two. Which is faster?

Based on this discussion the students decide it is faster to look at groups of numbers.

Paula continues to just put the big idea of the lesson “out there” to see what the students know about it and what they think. In the exit interview Paula stated “the roles of teacher and student have not changed because of the algebraic thinking class” and neither have her instructional strategies. However, there have been subtle changes in the mathematical content. Number sentences are more prevalent in the activities that support the big idea that she is teaching. Ms. Whitford used the following problem to reinforce multiplication facts and to continue exploring the idea of a variable. She tells the
students that she wants them to write their solution strategies on paper. To a mixture of 
“groans” and “yeas” from the class she writes the following problem on the board:

\[
\begin{align*}
4 \\
\times h \\
24
\end{align*}
\]

Scattered student comments are: Numbers scared me.
Letters scared me.
Christine: Call the letter a variable.
Melody: It is like \(4 \times 5 = 20\) and the 20 is on the bottom.
Do not know the five. Pretend it is not there.
Another student: Yes.
Ms. Whitford: Can we put any number in for “h” if we 
want a true statement.
Student response: No.

Students are given time to write some solution strategies on their papers. Ms. Whitford 
circulates around the room to monitor student work. She stops to question one group 
about how they are solving the problem.

Ms. Whitford: Okay, where did you start? What number goes in? 
How do you do it? Will the number ten fit? 
Stanton: No, it will be 40. That is too high.
Ms. Whitford: Add or multiply? Add what? What you did on 
your fingers, you need to put on the paper.
Stanton: That’s difficult.

The five boys in the group are heard counting 1, 2, 3, 4, and then 5, 6, 7, 8. They decide 
that they count by 4’s to get 24. Thus 4, 8, 12, 16, 24 is the decision they say they have 
made. They present their strategy to the class. Alan displays the group results.

\[
\begin{align*}
4, 8, 12, 16, 24, \\
4 \times 4 = 16 \\
4 \times 5 = 20 \\
4 \times 6 = 24
\end{align*}
\]

Skip count by 4’s. Answer is 6.
Ms. Whitford: What is put in for “h” to make it true?
Alan: Six.
Ms. Whitford: When you skip count you do not hit the number six.
Alan: After we got 24, we counted all the numbers and got 6.
Ms. Whitford: Counted the multiples of four and got six?
Tasha: I have a new strategy. I remembered that two 12's make 24.
Break the 12 up into 6 + 6 because 6 is one-half of 12. Tasha writes:

\[
\begin{array}{c|c|c|c|c}
12 & 6 & 6 & 6 & 6 \\
\hline
+12 & & & & \\
24 & 12 & 12 &
\end{array}
\]

four 6’s = 12

Ms. Whitford: I follow what you did. I could not have come up with that on my own.

Other strategies are presented. Not all of the solutions are correct. One group initially suggests \(4 \times 15 = 24\) and states that 24 is the answer. When asked if four groups of 24 is the same as 24, they go back and recount to find a new solution. Another group draws four circles and places six dots in each circle to represent the fact that \(h = 6\). The big idea of finding the value of a variable was the objective of the group work on the problem and the presentation of student solution strategies.

The teachers have demonstrated that they value the teaching of algebraic thinking based on student engagement in the learning process, but there are some factors that discourage this type of instructional practice. The teachers created their own number sentences based on the examples in Thinking Mathematically (Carpenter, et al., 2003) or modified problems found in workbooks or textbooks in order to teach algebraic thinking. There are no materials in the approved textbooks that use number sentences to teach relational thinking. All of the number sentences in the printed materials represented equivalence as \(2 + 3 = 5\), not as \(5 = 2 + 3\). Josh Abernathy uses a website that provides templates for teachers to create student worksheets. Problems typed on the website template had to be written in the form \(a + b = c\). The template would not accept any number sentence in the format \(c = a + b\). Josh also pointed out that the calculators used
in the elementary classroom have the same limitation when student enter problems.
Elementary students cannot enter \( 5 = 2 + 3 \) in those calculators. There are worksheet or
book problems with a blank or a box such as in \( 3 + \Box = 8 \), but the problem never appears
as \( 8 = 3 + \Box \). Sonya, Josh, and Paula articulated that access to instructional materials that
support the teaching of algebraic thinking would be helpful in several ways. They would
have to spend less time planning and designing their own instructional activities and it
would be external evidence that developing student algebraic thinking is valued in the
elementary school curriculum.

In each of the three cases the school principal fully supports the work of the teachers
to implement teaching strategies that foster the development of algebraic thinking.
However, all three teachers stated that they would like more collaboration with others in
teaching and learning this mathematics that emphasizes the development of algebraic
thinking. Sonya states that she would like to have a group of teachers involved because it
is better to have someone to work with than to implement new strategies in isolation.
Josh is the teacher leader of the book study group at his school, but would still like to
share questions or concerns about the teaching of algebraic thinking with teachers at the
same grade level at different schools. Paula would like to have a person to talk with
personally or online about what seems to work or does not work in the classroom as she
implements the teaching of algebraic thinking. Paula adds that a professional
development experience focused on algebraic thinking, offered during the school year,
would be optimal. Then the teachers could field test activities in the classroom and then
come back to discuss the results as a group. Although participation in the case study was
positive pressure to ensure that the teaching of algebraic thinking was an ongoing
experience for all three teachers, they have communicated the need to have a resource person available to answer questions and just generally support the teaching of algebra in the elementary grades.

These comments reflect the Lieberman and Miller (1990) concept of continuous inquiry into practice. Sonya, Josh, and Paula are teaching and learning in the context of the actual classroom, restructuring their leadership roles as teacher, and taking time for inquiry into their own teaching practice. However, they are still searching for opportunities to collaborate with peers in their own school and the district as a whole. They lack a professional community in which to examine, discuss, and evaluate the actions taken to reform their teaching practice (Dana, Campbell, & Lunetta, 1997).

Student Discourse as the Basis for Instruction

Sonya, Josh, and Paula are products of traditional mathematics teaching, but all three teachers have changed their beliefs about the nature of mathematics teaching and learning. These teachers have voluntarily sought to implement new instructional strategies in their practice, are exploring different patterns of authority and control over student learning and behavior, and are continually adjusting how to implement these changes in their classrooms because they have been provided with teaching practices that align with the new beliefs (Pradl, 1993; Richardson, 1994a). The pedagogical basis for the new instructional strategies is the emphasis on engagement with student mathematical thinking. Prior CGI research (Franke, et al., 1997) established that a change in beliefs must precede or accompany change in practice for any teacher to move to the highest level of engagement with children's mathematical thinking (Appendix E). The
incorporation of student discourse and student explanations into instruction is documented by examining each teacher’s level of engagement with student mathematical thinking.

All three case study teachers have surpassed Level 3 (Appendix E) use of student mathematical thinking in instruction. In the described lessons on algebraic thinking the teachers allowed students to solve problems in their own way, provided a variety of different problems for the students to solve, provided an opportunity for the students to discuss their solutions, and listened to the students talk about their thinking. Sonya has demonstrated some of the Level 4A (Appendix E) characteristics of engagement with student thinking. She provides opportunities for children to solve problems and elicits their thinking. Evidence of this level of engagement was found in the student discussion of the number sentence $8 + 2 = 5 + 5$. This was the first time that students had examined the validity of a number sentence that had more than one term on each side of the equal sign. Eight different types of student thinking were revealed during the presentation of student solutions.

Carl: False. Supposed to be right there. The plus should be there. He demonstrates that $8 + 2 + 5 = 5$, but says that yeah, it is still false. Counts eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen. Should be $8 + 2 + 5 = 15$.

Paul: True. $8 + 2 = 10$. This is true that $8 + 2 = 5 + 5$, but then adds it is actually false.

Priscilla: True. I think it is false. Eight and two is not five.

Thurman: Backwards or frontwards. Doesn’t matter. $8 + 2 = 10$ and $5 + 5 = 10$.

Ms. Henderson: Very interesting. Any other ideas?

Arlene: Think it is true. Kind of backwards. $8 + 2 = 10$ and $5 + 5 = 10$.

MaryAnn: Right up there. Think it’s different because it is plus. $8 + 2 = 10$ or $5 + 5 = 10$.

Ms Henderson writes the following and asks is ten equal to ten? $8 + 2 = 5 + 5$. 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
10 = 10
Responses included: No. Yes, it’s a double.
Ms. Henderson: Are there two different ways of making ten?
Kelly: I want to show you something. He says they can both be ten and he writes: 8 + 2 = 10 = 5 + 5

Ms. Henderson then moved on to a different problem. She later commented that she is learning not to tell the students that they are right or wrong, but to let the students leave the lesson thinking about the ideas.

The mathematics curriculum in her classroom is slowly evolving based on the student explanations during the number sentence discussions. Sonya stated that the school district requires her to give written tests as part of the assessment process, but she is changing the format of her tests to include student explanations of solutions, not just the answers themselves. In the coming school year the teaching and assessment of student number fact knowledge will be integrated into work on number sentences.

Sonya works with every student individually at least once in each 50-minute afternoon mathematics session. The student is required to show a completed paper to her before being permitted to select a math game or work on a math puzzle. Ms. Henderson checks each assignment over as the student is standing next to her. If there are incorrect answers, then she asks the student to explain how the solution was derived. If the student is uncertain, she guides the discussion with specific questions. Sonya knows the problem areas for each of her students, but uses the knowledge of the thinking of the students as a group to make instructional decisions. This is also evident in her description of the student results on the four-problem number sentence assessments. She lists the number of times the correct true or false response is given, but does not provide any information
about the student explanations or reasons behind the incorrect responses. The results of the number sentence assignment containing the problems:

1. \(6 + 1 = \square + 5\)
2. \(5 + 1 = 3 + 4\)
3. \(4 = 3 + 1\)
4. \(5 = 3 + 2\)

were summarized in quantitative terms. Ms. Henderson wrote that out of the fifteen students who completed the assignment that six missed none of the problems, four missed one problem, three missed two problems, and two missed three problems. In addition, she stated the first problem was missed four times, the second problem was missed twice, the third problem was missed four times, and the last problem was missed six times. Sonya did remark that the format of the last two problems was new to the students. She wanted to have some measure of their initial ideas before the topic was discussed in class. However, there is no detailed description of the individual student responses.

Josh Abernathy is also expanding his engagement with student thinking to include many characteristics of Level 4A (Appendix E). He provides opportunities for children to solve problems and elicits their thinking. Toward the end of a mathematics session Mr. Abernathy walked to the board and wrote \(0 + 5 = \square\). He introduced this number sentence by stating that he noticed several students working with problems like this and thought it was very interesting. He asked someone to complete the problem and the following discussion transpired.

Mr. Abernathy: \(0 + 5 = \square\).
Student responds: Five. Still have five.
Mr. Abernathy: \(0 + 4 = \square\).
Student: Four.
Mr. Abernathy: $7 + 0 = \square$.
Student: Seven.
Mr. Abernathy: What did all the problems have?
Student: Zero.
Mr. Abernathy: What happens when you add zero to any number?
Martha: $0 + 5 = 5$ and write it backwards $5 + 0 = 5$.
Mr. Abernathy: What happens with $22 + 0 = \square$.
Student: Get twenty-two.
Mr. Abernathy: $0 + 349 = \square$.
Student: Three hundred forty-nine.
Mr. Abernathy: $0 + 17,963 = \square$. There is a loud chorus of "Oh's" when he writes this problem.
Bruce: Correctly says seventeen thousand nine hundred sixty-three.
Mr. Abernathy: That is hard to say. What is the rule?
Karoline: $0 + 5 = 5$. Adding zero to any number.
The number doesn't change.
Mr. Abernathy: Does it work all the time? Try some more problems.
$39 + 0 = \square$.
Student: Thirty-nine.
Mr. Abernathy: $211 + 0 = \square$.
Student: Two hundred eleven.
Mr. Abernathy: What can we say is the rule?

The class decided that if you add zero to any number, then you still have the same number. The student responses and the sharing of solutions produced a conjecture about adding zero to a number.

Josh keeps a file of student work so he can refer to the papers to describe the problem solving strategies used by particular students. The assignment in which the students used their dot cards to represent the sums of 4, 7, 10, and 12 included writing the numerical representations of the solutions on the back of the paper. Reviewing the symbolic solutions revealed that one student interpreted the directions to mean redrawing the dot cards. Another student wrote each answer in the $c = a + b$ format. One student wrote each solution as the sum of the original number and zero, that is $4 + 0 = 4$, $7 + 0 = 7$, and so on. Then this same student tried to erase these solutions and replaced them with $4 = 4$, $7 = 5 + 2$ accompanied by the comment "not sure", $10 = 5 + 5$, and $12 = 5 + 4 + 1 + 2$. If
the solution involved more than two numbers, then one student wrote the first two numbers equal to the third number. For example, $3 + 3 = 1$ instead of $3 + 3 + 1 = 7$ and $5 + 4 = 3$ instead of $5 + 4 + 3 = 12$. Mr. Abernathy uses this information to guide his choice of number sentences in subsequent "number talks" with the class. Thus, the knowledge of the thinking of individual students is used to guide instruction for the class as a whole. Josh believes his teaching of algebraic thinking based on the notion of equivalence and relational thinking in number sentences promotes deeper conceptual understanding of patterns and relationships. However, he has still expressed concern about teaching all of the required mathematics content in the curriculum by the end of the school year. Thus, the progression or evolution of his mathematics curriculum is not totally determined by the student's mathematical thinking.

Paula Whitford exhibits the qualities of a teacher transitioning from Level 4A to Level 4B (Appendix E) as she bases her instruction on engagement with student thinking. Ms. Whitford creates opportunities to build on student mathematical thinking as she facilitates the student discussions about mathematics. Evidence of how she creates opportunities for student learning was present in the word problems about the tomato plants. The original problems did not include rewriting the solutions so that the students had to solve for the value of the variable "L". Paula encouraged the students to link the arithmetic word problems to equations with variables.

Ms. Whitford appreciates the way in which the knowledge of one child fits in with how mathematical understanding develops, and can provide detailed descriptions of the thinking of most of her individual students. She tracks the understanding of her students by maintaining a checklist of which types of problems individual students explained
correctly. For each student she marks a check to indicate which of the four number
sentences in the following list were correctly identified as true or false.

\[
\begin{align*}
14 &= 7 + 7 \\
6 + 2 &= 8 + 1 \\
6 + 2 + 1 &= 2 + 1 + 6 \\
1 + 2 &= 3 + 2 = 5
\end{align*}
\]

A similar check sheet was completed after students worked on the pattern activity that
represented the sum of the consecutive odd integers from one to twenty. Ms. Whitford
states that this technique allows her to identify, early in the school year, the students who
do not have the faintest idea about what is going on, who understands the concepts, and
who is working toward understanding. In the journal reflections Paula wrote that there
are several ways she can identify a student with genuine understanding of a problem and
its solution. An efficient solution strategy, the creation of several different correct
strategies, and the ability to explain or clear up problems with another student’s solution
are evidence that a student understands the mathematics being taught. Paula also listens
carefully to student conversations and observes the body language of individual students.
Paula acknowledges that she is more successful probing the understanding of some
students, than others. She has to coax some of the students to write out explanations of
their solution strategies and there are limits to how much time she can spend working
with individual students. Within the limits of class size and time she endeavors to probe
the thinking of each student. Ms. Whitford bases her instruction on her knowledge of
student thinking of the students as a group and on her detailed knowledge of the thinking
of individual students.
Each of the three teachers has stated that it is important for students to communicate in mathematics, that is be able to explain the reasoning behind the selection of a particular solution. The teachers are striving to improve how they facilitate this communication about student thinking into instruction. The way that the teacher listens to student discourse is a significant factor in the use of student thinking to guide instruction. Davis (1997) has described three levels of teacher listening to student discourse. Evaluative listening is a passive taking in of what learners are saying that reflects a view of mathematics as a system of already established formal truths. Interpretive listening is more of a negotiatory process that does not assess student contributions as correct or incorrect, but allows for revision of mathematical ideas. Interpretive listening reflects the belief that there is still one right understanding of mathematics. Both evaluative and interpretive listening communicate the fact that authority in the class rests with the teacher.

Hermeneutic listening is necessary for inquiry mathematics in which the teacher listens in order to participate as a learner in the student exploration of mathematical concepts. This level of listening reflects the belief that mathematics is subject to change and the teacher’s role is to interpret, transform, and question student understanding. The responsibility for learning is dispersed among students and the teacher. Examining the level of teacher listening is one more way to document the change in teacher use of student discourse in instruction.

Both Sonya and Josh are in the process of transitioning from interpretive listeners of student discourse to hermeneutic listeners of student discourse. The initial classroom observations revealed the listening as a negotiatory process in which student
contributions were neither correct nor incorrect, but were open to re-presentation and revision. The teachers listened constructively to the knowledge being acquired and to the sense the students made of the knowledge. The focus was still on a preset list of learning outcomes. The outcomes were the similarity to the student responses demonstrated in the videos viewed in the summer course. As Sonya and Josh gained experience facilitating student discussions, that is learning to let go of the need to “talk” more or to insure that students left the class knowing which response was the correct response they moved closer to the definition of hermeneutic listeners of student discourse. They are striving to listen as a basis for an inquiry mathematics that lacks a structured format and a preset list of learning outcomes. Their listening reflects a vision of the teacher as a participant, interpreter, and interrogator of student understanding with authority dispersed among students and teacher. Sonya and Josh acknowledge that they are still developing the skills to listen to the students at this level.

Paula is refining her hermeneutic listening skills. Paula eagerly participates as a learner in student exploration of mathematical concepts. She is flexible in responding to changing circumstances in student understanding and contributes to this change with her constant requests for students to prove what they have said. She has commented how some of her activities take longer than she had originally planned because of the student reaction and the different directions students go with their solutions. There is no concern about covering all the material when a lesson takes additional days, because she believes that the way she teaches math is working. Paula sees her role as facilitator of student learning and the responsibility for that learning is distributed among the students and the teacher.
All three teachers are already planning how they will approach the teaching of algebraic thinking in the coming school year. Sonya Henderson plans to keep the number sentence work on equivalence and relational thinking and the sharing of student solutions and strategies in the math discussions. She will expand the number sentence work to include the learning of basic number facts and will devote more class time to the writing and posting of student conjectures. Sonya did not indicate concern about student performance on standardized assessments or how her students will adapt to more traditional teaching in the second grade. She believes they are learning mathematical facts and thinking skills that will be applicable in any teaching environment. Sonya stated,

*It is my opinion that it is better for me to expose kid to these types of things even if their future teachers do not. Hopefully, I can establish enough of a base so that students will continue to think in this manner even if their future teachers do not provide direct opportunities for them to do so.*

Josh Abernathy will also continue to explore instructional strategies to foster the development of student algebraic thinking. He will focus on equivalence, using number sentences to teach arithmetic properties, and student development of conjectures. He is interested in refining his algebra assessment activity and administering it to third grade and fifth grade students to understand what their beliefs are about equivalence. Josh is totally fascinated with the idea of understanding how students at different grade levels think about basic algebraic ideas. He does not express concern about student performance on standardized assessments or how his students will adapt to a more traditional second grade mathematics instruction.
Paula Whitford embraces the pedagogical basis for fostering the development of student algebraic thinking, and will expand the use of number sentences to demonstrate equivalence, relational thinking, and the concept of variable. She has already described the high ranking of her students on the problem solving section of the standardized assessment used in the district. Ms. Whitford is interested in tracking the mathematical progress of her students in the fourth and fifth grades because of the very traditional mathematics instruction in those classes.

Summary

These three exceptional elementary school mathematics teachers have made significant changes in the teaching of mathematics and implemented many of the strategies and content of the summer course into their practice. They plan to expand and sustain the implementation of instructional strategies to foster the development of algebraic thinking. Sonya and Josh are confident that they have the mathematical content knowledge to be successful first grade mathematics teachers. Paula feels confident about teaching the mathematical content of the third grade curriculum, but would be hesitant about teaching mathematics in fourth or fifth grade or at the middle school level.

In all three cases the written documents, interviews, and classroom observations corroborate the fact that the teachers are using the engagement with student thinking as the basis for developing student algebraic thinking. The journal writings and the interviews revealed that all three teachers expressed a sense of uncertainty about a precise definition of the nature of algebraic thinking. However, the classroom observations and examination of related student assignments represented an understanding of algebra as
described in the *Principles and Standards of School Mathematics* or as a "web of knowledge and skill" (Kaput, 1995, 2000a, 2000b). Journal writings and interviews contained expressions of frustration over the lack of curriculum resources to support the teaching of algebraic thinking. The same data sources disclosed the common lack of opportunities for collaboration and communication with other teachers working to reform the teaching of mathematics. The exit interviews with Sonya, Josh, and Paula documented the desire of these teachers to sustain their efforts to explore instructional strategies that promote student algebraic thinking.
CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

The aim of this research study was to combine the ideas in the learning to teach aspect of change in teacher practice with the research on the teaching and learning of algebra to document and analyze the process of teacher efforts to facilitate the development of student algebraic thinking. The catalyst for this process was the graduate course “The Development of K-8 Algebraic Thinking.” The research study focused on investigating how this professional development experience influenced the teachers’ understanding of the nature of algebraic thinking and affected both the content and the pedagogy of the teachers’ mathematics instruction.

Examining the teacher understanding of the nature of algebraic thinking required establishing what aspects of each teacher’s vision of the nature of mathematics supported their interest in teaching to develop algebraic thinking. In addition, they communicated their understanding of the nature of algebraic thinking at the beginning and the end of the research study as documentation of any changes in this understanding. Evidence of how the teachers value the teaching of algebraic thinking was found in the extensive use of arithmetic-based number sentences to teach algebraic ideas. The implementation of this new mathematics content was accompanied by the use of student discourse and examination of student thinking to guide instruction. The changes in levels of teacher use
of student discourse and teacher comprehension of the nature of student learning are documented in the descriptions of classroom dialogue, teacher comments after the lessons, and formal interviews.

The three teachers who participated in this case study successfully added to their mathematics content knowledge base and either incorporated new pedagogy into their practice or refined an existing constructivist approach to teaching and learning. Their interest in the subject of student algebraic thinking and the willingness to participate in the research study represented an ongoing evaluation of values and continuous inquiry into practice that is characterized in the literature as "reflective practice." The summer professional development experience supported this teacher inquiry into practice by requiring the teachers to be actively engaged as learners during the course, focusing on subject-matter content, encouraging the teachers to share their experiences, providing multiple opportunities for reflection, and providing follow-up support as they implemented the teaching of algebraic thinking in the classroom.

The vision each teacher has of the nature of mathematics was a critical element in their ability to initiate the teaching of algebraic thinking and to sustain these efforts over the entire school year. Sonya, Josh, and Paula each demonstrate a relational view of the nature of mathematics (Skemp, 1978) in that their conceptions of mathematics allow for many different ways to complete a problem or mathematical task. Their collective vision of algebraic thinking as the examination of patterns or relationships and the use of these relationships to solve problems and analyze changing mathematical situations epitomizes the problem-solving view of the nature of mathematics (Ernest, 1989; Thompson, 1992).
The teachers have identified connections between the content that fosters student algebraic thinking and the pedagogy of basing instruction on levels of student engagement. They also understand there are tensions or contradictions that exist in the teaching of mathematics. Sonya remarked there is what she has to do in her teaching and what she wants to do and Josh expressed concern over the required curriculum content he has yet to teach in the semester. The teachers have identified the contradictions between their beliefs about the nature of mathematics teaching and learning and the traditional practice of mathematics teaching. They have worked to resolve the differences between beliefs and practice to accommodate the new pedagogy of using student thinking to drive instruction. Cooney (1999, 2001) describes this type of teacher as a reflective connectionist.

The three case study teachers view mathematics as a changing and expanding body of patterns whose study requires hands-on inquiry, experimentation, discussion, and reasoning. Their broad vision of the nature of mathematics is reflected in their understanding of algebraic thinking. The teachers have expanded their view of algebra as the study of patterns and relationships to include understanding the structure of number systems, justifying the use of particular operations or strategies to solve problems, understanding the meaning of equivalence, using and understanding the concept of variable, working with functions, facilitating student communication about their understanding of mathematics, and using algebraic symbols to represent mathematical situations. All three of the elementary teachers effectively implemented the teaching of equivalence, relational thinking, and justification or proof of student solution strategies into their mathematics teaching. Arithmetic-based number sentences such as $5 = 3 + 2$, 

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
$37 + 18 = 36 + 17$, and $75 + 5 = 73 + □$ were the vehicles for instruction on these topics. The third grade teacher added an emphasis on the understanding and use of variables in the writing of number sentences and application problems.

The development of student-generated conjectures was an integral part of the mathematics discussions and number talks. The formalization of the process by writing the conjectures for display in the classroom was not viewed, by the teachers, as the most significant part of the student learning process. Students developed conjectures, such as the statement that the addition of zero to any number does not change the number, but both Josh and Sonya stated that writing and displaying these conjectures in the classroom was an aspect of the teaching of algebraic thinking that they were still working to integrate into their practice. There were two conjectures displayed in one first grade room and four conjectures posted in the other first grade room. The third grade teacher asserted that she did not believe that writing and displaying student conjectures was the notable aspect of generating conjectures. It was important that students could recognize patterns or properties, verbalize these ideas, and use them to justify solutions to problems.

The teachers demonstrated the value they placed on the teaching of algebraic thinking by integrating student evaluation of number sentences into their lessons and by restructuring the format of their mathematics lessons to accommodate this type of instruction. Sonya initiated mathematics discussions based on the examination of number sentences, during the daily morning mathematics session, in place of the usual review of math facts. She plans to use this approach to teach number facts next year. Josh is implementing number talks several times a week. Students in his class are encouraged to contribute more to the class dialogue. He is trying to facilitate more student-to-student
talk and less teacher-to-student talk. Paula explained to her principal why the approved
elementary mathematics textbook does not support her approach to the teaching of
mathematics. The principal honored Paula’s request to supplement the textbook with
teacher-made instructional materials that support the teaching of mathematics in general
and algebraic thinking in particular.

All three individuals are restructuring their approach to student assessment. The
traditional tests in which students just write their answers are altered to include student
explanations of the solution strategy. Assignments also require students to state the
answer and a solution strategy. Mathematics discussions or number talks allow the
teachers to informally assess which students understand the concepts and which students
are still struggling to understand the mathematics. An integral part of the assessment
process is determining the nature of student learning that is occurring, particularly with
regard to student understanding of algebraic thinking.

The teacher perceptions of student learning with regard to algebraic thinking are
equivocal. Sonya states that evidence of student learning or understanding of algebraic
ideas is reflected in the ability to recognize relationships among numbers, use these
relationships to solve problems in a variety of ways, discover these solution strategies on
their own, and to justify the use of particular operations or properties to solve problems.
Josh acknowledges that he is not completely sure what his students understand about the
nature of algebraic thinking, but he believes an understanding of basic algebraic concepts
can be demonstrated in three significant ways. The student would be comfortable
working with number sentences written as $12 = 9 + 3$ and $9 + 3 = 12$, be able to justify
their theories, and support their thinking with examples, conjectures, or number
properties formulated in class. Paula describes algebraic thinking as a way to represent mathematics using symbols, numbers, and patterns. She fosters student understanding of multiple representations of problem situations with the mathematics activities she creates. All of the lessons observed in her classroom required students to use concrete objects, pictures, and symbolic mathematical notation in the descriptions of their solution strategies. The teachers are still in the process of refining their own definitions of algebraic thinking and examining student work as well as relying on student discourse to determine the nature of student learning.

The three teachers complained about the lack of peer support for their efforts to reform mathematics teaching. There was administrative support by the respective principals for the teaching of algebraic thinking, but no opportunities to collaborate with peers at their own schools or in the district as a whole. Sonya reiterated that it is better to have a group of teachers to work with than to implement new strategies in isolation. Josh is the algebraic thinking teacher leader in his school, but this role essentially has him marketing the beliefs and strategies that he has incorporated into his teaching practice. The teachers in his book study group are not true collaborators in reform of practice. Paula is building a collaborative network with the other three third grade teachers, but would like to be able to communicate with peers at other school sites, a math specialist, or a university instructor on a regular basis to discuss implementation issues. The lack of curriculum materials to support this type of teaching and the paucity of school district mathematics training in general are other factors that discourage the teaching of algebraic thinking in the elementary grades.
Sonya Henderson, Josh Abernathy, and Paula Whitford have made excellent progress in their efforts to implement instructional strategies that foster the development of algebraic thinking in their students. The beliefs that these teachers hold regarding the nature of mathematics and the nature of the teaching and learning of mathematics are the significant factors in determining the degree to which they have worked to accommodate reform and demonstrate reflective practice (Cooney, 1999, 2001; Cooney, et al., 1998). All three teachers cited their experiences in graduate mathematics education courses as influential in changing their beliefs about the teaching of mathematics. Sonya stated that in the mathematics methods course she realized that she could do the mathematics, but did not really understand the concepts behind the procedures. Thus, she was limited in how she could communicate the mathematics to her students in a meaningful way. Josh was introduced to the intriguing idea of teaching the ideas of algebra to first grade students in a graduate course. In addition, Josh credits the MASE training on the Investigations methodology with introducing him to the value of the hands-on experiences to teach mathematics. Paula stated that it was in the graduate mathematics education courses that she was introduced to the constructivist approach to teaching and had the opportunity to use a variety of manipulatives to teach arithmetic. These ideas became the foundation for her mathematics teaching.

Sonya, Josh, and Paula had already abandoned the belief that a traditional mathematics classroom based on direct instruction was an effective instructional environment prior to enrolling in the algebraic thinking course. Sonya and Josh implemented both the algebraic thinking content and the pedagogy focused on student thinking into their practice. Paula was already an advocate and implementer of teaching
based on the understanding of student thinking. She incorporated the algebraic ideas of equivalence, relational thinking, and proof into her teaching of arithmetic. The change in beliefs and the workshop focus on subject matter content, i.e. algebraic thinking, that was of interest to the teachers were the necessary elements for these teachers to move classroom practice to the highest levels of student engagement.

Recommendations

For these teachers to continue exploring the teaching of algebraic thinking they need opportunities for collaboration and communication with other teachers seeking to reform their practice. The algebraic thinking class will be taught again in the coming summer. Each of the three case study teachers should encourage another teacher at their school to attend the upcoming algebraic thinking class. In addition, Sonya, Josh, and Paula should share their experiences with the teachers in the class. In this way they create a collaborative partnership with another teacher at their school site. The school district could also be contacted about using grant money to pay for the tuition of teachers in the summer class. Encouraging Sonya, Josh, and Paula to share some of their lessons or activities as individual speakers or as part of a panel at local or national mathematics conferences would be another way for them to share and communicate with their peers.

The school district needs to provide more professional development opportunities focused on the teaching of a mathematics that fosters student algebraic thinking. The elementary mathematics specialist or the teacher leaders responsible for the district mathematics training should be invited to participate in the summer algebraic thinking class. Following participation in the course, these teacher leaders should observe Sonya’s
mathematics discussions and the number talks in Josh's classroom. Short videotapes could be made of the mathematics discussions and viewed at a school-level or district-level staff development session. Josh and the MASE coordinator for his school are preparing an article for publication. The article will highlight lessons that promote algebraic thinking. He and the MASE coordinator have videotaped one of the lessons as part of the article preparation. The videotaped lesson could also be shared with teachers in the book study group or at a district-level inservice training. Based on participation in the algebraic thinking class, observations of the case study teachers, and viewing of the videotapes, the district teacher leaders could plan and develop more mathematics training emphasizing algebraic thinking.

The school district has implemented the teaching of three levels of algebra as outlined in the middle school curriculum guide. Completion of an algebra course is required for high school graduation. This is evidence of administrative interest in student learning of algebra and puts pressure on mathematics teachers to find instructional strategies that improve the rates of student success in the algebra classes.

The university should continue to offer the algebraic thinking course. Other graduate mathematics education courses should include the topic of student algebraic thinking and explorations of the content and pedagogy described in the Thinking Mathematically textbook. The undergraduate mathematics methods classes for elementary, middle school, and secondary preservice teachers should include an introduction to the algebraic thinking content and corresponding pedagogy. Mathematics education course offerings could be expanded to include a one-week class that is geared toward the development of algebraic thinking in grades 7-12 or 9-12. All three teachers stated that they would
generally like the opportunity to take more mathematics content courses and would be particularly interested in mathematics with an emphasis on algebraic thinking. Collaboration with the mathematics department to offer a mathematics course such as “Algebra for Middle School Teachers” or “Algebra for K-8 Teachers” could address this interest in furthering mathematics content knowledge of algebra.

Implications for Further Research

There are some interesting possibilities for further research with these case study teachers and for designing new research studies. A longitudinal study of how these three teachers sustain the changes they have made in the teaching of mathematics over the next two to three years is a possible research opportunity. What changes in content and pedagogy did they make compared to the first year of teaching algebraic thinking? Did the teachers volunteer to participate in the algebraic thinking class? Did they speak at local, state, or national mathematics conferences? Has the district used their experiences to develop more and varied mathematics training? Have they sought out other teachers for collaboration and sharing of experiences in the teaching of algebraic thinking? Three new teachers, in the next algebraic thinking class, could be chosen for a similar study. It would be interesting to compare the experiences of the new case study teachers with those of the present case study teachers.

As Sonya, Josh, and Paula enter the second year of teaching algebraic thinking, student achievement could be examined or measured. Did their students demonstrate any significant difference in mathematics scores on the annual district standardized assessment compared to other students in the same grade at the same school? In
particular, how did the problem-solving scores compare? Paula has already done this type of examination of student achievement with her students. The results showed that her students scored higher on problem solving than the other third grade students. Students in the CGI study of the teaching of addition and subtraction facts also performed better in an assessment of problem solving ability. Informal assessment of student progress or achievement in the next grade level could be done by simply questioning the teacher in the new grade level about the mathematics performance of the students.

Another implication of the research with the three elementary teachers is to move this type of study to the examination of the algebra instruction of middle school and high school teachers. A similar type of summer course would need to be designed. Based on the ideas in Thinking Mathematically, the idea of equivalence could be taught using number sentences that incorporate the mathematics content of the middle school and high school curriculum. Arithmetic operations with decimals and fractions, order of operations problems with grouping symbols and exponents, and properties such as commutativity, associativity, and the distribution of multiplication across addition could be represented in number sentences. The students would be asked to determine if the statements were true or false and then be required to justify the response. Relational thinking and proof or justification would be taught in the same fashion.

Case study teachers would be selected and a yearlong commitment would be required. Based on the current study, it would be important to include a way for the teachers to collaborate and communicate with each other. The study could be linked to a university class or a district inservice with the use of grant money to either pay for course tuition or for the teachers to participate in regularly scheduled meetings. The scheduled
meetings or course times would provide opportunities for teachers to share materials and to discuss their experiences. The researcher could also work with the teachers to develop materials or find resources that support the teaching of algebraic thinking. The course instructor, researcher, and all the teacher participants would also be accessible electronically. Email could be used as part of the support mechanism. The same research design could be applied to the new group of elementary case study teachers.

The fact that access to instruction in algebra for every student is a significant issue in the overall movement for reform in mathematics teaching creates opportunities for teachers, administrators, and mathematics educators to investigate mathematics content and pedagogy that support student development of algebraic thinking. The three teachers in this study are rethinking the definition of algebra, exploring the nature of algebraic thinking, understanding more about the nature of student learning, and using this student understanding or thinking to guide classroom instruction. They are striving to be the kind of teachers that produce students who think instead of just applying algorithms. They describe good teaching as teaching for conceptual understanding, creating a learning environment where students are actively engaged in the mathematical task and talking about it, and most importantly teaching that is open to change. Research studies, such as this one, that examine the process of change in the individual teacher's classroom and the impact of that change on student learning can contribute to the professional dialogue about effective reform of mathematics teaching.
EXHIBITS
Table 1

Survey Response Table

<table>
<thead>
<tr>
<th>Question</th>
<th>SA</th>
<th>A</th>
<th>MA</th>
<th>MD</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>*1. Knowledge of arithmetic computations and procedures and repeated practice with the procedures is a prerequisite for success in algebra.</td>
<td>9</td>
<td>15</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td><strong>2. Elementary students usually understand the significance of an equal sign (&quot;=&quot;) and are able to correctly complete problems such as $8 + 7 = \Box + 5$.</strong></td>
<td>1</td>
<td>6</td>
<td>13</td>
<td>6</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td><strong>3. Elementary students should focus on the computational procedures in arithmetic and wait until middle school to explain and justify the properties of numbers.</strong></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td><strong>4. A required algebra course is not an appropriate class for most middle school students. Mathematics that requires algebraic thinking should be reserved for the high ability middle school students or high school students.</strong></td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>5</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td><strong>5. Student mistakes in mathematics should be resolved when they occur or at least before the student leaves the class for the day.</strong></td>
<td>3</td>
<td>13</td>
<td>9</td>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

(Table Continued.)
6. A thorough understanding of why arithmetic procedures work and the opportunity to apply these procedures in a variety of problems is a prerequisite for learning algebra.

7. An effective teacher provides the correct answer or explains the appropriate procedure when students make mistakes in mathematics.

*8. Elementary and middle school students work with equalities such as $8 + 7 = x + 5$, so the algebra curriculum could effectively begin with operations on equations, eliminating practice on expression such as $x + 3$, $5x$, or $x/3$.

9. Student discussions that include correct and incorrect strategies to solve math problems are a valuable part of mathematics instruction.

10. Algebra students who can successfully solve the equation $3x + 25 = 7$ can easily transfer this ability to an equation such as $3x + 25 = x + 7$.

11. Student mistakes in mathematics can be effectively resolved by student explanation of the work and input from other class members.

(Table Continued.)
12. The following list of number sentences has been used to demonstrate equality. How would you use this with your students?

- $3 + 5 = 8$
- $8 = 3 + 5$
- $8 = 8$
- $3 + 5 = 3 + 5$
- $3 + 5 = 5 + 3$
- $3 + 5 = 4 + 4$

Responses included:
- Variety of ways to write equations.
- Show both sides balance.
- Demonstrate properties.
- Practice math facts.
- Exploration – how else get this number?
- Show many ways to represent a number.
- Focus on “=” meaning “same as.”
- Discuss why equations are true.

*Indicates only 34 responses to this question.

S indicates that Sonya’s response is counted in the number of responses in this category.

J indicates that Josh’s response is counted in the number of responses in this category.

P indicates that Paula’s response is counted in the number of responses in this category.
Table 2

Ethnic Distribution of Student Population

<table>
<thead>
<tr>
<th></th>
<th>American</th>
<th>Asian or Pacific</th>
<th>Hispanic</th>
<th>African American</th>
<th>White, not of Hispanic</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Native</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hilltop</td>
<td>1.2%</td>
<td>7.3%</td>
<td>38.6%</td>
<td>9.3%</td>
<td>43.5%</td>
<td>99.9%</td>
</tr>
<tr>
<td>Grandview</td>
<td>1.1%</td>
<td>16.4%</td>
<td>29.8%</td>
<td>13.4%</td>
<td>39.2%</td>
<td>99.9%</td>
</tr>
<tr>
<td>Evergreen</td>
<td>1.0%</td>
<td>6.4%</td>
<td>20.8%</td>
<td>7.1%</td>
<td>64.7%</td>
<td>99.9%</td>
</tr>
<tr>
<td>District</td>
<td>0.8%</td>
<td>7.9%</td>
<td>33.4%</td>
<td>14.2%</td>
<td>43.7%</td>
<td>100%</td>
</tr>
</tbody>
</table>

(Clark County School District Nevada Ethnic Report, 2004)
Table 3

School Characteristics

<table>
<thead>
<tr>
<th></th>
<th>Special Education</th>
<th>English Language</th>
<th>Free or Reduced Learner</th>
<th>Transiency Rate</th>
<th>Enrollment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hilltop</td>
<td>10%</td>
<td>15%</td>
<td>37%</td>
<td>38%</td>
<td>728</td>
</tr>
<tr>
<td>Grandview</td>
<td>10%</td>
<td>11%</td>
<td>21%</td>
<td>31%</td>
<td>707</td>
</tr>
<tr>
<td>Evergreen</td>
<td>7%</td>
<td>4%</td>
<td>19%</td>
<td>24%</td>
<td>807</td>
</tr>
<tr>
<td>District</td>
<td>11%</td>
<td>16%</td>
<td>42%</td>
<td>39%</td>
<td>268,357</td>
</tr>
</tbody>
</table>

(Clark County School District Accountability Report, 2001-2002)
<table>
<thead>
<tr>
<th>School</th>
<th>Reading</th>
<th>Language</th>
<th>Math</th>
<th>Science</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hilltop</td>
<td>53</td>
<td>57</td>
<td>59</td>
<td>58</td>
</tr>
<tr>
<td>Grandview</td>
<td>60</td>
<td>69</td>
<td>70</td>
<td>65</td>
</tr>
<tr>
<td>Evergreen</td>
<td>66</td>
<td>78</td>
<td>69</td>
<td>70</td>
</tr>
<tr>
<td>District</td>
<td>49</td>
<td>53</td>
<td>52</td>
<td>54</td>
</tr>
</tbody>
</table>

(Clark County School District Accountability Report, 2001-2002)
Table 5

Algebraic Thinking Assessment

<table>
<thead>
<tr>
<th>Symbol assessment</th>
<th>Solve the problems: TRUE or FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>What does each sign mean?</td>
<td>2 + 2 = 4</td>
</tr>
<tr>
<td>+</td>
<td>4 - 1 = 3</td>
</tr>
<tr>
<td>-</td>
<td>10 = 5 + 5</td>
</tr>
<tr>
<td>=</td>
<td>6 = 6</td>
</tr>
<tr>
<td>□</td>
<td></td>
</tr>
<tr>
<td>Read the following equations:</td>
<td>4 + 1 = □</td>
</tr>
<tr>
<td>4 + 3 = 7</td>
<td>6 - 3 = □</td>
</tr>
<tr>
<td>6 - 2 = 4</td>
<td>5 + 5 = □ + 9</td>
</tr>
<tr>
<td>8 = 5 + 3</td>
<td>□ = 3 + 1</td>
</tr>
<tr>
<td>10 = 10</td>
<td></td>
</tr>
<tr>
<td>5 + 5 = □</td>
<td>Side 1</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Appendix A

Teaching of Algebra Teacher Opinions

Fill out as completely as possible. Please add any additional comments or information that you feel would assist in documenting a complete picture of your background, experience, and vision of effective mathematics instruction that promotes the teaching of algebra.

Teaching Experience and Educational Background

Number of years teaching in CCSD. __________

Total number of years teaching experience. __________

Highest degree attained.

_____ Bachelor’s _____ Master’s _____ Ed Specialist, Ed.D, or Ph.D.

Total number of college credits in mathematics. __________

Current Teaching Assignment

_____ Elementary Indicate grade level or specialization _________________________

_____ Middle School Indicate grade and/or subject ______________________________

_____ High School Indicate grade and/or subject ________________________________
Teacher Opinions

Circle the response that best represents the degree to which you agree or disagree with each statement.

SA  A  MA  MD  D  SD
Strongly Agree  Moderately Agree  Moderately Disagree  Disagree  Strongly Disagree

1. Knowledge of arithmetic computations and procedures and repeated practice with the procedures is a prerequisite for success in algebra.

2. Elementary students usually understand the significance of an equal sign ("=") and are able to correctly complete problems such as $8 + 7 = \square + 5$.

3. Elementary students should focus on the computational procedures in arithmetic and wait until middle school to explain and justify the properties of numbers.
4. A required algebra course is not an appropriate class for most middle school students. Mathematics that requires algebraic thinking should be reserved for the high ability middle school students or high school students.

5. Student mistakes in mathematics should be resolved when they occur or at least before the student leaves the class for the day.

6. A thorough understanding of why arithmetic procedures work and the opportunity to apply these procedures in a variety of problems is a prerequisite for learning algebra.

7. An effective teacher provides the correct answer or explains the appropriate procedure when students make mistakes in mathematics.
8. Elementary and middle school students work with equalities such as $8 + 7 = x + 5$, so the algebra curriculum could effectively begin with operations on equations, eliminating practice on expressions such as $x + 3$, $5x$, or $x/3$.

9. Student discussions that include correct and incorrect strategies to solve math problems are a valuable part of mathematics instruction.

10. Algebra students who can successfully solve the equation $3x + 25 = 7$, can easily transfer this ability to an equation such as $3x + 25 = x + 7$.

11. Student mistakes in mathematics can be effectively resolved by student explanation of the work and input from other class members.
12. The following list of number sentences has been used to demonstrate different perspectives on equality (Carpenter, et al., 2003). How would you use this with your students?

\[
\begin{align*}
3 + 5 &= 8 \\
8 &= 3 + 5 \\
8 &= 8 \\
3 + 5 &= 3 + 5 \\
3 + 5 &= 5 + 3 \\
3 + 5 &= 4 + 4 \\
\end{align*}
\]

13. Give a detailed description of how you would justify the solution \( d = 49 \) in the number sentence \( 57 + 46 = 54 + d \).

14. Please write any additional comments or ideas about teaching algebra in our school district.

Optional: Name and School ______________________________
Appendix B

Journal Topics

1. What was your motivation for taking this course in fostering the development of algebraic thinking?
2. What is your understanding of the nature of algebraic thinking?
3. How would you define good teaching, particularly in mathematics?
4. How do you determine when the students understand the math concept that you are teaching?
5. What are the three to five big ideas in mathematics that you want your students to know and understand at the end of the school year? (Schifter & Fosnot, 1993)
6. What aspects of this summer class do you think you will use in your teaching?
Appendix C

Interview Questions

Initial interview.

1. How would you characterize your own mathematics background and aptitude?
   Specifically, what is the best way for you to learn mathematics?
   Probe for view of nature of mathematics and vision of mathematics teaching and
   learning. Probe for elaboration on responses to specific survey questions.

2. What aspects of this summer class are you specifically planning to incorporate
   into your teaching? Provide details of your ideas to implement this in your class.

3. What is your understanding of the nature of algebraic thinking?
   Give examples to support this understanding.

4. What do you think is your students' understanding of the nature of algebraic
   thinking? How do you determine/document when the student has this
   understanding?

5. Describe a typical lesson in your mathematics classroom. Specifically discuss the
   role of the teacher and the involvement of the student in the teaching and learning
   process.

6. What are the three to five major ideas of algebra that you want the students to
   know and understand at the end of the school year?
Exit interview.

1. How has your view of mathematics teaching and learning changed?

2. What aspects of the development of algebraic thinking were successfully integrated into your teaching practice? Did this include the three to five major ideas of algebra that you identified previously?

3. Has your understanding of the nature of algebraic thinking changed? What do you think is the students' understanding of the nature of algebra?

4. In your classroom, how have the roles of both teacher and student changed or not changed?

5. What aspects of the teaching for the development of algebraic thinking will you maintain in your classroom? What would you like to expand or alter? How do you visualize doing this?

6. What would you change in the professional development experience? Workshop content or pedagogy? Follow-up support structure?
Appendix D
Levels of Teacher Listening to Student Discourse

Evaluative Listening

Listening as a largely passive taking in or absorption of what learners are saying.
Listening for something in particular such as a specific mathematical explanation.
Listening to evaluate the correctness of a student response by judging it against a preconceived standard or answer.
Limited and limiting type of listening.
Reflects belief that mathematics is a system of already established formal truths.
Reflects belief that teaching mathematics is a process that avoids ambiguity.
Authority rests with the teacher.

Interpretive Listening

Listening to what learners are saying and trying to make sense of what they are saying.
Listening more of a negotiatory process.
Listening without assessing student contributions as either correct or incorrect, but being open to re-presentation and revision of mathematical ideas.
Listening with an awareness that an active interpretation of student talk or reaching out for student explanations assists in the task of converging on some sort of correct understanding of the concept.
Reflects belief that mathematics exists independently of classroom experiences.
Reflects belief that there is still one right understanding of mathematics.
Listening constructively to knowledge acquired and the sense being made of the knowledge.

Authority rests with the teacher.

Hermeneutic Listening

Listening that is necessary for inquiry mathematics that lacks a clearly structured format and a preset list of learning outcomes.

Listening to enable a flexible response to changing circumstances in student understanding.

Listening as a starting place for bringing together insights into cognition and criticism of conventional school mathematics.

Listening to participate as a learner in student exploration of mathematical concepts.

Listening reflects the negotiated and participatory nature of interacting with learners.

Reflects belief that mathematics is subject to change and some of its conventions are arbitrary.

Reflects belief that the teacher’s role is one of participation, interpretation, transformation, and interrogation of student understanding.

Authority dispersed among students and the teacher.

(Davis, 1997)
Appendix E

Levels of Engagement with Children’s Mathematical Thinking

*Level 1:* The teacher does not believe that the students in his or her classroom can solve problems unless they have been taught how.

- Does not provide opportunities for solving problems.
- Does not ask the children how they solved problems.
- Does not use children’s mathematical thinking in making instructional decisions.

*Level 2:* A shift occurs as the teachers begin to view children as bringing mathematical knowledge to learning situations.

- Believes that children can solve problems without being explicitly taught a strategy.
- Talks about the value of a variety of solutions and expands the types of problems they use.
- Is inconsistent in beliefs and practices related to showing children how to solve problems.
- Issues other than children’s thinking drive the selection of problems and activities.

*Level 3:* The teacher believes it is beneficial for children to solve problems in their own ways because their own ways make more sense to them and the teachers want the children to understand what they are doing.

- Provides a variety of different problems for children to solve.
- Provides an opportunity for the children to discuss their solutions.
- Listens to the children talk about their thinking.
Level 4A: The teacher believes that children’s mathematical thinking should determine the evolution of the curriculum and the ways in which the teachers individually interact with the students.

  Provides opportunities for children to solve problems and elicits their thinking.
  Describes in detail individual children’s mathematical thinking.
  Uses knowledge of thinking of children as a group to make instructional decisions.

Level 4B: The teacher knows how what an individual child knows fits in with how children’s mathematical understanding develops.

  Creates opportunities to build on children’s mathematical thinking.
  Describes in detail individual children’s mathematical thinking.
  Uses what he or she learns about individual students’ mathematical thinking to drive instruction.

(Franke, et al., 2001, p. 662)
REFERENCES


Kaput, J. J. (2000b). Transforming algebra from an engine of inequity to an engine of mathematical power by “Algebrfying” the K-12 curriculum. Dartmouth, MA: National Center for Improving Student Learning and Achievement in Mathematics and Science. (ERIC Document Reproduction Services No. ED 441 664.)


Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula, T. J. Buttery, & E. Guyton (Eds.), *Handbook of research on teacher education*.


Tall, D. (1989). Different cognitive obstacles in a technological paradigm or A reaction to: “Cognitive obstacles encountered in the learning of algebra”. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of school algebra*.


VITA

Graduate College
University of Nevada, Las Vegas

Cynthia Ann Hernon

Home Address:
2248 Bowie Circle
Henderson, Nevada 89014-4909

Degrees:
Bachelor of Arts, History, 1970
University of Nevada, Las Vegas

Masters of Education, Curriculum & Instruction, 1975
University of Nevada, Las Vegas

Special Honors and Awards:
Second Place Award, Humanities Division, Poster Session,
Graduate & Professional Student Research Forum, 2004
University of Nevada, Las Vegas

Dissertation Title: Teacher Exploration of Instructional Strategies to Promote
Algebraic Thinking

Dissertation Examination Committee:
Co-Chairman, Dr. Jeffrey Shih, Ph. D.
Co-Chairman, Dr. William Speer, Ph. D.
Committee Member, Dr. Marilyn Sue Ford, Ph. D.
Committee Member, Dr. Kent Crippen, Ph. D.
Graduate Faculty Representative, Dr. Peter Shiue, Ph. D.