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GERMAN MATHEMATICS TEACHERS' SUBJECT CONTENT AND PEDAGOGICAL CONTENT KNOWLEDGE

by

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ABSTRACT

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by

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How required teacher knowledge is obtained is debated in today's educational context. This dichotomy in acquisition of teacher knowledge between university training including content and pedagogy versus classroom experiences combined with strong subject background has become particularly important as the U.S. seeks to find key components to increase student achievement and to improve education. International comparisons indicating the U.S. consistently lags behind top-performing countries have spurred such efforts.

Review of existing literature exposed differences in what is considered necessary knowledge for effective teaching, and where such knowledge can be developed. Types of knowledge and where such knowledge is acquired are examined. A gap in the body of knowledge is identified followed by a description to begin to fill it. An examination of international mathematical comparisons, typically resulting in an Asian-U.S. comparison, is included. Justification is provided to analyze Germany to challenge current assumptions concerning teacher knowledge and the role thereof on student achievement. German teachers receive increased content and pedagogy training, yet German students score only average on international mathematics comparisons.

To understand better the impact of reforms calling for increased teacher subject content knowledge, further investigation into teachers' understanding of subject content knowledge along with contributions to such knowledge was conducted. To investigate this issue three research questions emerged: Do German mathematics teachers possess the knowledge and skills to solve correctly basic mathematics problems? Can they translate this knowledge into accurate representations? According to them, what is the contribution of teacher education and classroom experiences in building teacher knowledge? A qualitative interview project approach involving surveys and interviews was utilized.

Findings indicate that German mathematics teachers possess the knowledge and skills to solve basic mathematical problems correctly implying solid subject content knowledge; however, are not as successful in generating accurate representations and explanations implying a limited pedagogical content knowledge. According to these teachers, teacher preparation courses contributed to pedagogical not content knowledge while classroom experiences were valued as contributing to both types of knowledge. Results can inform educational policies, practices, and reforms in the U.S., and provide a basis for further research, with increased student achievement the ultimate goal.

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CHAPTER 1

INTRODUCTION

Education in the United States has faced scrutiny through educational comparisons with other countries (Third International Mathematics and Science Study (TIMSS), 1995; Program for International Student Achievement (PISA), 2000). Evaluation, analysis and a steady stream of criticism of the education system are not unique to the United States ("Seeing to It", 2003). Other countries understandably also want to see their students score at the top in international comparisons. In response, several studies both in the United States and comparatively have been conducted in order to compare, contrast, and hopefully uncover aspects of education that can be changed in order to improve student achievement, including a focus on teachers nature and function of their knowledge for effective teaching (e.g. Shulman, 1986; Ball, 1990; Ma, 1999; Perry 2000). To explore better such issues of teacher knowledge, further investigation is made into what constitutes necessary teacher knowledge.

Teacher Learning

A current debate in education revolves around the professionalization versus deregulation of teaching (Angus, 2001; Cochran-Smith & Fries, 2001). The process of learning various components such as subject content and pedagogy that comprise teacher knowledge is debated in the literature and focuses on where and how such knowledge is developed. This professionalization/deregulation debate includes arguments concerning how and where teachers acquire such necessary knowledge and skills to become an effective teacher (e.g. Angus, 2001; Hill, Sleep, Lewis, & Ball, 2007; Cochran-Smith & Fries, 2001). Both sides claim that their stance is research based and necessary to improve education in the United States (Cochran-Smith & Fries, 2001). New standards and changes in the make-up of classrooms across the United States have prompted both sides to seek out new means for teacher preparation to equip teachers with the knowledge and skills necessary to be effective in reaching all students while meeting higher standards. Arguments made by each side of the professionalization/deregulation debate are reviewed next.

Professionalization

Those who call for professionalization point to the need for better-trained teachers who are equipped with the skills and knowledge unique to teaching (Cochran-Smith & Fries, 2001; Ballou & Podgursky, 2000). In order to accomplish the task of providing well-trained teachers to every student, those advocating professionalization recognize the importance of teacher education programs to prepare future teachers (Cochran-Smith & Fries, 2001). It is important to note that teacher education programs are just one component of teacher learning since teachers' learning begins before and continues after their enrollment in teacher education courses (Feiman-Nemser & Remillard, 1995; Ball & McDiarmid, 1990). Despite the realization that teacher learning involves more than teacher education programs, as well as taking into consideration differences between programs. At the heart of the professionalization versus deregulation discussion is whether or not teacher education programs effectively prepare teachers with appropriate knowledge for teaching, and whether or not they are a necessity to adequately prepare the teaching force (Cochran-Smith & Fries, 2001). There has been agreement that pre-service teachers must be exposed to the type of teaching they themselves will be expected to teach (Feiman-Nemser & Remillard, 1995). Teacher education programs are expected to provide pre-service teachers with the substantial knowledge and special skill set necessary to ensure all students are able to learn and achieve, which includes both subject content as well as pedagogy (Darling-Hammond & Cobb, 1996). University programs are seen as a source for helping teachers develop pedagogical content knowledge, or the ability to transform content knowledge in a manner so as to facilitate learning and understanding in others (Shulman, 1986).

Universities are in the midst of reform in terms of seeking accreditation, and assessing the manner in which they approach the education of future teachers including a greater emphasis on areas such as "learning theory, cognition, and learning strategies that has accompanied a deepening appreciation for content pedagogy and constructivist teaching strategies (Darling-Hammond & Cobb, 1996, p. 43)." Various organizations, such as the National Council of Teachers of Mathematics, the National Science Teachers Association, and the National Council of Teachers of English, have supported initiatives with the intent to create standards-based teaching (including standards for teacher knowledge) within the professionalization drive (Cochran-Smith & Fries, 2001; Darling-Hammond & Cobb, 1996). Darling-Hammond and Cobb (1996) contend that despite differences in teacher education programs, every program must teach teachers to build their knowledge and understanding through collaboration, inquiry, evaluation of new ideas, and reflection on the products of their work consistent with knowledge-*of*-teaching as discussed by Cochran-Smith and Lytle (1999), which includes research, universities, and classroom experiences all as sources of necessary knowledge for teaching.

Deregulation

Conversely, proponents of deregulation claim that teacher education programs are an unnecessary institution designed to promote traditions and structures that are inadequate in preparing teachers (Cochran-Smith & Fries, 2001). This side of the debate maintains that effective teaching skills are not the result of teacher preparation but of natural talent, subject training, plus field experience. A major premise of the deregulation agenda is that teachers should be tested for skills and knowledge rather than requiring certain courses and degrees since these keep some of what may be the most effective teachers out of the profession (Cochran-Smith & Fries, 2001). As a result of these beliefs and viewpoints, those in favor of deregulation are seen as advocates for alternate routes to licensure, emergency licenses, and even for the elimination of teacher education programs altogether (Cochran-Smith & Fries, 2001). In fact, whether intended or not, support of the deregulation of teaching has come from state legislatures in the form of granting what is often termed emergency licensing to individuals who have not completed a teacher preparation program (Darling-Hammond & Cobb, 1996). Consequently, in this model subject content knowledge is valued as a prerequisite for teaching, with such knowledge most likely coming from completing a university degree in content knowledge and management through one's own classroom experiences (Cochran-Smith & Fries, 2001).

The crux of the argument then is what knowledge is necessary, and where such knowledge is developed. On the side of professionalization is the contention that a professional body of knowledge in the form of content and pedagogy is necessary for effective teaching and that can be developed through teacher education programs. Opposite that stance are those favoring deregulation, that is, those who maintain subject content knowledge is what is necessary, and that no pedagogical training is needed for effective teaching. Teacher education programs seek to tie subject matter knowledge with pedagogy with universities favoring professionalism, while those in favor of deregulation focus solely on content knowledge.

Teacher Knowledge in an International Context

As the United States seeks to improve student achievement on international comparisons that indicate the United States is average at best (TIMSS, 1995; PISA, 2000; PISA 2003; PISA 2006), teacher knowledge has become one of several components compared and contrasted to other countries usually outperforming the United States, in the international comparisons (e.g. Perry, 2000; Stigler & Hiebert, 1999; Ma, 1999). These studies, along with studies specific to teachers in the United States (e.g. Ball, 1990; NCRTE, 1993) have found that teacher knowledge in the United States is not strong in terms of content or pedagogical content knowledge. A large study, Teacher Education and Learning to Teach (TELT), was conducted in the United States examining this phenomenon in American mathematics teachers by the National Center for Research on Teacher Education or NCRTE (NCRTE, 1993). The study involved both pre-service and in-service teachers across the United States with researchers interviewing participants at varying points in their respective programs (Kennedy, Ball, & McDiarmid, 1993). One of the in-service programs studied focusing on mathematics, and provided some of the basis for this study (Kennedy, Ball, & McDiarmid, 1993). The results indicated that most American mathematics teachers did not have a deep understanding of principles that are fundamental to their subject evidenced by the fact that they had difficulty explaining basic mathematical principles such as place value (NCRTE, 1993).

On the other hand, studies have shown that teachers in Asian countries such as China and Japan have a deeper knowledge and understanding of subject content, which in turn affects their pedagogical content knowledge and practice (Ma, 1999; Perry, 2000). The measurement from the TELT study was used to conduct a similar study by comparing teachers from both China and the United States (Ma, 1999). The results were that Chinese mathematics teachers (as opposed to American mathematics teachers) for the most part demonstrated a deeper and better mathematical knowledge that Ma termed as "Profound Understanding of Fundamental Mathematics" or PUFM. The conclusions reached by Ma (1999) showed that not only did Chinese mathematics teachers have a better understanding of fundamental mathematics, they were able to use this knowledge to provide a more pedagogically sound approach to teaching mathematics by connecting and revisiting concepts as well as to provide multiple instructional approaches to the concepts being taught.

Although studies including the TELT study and Ma's study showed differences in teacher knowledge and practice between the United States and various countries that typically tend to be Asian countries, such as China or Japan, who consistently score the highest on international comparisons (e.g. Ma, 1999; Stigler & Hiebert, 1999; Perry, 2000), what is missing from the research is an examination of other countries that could provide a challenge to the conclusions of these studies. Perhaps an examination of countries that score similarly to that of the United States but where teachers acquire different knowledge varies, could uncover differences leading to an increase in teacher knowledge and student achievement.

For example, German students perform similarly to the United States on international comparisons, yet teacher preparation requirements in both content and pedagogy exceed the requirements of teachers in the United States, with the typical German education degree consisting of five to six years focused solely on content and pedagogy in comparison with a typical four year degree in the United States that tends to include general education requirements in addition to course required for the major (Viebahn, 2003; Kolstad, Coker, and Kolstad, 1996; Darling-Hammond & Cobb, 1996). The unique case presented by Germany merits further investigation. Perhaps German teachers also possess similar teacher knowledge as what has been found in China and Japan; yet this would cause a re-evaluation of what is believed about teacher knowledge since student achievement does not match their Asian counterparts, including China, Japan, Korea, etc. Or, it could be that despite extensive training in both content and pedagogy German teachers do not possess similarly deep and complete understandings of content and pedagogy, which would support current views of teacher knowledge. The latter results however would raise questions concerning lengthy teacher preparation programs and professionalism if such an investment of time does not produce the type of knowledge considered necessary for teaching. At a time when education reform is seeking to improve student achievement through teacher learning (Kennedy, 1991) it seems prudent that a complete and broad picture of various components is obtained prior to advocating certain reforms. These conditions taken together warrant further research into this matter through the involvement of another country to provide further data and begin to broaden the research and understanding that currently exists. Further research will not only begin to fill in gaps in the literature, it will also enable more sound reform and provide additional data for the professionalization/deregulation debate in the United States.

CHAPTER 2

REVIEW OF THE LITERATURE

Teacher Knowledge

Teacher knowledge, including what teachers should know, and where such knowledge comes from, has been an issue for over one hundred years; and a debate is still revolving around subject content knowledge, pedagogical knowledge, and pedagogical content knowledge (e.g. Shulman, 1986; Cochran-Smith & Fries, 2001). An analysis of nineteenth century teaching tests to examine teacher knowledge and determine the qualifications of teachers required in the United States prior to licensure show that ninety to ninety-five percent of what was tested dealt with subject matter content, with just five to ten percent focused towards the pedagogy, or the "knowledge base" needed to teach (Shulman, 1986; Angus, 2001). Clearly the emphasis was on subject content and not pedagogy, or the knowledge and skills for specific to teaching.

In contrast to tests administered in the 1800's, an analysis of today's tests show a greater emphasis is now on ability to teach (pedagogy) with topics that include: organization in preparing and presenting instructional plans, evaluation, cultural awareness, understanding youth, management, educational policies and procedures, etc. (Shulman, 1986). Hill, Sleep, Lewis, and Ball (2007) also note the variations in the types and purposes of teacher testing.

Many states currently require a content portion of the praxis test, but again, most of what is tested deals with generalized pedagogy or skills, with such tests developed not by members of the education profession, but by commercial vendors or state agencies (Darling-Hammond & Cobb, 1996). Policymakers refer to research on teaching, which due to their scope and focus are necessarily limited to specific behaviors an effective teacher would demonstrate, to justify why most of what teachers are tested for deals with pedagogy (Shulman, 1986). This definite shift in what is tested perhaps mirrors what is now considered important for effective teaching.

As explained by Feiman-Nemser and Remillard (1995), there is a difference between knowing how teachers learn to think and act, and what good teachers actually do, how they think, or what they know. Current tests of generalized pedagogy or skills are not designed to measure what good teachers actually know or do. This is not to say that subject matter is discounted in these research publications as unimportant, but it does indicate a shift in focus for teacher knowledge. In addition to subject matter and pedagogy, Shulman (1986) contended that what is missing in research is an examination of how subject content knowledge is "transformed from the knowledge of the teacher into the content of instruction (p.6)", or pedagogical content knowledge. Pedagogical content knowledge is necessary for effective teaching because it is this knowledge that teachers use to decide what content to include in a lesson, what questions to ask, and what explanations to provide (Shulman, 1986). Despite contentions by some such as deregulationists who argue that knowledge of pedagogy is not required prior to entering the classroom. Such a separation of this knowledge into only content or only pedagogy becomes problematic due to the intertwining nature of the two types of knowledge;

neither type of knowledge by itself is adequate for teaching others (Shulman, 1986; Cochran-Smith & Lytle, 1999; Hill, Sleep, Lewis, & Ball, 2007). Thus, despite various terms assigned, there emerge three main prongs of teacher knowledge, which will be referred to as: subject content knowledge, pedagogical knowledge, and pedagogical content knowledge.

Subject Content Knowledge

Subject content knowledge is considered by Shulman (1986) to be knowledge that deals with the quantity and quality of knowledge of the teacher, and is considered "what teachers need to know" (Ball & McDiarmid, 1990, p.437). Knowledge is more than a collection of facts, figures, and memorization of procedures, and goes beyond simply being able to compute an answer to a given problem. Subject content knowledge includes knowledge of topics, procedures, concepts, and the relationship between each of these (Ball, 1991). Indeed, subject content knowledge equips teachers to know which topics of a given subject are most vital, and then enables them to choose appropriate assignments to reinforce specific concepts (Kennedy, 1991). Beyond avoiding misconceptions and mistakes, teachers are able to inspire and engage students based in large part on their own intellectual level and understanding of the subject (Ball & McDiarmid, 1990). A number of studies indicate, however, that many teachers believe they possess the necessary knowledge of facts, concepts, and skills for being an effective teacher simply from their years of schooling (Feiman-Nemser & Remillard, 1995). Based on their previous experiences prospective teachers believe that subject content knowledge is structured, and should be taught in the same manners in which they were taught (Feiman-Nemser & Remillard, 1995). Indeed, true teacher knowledge goes much further that a teacher can not only explain given components of the subject, but can also explain why such knowledge is important, valid, and how it relates to broader field of study (Shulman, 1986; Grossman, 1990). Adequate subject content knowledge is vital if teachers are to avoid incorrectly conveying the subject being taught (Ball & McDiarmid, 1990; Grossman, 1990).

The "keepers" of academic disciplines contend that American students do not receive adequate preparation in most subject areas, which is assumed to perhaps reflect questioningly into the subject content expertise of teachers (Kennedy, 1991; Stedman, 1997). Several professional organizations such as the National Council of Teachers of Mathematics, the National Science Teachers Association, and the National Council of Teachers of English have promoted and redefined standards of teaching and student learning (Darling-Hammond & Cobb, 1996). These standards provide certain expectations of the type of subject content knowledge teachers within the various disciplines should possess. Recent, renewed interest in teachers' subject content knowledge, as was a previous focus for teacher certification (Shulman, 1986; Angus, 2001) has provided surprising evidence of the lack of deep understanding of the subject, even when teachers have majored in the subject (NCRTE, 1993).

Teacher subject content knowledge, insofar as teaching the content is concerned, has become the focus of policies concerning education with implications for both teachers and students especially since subject content knowledge is often linked to student performance (Hill, Rowan, & Ball, 2005). Despite the current realization of the importance of teacher subject content knowledge there is still not enough known about what teachers actually know, and how such knowledge leads to student performance

(Kennedy, 1991; Ball & McDiarmid, 1990). These facts support the need for further research and understanding into teachers' subject content knowledge and the contribution of such knowledge to effective teaching. Additionally, some scholars assume in spite of its importance, content knowledge alone is not considered sufficient for effective teaching (Hill, Sleep, Lewis & Ball, 2007). Content knowledge alone does not provide teachers with the ability to interpret classroom events, how to respond to student questions or inquiries, what assignment will be most effective in promoting learning, what questions to ask, or how to coordinate the learning of the many different learners found in today's classrooms, rather what is needed is the knowledge and ability to combine these factors with subject content knowledge in order to foster student learning and comprehension (Kennedy, 1991).

Pedagogical Knowledge

Pedagogy is often defined as "the art, profession, or science of teaching" (Webster's Dictionary, 1996). This general definition thus would include the many different aspects of running a classroom, for teaching is much more than simply delivering or facilitating the delivery of the information for each lesson. Pedagogical knowledge, as explained by Carter (1990), is practical knowledge with several sub-categories, including broad knowledge of classroom situations, as well as dilemmas faced while executing purposeful action in the classroom. As with subject matter, research indicates that teachers developed their view of what teaching entails and what the role of the teacher is in facilitating learning, again from their childhood experiences in education (Kennedy, 1991).

One area of pedagogical knowledge discussed by Carter (1990) focused on developing specific pedagogical knowledge involving students in a lesson. Veteran teachers had developed pedagogical knowledge to deal more effectively with students who were resistant to working and had more effective skills in identifying and engaging such students, and were capable in knowing who "could not" and "would not" participate in the work. Another vital component of involving students in the lesson is the careful planning and preparation of the lesson to include activities, methods of instruction, assessment, etc.

Pedagogical knowledge thus appears to be a significant factor for effective teaching in that teachers must know more than simply subject content, they also must know about students and the various contexts found in the classroom (Ball & McDiarmid, 1990). However, pedagogy alone would appear insufficient to educate students fully in various subjects. Knowledge of how to manage a classroom and engage students without knowledge of content would not seem to allow for proper coverage of required content (Shulman, 1986). If this were the case then there would be no need for content preparation of teachers, which has been shown to be vital to student understanding and achievement as discussed in the previous section. How can teachers teach subjects they themselves are unfamiliar with (Kennedy, 1991)? Thus, being able to establish order in a classroom does not facilitate learning of subject matter by students (Kennedy, 1991). Likewise, while knowledge of pedagogy is needed for effective teaching, pedagogy alone will not produce desired results in student learning and achievement.

Pedagogical Content Knowledge

The pedagogical content knowledge aspect of teacher knowledge combines the previous two areas of knowing about subject matter and knowing about learning and teaching. According to Shulman (1986) pedagogical content knowledge includes knowledge of the best manner in which to represent a given subject in order for others to be able to understand it. Such knowledge is considered "specific to teaching particular subject matters" (Grossman, 1990, p. 7). Pedagogical content knowledge enables a teacher to represent accurately and effectively information to students in a manner that will allow them to understand the concepts and topics being learned (Wang & Odell, 2002). Pedagogical content knowledge can be thought of as the teachers' knowledge and skill that helps learners develop their own deep understanding of the subject (Putnam & Borko, 2000). According to Darling-Hammond and Cobb (1996), teachers need deeper understandings of their subject area, as well as how their discipline connects with others. Teachers need to be able to provide learning experiences that will allow students to construct, relate, and apply their own knowledge (Darling-Hammond & Cobb, 1996). A thorough pedagogical content knowledge includes an understanding on the teacher's part of what is easy or difficult to learn about a given topic, as well as a knowledge of the conceptions held by the learners of the topic (Shulman, 1986).

Integral to pedagogical content knowledge is the belief that different subjects require different pedagogical approaches, and that teachers must be able to combine their knowledge of the subject as well as their knowledge of effective pedagogies (Kennedy, 1991). For example, teaching history to students in a socio-economically advantaged area is different than teaching geometry in an urban setting (Kennedy, 1991). Indeed different pedagogies would be employed even within the same school between the various subjects being taught due to differences among subjects. Pedagogical content knowledge in mathematics enables a teacher to determine what aspects of certain concepts are likely to be interesting to a particular grade level, to be able to modify problems depending on the level of the students, as well as the knowledge of where students might have difficulties in solving the problem (Ball & Bass, 2000). Other evidence of pedagogical mathematical content knowledge displayed by teachers includes the ability to guide the course of mathematical discussion in the classroom in determining whose comments to include, explore, expand on, when to push students to continue, what explanations to provide, and thus ultimately helping students understand the content of the principle being studied (Bass & Ball, 2000). However, pedagogical content knowledge in teaching literature enables teachers to help students make connections between literature and their own lives (Grossman, 1990). When teaching writing, pedagogical content knowledge enables teachers to aid students to not only master grammar, but also to elicit student self-expression (Grossman, 1990).

A challenge with pedagogical content knowledge is that researchers do not know what such knowledge would look like (Ball & McDiarmid, 1990). In what manner does pedagogical content knowledge connect with subject? In what manner does pedagogical content knowledge connect to pedagogical knowledge? These questions have not been answered. It is not known what pedagogical content knowledge is exactly, how or where it is developed, or if and how it influences student performance.

Three different emphases on teacher knowledge lead to research and debate into what type of teacher knowledge is most effective and important, as well as the relationship between them. Such debate has continued throughout the past century with current positions of "it depends on both" harking back to Dewey's contention that both subject content and method are necessary, and that they must be learned in relation to each other (Ball & Bass, 2000). Maintaining this view, this study sought to investigate not only German mathematics teachers' subject knowledge, but also the methods and manners in which such knowledge would be used to represent knowledge to students. How participants understood topics and concepts, and underlying mathematical principles, as well as how they would use such knowledge to explain, generate representations, and handle scenarios that might be found in the classroom indicative of pedagogical content knowledge, and how these two areas interact furthers the discussion and research in this area of the literature.

Sources of Knowledge

Clearly evident is the difference in opinion of teacher knowledge. Another lens through which to view teacher knowledge are where the various types of knowledge originate. Three competing conceptions of teacher knowledge including knowledge-*for*-practice, knowledge-*in*-practice, and knowledge-*of*-practice according to Cochran-Smith and Lytle (1999) will be used to analyze and categorize teacher learning and the associated settings in which each of these three conceptions would most likely tend to be learned. These three conceptions describe the various areas in which a teacher's knowledge and perception lie, as well as where the source of teacher knowledge is produced. The areas examined also speak to the professionalization versus deregulation debate when analyzing where teacher learning occurs.

Knowledge-for-Practice

Knowledge-*for*-practice is based on the assumption that formal knowledge and theory that teachers use for practice are generated through university-based research. Therefore there exists a reliance on the knowledge obtained by university and other experts *for* teachers to use (Cochran-Smith & Lytle, 1999). This conception assumes that teacher knowledge originates from university experts whose outlet for disseminating such knowledge is through teacher education programs. It would be inaccurate, however, to state that teacher education programs work under the premise that knowledge-*for*-practice is the rationale for the existence of such programs, or that teacher education programs do not advocate other means of teacher learning. Rather, it is how teachers approach knowledge and where they look to in order to receive and construct knowledge. It is true that teacher education programs provide knowledge and skills through a variety of methods and approaches to enable teachers to gain an understanding of the knowledge base of teaching that has been developed through research and practice. Subject-specific methods courses either through university work or professional courses are one example of how teachers gain knowledge-*for*-practice despite the fact that not much is known about the content or effectiveness of such courses (Grossman, 1990).

If teachers absorb such information, but do not actively reflect and analyze on themselves, but on research knowledge and practice in general, it is likely they will remain in the knowledge-*for*-practice area of teacher knowledge. Without personal application and synthesis of knowledge gained from an outside source, teachers may not correctly and/or fully implement desired practices or reform (Fullan, 2001; Stigler & Hiebert, 1999). Knowledge-*for*-practice addresses teacher knowledge in both subject content knowledge, as well as pedagogical knowledge, but not necessarily a combination of the two. That is to say not all research and data available through knowledge-*for*-practice provides consumers of such data with information as to how particular information may relate to the inclusion of either subject content or the application of pedagogical knowledge. Additionally, subscribers to this source of knowledge may tire of the multitude of reforms seeming to constantly come from research and policymakers causing teachers to stop seeking additional knowledge to improve their teaching (Stigler & Hiebert, 1999). What sets knowledge-*for*-practice apart from other types of knowledge is that teacher knowledge comes from the university and other researchers, and that teachers strictly rely on such outside sources of information to inform them as to what leads to effective teaching. Teachers and classroom experiences are not seen as sources for teacher knowledge to develop. With teacher education programs considered a source for producing and disseminating teacher knowledge, proponents for professionalization would likely agree with the knowledge-for-practice model as a source for teacher knowledge.

Knowledge-in-Practice

Crucial teacher knowledge according to knowledge-*in*-practice comes from the reflection on practice by teachers. Teachers examine the knowledge *in* the products and works of effective teachers in order to improve their own practice, and as such are sometimes referred to as practical knowledge (Cochran-Smith & Lytle, 1999). Those favoring deregulation and alternate routes to the classroom would have similar notions of teacher knowledge as proponents of knowledge-*in*-practice. Grossman (1990) found that teachers acquired pedagogical content knowledge through their experiences as classroom teachers supporting the idea of knowledge-*in*-practice. Further findings, however, indicated that those teachers without teacher education geared their lessons towards high achievers, the type of student they remembered themselves to be, with subject matter knowledge being the most relied on knowledge, with much less thought given to pedagogy or pedagogical content knowledge (Grossman, 1990). Thus, classroom experiences alone were not sufficient to enable those teachers to effectively reach students at all levels. Knowledge-*in*-practice essentially focuses only on experiences teachers have in the classroom with accompanying reflections to build teacher knowledge of effective teaching. Findings of research, as well as knowledge from outside sources such as universities are not sought after.

Knowledge-of-Practice

The last conception, knowledge-*of*-practice does not delineate between formal and practical knowledge as separate entities of teacher knowledge exclusive to the other. Rather, knowledge-*of*-practice synthesizes aspects of both knowledge-*for*-teaching and knowledge-*in*-practice. This area contends that teacher knowledge must include a balance of equal consideration on the part of teachers between what is learned from their own experiences as a classroom teacher using such experiences as opportunities for investigation and the research and knowledge that is produced by others (Cochran-Smith & Lytle, 1999). Thus, teachers help produce knowledge of effective practice by participating in inquiry to connect their work to the larger issues of education (Cochran-Smith & Lytle, 1999). Grossman (1990) contended that this type of knowledge can also be developed at the pre-service level if novel teaching strategies combined with research are used to dispel prior beliefs about what teaching is based on previous experiences either as a student and/or in field experiences. Knowledge-*of*-practice is the type of teacher knowledge that those in favor of professionalization of education would advocate. Without proper instruction and encouragement as to how to best engage in the process of reflection and creation of knowledge it is likely many teachers will remain in either knowledge-*for* or knowledge-*in*-practice suggesting the need for reform-minded teacher education programs. This would lead to the conclusion that along with other evidence supporters of professionalization would be proponents for teacher education programs. Unlike either knowledge-*for*- or knowledge-*in*-practice, in knowledge-*of*-practice teacher knowledge comes from both researchers and university sources as well as from classroom experiences. Therefore, individual classroom experiences along with research-based findings combine to form teacher knowledge.

An examination into contributions of various sources to teacher knowledge according to teachers is vital in developing further an understanding of and the importance of each source in order to increase teacher knowledge (Smylie, 1989; Cochran-Smith & Lytle, 1999). Particularly given the difference in depth of teacher preparation between countries, further investigation into how teachers view various sources of knowledge contributing to their own teacher knowledge may be beneficial in analyzing and improving teacher education in the United States.

Justification for Research Using Mathematics

A varied view of teachers' knowledge and learning provides ample fodder for debate and conversation within the United States as to what is most effective and what should be advocated and promoted in order to improve education. A broad curriculum necessitates that a narrower examination be made in order to scrutinize various issues carefully. There simply is too much to study in one attempt. Narrowing research to just one subject does necessarily limit what is examined; however focusing on one subject provides a more manageable area within which to examine the issues of teacher education.

Mathematics is one subject that is often used in research for several reasons. Mastery of mathematics is seen as important, mathematics content does not change from one country to the next, and it is often a subject that is difficult even for teachers to understand or explain (e.g. Kennedy, 1991; Ball, 1990; Husen, 1967). These factors allow for research to be conducted in a variety of settings and contexts while still maintaining the ability to compare findings in search for components that could lead to better understanding of what is necessary for effective mathematics learning, teaching, and understanding.

Mathematics was selected for study by the International Study of Achievement in Mathematics (now also referred to as the First International Mathematics Study or FIMS) conducted by the International Project for the Evaluation of Educational Achievement (IEA) and has continued to be the focus of studies both within and between countries. Husen (1967) explained that mathematics was of major concern to those countries participating in FIMS; additionally, mathematics is with few, insignificant exceptions a universal subject not as affected by language and semantics as other content areas might be. It is evident that mathematics continues to be of primary concern and research as it is the focus of not only the major comparative studies (Third International Mathematics and Science Study (TIMSS), 1995; Program for International Student Achievement (PISA), 2003), but also of various studies of smaller scopes.

Leading researchers in the field of mathematical education area maintain that not enough is known or understood about the role of mathematical knowledge needed for teaching (Ball & Bass, 2000). Crespo and Nicol (2006) contend that more research must be done to examine teachers' disposition and mathematical ideas and how they work in conjunction with the teaching of mathematics of pre-service teachers. It can be argued that such research should also extend to inservice teachers. They contend that mathematics teaching incorporates content, students, and pedagogy leading to a degree of variability that teachers must be prepared for; thus, it is imperative to not only understand what teachers need to know, but also how they must be able to use such knowledge (Ball & Bass, 2000).

Comparative Mathematics Performance of Students in the United States

International comparison tests have been conducted for nearly four decades beginning with FIMS, which compared twelve countries (IEA Website, 2008; Husen, 1967). It included four different populations: 13 year-olds (1a), the grade most typical for 13 year-olds (1b), terminal year students on a mathematical track (3a), and terminal year students on a non-mathematical track (3b). Due to differences in years of schooling between countries, terminal year students were students in their final year of schooling in their respective countries, regardless of actual grade. A comparison of scores show that the United States scored not only below the total mean, but ranked second lowest for both population 1a and 1b, and lowest for populations 3a and 3b. The difference between the mean score for the United States and that of the highest scoring country was more than one standard deviation for all with the exception of population 1b where it was slightly lower than one standard deviation. Essentially, in most cases the highest scoring country had a mean nearly double that of the United States. In FIMS, the participating Asian country of Japan scored the highest for only one of the populations; however, their score was consistently near the top. Whereas the mean score for the United States was typically nearly one standard deviation below the total, the mean score for Japan was typically nearly one standard deviation above the total mean score. Appendix B illustrates the scores for the United States, the highest and lowest achieving country for each population, Japan, as well as the total score for all countries, along with standard deviations.

An increased number of countries participated in the Second International Mathematics Study (SIMS), with nineteen countries participating, including ten of the original twelve countries from FIMS (IEA Website, 2008). Once again two populations were tested including 13 year-olds and final year students, as with above these students were in their last year of schooling regardless of actual grade level taking at least five hours of mathematics per week (Brown, 1996). Results of SIMS revealed that Japan with a score of 60%, and Hungary and The Netherlands with scores of 56% ranked the highest (Brown, 1996). A group of twelve countries followed these three with France and Belgium at the top of the group with 52% and the United States and Israel at the bottom of the group with 45% (Brown, 1996). Just five countries, including three developing countries, scored lower than the United States and Israel (Brown, 1996). In comparison with FIMS, Japan improved to become the sole high-performer. The United States remained at the bottom. Israel, which had challenged many of the top spots in FIMS, fell to the same level of the United States. For final year students, Hong Kong and

Japan ranked highest in senior year advanced algebra, while the United States scored second lowest outscoring only Thailand (Stedman, 1997a).

The 1995 Third International Mathematics and Science Study (TIMSS) released through the National Center Educational Statistics (NCES, 1999) found that in keeping with previous international comparisons, the United States' performance was just average. With an ever-increasing number of countries participating, the top spots of TIMSS went to not one, but four Asian countries: Singapore, Korea, Japan, and Hong Kong (IEA Website, 2008). See Appendix C. A repeat study of the 1995 TIMSS conducted in 1999 to test students who were in fourth grade for the initial 1995 test, and who were then in eight grade for the 1999 repeat also indicated that the United States scored in the average range in mathematics for eighth graders, with the top scores again belonging to Asian countries, including Singapore, Korea, Chinese Taipei, Hong Kong (SAR), and Japan (IEA Website, 2008).

The most recent Program for International Student Assessment (PISA) study to focus on mathematics in 2003 (PISA tests are cyclical in nature and alternate focus from one cycle to the next, PISA 2006 focused on science) found that the United States scored below the Organization for Economic Cooperation and Development (OECD) average in the combined mathematics literacy, as well as in each of subscales measured within the mathematics test (NCES Website, 2006). Departing from most other recent comparative studies, the top performing country was not an Asian country, but the four top-performing countries still included two Asian countries. The top countries in order were Finland, Korea, The Netherlands, and Japan. Appendix D depicts the rankings of select countries of the PISA test. Included are select top performing countries, Germany and the United States. As the chart indicates, as far as math is concerned the United States and Germany are below average, and far behind the leading countries.

Scores for the United States on the Trends in International Mathematics and Science Study (TIMSS), formerly known as the Third International Mathematics and Science Study, on the most recent study conducted in 2003 showed the United States was above average in international rankings for both fourth and eighth grade students. See Appendix E and F. On the surface this may seem in contradiction to the findings of the PISA 2003 study. Further evaluation, however, indicates that when comparing the United States only with other Organization for Economic Cooperation and Development (OECD) countries that participated at the two levels the United States is average at the fourth grade level of the ten OECD countries participating, the United States outperformed five OECD countries, but was outperformed by the remaining five OECD countries. Once again, the four highest scoring countries were Asian countries including: Singapore, Hong Kong, Japan, and Chinese Taipei. At the eight-grade level, of the twelve OECD countries participating, the United States outperformed two OECD countries, but was outperformed by five. In keeping with performance at the fourth grade level, Singapore, Korea, Hong Kong, and Chinese Taipei, and Japan ranked as the highest countries at the eighth grade level (Korea did not participate at the fourth grade level). The average score of the remaining OECD countries did not significantly vary from that of the United States (NCES Website, 2006). TIMSS 2003 did indicate that the United States was making progress. Despite fourth grade scores remaining unchanged from 1995 to 2003, eight grade scores increases significantly.

The focus of the PISA 2006 test was on science; however, limited mathematical data was also collected. From the available mathematical data from PISA 2006, the United States continued to score behind not only the top performing countries, but also below the OECD average (OECD Website, 2008). The top performing countries included once again several Asian countries, including Chinese-Taipei, Hong Kong- China, Japan, and Korea, but this group also included countries such as Finland, and Switzerland. Germany's scores were above the OECD average, and somewhat better than that of the United States though still not near the top performing countries (OECD, Website, 2008). See Appendix G.

It is clear that throughout the history of international comparisons Asian countries have far outranked the United States. Typically Asian countries are at, near, or compose multiple spots at the top of the rankings. The United States meanwhile is average at best. The conclusion drawn from such comparisons is that Asian students achieve at a much higher level than their counterparts in the United States. Different hypotheses and investigations have examined what the cause of such a great disparity could be (NCRTE, 1993; Perry, 2000; Stevenson & Stigler, 1992). In comparing international test scores, one could hypothesize that it is this mathematical understanding on the part of the teacher that leads to increased student achievement.

Comparative Studies of Teacher's Knowledge and Practice

International comparison tests have opened the door to an examination of various factors that could be *the* key component to increasing student achievement. Teacher knowledge and practice, among other possible components, are assumed to be two major yet related components in student achievement and have been the focus of research studies following international comparisons. While an examination into the many different components affecting student achievement would be a worthwhile and interesting endeavor, based on the purpose and aims of this study, teacher knowledge and teacher practice are the two components highlighted and discussed.

Teacher Knowledge

One identified difference between teachers in the United States and Asia is teacher knowledge. This discrepancy is used to explain differences in student performance between the United States and Asian countries, specifically China and Japan (e.g. Ma, 1999; Stigler & Hiebert, 1999). Comparative studies have revealed that whereas teachers in the United States are most concerned with being patient and sensitive towards students, Chinese teachers are most concerned with being enthusiastic and being able to give clear explanations (Stevenson & Stigler, 1992). This finding leads directly into further study of the role of teachers' mathematics content and pedagogical knowledge in the current reform of mathematics education in the United States.

Arguably one of the first large-scale studies conducted in the United States examining teacher knowledge was the TELT study designed to explore the relationship between content and format of teacher education, and what teachers learn about teaching. It included traditional undergraduate programs, alternative routes, induction, and in-service teachers. The report on their findings was released in March 1993 through the National Center for Research on Teacher Education, (NCRTE, 1993). The results of this study served to dispel six common myths about teaching, two of which are: Myth #1: Majoring in a subject area fulfills requirement of subject content knowledge necessary for teaching; Myth #4: Good teachers can be produced by starting with people possessing subject specific degrees, and then give them classroom survival skills (NCRTE, 1993.) The TELT study summarized that teacher's understandings of subject matter, curriculum, learners, learning, and context are all interdependent and mutually supportive (NCRTE, 1993). If extensive subject training does not lead to knowledge necessary for teaching, why is there an increasing demand for greater content preparation of teachers? Does similarly extensive content training in other countries also result in an inadequate subject content knowledge for teaching? According to the TELT study, such in-depth content training would not in fact lead to more effective teaching.

In order to gauge mathematical understanding in pre-service elementary and secondary teachers, Ball (1990) reported on surveys and interviews of 252 prospective teachers drawn from the TELT study for further analysis. Ball (1990) reported that this study assumed that "the goal of mathematics teaching is for students to develop mathematical understanding" which "implies that pupils should acquire knowledge of mathematical concepts and procedures." Ball continued by saying that to understand mathematics also meant "learning about mathematical ways of knowing as well as about mathematical substance" (p. 457). It goes without saying that if this is what is expected of students, teachers too must possess these same types of knowledge, skills, and the ability to facilitate the development of such in others. Ball (1990) contended that mathematical teachers needed to have "substantive knowledge of mathematics" (p.458), or what has previously been referred to as subject matter knowledge. She then went on to specify various types of knowledge teachers need, such as: knowledge of concepts and procedures, understanding of the underlying principles and meanings, and appreciations and understanding of how mathematical ideas are connected. Much of the findings served to further general conceptions concerning mathematical understanding required for teaching in that those who majored in mathematics seemed confident in both their mathematical and teaching skills, some who had not majored in mathematics were confident in their mathematical ability but felt they now need to learn how to teach it, and finally those who felt they could do mathematics but may not have the correct understanding of mathematics to teach it to others. Various items measured such as fractions and division showed the prospective teachers possessed a limited understanding. Findings indicated that the prospective teachers in this study were deficient in understanding of concepts and principles despite being able to complete the calculations presented (Ball, 1990). A somewhat interesting, yet disturbing, finding is that only half of the prospective elementary teachers said they "enjoyed and were good at mathematics" (Ball, 1990, p.461). Overall results from Beswick (2006) on a survey administered to over 200 education students revealed that as with the participants in Ball's (1990) study, prospective mathematics teachers did not have an overall positive view of mathematics education and were not comfortable with mathematics themselves. Were exhaustive content training required, would prospective mathematics teachers feel more comfortable with, and not only feel more confident, but actually become more competent in mathematics? Going one step further from Ball's (1990) findings, are teachers who majored in mathematics able to provide clear, meaningful, and accurate representations of various mathematical problems grounded in basic mathematical concepts? Does confidence in computing and explaining mathematics increase with time spent in the classroom? These questions are left unanswered by these studies, and merit further investigation.

A common assumption among prospective teachers in this study, as well as in teacher education, is that if you can do mathematics you should be able to teach mathematics, thus, subject content training is not conducted in colleges or departments of education, but rather in the liberal arts courses of the college or university (Ball, 1990). Another common assumption is that what is taught in schools is largely common knowledge that most adults should be able to do; therefore mathematical understanding for teaching reverts back to what was learned by prospective teachers in their experiences in K-12 schools (Ball, 1990). This led to students expecting to teach just as they were taught with similar views of mathematics and beliefs about teaching methods (Cooney, 2001). Given that students in Ball's (1990) study often did not enjoy mathematics, nor felt they were good at mathematics, why is it that such prospective teachers expect to teach in the same manner they were taught when it is apparent that such teaching did not aid them in their understanding of mathematics?

Not surprisingly, Ball (1990) also found that pre-college mathematics classes did not provide the mathematical knowledge that would be required to teach. Others assume that proper knowledge for teaching mathematics is the result of college coursework and advocates with such assumptions lead calls for reform to center around prospective teachers majoring in their specific subject area (in keeping with those who favor deregulating the profession) despite the fact that studies have found such courses to be inadequate in preparing mathematical teachers (Ball, 1990). Cooney (2001) encountered similar findings, concluding that preservice teachers were not exposed to the type of mathematics at the collegiate level that would be needed to teach secondary school other than some skills and concepts used in calculus. Overall, Ball (1990) concluded that subject content preparation is not focused on enough in teacher education as most assume such training will occur elsewhere, and that further attention to this topic dictates that more must be learned about how teachers translate their mathematical understanding in order to teach mathematics effectively. Based on this research, there is a major problem in teaching mathematics, and a case for why further research into mathematics teachers' pedagogical content knowledge is vital: being able to do mathematics and being able to teach another to do mathematics are two separate things. How and where is such ability developed? What is the role of extensive subject content knowledge? Where are teachers exposed to the type of mathematics needed for teaching- is it in their prior schooling before entering the university, or through years in the classroom? Do teachers who are trained in other countries encounter similar shortcomings in their college coursework adequately preparing them insofar as content is concerned to be able to effectively teach?

As part of her dissertation work, Liping Ma (1999) used the TELT study measurement to conduct a similar study to assess both Chinese and United States teacher's mathematical teaching knowledge. The conclusions reached by Ma (1999) were that teachers in the United States were procedurally focused, in that they focused on steps and processes rather than on the larger mathematical principle at hand, while Chinese teachers exhibited algorithmic competence and conceptual understanding; Chinese teachers' knowledge was comprehensive while American teachers' knowledge was fragmented (which coincides with curriculum in the United States); Chinese teachers possessed "knowledge packages" that were interconnected from topic to topic; and that teachers with what Ma termed a "profound understanding of fundamental mathematics" or PUFM demonstrated connectedness, promoted multiple approaches, revisited and reinforced basic ideas, and had longitudinal coherence. How and where is deeper and thorough mathematical knowledge, perhaps even PUFM obtained? Does university training, classroom experience, or prior K-12 education most likely to lead to such teacher knowledge? Do teachers in other countries who have an extensive background in mathematics possess exhibit similar types of mathematical knowledge? These questions will constitute the major issue of explanation in this study. Germany provides a useful context to explore this issue because teacher preparation programs in Germany are more extensive, and arguably intensive, than those found in the United States; however, what is not known is whether German mathematics teacher have deeper content knowledge than has been found in the United States (Ma, 1999), or whether teacher preparation programs are in fact the source of such knowledge.

In similar fashion to Ma, research by Zhou, Peverly, and Xin (2006) also investigated differences in teacher knowledge between China and the United States; however they focused only on the area of fractions with analysis into subject content knowledge, pedagogical content knowledge, and general content knowledge. The first area of subject content knowledge involved basic, computation, and word problems (Zhou, Peverly, & Xin, 2006). Results of this study supported Ma's (1999) findings in that all types of problems were difficult for teacher from the United States, while none of the areas proved difficult for Chinese teachers (Zhou, Peverly, & Xin, 2006). To assess pedagogical content knowledge, participants were questioned as to what the important concepts for the topic were; overall findings in terms of pedagogical content knowledge indicated that Chinese teachers regardless of experience had a deeper knowledge than teachers in the United States, with the biggest difference seen in teachers with 11-20 years teaching experience (Zhou, Peverly, & Xin, 2006). Teachers in the United States outscored Chinese teachers in every area of general pedagogical content knowledge; however, the authors urge caution in interpreting this data due to internal inconsistencies (Zhou, Peverly, & Xin, 2006). Based on this study, it would seem that Chinese teachers do, in fact, have a deeper subject content and pedagogical content knowledge than teachers in the United States, as has been reported in other studies.

Teacher Practice

Another area focused on by comparative studies was the differences exhibited in teacher practice. As another component of effective teaching, the practices employed by teachers have been examined in various countries, often with the comparison including at least one Asian country along with the United States.

Video-analysis of mathematic lessons collected as part of TIMSS served as the starting point for work done by Stigler and Hiebert (1999). Their work included not only an analysis and report of the video-taped mathematical lessons in Germany, Japan and the United States, but extended further to include commentary on teaching and reform from a comparative view contending that it is "teaching not teachers that is the 'critical' factor" (Stigler & Hiebert, 1999, p.10). In comparison with the other two countries, mathematic teachings in the United States were procedurally based and narrow (Stigler & Hiebert, 1999) with differences far greater between countries rather than within. Stigler and Hiebert (1999) developed mottoes to summarize the mathematical learning that took place in each of the countries; for Germany the motto was "developing advanced procedures"; for Japan "structured problem solving"; and for the United States "learning terms and practicing procedures" (p.27). What are the implications of such mottoes in regards to teacher knowledge in the areas of subject matter knowledge, pedagogy, and pedagogical content knowledge? When categorizing manners in which teachers facilitated student learning, Stigler and Hiebert (1999) found that the percent of topics developed (process of teachers helping students develop a definition and understanding) versus stated (simply providing a definition) varied a great deal with a majority of topics in Germany and Japan developed (76.9% and 83% respectively) compared to a majority of topics in the United States being stated (78.1%). While some of this may be attributed to curriculum or content, could part of this discrepancy be in teacher pedagogical content knowledge and their ease and confidence in helping students develop their own mathematical understanding?

While the previous statistic may not relate entirely to teacher knowledge, lessons were also rated for quality of mathematical content (Stigler & Hiebert, 1999), which found that lessons in the United States were predominately low in mathematical content quality (89% low, 11% medium, 0% high), those in Germany were fairly balanced in terms of mathematical quality (34% low, 38% medium, 28% high), while most of those in Japan were of medium or high quality (11% low, 51% medium, 39% high). This fact could speak more directly to teacher knowledge, as quality of mathematical content could be high despite the level of curriculum used in varying countries. It could be argued that increased mathematical pedagogical content knowledge affects quality of mathematical instruction. When asked what key point teachers wanted students to learn, 61% of teachers in the United States talked of skills, while 73% of teachers in Japan wanted students to consider things in novel manners, similar data for key points German teachers was not provided (Stigler & Hiebert, 1999).

Stigler and Hiebert (1999) described various aspects that differ from one country to the next in terms of teacher practice such as how topics are developed and quality of mathematical content in a lesson. Are these differences attributed to level of the teachers' own mathematical understanding, or are these differences due to another factor? Based on findings from Stigler and Hiebert (1999) it would seem that German mathematics teachers had a deeper mathematical understanding than their counterparts in the United States; however, this is not reflected in international comparisons (e.g. PISA 2003, TIMSS 2003, PISA 2006). Further research is needed into German mathematics teachers' knowledge to determine not only their mathematical knowledge, but also their ability to create accurate representations. Beyond classroom observations and the types of explanations provided, this study seeks to uncover the depth of German mathematics teachers' knowledge first in the area of subject content knowledge, which would be the basis for ability to provide mathematically sound explanations, and also an examination into the ability of these same teachers in generating accurate representations and discussing underlying mathematical principles. Both research areas provide the basis for a teacher's ability to provide mathematically sound explanations in the classroom, as was researched by Stigler and Hiebert (1999).

Following the assumption that good instruction should affect positively student achievement, along with findings that most new mathematical content is delivered through teacher explanation, Perry (2000) carefully examined and analyzed the quantity and quality of mathematical explanations by teachers in Japan, China, and the United States. Observations included ten schools in Japan and China, along with twenty schools in the United States (to account for diversity) at both the first and fifth grade levels with two teachers per school participating. Each teacher was observed four times towards the end of the school year to ensure a high level of familiarity between teachers and students (Perry, 2000). Observers in the classes maintained a written record containing the remarks of the classroom (Perry, 2000). These field notes were read and summarized before being coded for various aspects of a lesson. Explanations that emerged from the coding were then categorized by type of explanation. Summaries of the classroom segments were also compiled (Perry, 2000). Overall findings indicated that in comparison with their Japanese and Chinese counterparts, students in the United States received shorter and fewer explanations that were not as useful because they were not generalizable across a variety of problems (Perry, 2000). Although duration of explanations was not always significant, total time differences when considering the frequency of explanations led to a considerable difference in total amount of time spent explaining mathematical concepts. Perry (2000) went on to state that perhaps the reason for better explanations by Chinese and Japanese teachers was their deeper understanding of the mathematical concepts being taught. Speaking directly to the fourth area discussed by Romberg's (1999) discussion of NCTM standards student use of mathematics must go beyond mere memorization, and in keeping with the fifth point raised by Romberg (1999), Perry's (2000) overall conclusions admonished teachers in the United States to involve students in mathematical discussions to improve the mathematical explanations in the classroom. An assumption of this study was that better teacher explanations result from a better understanding of the subject matter. What is missing are data from teachers who may also contain deep understanding of mathematical subject matter, but who may not have the types of explanations Perry (2000) discovered in the study of Chinese and Japanese classrooms. Also, no explanation is given as to how these teachers developed their ability to provide more significant explanations, or their rationale for doing so. Along similar lines, further investigation is necessary to determine if increased subject matter knowledge or improved pedagogical content knowledge enables teachers to provide sustained and meaningful explanations to help student understanding and achievement. Thus, examinations of German teachers can help answer these questions by investigating if German mathematics teachers not only possess a similarly deep understanding as was found by Perry (2000) in China and Japan, but if and how they are then able to use such knowledge to provide accurate representations and explanations of basic, yet fundamentally important mathematical principles.

Spurred in part by calls for reform by the National Council of Teachers of Mathematics (NCTM), another study aimed at improving student achievement focused on professional development of inservice teachers' methods and understanding of mathematical concepts and procedures (Mistretta, 2005). A major focus of this study centered on the extent teachers used pedagogical practices that promoted conceptual understanding of mathematics (Mistretta, 2005). The 86 participating teachers completed surveys concerning pedagogical practices and were observed on three different occasions both before and after participation in the professional development. Teachers participated in bimonthly-modeled lessons that included focus sessions after these lessons and 2-hour hands-on workshops also held bimonthly. Modeled lessons, focus sessions, and workshops were intended to expose teachers to the types of learning environments that included engaging students in constructivist learning to achieve the goal of the study to improve teachers' mathematical understanding and approaches to mathematics (Mistretta, 2005). Teachers initially admitted their limited knowledge in using manipulatives, stated that they did not have time for creativity, and tended to use explanatory approaches with an emphasis on procedures. Initial observations also revealed that teachers in grades 6 to 8 were spending time on algorithmic procedures that should have been mastered in previous grades, but in areas students were obviously still lacking mastery rather than extending student understanding and creating realistic applications (Mistretta, 2005). Following the professional development, teachers indicated greater use of pedagogical approaches intended to promote conceptual understanding of mathematics (Mistretta, 2005). Results indicated that focused, sustained professional development for teachers does have a positive impact on the understanding of conceptual based mathematics, while stating the positive impact of the professional development provided these teachers the conclusion points to the need for even more frequent exposure to increase the positive effects found in this study (Mistretta, 2005). Although this study worked to incorporate both teacher understanding of conceptual mathematics and how to incorporate this into the classroom through pedagogical changes it is unclear exactly what mathematical principles were targeted, and if such principles were chosen due to the inclusion in the curriculum, or because they were deemed to be a foundational principle key to the understanding of both basic and advanced mathematics. Studies such as this indicate the dire need for improvement (as well as ability to foster change) in mathematics teachers' knowledge both in content and pedagogy. In addition to working to help inservice teachers through better preparation, further research into the mathematics content and pedagogical content knowledge necessary to teach effectively is crucial in helping not only inservice, but also preservice teachers. Would better pedagogical content preparation of prospective teachers curtail the need for professional development aimed at increasing teacher use of approaches designed to increase student achievement? Are teachers in need of such training due to lack of content knowledge, pedagogical knowledge, pedagogical content knowledge or a combination thereof? Do teachers in other countries face similar dilemmas in needing to have more of a support group approach to improving teaching and learning? Not enough is known about the preparation of teachers in Germany in comparison with the preparation of teachers in the United States, or the effects of their preparation on their knowledge and the representations and approaches used by German teachers in the classroom. Are German mathematics teacher better prepared in terms of pedagogical content? Are German inservice teachers lacking in terms of content or pedagogical content knowledge, or a combination? These questions are addressed in this study through an examination of German mathematics teachers' subject content and pedagogical content knowledge. These issues are important not only for broader understanding of the effects of teacher preparation, but also to help answer questions left unresolved by studies such as this by Mistretta (2005).

It seems clearly evident teachers in Asian countries like China and Japan seem to have a more in-depth and complete knowledge of subject content (e.g. Ball, 1990; Stevenson & Stigler, 1992; Ma, 1999). Other studies have revealed differences in teacher practices also presumed to possibly affect student learning (e.g. Stigler & Hiebert, 1999; Perry, 2000; Mistretta, 2005). A major weakness in the above Asia-United States comparisons is that most of the studies exclude the rest of the world. Other countries share certain aspects with the United States and the various top-performing Asian countries, and yet these countries have not received in-depth analysis either to bolster or to challenge the conclusions drawn from these studies. Were an outside country brought in, would similar conclusions stand or would they be challenged resulting in more careful analysis of what might be best to improve education in the United States?

Possible Problematic Situation

It seems somewhat indicative then that differences in teacher knowledge and teacher practice could likely account for differences in student achievement as evidenced on international comparisons such as TIMSS 2003 and PISA 2003. The comparisons examined between the United States and Asian countries seem to provide reasonable and plausible explanations for the differences in student achievement. For every factor examined significant differences were found between the United States and Asian countries. Were comparisons limited to include only the United States and countries in Asia, the explanations given seem to provide evidence of changes that the United States should take into consideration in order to improve its educational system. However, if other countries are brought into the comparisons some of the conclusions and hypotheses of previous research could become problematic. Based on the review of the literature presented, the research and literature in the area of mathematics teachers' knowledge is still incomplete with many unanswered questions. This is in keeping with findings from Wang and Lin (2005) whose review of existing literature also found that current studies do not adequately explain the disparities in student achievement between the United States and China, including stating that there is not enough evidence of a positive relationship between teachers' mathematical knowledge and student achievement.

A context for studying this problem could be created through the inclusion of another country such as Germany in an examination of teacher knowledge, as indicated previously. Similar to the United States, with each international comparison, German scores continue to lag behind the best of the countries (TIMSS 1995, PISA 2003, PISA 2006.) For example, in TIMSS 1995 Germany's mean scores were within ten points of the United States- both average, and both well behind those of the leading countries. In PISA 2000, Germany scored below both the United States and the OECD average. This has caused serious concern among the German general population as well as the educational policymakers. Also similar to the United States are efforts in Germany to improve education achievement. Reform movements in Germany have begun with the goal of being back on top in ten years' time ("Seeing to it", 2003). Evidence of such reforms may perhaps be partially evident in Germany's performance in PISA 2006 with German students scoring higher than not only the United States, but also above the OECD average. Despite the increase in German scores from PISA 2003 to PISA 2006, they continue to lag behind the top performing countries. Although students in both the United States and Germany fare relatively similar on international tests (TIMSS, 1999; PISA 2003), the road to becoming mathematics teachers of these students varies significantly. This difference may not be seen as important were it not for the efforts of other researchers whose work concluded that mathematics teachers' understanding of fundamental principles in mathematics may be a significant factor in students' learning (e.g. Ma, 1999). In comparison with counterparts in Germany, mathematics teachers in the United States typically receive less training in mathematics ("Studienpläne", 2004).)

A typical mathematics teaching degree (as well as other subject areas) in Germany is composed of two phases- the first academic phase consists of seven to nine semesters of academic work in the subject area followed by the second practical phase (separate from the university) which consists of an additional 18-24 months to become qualified as a teacher (Viebahn, 2003.) Kolstad, Coker, and Kolstad (1996) focusing on the state of Rheinland-Pfalz but maintaining remarkable similarity to all states provided a succinct summary of the preparation of German teachers beginning with elementary school leading into a description of their university education. According to Kolstad, Coker, and Kolstad (1996), students in the elementary education program took six semesters of courses that included 28 hours of education courses both subject-specific, content, theory, and psychology, and also completed student teacher practical type training during academic holiday times. In addition students were required to take 84 hours of two main subject areas with other non-credit courses in health education, speech therapy, etc. Following this coursework students then entered field experiences or on-site learning that lasted anywhere from one to three years. It was the assertion of Kolstad, Coker, and Kolstad (1996) that due to the quality of previous education German students did not have the need to take such courses as algebra, history, or composition upon entering the university because they were already adequately prepared and were able instead to enter directly into their chosen professional field for specific training. Jones (2000) reporting on the process of becoming a secondary teacher in Germany confirmed the length and depth of teacher preparation. Typically students spend five to seven years to complete a combination of three to four years of academic work at a university/teacher's college followed by 18-24 months of a practical teaching experience (Jones, 2000). To qualify to teach at the higher secondary levels (Realschule and Gymnasium) further content specialization is required resulting in four to five years of course content for Realschule and five to six years for Gymnasium with teacher candidates focusing on two or three subjects respectively (Jones, 2000).

In comparison, most education candidates in the United States complete a four-year program that consists of three different components comprising a total of 135 credit hours for secondary education majors, and 125 total credit hours for elementary majors (Darling-Hammond & Cobb, 1996). The first area of this training, which also comprises the bulk of the requirements for graduation are courses in the liberal arts that for secondary teachers focus on one area in which the prospective teacher will become licensed (Darling-Hammond & Cobb, 1996). The second area of teacher preparation in the United States is in the form of about 26 credit hours of education classes for secondary majors and 50 for elementary majors that comprise the remainder of the required coursework (Darling-Hammond & Cobb, 1996). Field experiences, including student teaching at the end of required coursework, make up the final area of teacher preparation (Darling-Hammond & Cobb, 1996).

Training and preparation as required in Germany can be viewed as not only more extensive than in the United States, but perhaps more rigid due to the fact that there are no general education course requirements to fulfill, such as physical education, basic language courses, basic history courses, etc. (Kolstad, Coker & Kolstad 1996). Under this model, all students should theoretically be better prepared for advanced study in areas such as mathematics since they have had extensive training before entering the university. Although only a small percentage of the German population is able to attend universities, it can also be argued that though not as pre-determined in the United States, the segment of the American population attending higher education is also in the minority.

Compared to typical mathematics education requirements in the United States, German requirements are substantially more comprehensive, leading to important research questions left unanswered by current research. These questions include: What is the depth of German teacher knowledge, in terms of both subject content and pedagogical content, at the conclusion of such a lengthy and intensive teacher program? Is increased training in both content and pedagogy manifest in teacher ability to compute and solve basic mathematics in addition to the ability to generate accurate representations for others? Given the difference in approach to teacher training, why do students in both Germany and the United States score similarly? Perhaps American teachers need more subject matter preparation. However, this is currently still an unresolved issue. How can teachers effectively teach students when they themselves to do not have a firm command of their subject matter? Past research conducted by the TELT study found that teachers in the United States do not have a deep understanding of fundamental mathematics, even when teachers have majored in the subject they teach they are unable to explain basic concepts (Kennedy, Ball, & McDiarmid, 1993). It is evident that the points of reform advocated by the NCTM, such as learning how to do more with mathematics knowledge other than manipulating arithmetic routines and students learning a greater variety of mathematics at more complex levels, aimed at increasing student achievement required certain levels of teacher content and pedagogical knowledge to be met (Romberg, 1999). Without proper teacher knowledge in both subject content and pedagogical knowledge, including pedagogical content knowledge, meeting such reforms would be difficult if not impossible. What teacher knowledge is, where it is developed, and how it impacts student achievement is a topic of great importance to the knowledge and current practice in education found in the United States. Currently there is no serious focus on intensive subject matter knowledge in teacher candidate requirements, where effective knowledge comes from, and to what extent various types of teacher knowledge has on student achievement. In an effort to improve teacher education and positively impact the education system in the United States, it is vital to understand variables and components that could determine the success of the future of this system. Additional research is required to ascertain what type(s) of knowledge on the part of the teacher are likely to increase student achievement, and where such knowledge is developed before effective reforms aimed at meeting new standards can or should be implemented.

The fact that drastic difference in the preparation of teachers in Germany and the United States is not evident through student achievement should alert the educational community and cause further examination before advocating and requiring substantial amounts of content training be added to teacher preparation courses in this country. At a time when educators, policymakers and the general public alike want education in the United States improved, it is important to evaluate and plan carefully the reforms that must be undertaken. Increasing teacher's subject content knowledge seems a logical area that could improve student achievement.

Previous research (NCRTE, 1993; Ma, 1999) seems to indicate that teacher knowledge is at least one of the causes American students do not perform as well in mathematics. However, when comparing the content training to other countries including Germany who does require a great deal of content preparation, it must be considered that perhaps this is not the most important area of concern. Do German mathematics teachers possess a similarly deep understanding of mathematics as examined by Stigler and Hiebert (1999) to the extent that they display thorough content understanding and perhaps PUFM as defined and discovered by Ma (1999) with Chinese teachers? Are German mathematics teachers with a greater pedagogical content knowledge able to approach mathematics in such a way as to provide a higher quality of mathematical teaching enabling students to develop a greater amount of understanding despite differences in educational systems and curriculum as discussed by Stigler and Hiebert (1999)? There simply is not enough known about German mathematics teachers' knowledge. Especially because Germany provides a potentially problematic challenge to currently held beliefs and assumptions about necessary teacher knowledge, research investigating German mathematics teachers' content and pedagogical content knowledge is necessary to help answer some of the questions surrounding teacher knowledge. Wang and Lin (2005) contend that future studies should focus on specific areas of mathematical ability, which concern would be addressed in a study such as this aimed at exploring the pedagogical content knowledge of German mathematics teachers focusing on specific aspects of mathematical concepts and abilities. Such research is not only important in developing better understanding of the relationship between teacher knowledge and student performance, but has policy implications. Depending on the depth of German mathematics teachers' knowledge, and sources attributed to such knowledge, findings would contribute to the professionalization/deregulation debate. Should German mathematics teachers have deep and thorough subject content knowledge and be able to generate accurate representations, and if such knowledge were attributed to teacher preparation programs, these data would support the need for professionalization. However, if German mathematics teachers do not display a deep and thorough knowledge in either subject content or in pedagogical content knowledge, or if such knowledge is present but not attributed to teacher preparation programs, these data would serve to further the deregulation view of education.

The contention of this study is that not enough is known about the subject content training of mathematics teachers, and the effect thereof on pedagogical content knowledge. In comparing international test scores, one could hypothesize that it is this understanding on the part of the teacher that leads to increased student achievement (Ma, 1999). Given the enormity of education reform in the United States attempting to increase student achievement, further research into teacher knowledge may help to ascertain the effectiveness of potential future reforms and requirements for teacher certification. What subject content and pedagogical content knowledge are necessary to teach mathematics effectively and in such a way as to meet policies and standards such as those of the NCTM? Where is such knowledge developed? Is extensive university training in subject content matter necessary to develop the needed knowledge base of a mathematics teacher?

In comparison with counterparts in other countries, mathematics teachers in the United States receive far less training in mathematics ("Studienpläne", 2004.) It has already been found by the TELT study that teachers in the United States do not have a deep understanding of fundamental mathematics (NCRTE, 1993). Alternatively, Ma (1999) discovered that Chinese teachers do have a significantly deeper understanding of fundamental mathematics. Ma (1999) contended that deeper subject matter knowledge in mathematics is a key factor in the ability of teachers to transform content knowledge into pedagogical content knowledge so as to provide realistic and diverse ways of representing mathematics to students. More information examining the role of deep subject matter knowledge is needed with the ultimate goal being to improve education. This gap in the knowledge must be addressed, and this research works to start filling in that gap. Inclusion of another aspect of international comparison could perhaps prove insightful for a number of reasons.

Ma (1999) provided valuable data concerning mathematics teachers in China and the United States. With the consistently high performance on international comparisons tests, it is easy to see why all aspects of the Chinese education system, including teacher knowledge, are of interest to those countries desiring to improve student achievement. What should also be of interest is an examination of countries scoring close to the United States to ascertain how the education system, including teacher knowledge, is developed and implemented. Such information could lead to insights into what should or should not be changed if the goal is increased student achievement. Germany's drastically different approach to teacher preparation from that of the United States and China provides an opportunity to examine the effects of extensive subject content matter training followed by considerable pedagogy training on teacher knowledge and how this translates into student achievement. Particularly at a time when education reform is seeking effective means to improve teacher knowledge and student achievement, those countries with systems similar to what is being proposed should be analyzed. Other countries constantly outscore the United States, with China often at the top of that list while Germany scores much closer to the United States (e.g. TIMSS, 1995; PISA, 2000; TIMSS 2003.) Perhaps careful analysis of a country whose approach to teacher preparation consists of extensive subject training (which might lead to deep and thorough content knowledge, and perhaps even PUFM similar to what Ma (1999) found in China), yet whose student achievement is average (similar to the United States) could help shed light on this gap in the knowledge. Relating back to assumptions by detailed by Cochran-Smith and Lytle (1999) as to sources for knowledge, findings from German mathematics teachers could help clarify the role of teacher preparation programs and classroom experiences in building necessary knowledge for effective teaching. Particularly given that German teachers already complete preparation programs with similarities now being promoted by some in the United States, such as additional content training and extended pedagogical training, that are assumed to lead to increased teacher knowledge (Cochran-Smith & Fries, 2001), an examination into possible contributions and effects of sources of knowledge is especially timely.

CHAPTER 3

METHODOLOGY

International comparisons indicate that students in both the United States and Germany rank average on mathematics achievement tests, scoring below the highest achieving countries, including China. Moreover, previous studies in the United States and China have shown a discrepancy in mathematics teachers' understanding of basic mathematical principles both in the ability of the teacher to compute the problem as well as how to represent the information to students (Kennedy, Ball, & McDiarmid, 1993; Ma, 1999). Germany provides a unique case that merits consideration and further study to understand this discrepancy better, as well as the impact of teacher education programs in preparing prospective teachers. Requirements for German mathematics teachers are drastically different from those in the United States, requiring substantially more mathematical courses before commencing 18-24 months of teacher training. Despite such extensive training in both content and pedagogy, student achievement in Germany remains similar to that of the United States. Guidelines for conducting a study revolving around this issue are examined and explained (deMarrais, Preissle, & Roulston, 2004), including further examination into the problem, purpose, and actual construction of a research study to investigate this potentially challenging variable to current understanding of teacher knowledge.

Theoretical Framework

Teacher knowledge involves different viewpoints concerning what teacher knowledge is and where it is obtained. Previous work done in the field of teacher education, particularly findings from the TELT study (1993) and Ma (1999), in addition to other studies investigating teacher knowledge particularly by Shulman (1986) contributed to the theoretical basis for this study. According to results from the TELT study (NCRTE, 1993) along with further investigation into the topic by Ma (1999), deep understanding of mathematics subject content seems to affect teacher ability to translate such knowledge into pedagogical content knowledge as defined by Shulman (1986). Such knowledge on the part of the teacher allows for multiple forms of representation to students, which presumably aids student understanding and construction of mathematics knowledge resulting in higher achievement scores on international comparisons. Such views on the nature of mathematics teachers' knowledge has been reaffirmed by Hill, Sleep, Lewis and Ball (2007) who maintain that teachers must not only know how to do the math they are teaching, but they must also be able to explain and represent ideas in various ways to students. Ma (1999) found that deep and thorough mathematical understanding leads to a more complete knowledge of mathematics, which on the part of teachers enables them to provide students with mathematical competence.

Based on this theoretical framework, this study sought to investigate the mathematical subject knowledge of German mathematics teachers, as well as their ability to utilize this knowledge in a manner necessary for teaching. In order to gauge German mathematics teachers' subject knowledge and its assumed effects on teacher ability to represent knowledge to others, this study first explored German mathematics teachers' ability to compute problems of basic mathematics, and also to solve word problems to uncover the extent to which they understand underlying mathematical principles, followed by how they might represent knowledge to their students, how they would know if students understood a given concept, and approaches they would use in teaching in four content areas, including: multi-digit subtraction, multi-digit multiplication, division with fractions, perimeter/area, which cover basic and fundamental areas of mathematics. In this fashion, this study sought to add to the knowledge base of mathematics teachers' knowledge and how subject content knowledge affects ability to represent knowledge to others through pedagogical content knowledge.

Cochran-Smith and Lytle (1999) detail three different types of teacher knowledge as knowledge-*for*-practice, knowledge-*in*-practice, and knowledge-*of*-practice. These different views of teacher knowledge and where and how it is developed also provide a portion of the theoretical foundation for this study. Where the type of deep mathematical knowledge necessary for teaching mathematics comes from is debated. Some contend that required knowledge for effective teaching is learned through classroom experience (Cochran-Smith & Lytle, 1999; Cochran-Smith & Fries, 2001) while others maintain it also relies on knowledge provided by researchers, universities, and schooling (Cochran-Smith & Lytle, 1999; Ma, 1999).

Following the framework, this study investigated mathematics content knowledge of German teachers. Working with the theoretical framework of various views on sources for teacher's knowledge, this study explored the contributions of factors such as teacher preparation and classroom experience to teacher's subject content and pedagogical content knowledge. Participants were asked to recall their teacher preparation experiences and to discuss contributions thereof to their knowledge. That is the extent to which participants' knowledge is developed through knowledge-*for*-practice, knowledge in practice, or knowledge of practice. Data were viewed and analyzed through a lens of teacher knowledge. In short, the role of content knowledge and pedagogical content knowledge was examined from the perspective of what teacher knowledge is necessary for mathematics teachers, as well as from views of teachers as to the contribution of teacher education preparation and classroom experiences in forming such knowledge.

The purpose of this study was to investigate German mathematics teachers' mathematical subject content knowledge as well as how this might be translated into pedagogical content knowledge, as well as how their knowledge is generalized. This study in conjunction with previous work that serves to make this a comparative analysis has provided additional information related to this gap in the knowledge about mathematics teachers' knowledge, how/where it might be developed, and what role extensive subject content knowledge plays in effective mathematics instruction.

Research Questions

Three research questions were formulated in relation to the proposed study. Each of them is interconnected, and helped frame the analysis of findings into a recommended course of action teacher education and education requirements in the United States could follow. The questions that emerged were: 1. Do German mathematics teachers possess the knowledge and skills required to solve basic mathematics problems successfully? 2. Can German mathematics teachers take this knowledge of basic mathematics principles and translate it into accurate representations and explanation to others (i.e. students)? 3. According to German mathematics teachers' perception, what contributions do teacher preparation and classroom experiences have on developing teacher knowledge?

These three questions were important to increasing the understanding of teacher knowledge and the role thereof for effective teaching. The first question spoke to German mathematics teachers' subject content knowledge, which as has been shown by research (e.g. Ball, 1990; Ma, 1999) to be important for teaching mathematics to others. The second question addressed pedagogical content knowledge because teachers must take their knowledge of the content and combine it with pedagogical knowledge to form an accurate representation. Posing this question not only examined knowledge of pedagogical content knowledge, but also further probed the depth of teacher understanding of the math principle being discussed. It is one thing to be able to compute a mathematical problem correctly, and quite another to be able to understand underlying mathematical principles in order to explain and provide an example correctly (e.g. NCRTE, 1993; Ma, 1999). Both of the first two questions lead to the third question to discuss adequacy and location of when the content preparation was developed. The third question relates to knowledge-*for*, knowledge-*in,* and knowledge-*of*-practice. Investigation into perceived contribution to teacher knowledge helps in determining the importance and effects of different sources of knowledge, and where such knowledge might be learned best (Cochran-Smith & Lytle, 1999; Ball, 1990). It also sought to provide information useful to the United States as to some of the effects and implications of requiring much more extensive content training for teachers.

An assumption of this study was that due to similarity in education and training German teachers generally as a group either would or would not have a deep and thorough mathematical knowledge. A second assumption was that it is the mathematical preparation of German mathematics teachers through university work that either does or does not provide them with deep subject content knowledge. Perhaps it was actually in their previous schooling that this understanding comes, which would render the second assumption incorrect. In Germany only a small percentage of students attend Gymnasium (upper level high school) and therefore are eligible for university, it could be argued that deep and complete subject knowledge comes from Gymnasium before entering the university.

One possible outcome from further study of German mathematics teachers' knowledge would support the finding of the TELT study (NCRTE, 1993) and Ma (1999) by discovering that although German teachers have substantial mathematical backgrounds, this knowledge does not necessarily translate into teaching craft knowledge; therefore, resulting in German students still lagging behind their Chinese counterparts, instead scoring closer to their American counterparts on international comparison tests. Thus, pure subject content knowledge does not result in a deep and complete knowledge similar to PUFM, and is not the key to student achievement. Perhaps extensive content training does not in fact lead to teachers acquiring deep and complete content knowledge such as PUFM enabling them to better teach their students. This would account for Germany scoring close to the United States on international testing. If this is the case, what is the purpose of having such a lengthy and timeconsuming process of content development if it does not have any measurable effect on student learning?

Another possibility is that although throughout their extensive training German teachers have in fact acquired a thorough subject content knowledge it is not the key to improve student learning and achievement, and Germany's average mathematics rankings are attributable to a factor other than teachers' knowledge. Perhaps there is some other factor that must be considered and researched that is the key to helping students learn better and achieve more. Further research will provide more additional information about the effects of teacher's knowledge. If German teachers had displayed a complete subject content as well as pedagogical content knowledge, other factors influencing average-test scores would need to be researched.

If German teachers did not display a deep and thorough knowledge of both content and pedagogy, this study would serve to strengthen the argument that teachers' understanding is indeed a necessary component to student achievement. Examination of another country that seems to have an extensive emphasis on content similar to knowledge found in teachers from Asian countries, yet scores closer to the United States was needed to further explore teacher knowledge. The answers to each of these questions could help guide the recommendations for reform in the United States in the area of teacher knowledge.

Study

Mixed Methods Design

The original research design for this study was a mixed methods sequential explanatory design consisting of two separate phases; a quantitative phase followed by a qualitative phase (Creswell & Plano Clark, 2007). In this design, the quantitative data from surveys were to be collected from German mathematics teachers in the German state of Niedersachsen to measure the effects of teacher preparation programs and classroom experiences on teacher knowledge, as well as to ascertain levels of mathematical understanding on the part of the participants. It was the intent of the researcher to analyze the quantitative data to answer research questions, but also to enable purposeful sampling of the qualitative phase. Such sampling would allow selection of participants that would include a group of teachers that would ensure certain types of aforementioned characteristics and attributes of teachers were included (Berg, 2007). Purposeful sampling for the qualitative phase sought to include components that would lead to the inclusion of a variety of participants to allow for an analysis of factors that may or may not affect mathematics teachers' knowledge. These characteristics included: years teaching experience, gender, type and location of teacher preparation program, current grade taught, previous teaching experience in terms of grade(s) taught, and location of school currently taught at. The second phase was to consist of collecting qualitative data through interviews with German mathematics teachers, again in the state of Niedersachsen based on, and selected through purposeful sampling following the quantitative data analysis. This design should have allowed for broad sample representative of differences in German mathematics teachers (years of experience, location of teacher education, type of school taught at, etc and in.) for both the larger quantitative and more in-depth qualitative data. The qualitative data were then intended to be used to expand upon and explain the data collected from the quantitative phase. Interview questions expanded upon the survey and explored in greater depth teacher knowledge of fundamental principles of mathematics. The two phases were designed to be interconnected in the intermediate portion of the study, meaning analysis of the quantitative data would help determine the focus of the type of data to be collected for the qualitative phase (Creswell & Plano Clark, 2007). The rationale for this original research approach was that the quantitative data would provide a large base of data on the subject of teacher knowledge, while the exploratory qualitative follow-up data would provide a more detailed account of the findings through in-depth interviews and explanations of purposefully selected participants.

The original research plan of this study was soundly designed, based on following the recommendations of Creswell and Plano Clark (2007), and provided a good framework for answering the research questions. However, lower participation rates, particularly in the area of surveys, as well as the limitations and challenges when faced by one student researcher conducting an international study resulted in the need for modifying this design. Rather than upwards of 100 surveys, and the anticipated 27-45 interviews, collected data consisted of twenty interviews and eleven surveys. Two of the surveys were incomplete, and consequently did not provide data in all areas. Thus, an emergent design that focused on the qualitative data rather than a reliance on both qualitative and quantitative was necessary for this work to proceed. Such flexibility in the research design with decisions being made as the work progresses is considered an acceptable component of qualitative research (Bogdan & Biklen, 2007).

Qualitative Interview Project

In order to more deeply represent the collected data, the research design shifted to that of a qualitative interview project (Rubin & Rubin, 2005). In this model researchers rely on in-depth interviews allowing participants to explain their answers as well as their experiences. According to Willis (2007), interview research is an established qualitative research method that can be structured, semi-structured, or open ended in nature with interviews and surveys given in person or via the Internet. All of these characteristics were already an integral part of this study prior to the conclusion that a mixed methods study would no longer be a suitable research design.

In keeping with the original design described above, both interview and survey data were analyzed and are reported on; however, the main focus of data analysis necessarily relied on collected interview data. Survey data was still viewed as informative and insightful, particularly in areas not addressed by the interview, but were used in a more supportive role to the interview data. These supporting documents were also analyzed and included in the findings.

Instruments

The purpose of the survey (designed by the researcher, but grounded in work by Ma (1999), Pehkonen and Toerner (1999), and the TELT study) was to obtain general background information about the teacher such as years of experience, gender, type and extent of training in both content area and pedagogy (See Appendix H). Other than collecting background data, another purpose of the survey was to focus on teacher knowledge. University training and experience are often used to form categories of teachers to study differences that may be attributed to these two factors. The survey provided information into how teachers viewed their learning and knowledge. In order to ensure reliability and validity, mathematical portions of the survey were based on questions from TIMSS and PISA due to the careful manner in which these organizations have worked to ensure reliability and validity within their own work. Survey data, though limited, enriched data specifically to answer the first and third research question.

The interview protocol was based on work by the TELT study, which was also used by Ma (1999), and was aimed in part to allow for comparison of results to those of previous studies in the U.S. and China. The interview questions were pre-determined to ensure desired topics were uniformly discussed; however, the participants were welcomed to add anything else they felt would be applicable and helpful. (See Appendix I.) In addition to providing some background data and computational ability analysis, the interview questions asked participants to think of a way of explaining, representing, or responding to the basic principles in a unique manner similar to what they would use when teaching the concept to students. Confidence and ability to complete this for each of the basic principles of mathematics provided data especially relevant to answering the second research question concerning whether or not German mathematics teachers can translate deep mathematical knowledge into representations for teaching (i.e. pedagogical content knowledge).

This study followed the same methods, and used the same measurements as those developed by, and used in the TELT study and by Ma (1999). These materials were collected by the National Center for Research on Teacher Learning (NCRTL, now

NCRTE) funded through a grant from the federal government that makes the materials usable by other individuals wishing to utilize the material for other studies. According to Hill, Sleep, Lewis, & Ball (2007), several methods for gathering data on teacher's mathematical knowledge exist and have been used to conduct research, including analysis of teacher certification scores, content tests, observation methods, and math interviews and tasks. Given the myriad of possibilities, each with their own benefits and drawbacks and despite the shift away from some of these methods (Hill, Sleep, Lewis & Ball, 2007), utilizing the math interview and tasks method was selected for this study. Several reasons exist to justify using the same instruments and methods as the two previous studies mentioned.

Experts in the teacher education field carefully developed the interview questions to explore basic concepts of mathematics that are not only necessary for future mathematical understanding, but are also areas in which students (and sometimes teachers) are likely to struggle (Ball, 1990; Kennedy, 1991; Ma, 1999). Each of the mathematical principles covered will be discussed further below. Interview methods were carefully analyzed and refined by the National Center for Research on Teacher Education (NCRTE), forerunner to NCRTL, and TELT. The expertise of these members serves as an example to the education research field. TELT and Ma (1999) successfully used these methods, and since the purpose of the study is to provide a challenge to, as well as further this line of research, it was necessary to use these methods so the findings can be directly compared to those results found in the United States and China. Each teacher participant was interviewed based on the outline obtained from the TELT study.

Mathematical principles covered in the interview included a broad spectrum of elementary mathematics allowing for an inclusive analysis of teacher knowledge (Ma, 1999). Mathematical topics covered included subtracting with regrouping, multi-digit multiplication, division with fractions, and response to novel mathematical theories. (See Appendix I for interview questions.) Students frequently struggle with these topics (Ball & Bass, 2000), and teachers often have difficulty providing realistic representations to aid student learning, and yet teacher flexibility in providing representations can be crucial to student learning (Ma, 1999). Division is considered a main mathematical concept at all stages of mathematical learning that helps students understand about other concepts as well, such as rational and irrational numbers, place value, and basic operations (Ball, 1990). Work by Perry (2000) revealed that fractions are a topic many teachers spend a great deal of time explaining to students, thus an analysis of teachers' understanding of fractions is also justified as an area for this research. Yang and Cobb (1995) contended that these mathematical principles, specifically discussing place value and multi-digit subtraction, are the types of mathematics children should be exposed to, and that teacher knowledge of such principles affected student understanding and development of such concepts. The manner in which teachers were able to respond to a novel theory or idea about mathematics serves to not only show their knowledge of the principle in general, but also their understanding of mathematical evidence and how to relay such understanding (Kennedy, Ball & McDiarmid, 1993). Each of the mathematical principles investigated is central to basic yet fundamental understanding of mathematics, and as such are areas teachers should possess in-depth knowledge.

Site Selection

Niedersachsen was chosen as the basis for site selection for this research based on state rankings ("Länder Ranking," 2002). Based on the recent rankings of the German Länder (or states) on the PISA study ("Länder Ranking," 2002), one state that fell in the average portion of the scores was targeted. The overall mean of the PISA scores used to rank the German states was 572, with a standard deviation of 13.7768. Once outlier states (those exceeding the standard deviation) were removed, the mean of the remaining states dropped to 571, but the standard deviation was significantly reduced to 5.2345. A geographical examination of state rankings revealed that rankings did not appear to be influenced by geography. There were portions of the country north and south, east and west that scored at the top and at the bottom. Niedersachsen with a score of 575 is a state that fell within one standard deviation of the mean for both the total population, as well as for the modified analysis of states, and was the target of this study. Niedersachsen is the second largest state in terms of size, and fourth largest in terms of population (about eight million inhabitants). It is located in the western portion of Germany, stretching from central Germany into the north. Although it would have been ideal, and was the original intent of the study, teachers participating in the study taught at schools located primarily in smaller towns rather than in locations of varying size.

Sample Selection

Initial contact with potential participants, principals of schools, and other contacts in Niedersachsen primarily via email began a few months prior to traveling to Germany. In addition to information regarding the study, preliminary contact also included an email link to the online survey to allow potential participants to complete the survey prior to the interview if they so chose. It would be very difficult to state how many potential participants were initially contacted since many of the emails sent to principals in Germany ended up in the junk folder as evidenced by the emails received by the researcher stating that the sent message had been deleted from the junk folder. However, it is known that in addition to the twenty participants who were involved in the study and were encouraged to complete the survey, other principals and teachers also received the email. The researcher spoke with school principals/directors of fourteen different schools in Niedersachsen explaining the study and asking if an email could be sent to them with further information and a link to the online survey. All but one school agreed to this; however, it is unknown what, if anything, the principals did after the phone call and receiving the email. What is known is that some teachers did receive the link presumably from their principal. In total nine additional participants completed the survey to varying degrees, though their data are not included since there were no interview data to accompany their survey data. The overarching criterion for sample selection was that the participants be mathematics teachers who received their education and training in Germany. Participants were found and selected primarily through referrals from three initial contacts. A total of twenty participants were interviewed throughout the study. This number is comparable to that of the TELT study (23), but significantly lower than Ma (72); however, this sample size still yielded rich data. Of the twenty participants who were interviewed eleven completed the survey although two of the eleven did not complete any of the mathematical portions of the survey. The response rate was low, but also still vital and with various subgroups of the overall participant sample represented (age, years of experience, institutions attended, grade level taught, etc.).

Participants taught in eight different small towns in southern Niedersachsen. Participants included two male and eighteen female teachers, with an average age of fifty according to limited survey data. Participants taught at Grundschulen typically comprised of grades 1-4 or Hauptschulen typically comprised of grades 5-10. Participants taught at both ends of the spectrum from 1st grade to 10th grade, with 40% of the teachers teaching mathematics at more than one grade level, and 80% of the participants teaching at least one grade that would be considered elementary school in the United States. It was confirmed during the interview that all of the teachers who taught at a grade level not considered elementary in the United States had teaching certification that covered the broad range of grades. According to expectations for student knowledge, multi-digit subtraction and multi-digit multiplication should both be mastered by the end of fourth grade, figuring with area and perimeter is an expectation at both the fourth and sixth grade levels, while division with fractions is to be mastered by the end of sixth grade (Niedersachsen Bildungserver, 2008). As a point of reference, 30% of participants taught at least some 4th grade classes, and 10% taught at least some 6th grade classes. Participants also had a wide range of teaching experience from 1-42 years, and had attended a variety of teacher preparation programs. One teacher had originally studied to become a teacher in Russia; however, she had also completed teacher preparation in Germany and indicated that she ran her classroom in similar fashion to her colleague. Another teacher was originally from the former East Germany, but technically met the requirements of having completed teacher preparation in Germany and to be teaching mathematics in Germany. The research was conducted in both English and German depending on the comfort level of the participants in using English. The researcher has lived in Germany, and was fairly fluent in German, which helped when instances arose where the participants felt more comfortable using German. Additionally, the researcher collaborated and worked with a native-speaker of German to ensure that translations were accurate.

Implementation

Once access was gained to necessary individuals, surveys were sent with follow-up interviews scheduled at a later date. The extremely short amount of time the researcher was in contact with the German teachers somewhat limited the ability to build necessary and desired rapport with the subjects (Bogdan & Biklen, 2007). To counteract this, the researcher attempted to establish email contact with as many teachers as possible throughout the data collection phase to build a relationship with a maximum number of teachers. This was done to build rapport and to gain confidence of possible participants, as well as to determine who potential gatekeepers might be that could possibly help in locating other teachers who would be willing to participate through the snowball or chain method described by Marshall and Rossman (1999). Additionally, the researcher used doctoral student status as a means of being perceived as non-threatening and relatable as recommended by Bogdan and Biklen (2007). In keeping with the naturalistic setting described by Bogdan and Biklen (2007), the interviews were conducted at a location mutually agreed upon by the researcher and teacher, which most often was at the participant's school.

Altogether, approximately two weeks were spent in Germany in order to collect the necessary data for this research study; however, interviews could only be conducted on weekdays due to the schedules and availability of teachers. On days interviews were conducted, anywhere from one to five interviews were held based on teacher availability. The remaining time was spent working to find additional participants, scheduling other interviews, and transcribing and analyzing data obtained.

Data Collection

Data collection consisted of concurrent collection of both survey and interview data. A link to the online survey was sent to identified gatekeepers and to teachers who had agreed to be interviewed or as they were interviewed in various towns in Niedersachsen. (See Appendix H). Responses to surveys were categorized into knowledge-*for*-practice, knowledge-*in*-practice, and knowledge-*of*-practice as explained by Cochran-Smith and Fries (2001) to analyze more carefully grouping of teachers in terms of beliefs about teacher knowledge as well as other apparent categories that might be manifest that may have seemed to provide a varied population such as age, years of teaching experience, etc. Teachers were also asked to provide answers to four mathematical questions based on basic principles of mathematics. The ability to compute the answers correctly helped indicate whether teachers possessed knowledge and skills to compute and understand basic mathematical principles necessary to teach. Additionally, participants were presented with word problems in each of the four mathematical areas and asked to solve and explain the procedure for solving allowing for further analysis of the understanding on the part of the participants in each of the four areas. The results of the survey were used to support and confirm data from the interview portion of the study.

Utilizing a true snowball effect (Marshall and Rossman, 1999), teachers willing to participate were contacted to arrange for an audio taped interview. Interviews were audio taped to provide a lasting documentation of the interview. This also allowed for the interviews to be accessible at a later point in time for further reference in the analysis of the data, or should a new theory or question arise that could be aided by listening to these tapes. Each interview was later transcribed, and where necessary translated, and the results analyzed to ascertain to what degree, if any, the participants displayed necessary knowledge and skills as evidence of deep and complete mathematical knowledge. Each interview lasted approximately 30-45 minutes depending on the participants' length and detail of answers, as well as in part on their mathematical ability and confidence.

Data Analysis

Berg (2007) admonished that researchers keep the original study aim in mind, while still making allowances for unanticipated results that may emerge from the data. This was done to ensure that any and all valuable information concerning the topic of teacher knowledge and the roles of subject content knowledge and pedagogical content knowledge was incorporated and used to provide a more complete picture by allowing participants to extend their line of answering or to engage in discussion revolving around the topics in the study, but not necessarily delineated in the formal interview guide. Collected data were coded and analyzed to include both the original aims of the study, as well as other interpretations that emerged. The data were coded and analyzed for themes revolving around mathematical ability and confidence, in conjunction with the various types and interactions of teacher knowledge using the guidelines set forth by Marshall and Rossman (1999), and then used as verification for information gathered from the interviews.

Content analysis revolved around the focus of the study in terms of teacher knowledge, and was determined by the participant's ability to not only compute the answers to the mathematical questions in the survey, but also the confidence and ability to translate such knowledge into an original representation that might be used in the classroom. The responses given in the interview allowed the researcher to answer directly the research questions posed in regards to German mathematics teachers' ability to compute mathematical problems centering on basic principles of mathematics, as well as their ability to represent the problem accurately.

For both interview and survey data, content analysis was used with a deductive approach to develop categories in order to compare the data not only to previous studies, but also to compare the two data sources from within the current study. As explained by Berg (2007), a deductive approach to category development results in the researcher basing categories on a scheme from a theoretical perspective allowing data to be used to assess a hypothesis. In the case of this study, these analyses were conducted using categories similar to those found in the TELT study and work by Ma (1999) in order to provide comparable data to challenge and/or verify previous conclusions. A brief description of the four areas of study is provided after an explanation of the analysis process.

Three phases of analysis were conducted. The first phase consisted of content analysis of the survey with a focus on manifest content, which as described by Berg (2007) is analysis of elements in the data that are physically present. In the case of the survey, questions and answers were straightforward in nature (e.g. years taught, grade level currently teaching, answers to mathematical problems, etc.) The second phase of analysis was a manifest content analysis of interview data. This initial analysis of interview data included categorizing answers based on terminology used (such as place value) by the participants and clearly evident in the answers. The third phase of analysis once again involved the interview data, but this time with a focus on latent content analysis. Previously coded answers were further dissected for meaning behind the terminology, and for a careful examination of what participants were discussing and seemed to understand about a given topic. Previous studies were once again examined in order to determine making transitions from the manifest to latent level, and for making (Berg, 2007). For example during the third phase of analysis, not all teachers who used the term "place value" were deemed to have similar understanding of the mathematical principle and thus were placed into different groups. To support the latent content analysis conducted, detailed examples are provided throughout the study to illustrate and detail the different interpretations assigned to each group of participants. When reporting on individual participants pseudonyms were used in order to ensure anonymity.

Content analysis for the survey data did not involve categories used by previous studies, as the survey component was unique to this study, rather these data were categorized into knowledge-*for*-practice, knowledge-*in*-practice, and knowledge-*of*-practice as explained by Cochran-Smith and Fries (2001) to analyze more carefully grouping of participants in terms of contributions of teacher preparation program and classroom experiences, mathematical knowledge and ability, as well as other comparison categories that arose that provided a varied population such as age and years of teaching experience to explain and understand the sampling of participants. For example, participants answered the following question on the survey: "What contributions did your university education make in terms of helping you teach mathematics? What was the role of this training in helping you know how to represent mathematical concepts to students?" Participant responses were categorized based on whether they viewed these contributions to have been helpful in terms of mathematical knowledge, helpful only in terms of methods or pedagogy, or not helpful at all. To illustrate this process further, Ms. Richter's response that various math methods had helped her was coded as knowledge-*for*-practice since she viewed her teacher preparation program has having positively contributed to her knowledge. A limitation of this study is that collected data did not provide data to uncover knowledge-*of*-practice. Additionally, survey data were analyzed for connection between the aforementioned categories. For example, to determine if various categories such as years teaching experience had an affect on participant responses and performance, participant responses were compared based on years taught to uncover what, if any, effect this variable might have. Due to the small survey number no specific categories of years teaching experience were assigned, rather each participant and their years taught were compared against the other data.

Interview data were coded and analyzed for themes revolving around mathematical ability and confidence, in conjunction with the various types and interactions of teacher knowledge using the guidelines set forth by Marshall and Rossman (1999). This analysis provided insight into patterns that existed within the participant population, and also highlighted differences among the various members to provide some explanation into possible differences that became evident within the population. As previously mentioned these categories were based on work by TELT and Ma (1999). Each question in the interview was coded and analyzed, though only data directly related to answering the research questions are included. Explanations of the broad and manifest categories used

Figure 1
Coding Categories for Interview Content Analysis of Mathematical Principles

Mathematical Principle	Content Analysis Categories
Multi-Digit Subtraction	Answer provided though unsolicited What would be seen if researcher came to their classroom Type of approach to teaching Discussion and inclusion of place value, further analyzed for responses limited in such discussion Problems student have with concept
Multi-Digit Multiplication	Re-writes/reference German method for multi-digit multiplication Steps taken if students had problem Answer provided though unsolicited Explanation of why numbers should be moved over
Dividing with Fractions	Ability to recall how they were taught to divide fractions Answer in solving the given problem View on what makes dividing fractions difficult Ability to generate accurate story problem View on why generating story problem is difficult
Novel Theory: Perimeter and Area	Type of response How they would react if unsure of mathematical principle

for analyzing the mathematical portions of the interview included in this study, that are based on previous studies are illustrated in figure 1. See Figure 1.

Data were interpreted based on the analysis to determine the depth of German mathematics teachers' knowledge both in terms of the subject and ability to teach, as well as what, if any, effect various combinations of factors have on this topic. Such interpretations were then examined comparatively against data from the United States and China to aid in forming conclusions regarding the newly acquired data on teacher knowledge.

Reliability and Validity

Insofar as a similar population of participants is studied, there should not be any issue in terms of reliability when referring to the results since similar findings would be able to be obtained from both perspectives as described by Bogdan and Biklen (2007). The instruments used in this study did indeed test what they were designed to test. Although the sample of this study included diversity in terms of experience, gender, current grade taught, etc. this study reports on the knowledge of this group of participants. It cannot be said that these results necessarily apply to all German mathematics teachers. Despite this, based on the use of proven instruments accompanied by careful and thorough research design planning, for this particular group of participants the data are both reliable and valid.

Bias and Limitations

Biases exist when assumptions are made that teacher knowledge is a key factor in student comprehension and achievement. Further exploration into this topic should provide further insight into the matter to explain what the effects of mathematics teacher knowledge are. Another bias is the assumption that deep understanding of subject matter content comes from extensive university training in both content and pedagogy, and that this training should result in the development of PUFM. Continued research such as this should serve to help explain and dispel some bias and unknown variables centering on the issue of teacher knowledge.

A major limitation of this study is the lack of classroom observations of participants that would enrich the data to analyze participants' knowledge. Also, the fact that some participants were teaching the actual content included on the survey and in the interview, while others had perhaps not taught them for quite a while is also a limitation as this could have affected their ability to correctly solve the mathematical problems and story problems, and their discussion related to how they might teach the subject, especially in the relatively short span of time that the interview took place. Primarily, the data consisted of twenty interviews, supplemented by the completion of the survey by nine of the participants and an additional two partially-completed surveys. Without classroom observations the data relied on participants to report what they would do when teaching, with no way to verify the statements. Ball and Bass (2000) would describe this as incomplete data; however, this is a limitation that could perhaps be explored in future studies.

Another limitation is the sample in terms of size as well as the fact that only one of sixteen German states is included. The anticipated sample size was comparable to that of the TELT study (Kennedy, Ball, & McDiarmid, 1993), but significantly smaller than that obtained by Ma (1999). Despite this, it was large enough to achieve a representative sample of this group of participants' knowledge in Germany for the areas to be studied. The choice of mathematical principles to be studied could be viewed as a limitation; however, the careful selection of these key principles has been documented in numerous studies as being vital to student understanding of mathematics (Ball & Bass, 2000). Additionally, recent studies focusing on mathematics teachers' subject content knowledge as it relates to pedagogical content knowledge is somewhat scarce: this limitation also serves as a catalyst for further research in this area.

Explanation of Data

Survey data were used to verify data collected as part of the interview; however, only eleven of the twenty participants interviewed completed the survey. This disparity does not seem to allow for a completely equal comparison; however, it does help provide additional data that is essential in exploring each of the mathematical concepts and whether or not the German mathematics teachers were able to correctly solve the problems. In order to avoid confusion in reporting numbers, when discussing data from the interview percentages will be used, but it must be noted that these percentages do not have the same sample size. In all cases, an explanation is provided to clarify what number of participants the percentage comes from. When discussing survey data a number out of nine will be given, except for the fractions which will only be out of eight since one teacher did not answer any questions dealing with fractions. Although eleven of the twenty interview participants completed portions of the survey, this portion of the discussion will rely on only nine of the survey participants (eight for fractions). Of the remaining two participants, one did not respond to any of the mathematical computations portions of the survey, and the other gave one partial and one complete answer (which was correct) to this portion of the survey (there were twelve questions in all). It is impossible to know whether they would have answered correctly or incorrectly. Thus, they were not counted as having completed the survey insofar as discussion of mathematical computations is concerned. This meant that although eleven participants took the survey, survey data to investigate German mathematics teachers' ability to correctly solve basic mathematics relied only on the nine who completed this portion. Reporting on the number of participants who provided and/or attempted an answer rather than the entire research sample is in keeping with other studies of this type (Ma, 1999) that reported only on the number of participants who attempted an answer. Since these nine survey participants also completed the interview, this extension of results serves as further data and verification rather than an entirely new set of data with different participants. A much more complete picture would have emerged had all twenty participants completed the survey; however, due to the important data from the survey this portion of the discussion necessarily relies heavily on the results of the participants who did participate in the survey. Although the topics covered in the interview and on the survey were identical, the problems themselves differed in order to provide additional assessment and data. The areas assessed were multi-digit subtraction, multi-digit multiplication, dividing by fractions, and responding to a novel theory dealing with perimeter and area. In the interview, participants were only asked to specifically provide an answer when dividing by fractions. Some participants supplied answers to problems in the other areas, and those results are included, but again the data are discussed based on the number of participants who attempted and/or did provide an answer.

This study is comparative in nature. A comparison between teachers from Germany, the United States and China is an important part of this research. As mentioned above though, participants were only asked to specifically answer the problem dealing with dividing with fractions. Thus, a completely accurate comparison of teacher knowledge insofar as computation is concerned is necessarily limited. Data from both the United States and China is limited to the section dealing with division by fractions since this was the only area of the interview that specifically asked participants to compute an answer.

CHAPTER 4

RESULTS: ABILITY TO SOLVE BASIC MATHEMATICS CORRECTLY

Data described in this chapter seeks to answer whether or not German mathematics teachers possess the knowledge and skills necessary to solve correctly basic mathematical problems. While there are many possible means for exploring teacher knowledge (see previous discussion about various methods used to gauge teacher knowledge), in a dissertation it is impossible to explore every area. The focus for gauging participant's mathematical knowledge in this study focused on a combination of computation and word problems. The justification for this approach is that mastery of computation skills is seen as a focus in elementary school and beyond, and that word problems are used at all levels of mathematics to push the student's understanding of mathematical principles to be learned. According to Hill, Sleep, Lewis and Ball (2007), teachers must not only know how to compute the mathematics they are teaching, but they must be able to spot errors, represent the material in various ways, and must also be capable of determining and explaining methods and approaches that might be used by students. Thus, in order to do determine participant's mathematical ability in this view, participants were asked to complete both computation and word problems in the four mathematical areas of this study. This allowed for discussion of both computational skills, as well as further analysis into understanding of underlying principles to check for understanding deeper than the computational level. Data from both the interview and survey were used to explore this issue.

Each of the following sections reports on the findings of one of the four mathematical principles studied. In each case, participants were asked to complete a computation as well as a word problem as part of the survey. In each instance, participants were asked to show the steps of their calculation to allow for insight into the approach and understanding on the part of the participants. Interview data only specifically asked for a computation answer in the area of dividing with fractions, though participants also offered unsolicited answers to computations involved in the interview discussion. Reports on these data are included. As discussed in the previous chapter, computation and word problems were based on previously established and proven research to aide in comparison, as well as to ensure reliability and validity. See Appendix H and I for survey and interview instruments respectively. The chapter concludes with a discussion of the findings, as well as a comparison among Germany, China and the United States.

Multi-digit Subtraction

Computation

The first area explored for multi-digit subtraction dealt with simple computation skills. When asked on the survey to compute the following question without the use of a calculator, subtract: 6000 –2369, data showed that eight of nine participants correctly answered the subtraction problems, while the remaining participants did not provide an answer to this question.

One participant did not show any steps, rather only provided an answer. The remaining seven respondents were divided into two groups based on how they explained they would solve the problem. The first group explained they would decompose higher levels on three separate occasions in order to complete the subtraction process. All of these participants explained and/or showed how they would accomplish the process of decomposing a higher number in order to be able to complete this problem. Although these participants discussed decomposing higher units, survey data does not necessarily allow for further analysis as to how concretely these participants understood the principles underlying this process.

The steps shown by the second group indicated that they would break apart the problem into smaller, separate problems. The work of two of the participants was rather similar, but although these participants used essentially the same method, there was some variation. One participant in this category went a bit further in showing the exact breakdown of the problem. Figure 2 shows the number of participants in each category.

Figure 2
Approaches Used by Participants for Multi-digit Subtraction (n=9)

No answer or approach given	1
Decomposing higher units	4
Break apart problem into smaller, separate problems	3

Ms. Schultz, a participant in the first group, that discussed decomposing higher numbers, explained:

> First I look at the ones. Nine from zero doesn't go. I have to add to the zero ten ones. The difference between nine and ten is one. I write this number down at the bottom. Because I added from the top number (6000), I must do the same with the bottom number (3369) to even it out. You write a small tens by the 6 in 2369. Then there are 7 tens there. From the 7 to the zero doesn't go, so I must add a tens. And so on.

In contrast, participants from the second group who discussed breaking apart the problems showed work such as Ms. Muller:

```
  6000
- 2000
  4000
-  300
  3700
-   60
  3640
-    9
  3631
```

Ms. Lowe, also from the second group illustrated the exact break down of the problem she would use.

6000	4000	3700	3640
- 2369	- 300	- 60	- 9
	- 60	- 9	3631
	- 9		

By way of discussion, during the interview, participants were asked about multi-digit subtraction, but not to actually solve the sample problem provided (64-46). Though not specifically asked to provide an answer to the simple multi-digit subtraction problem, 55% of the participants did provide the correct answer and another 10% alluded to the correct answer without actually stating what the answer was. These participants explained how they would solve the problem; however, when they got to the end of the explanation they did not actually state the answer. As with the survey data, participants in the interview also discussed a variety of procedures they would use in solving this problem. The other 35% focused on methods of instruction, which is what the question focused on. No one provided a wrong answer. Due to the fact that answers were unsolicited, and steps not necessarily shown, further discussion as to specific approaches used by these participants is discussed in the next chapter.

Word Problem

The next survey item was a word problem that focused on not simply computation, but further understanding of multi-digit subtraction. Participants were presented with the following word problem dealing with multi-digit subtraction: There are 30 people in the music room. There are 74 people in the gymnasium. How many more people are in the gymnasium than the music room?

All nine participants correctly answered this question, although it seems one participant perhaps did not fully understand the question. This participant explained there were 30 students in the music room and 44 students in the school so there were 14 more students in the school than in the gymnasium. Despite this discrepancy, the participant did provide the number 44 and also indicated the difference between 44 and 30 accomplishing the task of multi-digit subtraction.

Those participants completing the survey all provided the steps they would take in solving this problem. These approaches ranged from solving the problem of 74-30 in their head (Mr. Pfeiffer) to the more typical answer of 74-30=? or 30+ ? = 74. In this manner, participants illustrated that not only were they capable of performing the computations, but also that they understood how to approach the problem for successful completion.

Results and Discussion

The responses to the survey, and also data from the interview particularly when taken together seem to indicate that this group of German mathematics teachers was able to compute this basic, yet fundamental mathematical problem correctly. Figure 3 illustrates the performance of the participants in solving multi-digit subtraction problems.

Figure 3
Participant Performance Solving Multi-digit Subtraction

Number Correct in Answering Survey Computation (n=9)	8 *1 did not provide answer
Percentage Providing Correct Computation Answer on Survey	55%
Percentage Correct of Attempted Interview Computations	100%
Percentage Alluding to Correct Answer in Interview Computation	10%
Number Correct in Answering Survey Word Problem (n=9)	9
Total Overall Number Providing Incorrect Answer to Multi-digit Subtraction Problem	0%
Total Overall Percentage of Participants Providing Answer to Multi-digit Subtraction on Survey or Interview	75%
Total Overall Percentage Providing Correct Answer to Multi-digit Subtraction Problem	100%

Despite not everyone providing answers, no one provided an incorrect answer either. Analysis of interview and survey data showed that 75% of participants answered correctly in one or more instances problems dealing with multi-digit subtraction. Participants who provided an answer either in the interview or on the survey did so at a successful completion rate of 100%.

These results in the area of multi-digit subtraction indicate that participants have sufficient subject content knowledge in this area. Based on ability to in not only computation, but also in demonstrating further understanding and knowledge required for word problem completion, this group of participants demonstrated strong subject content knowledge in regards to this topic.

Multi-digit Multiplication

Of the four mathematical principles studied, multi-digit multiplication revealed perhaps the most interesting data, in that the process of multiplication is completely different than in the United States. It quickly became apparent that none of the participants would follow the same steps to compute multi-digit multiplication as is seen in the United States or as was posed in the interview. In the interview, participants were presented with this scenario:

> Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate 123 x 645 the students seemed to be forgetting to move the numbers over on each line. They were doing this, instead of this.

```
  123            123
x 645          x 645
  615            615
  492           492
  738          738
 1845          79335
```

While the teachers agreed this was a problem, they couldn't agree what was the best way to solve the problem. What would you do if you were teaching sixth grade and you noticed your students had this problem?

In nearly every instance the participant being interviewed would interrupt and either point out that the problem was written completely wrong and that it would never be seen in this format in Germany or would in a puzzled manner ask about the format. In Germany, multiplication is written and computed with the numbers written side by side, in this fashion:

123 x 645

According to German methods, the problem would then be computed as follows:

```
    123 x 645
      738
       492
        615
    ---------
      79335
```

The participants explained that this problem dealt with place value, and that in order to keep the numbers properly aligned one would write the answer to the separate steps of the problem under the place value being multiplied. For instance, when multiplying 123 x 6(45), the numbers would begin to be written directly under the hundreds place, in this case under the number six. The next step would be to multiply the tens place with the answer being aligned with the tens place directly under the four. Finally, the ones place, in this case the five, would be multiplied with those numbers being written under the ones place. Once this was accomplished the numbers would be correctly aligned for the final step of addition.

Computation

As with multi-digit subtraction, participants were not asked to provide an answer to the multiplication problem during the interview, and 50% of them did not. Of the remaining participants, 25% provided a correct answer with the remaining 25% providing an incorrect answer resulting in a 50% success rate. Minor calculation errors accounted for 30% of the participants providing an incorrect answer. Responses to providing an answer to the multi-digit multiplication problem in the interview are seen in Figure 4.

Figure 4
Participant Responses on Interview Multi-digit Multiplication (Percentage of participants, n=20)

No answer provided	50%
Correct answer provided	25%
Incorrect answer provided, minor calculation error	15%
Incorrect answer provided, other reason	10%

Assuming the participants who made minor calculation errors, such as forgetting to add the carry-over from the previous place value resulting in an answer of 79,235 instead of 79, 335, knew how to compute correctly the problem, but made minor mistakes in the interview it would then appear that of those participants providing a response, 80% of the participants interviewed have the knowledge and skills to solve correctly multi-digit multiplication. Though this cannot be assumed, two of the three participants making minor calculation errors provided correct answers on the survey. Additionally, one participant who provided an incorrect answer in the interview computed correctly the answer on a similar type problem on the survey, which would result in a total successful completion rate of 90%. An even higher successful completion rate in the survey data affirms this conclusion further.

Presented with the following problem: multiply: 345 x 476, all nine participants provided the correct answer on the survey. Again, participants were asked to show the steps of their calculation. As with multi-digit subtraction, participants fell into two separate groups. The first group multiplied the problem in a similar fashion as was shown above. Although the remaining participants indicated they could complete the problem through writing the multiplication out or in other words, figuring with the multiplication cross (referring to the German method of solving multiplication problems), they showed how they would break apart the problem and multiply each part of the problem separately.

Thus, the first group used the approach discussed above, and completed their computation in a manner such as shown by Ms. Riese:

```
345 x 476
  1380
   2415
    2070
  164220
```

Ms. Kuhn demonstrated how the second group computed the answer to this problem:

```
300 x 400= 120000
300 x  70=  21000
300 x   6=   1800
 40 x 400=  16000
 40 x  70=   2800
 40 x   6=    240
  5 x400=    2000
  5 x  70=    350
  5 x   6=     30
Add Results  =164220
```

Regardless of the method shown in the calculation, all survey responders correctly answered the problem. Figure 5 illustrates the break down of participants according to the approach they used for multi-digit multiplication.

Figure 5
Participant Approach to Multi-digit Multiplication (n=9)

German method of multiplication cross	7
Break apart problem, multiply each part separately	2

Word Problem

Following this multiplication computation problem, participants were then presented with this word problem: A person's heart is beating 72 times a minute. At this rate, about how many times does it beat in one hour? As with the first multi-digit multiplication problem, all nine participants correctly computed this problem. Three groups emerged, though the first two were quite similar. The first group simply indicated they would multiply the two numbers together. No explanation was given as to how they would arrive at the second number to be included in the computation. Perhaps as adults they felt it would be unnecessary to explain this step to another adult. Interesting to note is that although this is a multi-digit multiplication problem, participants still wrote the numbers next to each other, but no participant wrote their result below. Rather, all of the answers were written at the end of the computation.

The second group also indicated they would multiply two numbers together to solve the problem. This group, however, showed how they would get the second number by explaining that there are 60 minutes per hour. After obtaining the second number for their problem they could multiply to solve.

In similar fashion to approaches used in previous sections, the third group broke apart the multiplication problem into a series of smaller problems to varying degrees. One approach within this group separated the 72 into two separate parts to be multiplied by 60. Another approach in this group skipped separating the 72, but still multiplied 60 by each part of 72.

Ms. Hoffman, along with the rest of the first group showed their steps to calculation as follows:

72 x 60 = 4320

As part of the second group, Ms. Muller illustrated the additional steps characteristic of their approach:

1min=72times

60min= ?

72 x 60=4320

Finally, Mr. Pfeiffer's approach demonstrated how the third group approached this problem:

72 x 60=

70 x 60+2x 60=

4200+120=

4320

The composition of participants based on approach to multi-digit multiplication is shown in Figure 6.

Figure 6
Participant Approach to Multi-digit Multiplication Word Problem (n=9)

Multiply two numbers, no explanation of second number	5
Multiply two numbers, explain second number	2
Break apart problem into smaller, separate problems	2

Results and Discussion

Obviously, the participants completing the survey were all able to solve multi-digit multiplication problems correctly. As discussed above, this group included participants who did not provide and/or compute correctly answers in the interview. There was a 100% successful completion rate on the survey. By combining the data of the survey and the interview, the total number of participants who correctly computed a multi-digit multiplication problem at least once either during the interview, or on the survey was 60% overall, with an overall successful completion rate of 92% (n=12). Figure 7 details the performance of participants in the area of solving multi-digit multiplication.

Figure 7
Participant Performance on Multi-digit Multiplication

Number correct in answering survey computation (n=9)	9
Percentage providing answer in interview (n=20)	25%
Percentage correct of attempted interview computations (n=9)	55%
Percentage incorrect answer in interview, but correct on survey	33%
Number correct in answering survey word problem (n=9)	9
Total overall percentage providing answer to multi-digit multiplication problem	60%
Total percentage of participants providing answer survey or interview, that were correct	92%

The participants interviewed indicated that by using both the method taught in Germany with the emphasis on place value, as well as ensuring one uses graph paper, the problems posed in the hypothetical situation would likely be avoided. Relying primarily on survey data, but also taking into account interview data, it appears German mathematics teachers involved in this study do have the knowledge and skills necessary to solve multi-digit multiplication problems.

Delving deeper into a discussion of these findings, it is important to note that the relation of these data to subject content knowledge goes beyond ability to complete such problems. The focus of this investigation is whether or not participants truly understand the mathematical principles involved. Based on data from this section, it does seem apparent that not only are these participants able to compute and solve mathematics dealing with multi-digit subtraction, but that they understand the involved principles such as place value. Further discussion about participants' understanding is explored in the next chapter.

Further Discussion

It may be useful to discuss briefly the German approach to mathematics a bit further at this point. Although participant approaches to teaching, representation, and explanations are in the next chapter, this approach may also have an affect on participant ability to solve such problems. As explained by the participants, the German approach may actually eliminate much of the problems that were presented in the interview scenario, including confusion of place values within the numbers, elimination of using zeroes or needing to "move over" numbers within a problem, and the difference between addition/subtraction and multiplication. The approach of the participants learned from elementary school and beyond, facilitates an understanding of place value and may impact their successful performance. Rather than simply carrying out a set of learned steps for either the computation or word problem, perhaps this unique approach fosters a greater comprehension of the mathematical concept and the mathematical procedure being conducted. Further investigation into this approach and consequences for performance and understanding of the mathematical principles was not addressed by this research beyond what is reported in this chapter and the next in regards to teacher representation, but would certainly be worthy of additional research.

Division with Fractions

The type of mathematical computation met with the most amount of anxiety as perceived by the researcher was in the area of dividing with fractions. The dividing with fractions portion of the interview began with asking the participants to think back about how they were taught to divide with fractions. From this question, answers such as: "I know that I was really confused about it, but I can't remember anymore. I just remember the feeling of stress" (Ms. Muller) were noted. Participants were asked to provide answers to different problems dealing with division with fractions in both the interview and in the survey.

Computation

In the interview, participants were asked: How would you solve a problem like this: 1¾ divided by ½? After varying lengths of deliberation, participants fell into one of three groups. The first group was able to correctly solve the problem. Participants in this group stated the rules involved with multiplying with the reciprocal or simply solved the problem. This group converted the mixed fraction into an improper fraction then multiplied by the reciprocal. For the most part this group explained the steps they were taking as they completed the process.

A smaller number of participants comprised the second group that correctly explained the necessary steps to solve the problem, but did not ever state the actual answer to the problem. These participants were able to state the rules they had learned, which were correct, but they did not go on to finish solving the problem. While it seemed these participants were knowledge in this area no conclusive data about their abilities can be determined, although it would seem to be that these participants would have the knowledge and skills to solve problems dealing with division with fractions. Two participants in this second group approached the question with how they would teach or represent the information to students, but still did not provide an answer. Further discussion about their approach is examined in the next chapter.

The third group was comprised of participants who stated they could not solve the problem or would have to brush up on it before being able to do so. Ms. Muller attempted to solve the problem, but did not. Ms. Schultz explained how she had learned, and that she would need to brush up. Neither participant attempted an incorrect guess.

Participants in the first group took a fairly uniform approach in solving the presented problem. Ms. Schneider's response seems to convey the ease with which this group solved the problem:

> "1 ¾ divided by ½ (Solved problem quickly in head.) 3.5. Right?!"

Ms. Riese also from the first group provided a detailed description of how she would solve the problem, but which also showed the depth of her understanding and how she might represent the concept to students. Ms. Riese said:

> "I would say I have one pizza and ¾ of a pizza. And I would divide it so two students could have the same amount. How do you divide it up? So, if it works with pizza that's a good thing. So, students who already know fractions I would tell 7/4 and then divide it by ½. You have to take the reciprocal and then I would say you times by 2/1 so 14/4 and then 3 ½."

This group of participants indeed had the knowledge, skills, and understanding of division with fractions. Not only could they solve correctly the problem, they were able to state the rules and even generate a representation while solving the given problem. While not all participants in this group went to the same extent as Ms. Riese, certainly this group of teachers demonstrated a sound subject content knowledge of the mathematical principle involved.

Ms. Schwab from the second group aptly represented the whole group with her quick summation of procedures:

> Take the reciprocal, multiply. You just have to memorize it, the end. You have to combine the mixed fraction, the one. And then turn the two over. Well, first convert it to a fraction and then the one and the two and turn it over.

Two additional participants in this group offered similar explanations:

> First I would convert the mixed number into an artificial fraction, so that I have 7/4 and then the reciprocal of ½, so divided by ½ that means multiply by two. (Ms. Gauss)

> Then I explain 7/4 divided by ½ is 7/4 multiplied by 2. That's how it's written. It is difficult. (Ms. Bock)

Again, this group had a sound approach were they to actually work to solve the problem. However, with no answer provided these participants formed a group of their own.

Ms. Muller, from the third group, converted the mixed fraction and repeated the problem softly, but finally stated she could not finish. Upon asking her why she thought this was difficult to do she stated that she was unsure exactly how to proceed. Analysis of what she was discussing revealed that in actuality she was trying to reconcile dividing by two with the stated problem. She said,

> Well, because I go back and forth between making it a whole number… like I just started one is 4/4 plus the ¾ is 7/4. So then I looked at dividing it by ½ is the half. Then I could just take the half from the one, that is a half, but then what do I do with the ¾? I didn't know how I would make that line up. (Ms. Muller).

Another participant in this group explained that she had learned through memorization, and knew it was the opposite of multiplication, but that as a student she had not really looked at the fraction and did not understand "why those strategies were they way they were" (Ms. Schultz). Ms. Schultz explained, "Well, if I ever had to teach it to a class I'd have to brush up on how to do it.... I notice I haven't really learned it in school myself." Although this same participant did, however compute correctly a similar problem on the survey such comments highlight the idea that teacher knowledge is conditional and based on current and/or recent experiences. Ms. Schultz acknowledged that were she to be teaching this topic she would need to update and perhaps reactivate her knowledge of the subject. See Figure 8 for participant approaches to dividing with fractions.

Figure 8
Participant Approach to Dividing with Fractions (n=20)

Correct answer provided (n=14)	70%
Correct explanation, no answer (n=4)	20%
Incorrect answer (n=2)	10%

If it were assumed participants who provided correct explanations but no answers have the knowledge and skills necessary to solve the problem, particularly since half of them provided correct answers on the survey, then presumably 90% of participants interviewed would have been able to compute the fractions problem correctly.

Survey data provided a similar picture. On the survey, participants were posed with the following problem: Divide: 8/35 by 4/15. Four groups emerged from survey data, although it could more simply have been correct or incorrect, further explanation into approaches is provided. The first group correctly solved the problem, the second group was unsure how to correctly solve the problem, another began to answer the question but from the work shown was not using correct methods, and one did not answer the question.

Although participants in the first group all arrived at the correct solution, each participant approached the problem slightly differently. The biggest difference between participants was where in the process they decided to reduce the numbers being worked with. Ms. Schultz provided the most in-depth explanation of how she solved the problem.

1. 8/35 x 15/4 = 120. I divided the two fractions by taking the reciprocal of the second fraction and multiplying.
2. 120/140=12/14. I reduced the fraction by dividing both numbers by 10.
3. 12/14=6/7. I reduced once again by dividing both numbers by 2.

Not only did Ms. Schultz correctly solve the problem, she also explained each step. Three additional participants' approach was very similar to Ms. Schultz. Ms. Kuhn, on the other hand, reduced the numbers prior to multiplication:

8 x 15/ 4 x 35=

2 x 15/35=

30/35

Ms. Kuhn did not simplify the fraction, so although technically an incomplete answer it is evident she had understanding and knowledge of this mathematical concept. Finally, Mr. Pfeiffer's approach appeared quite different, but as with Ms. Kuhn, his approach simplified the fraction prior to multiplication. Earlier simplification may be key to student success since it would lead to easier multiplication steps in the problem. Mr. Pfeiffer reduced across the reciprocal prior to multiplying, resulting in an approach such as this:

8/35 / 4/15=

8/35 x 15/4=

2/7 x 3/1=

6/7

Thus, within this first group there were two participants who reduced and simplified earlier in the process, and three who waited until the end to reduce and simplify. See Figure 9 for a break down of the performance of participants in dividing with fractions on the survey.

Figure 9
Participant Approach to Dividing with Fractions on the Survey (n=8)

Correct answer provided	5
Began to solve, but used wrong methods (did not finish)	1*
Unsure how to solve	1
Did not answer	1*

Of the five participants who did correctly answer the first problem, one did not simplify the answer thus the answer was technically incomplete. This same participant did completely and correctly solve the problem posed in the interview. Two of the participants that provided correct answers on the survey had provided a correct explanation but no answer in the interview, and the third had provided an excuse as to why she could not solve the problem in the interview. The participants who did not finish and did not answer the question did provide correct answers in the interview.

Word Problem

Following the more computationally oriented question, participants were presented with the following word problem on the survey: "Kurt had $240. He spent 5/8 of it. How much money does he have left?" As with the first problem, some participants used correct steps and methods, but did not provide a complete answer. In the word problem, the question asked how much money Kurt had left. In contrast to the previous problems that involved primarily just computation skills, this problem extended the depth of understanding required to correctly solve. Although this problem does not necessarily involve division of one fraction by another, it does provide insight into how participants understand and solve fraction problems. Participant knowledge concerning division with fractions both computationally, as well as in terms of representation was explored in the interview. What is particularly interesting is that in this case all participants used the correct computational methods despite not all of them arriving at the correct solution.

The first group included those participants who solved correctly the problem. All but one of these participants had the same approach of figuring what 5/8 of $240 is and then subtracting that amount from $240 to determine how much money Kurt had left. One participant in this group indicated through her steps that what she needed to know was 3/8 of $240. She solved the problem not in the lengthier approach used by the other participants, but rather simply by finding 3/8 of $240.

The second group also included participants who demonstrated they could correctly divide with fractions. These participants did correctly figure how much Kurt had spent, but did not complete the final step of stating how much money Kurt had left. This may have been a simple omission of the final result, or it may be indicative that of a lack of more extensive understanding of these concepts. Both participants left their final answer at $150, which is what Kurt would have spent, not how much money he would have remaining.

The final group also used correct methods to divide with fractions. However, this group either did not understand the word problem or simply used the wrong numbers in solving the problem. Rather than arriving at a final answer of 3/8 of $240, the participant in this group arrived at a final answer, which was correct, of 1/5 of $240.

Representative of the majority of the first group, Ms. Kuhn's work indicated the knowledge and understanding to convert the word problem into a computation, which she then correctly solved.

$240 / 8= $30.00

$30.00 x 5= $150.00

-$150.00= $90.00

The only participant in the first group to actually directly solve for 3/8 was Ms. Riese. Her work showed the steps she took to solve this problem:

8/8- 5/8= 3/8

240 / 8 = 30

30 x 3= 90

Both approaches allowed the participants to arrive at the correct solution, and while Ms. Riese's approach may seem quicker both indicate a deeper understanding of division with fractions because participants had to translate the word problem regardless of the final approach they used.

Participants in the second group also used correct methods, but did not finish the entire problem. Ms. Schwab's calculations demonstrated how she arrived at $150, but lack of extension to complete the problem is evident by the absence of further work.

240.00 x 5/8 = 150.00

Mr. Pfeiffer also had an approach similar to Ms. Schwab, without actually answering the problem.

As with the previous groups, the final group used correct methods to figure fractions. The problem is that the numbers used and answer provided do not relate to the word problem given. Ms. Muller's work is shown below:

8/8 is 240. A fifth of that is 48.

Ms. Muller is correct that one-fifth of 240 is 48; however, the problem did not involve fifths, and it is difficult to see the connection between the two parts of the work she provided. Ms. Muller did not correctly answer any other problem dealing with dividing by fractions, often stating she was unsure. It seems she may have some understanding of the concept, but it would seem to be quite limited at both the computational level, and in areas requiring further depth of understanding. Figure 10 illustrates participant performance on solving the word problem dealing with division with fractions.

Figure 10
Participant Performance on Word Problem: Dividing with Fractions (n=8)

Correct answer provided	5
Correct method, no complete answer (how much spent, not left)	2
Correct method, incorrect numbers	1

As a reminder, the ninth participant participating in survey data did not complete any of the fractions portions of the interview, though this same participant did provide the correct answer to a dividing with fractions question in the interview demonstrating and verifying knowledge and ability to compute correctly such problems. If these two participants are considered to have the knowledge and skills to correctly divide with fractions, seven of nine participants arrived at the correct answer. Thus, overall seven of eight correctly computed a fractions question on the survey.

Results and Discussion

A precursory glance between the data sets may seem to indicate that participant performance in dividing with fractions was not verified when comparing survey data with interview data. However, data are actually confirmatory, and even more revealing. In addition to the participants who correctly solved the dividing with fractions problem in the interview, three additional participants correctly solved such a problem on the survey.

In all, when considering combined data from the interview and survey, seventeen of twenty participants, or 85% correctly computed a problem dealing with division with fractions. Only one participant was not able to provide an answer, or even attempted to discuss how one might go about solving a problem such as this. This conclusion is reinforced by the fact that this participant reacted similarly in both the interview and on the survey. Two additional participants explained how to solve the problem, but never provided an answer. If it is assumed these participants could follow their own steps to solve the problem then a total of 95% of participants in this group have the knowledge and skills to divide by fractions. Since it appears 95% of participants in this study had knowledge to at least explain, but for the most part also correctly solve problems of this type, it does indeed seem that they possess the subject content knowledge and skills necessary to understand and solve division with fractions.

Novel theory: Perimeter and Area

As with survey data regarding multiplication, problems involving perimeter and area had a high success rate by participants. Participants were asked: Please compute the perimeter for a rectangle with the following dimension, 8 m wide x 5 m long. In this case, eight of nine participants correctly computed the answer with the ninth making a computational error after showing the correct steps to complete the problem. Ms. Muller, the only participant to provide an incorrect answer showed the following steps:

> 16 plus 10 = 36
> The area is 36m

Ms. Muller was able to correctly figure the numbers to be added, but then added incorrectly. This same participant did solve correctly the story problem, so it can be assumed that Ms. Muller does have the knowledge and skills to solve correctly perimeter and area. The eight participants who did provide correct answers to the problem showed different steps in solving the problem. Four different approaches were used. Examples of each approach include:

(Length + Width) x 2
(8+5) x 2 =
13 x 2= 26
The perimeter totals 26 m.
(Ms. Huber)

5m+5m+8m+8m= 26m
perimeter=2(a+b)
area=a x b
(Ms. Richter)

$A=2\cdot a+2\cdot b$ or
$A=2\cdot(a+b)$
$A=2\cdot 8\text{ m} + 2\cdot 5\text{ m}$
$A=16\text{ m} + 10\text{ m}$
$A=26\text{ m}$
(Mr. Pfeiffer)

8 x 2 + 5 x 2 =
16 + 10 = 26
(Ms. Schwab)

Word Problem

The next survey question also dealt with area and perimeter, and presented the following word problem: A thin wire 20 centimeters long is formed into a rectangle. If the width of this rectangle is 4 centimeters, what is its length? As with the first problem, eight of nine participants computed the correct answer. The ninth participant, whose solution to the first problem is the last example shown above, solved the problem making the shape into a square instead of a rectangle. Ms. Schwab solved the problem with these steps:

20 = 4 x X
20 / 4 = 5 cm

As with the participant who incorrectly computed the first problem, it seems that Ms. Schwab did actually understand perimeter and area, but in this case did not use all of the information provided to solve the problem correctly. Steps the other eight participants took include some of the following examples:

> 20 –2 x 4=12
> 12/2=6
> (Ms. Kuhn)
>
> 20-4-4=12
> 12/2=6
> (Ms. Lowe)
>
> 20 cm / 2 = 10 cm
> 10 cm - 4 cm = 6 cm
> (Ms. Riese)

Results and Discussion

Since all participants answered one of the questions correctly, either computation or word problem, and especially since an examination of the steps participants used to solve the problems show an understanding of the concepts even when the correct answer was not provided, it can be concluded that this group of participants has the knowledge and skills needed to solve basic mathematics dealing with perimeter and area correctly. Rather than all participants solving the problems using the exact approach, which might indicate memorized rules and procedures, the variety in approaches taken to solve the problems seemed to indicate a deeper understanding of what perimeter and area entail. How these participants are able to utilize their knowledge of what perimeter and area are, and how they relate to one another is further examined in the next chapter.

Comparison

Data to compare computation and solving skills between Germany, the United States, and China is limited to the division with fractions portion of the interview, as this was the only portion of the interview that specifically asked for an answer to be computed, and is the only area where data for the other two countries is available. The performance of the German participants in dividing with fractions far surpassed the ability of teachers from the United States, and was near the same level of performance of Chinese teachers, based on ability to answer the problem posed in the interview. In previous studies only 43% of teachers from the United States could solve the problem correctly with an additional 9% discussing the correct procedure but not providing an answer, while 100% of Chinese teachers computed the same problem correctly (Ma, 1999). Even when combining the groups from the United States for a total of 52% who presumably possessed the knowledge to solve division with fractions problems correctly, that is still significantly lower than the combined figure of 95% for German participants. See Figure 11.

Figure 11
Comparison: Teacher Ability to Solve Division with Fractions

	Correct Answer	Correct Explanation, No Answer
United States (n=22)	43%	9%
Germany (n=20)	85%	10%
China (n=72)	100%	

Another area that yielded interesting data for comparison was in the area of multi-digit multiplication. It is apparent that approaches in mathematics are not the same from country to country. A seemingly apparent strength of the approach used in Germany to solve multi-digit multiplication problems is the obvious difference between multiplication and addition/subtraction. Rather than students having to remember when to “line up” the numbers, and when to “move the numbers over” is virtually eliminated. Students still benefit from an understanding of place value. Indeed, on a surface level students must still have knowledge of the correct location of the various places, and to a certain degree must still learn where the numbers are intended to be written; however, the basis of how students have been instructed in mathematics does not change. Students are not told to line up the numbers only to be told later not to line up the numbers. The elimination of this potentially confusing aspect as a result of the two topics being written and approached in a similar manner in other countries, including the United States, may be an important factor in the success of German teachers and students in solving such problems.

Conclusion

An examination of the cumulative data from both interview and survey indicates that for the most part these German teachers do possess the knowledge and skills to compute basic mathematical problems correctly. This conclusion is based on the participants who provided an answer whether solicited or not. Ideally survey data would have been available for all twenty participants in order to verify further the knowledge of all the participants who were interviewed; however, the group of participants who did complete the survey provided a much clearer picture since there was more than one data source. Of this group of nine participants all were able to solve correctly at least one question in the areas of multi-digit subtraction, multi-digit multiplication, and area/perimeter, while seven of the participants in this group also solved correctly at least one dividing by fractions question. While there were some instances when participants made minor calculation errors it does seem this group of participants understands the mathematical principles that were investigated.

The compilation of data provides a basis for an examination of this group of German mathematics teachers' knowledge. For the smaller group of nine participants who completed the survey data and provided a closer examination into whether or not participants were able to successfully solve basic mathematical problems it is apparent that this group of participants is in fact able to do so. Examining the larger group as a whole, other participants were also able to provide correct answers, particularly when specifically asked to do so. Although some participants were not able to solve the different types of questions correctly, it still seems that overall this group of German mathematics teachers does possess the knowledge and skills to solve basic mathematical problems successfully. Overall, three participants did not solve correctly a divisions by fractions problem. Survey data is available for only one of these participants, with such data confirming her performance in the interview in that she did not solve correctly fractions on the survey either. However, survey data did indicate that this participant was able to compute and solve word problems correctly in the other three mathematical areas. Further data are not available for the other two participants. Beyond merely solving a problem that was computational in nature, participants demonstrated the ability to extend computational knowledge in order to understand, formulate, and solve word problems. Participants were successful in both areas. Comparatively, German participants were more successful in solving and generating accurate representation than teachers in the United States, but not as successful as teachers in China (Ma, 1999). Having established the fact that participants do indeed possess the knowledge, skill, and ability to both compute and solve word problems in these areas correctly, discussion now turns to whether German mathematics teachers can use the knowledge and skills to correctly solve basic mathematics into the ability to represent such knowledge to others correctly.

CHAPTER 5

RESULTS: ABILITY TO GENERATE ACCURATE REPRESENTATIONS

Possession of the knowledge and skills necessary to compute basic mathematics and to also solve word problems in these same areas correctly would seem to be essential for teachers; however, knowing how to compute and solve basic mathematics problem does not necessarily translate into the ability to represent such knowledge to others correctly. Having gauged the knowledge and skills of the participants in this study, and having found this group of German mathematics teachers capable of computing and solving word problems in basic mathematics successfully, it must now be examined how this group of participants understood each of the four areas, and how they would represent these topics to students. Participants were again questioned in the areas of multi-digit subtraction, multi-digit multiplication, dividing with fractions, and dealing with a novel theory involving perimeter and area. Survey data were collected in this area, but in contrast to the previous section that relied more on the survey data, the primary data for examining whether or not participants can represent their knowledge to others comes from the interview. As with the previous discussion, nine participants completed the survey and will once again be referenced, but on a more limited basis. The interviews with the twenty participants will provide a broader view of how these participants represent their knowledge, based off the confirmation from the smaller group of participants completing the survey, that they do possess the necessary knowledge and skills.

Multi-digit Subtraction

It may seem at first glance that subtraction may very well be the simplest of all the areas this study addresses. However, there are larger underlying mathematical principles, namely that multi-digit subtraction problems deal with place value (Kennedy, Ball, & McDiarmid, 1993) that must be understood by teachers if they are to represent their knowledge in this area to students so they too can understand the principles rather than merely perform steps to complete a problem. As with past studies (Ma, 1999; Kennedy, Ball, & McDiarmid, 1993), participants were asked about the teaching aspect of multi-digit subtraction, with their answers analyzed for further understanding of the principle. In the interview, participants were asked

> How would you introduce double-digit subtraction to your students, for example 64-46? If I were to come to your classroom, what would I see? What problems do students have with this type of problem? How would you know your students understand the principles involved with this type of math?

Again, as with previous studies, key points included in analysis were an examination of whether participants referred to the process as borrowing or regrouping, if discussion centered on place value or simply a series of steps, what approaches they would take in introducing this problem, and also what they perceived to be the most common problem with multi-digit subtraction (Ma, 1999; Kennedy, Ball, & McDiarmid, 1993). Rather than simply coding a response based on terminology used, each participant's answer was carefully evaluated based on the context and meaning of the answer, in keeping with previous studies (Ma, 1999). For example, not all participants who used terms such as place value or regrouping were categorized together. Instead careful analysis to determine the meaning of what was said was used to determine category placement. It is one thing to know a term, and another to understand and apply that term correctly. Employing these methods allows for a better discussion of how participants would represent knowledge and their understanding of the mathematical principles employed.

As students progress from single-digit subtraction to multi-digit subtraction, teachers typically use numbers that will not require students to deal with other place values in the number. For example, students may solve problems such as the following problems that one of the participants gave as examples that he would start with where the subtraction is still relatively straightforward.

$$\begin{array}{r} 17 \\ \underline{-12} \end{array}$$

or

$$\begin{array}{r} 19 \\ \underline{-16} \end{array}$$

In these problems, the ones place value is subtracted from the ones place above it, and the tens place value is subtracted from the tens above it. However, when the numbers in the subtrahend (number in the lower portion of the problem) are larger than the corresponding place value in the minuend (number on top), students must learn how to compute such a problem by regrouping place values in the minuend. A brief explanation of the role and importance of place value precedes discussion of participants' approaches to multi-digit subtraction and place value.

Place value vs. steps

An important principle of multi-digit subtraction is the importance of place value, and the understanding of numbers as a whole, as well as recognizing the role of the various components of the numbers. Place value is important not only in multi-digit subtraction, but also later in other mathematical areas, including multi-digit multiplication, which is discussed later in the chapter. A sound understanding of place value is necessary to understand both.

Based on responses from the interview, participants were grouped according to whether their discussion about multi-digit subtraction included place value, and if so to what degree, or whether their primary focus was on steps of the procedure. A final group consisted of one participant who discussed approaches and focused a great deal on the use of manipulatives, but did not include in his discussion any mention of either place value or steps.

The first group of participants shared in common the fact that they included place value in their discussion of this topic; although it was evident they understood to varying degrees the importance and role of place value in this type of problem. In fact, nearly half of the participants in this group referred to place value, but their explanations and use of place value appeared limited as their responses did not seem to truly delve into the importance of place value. For example, one participant explained that students would,

> … see that they have to break apart one of the bundles. And then to make it visual I'll take the rubber band of one of the bundles and count how many they have to remove. Then I can explain relatively well that I can't count to the four but I have to

> borrow a bundle so I have to count to 14. And then I have to cross the ten because I borrowed a ten bundle. (Ms. Bock)

This participant used place value but did not elaborate on the role of the bundle of the ten within the number; simply that one bundle had to be borrowed in order to make the other number sufficiently large to carry out the computational procedure. Another participant also used the term "place value" when she referred to the need to convert (a ten to ones), but failed to go further with the explanation. "I would tell them that you have to go from the 6 to the 14 because you can't go 6 to 4. So, I have to borrow a ten. And I have to convert it." (Ms. Schwab). Instead of discussing place value most of the discussion of these participants involved explaining the steps required to complete problems such as this. Almost as if explaining to themselves and the researcher how to complete the problem, some participants went through the steps one by one with no mention of place value.

> First I would explain to them add to fourteen and then write the eight under here. And because I already took the fourteen I have to remember a number. Then I figure one plus four is five and then from five to six is one. (Ms. Sanger)

On the other hand, the other half of participants that discussed place value did refer to, and expound on their explanation of place value and the importance thereof in understanding these types of problems. Besides mentioning place value and describing activities such as breaking apart bundles or having to convert a ten, these participants went further. This group of participants discussed place value as an important concept for students to understand as a basic and necessary component. Ms. Huber said:

I would say that either way [you choose to solve the problem] you have to pay attention to the values. Are you working with tens or with ones? I think that it is very important that from the beginning we tell them that numbers really don't tell us anything but the position within the number is extremely important and that if there is one number and that one number can encompass a wide range of value. And I think that lays the ground.

She wanted to ensure her students not only understood the number, but that any given number can be regrouped in myriad of ways. What was important according to Ms. Huber was that students pay attention to the place value within a given number and when solving problems remember if they are working with ones or tens. Another participant who went further in explaining place value said,

To understand is difficult that they can visualize I have tens blocks and ones blocks. That's difficult. That's why it is important to break it out into ones, tens, and hundreds. To go to the place value table. That's difficult to visualize. It's important to say we have to break it out into ones, tens, and hundreds. You have to be able to grasp the numbers, that's essential. (Mr. Reiman)

Yet another participant explained the difficulty students have of understanding place value, but the importance thereof.

You explain you go from 6 to 4, and if something is missing you write a little one in front of it and that is the 14 and then figure from 6 to 14. At first the children really don't understand where that 1 comes from. What are we doing? What are we trading in? How can we just take one from here and put with this one? That's very difficult to understand. Taking a ten and trading it in for ones. So, you really have to do it or

> else they will just write the 1 somewhere and have no idea where it came from, or they forget it. (Ms. Schultz)

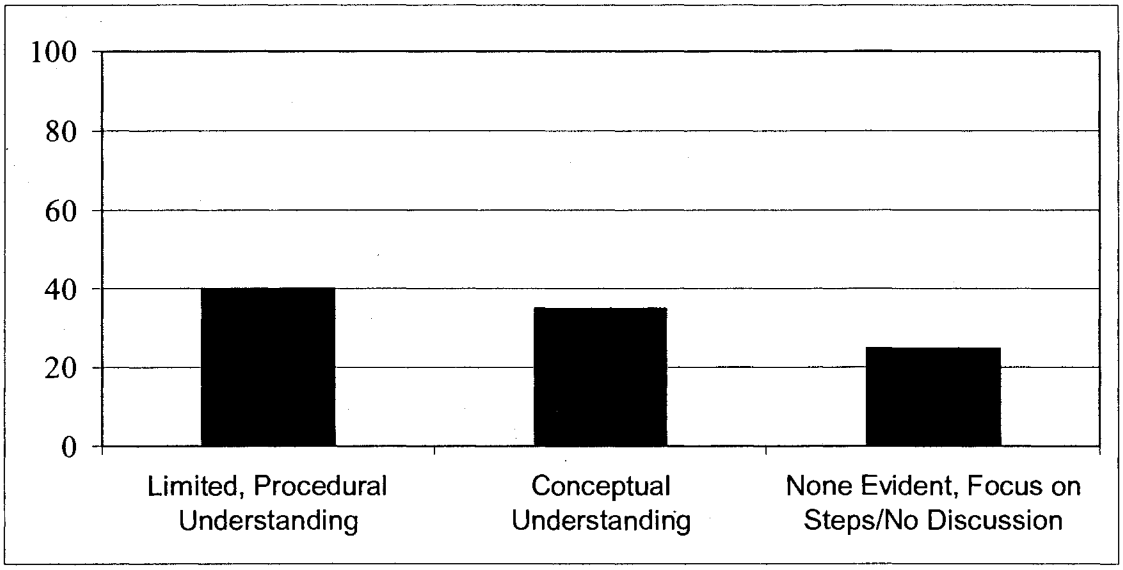
This group of participants not only used place value in approaching teaching multi-digit subtraction, but they also emphasized the importance of having students really grasp and understand the concept. They realized the difficulties students have with this underlying principle, and explained they would need to help the students develop a comprehension of place value otherwise they would "have no idea where it came from or they forget it" (Ms. Schultz).

In contrast to the group who referenced place value, the second group of participants focused more simply on the steps involved in completing multi-digit subtraction problems. Ms. Trachsel's explanation illustrated her focus on steps when teaching and learning this concept:

> First I would start with ones, without the tens transition. And when that is mastered, then we go to the second step. In any case every child comes to the board, so I can see that they can do it. While they do it they have to say the steps out loud. Also I would have lots of practice so they can be proficient in it. First I would explain it on the board and what seems to work is that every child would show the steps and do it in front of me. Mostly I do it like this: We figure it together and then those children who can do it by themselves are allow to work at their own pace; with the others we do it together on the board.

It may be that participants in this group had an understanding of place value as it relates to multi-digit subtraction, but based on responses that focused solely on steps to be completed these participants were grouped accordingly. The percentage of participants in each group is shown in Figure 12.

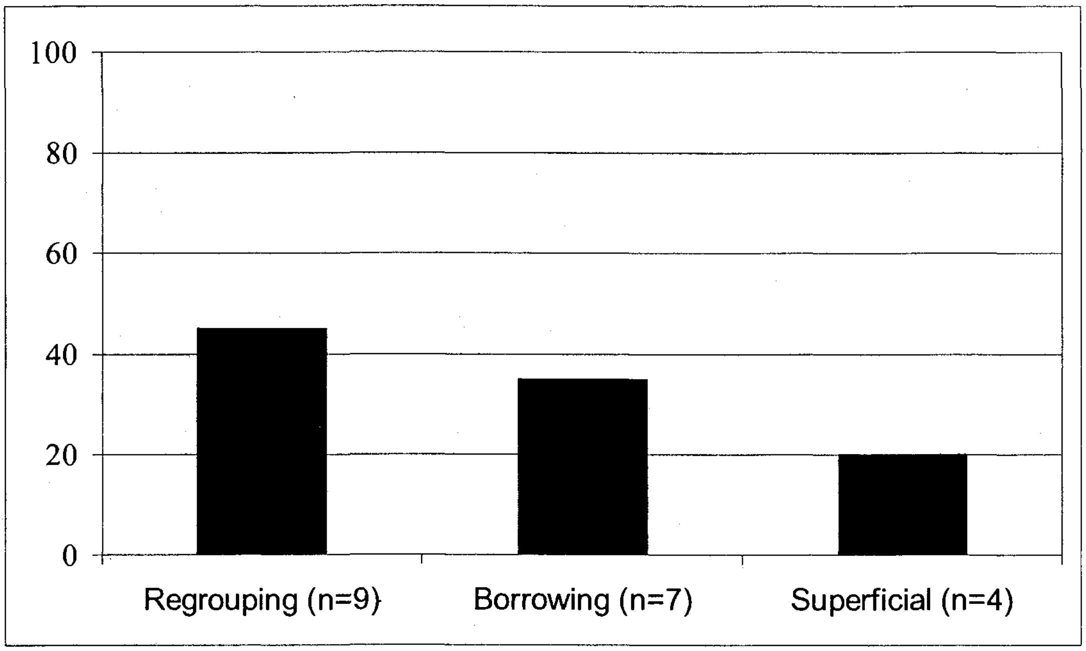
Figure 12
Participants' Understanding of Place Value



Regrouping vs. borrowing

Whereas 100% of responding participants in the previous portion of the study were able to compute and solve word problems in the area of multi-digit subtraction correctly, the same robust performance in understanding the underlying principles was not evident when the participants were asked about the approaches they would take to introduce multi-digit subtraction. In fact, participants fell into one of three categories. As discussed in the methodology section, similar categories were used in previous studies (Ma, 1999); however the third category of "superficial" was not used previously. Four participants from the German sample could not be placed in either the regrouping or borrowing category. These participants discussed different approaches and types of exercises, as well as problem areas for students, but their answers could not be categorized into either group and thus were termed as superficial since data could not determine type or depth of knowledge in this area. The group classification breakdown is seen in Figure 13.

Figure 13
Participants' Approaches to Multi-digit Subtraction



The first category included those participants who explained they would approach teaching this problem through regrouping the numbers involved in the problem. Previous studies have examined and explained the importance of this type of understanding and approach to fully represent to students the principles of multi-digit subtraction (Ma, 1999; Schram, Feiman-Nemser, & Ball, 1989). Only nine of the participants, or 45%, fell into this category. Ms. Riese summed up the root of the mathematical principles involved by stating that, "You have to have an understanding of the number 64 ... They have to have an understanding of the numbers first." While regrouping indicates a deeper understanding of the principle than the other categories, it cannot be said that classification within this category necessarily indicated that all participants in this category possessed the same understanding of the mathematical principles involved in multi-digit subtraction.

More than one participant commented that one could not introduce multi-digit subtraction with the sample problem provided, but rather the introduction would need to begin with smaller numbers that still require regrouping, but at a more basic level. This group of participants went on to explain that given the sample problem there would be a variety of approaches that could be used. These approaches typically entailed regrouping the number(s) so that the computation could be completed successfully. As was seen in the computational portion of this study, this did not always mean that the minuend was regrouped to allow the subtrahend to be subtracted place value by place value as might be seen when merely "borrowing." In fact, it should be noted that only one of the nine participants, or 5% overall, discussed what would be considered the standard method for regrouping numbers by decomposing a ten into the ones place allowing the six to be subtracted from fourteen. Ms. Schultz explained that,

> I do this with a math board, a Roman math board and on the top it shows the ones and the tens. So we have tiles and we put them on there. So, you put four tiles in the ones and six in the tens. Then we think how many are missing from six to four. How many are missing? You move the tiles from the tens to the ones. You have to break down the tens into ones.

The remainder of this group discussed a variety of approaches to regrouping numbers, with six of the nine, or 30% overall, discussing a similar approach to how they would regroup the numbers. For example, Ms. Richter explained and showed,

Very simple. First I would split up the 64 equals 60 plus 4. So and 46 that they can see it again like this, this would be the introductory example. Very important- 40 plus 6,	64= 60 + 4 46= 40 + 6
Now I can for example say from 64 I first subtract the 40. So, they know I figure 60 minus 40 equals 20.	60-40=20
I have the 24 faster here, I am a step ahead in writing…	=24 (added the 4 left from 64)
Now minus six. Now we are at the problem spot, here many children do minus four plus two. That is where the mistakes creep in.	=24-6
So then I figure 24 minus 4 equals 20	=24-4
Minus 2- that is very important, this step is usually where the mistakes get made-	=20-2
Equals 18.	=18

Ms. Richter not only regrouped the minuend, but also the subtrahend allowing students to see that both numbers involved in the computation are able to be regrouped. Although this would not be considered the standard way to regroup the numbers, which would have resulted in 64 being regrouped into50 and 14, Ms. Richter nonetheless was able to illustrate how she would regroup the numbers in a manner that according to Ma (1999) might be better suited for a given problem.

Those participants who fell into the borrowing group also discussed converting and changing tens to ones, but they spoke about "borrowing" or "getting" a ten. This group included seven participants, or 35% of the group. These participants made comments such as "From 6 to 4 doesn't work. We have to get a ten, and then we have 14. (Ms. Roth)" while also explaining that "If we need a ten we get a ten and change it to ten ones." (Ms. Roth). Thus, on the one hand they discussed the fact that a larger number cannot be subtracted from a smaller number necessitating the need to "get" or borrow a ten while on the other hand they discussed that the process of getting a ten also required that that ten be changed into ten ones. There appeared to be an inconsistency with how deeply this group of participants understood, and perhaps approached multi-digit subtraction. Certainly there was a focus on converting or changing a tens into ten ones; however, there seemed to be a lack of explanation how and if this affects the overall number. Another participant in the borrowing group, Ms. Trachsel, stated that:

> First I would explain: ones minus ones; tens minus tens; here you already have a ten transition. So I learned it this way and I would teach it the same way to my students. From six to four doesn't work, so six to fourteen, that would be eight, so I borrowed a one here so I have to remember the one here; then this is five then from five to six you need one, so add.

A slight difference from the United States, rather than crossing out the number and writing the next smaller number (i.e. cross out 6 and write 5), participants would write a small one near the four and then add those two numbers together to be subtracted from the six.

$$\begin{array}{rr} ^{5}\cancel{6} & ^{1}4 \\ \underline{-\ 4} & \underline{6} \\ 1 & 8 \end{array} \qquad\qquad \begin{array}{rr} 6 & ^{1}4 \\ \underline{-_{1+}4} & \underline{6} \\ 1 & 8 \end{array}$$

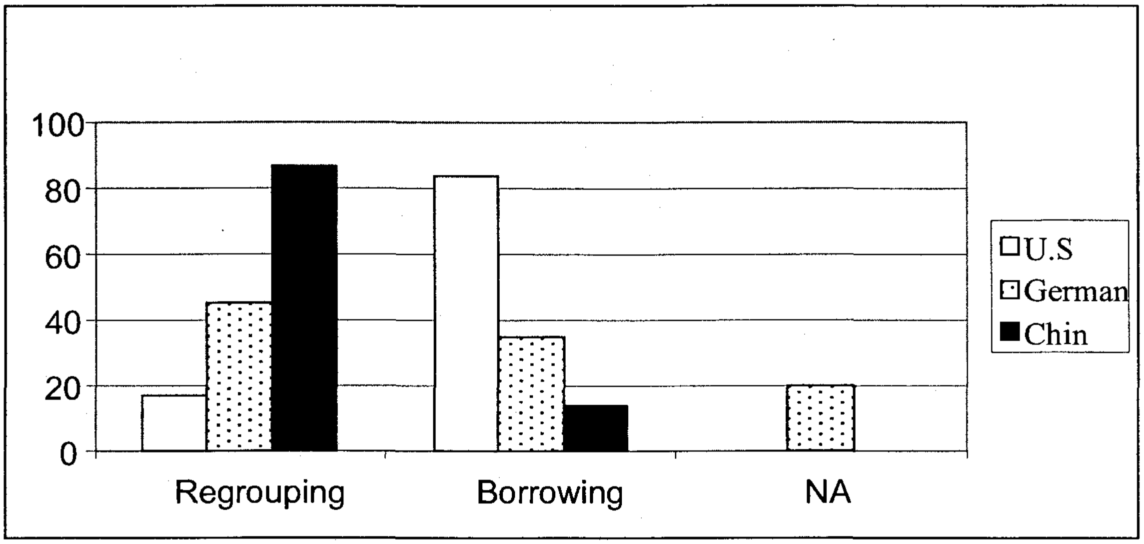
Another participant in this group also employed the approach of borrowing, but through the use of addition to solve subtraction. Ms. Schneider said, "To complete we calculate. How many are missing between 64 and 46? How many are missing between 6 and 14—8. Then we're at 54. How many from 5 to 6- 10, so 18." Similar to results of some teachers in the United States, (Schram, Feiman-Nemser, & Ball, 1989), this group did not discuss the process of regrouping, but rather stayed at a superficial level of borrowing from one number to allow the other number to be large enough to be subtracted from. The approaches used by these participants were procedurally focused in remembering that one was taken from the tens and that must be remembered when finishing the calculation. This is evident in Ms. Trachsel's explanation of her approach. It would seem the approach used by these participants to "borrow" rather than regroup the numbers indicates a deeper understanding of the mathematical principles than those who used regrouping.

The third and final group was made up of four participants, or 20% of the study's participants. These participants did not discuss regrouping or borrowing when explaining the approaches they would use to introduce multi-digit subtraction. One of these participants, 5% overall, also discussed using addition to solve subtraction as the approach they would use, but only gave examples of problems. Not enough information was provided to ascertain whether this participant would fall into the borrowing group since the problems discussed were basic beginning problem and the multi-digit subtraction problem posed was not addressed. It cannot be determined which of the two previous groups these participants would fall into; however, the fact that these participants did not discuss vital aspects of multi-digit subtraction may be an indication of superficial knowledge in this area.

Previous studies in the United States and China also grouped participants in the respective countries according to whether or not they approached multi-digit subtraction by regrouping or by borrowing (Ma, 1999; Schram, Feiman-Nemser, & Ball, 1989). The results of this study serve to broaden understanding about teacher knowledge and the approaches they would use to represent such knowledge by providing an additional data set, and thus this group of German mathematics teachers will be compared to the data gathered from teachers in China and the United States.

Discussion may seem somewhat narrow in scope, focusing on the German data; however, since this study focuses on findings from this group of German mathematics teachers, so necessarily does the discussion. As a point of reference though, it is important to illustrate the differences in results from each of the three countries. A comparison of teachers from China, the United States, and Germany and the percentage of teachers from each country that approached multi-digit subtraction with regrouping or borrowing can be seen in Figure 14.

Figure 14
Comparison: Approaches to Multi-digit Subtraction



It has already been documented that Chinese teachers seem to have a deeper understanding of multi-digit subtraction, with a majority of teachers, 86%,.using a regrouping approach to represent and solve multi-digit subtraction (Ma, 1999). Furthermore, 35% of Chinese teachers also expounded on regrouping by providing multiple ways of representing how the numbers could be regrouped and were also able to explain why the different ways of regrouping would be beneficial based on the specific numbers contained in the problem (Ma, 1999). Chinese teachers were able to provide lengthy and detailed explanations on the topic including phrases such as, "I will explain to them that we are not borrowing a 10, but decomposing a 10. (Ms. S)" (Ma, 1999, p. 9). Teachers from the United States approached subtraction with regrouping at a much lower rate of 17%; however, statements of these teachers indicating understanding of regrouping, "have to understand how exchanges are done (Faith)" (Schram, Feiman-Nemser, Ball, 1989, p. 5) also indicate a conceptual understanding of subtraction with regrouping (Schram, Feiman-Nemser, Ball, 1989).

In contrast, a majority of teachers, 83%, in the United States used a borrowing approach indicating a shallower understanding of this topic (Ma, 1999; Schram, Feiman-Nemser, Ball, 1989). As with some German participants, comments by participants in this group focused on the steps, the procedure, and that "you must borrow (Fay)" (Schram, Feiman-Nemser, Ball, 1989, p. 4) when computing and solving these types of problems. Only 14% of Chinese teachers took a borrowing approach to multi-digit subtraction, including phrases such as "you should borrow (Ms. Y)" (Ma, 1999, p. 7). Teachers from all three countries who approached subtraction with borrowing seemed to focus on procedure rather than underlying principles.

The German participants were more evenly disbursed, without a majority using regrouping or borrowing. The fact that 20% of German participants could not be grouped though frustrating does not necessarily change the fact that German participants are nearly evenly split between the two groups. It can be concluded from comparing these three groups of teachers that more German participants seem to have a better understanding of the underlying principles involved in multi-digit subtraction than those teachers from the United States' sample, but that more Chinese teachers have a deeper understanding than participants from either Germany or the United States. Participants in this study also did not go the extra step as 35% of the Chinese teachers did to explain multiple ways of regrouping numbers or why/how this would be beneficial in computing problems of this type.

Approaches

Interview data revealed that there were myriad of varieties German participants would employ to introduce and teach multi-digit subtraction. Most of them also discussed more than one approach they themselves would use. These approaches ranged from using addition to complete subtraction problems to introducing the topic then letting students experiment with it to find ways of solving, to workbook practice, to problems on the board, and finally the use of manipulatives. Over half of the participants, 60%, specifically mentioned manipulatives they would use.

Manipulatives

For the most part, the use of manipulatives entailed bundles of sticks or toothpicks that could be broken down and counted into bundles. Ms. Gauss explained that,

> At the very beginning when I introduce it then we would see on the board a place value system, you would see symbols for the ten, we would do a lot of dissolving of sticks of tens, we would separate a stick of tens, as this problem requires, and convert this stick to ten ones.

A similar type of manipulative discussed by other participants was described by Ms. Schultz.

> Yeah, every student has one of these math boards, and the tiles- tens and ones tiles and we practice those kinds of problems always with the tiles. They have to trade the ten tiles for ones tiles so they actually do it. They trade it, bundle it again. That's why everyone has such a board. It's a process. We have a big one for the board, but every student has their own so they can try it for themselves.

Ms. Schultz, Ms. Gauss, and the other participants who discussed using these types of manipulatives seemed to believe that providing concrete experiences of this nature would help students truly visualize and grasp the concept of place value by being able to physically break down and trade in ones and tens as the problems might require. In this manner students would understand not only the components of a given number, but also how one would then complete computations of this nature. This group comprised 50% of the overall group of participants, and almost all of the participants who specifically mentioned manipulatives.

There were, however, other approaches and manipulatives described that participants would use that would not seem to help students understand underlying principles, rather the use of this second type of approaches and manipulatives seemed to only help students complete computations. Ms. Kuhn described one such example.

> Well we have a very clever math book. And I know that we would work with a driver's tachometer. You drive, for instance 40km, or you already drove 40 km, problems of this kind and how many more will you be driving? We would do it as an add- to problem. The add-to problems are included in this problem. That is how we begin. We would build ourselves a disc. Every child has his or her own on their desks or together. So that everyone has something they can turn themselves. From here to there, how far do I have to turn?

The use of this type of approach and manipulative does not seem to address any of the underlying principles of place value or regrouping. In fact, the entire approach is not subtraction at all, but addition to solve subtraction. Using addition to solve subtraction was one of the methods and approaches 25% of the participants said they would use to teach multi-digit subtraction. Only Ms. Kuhn, who represents 5%, did not elaborate or include other types of approaches, but at the same time only one other participant in this group described other manipulatives with the remainder discussing alternate approaches but not other types of manipulatives. Thus, while this type of approach and manipulative does not seem to address important mathematical principles involved in multi-digit subtraction all but one of the participants also said they would employ other approaches. The last participant who specifically mentioned manipulatives simply mentioned she would use manipulatives, but did not elaborate on the type stating that it would depend on the school. Use of manipulatives by teachers from the United States also seemed to range from a classroom tool to help students complete computations to a meaningful lesson on representing the concept, and to aid students visually (Schram, Feiman-Nemser, Ball, 1989).

Relying solely on interview data necessarily limits analysis and discussion about types of approaches this group of participants might employ when teaching multi-digit subtraction; however, based on the data these participants did provide it would seem that virtually all of the participants uses more than one approach to introduce and teach multi-digit subtraction. When manipulatives were specifically discussed it appeared that the participants are using the types of manipulatives that can lead to an understanding on the part of students of important underlying principles involved in multi-digit subtraction. The fact that only 50% discussed these manipulatives might be an indicator that not all German mathematics participants represent these underlying principles in ways that would lead to better understanding on the part of students; however, further study and observations would be necessary to provide a more accurate evaluation.

Experimentation

Another common topic discussed by 35% of the participants was that there is not necessarily only one correct way to compute multi-digit subtraction correctly. These participants as a group explained that in addition to explanations and practice they would either let their students experiment or come up with their own methods so long as they could explain their steps and methods in solving the problem correctly. Ms. Heinz was one participant who discussed having her students experiment with the topic.

> Before I do an example on the board I let the children experiment. I always teach the children that there are different ways to solve the problem and sometime during this they will recognize; this is the easiest way. I would do it step by step, think about one place value, ten place value and always from the bottom to the top, that is very important. And always before I introduce something like written addition, which we just introduced; always the place value table. That is the Alpha and Omega.

Because Ms. Heinz is apparently comfortable in allowing students to experiment before even introducing the subject, it seems indicative of a deep understanding of the subject matter and underlying principles. She went on to explain not only experimentation, but also how she would introduce the subject and constantly refer back to place value since the place value is really the basis for these problems. Ms. Huber was another participant who discussed allowing students to find and use their own method.

> And there isn't one right way that works for everyone. It's very important that the weaker children have rules that they can repeat step by step. But, it's just as important that through these rules we don't inhibit the more advanced students to often times work in different ways more efficiently.

Of particular interest was the fact that she felt that "weaker" students needed to rely on rules and steps in order to correctly compute these types of problems. The dichotomy in encouraging advanced students to find their own methods yet having weaker students rely on steps seems to evidence that not all students are brought to the same deep understanding of underlying principles. This could be an indication of the ability of certain participants to represent mathematical knowledge in various ways in order to help all students fully grasp mathematical principles.

Difficulties

The main difficulty with multi-digit subtraction discussed by the German participants was the crossing of the tens, or the carry-over. Although 5% of participants stated there would not be any problems unless students had no mathematical understanding, and 10% discussed students mixing up numbers and making systematical errors, 50% of the participants (including the previous 10%) explained the problem with crossing the tens, the carry-over, or differences in place value. These participants expressed not only the difficulty, but how they might also address these problems. For instance, Ms. Muller said

> You lay your numbers out. And so when it crosses the ten then you practice taking things away. So you can see because crossing the tens with a minus in math, is from the thought process very difficult. So, they have to actually do it so they can see that you have to divide the ten so you can borrow. We show there are different possibilities how a person can do this.

While Ms. Muller discussed the difficulty with crossing the tens, Ms. Schneider explained systematical errors: "Or, they say 6 from 14 is 8 and then forget the carry-over and from 4-6 is 2 so they answer 28. Those are two common mistakes. Systematical errors." Of the participants who mentioned specific difficulties students are likely to have, the consensus seemed to be the carry- over from one place value to the other.

Summary and Comparison

A majority of German participants discussed multi-digit subtraction in terms of place value; however, it was evident that not all participants who discussed place value had the same understanding and regard for the role place value plays in multi-digit subtraction. Their explanations of how to compute as well as how they would approach teaching this principle indicated various levels of deep understanding of place value. While 75% of participants discussed place value, 35% seemed to have a limited understanding of the importance of place value. Additionally, only 45% of the participants discussed multi-digit subtraction in terms of regrouping. These same 45% of the participants were also in the group who had discussed place value. Further analysis revealed that 20% of the participants who appeared to have limited place value understanding, but who had discussed place value, approached the problem not through regrouping, but by "borrowing." This would seem to indicate that roughly half the participants understood the importance of place value and approaching this problem with regrouping. While all the participants were able to solve this type of multi-digit subtraction problem correctly, only about half had a deep understanding of the underlying principles of place value and regrouping.

These results differ from findings in both the United States and in China. While only about half of the German participants discussed and approached multi-digit subtraction through place value and regrouping, this was more than teachers in the United States. Studies indicated that in contrast to the 45% of German participants, who held a conceptual understanding of this concept, only 17% of teachers from the United States did and a much greater 86% of Chinese teachers did. Although participants had the subject knowledge and skills to compute multi-digit subtraction, the knowledge to represent it to others seems to be lacking for teachers from both Germany and the United States.

This comparison would seem to indicate that German teachers are somewhat more prepared to represent mathematical knowledge to students than their counterparts in the United States, but less prepared than their Chinese counterparts. At this point in the discussion, this could lead to an assumption that German mathematical performance should perhaps exceed that of the United States but not of China. As the discussion of the data continues it remains to be seen whether this continues to hold true.

Multi-digit Multiplication

Multi-digit multiplication is another area considered to be a basic part of elementary mathematics. As with the previous section on multi-digit subtraction, place value is an important underlying concept of this mathematical principle, and it must be understood that the problem entails multiplying the ones, tens, and hundreds of the parts of the problem using the distributive property (Kennedy, Ball, & McDiarmid, 1993). During the interview, participants were posed with a problem dealing with multi-digit multiplication written in the format used in past studies (e.g. Ball, 1989; Ma, 1990; Kennedy, Ball, & McDiarmid, 1993). Participants were presented with the following scenario.

> Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate 123 x 645 the students

seemed to be forgetting to move the numbers over on each line. They were doing this, instead of this.

$$\begin{array}{r} 123 \\ \underline{X645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array} \qquad\qquad\qquad \begin{array}{r} 123 \\ \underline{\text{x } 645} \\ 615 \\ 492 \\ \underline{738} \\ 79335 \end{array}$$

While the teachers agreed this was a problem, they couldn't agree what was the best way to solve the problem. What would you do if you were teaching sixth grade and you noticed your students had this problem?

Responses to this scenario proved quite interesting, and also very different from what has been found in previous studies.

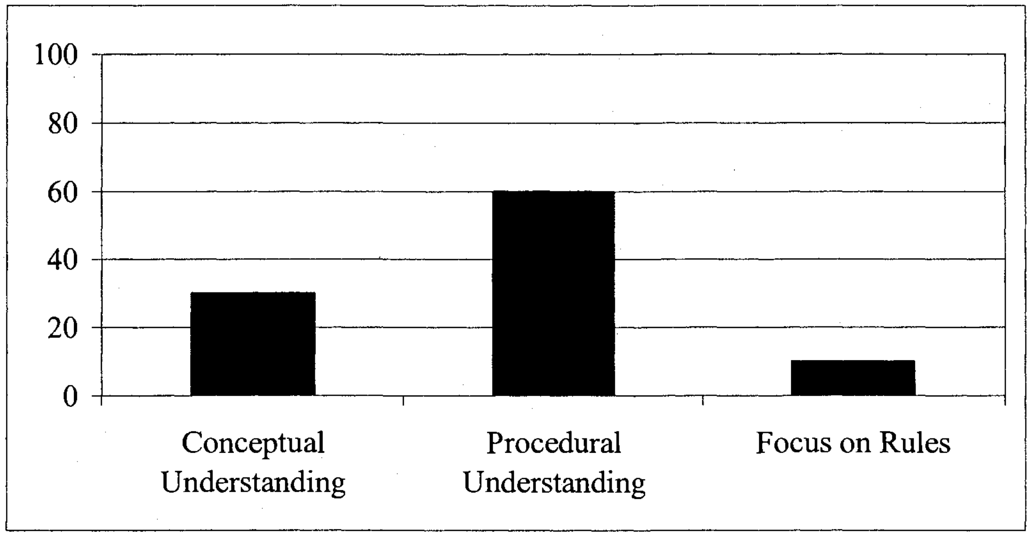
Not only did the participants feel they would probably not have such problems, the entire approach and process for multi-digit multiplication in Germany is completely different from that in the United States. As was discussed in the previous chapter, almost without fail the question could not be finished without interruption on the part of the participant being interviewed to question the format of how the problem was written, to state that that is not how such a problem would be written in Germany, and also to call into question the logic of writing it in such a manner. An overwhelming 90% of participants referenced and rewrote the problem into the German method before proceeding. These participants felt quite certain that by teaching using the German methods most, if not all, potential problems presented in the scenario would be avoided. Again, as discussed in the previous chapter, participants would represent this type of mathematical problem in the same manner they themselves solved it. If, for some reason, problems of a similar nature should arise, or students had difficulty with a multi-digit multiplication problem, this group of participants said they would simply go back a step and review fundamentals of the procedure.

Place value

Understanding and explanation of the role of place value in this type of procedure was also interesting. Whereas only 75% of German participants discussed place value with varying degrees of understanding when faced with multi-digit subtraction, 95% of participants demonstrated an understanding of place value in order to represent knowledge of this procedure to others. This interesting discrepancy will be discussed further at a later point. The fact that nearly all of the German participants referred to place value when explaining how they would approach this topic spoke to their understanding of underlying principles involved in multi-digit multiplication.

As with subtraction not all participants who spoke of place value seemed to have the same deep understanding of the importance of place value. Participants seemed to fall into two categories: limited, procedural discussion of place value (30%), and more conceptually based understanding with extension of applicability (60%). See Figure 15.

Figure 15
Participants' Understanding of Place Value in Multi-digit Multiplication



These categories again are similar to those used in previous studies (Ma, 1999). Thus it seems evident that although this group of German participants understood to various degrees the importance of place value in multi-digit multiplication, only two-thirds of the group had the deepest type of ability to understand, apply, and explain how place value was important. It should be noted that the remaining 10% of participants focused solely on rules to be remembered. One of these participants stated that:

> The number I start with, underneath that number is where I have to start writing the results. I would say that's the rule. You cannot discuss why you drive on which side of the road, it's simply so. You simply remember that the first number you start with that's the number you write the result under. (Ms. Muller)

Discussion thus focuses on the remaining 90% of the participants who did use place value when representing knowledge to others.

Participants from the first group used phrases and terms dealing with place values, but as with their counterparts in the United States (Ball, 1991) discussion seemed limited to places rather than place value. Participants had explanations such as:

> I start with the hundreds, but you have to be careful to write it precisely under where it goes. It's also important to use graph paper. Then, the next number, 492 underneath. And also here, figure the ones. The 615, that's right. 615. Put it under here. So, you have the hundreds, the tens, and the ones. Then you can add and then you figure this way. (Mr. Reiman)

This response indicated some understanding of place and perhaps even place value, but with no further elaboration it would seem that the focus is on places rather than the actual place value, particularly when referencing the importance of writing the numbers in the correct box of graph paper, which was cited as being used by virtually every German participant in the study. Another example of limited understanding and/or use of place value is evident in the statement of another participant that indicated place value is more of a help to know where to start.

> And I start with the six, then with the four and the five and then I have the one place value already here as the one place value and then they just have to only figure it under each place. You just have to take care to write in the proper box. (Ms. Kuhn)

One participant in this group made an interesting comment on the procedural differences concerning not only the difference in writing multiplication between the United States and Germany, but also possible effects on multiplication and other areas of mathematics as well. Mr. Pfeiffer said,

> When I take a problem like this and have the students write it on graph paper they can remember much easier how. I have experience with this. I could imagine that this could be difficult. [Referring to the style of writing multi-digit multiplication in the United States.] This is also the way we write it when we add or subtract. Then we write this way. It's a different math problem and a different way to write it. In that case, well, you do an x here you write it like this and it would be a multiplication problem. Then many would simply ignore the multiplication sign or forget to write it. And then they would start to add or to subtract. So, it is easy to confuse. So, from this angle I would not do this.

Although his comments did not focus on place value or other underlying mathematical principles, his observation holds valid points for why some students may have a difficult time remembering to "move over" the numbers when multiplying.

Of the participants in this group, all but two stated that they did not teach this subject, could not remember how they had taught it in the past, or had taught at a higher-grade level that would not have multi-digit multiplication introduced in. One of the participants who did work at the level at which multi-digit multiplication is taught explained her limited knowledge:

> … my elementary school experience is limited to these two years. In the fifth and sixth grade they just said this is how we do this. So this was our method to explain and make it clear to the children that if they do it this way they can avoid this mistake. (Ms. Gauss)

While the underlying principle of place value is of course a basic mathematical principle, a commonality of this group of participants was that for all but one they were not teaching place value and felt that they could not explain it as well. Place value for them was not contextually situated in the curriculum they were teaching. Although two participants were indeed teaching levels that would include concepts requiring place value, one felt her previous teaching experience had not required her to teach multi-digit multiplication (or subtraction) and she expressed she simply was following the outline provided. For this group of participants the extent of place value did not seem to play a large role in representing their own knowledge to others. Perhaps they did not have a deep understanding of place value, and perhaps their knowledge of place value was not evident due to the fact that it was not contextualized into the type of teaching knowledge they were currently utilizing. Other participants who did not teach this particular concept did, however, understand place value despite not necessarily being in a position to teach multi-digit subtraction with the underlying principle of place value in their current level.

The second group of participants comprising 65% of the participants in the study appeared to have a deeper understanding of place value as evidenced by the manner in which they would represent such knowledge to others. Rather than simply using terms such as "ones" and "tens" to refer to a place within a problem or where numbers should be lined up, these participants expounded on what the places meant, and how they would both solve and teach such a problem. One participant frankly stated that not understanding place value was problematic. Ms. Riese said, "So, maybe they don't recognize ones, tens, and hundreds. And that's the problem." These participants also discussed the importance of students understanding what the number represented. Ms. Huber explained this quite nicely, "The problem is not moving the numbers over, but learning where the numbers have to be written. You have to understand what this number represents. What value is assigned to the number." Ms. Roth also explained understanding what the different numbers within a number represent.

> Now times a hundred this is 600. Then we first put in the zero and then start to figure. Times 40 this is a tens, the 40. And thus we avoid making the mistakes. Mmmh?! This is very important, the place value table. We orient ourselves very strongly...

Another commonality of these participants was not only explaining what they were multiplying, but that they would all also break the problem apart into the various steps so students could see they were multiplying ones or tens or hundreds. "You'd have to explain perfectly to them that they are multiplying times ten, times one hundred, times one thousand …(Ms. Heinz)." Particularly if students were having trouble with this type of a problem, this group of participants referred back to place value and needing to go back to dissecting the problem into the various components (with the literal translation of this approach being half-writing) so that students truly understood how place value affected what was being multiplied.

> I multiplied the hundred, then the ten, then the ones, and then added. Before we do the writing down of the multiplication problem we do it in half writing way. For instance, we do it first with the tens and then with the ones and later with the bigger numbers, maybe with the hundreds. And then this half writing way you learn where to put the numbers correctly so you can later add them together. (Ms. Huber)

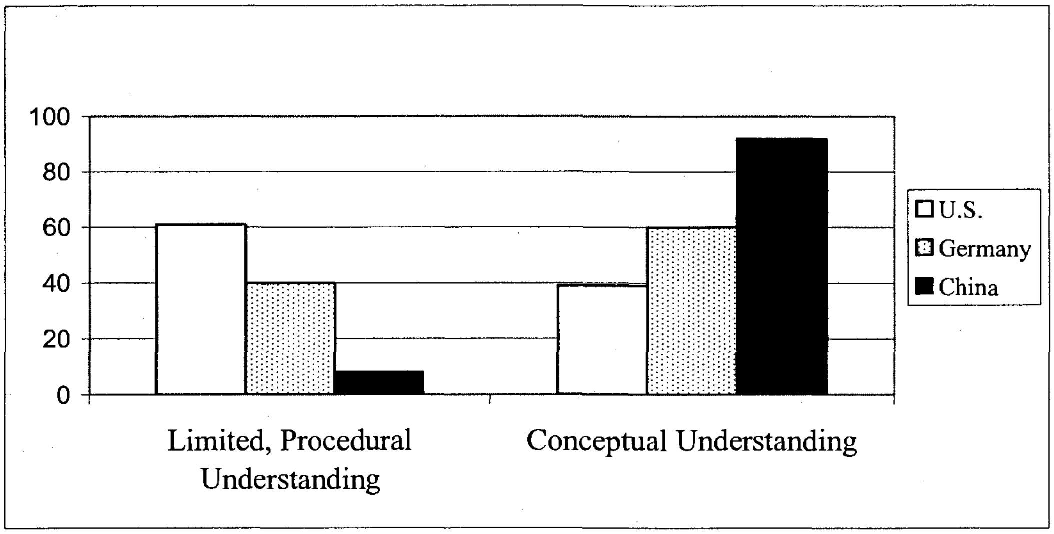
The explanations provided by this second group of participants indicated that besides merely learning to compute multi-digit multiplication in the prescribed German format, they would represent their knowledge to students so that the students would truly understand the various components within a number and why certain digits within a number had a higher value. How these participants understood place value went far beyond a procedural approach to completing multi-digit multiplication, and truly evidenced a thorough and conceptual understanding of place value. By including in their representation of their knowledge a discussion of place value, this group of participants could presumably help students learn not only to correctly compute problems of this nature, but to also grasp underlying concepts that explain the why and the how of this mathematical principle. However, does representation by the teacher that involves place value necessarily lead to better student achievement?

Summary and Comparison

All participants in this study were able to compute the provided multi-digit multiplication problem correctly, as well as solve the provided word problem also in multi-digit multiplication; however, it is the underlying understanding and approach to representing such knowledge that seems to vary quite a bit both within this study, and between previous studies (Ma, 1999). As reported in this study, 60% of German participants had a conceptual understanding of place value and 30% had more of a procedural understanding of place value. The remaining 10% of German participants did not discuss place value at all, which can perhaps be taken as a lack of the importance and conceptual understanding of place value. Thus, for purposes of comparison these participants will be grouped with the procedural understanding group brining the procedurally focused group of German participants to 40%.

In comparison, 92% of Chinese teachers and 39% of teachers from the United States demonstrated conceptual understanding while 8% of Chinese teachers and 61% of teachers from the United States did not provide evidence of conceptual understanding of this topic (Ma, 1999). See Figure 16.

Figure 16
Comparison: Teachers' Understanding of Place Value in Multiplication



Teachers from the United States and China who seemed to have a procedural understanding of place value and multi-digit multiplication discussed similar approaches and reasoning as was found from the German participants (Ball, 1991; Ma, 1999). Procedurally focused participants made statements about the importance of "lining up" and "shifting over" (Ball, 1991); similar to concerns of the German participants to write directly underneath the number they were dealing with. On the other hand, conceptually directed participants from the United States demonstrated place value versus simply place by stating the need to explain the difference between multiplying 123 x 4 and 123 x 40 (Ball, 1991). The overwhelming majority of Chinese teachers who demonstrated a conceptual understanding were further categorized into three groups based on their explanations (Ma, 1999). Teachers were grouped into distributive law (similar to the

German participants who "broke apart" the problem), place value (participants who did not see the need to reveal then eliminate zeroes), and teachers who combined both distributive law and place value (Ma, 1999). Teachers in the first group had similar demonstrations as shown by the German participants in completing the multiplication problem utilizing distributive properties and clearly showing students how the various place values worked in completing the computation (Ma, 1999). The second group's focus on place value also produced detailed and specific comments about the importance of students understanding place value (Ma, 1999). Perhaps the reason German participants did not have similar discussions comes from the fact that it is not common practice to introduce then eliminate zeroes when using the German method of multi-digit multiplication. Nonetheless, it does demonstrate deep understanding on the part of the Chinese teachers. The final group's combination of the distributive law and place value would seem to show the deepest understanding of all. As with multi-digit subtraction, the German participants from this study fall in between findings from China and the United States. A greater percentage of German participants would seem to have a deeper understanding of place value and multi-digit multiplication and to then be able to represent this to students; at the same time this percentage is lower than that found of the Chinese teachers (Ma, 1999). In comparing not only the percentage of teachers with a conceptual understanding, but also the different types and depths, it seems that some Chinese teachers possess different and at times more in-depth understanding of place value and multi-digit multiplication than teachers from either the United States or Germany.

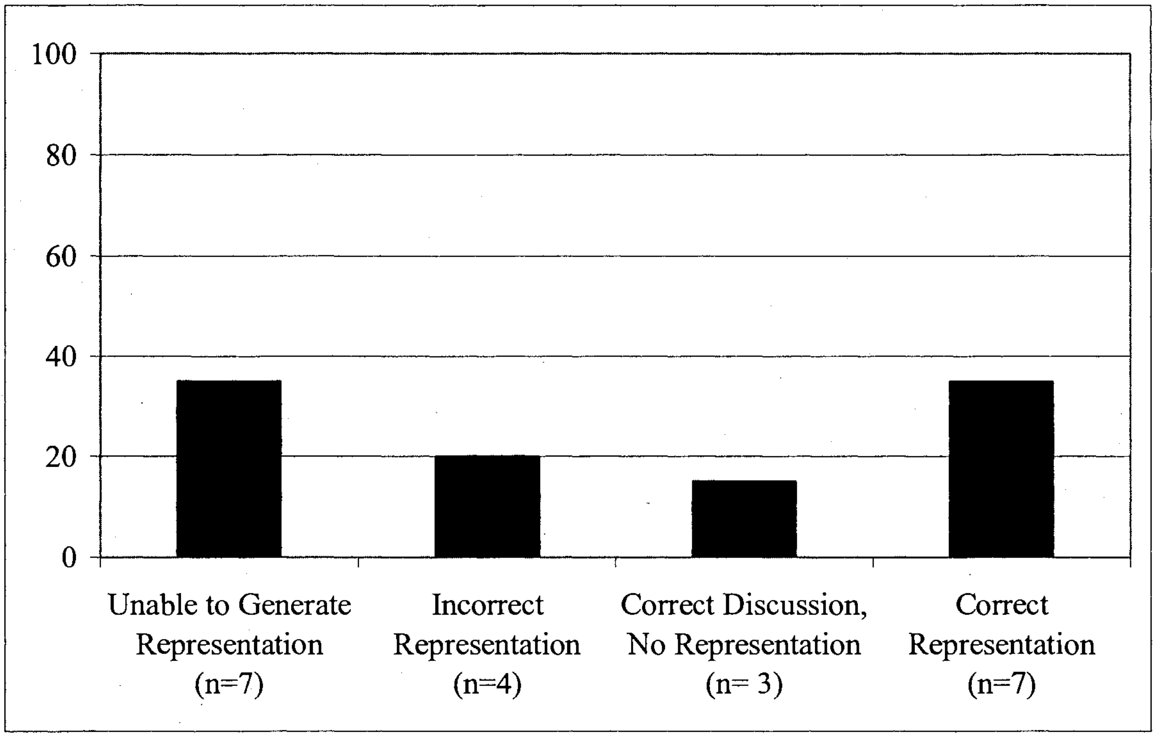
Division with Fractions

Participants who appeared stressed about answering problems dealing with dividing with fractions seemed equally or more so flustered when attempting to correctly represent such a problem. Performance in developing a representation was not nearly as good as in computing. Granted, these participants were put on the spot during the interview to try to think of a representation off the top of their heads. This could prove difficult for someone who under other circumstances might have been able to complete this task. On the other hand, it would seem that if a teacher truly understood the mathematical principle they would be able to generate an acceptable representation. After asking participants to compute a problem involving dividing by fractions, they were asked

> Something many teachers do is try to relate a problem to a real-world situation. This can be very difficult. Can you think of a story problem or real-world situation this might apply to that you could use to help teach your students? Many people find this difficult to do. Why do you think it is difficult to do? (To come up with a story problem that fits this type of fraction problem.)

If a participant did not explain how their representation would fit with the problem (1 ¾ divided by ½) they were asked to elaborate. Participants were placed into one of four categories, based on their responses. Two categories of participants did not display the ability to generate a correct representation during the interview and fell into one of the following categories: can't think of a realistic representation/disconnect between the problem and real-life, and incorrect representation. These two groups combined to account for 55% of the participants. The remaining 45% of participants were also divided into two categories with varying indicators of ability to correctly generate a representation. These categories included: correct discussion of how to solve division with fractions but no actual representation provided and correct representation. See Figure 17.

Figure 17
Participants' Ability to Generate an Accurate Representation



Unable to Generate Representation

Perhaps somewhat surprising was the fact that of the 55% of participants who could not generate an accurate representation, the majority of them simply could not generate a representation at all. Although the responses of participants who did not display the ability to accurately represent the given problem varied, there were also many similarities. Despite the fact that this group of participants that comprised 35% of the group did not generate accurate representations of the problem, four of them had correctly computed the problem, two more had correctly explained how to compute the problem without providing an answer, and only two had been unable to correctly compute the problem. Thus, four participants who appeared to have the knowledge and skills to compute division with fractions- two participants who had correctly computed the problem, and two more participants who had correctly explained how to compute but who had not provided an answer- did not explore the discrepancy between their abilities to solve the problem but not being able to generate a representation. To these participants, there simply did not seem to be a realistic story problem they could provide to their students. Some participants did extend their line of thought to discuss the discrepancy or what they might do. These three participants included two who had correctly computed the answer and one who had not.

A common perceived problem was that it was impossible to have a half of a person or some other realistic representation that students would understand. These participants for the most part did attempt to generate a representation, but in their minds it simply could not work. While working on a representation, participants from this group made comments such as:

> I could take 1 ¾ tons of seed and divide it by ½ garden but that doesn't make sense either. (Ms. Schwab)

> I find this very difficult. Divided by ½. Where do you have a half that you want to divide? You do not have half of a child, so this.. half with candy or divide with a pizza. But you don't divide by half a person. (Ms. Kuhn)

> You can show ¾ of a cake or ½ a cake, but you can't divide by ½ in a realistic situation. And then you can't visualize it because you don't really encounter it in everyday life. It would be difficult to explain. (Ms. Schultz)

These participants truly seemed unable to generate a representation that involved the problem, and while most of them concluded there was no realistic representation to be made some of the participants in this group stated they would have to ponder the question in order to develop a realistic story problem. As with previous topics, some participants in this group also mentioned that they did not teach fractions and thus had not dealt with them for a long time. Rather than accepting there was no accurate representation that would also be realistic, they discussed the implications of not having worked with fractions and explained what they might need to do in order to be able to not only generate an accurate representation, but also what they might need to do to be able to be prepared to teach the concept. One of these participants had also been unable to correctly compute the problem, but discussed what she would do to improve her knowledge and skills to both compute and to represent her knowledge to others.

> Well, if I ever had to teach it to a class I'd have to brush up on how to do it. One thing that I've noticed about myself is I've done a lot of things automatically because that's just how you do it. That's how I learned it, I don't really know why but that's how you do it. But, if I have to teach it now and see where the problems lie I would have to go ahead and study up on it. I would have to think about why. Why is it this way? And then I notice I really haven't learned it in school myself. I can't really think of anything right now how I could explain this math problem. (Ms. Schultz)

One participant who although she had provided a correct answer to the computation, simply did not think there was a believable representation to be had asked for an example. After further discussion, it is apparent that Ms. Baum believed that it is a disadvantage to not be working with and teaching a given topic. She said, "They probably teach it right now. When you do not teach it then it is so far removed." Once again it seems teacher knowledge may be contextualized to the topics and concepts of the particular level participants are teaching, and perhaps to an even greater extent to the specific concepts currently taught by participants. Indeed, it may seem that teacher knowledge is in the moment and limited to material currently being taught. The two participants, Mr. Pfeiffer and Ms. Hoffman, who taught at the level that included division with fractions in the curriculum, were indeed both able to generate correct representation. This would seem to support the view that teacher knowledge is momentary; however, given that other participants were able generate accurate representations who did not teach that level further research would need to be conducted.

Incorrect Representation

Those participants who generated a representation, but an incorrect representation represented 20% of the participants. These participants confused dividing by ½ with dividing by 2. Of these participants, one had correctly computed the problem after reworking it, one was unsure and stated that she could not, and although neither of the other two provided the correct answer when computing the problem, one correctly explained the process without stating the final answer and the other made a mistake when simplifying and ended with 3 ¼ instead of 3 ½. The representations of these participants clearly showed a discrepancy between how they had just explained how to correctly complete the computation and how they would then represent this problem.

> You have 1 ¾ liters of milk and you have to divide this up into two containers. The amount should be equal in both containers. How many liters does each container hold? (Ms. Gauss)
>
> Maybe, we are having a spring tournament and we will need one whole class and only ¾ of another class and because we need two teams we have to divide by two. (Ms. Heinz)
>
> With a cake or something. Cut it in pieces. That would work pretty well. Well, maybe not because it would be a whole cake and ¾ of a cake and then to divide between two people. (Ms. Reinhardt)
>
> Yes, if I say for instance I have 1 ¾ liter and from that a half. So, I have a unit but I need for instance a recipe I'm going to half it and need to figure out how much it is. So, I just need half of this. I find this easier than to figure the actual numbers. I look half of a liter is 500ml, and from ¾ liter is 375 divided [figures 375 divided by 2]- it's hard to do on the spot when someone is watching...then I have a number that I can add together and then I would have the solution. (Ms. Muller)

Ms. Muller had stated she could not compute the problem, and in attempting to represent it, she had confused dividing by ½ with dividing by 2. Of the participants in this category she seemed least sure of herself although she certainly was no more incorrect than the others in the group.

Correct Discussion, No Representation

While 10% of German participants were able to discuss the problem and what a representation would perhaps look like, neither of these participants were able to finish the process of generating a representation. The immediate reaction of one participant was that the problem was "dumb" because it used 1 ¾, but continued her train of thought to discuss what a representation might entail.

> I would have to find a situation where you would divide by ½. No, I can't. For example, if you have a pie or a pizza- say you have 2 because 1 ¾ is dumb, and you divide it by 2 each gets 1, but if you have half as many people you can twice as much. Something like that. There would be twice as much if you had half as many people. I can't come up with something right now. I don't have to teach that right now so I can't come up with an example right now. Maybe in a minute. (Ms. Schneider)

> Here one pie and here ¾. So, here a pie and ¾ of a pie. And then you can explain to the children divided by half. Uh, mmh. The half will have to be bigger. The half. Mmmh. So 1 ¾ divided by ½ that would be the half, a ½ a ½ [writing on paper] and so, so that would be ¼ and ¼ and divide it again then it would be eighths. Yes. So you could explain it to the children. Ms. Richter

It seems Ms. Schneider also struggled with dividing by two, particularly when trying to fit half of a person into her thoughts. She did explain correctly that the representation would end with "twice as much". (Ms. Schneider also vocalized once again, the notion that teacher's subject content knowledge may be something that is fluid and changing based on what content a given teacher is responsible for teaching.) On the other hand, Ms. Richter visually created what including a pie in her representation might look like. She concluded you would be able to explain it to children, but as is evident she herself did not create an entire story that would represent the problem accurately. These two participants had computed the problem correctly, and seemed to have the basic understanding necessary to create a representation although they did not actually complete the representation. As Ms. Schneider stated, she did not teach fractions and was not able to come up with an example. Ms. Richter did not discuss the fact that she did not teach fractions, but as a first grade teacher it is unlikely she would be working much with fractions in her normal curriculum. For these two participants it seems again that perhaps knowledge is strengthened when it is used, and despite the fact these participants could complete the problem they could not extend that knowledge into a representation.

Correct Representation

Only 35% of German participants were able to generate an accurate representation for 1 ¾ divided by ½. All but one of the participants in this group had computed the problem correctly, with the remaining participant explaining the process correctly but not providing an answer. While they did come up with representations, some still seemed convinced there was a disconnect with reality. It is evident from the representation that some participants still seemed hesitant.

> The problem would look like this for me. 1 ¾ pizza, anyone can picture that, and then it is divided into halves. And then it starts. One half, then another half, now I have

> three halves and the leftover is a half of a half. That would be a possibility, and then it starts into reality. All of a sudden I have three halves, but I only have one three-quarters. It's very difficult to show those kinds of things. A half, and a half, and a half, and a half of the half. That's what I said before. To explain these things for everyday living just doesn't make much sense. (Mr. Reiman)

It seemed Mr. Reiman used a visualization to solve the problem, rather than writing it out. Although his answer may be somewhat difficult to follow, it is, in fact correct. If the whole pizza were divided in half that would lead to two halves; if another half is taken from the remaining three-fourths of a pizza that brings the total number of halves to three with a remainder of a half of a half (or the remaining fourth of a whole pizza), with a final answer of 3 ½ halves.

Other participants were quickly confident in the representations they developed. More than one participant used a liquid of some sort either milk or juice in creating a realistic word problem they might use in the classroom.

> Okay. A mother has orange juice, o-juice. She has 1 ¾ liter. So, then. 1 liter and ¾ liter, about like this. Uh, glasses. (Draws representation on paper as she explains). The glasses hold ½ liter, or some other container that will hold ½ liter. How many ½ liters can I get from this? If students didn't understand this, I might bring it to class and then show it to them. How many ½ liters can I get? I would pour it into the glasses to see how many. So they could see. (Ms. Hoffman)

As a teacher in elementary school I would draw pitchers. Juice. 1 ¾ juice and I have cups. In each cup I will put a ½ liter and then we'll see how many cups can we fill. (Ms. Lowe)

Both Ms. Hoffman and Ms. Lowe created representations of the problem similar to the approach used by Mr. Reiman. Both participants created a mathematical representation to illustrate how many ½ liter servings of orange juice could be derived from 1 ¾ liters of juice.

A common visual participants liked to use was cake or pizza, which was used successfully by Mr. Reiman above when he divided the pizza in half. Such visuals can also be used in a different manner, as shown by Ms. Riese. While the above representation was focused on dividing a pizza into halves, the following representation illustrated a correct method in which people can be used as part of the representation, which might be more relatable to students. Rather than attempting to divide the pizza between the two people (which would represent divided by 2 not by ½), Ms. Riese's representation asked what portion two people would receive, which does represent division by ½. Although quite similar to Mr. Reiman, the fact that Ms. Riese was able to involve a more realistic element into her representation indicated an ability to create means of sharing her knowledge in a manner interesting to students. Students may wonder why it is important to know how many halves of a pizza one could get from 1 ¾ pizzas, but students would definitely relate to how much pizza each person would be able to get eat if two people equally divided 1 ¾ pizza.

I would say I have one pizza and ¾ of a pizza. And I would divide it so two students could have the same amount. How do you divide it up? So, if it works with pizza that's a good thing. (Ms. Riese)

One participant was able to generate more than one representation for dividing by ½. His first example did not use 1 ¾, but his second example did. According to Mr. Pfeiffer, he simply needed to "detach" himself from the numbers. Generating representations for dividing by ½ did not seem to be difficult for him at all.

Well, dividing by ½ is very easy. I take any amount of money. It would be better with a natural number. So, 12 Euros and I count them out in 50-cent pieces. How many pieces do I end up with? That is divided by ½. I can do it with weights. I just have to detach myself from the numbers. I take 1 ¾ of an hour if I have to I can convert it into minutes and ask how many ½ hours do we have. (Mr. Pfeiffer)

Participants who were able to generate an accurate representation of 1 ¾ divided by ½ did not all teach the grade level at which division by fractions would be taught, which might seem to contradict that teacher knowledge may be contextualized. Four of the seven participants who generated a correct representation also completed surveys. This data revealed that three of the four had previously taught the level at which mastery of division with fractions is expected. The fourth participant who completed the survey had and was teaching the level where fractions are part of the curriculum. Additionally, although background data is not available for all of the participants who did generate an accurate representation, at the time of the study all of these participants taught at least one class where fractions are part of the curriculum, and all but two taught at or above the level where mastery of division with fractions is expected. It may be that these participants simply had a deeper knowledge and understanding of fractions that enabled them to complete this process despite not currently teaching this concept, but all did have some teaching experience with this concept.

Both participants who correctly discussed the process of solving such a problem but did not generate a correct representation taught at the first grade level, and did not have fractions as part of their curriculum. One of these participants had completed the survey and it is known that she had previously taught a higher level that included fractions as part of the curriculum, but not at the higher level where division with fractions was included. Based on available survey and interview data, three of the four participants who generated incorrect representations had or were teaching at a level where fractions are part of the curriculum. Survey data were available for two of these participants, but neither had taught above a more introductory level for fractions. Thus, none of the participants who failed to generate an accurate representation seemed to have taught at the level where such instruction would take place while all of the participants who were successful in generating an accurate representation did have teaching experience with fractions. These finding would then in turn support the view that teaching knowledge can be momentary and/or based on previous teaching experience.

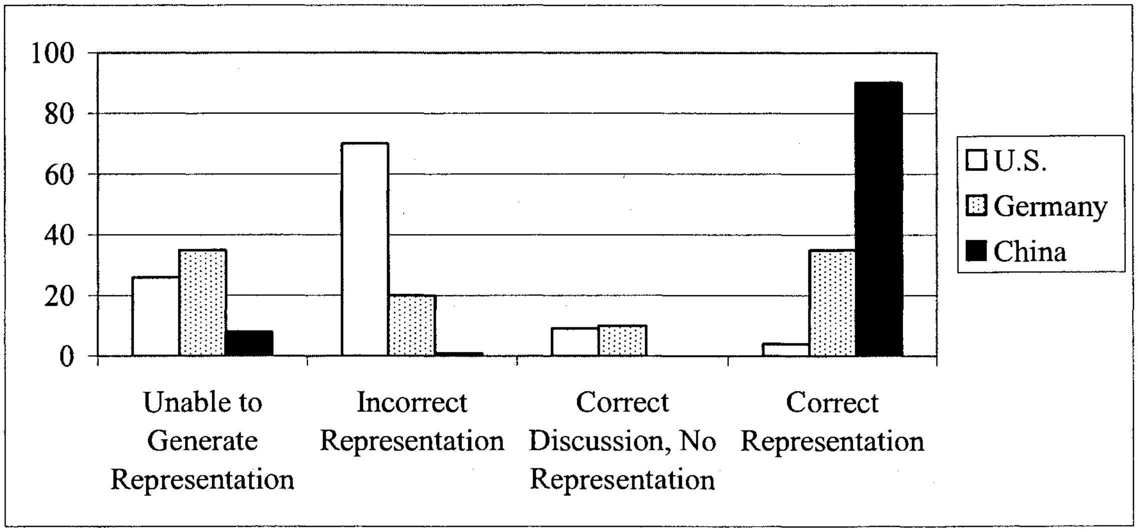
Difficulties in Generating an Accurate Representation

Regardless of whether participants were able to generate an accurate representation, all participants were asked what they thought made it difficult to do so. The most common response, involving 60% of the participants, was that there was a disconnect with the real world, that it would be difficult to come up with a believable story. One participant who had generated a correct response still maintained it would not make much sense. For many dividing by ½ was the root cause of the problem, even if they themselves had been able to correctly compute. Ms. Huber said, "The divisor is certainly one of the problems. I think that's the main problem. In daily life you just don't have half things. You have a whole bowl, or a whole person, or whatever." Contrary to this view, one participant stated that the difficulty with developing a representation was, "Maybe they don't have enough imagination to package a mathematical problem into a story. I find this very important. If you always just write the numbers it is difficult to understand why you even have to learn all these things." Ms. Gauss. This participant happened to have confused dividing by 2 with dividing by ½, but in her view lack of imagination was problematic. Some participants also felt that an obstacle to both solving the problem and generating a representation was the fact that when dividing with fractions the answer is larger rather than smaller as is typical when dividing with whole numbers. Ms. Riese said, "Why is it more when I divide? Usually when I divide it gets smaller, but with fractions you get more. That's the problem to understand."

Summary and Comparison

In comparing participants from Germany to those in previous studies from the United States and China, there were similarities in how participants were or were not able to generate a comparison, but there were noticeable differences as to how many participants from each country fell into these categories. See Figure 18.

Figure 18
Comparison: Generating an Accurate Representation



Participants who were unable to develop a representation included 26% of teachers from the United States and 8% of Chinese teachers (Ma, 1999). In this instance, the 35% of German participants who were unable to generate a representation exceeded that of either country. At first glance this may seem inconsistent with perceived knowledge and skills involved with division with fractions. After all, more German participants were able to complete the computational portion than teachers from the United States, and only slightly more Chinese teachers were able to complete it than German participants. Further analysis of interview data showed, however, that 15% of German participants thought about the representation but ended by stating they would have to think and ponder more before finishing. Without this group, the number of German participants who were unable to generate a representation would have dropped to 20%, which is lower than that of the United States, higher than China, and in keeping with other findings. Regardless of whether these 15% of participants generated accurate or inaccurate representations the reassignment of this group to either of the remaining groups would once more place German performance in between that of the United States and Germany.

Only 20% of German participants created an incorrect representation compared with 70% of teachers from the United States and a mere 1% of Chinese teachers (Ma, 1999). Rather than floundering and attempting to generate a representation that might be wrong, perhaps German participants felt more comfortable in stating they did not think it could be done rather than attempting something they could not quickly see would work. Although overall the German participants were, for the most part, able to solve the problem, perhaps their skills and knowledge of division with fractions was not a deep enough understanding that would allow them to represent it to others. More certainly felt it was unrealistic than those who simply were wrong. All of the German participants who created an incorrect representation confused dividing by 2 with dividing with ½. Some participants from the United States made the same mistake (43%) while others confused dividing by ½ with multiplying by ½ (26%), and still others confused dividing by ½, dividing by 2 and multiplying by ½ all in the course of the same problem (9%), (Ma, 1999). It would seem that not only were more teachers from the United States unable to correctly represent the problem, but the types of problems they had varied a great deal as well. The one Chinese teacher (1% of overall sample) who incorrectly represented the problem created a representation for ½ divided by 1 ¾ rather than the other way around (Ma, 1999).

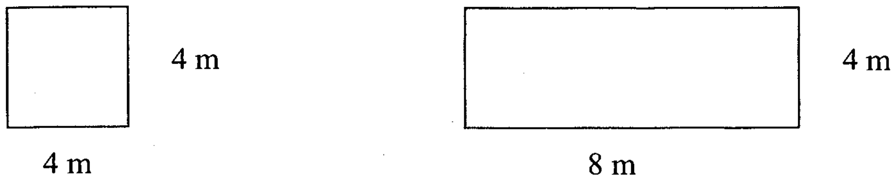
Participants who did not seem to be confused, and were able to discuss the scenario but not provide a representation included 10% of German participants, 9% of teachers from the United States, and no Chinese teachers (Ma, 1999). Participants in this category from both Germany and the United States expressed that they would need to further explore and think about the problem (Ma, 1999). As with the German participants, teachers from the United States in this category said they could not think of a realistic representation and could not conjecture the meaning of dividing by half (Ma, 1999). Teachers from both countries avoided misrepresenting the problem, but at the same time were unable to conceive of a representation for the problem.

A significant difference is evident when comparing participants who did in fact generate a correct representation of 1 ¾ divided by ½. Only 35% of German participants were able to generate an accurate representation compared to a mere 4% of teachers from the United States, and an overwhelming 90% of Chinese teachers (Ma, 1999). This last comparison may be the most telling in discussing division with fractions. In the category that truly matters most; the ability to represent teacher knowledge to others correctly, the discrepancy is significant. One teacher from the United States was able to correctly represent the problem; however, her representation left her with half of a person, though unrealistic conceivably she would have been able to work through the symbols in her representation and found a more realistic representation. Although more German participants were able to develop a correct representation than teachers from the United States, far more Chinese teachers, in fact nearly all of the Chinese teachers were able to do what the German participants could not. Not only were Chinese teachers able to develop an accurate representation on a larger scale, Chinese teachers often developed more than one representation, and were also able to generate representations using the given problem in various means (Ma, 1999). Several stories were constructed along similar lines of those developed by the German participants in which the crux of the story was figuring how many ½ s there are in 1 ¾, for example the representation by Ms. Bock dealing with orange juice. In addition to this approach, Ma (1999) reported that Chinese teachers also developed stories to find a number whose half would be 1 ¾ and finally stories that led to models to compute what would need to be multiplied by ½ to end with 1 ¾. Despite nearly as many German participants computing the problem correctly as their Chinese counterparts, not nearly as many could generate a representation they might use to explain the concept to their students, let alone for the most part an additional representation, or a representation with a different approach. The subject content knowledge was evident, but the ability to translate this information to represent it to others through pedagogical content knowledge was noticeably absent.

Novel Theory: Perimeter and Area

In asking participants how they would respond to a student's claim of having "discovered" a new mathematical theory, it was still possible to examine another area of mathematical understanding and how they might represent such knowledge to students, but it was also possible to ascertain how participants would theoretically indeed act towards a presentation of "new knowledge" (McDiarmid & Ball, 1989). Participants were presented with the following scenario:

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:



perimeter= 16 m
area= 16 square m^2

perimeter= 24 m
area= 32 square m^2

How would you respond to this student?

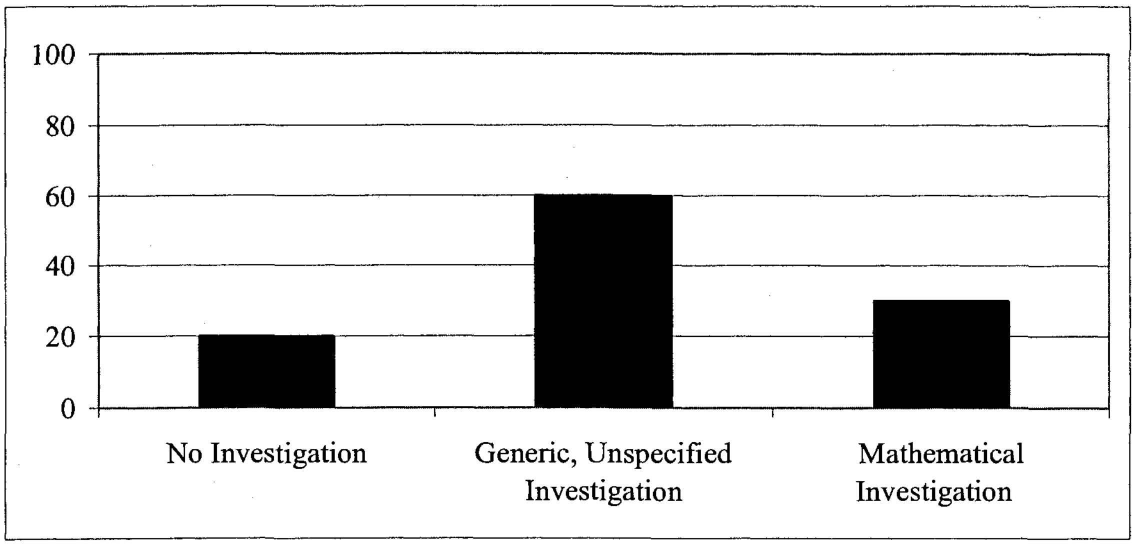
Occasionally it could be that something comes up where you are not sure yourself about whether the mathematics is correct or not. I'm interested in how you think you would respond in such a situation. What would you do with or say to the student?

Would you say or do anything else?

This scenario also provided an insight into approaches participants might take if they themselves were unsure of a mathematical concept. None of the German participants indicated they would need to investigate further on their own until after asked what they would do if they unsure about a mathematical concept. In this scenario, participants provided emotional support and/or indicated they would investigate with the student or class. In keeping with previous studies, participants' responses were categorized based on whether they would investigate the claim on their own or whether they would investigate with the student and/or class; also taken into account was whether the teacher offered some sort of emotional support for the student's efforts (McDiarmid

& Ball, 1989; Ma, 1999). See Figure 19 for participants' reactions and plans for proceeding.

Figure 19
Participants' Response to a Novel "Theory"



Emotional Support

The initial response of a majority of participants, 75%, included some sort of praise and emotional support for a student having taken the initiative to work on something of this nature on their own. Participants made comments to show not only praise, but also their own enthusiasm.

> That's great that you figured this out. You would have to go ahead and make it a bit dramatic because she worked on this herself. Sometimes they come with theories that are not necessarily correct, but at least they are thinking about things. If someone comes with a theory that he figured out himself, I think that's worthy of praise and recognition. (Ms. Muller)

> I would tell her great that she thought about this at home and that she pursues such ideas. (Ms. Schultz)

Some participants used praise and emotional support of the student's efforts as a transition to explore the "theory" the student believes she has discovered further.

> That's an academic question. First I would say that's really great that you discovered this. Then I'd say let's do this on the board and see if it's always like this. (Ms. Huber)

> I would go to the proving step. In math we always prove through evidence, examples and I would do at least, how my colleague said, three examples to see if it proves correct. I would share her joy and praise her for her efforts. That I am happy that she… also, first that I am very happy that she is looking so intensive at a math problem and that together or in group work we would look if her theory is correct. And I would certainly express my joy that she is thinking about and working with these things so intensely. Naturally, I would ask her in the first place how she found out about this theory and then go through the proof process to see if it is correct. (Ms. Heinz)

These participants wanted to provide support and encouragement to the students, but at the same time turn their attention to a focus on the subject matter. Although a majority of participants did react with emotional support, the response of 15% of the participants was limited to only emotional support. These participants did not expand on what they might do until later prompted; other than praising the student these three participants did not indicate how else they might react. The remaining 85% of participants discussed how they would proceed in investigating the claim with the student. Two groups emerged.

Investigates with student: no specific mathematical approach

Participants who would investigate but who did not explain specifically how they might go about this, and who if they mentioned certain examples did not necessarily seem to have a mathematical reasoning for proceeding in that manner fell into this category. This group included 50% of the German participants. Overall these participants discussed "proving" whether the theory was always correct and for the need for additional, non-specific examples. Some of these participants, 20% overall, said that not only would they investigate further with the student, they would involve the entire class in exploring and proving the "theory", which might be an indication these participants felt comfortable exploring mathematical knowledge either familiar or unfamiliar with not only one student, but in front of the entire class due to either the rapport they have with the class, how comfortable they feel admitting they do not know an answer, or simply that they do feel confident in their knowledge on the topic. Participants with no apparent specific mathematical basis for their investigation explained their approaches:

> I would have them explain to me what they discovered and then I would say let's prove if this is always so, find more examples. Later, find a rule. (Ms. Schneider)

> That she needs to show me examples to try it out and to look if this is always true. (Ms. Reinhardt)

These samples seem to indicate that although these participants would investigate they perhaps did not have the mathematical understanding to know immediately what path such an investigation should take. Ms. Kuhn seemed at first to understand the flaw in the "theory", but then changed her mind and decided it was in fact correct after all. These types of responses hint at a lack of not only subject content knowledge, but also the pedagogical content knowledge necessary to represent knowledge to others. With little or no mathematical reasoning in the approach these participants intended to take, it may be a fair assumption that not all of them would be able to lead the student and/or class in discovering the actual truth behind the relationship between perimeter and area.

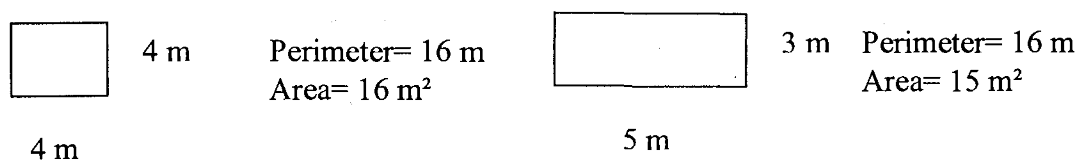
While most participants in this group did not mention a mathematical reasoning for how they would investigate with the student, 15% of the participants hinted at or gave strategies that were mathematical in nature, but not specific enough to ascertain either their own understanding of the relationship between perimeter and area or whether their investigation would help clarify the "theory". The following are examples of how these participants would proceed:

> And then you could use it as a theme for the class right now so she could explain it to her classmates. And then we would think together if this theory is correct, and have the other students draw different size squares and rectangles to explore this theory. We would think together if this theory is correct. (Ms. Schultz)

> First I would ask her to show me exactly how she came to this theory and would ask her to show me a few more examples. So we can try to prove if her theory is always correct. It is an awesome thing if she had such an idea and we have to prove it. First

we try to prove it in a practical way and then possibly in a mathematical way. So if this student already knows the rule about perimeters and area you can come up with a variety of examples to see if the theory always holds true. (Ms. Gauss)

While these participants appeared to be taking a mathematical approach, the lack of specific details makes it difficult to gauge their understanding. Ms. Schultz may very well indeed have a sound understanding of perimeter and area; however, if all of the different sizes of squares and rectangles do not disconfirm the theory she and her students may in fact hold the theory to be true. Ms. Schultz would need to guide the exploration to include a specific change in the size of the squares and rectangles, namely one side would need to decrease such that while perimeter might stay the same or increase, area would actually decrease.



Ms. Gauss on the other hand talked about proving the theory first practically and then mathematically, and yet neither of these avenues was more clearly explained. However, the fact that Ms. Gauss did mention proving the theory mathematically is perhaps an indication that her own understanding of perimeter and area is somewhat better defined than a teacher who simply calls for additional examples. It also would appear that the approaches she would use with her students might be more systematic and intentional.

Investigates with student: mathematical approach

The second, smaller group included participants who discussed specifically how they would investigate the topic. Rather than calling for more examples at a general level, this group comprised of only 30% of the participants provided more specific examples, and in some cases a series of specific examples they would complete with the student. Another difference between the previous group and this one is that none of these participants mentioned investigating with the entire class, giving the impression these participants would investigate with just the student who came with the new "theory". Two participants from this group, Mr. Pfeiffer and Ms. Lowe, noted immediately that the "theory" was incorrect.

> The possibility exits that if you have ... the perimeter and the area do not change always in proportion. It is possible to have the same area and a different perimeter. That is something that the students would have to still get to know. (Ms. Lowe)

> It's wrong. She should show me an opposite example. So, I would draw a very simple figure. One side is minimal and the other side is extremely large. Then, it doesn't work anymore. The perimeter is very large- infinitely so and the area is still the same. (Mr. Pfeiffer)

While both Ms. Lowe and Mr. Pfeiffer were concise and direct in their assessment and direction, other participants in this group provided much longer and detailed explanations about how they would proceed. By providing a series of example problems to work through with the student, these participants felt a better understanding of perimeter and area could be attained.

> Then we would maybe shorten one side and then lengthen the other one and figure out if this is always the rule. I would show on the board different squares all with the same perimeter of 20 cm. (Ms. Huber)

For Example: 20 cm Perimeter

$1 \text{ cm} \times 9 \text{ cm} = 9 \text{ cm}^2$
$2 \text{ cm} \times 8 \text{ cm} = 16 \text{ cm}^2$
$5 \text{ cm} \times 5 \text{ cm} = 25 \text{ cm}^2$

Ms. Huber approached the problem by maintaining the same perimeter, but showing differences in area, while Ms. Richter altered both the perimeter and the area to show that an increase in perimeter does not necessarily equate an increase in area. It must be kept in mind that the original problem had sides whose length was 4 m (although these participants used cm in their examples the numbers remain the same). It may appear at first glance that Ms. Richter's example confirms the theory; however, her third example also results in a perimeter of 16 while the area decreases to 15. Thus, when comparing these three additional examples to the original two samples students would be presented with three sample problems that confirm and one sample problem that disproves the theory.

So I would, for instance, always take the theory the student came up with and try to prove it. I would give them one, two, three- always at least three examples. [Points to examples she came up with.] Then we would always figure it. Perimeter, area, so then I would always compare it in a table because it would make it clear optically. Here it is too far apart. This is here. And then maybe even arranged in order for the really weak students. Lay out the sequence, the perimeter was twelve, and then the children can come to their own conclusion, yes. I would do it differently this way. Ms. Richter

For Example:

Perimeter $= 2\,(a + b)$
$= 2\,(4 \text{ cm} + 2 \text{ cm})$
$= 12 \text{ cm}$

Area $= a \times b$
$= 4 \text{ cm} \times 2 \text{ cm}$
$= 8 \text{ cm}^2$

Perimeter $=2(a+b)$ $=2(6cm+3\text{ cm})$ $=18\text{ cm}$ — Area $=a \times b$ $=6\text{ cm} \times 3\text{ cm}$ $=18\text{ cm}^2$

Perimeter $=2(a+b)$ $=2(5\text{ cm}+3\text{ cm})$ $=16\text{ cm}$ — Area $=a \times b$ $=5\text{ cm} \times 3\text{ cm}$ $=15\text{ cm}^2$

	Perimeter	Area
1st Example	12 cm	8 cm²
2nd Example	18 cm	18 cm²
3rd Example	16 cm	15 cm²

Perimeter	Area
12 cm	8 cm²
16 cm	15 cm²
18 cm	18 cm²

A more complete understanding of the interaction between perimeter and area is evident in the sample problems that these participants were able to develop. The mathematical purpose behind their further investigation would allow the student to not only see that the "theory" did not hold true, but would also help them visually understand various interactions between perimeter and area.

Another participant in this group explained that grade level would determine how far he would go in proving the theory. For younger grades (5^{th} or 6^{th}) he would figure through the problem, while for older grades (8^{th} or 9^{th}) he might take it a step further. Regardless of grade Mr. Reiman clearly showed that he would converse with the student and would ask questions to help the student understand what was happening. It is also evident that Mr. Reiman enjoys mathematics and that for him it is "fun".

> Well, no, if a student would come to me with this I would go and compare the numbers. You can do that and think about what made her come to this conclusion. So, does the student have a talent for figuring out rules or formulas or was this discovery by accident? There's a difference. I could say I'll look at it and that's okay. When they say the perimeter is 16 meter and then 20 m for perimeter. And

now you have to say look what happens. The perimeter isn't doubled. This is why math is fun for me. You did a good job thinking, and let's think further. What stands out? What did you notice? What is the area now? How big is the area with this one, and how big is the area with this one? With these kinds of things I think this is great. You can start a conversation with a student- are there any mathematical laws that apply? Or I can make it into a game. It depends on the grade level. In the 5th and 6th grade I would say well, start and figure it through. What is it here? What is it here? [Pointing to sample figures on paper.] And then you can think about why the perimeter did not double but the area did. And then you can think further what happens when I change the perimeter again? So, you can try with things like this to establish the mathematical rule. You can do it as a game depending on the age of the student, but if it were in the eighth grade I would take it a step further to see what else there is. What can we find about this rule? (Mr. Reiman)

Participants in this group not only appeared to have a firm understanding of perimeter and area; they also seemed to easily and comfortably develop an approach to investigate the "theory" with the student. They had specific problems they would use, and also strategies for questioning and conversation. The proving or disproving of the "theory" was not left to random additional examples that may or may not have led to the desired outcome.

Simply accepts claim

Although most of these participants indicated they would investigate further with students, 20% of the participants also simply accepted the claim as true. "So, the student theorizes that when the perimeter is bigger then the area is also bigger. Isn't it correct?

Yes." (Ms. Reinhardt). Ms. Reinhardt sought confirmation from the researcher, and when explained that it was in fact not a correct "theory" she went on to explain she would work with the student with more examples. Ms. Riese, however, accepted the "theory" as correct and continued with her response accordingly.

> Well, she made a connection that even in class we would not necessarily make this way. The bigger the perimeter, the bigger the area. So, I would say are there exceptions? The bigger the perimeter the area- I think that's right. Well, what would I tell her? Excellent. So, you could double-check yourself and say if I come out with a smaller area I must have made a mistake. Can't think of anything better. You recognized this well. So, in the future you can double-check your solutions with this theory. (Ms. Riese)

The fact that these participants simply accepted a "theory" presented to them to be true seemed an indication that their own understanding of these mathematical concepts were limited, which would in turn affect their ability to represent such knowledge to others. Ms. Riese particularly was quick to say she would tell the student to use the "theory" in the future to double-check her work against. For obvious reasons this could cause problems in the future.

No investigation

As previously mentioned, three participants limited their response to only emotional support and praise. These participants did not discuss investigating the "theory" further. Additionally, one participant who simply accepted the claim also did not indicate she would investigate further. Thus, 20% of the participants did not appear to be inclined to investigate further the claim of the student.

Reaction if unsure about mathematical knowledge

Only when further questioned as to what they might do if they were unsure about a mathematical concept, did some participants express that they would investigate on their own and then report back to the student. As the participants had already discussed how they would respond to the student above, it must be taken into account that the participants likely felt that this question was not necessarily limited to the above scenario. Some participants who had explained investigating the above scenario further with the student indicated that with an unfamiliar concept they would first study individually and then report back to the student(s). This group included 50% of the participants from the study. These participants seemed to want to research and learn before coming back to discuss the concept with the student or class. These participants stated that

> Well first it looks correct, I if I am not sure I would take it with me and tell the child that it is great that they thought about this and then at home to really look if it is correct or not. (Ms. Trachsel)

> If I weren't sure the theory was correct I would say that's great and very interesting, but that I want to prove the theory, so I will take it home and work it over and discuss it tomorrow together. Just to be sure. (Ms. Muller)

> I would tell the children that I cannot solve this right now and that I have to inform myself and that we will discuss it the next day again. (Ms. Baum)

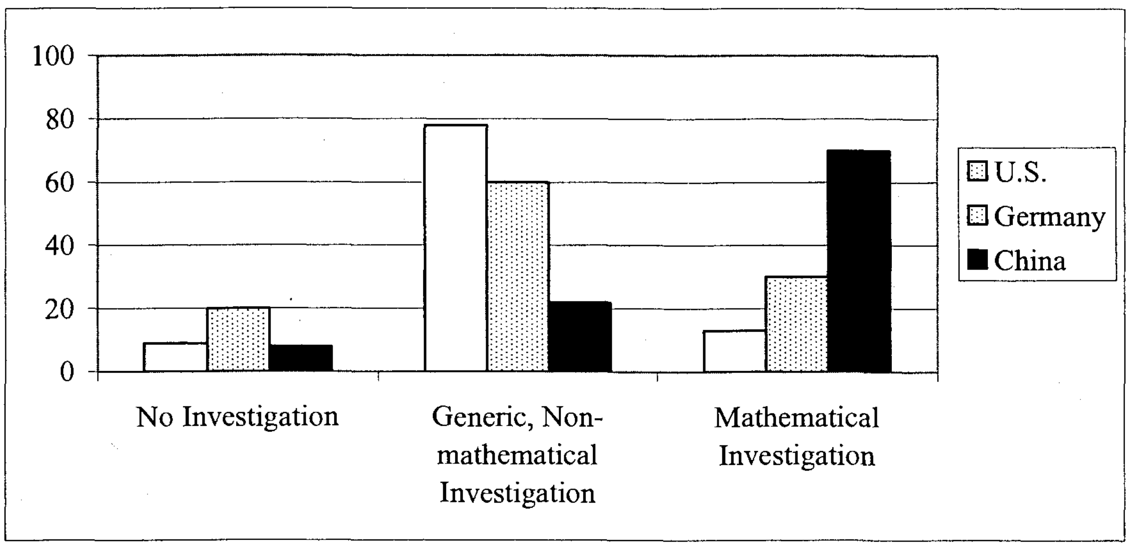
The views of these participants may indicate that they feel they have limited mathematical knowledge insomuch that they are not comfortable investigating unfamiliar mathematical concepts with students, and so prefer to do so individually. It could, however, also be an indication that they prefer to spend class time on prepared lessons and material. More than one participant expressed that it was good for students to see that no one can know everything, and that even teachers must continually learn and seek out information. Sources cited by participants that they would turn to include: the Internet, colleagues, and books.

Summary and Comparison

Participants showed an overwhelming desire to offer praise and emotional support to a student undertaking mathematical investigations in their own time. Participants were pleased that a student would develop a new "theory" and for the most part wanted to foster that excitement and enthusiasm. The next step of the participants, however, showed differences in not only their approach, but also their understanding of perimeter and area. Some participants offered only emotional support and did not indicate they would investigate further with the student. One participant who simply accepted the claim also fell into this group. Other participants who also simply accepted the claim did indicate they would proceed with further investigation, along with the rest of the group. At this point, participants were categorized either as continuing with a generic, non-mathematical approach, or as employing mathematical strategies in their investigation. Similar reactions and responses were found in previous studies (McDiarmid & Ball, 1989; Ma, 1999) involving teachers from the United States and China. As with previous areas examined, German participants seemed to have a more thorough understanding at increased rates than teachers from the United States, but still not as complete and not as many teachers as those from China.

Nearly twice as many German participants did not indicate they would investigate further than did participants from either the United States or China. Unlike participants from the latter two countries, the primary reason for German participants to not investigate further was the focus on emotional support rather than simply accepting the "theory". However, insofar as further investigations are concerned the same pattern as has been evident in the previous sections was manifest once again. German participants were less likely to investigate with a generic, nonmathematical approach than participants from the United States, but more likely to do so than participants from China; and more likely to investigate with a mathematical approach than participants from the Unites States, but less likely than participants from China. See Figure 20. It seems yet again that German participants' ability to represent knowledge to others falls in between that found of participants from the United States and teachers from China.

Figure 20
Comparison: Teachers' Response to a Novel "Theory"



Teachers from the United States and China who investigated further, but with no specific mathematical approach made similar comments as those of the German participants. These teachers also called for additional, but not specific examples, that perhaps they needed "enough examples, Tr. Blanche" (Ma, 1999, p. 86), stated they would see if the theory "proves true in every situation, Ms. Florence" (Ma, 1999, p. 86), and also how they thought the "theory" was true, "Let's have a look at how it is true, Mr. Felix" (Ma, 1999, p. 92)." It seems participants from all three countries who took this approach to investigate further had common misconceptions about perimeter and area or may have simply not known how to proceed to discover whether or not the "theory" was indeed correct.

Further investigation with a mathematical approach also highlighted similarities of teachers from all three countries. As with the German participants' responses, teachers from the United States and China also appeared to have similar knowledge and purpose for their investigation. These participants also discussed specific examples to help the student understand perimeter and area. Purposeful examples included comments such as "what happens when you have got 2 inches on the one side and 16 inches on the other side, Ms. Faith" (Ma, 1999, p. 87) and "I may want to draw a rectangle with the length of 8 cm and the width of 1 cm, Ms. I." (Ma, 1999, p. 93). A difference noted by Ma (1999) was that the one teacher from the United States who did successfully develop an accurate mathematical explanation, as well as 19% of the Chinese teachers limited their explanations to disproving the student's claim. Similarly, 15% of German participants seemed to limit their investigations to disproval of the "theory". On the other hand, 11% of Chinese teachers and 15% of German participants also included in their investigations examples that both proved and disproved the "theory". As with the German participants, who provided a series of examples, some that confirmed and some that disproved the concept, Chinese teachers also discussed various ways that this could be accomplished (Ma, 1999). While the extent of the German participants' investigations seemed to end at that point, 36% of Chinese teachers continued to expand their investigation with discussions about perimeter, area, and specific examples, and some Chinese teachers continued further to explain why the "theory" was true some of the time, and why it was not true other times (Ma, 1999). Thus, while more German participants were able to discuss a mathematical approach to investigating the "theory" none of them went into the further stages of investigation, as did the Chinese teachers. Despite a greater number of participants using a mathematical approach than teachers from the United States, and some of the German participants delving further into the investigation the Chinese teachers not only surpassed in terms of number teachers from both Germany and the United States, but the depth of their knowledge and representation also exceeded that of teachers from the other countries as well.

Conclusion

An examination into how German mathematics teachers would represent knowledge of four basic mathematical concepts, multi-digit subtraction, multi-digit multiplication, division with fractions, and perimeter/area, to students revealed interesting data. While German participants were able in nearly every instance to correctly compute and solve word problems of these types of basic mathematics, data indicated they were not as strong in representing their knowledge to others (i.e. students) as they were in using the knowledge themselves. The degree to which they seemed to be able to represent knowledge to others accurately does seem to depend on the mathematical principle.

The mathematical principle German participants seemed strongest in both for their own use and in representing knowledge to others was multi-digit multiplication, with 60% of the participants displaying a conceptual understanding of the topic. This, however, may be in part due to the fact that the process of completing multi-digit multiplication is entirely different than what is used in either the United States or China. An interesting finding was that more German participants discussed the importance and role of place value in multi-digit multiplication than did for multi-digit subtraction. Both concepts are founded in place value, but obviously multi-digit subtraction is a concept taught before multi-digit multiplication. It would seem that teachers with a deep understanding of place value would recognize and explain the importance and role for both concepts. While this area seemed to be the strongest for the most number of participants, 60% of participants are still just more than half of the entire group, certainly not ideal.

In all areas there were participants who had deep, thorough, and conceptual understandings of the given concept. However, in all areas there were also participants who were procedurally focused and did not seem to fully understand the concept or underlying principles. The affect of this on their ability to represent knowledge was apparent. A comparison across concepts revealed that only 10% of participants were able to accurately represent their knowledge in all four areas. Another 20% were able to do so in three of four areas, with different trouble areas, and 15% more were able to do so in two areas. One of the participants able to represent knowledge in two areas could possibly have been categorized with the participants able to represent knowledge in three areas since she used some mathematics in her approach to investigate the "theory", however because she did not provide clearly specific examples she was earlier categorized as not using a mathematical approach. Thus, based on this sampling less than half of the participants were able to represent their knowledge accurately in more than one area.

The participants who could represent knowledge in more than one area varied in teaching experience of 1 to 42 years, a broad spectrum indeed. What might these participants have in common that allow them to be successful across topics? Is the lack of ability to represent knowledge accurately to others an indicator of shallow subject content knowledge understanding, or does it deal more with pedagogical content knowledge shortcomings? Or, could it be that teachers do not necessarily retain all necessary knowledge at all times, but rather "brush up" as it were as needed? Participants commented that they were not currently teaching a specific topic or had not dealt with it for some time as an explanation for their performance. Further examination into the contributions of university preparation as well as classroom experiences follows, accompanied by a discussion of timeliness and contextualization of teacher knowledge.

CHAPTER 6

CONTRIBUTIONS TO KNOWLEDGE

Data from the previous chapters highlighted the knowledge and skills of participants and provided a comparison to what has been found in previous studies (e.g. Ma, 1999); however, to use this data to the fullest extent possible, examination must turn to how and where participants' believe their knowledge comes from, as well as an analysis of the adequacy of such knowledge. This discussion contains information and analyses intended to be useful to educators and policymakers in the United States who through a lens of teacher knowledge aim to improve education here in the United States. Participants' knowledge was categorized into contributions of teacher education training, or knowledge-for-practice (Cochran-Smith & Lytle, 1999) and into contributions of classroom experiences, or knowledge-in-practice (Cochran-Smith & Lytle, 1999). The nature of teacher knowledge is also examined, as well as an overall analysis of teacher knowledge as found in this study. Except where otherwise noted, data represents all twenty participants.

Contributions of Teacher Education Training

View of contribution of teacher preparation courses

Of this group of participants, some indicated they did not study mathematics as part of their teacher preparation at all, and one could not recall any specific mathematics.

Without prompting, some participants indicated the math they learned was relevant to the math they are currently teaching while one said that most of the math learned at the university was not important for teaching. When asked what stood out about their university mathematics courses, additional participants said it was applicable to teaching while others said it was not useful for teaching. Thus, some participants interviewed felt the mathematical content of their teacher preparation contributed to the knowledge they use for teaching while others felt it did not contribute to their teacher knowledge. The remaining participants mentioned a favorite instructor, that mathematics was fun and interesting again (as opposed to responses concerning the secondary level), specific approaches, working with students and so on. See Figure 21 for categorization of participant responses in this area.

Figure 21
Participant Recollection of Teacher Preparation Mathematics (Percentage; n=20)

Math was relevant to teaching	25%
Math not important to teaching	15%
Other: did not study, could not recollect specific math, favorite instructor, approaches	60%

While these responses answered the question posed, they did not speak to as to whether or not these courses contributed to their teacher knowledge. A similar proportion of participants in the survey reported mathematics courses contributed to their teacher knowledge as in the interview, although these participants specified they found their mathematics courses helpful in terms of methods, not necessarily in mathematical content or knowledge.

Survey data involving ten of the participants (the eleventh survey participant did not answer this question) revealed that seven of participants did not feel the mathematics courses at the university were helpful in preparing them to teach. The remaining three participants indicated that what they found helpful from their teacher preparation mathematics courses was the methods aspect of the courses, and the relation to practical school experiences.

Survey data also asked participants about what contributions they felt pedagogical courses in their teacher preparation programs had on their teacher knowledge. In contrast to views about mathematics preparation, six of ten participants indicated they felt the pedagogical coursework from teacher preparation courses was helpful, while only two did not find it helpful in building teacher knowledge. Two participants had both a positive and negative view of their teacher preparation training in that they found it to be helpful for pedagogy and psychology, but not concrete enough for one of the participants, and for the other it was already an actual teaching experience instead of coursework. Overall though eight participants found the pedagogical aspects of their teacher preparations contributed to their knowledge. An extension not found in the interview was that a majority of participants felt that teacher preparation courses dealing with pedagogy did in fact contribute to their knowledge for teaching

Relationship between contributing coursework and mathematical performance

Are teachers who feel their teacher preparation coursework contributed positively to their teacher knowledge better able to correctly compute basic mathematical problems and to then represent knowledge to others? Are the subject courses sufficient for building effective teacher knowledge? Except in the area of fractions, participants had virtually no problems in computing basic mathematics correctly. Of the three participants who indicated on the survey that their teacher preparation courses had contributed mathematically, albeit primarily to mathematical methods, two were able to solve correctly all four types of mathematical problems presented both in terms of computation and word problem, including fractions, which was the only area participants struggled with in completing their computations and word problems. The third participant was successful in computing and solving included problems with the exception of fractions. Insofar as mathematics is concerned, it would seem that these participants did for the most part have the proper mathematical experiences to enable them to compute mathematics correctly. Mathematical coursework may have played a role in contributing to participants being able to compute these mathematical problems correctly; however, this connection cannot be confirmed. While participants may or may not have gained mathematical content knowledge from their teacher preparation programs, data from the participants did not establish this connection. It may very well be that such knowledge was acquired prior to or after the teacher preparation programs. In fact, when asked, "What do you remember about learning math in elementary school?" 65% of participants included in their response a reference to the basics of math- addition, subtraction, multiplication, and division. Perhaps mathematics taught and learned at the elementary level is the main influence for the mathematical knowledge of the participants, along with other German mathematics teachers. It is difficult to show such a relationship without data to verify what these participants learned.

Types of courses

During the course of the interview, participants were asked what mathematical courses they remembered taking as part of their teacher preparation program either at the university or teacher's college level. While responses to this question are by no means an accurate representation of what courses these participants may have actually taken, it established a baseline and also perhaps an indication for courses that may have been particularly memorable. It is not assumed these participants were limited only to these courses; however, it does provide a glimpse into the types and in some instances numbers of courses taken.

Participants in this study recalled having taken 24 different math courses, but for the most part, only one participant mentioned each course. Some of the courses mentioned included: introduction to mathematics, formulas, analytical geometry, number range, structure of number system, etc. Some courses that were mentioned by more than one participant included 30% recalling set theory (mengenlehre), an approach that was widely emphasized but is no longer used and 40% who remembered taking mathematics courses related to mathematics methods/didactics.

Relationships between contributing coursework and pedagogy

Possible relationship between teacher preparation courses contributing to pedagogical knowledge was not quite so clear. Participants who indicated a positive pedagogical contribution from teacher preparation courses in survey data performed as follows in accurately representing knowledge to others: two participants generated correct representations in all four areas, two participants generated correct representations in three areas, and one participant generated correct representations in two areas, with the remaining three participants only able to generate correct representations in one area.

Although a majority of participants who were able to generate correct representations in more than one area had mentioned a positive contribution from teacher preparation courses in the area of pedagogy, it would seem that they did not necessarily contribute to teacher knowledge in the manner these participants might believe. Phillip (2007) explores the notion that there can be inconsistencies in teachers' beliefs and views between what they say and what is actually observed. Thus, courses aimed at building teachers' pedagogical content knowledge, such as subject specific methods courses, to aid accurate representation of knowledge to others did not necessarily provide all of these participants with the knowledge and skills necessary for effective teachers. Certainly some performed better than others.

From this limited sample, location of teacher preparation did not seem to necessarily be a factor. Ms. Richter, who was able to represent all four areas accurately, attended a specialized training school for teachers in the former East Germany (Fachschule Institut for Lehrerausbildung Weimar) while Ms. Lowe, also able to represent all four areas accurately attended a teachers college (Pedagogische Hochschule Göttingen). Ms. Richter expressly stated that her teacher preparation consisted of a great deal of methods, but that for someone wanting to teach Grundschule (typically grades 1-4) the methods were specialized specifically for Grundschule since it is so different from Hauptschule (typically grades 5-10). Of the remaining four participants who correctly represented multiple areas, three had attended teachers colleges and one had attended a university; the same is true for participants representing only one area correctly. Counteracting the 10%

who successfully represented their knowledge in all four areas were 15% of participants who were not able to represent their knowledge accurately in any of the areas. Each of these participants attended a university for their teacher preparation, but at least as participants in this research did not evidence ability to represent their knowledge to others.

Accepting that knowledge required for effective teaching comes from university preparation or other research entities is consistent with knowledge-*for*-practice as discussed by Cochran-Smith and Lytle (1999). Based on the self-reported data of these participants compared to actual performance, it would appear that knowledge-*for*-practice is limited in nature. It may have helped with mathematical knowledge for participants'; however, it did not seem to have provided all of these participants with the knowledge necessary to represent such knowledge to students accurately. Again, it must be acknowledged that a limitation is that no data are available to indicate what content these participants learned as part of their pedagogical training, making such connections difficult to illustrate.

Contributions of Classroom Experience

Due to the wide range of years teaching, the range of classrooms experiences is necessarily quite broad. The most novice participant was in her first year of teaching, while the most experienced participant had been in the classroom for 42 years. See Figure 22 for classroom teaching experience. The data for this figure were derived from the responses of eleven survey participants plus three additional participants who discussed when they started teaching

Figure 22
Participants' Years Classroom Experience

	Ms. Schultz	Ms. Muller	Ms. Kuhn	Mr. Reiman	Ms. Lowe	Ms. Heinz	Ms. Roth	Ms. Gauss	Mr. Pfeiffer	Ms. Schwab	Ms. Richter	Ms. Baum	Ms. Riese	Ms. Huber
Years Teaching Experience	1	8	13	14*	15	21	26*	31*	32	32	34	37	38	42

The survey also asked participants how many years they had been teaching mathematics. For all participants it was the same number of years except for Ms. Kuhn who taught mathematics for only eleven years compared to thirteen overall, and for Ms. Lowe who had taught mathematics for only ten years compared to fifteen overall. Participants with an asterisk by years taught are an estimate based on interview discussion. For example, Ms. Roth mentioned she had been teaching at that school for 26 years, thus it is known she has at least that much teaching experience, if not more. Because the interview did not examine classroom experiences, this section relies on data from the survey. Participants were asked to explain what contributions classroom experiences made in obtaining necessary knowledge and skills for both effective teaching.

Participants reported a wide array of contributions that classroom experiences made in helping obtain knowledge and skills for effective teaching. Responses were separated into answers dealing with pedagogy and answers addressing mathematics specifically. Examples of contributions to effective teaching in general included trying out many methods to be able to help students, learning how to pose problems, to avoid mistakes, and methodical use of materials, learning from novice mistakes, gaining knowledge of problem areas for students, knowing mistakes teachers make, learning how kids think and learn effectively, awareness of student problems, etc. From these views on how classroom experiences had contributed to their knowledge, it would seem that these participants should know effective methods, potential problem areas and effective ways to help students. The participants also provided specific contributions of classroom experiences to effective mathematics teaching. Statements highlighted these benefits. Participants expressed that experience had shown them how children learn best, that they had learned from novice mistakes, through their teaching experience they had learned how to explain different ways to solve problems, had learned the importance of views and illustrations in mathematics, that you cannot be afraid you do not know something. These statements would seem to indicate that participants had developed knowledge-*in*-practice (Cochran-Smith & Lytle, 1999) through their classroom experiences.

In order to gain a further understanding of what might lead to knowledge-*in*-practice, participants were also asked how often they collaborate with colleagues on topics dealing specifically with teaching mathematics, and also how helpful they found these interactions to be. Answers included: never (1), less than once a month (3), once a month (1), once a week (3), and multiple times a week (3). The helpfulness of these meetings was classified by seven of the participants as "very helpful" and by four of the participants as "sometimes helpful". Ms. Kuhn earlier indicated she never collaborated with colleagues; however, when indicating the helpfulness of meeting with colleagues classified meetings as sometimes helpful and stated that sometimes she was able to meet with colleagues during breaks. Participants able to represent knowledge correctly in all four areas, and who presumably had a better understanding of the topics indicated they collaborated once a month and once a week and found these interactions somewhat and very helpful respectively. This compared to less than once a month and once a week for participants able to represent three of four areas correctly with these participants finding the interactions very and somewhat helpful, and finally less than once a month and multiple times a week for those participants representing two of four areas correctly with both of these participants finding the collaboration very helpful.

In comparison, Ma (1999) reported on Chinese teachers' collaboration, which was typically once a week for at least one hour during which time Chinese teachers studied material related to what was being taught. In addition to these weekly meetings, Chinese teachers also had a shared office room where they corrected homework and worked on lesson plans (Ma, 1999). This setting allowed for increased teacher collaboration and contact on an informal basis. Both novice and veteran Chinese teachers indicated they benefited from this type of interaction with their colleagues, which contributed to their knowledge both in terms of learning the content and how to work with students. It is obvious Chinese teachers collaborated on a more frequent and consistent basis with a more common purpose. Perhaps it is this extended collaboration with a specific purpose that has helped Chinese teachers develop a deeper understanding of mathematics in terms of both content and representation.

Relationship between classroom experience and mathematical abilities

If knowledge-*in*-practice with its classroom experience were truly effective, is this manifest in the ability of teachers to accurately represent knowledge to others? Does this knowledge increase over time? (Again, the two participants who were able to represent knowledge correctly in all four areas in the study had extensive classroom experience, having taught for 15 years and 34 years.) Length of time in the classroom alone does not seem to be an indicator of depth of understanding. Two more participants from the survey were able to represent knowledge correctly in three of the four areas, with teaching experience of 38 and 42 years respectively. Finally, two participants with 1 year of experience and 37 years experience each represented correctly two of four concepts.

It might seem that longer time in the classroom and more frequent opportunities to collaborate with colleagues would most likely lead to increased knowledge-*in*-practice; however, even experienced participants who could only represent correctly one of the four areas had teaching experiences of 8, 13, 21, 32, and 32 years. This is in keeping with previous findings that found that although more experienced teachers may have a more developed teaching script, their scripts are not necessarily better (Schram, Feiman-Nemser, & Ball, 1989). These participants met anywhere from multiple times a week to never, and although some of these met more often than others they also could represent correctly only one of four areas. While the main focus of such meetings varied, some topics mentioned included, among other things: discussing lessons and tests, helping one another with difficulties and challenges both in terms of the curriculum and students, sharing experiences for the benefit of others, discussing the standardization of the degree of difficulty for tests, discussing how they can teach certain subjects to specific students, and discussing the teacher's handbook. A participant with very little experience comparatively speaking, one year, seemed to have more knowledge-*in*-practice than others, and a participant with what might be perceived as mid-level experience, 15 years, was able to perform equally well as another participant with nine years more experience.

It must be concluded that time in the classroom and classroom experiences alone do not produce deep, thorough knowledge.

Views on Mathematical Ability

To gain additional insight into participants' views on mathematical knowledge, which might affect their approaches in the classroom, a series of questions in the interview were posed to uncover what participants believed contributed to a person being either good or not good at math. (See Appendix I). Views on attributes or characteristics that make a person good at mathematics, also added to data on what participants consider sources of knowledge, both for students and for teachers. Various past studies have detailed differences between what factors are believed to be the reason for mathematical achievement (see Wang & Lin, 2005 for an overview). Data to understand how German participants' views compared to previous findings was gained from these questions.

Participants tended to have more than one reason why they thought a person was good at mathematics, and thus more than twenty reasons were provided. Only the most commonly reasons, which were mentioned by all twenty participants are discussed further. The most often cited quality that participants claimed was a determining factor in someone being "good" at math was their mathematical knowledge. The next frequently mentioned characteristic was that of natural talent/ability, followed by the ability to teach or explain mathematics. Four participants mentioned a combination of the first two characteristics, citing both knowledge/effort and ability.

It could be assumed that participants who view mathematical achievement in terms of mathematical knowledge or effort might take a different approach to instruction than participants who indicated talent or ability is the primary factor involved in determining a person's success in mathematics. The third group who cited ability to teach would presumably be more similar to the first group since the ability to explain and teach requires effort and deeper understanding of the material, while those participants who view mathematical success as a combination of talent and effort may in fact approach different students in different ways with different expectations. Indeed, if achievement is attributed to knowledge or effort a teacher and/or student may work harder to master mathematical principles while those whose view entails mathematical success due to ability or talent may simply give up if mastery is not achieved. However, with the lack of classroom observations, data from this study cannot these assumptions.

Comments made by participants who viewed mathematical knowledge as a determinant in being good at mathematics varied from simplistic to more in-depth. Some explanations from participants in this group included:

> Because he possessed subject knowledge. His knowledge has foundation. (Ms. Richter)

> You have to understand numbers and also space orientation. You just have to be able to grasp it. (Ms. Roth)

> Because they have a good understanding of numbers. They have oversight of numbers. They can quickly recognize number combinations and decide on the important things. They can clearly and logically think in a direction, and are not persuaded to move to the left or right. They have a mathematical understanding. (Ms. Riese)

Nearly as many participants also mentioned a natural talent or ability as a primary reason a person might be good at mathematics. Comments involving talent and/or ability included the following:

> It comes naturally, because she has the corresponding knowledge. (Ms. Heinz)
>
> I believe it comes naturally. (Ms. Sanger)
>
> That's difficult. I have the feeling that there is a certain talent that you understand numbers. You have a relationship to numbers, something that I can't duplicate. (Ms. Muller)

While many of these participants also gave other reasons for a person being good at mathematics, the fact that talent and ability is in their opinion a component to being good at mathematics may or may not affect not only their own approach to learning, but also their approach to teaching. Further examination into this area would need to be done to make such connections.

The third most often mentioned attribute of a person good at mathematics was the ability to explain and teach mathematics. Quite often this characteristic was described in conjunction with mathematical knowledge. Examples of these explanations included:

> Every hour was a highlight. Yes. Every hour we added to our knowledge. You could tell he loved what he was doing. Not just for money. Many people become teachers because of the money and the security but I must say, thank goodness, all the teachers I had were teachers because they felt it as a calling. That is important. (Ms. Richter)

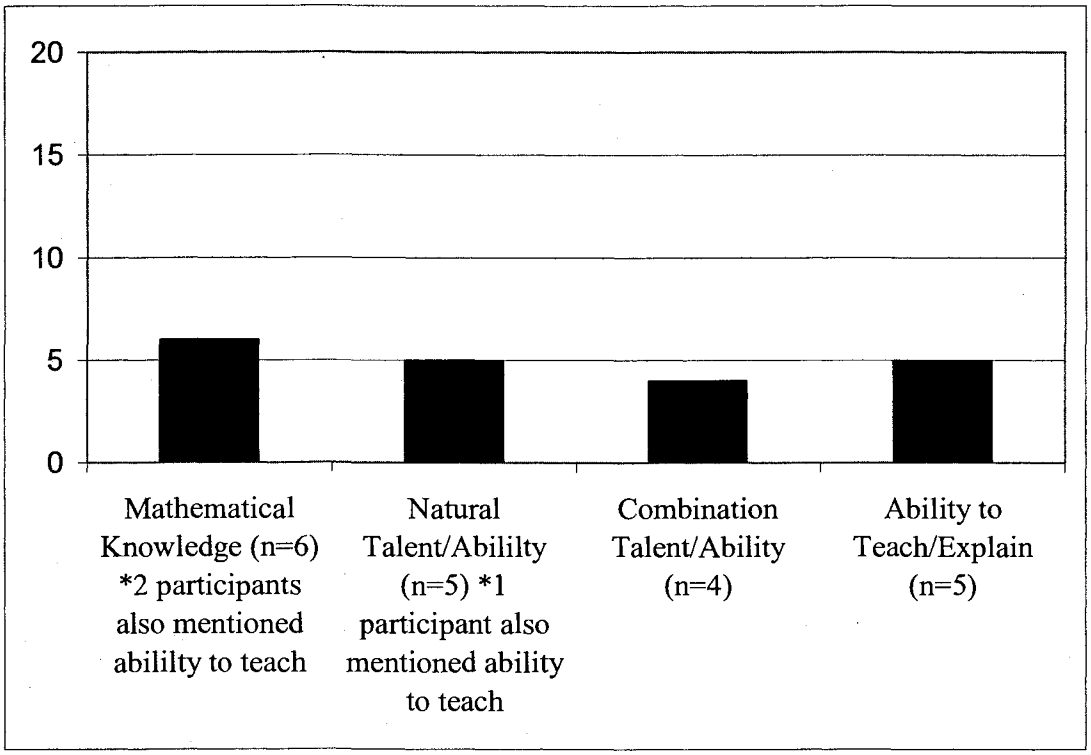
She could explain things very well. I think of my Gymnasium teacher, he could explain things very well also. So I was able to understand very well. (Ms. Trachsel)

There was a teacher and he was good in math instruction. It was clear and comprehensible and I did not have any problems, he did such a good job. Clear and insightful. (Ms. Baum)

He has good structure, he's organized. He can paint a picture so you can visualize. Lots of diagrams and sketches. He knows how to apply the math in everyday life, and comes up with examples. He comes up with situations where you can use math. (Ms. Hoffman)

From the detail and length of the comments from this third group, it seemed that although this characteristic was not the most mentioned, participants could provide more specific reasons for why they thought the ability to teach or explain mathematics made someone good at mathematics. Figure 23 shows the number of participants who mentioned these attributes of being good at mathematics.

Figure 23
Participants' Views on Characteristics of Someone Good at Mathematics



While German participants viewed knowledge and ability nearly evenly in terms of knowledge/effort, talent/ability, or a combination thereof, if ability to teach is categorized with knowledge/effort this category then includes 80% of participants. In comparison, data reported on by Wang and Lin (2005) detailed numerous studies indicating that Chinese and Asian American students view their mathematical success as dependent on their effort while students in the United States attribute mathematical success to ability. Thus, for the most part participants held a view more similar to Chinese and Asian American students than those in the United States. This might provide yet another factor that might lead to queries as to why German students do not perform as well as their Chinese counterparts.

At the teacher level, with a majority of teacher viewing mathematical success due to mathematical knowledge, a combination of knowledge and talent, or the ability to teach, why is it that these participants have not developed a stronger conceptual understanding and ability to represent knowledge to others? Participants have mastered computation and problem solving skills, yet they have not deepened their knowledge to be able to use it as successfully to explain and represent it to others. Of the five participants who only discussed talent/ability, two displayed conceptual understanding or were able to generate accurate representations in three of four areas, two did so in one area, and one did not display such abilities in any of the areas. Likewise, of the six participants who only discussed knowledge/effort two displayed conceptual understanding or were able to generate accurate representations in all four areas, one in three areas, one in two areas, and two in one area. Thus, it may be that view of what makes a person good at mathematics may not be a factor in knowledge acquisition on the part of teachers. Further exploration into these topics is necessary to fully explore views and beliefs and how they might affect classroom approaches.

The Case of Ms. Riese

Contrary to the experiences of most of the participants who had mathematical training as part of their participant preparation, one participant stated that she had not studied mathematics during her teacher preparation at all. As she completed her teacher preparation it included "three areas of emphasis- lots of pedagogy and sociology, psychology. And I did other things- geography, P.E., and physics" (Ms. Riese). Through both survey and interview data, Ms. Riese demonstrated that she was able to solve correctly problems in three of the four areas (in the area of novel theory involving perimeter and area she simply accepted the claim). Having not had any mathematics as part of her teacher preparation, it must be concluded that Ms. Riese developed her mathematical understanding and knowledge elsewhere. Ms. Riese, who did not study any mathematics, but who had indicated she studied a lot of pedagogy in her program, was able to generate accurate representations in three of the four areas. This may, in fact, be an indication of the effectiveness of pedagogical teacher preparation. However, according to Ms. Riese, "Throughout the years I learned and kept up and added to my math knowledge, so I feel fit in it. Even though I didn't ever study it." From Ms. Riese's experiences it is evident that no teacher preparation contributed to her mathematical knowledge since she did not study mathematics. Ms. Riese did not provide an exact number of pedagogical courses taken as part of her teacher preparation; however, two fellow participants completed their teacher preparation program at the same institution as Ms. Riese. Both of these participants referenced their transcripts in answering this question on the survey, and both indicated that according to their transcripts they had taken 74 pedagogical credits. One participant had graduated the same year as Ms. Riese, the other participant five years later. Incidentally, both of these participants had represented correctly only two of four areas. If it is assumed her pedagogical training was similar to that of fellow contemporary graduates of the institution, Ms. Riese's extensive pedagogical training in addition to her 38 years in the classroom do appear to have contributed to her knowledge of effective teaching. In similar fashion to participants of Ma's study (1999) Ms. Riese and Chinese teachers have a more limited teacher training than what is perhaps seen in the United States in terms of content, and yet both Ms. Riese, as an example of German teachers, and the Chinese teachers have a seemingly deep understanding of mathematics, and are able to take such knowledge and provide conceptual explanations and generate accurate representations.

Teacher knowledge

Neither an examination of contribution of teacher preparation courses nor classroom experiences has provided a clear view of what teacher knowledge entails, what factors improve it, or causes of inconsistency from one teacher to another. Comments made by participants during the course of this study indicated that knowledge might in fact be momentary, unique, and contextualized to material currently being taught. Participants indicated that not teaching a specific level affected their ability to accurately recall knowledge they felt they had previously known. Statements to this effect by participants included:

> Dividing fractions is difficult because I don't have to teach it in elementary school I really haven't thought about it. (Ms. Schultz)

> I can't come up with something right now. I don't have to teach that right now so I can't come up with an example right now. Maybe in a minute. (Ms. Schneider)

> Now I don't know if I were still teaching Grundschule [typically comprising grades 1-4] I would have more visual aides. They have a lot of materials for these things. Those are their tools. And in Grundschule there are different boards where you can see how many tens there are and that I really have to borrow one.

> So, you can see optically and not just in numbers. So, they can touch it. Really can't remember. They have math boxes. (Ms. Schwab)

> I can't remember how they taught it to us at the university… (Ms. Hoffman)

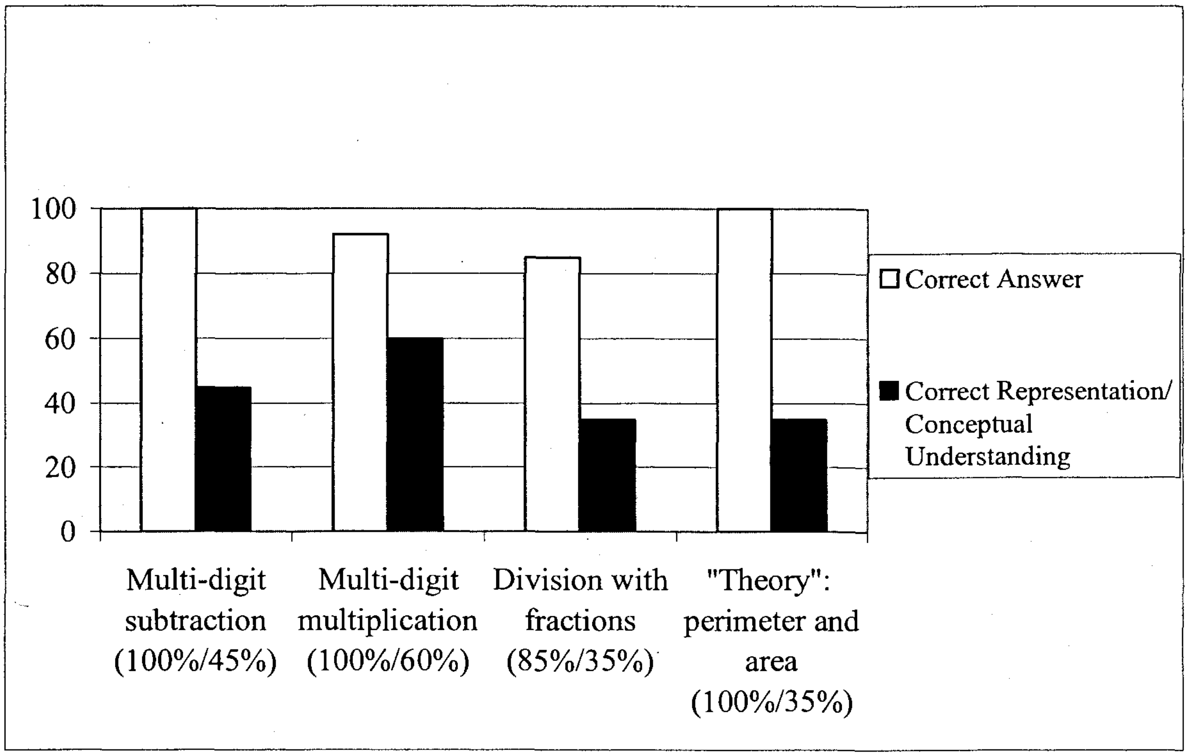
> It's been a long time since I taught Grundschule. I don't know anymore how I would do that. Mmmh. (Ms. Sanger)

These participants did not state they did not know, or had not learned principles and concepts being discussed, but rather that they could not "remember" or that they had not "thought about it" because it was not a part of their normal teaching curriculum. This evidence seems to support the notion that teacher knowledge is not necessarily a constant, unchanging body that once obtained remains with the teacher. If this were accepted as true, then examinations of this nature would necessarily be limited in scope. Some teachers apparently do in fact retain knowledge across various areas as evidenced by participants who were able to successfully represent all four areas; however, this may be due to having taught it more recently than others. It might seem that for a majority of participants the extent of their ability to correctly represent knowledge might be limited to the topic(s) they are currently or have recently covered with their classes. Data collected as part of this study does not address this issue; further research would be needed to confirm this, but it does seem a very plausible explanation that might help account for participants being able to compute basic mathematics correctly, but not being as successful in represent it to others correctly.

Analysis

It has been established this group of participants who teach mathematics in Germany do possess the knowledge and skills necessary to compute and solve word problems dealing with basic mathematics problems correctly; however, it seems they are much less capable in generating accurate representations or in displaying conceptual understanding of underlying mathematical principles. Figure 24 contrasts participant's correct answers with correct representations.

Figure 24
Comparison: Correct Answers vs. Correct Representation



These German participants viewed teacher preparation programs as primarily helpful only in terms of methods and pedagogy, but not particularly with mathematics content. According to these participants, classroom experience contributed to knowledge of effective teaching in terms of both pedagogy and mathematics content. The question remains where did these participants acquire the knowledge and skills to compute basic mathematics? Also, would more of these participants be able to represent knowledge accurately given their content knowledge if the concept was a topic they currently and/or recently taught? Where and how did the participants who were able to represent more than one area correctly learn to do so? It seems these participants were better prepared mathematically than in representing that same knowledge. Perhaps teacher preparation courses were not sufficient, perhaps they have not had enough classroom experience with the topics, or perhaps as alluded to earlier their knowledge is based in the context of what they are teaching.

Conclusion

The German participants outperformed their counterparts from the United States both in solving problems, as well as in generating representations in all four areas researched; however, the Chinese teachers outperformed teachers from both the United States and Germany. It seems that some mathematical areas are more difficult than others. Participants had a harder time correctly computing fractions for instance than subtraction. The same seeming difficulty is manifest in the number of participants able to represent fractions accurately in comparison with the other areas. According to the participants in this study that may be due to the fact that division by fractions is not something typically encountered in every day life, or at least that is the perception held by many.

Based on these data it seems that German participants viewed teacher preparation courses as helpful primarily only in terms of pedagogy and methods. Insofar as mathematics is concerned, it does not seem that teacher education training programs helped prepare participants for their experiences as teachers. Based on the types of courses they remember taking at the university level for mathematics, perhaps different types of courses would have been more useful. Only 5% (or one participant) remembered a course dealing with fractions; these participants were able to solve, but not represent such problems.

Given that more courses are required in both content and pedagogy for German teachers than those in the United States, the push for more courses under the assumption that this would lead to increased teacher knowledge may need to be more carefully researched and analyzed. According to their own perceptions and views, these participants did not consider teacher preparation programs as having contributed positively to their teacher knowledge. This finding speaks not only to sources of knowledge, but also to the debate concerning deregulation and professionalization (Hill, Sleep, Lewis, & Ball, 2007; Angus, 2001; Cochran-Smith & Fries, 2001; Cochran-Smith & Lytle, 1999).

All participants who answered the survey questions indicated different ways that their experiences in the classroom had helped them gain the knowledge and skills necessary for teaching, both in terms of pedagogy and in terms of mathematical content. These comments provide useful insight to help answer the third research question concerning sources of teacher knowledge. Based on this data it would appear that classroom experiences have a greater contribution to participants' knowledge and skill set as teachers than their teacher preparation programs, but again this differed from pedagogy to content. Additionally, comments analyzed concerning views on attributes and characteristics of persons good at mathematics on the part of the participants indicated that a majority consider knowledge/effort or ability to teach as at least contributing to mathematical success; however, differences in views were not necessarily manifest since participants with different views displayed conceptual understanding regardless of expressed views. The knowledge of these participants has now been established. What is now needed is a determination of where such knowledge is developed.

CHAPTER 7

DISCUSSION OF RESULTS, IMPLICATIONS, FUTURE STUDIES, CONCLUSION

The purpose of this study was to contribute to what is currently known concerning teacher knowledge in terms of both subject content knowledge and pedagogical content knowledge, as well as where such knowledge might be developed in a comparative manner. Investigation into German mathematics teachers' knowledge was the vehicle chosen for this examination given the similarities and differences between the United States, as well as Asian countries such as China. This study sought to achieve the ability to add to the knowledge base being used by those seeking to improve teacher education in the United States, whether that is policymakers, other researchers, or institutions preparing teachers.

Discussion of Results

Subject Content Knowledge

This study sought and established the subject content knowledge of German mathematics teachers. Conclusions of this study maintain that participants of this study have a sound subject content knowledge as demonstrated by their ability to solve both computation and word problems in four areas basic and fundamental to mathematics. Participants were particularly strong in multi-digit subtraction, multi-digit multiplication, and dealing with perimeter/area. A weakness of these participants was evident in problems involving division with fractions. Both computationally based and more in-depth word problems were posed to participants for completion. Data collected from those participants providing answers to problems in each of these areas revealed that 100% of participants were able to produce correct answers to multi-digit subtraction and perimeter/area, 92% produced correct answers in multi-digit multiplication, while 85% of participants solved problems involving division with fractions correctly. Survey data alone, which specifically asked for answers indicated 100% success rate in the area of multi-digit multiplication as well. Participants demonstrated they indeed have the knowledge and skills to solve correctly such problems, indicating they possess a sound and complete subject content knowledge.

Comparatively speaking data for computation/word problems was available only in the area of division with fractions. Comparisons in this area revealed that participants of this study were more successful in completing division with fractions problems than teachers in the United States, but not as successful as teachers from China (Ma, 1999). While 85% of German participants solved these problems correctly, only 43% of teachers from the United States correctly solved division with fractions, but 100% of Chinese teachers demonstrated ability to solve these problems correctly. While this was the weakest area for participants, it likely was also the weakest area for teachers from other countries as well. Such findings are also consistent with data from Zhou, Peverly, and Xin (2006) who found that Chinese teachers were much more able in terms of subject content knowledge than their counterparts in the United States. German participant performance would seem to rank quite a bit higher than the United States based on both

Ma (1999) and Zhou, Peverly, and Xin (2006), but still not to the same consistently high levels of knowledge found in Chinese teachers, especially in the area of fractions.

Participants seemed to possess the knowledge spoken of by Ball & McDiarmid (1990) and Grossman (1990) in that they displayed a deep knowledge of content as far as computation and word problem solving was concerned. For the most part these participants were capable and knowledgeable in providing answers. Additionally, participant's data supports contentions by Feiman-Nemser and Remillard (1995) because participants seemed to believe that their subject content knowledge was not gained in teacher preparation, but from previous schooling or during their time in the classroom.

Pedagogical Content Knowledge

Whether these participants could transform their subject content knowledge into accurate representations to others was the focus of the second research question. For each of the above-mentioned mathematics areas, participants were asked additional questions aimed at uncovering not only if and how participants would accurately represent knowledge to others, but also to allow participants to explain and discuss underlying and fundamental principles necessary to understand these areas. In contrast to strong performances in computation and solving word problems, this group of German participants was not as successful in generating accurate representations or discussing underlying principles. Participants were strongest in the areas of multi-digit multiplication with 60% successfully discussing underlying principles and generating representations dealing with place value and conceptual understanding versus a focus on steps and procedural understanding. Though dealing with some similarities in underlying mathematical principles such as place value, only 45% of participants displayed a conceptual understanding of place value and used a regrouping approach when discussing multi-digit subtraction. Participants' reaction to a novel "theory" dealing with perimeter and area was also difficult for participants with 30% of the participants able to provide mathematical approaches for investigating the validity of the so-called theory presented by a student, with an additional 15% discussing the need to prove mathematically, but not presenting any evidence of their own knowledge of what that might entail. Division with fractions was the weakest area for these participants with only 35% of participants generating an accurate representation of the given problem successfully. Many participants spoke of a disconnect between this principle and real life. Some areas of mathematics were more difficult for these participants than others; however, even with 60% success rate in the area with the strongest performance is presumably still not what would be considered ideal. Only 10% of the participants were able to represent accurately and conceptually their knowledge in all four areas, while 15% of participants were not successful in any of the areas. These participants explained that they either had not taught the concept or level.

To summarize comparative findings, German participants were more successful in generating accurate representations and displaying a conceptual understanding of mathematical principles than teachers in the United States, but not as successful as Chinese teachers (Ma, 1999). German participants outperformed teachers in computation and solving word problems by a much greater margin as teachers in the United States than in representing such knowledge. Despite a seemingly strong subject content knowledge on the part of these participants it seems that overall there appear to be substantial gaps in their pedagogical content knowledge. These findings indicate that

German participants have pedagogical content knowledge much more similar to teachers in the United States as found by Ma (1999) and Zhou, Peverly, and Xin (2006) due to the fact that a majority of teachers in both countries were unable to generate representations or discuss underlying principles and concepts.

A strong subject content knowledge but weak pedagogical knowledge would not lead to the type of pedagogical content knowledge that would allow for teachers to represent accurately information in a manner that students would understand the material being presented (Wang & Odell, 2002). This may be a reason that German students perform similar to students in the United States. As maintained by those on the side of professionalization, German participants illustrated what might happen with the presence of subject content knowledge alone and no pedagogical content knowledge to help represent knowledge to students. Participants in this study did indeed manifest strong subject content knowledge, nearing levels shown by Chinese teachers (Ma, 1999), and yet German students' achievement does not match that found in China or other Asian countries. However, given the fact that participants did not attribute their subject content knowledge to teacher preparation programs this finding would actually not necessarily support professionalization. Further research into where these participants developed their knowledge is necessary. Assertions by Ball and Bass (2000) contend that pedagogical mathematics content knowledge would allow teachers to explore, expand, and know when to push students, what explanation to provide, and finally help students understand the content. Besides ability and knowledge to compute and solve problems as a teacher or the ability to find errors in student work, teachers must have a deeper knowledge to understand approaches and methods used by students (as seen in the novel theory situations), and must be able to explain in various ways for students to understand (Hill, Sleep, Lewis, & Ball, 2007). It seems this type of pedagogical content knowledge is lacking for many of these participants. The inability to discuss underlying principles or generate accurate representations leads to the conclusion that as a group these participants would not be able to use their knowledge in the manner discussed by Ball and Bass (2000) or Hill, Sleep, Lewis, and Ball (2007).

Sources for Knowledge

To assess the contributions of teacher preparation and classroom experiences on the knowledge for effective teaching of these participants was the aim of the third research question. According to these participants, teacher preparation courses were primarily only helpful in terms of methods and pedagogy. Surprising about this view held by the participants is that participants demonstrated a much better subject content knowledge than they did pedagogical content knowledge. This would lead to the assumption that subject content knowledge and skills were obtained elsewhere, whether prior to the university, through teaching experience, or elsewhere. It seems the perceived contributions of teacher preparation by these participants were perhaps limited in nature. Participants who were able to represent their knowledge accurately to others in all four areas and participants who were not able to provide evidence of being able to represent their knowledge in any of the areas had in some instances attended the same teacher preparation program. While participants indicated that teacher preparation courses contributed to their preparation to become a teacher, many struggled with representing their knowledge. Classroom experience was viewed as having contributed to teacher knowledge in terms of both mathematics and pedagogy. However, as with teacher preparation years experience did not seem to be an indicator for increased pedagogical content knowledge. Novice and experienced participants were able to represent their knowledge while a wide range of experienced participants had trouble representing their knowledge. Previous studies have also found that both novice and veteran teachers in both the United States and China are able and not able to represent their knowledge accurately (Ma, 1999). A limitation of both this study and of Ma's study (1999) is that classroom observations were not conducted in order to verify and confirm data collected through interview, and in this case survey. Were knowledge-*in*-practice in fact the key to knowledge acquisition, one must wonder why the many years of teaching on the part of these participants who have the subject knowledge did not increase their pedagogical content knowledge. The ability to assess the full extent of contributions of teacher preparation and classroom experience seems somewhat limited given the discrepancy between views of the participants and skills and knowledge evidenced during the course of the study.

Another facet of sources for knowledge examined through questioning to determine attributes and characteristics that participants believed made a person good at mathematics. With a majority of participants indicating that knowledge/effort, or the ability to teach (which would seem to indicate the need for both knowledge and effort) it is surprising that these participants themselves have not developed deeper understanding of mathematics and pedagogical content knowledge to facilitate better representations to others. If effort and knowledge are what is deemed by these participants, who do have the knowledge and skills to compute and solve word problems, to be essential for a person to achieve in mathematics, it would seem they would be able to gain the knowledge and skills necessary to generate accurate representations. Participants from both groups were among those able to generate accurate representations and discuss underlying principles; however, views held by participants, whether in favor of knowledge/effort or talent/ability may influence both their own approach to learning, but also their approach to teaching, which in turn may impact student achievement. A more complete analysis of such views and how they may or may not influence approaches to learning and teaching would be necessary to conclude whether or not this is an important factor to consider when researching teacher knowledge and student achievement.

Comparative Studies

Given the debate concerning what type of knowledge is necessary for effective teaching and where such knowledge is obtained (Shulman, 1986; Angus, 2001; Cochran-Smith & Fries, 2001; Sleep, Hill, Lewis, & Ball, 2007), this study sought to uncover data to broaden the knowledge base to understand better teacher knowledge, both subject content and pedagogical content, as well as the contributions of teacher preparation programs and classroom experience to such knowledge. Data from this study indicated a deep subject content knowledge, but lack of pedagogical content knowledge, which when considering German achievement scores would support work by Shulman (1986) that both types of knowledge are necessary for effective teaching. International comparisons indicate that the United States performs only average (NCES Website, 2006). Understandably various groups in the United States including lawmakers and educators have sought for key components to improve education and student achievement (Hill, Rowan, & Ball, 2005). As a result, numerous comparative studies have been conducted to uncover such components. Typically past comparative studies have resulted in an

Asian-United States comparison, usually involving China and Japan, and have included topics such as teacher knowledge and teacher practice (e.g. Ma, 1999; Stigler & Hiebert, 1999; Perry, 2000). What has been missing in this quest for deeper understanding is analysis of countries that score similarly to the United States, but who may have different approaches to or components of education. Data from this study now begins to fill this gap by providing further insight into an additional country. Germany is a country, which although their approach to teacher education varies from that in the United States with German teachers receiving more pedagogical and subject training, performs similarly to the United States (Kolstad, Coker, and Kolstad 1996; Darling-Hammond & Cobb, 1996). To uncover further data on teacher knowledge, specifically whether German mathematics teachers had the knowledge and skills to solve correctly basic mathematics, whether they could represent accurately such knowledge to others, and according to participants what contributions teacher preparation and classroom experiences made to teacher knowledge, a qualitative interview project utilizing survey and interview were used in order to obtain and verify the data.

Implications

Either through teacher preparation, classroom experience, or some other means this group of German mathematics teachers was able to obtain the knowledge and skills to compute and solve basic mathematics. Certainly these participants displayed a better ability to solve basic mathematics than their counterparts in the United States, but they did not state that their teacher preparation contributed to their mathematical knowledge. Cooney (2001) found that preservice teacher in the United States were not exposed to the type of mathematics needed for teaching at the collegiate level. Despite such connections not made by participants of this study, it may very well be that they were exposed to the types of mathematics needed for teaching. Otherwise, these German participants had developed their subject content knowledge through their years in the classroom either as a student (Feiman-Nemser & Remillard, 1995) or a teacher. Given that years in the classroom did not seem to determine success in correctly solving mathematical problems, it might be assumed that such knowledge is developed before entering the classroom as a teacher. However, as a group these participants were not able to represent accurately their knowledge to others at a consistently high rate. Therefore, without indicators that mathematical subject knowledge was developed through teacher preparation, mandating increased content preparation in teachers at the university level will not necessarily lead to an increase in teacher knowledge and/or understanding of the fundamental mathematical principles that might have a positive impact on student achievement. This finding is in keeping with conclusions by Ma (1999) that indicated that Chinese teachers with far less teacher training actually had better developed teacher knowledge in both content and pedagogical content than teacher in the United States, and now also in Germany.

Data in this area of the study did not clearly support any of the sources of knowledge (Cochran-Smith & Fries, 2001) above another, nor was there support for the professionalism view of current debates (Hill, Sleep, Lewis & Ball, 2007; Angus, 2001). Further research into where knowledge is developed is necessary to address these issues; however, until these issues are resolved, advocating a substantial increase in teachers' content preparation would seem premature. Teacher preparation was viewed to have a positive impact on pedagogical content knowledge for these teachers. These findings would support the professionalization view of teacher knowledge (e.g. Hill, Sleep, Lewis, & Ball, 2007; Angus, 2001), were such a connection made; however, with a much weaker performance in demonstrating pedagogical content knowledge, these data does not serve to promote the professionalization approach to teacher education. It seems subject content knowledge is learned elsewhere, and that pedagogical content knowledge may or may not be learned in the classroom, but certainly with the decreased success rate exhibited by participants in generating accurate representations and discussing underlying principles it appears the years spent in teacher training perhaps did not contribute to knowledge in the ways these teachers needed. Given the short amount of teacher training Chinese teachers receive, data would suggest that Chinese teachers also obtain and build their pedagogical knowledge outside of teacher preparation programs. Despite the fact that this knowledge seemed weaker than subject content knowledge, participants did feel they benefited pedagogically from pedagogical training. Although not as capable at representing knowledge as Chinese teachers, German participants did outperform the teachers from the United States (Ma, 1999). Findings seem to confirm the mottoes assigned by Stigler and Hiebert (1999) in that the German participants were more successful and did indeed seem to be "developing advanced procedures", as opposed to the motto for the United States of "learning terms and practicing procedures." Uncovering commonalities and differences between teacher preparations in all three countries may help teacher preparation programs in the United States to evaluate and refine teacher preparation to improve teacher knowledge. Such research and analysis could prove mutually beneficial to both the United States and Germany.

Overall, results in both subject and pedagogy revealed that German participants outperformed teachers in the United States but also did not equal performance by Chinese teachers in terms of representing knowledge while they do come close to rivaling Chinese teachers' content performance. Increased time spent in teacher preparation may or may not have contributed to stronger content knowledge and a slightly increased pedagogical content knowledge; however somewhere these German participants were exposed to necessary mathematics for teaching. Further research is required to uncover where content knowledge is developed by German mathematics teachers; however, because teachers in the United States do not seem to have strong content knowledge further research is also required to determine whether teacher education or another source should be relied on for teachers in the United States to develop the type of computational skills displayed by teachers from both Germany and China. Results of this study do not indicate that strong content knowledge necessarily allows for accurate representations to others. Ability to compute and solve word problems does not lead to increased ability to represent knowledge to others, which indicates that reform in the United States should not necessarily focus solely on substantially greater amounts of mathematics courses. Thus subject content knowledge should not be looked to as the only type of knowledge that must be addressed in the United States in order to improve education.

German participants spent greater amounts of time in arguably more in-depth teacher preparation programs, and yet the differences between the United States and similarities with China do not seem to be evident of this investment. Although this preparation was viewed as having contributed to teacher knowledge in support of professionalization, without further research these data would seem to support deregulation. Participants' strong subject content knowledge was presumably developed outside of teacher preparation programs, and pedagogical content knowledge was weaker and lacking the same strong performance seen in computation and word problem solving. Perhaps if teacher preparation were structured differently to draw on the strength of what may be an already established subject content knowledge, and then focused on building and developing pedagogical content knowledge German mathematics teachers would have a deeper knowledge and have a more positive impact on student achievement.

The relatability of this study to the TELT study and further findings by Ma (1999) has provided valuable information to broaden understanding of teacher knowledge and to guide educational research. First, the challenge to such influential studies will have an impact because as a third country that exhibits components of both of the first two countries studied (emphasis on subject knowledge similar to China, and score rankings similar to the United States) this study provides another view on the subject of teacher subject content knowledge. This study could and should be a catalyst for further investigations that continue to broaden the understanding of teacher knowledge, what it looks like, where it is obtained, and what the effects thereof are on student achievement.

Perhaps most importantly, results of this study can be used to improve the effectiveness of teacher education that will hopefully have a direct impact on the quality of education programs that would in turn improve the quality of education and level of achievement of students in the United States. While much has been gained from this study in terms of further understanding teacher knowledge, much remains to be uncovered and understood.

Future Research

This study has contributed to the body of knowledge concerning the various forms of teacher knowledge and contributions to such knowledge. Additional research is needed to continue to understand fully the depth and breadth of teacher knowledge. The inclusion of a country that performs similarly to the United States on international comparisons undoubtedly would yield further data of benefit to the United States, but perhaps also to the broader educational international community.

Examining differences in pedagogical requirements and training between Germany and the United States would be helpful in uncovering more specific data concerning what types of methods courses are required and the impact on teachers' ability to represent such knowledge. A possibly revealing study would be further analysis of pedagogical training at specific institutions. In this study graduates from the same teacher's college were able to represent knowledge in all four areas while some of their fellow graduates were not able to provide evidence of ability to represent their knowledge in any of the areas. Similar performances were evident between institutions; isolated analysis of one institution might yield highly valuable data.

Insofar as subject content knowledge is concerned further research must be conducted to attempt to determine at what stage German mathematics teachers develop the knowledge and skills to compute basic mathematics correctly. It may be a product of the school system prior to higher education, a result of teacher preparation, or knowledge that is developed in part through classroom experiences. In conjunction with further research on subject content knowledge, further research examining the different approach Germany uses towards multi-digit multiplication could uncover important data to improve understanding of and ability to compute such problems. Different approaches in the process of completing multi-digit subtraction and multi-digit multiplication could be key in helping students understand the different computations being performed, and to avoid some of the common mistakes of not "moving numbers over." While it may not be realistic to expect that the United States would entirely change the approach used in multi-digit multiplication, perhaps further research could reveal certain aspect that could be incorporated to improve both teacher and student knowledge and understanding of this concept. In the area of subject content knowledge, further study to check and compare student performance in relation to teachers' knowledge to extend understanding as to effects of teacher knowledge on student achievement is another important topic for future research.

Particularly valuable would be further studies examining teacher knowledge as something that may be momentary or based on the context within which a given subject is taught. Evident from study participants was the thought and belief that their inability to accurately represent knowledge was due in part to the fact that the given concept was not what a part of what was being taught at the moment or grade level of these teachers. Studies that examined teachers' ability to accurately represent knowledge to others as such concepts are being introduced and taught in the classroom may reveal significant findings about the nature of teacher knowledge, how it might change from context to context within a given subject, and how teachers acquire and perhaps later restore vital knowledge for effective teaching. Researchers and experts do not agree on what pedagogical content knowledge would look like (Ball & McDiarmid, 1990). Further study into how context and immediate use may contribute to pedagogical content knowledge would perhaps help to define better various aspects of such knowledge.

Finally, studies of this type, but in other countries, would provide an even greater perspective on the impact of teacher subject content knowledge on teachers' ability to accurately represent such knowledge and also on student achievement. A main premise for conducting this study was that while most comparative studies have resulted in an Asian-United States comparison, surely there is much to be learned from other countries who are perhaps more similar than different from the United States. Countries who score similarly to the United States but whose approach to teacher education is different or countries who score differently but whose approach to teacher education is similar can also be sources of tremendously useful data. Furthering work such as this can provide necessary information to improve teacher knowledge and student achievement.

APPENDIX A

ACRONYMS

FIMS: Used to refer to the initial international comparison, now commonly referred to as the First International Mathematics Study

NCRTE: National Center for Research on Teacher Education

NCTM: National Council of Teachers of Mathematics

OECD: Organization for Economic Cooperation and Development

PISA: Program for International Student Achievement

PUFM: profound understanding of fundamental mathematics

SIMS: Second International Mathematics Study

TELT: Teacher Education and Learning to Teach

TIMSS (1995 and 1995): Third International Mathematics and Science Study

TIMSS (2003 and beyond): Trends in International Mathematics and Science Study

APPENDIX B

1967 IEA COMPARISON

Select performances 1967 IEA test including the U.S., Germany, high and low achiever (if applicable) and total mean for the combined scores of the countries. Scores indicate percent correct on the administered test, with standard deviation within that country.

Population 1a	Lowest Achieving (Sweden)	United States	Germany	Highest Achieving (Japan)	Japan	Total (All Countries)
Mean Score	15.7	16.2	Did not participate	31.2	31.2	19.8
Standard Deviation	10.8	13.3		16.9	16.9	14.9
Population 1b	Lowest Achieving (Sweden)	United States	Germany	Highest Achieving (Israel)	Japan	Total (All Countries)
Mean Score	15.3	17.8	25.4	32.3	31.2	23.0
Standard Deviation	10.8	13.3	11.7	14.7	16.9	15.0
Population 3a		United States	Germany	Highest Achieving (Israel)	Japan	Total (All Countries)
Mean Score		13.8	28.8	36.4	31.4	26.1
Standard Deviation		12.6	9.8	8.6	14.8	13.8
Population 3b		United States		Highest Achieving (Germany)	Japan	Total (All Countries)
Mean Score		8.3		27.7	25.3	21.0
Standard Deviation		9.0		7.6	14.3	12.8

APPENDIX C

1995 TIMSS COMPARISON

Select Results of 1995 TIMSS Study. Included are average achievement scores for seventh and eighth grade for the top performing countries, Germany, and the United States.

	8th Grade	7th Grade
Singapore	643	601
Korea	607	577
Japan	605	571
Hong Kong	588	564
Germany	509	484
United States	500	476

APPENDIX D

2003 PISA COMPARISON

Select Results of PISA 2003 Test. Included are the top-performing countries, Germany, and the United States.

Select PISA 2003 Scores	
Finland	544
Korea	542
The Netherlands	538
Japan	534
Germany	503
The United States	483

APPENDIX E

2003 TIMSS FOURTH- GRADE COMPARISON

Average mathematics scale scores of fourth-grade students, select countries: 2003 TIMSS. Included are top performing Asian countries whose average score is higher than the United States and the United States. Note: Germany did not participate.

Country	Average Score
Singapore	594
Hong Kong SAR	575
Japan	565
Chinese-Taipei	564
United States	518

APPENDIX F

2003 TIMSS EIGHTH-GRADE COMPARISON

Average mathematics scale scores of eighth-grade students, select countries: 2003 TIMSS. Included are top performing countries, whose averages are higher than the United States, and the United States. Note: Germany did not participate.

Country	Average Score
Singapore	605
Korea, Republic of	589
Hong Kong SAR	586
Chinese-Taipei	585
Japan	570
United States	504

APPENDIX G

2006 PISA COMPARISON

Select Results of PISA 2006 Test. Included are the top-performing countries, Germany, and the United States.

Select Scores PISA 2006	
Chinese-Taipei	549
Finland	548
Hong Kong- China	547
Korea	547
Switzerland	530
Macao-China	525
Japan	523
Germany	504
United States	474
OECD Total	484
OECD Average	498

APPENDIX H

TEACHER SURVEY

German Mathematics Teachers' Knowledge
Background
1. Please enter today's date. (Day.Month.Year)
2. Please enter the month and year you were born (Month.Year).
3. Gender Male Female

Education Background
4. What is the name of the university/universities you attended for your teacher preparation?
5. In what year did you complete your teacher training?

Mathematical Background
6. To your recollection, how many mathematics courses did you complete as part of your university training?
7. To the best of your knowledge, which of the following mathematical courses did you complete as part of your university training? Please check all that apply.
Analysis I Analysis I Exercises Linear Algebra
Linear Algebra Exercises Subject Knowledge Preparation Other
8. Please list any other mathematics courses you can recall taking as part of your university training that contributed to your mathematical knowledge. Please explain.

Teacher Preparation Background
9. To your recollection, how many pedagogy/teacher preparation courses did you complete at the university?
10. To the best of your knowledge, which of the following teaching courses did you complete as part of your university training? Please check all that apply.
Math Pedagogy Math Pedagogy II Math Pedagogy III
Subject Pedagogy Preparation Other
11. Please list any other teaching/pedagogical courses or other courses you can recall taking at the university that contributed to your knowledge of teaching.

Teaching Background
12. How many years have you been teaching?
13. How many years have you been teaching mathematics?

14. What level(s) have you taught? 1 2 3 4 5 6 7 8 9 10 11+ University
15. What level are you currently teaching?
16. How many years have you taught at this level? 1 2 3 4 5 6

Teaching Context
17. In what city is the school located where you currently teach?
18. To the best of your knowledge, where are students in your class from?
All are from Germany Most are from Germany
About half are from Germany Most are from countries other than Germany
All are from countries other than Germany
19. What is the native language(s) spoken by the students at your school?
20. Where are teachers at your school from?
All are from Germany Most are from Germany
About half are from Germany Most are from countries other than Germany
All are from countries other than Germany

Contributions to Teacher Knowledge of University Training
21. What contributions did your university education make in preparing you for your experiences as a teacher?
22. What contributions did your university education make in terms of helping you teach mathematics? What was the role of this training in helping you know how to represent mathematical concepts to students?

Contributions to Teacher Knowledge of Classroom Experiences
23. What contributions have your classroom experiences made in obtaining necessary knowledge and skills for effective teaching? Please explain.
24. What contributions have your classroom experiences made to your mathematical knowledge and teaching knowledge as far as teaching mathematics is concerned?

Contribution to Teacher Knowledge through Working with Colleagues
25. How often do you work with other teachers at your school on topics specifically dealing with mathematics, such as: increasing your personal mathematical knowledge, developing mathematics lessons, or discussing other mathematical issues?
Multiple times a week Once a week Multiple times a month
Once a month Less than once a month Once or twice a year Never
26. What is the primary focus of such interaction? What do you find most beneficial from such interaction in helping you as a mathematics teacher?
27. How helpful do you find working with other teachers to be in terms of building your mathematical teaching knowledge either in terms of actual math knowledge and/or approaches to teaching math?
Very helpful Somewhat helpful Sometimes helpful
Rarely helpful Never helpful
28. In working with other teachers, what type(s) of interaction have you found most helpful in contributing to your knowledge of mathematics and your knowledge of teaching mathematics?

29. How could interactions with colleagues be more helpful in terms of developing your math knowledge and/or approaches to teaching mathematics?

Math Knowledge

Please answer the following mathematical questions to the best of your abilities without using a calculator. Click "Next" for the first mathematical question.

Fractions

30. Divide: 8/35 by 4/15

Please show the steps of your calculation.

31. Kurt had $240. He spent 5/8 of it. How much money does he have left?

Please show the steps of your calculation.

32. As a teacher what types of common problems do you anticipate students might have when learning to divide fractions (such as the problem below)? How would you deal with these obstacles?

1 2/3 divided by 1/4

Multi-Digit Subtraction

33. Please compute the answer to the following problem:

Subtract: 6000 –2369

34. There are 30 people in the music room. There are 74 people in the gymnasium. How many more people are in the gymnasium than the music room?

Please show the steps of your calculation.

35. If you were evaluating and grading student work on the following problem, on a ten-point scale what grade would you assign? Ten points is the maximum possible, six points would barely pass, one point is the lowest possible. Please explain the grade you would assign and why.

236 -179 = 65

Multi-Digit Multiplication

36. Please compute the answer to the following problem:

Multiply: 345 x 476

Please show the steps of your calculation.

37. A person's heart is beating 72 times a minute. At this rate, about how many times does it beat in one hour?

Please show the steps of your calculation.

38. What approaches do you use as a teacher when teaching multi-digit multiplication, such as the problem below?

123 x 645

Perimeter

39. Please compute the perimeter for a rectangle with the following dimension:

8 m wide x 5 m long

Please show the steps of your calculations.

40. A thin wire 20 centimeters long is formed into a rectangle. If the width of this rectangle is 4 centimeters, what is its length?

Please show the steps of your calculations.

41. How would you respond in the following scenario?

Your student says that he has "discovered" a "new theory" which states as the perimeter of a rectangle increases, so does the area of the rectangle.

APPENDIX I

INTERVIEW QUESTIONS

First I wanted to ask you about why you came into teaching.

When did you first start thinking about being a teacher?

Why were you interested in teaching?

When you think back to your own experience in elementary school, what stands out to you?

What are the major differences between your own experience as elementary student, and elementary teacher?

- What do you mean?
- Can you give me an example?
- Is there anything else you remember?
  About teachers? What you learned? How you felt about different subjects?

Do you remember anything about the different subjects you learned in elementary school?

My research is focusing specifically on math, so I'm interested in your own past experiences with math and math teaching.

What do you remember about learning math in elementary school?

What about at the high school level?

What about at the university level?

What courses did you take at the university?

What stands out to you about math at the university?

Now, I'd like for you to think about someone who is good at math.

- Who is that person?
- Why do you think ____________ is good at math? What does he\she do?
- What is your hunch\idea about why this person is good at math?
- What do you mean? Can you give me an example? What does x have to do with being good at math?

What about on the other side? Do you know anyone that is not good at math?

- Why do you think of ___________ as not very good at math?
- Do you have any ideas about why______________ is not good at math?
- If says self, what explanation do you give yourself for not being good at math?

Do you have a favorite subject or favorite area within a particular subject?

Are there some things in math you especially like\enjoy?

For this question, think of the grade you are currently teaching. What grade is that? Let's say early in the fall the principal comes to you and asks you what your goals are for your students. What would you say in describing the most important things you'd be trying to accomplish across the year with your _____ grade students?

Some sixth grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate 123 x 645 the students seemed to be forgetting to move the numbers over on each line. They were doing this, instead of this.

```
  123                    123
 X645                  x 645
  615                    615
  492                   492
  738                  738
 1845                  79335
```

While the teachers agreed this was a problem, they couldn't agree what was the best way to solve the problem. What would you do if you were teaching sixth grade and you noticed your students had this problem?

Where did you get this idea?

What if some students ask why they need to move the numbers over? How would you explain this?

Division by fractions is often confusing. People seem to have different approaches to solving problems involving division by fractions. Do you remember how you were taught to divide fractions?

How would you solve a problem like this?

$1\frac{3}{4}$ divided by $\frac{1}{2}$

Many people find this difficult. What do you think makes this difficult?

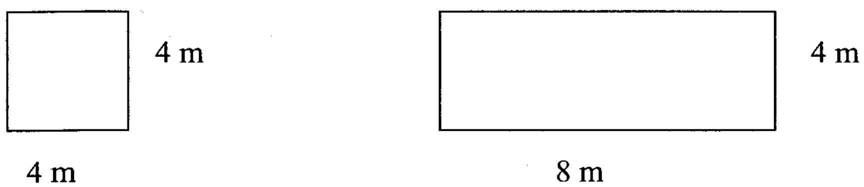
Something many teachers do is try to relate a problem to a real-world situation. This can be very difficult. Can you think of a story problem or real-world situation this might apply to that you could use to help teach your students?

How does that fit with $1\frac{3}{4}$ divided by $\frac{1}{2}$?

Would this story fit well with this problem?

Many people find this difficult to do. Why do you think it is difficult to do? (To come up with a story problem that fits this type of fraction problem.)

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:



perimeter= 16 m
area= 16 square m

perimeter= 24 m
area= 32 square m

How would you respond to this student?

Occasionally it could be that something comes up where you are not sure yourself about whether the mathematics is correct or not. I'm interested in how you think you would respond in such a situation. What would you do with or say to the student?

Would you say or do anything else?

How would you introduce double-digit subtraction to your students? (For example 64-46)

If I were to come to your classroom, what would I see?

- How did you come up with this method?
- Can you think of another approach?

What problems do students have with this type of problem?

How would you know your students understand the principles involved with this type of math?

If parents were to ask you why so much time is spent on basic math- addition, subtraction, multiplication, division- how would you respond?

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