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Estimates at infinity for positive solutions to problems involving the p-Laplacian

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ESTIMATES AT INFINITY FOR POSITIVE SOLUTIONS TO PROBLEMS INVOLVING THE p-LAPLACIAN

by

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ABSTRACT

Estimates at Infinity For Positive Solutions to Problems Involving the p-Laplacian

by

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There has been much study of finding positive solutions to various logistic problems involving the Laplacian and the p-Laplacian; problems which, loosely speaking, contain a nonlinear term that behaves like $\lambda u^{p-1}(1-u^{\gamma})$.

Du and Ma [12] studied a logistic problem for the Laplacian in $\mathbb{R}^N$, looking for positive solutions (λ, u) to

$$-\Delta u = \lambda a(x)u - b(x)u^{\gamma} \text{ in } \mathbb{R}^N$$

where $\gamma > 1$, $0 < b(x) \in C^{\infty}(\mathbb{R}^N)$, $0 < a(x) \in C^1(\mathbb{R}^N)$ and $a(x) \leq P(x)$ for a radially symmetric $P(x)$ satisfying

$$\int_{\mathbb{R}^N} \frac{P(x)}{|x|^{N-2}} < \infty.$$

Next, there has been study of logistic problems with harvesting, where we subtract a harvesting term of the form $\mu h(x)$. These arise from problems in fishery or hunting

a harvesting term of the form $\mu h(x)$. These arise from problems in fishery or hunting management, and in those circumstances one is interested in finding positive solutions. In [22], Oruganti, Shi and Shivaji looked for results in bounded domains, finding positive solutions to

$$\begin{cases} -\Delta u = au - bu^2 - ch(x) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $a, b, c > 0$ are constants, Ω is a smooth bounded region with $\partial\Omega \in C^2$ and $h \in C^\alpha(\bar{\Omega})$ satisfies

$$h(x) > 0 \text{ for } x \in \Omega, h(x) = 0 \text{ for } x \in \partial\Omega, \max_{x \in \bar{\Omega}} h(x) = 1.$$

They proved that if $a > \lambda_1$ then there exists a constant $c_1 = c_1(a, b)$ such that for $0 < c < c_1$, (0.1) has a positive solution $u \in C^{2,\alpha}(\bar{\Omega})$ satisfying $au(x) - bu^2(x) - ch(x) > 0$ for all $x \in \Omega$. In addition, there exists $c_2 = c_2(a, b) > c_1$ such that:

(i) for $0 < c < c_2$, (0.1) has a maximal positive solution u, so that for any other solution v of (0.1) we have $u \geq v$ on Ω,

(ii) for $c > c_2$, (0.1) has no positive solution.

In addition, Oruganti, Shi and Shivaji in [23] were able to extend the above problem to the p-Laplacian, finding positive solutions to

$$\begin{cases} -\Delta_p u = au^{p-1} - u^{\gamma-1} - ch(x) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{0.2}$$

where $a, c > 0$ are constants, $\gamma > p$, Ω is a smooth bounded region with $\partial\Omega \in C^{1,\beta}$ and $h \in C^{\alpha}(\bar{\Omega})$ is a nontrivial function satisfying

$$h(x) \geq 0 \text{ for } x \in \Omega, h(x) = 0 \text{ for } x \in \partial\Omega, \max_{x\in\bar{\Omega}} h(x) = 1.$$

For this problem they showed, for $a > \lambda_1$:

(i) there exists $c_1 = c_1(a) > 0$ such that for $0 < c < c_1$, (0.2) has a weak positive solution $u \in C^{1,\alpha}(\bar{\Omega})$ satisfying $u(x) \geq \left(\frac{ch(x)}{\lambda_1}\right)^{1/(p-1)}$ for $x \in \bar{\Omega}$.

(ii) there exists $c_2 = c_2(a) > c_1$ such that for $0 < c < c_2$, (0.2) has a maximal positive solution, and for $c > c_2$ (0.2) has no positive solutions.

Costa, Drabek and Tehrani also extended the result of Oruganti, Shi and Shivaji for the Laplacian in bounded domains, however instead of extending to the p-Laplacian, they instead extended the result from a bounded domain Ω to all of $\mathbb{R}^N$, finding positive solutions to

$$-\Delta u = a(x)(\lambda u - u^{\gamma}) - \mu h(x) \text{ in } \mathbb{R}^N,$$

where $\gamma > 1$, $\lambda > \lambda_1$, $0 < a(x) \in L^{N/2}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $0 < h(x)$ is a rapidly decreasing function in $\mathbb{R}^N$. They showed that there exists $\hat{\mu} = \hat{\mu}(\lambda) > 0$ such that for all $0 < \mu < \hat{\mu}$ there exists a solution $u_\mu > 0$ in $\mathbb{R}^N$ satisfying

$$u_\mu \geq \frac{C}{|x|^{N-2}} \text{ for } x \text{ large.}$$

In this paper we will extend the results above. We will first look for positive solutions to

$$-\Delta_p u = a(x)\lambda |u|^{p-2}u - a(x)g(u) \quad x \in \mathbb{R}^N \tag{0.3}$$

where $g(s)$ behaves like $s^{\gamma-1}$, $\gamma > p$, for s large. We will employ many of the same methods as Costa, Drabek and Tehrani, and in doing so will not only prove the existence of positive (weak) solutions, but will also have estimates for the behavior of these solutions at infinity. Namely we will show that a solution u_0 satisfies

$$u_0(x) \geq C|x|^{-\frac{N-p}{p-1}} \text{ for } |x| \text{ large.}$$

In addition, by an appropriate modification of our assumptions on $g(u)$ and $a(x)$, we prove that u_0 is the unique positive solution to (0.3), and that the above estimate at infinity is sharp.

Second, we will generalize our first scalar result to a system result, finding positive solutions (u_0, v_0) to

$$\begin{cases} -\Delta_{p_1} u = a_1(x)(\mu_1 |u|^{p_1-2}u - g_1(u)) + F_u(x,u,v) \\ -\Delta_{p_2} v = a_2(x)(\mu_2 |v|^{p_2-2}v - g_2(v)) + F_v(x,u,v) \end{cases}$$

satisfying

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p_1}{p_1-1}}} \text{ for } |x| \text{ large,}$$

and

$$v_0(x) \geq \frac{C}{|x|^{\frac{N-p_2}{p_2-1}}} \text{ for } |x| \text{ large,}$$

where the interaction term behaves like $F(x,s,t) = b(x)s^{\frac{p_1}{m}}t^{\frac{p_2(m-1)}{m}}$ with $m \leq p_1$ and $\frac{m}{m-1} \leq p_2$.

Finally, we will add harvesting terms to our system equations, finding solutions (u_0, v_0) to

$$\begin{cases} -\Delta_{p_1} u = a_1(x)(\mu_1|u|^{p_1-2}u - g_1(u)) + F_u(x,u,v) - \nu_1 h_1(x) \\ -\Delta_{p_2} v = a_2(x)(\mu_2|v|^{p_2-2}v - g_2(v)) + F_v(x,u,v) - \nu_2 h_2(x) \end{cases}$$

In this case, we must have $p_1 = p_2 = 2$ to prove that the solution is positive and satisfies the same behaviors at infinity as above.

We note that the method used by Oruganti, Shi and Shivaji was sub and super solutions, but that all the other problems above (Du and Ma, Costa, Drabek and Tehrani, and the problems in this paper) are solved by applying minimization methods to the underlying functional.

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On a personal note, I want to thank my family and friends who supported me all these years. My sons Jacob and RJ, my mother Rita Thomas, my sisters Katy, Sari and Colleen, and my very close friends Dan and Gary Mack.

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Finally, I really owe all of this, and all that I have in life, to my father Tom Thomas. Words simply cannot describe all he has done for me. A more supportive man does not exist in this world.

CHAPTER 1

BACKGROUND

We begin by discussing some of the origins and applications of the problems contained in this dissertation.

There has been much study of finding positive solutions to various logistic problems involving the Laplacian and the p-Laplacian; problems which, loosely speaking, contain a nonlinear term that behaves like $\lambda u(1-u)$.

In [22], Oruganti, Shi and Shivaji examined a logistic problem with harvesting in bounded domains, finding positive solutions to

$$\begin{cases} -\Delta u = au - bu^2 - ch(x) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $a, b, c > 0$ are constants, Ω is a smooth bounded region with $\partial\Omega \in C^2$ and $h \in C^\alpha(\bar{\Omega})$ satisfies

$$h(x) > 0 \text{ for } x \in \Omega, h(x) = 0 \text{ for } x \in \partial\Omega, \max_{x \in \bar{\Omega}} h(x) = 1.$$

As noted in their paper, this problem arises from the population biology of one species, which we describe here.

Let $u(t,x)$ be the concentration of the species or the population density. We assume that (a) the species disperses randomly in the bounded environment Ω; (b) the reproduction of the species follows the logistic growth; (c) the boundary $\partial\Omega$ of the environment is hostile to the species; and (d) the environment Ω is homogeneous (i.e., the diffusion does not depend on x). Then it is well known that $u(t,x)$ satisfies the reaction-diffusion equation

$$\frac{\partial u}{\partial t} = D\Delta u + au\left(1 - \frac{u}{N}\right), \quad (t,x) \in (0,T) \times \Omega,$$

with the initial and boundary conditions

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega,$$
$$u(0,x) = u_0(x) \geq 0, \quad x \in \Omega,$$

where $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate and $N > 0$ is the carrying capacity of the environment. (See Murray [21] for details). Now, in many ecological systems, harvesting or predation of the species occurs. For example, fishing or hunting of the species u could happen. Hence it is natural to add a harvesting term to the right-hand side, and the equation becomes

$$\frac{\partial u}{\partial t} = D\Delta u + au\left(1 - \frac{u}{N}\right) - p(t,x,u),$$

where $p(t,x,u) \geq 0$ for all possible (t,x,u) values. In (1.1) is considered the case of constant yield harvesting (not dependent on the density u or on t), and in particular the case

$$p(t,x,u) \equiv ch(x),$$

where $c > 0$ is a parameter which represents the level of harvesting. So $ch(\cdot)$ can be understood as the rate of the harvesting distribution, and the harvesting only occurs in the interior of the environment. Such a harvesting pattern arises naturally from fishery management problems, where $ch(x)$ is related to the fishing quota imposed by regulating authorities. Finally, the problem is reduced to (1.1) by applying a standard non-dimensionalization process and then looking for steady state solutions.

In addition, Oruganti, Shi and Shivaji in [23] were able to extend the above problem to the p-Laplacian, finding positive solutions to

$$\begin{cases} -\Delta_p u = au^{p-1} - u^{\gamma-1} - ch(x) & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $a, c > 0$ are constants, $\gamma > p$, Ω is a smooth bounded region with $\partial\Omega \in C^{1,\beta}$ and $h \in C^{\alpha}(\bar{\Omega})$ is a nontrivial function satisfying

$$h(x) \geq 0 \text{ for } x \in \Omega, h(x) = 0 \text{ for } x \in \partial\Omega, \max_{x\in\bar{\Omega}} h(x) = 1.$$

The extension of their result from the Laplacian to the p-Laplacian is a natural mathematical one, however the physical relationship between the problem and the study of population density becomes severed. One may ask if the p-Laplacian has any physical applications? The answer is affirmative, as the p-Laplacian arises in the study of non-Newtonian fluids, reaction-diffusion problems, flow through porous media, petroleum extraction, torsional creep problems, and glaciology. In fact, we conclude this chapter by presenting a physical model that involves the p-Laplacian, from Drabek [9].

We present a mathematical model of the behavior of compressible fluid in a homogeneous isotropic rigid porous medium. Let $\rho = \rho(x,t)$ denote the density, ϕ be a volumetric moisture content and $\overrightarrow{V} = \overrightarrow{V}(x,t)$ be a seepage velocity. Then the continuity equation reads as follows:

$$\phi\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\overrightarrow{V}) = 0. \tag{1.3}$$

In the lamizar regime through the porous medium the momentum velocity $\rho\overrightarrow{V}$ and the pressure $P = P(x,t)$ are connected by the Darcy law

$$\rho\overrightarrow{V} = -\lambda\nabla P. \tag{1.4}$$

In turbulent regimes, however, the flow rate is different and several authors proposed a nonlinear relation instead of (1.4). Namely, the nonlinear Darcy law of the following form is often considered:

$$\rho\overrightarrow{V} = -\lambda|\nabla P|^{\alpha-2}\nabla P, \tag{1.5}$$

where $\alpha > 1$ is a suitable real constant. Taking into account the equation of state for the polytropic gas

$$P = c\rho$$

with some constant of proportionality $c > 0$, we get from (1.3) and (1.5) the equation

$$\phi \frac{\partial \rho}{\partial t} = c^{\alpha-1} \lambda \mathrm{div}(|\nabla \rho|^{\alpha-2} \nabla \rho).$$

After the change of variables and notations this equation becomes

$$\frac{\partial u}{\partial t} = \mathrm{div}(|\nabla u|^{p-2} \nabla u),$$

where $p > 1$, giving us an application of the p-Laplacian, which is defined by:

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u).$$

CHAPTER 2

PRELIMINARIES

All integrals will be assumed to be taken over $\mathbb{R}^N$ unless otherwise stated. Define $D^{1,p} = D^{1,p}(\mathbb{R}^N)$ to be the completion of $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\| = (\int |\nabla u|^p)^{1/p}$, and define, for a bounded domain $\Omega \subset \mathbb{R}^N$, $W^{1,p}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ under the norm $\|u\|_{W^{1,p}(\Omega)} = (\int_\Omega |\nabla u|^p + |u|^p)^{1/p}$. Define the norm on $L^r = L^r(\mathbb{R}^N)$ by $\|u\|_r = (\int |u|^r)^{1/r}$, and define the norm on the weighted L^r space $L^r_{a(x)} = L^r_{a(x)}(\mathbb{R}^N)$ by $\|u\|_{r,a(x)} = (\int a(x)|u|^r)^{1/r}$. We denote $p^* = \frac{Np}{N-p}$ and note that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. We let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ be the p-Laplacian.

Theorem 2.1. *(Sobolev Embedding Theorems)*

(i) $D^{1,p}$ is continuously embedded into L^{p^*}. In other words, there exists a constant C depending only on p and N such that $\|u\|_{p^*} \le C\|u\|$ for all $u \in D^{1,p}$.

(ii) For a bounded domain $\Omega \subset \mathbb{R}^N$ and for $p \le q < p^*$, $W^{1,p}(\Omega)$ is compactly embedded into $L^q(\Omega)$, and $W^{1,p}(\Omega)$ is continuously embedded into $L^{p^*}(\Omega)$.

Proof: See, for example, McOwen [20]. □

Lemma 2.2. *For $a(x) \in L^{N/p}$ we have $D^{1,p} \subset L^p_{a(x)}$. In particular there exists a constant $C > 0$ such that*

$$\int a(x)|u|^p \le C\|a\|_{N/p}\|u\|^p$$

for all $u \in D^{1,p}$.

Proof: Let $u \in D^{1,p}$. Then we have, by Holder's Inequality and the Sobolev Embedding Theorem,

$$\begin{aligned} \|u\|^p_{p,a(x)} &= \int a(x)|u|^p \\ &\leq \|a\|_{N/p}\|u^p\|_{N/(N-p)} \\ &= \|a\|_{N/p}\|u\|^p_{p^*} \\ &\leq C\|a\|_{N/p}\|u\|^p < \infty, \end{aligned}$$

proving the lemma. □

Lemma 2.3. *(Hardy's Inequality)*

For $u \in D^{1,p}$ *we have*

$$\int \frac{|u|^p}{|x|^p} \leq \left(\frac{p}{N-p}\right)^p \int |\nabla u|^p.$$

Proof: Let $u \in C_0^\infty$, $1 < p < N$. Then

$$\int \operatorname{div}\left(\frac{x|u|^p}{|x|^p}\right) = 0$$

which implies that

$$\int \frac{(N-p)|u|^p}{|x|^p} + p|u|^{p-2}u\frac{x \cdot \nabla u}{|x|^p} = 0,$$

and hence, by applying Holder's Inequality,

$$\begin{aligned}\frac{N-p}{p}\int\frac{|u|^p}{|x|^p} &= \left|\int\frac{|u|^{p-2}u}{|x|^{p-1}}\cdot\frac{x\cdot\nabla u}{|x|}\right| \\ &\leq \left(\int\left(\frac{|u|^{p-1}}{|x|^{p-1}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int|\nabla u|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore

$$\frac{N-p}{p}\left(\int\frac{|u|^p}{|x|^p}\right)^{\frac{1}{p}}\leq\left(\int|\nabla u|^p\right)^{\frac{1}{p}}$$

i.e.

$$\int\frac{|u|^p}{|x|^p}\leq\left(\frac{p}{N-p}\right)^p\int|\nabla u|^p.$$

The result now follows by approximating any $u\in D^{1,p}$ by C_0^∞ functions. □

Theorem 2.4. *(Vitali's Convergence Theorem)*

Let (X,μ) be a positive measure space with $\{f_n\}\subset L^1(\mu)$ satisfying

(i) $\mu(X)<\infty$,

(ii) $\{f_n\}$ is uniformly integrable,

(iii) $f_n(x)\to f(x)$ a.e. as $n\to\infty$, and

(iv) $|f(x)|<\infty$ a.e. Then $f\in L^1(\mu)$ and

$$\lim_{n\to\infty}\int_X|f_n-f|d\mu=0.$$

Proof: See, for example, Rudin [24]. □

Lemma 2.5. *(i) $\|\cdot\|^p$ is continuous and convex for $p>1$.*

(ii) A functional defined on a reflexive Banach space that is continuous and convex

is weakly lower semi-continuous (l.s.c.)

(iii) A functional ϕ defined on a reflexive Banach space E that is weakly l.s.c. and coercive (i.e. $\phi(u) \to \infty$ as $\|u\|_E \to \infty$) must be bounded from below. In addition, there exists $u_0 \in E$ such that $\phi(u_0) = \inf_{u\in E} \phi(u)$.

(iv) $D^{1,p}$ is a reflexive Banach space.

Proof: See, for example, Costa [6]. □

Lemma 2.6. *Define $G : D^{1,p} \to (D^{1,p})^*$ by $\langle G(u), v\rangle = \int a(x)|u|^{p-2}uv$, where $a(x) \in L^{N/p} \cap L^\infty$. Then G is compact.*

Proof: This follows from Lemma 2.2ii of Drabek and Huang [10]. We include the proof for the reader's convenience.

Let $\frac{1}{p} + \frac{1}{p'} = 1$. We first claim that for any $\epsilon > 0$ there exists $K > 0$ (depending only on ϵ and a) such that

$$\sup_{\|v\|\leq 1} \int_{|x|>K} a|\phi|^{p-1}|v| \leq \epsilon\|\phi\|^{p-1}$$

for all $\phi \in D^{1,p}$. To see this, we have by Holder's inequality and Sobolev Embedding

$$\begin{aligned}
\sup_{\|v\|\leq 1} \int_{|x|>K} a|\phi|^{p-1}|v| &\leq \sup_{\|v\|\leq 1} \left(\int_{|x|>K} a|\phi|^p\right)^{1/p'} \cdot \left(\int_{|x|>K} a|v|^p\right)^{1/p} \\
&\leq \sup_{\|v\|\leq 1} \left(\int_{|x|>K} a^{N/p}\right)^{(p-1)/N} \left(\int_{|x|>K} |\phi|^{p^*}\right)^{(p-1)/p^*} \\
&\quad \cdot \left(\int_{|x|>K} a^{N/p}\right)^{1/N} \cdot \left(\int_{|x|>K} |v|^{p^*}\right)^{1/p^*} \\
&\leq C \sup_{\|v\|\leq 1} \left(\int_{|x|>K} a^{N/p}\right)^{p/N} \cdot \left(\int_{|x|>K} |\phi|^{p^*}\right)^{(p-1)/p^*} \|v\| \\
&\leq \epsilon\|\phi\|^{p-1},
\end{aligned}$$

since $a \in L^{N/p}$. Now suppose that $u_n \rightharpoonup u_0$ in $D^{1,p}$. Then we have

$$\begin{aligned}
\|G(u_n) - G(u_0)\|_{(D^{1,p})^*} &= \sup_{\|v\|\le 1} |\langle G(u_n) - G(u_0), v\rangle| \\
&= \sup_{\|v\|\le 1} \left| \int a(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)v \right| \\
&\le \sup_{\|v\|\le 1} \left| \int_{|x|\le K} a(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)v \right| \\
&\quad + \sup_{\|v\|\le 1} \left| \int_{|x|>K} a(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)v \right| .
\end{aligned}$$

Now, given $\epsilon > 0$ we can choose $K > 0$ such that the integral over $\{|x| > K\}$ is less than $\epsilon/2$ for all n, while for this fixed K, by strong convergence of u_n to u_0 in L^{p^*} on any bounded region, the integral over $\{|x| \le K\}$ is less than $\epsilon/2$ for n large enough. Therefore $G(u_n) \to G(u_0)$ in V^*, i.e. G is compact. □

Theorem 2.7. *Consider weak solutions to*

$$\text{div } A(x, u, \nabla u) + B(x, u, \nabla u) = 0 \tag{2.1}$$

where for some domain $\Omega \subset \mathbb{R}^N$ *we have*

$$A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N, \; B : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

satisfying for all $M < \infty$ and all $(x, u, v) \in \Omega \times (-M, M) \times \mathbb{R}^N$:

$$
\begin{aligned}
|A(x,u,v)| &\le a_0|v|^{\alpha-1} + |a_1(x)u|^{\alpha-1}, \\
v \cdot A(x,u,v) &\ge |v|^{\alpha} - |a_2(x)u|^{\alpha}, \\
|B(x,u,v)| &\le b_1(x)|v|^{\alpha-1} + (b_2(x))^{\alpha}|u|^{\alpha-1} + (b_3(x))^{\alpha},
\end{aligned}
$$

where $\alpha > 1$, a_0 are constants, $a_i(x)$ and $b_i(x)$ are non-negative functions in $L^\infty(\Omega)$ with $\|a_i\|_\infty, \|b_i\|_\infty \le \mu$, and α, a_0, b_0, $a_i(x)$, and $b_i(x)$ all may possibly depend on M. Then we have:

(i) Let $u(x)$ be a weak supersolution of (2.1) in a cube $K = K(3\rho) \subset \Omega$, with $0 \le u$ in K. Then, with χ_K denoting the usual characteristic function of a set K,

$$\rho^{-N/\gamma}\|u(x)\chi_{K(2\rho)}\|_\gamma \le C \min_{K(\rho)} u(x)$$

for any

$$\begin{cases} \gamma < \frac{N(\alpha-1)}{N-\alpha} & \text{if } \alpha \le N \\ \gamma \le \infty & \text{if } \alpha > N \end{cases}$$

where $C = C(\alpha, N, a_0, \mu\rho)$

(ii) Let $u(x)$ be a weak subsolution of (2.1) in a cube $K = K(3\rho) \subset \Omega$, with $0 \le u$ in K. Then

$$\max_{K(\rho)} u(x) \le C\rho^{-N/\gamma}\|u\chi_{K(2\rho)}\|_\gamma$$

for any $\gamma > \alpha - 1$, where $C = C(\alpha, N, a_0, \mu\rho)$.

Proof: This is proved in Trudinger [27], Theorems 1.2 and 1.3, where we apply the case $b_0 = 0$ (see the note between Theorems 1.1 and 1.2) $\square$

Corollary 2.8. *Consider weak solutions to*

$$-\Delta_p u = c(x)|u|^{p-2}u + d(x) \tag{2.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$, *where* $u \in D^{1,p}$, $1 < p < N$ *and* $c(x), d(x) \in L^\infty(\Omega)$. *Suppose* $K = K(3\rho) \subset \Omega$ *is a cube and that* $u \geq 0$ *in* K. *Then*

(i) If u *is a weak supersolution of (2.2) in* K *then*

$$\rho^{-N/\gamma}\|u\chi_{K(2\rho)}\|_\gamma \leq C \min_{K(\rho)} u(x)$$

for any $\gamma < \frac{N(p-1)}{N-p}$, *where* $C = C(p, N, \|c\|_\infty, \|d\|_\infty)$.

(ii) If u *is a weak subsolution of (2.2) in* K *then*

$$\max_{K(\rho)} u(x) \leq C\rho^{-N/\gamma}\|u\chi_{K(2\rho)}\|_\gamma$$

for any $\gamma > p - 1$, *where* $C = C(p, N, \|c\|_\infty, \|d\|_\infty)$.

Proof: Since $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, we have the following

$$A(x, u, v) = |v|^{p-2}v, \;\; B(x, u, v) = c(x)|u|^{p-2}u + d(x)$$

so that $A(x, u, v)$ and $B(x, u, v)$ satisfy the conditions of Theorem 2.7 with $\alpha = p$, $a_0 = 1$, $a_1(x) \equiv 0$, $a_2(x) \equiv 0$, $b_1(x) \equiv 0$, $b_2(x) = |c(x)|^{1/p}$ and $b_3(x) = |d(x)|^{1/p}$. The result now follows. $\square$

Theorem 2.9. *Suppose* $\Omega \subset \mathbb{R}^N$ *is open and* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $1 < p < \infty$, *is a weak solution to*

$$-\Delta_p u = a(x, u, \nabla u)$$

in Ω, *where* $|a(x, u, \nabla u)| \leq \Gamma(1 + |\nabla u|)^p$ *for some constant* $\Gamma > 0$ *and all* $x \in \Omega$. *Then there exists* $0 < \alpha < 1$ *such that* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

Proof: This is proved in Theorem 1 of Tolksdorf [26]. □

Theorem 2.10. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain of class* $C^{1+\alpha}$, $0 < \alpha < 1$ *and let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

(i) Let $u \in C^1(\Omega)$ *satisfy* $u \geq 0$ *in* Ω *and* $-\Delta_p u \geq 0$ *a.e. in* Ω. *Then either* $u \equiv 0$ *or* $u > 0$ *on* Ω. *Moreover, if* $u \in C^1(\Omega \cup \{x_0\})$ *for any* $x_0 \in \partial\Omega$ *that satisfies an interior sphere condition and* $u(x_0) = 0$, *then* $\frac{\partial u}{\partial \nu} > 0$ *where* ν *is an interior normal at* x_0.

(ii) Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to*

$$\begin{cases} -\Delta_p u = f(x) & \text{in} \quad \Omega \\ u = f_1 & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^\infty(\Omega)$ *and* $f_1 \in C^{1+\alpha}(\partial\Omega)$. *Then there exists* $0 < \beta < 1$ *such that* $u \in C^{1+\beta}(\bar{\Omega})$.

(iii) (Maximum Principle) Assume that $u \in W^{1,p}(\Omega)$ *satisfies*

$$\begin{cases} -\Delta_p u = f(x) & \text{in} \quad \Omega \\ u \geq 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $f \in W^{-1,q}(\Omega) \cap L^{\infty}_{loc}(\Omega)$, $f \geq 0$. *Then either* $u > 0$ *in* Ω, *or* $u \equiv 0$ *in* Ω.

(iv) (Weak Comparison Principle) For $i = 1, 2$, *suppose* $u_i \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ *satisfy* $\Delta_p u_i \in L^{\infty}(\Omega)$, $u_i|_{\partial\Omega} \in C^{1+\alpha}(\partial\Omega)$ *together with the inequalities*

$$\begin{cases} -\Delta_p u_1 \leq -\Delta_p u_2 & in \quad \Omega \\ u_1 \leq u_2 & on \quad \partial\Omega. \end{cases}$$

Assume in addition that $-\Delta_p u_2 \geq 0$ *in* Ω *and* $u_2 \geq 0$ *on* $\partial\Omega$. *Then*

$$u_1(x) \leq u_2(x) \quad \textit{for each } x \in \Omega.$$

Proof: (i) This follows from Theorem 5 of Vazquez [28].

(ii) This follows from Lemma 2 of Garcia-Melian and de Lis [15].

(iii) This follows from Theorems 1 and 5 of Garcia-Melian and de Lis [15]. For the reader's convenience, we present a proof here as well.

We have $u \in W^{1,p}(\Omega)$ and $\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega fv$ for all $v \in W^{1,p}(\Omega)$. Thus we can let $v(x) = u^-(x) = \min(u(x), 0)$ and get

$$\int_\Omega |\nabla u^-|^p = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- = \int_\Omega fu^- \leq 0.$$

Now, let $\lambda_1(\Omega)$ be the first eigenvalue of the equation $-\Delta_p u = \lambda|u|^{p-2}u$. It is well known that

$$\lambda_1(\Omega) = \inf_{u \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} > 0,$$

and in fact we prove this result for the case $\Omega = \mathbb{R}^N$ in the next chapter. Therefore we have

$$\lambda_1(\Omega)\int_\Omega |u^-|^p \le \int_\Omega |\nabla u^-|^p = \int_\Omega f u^- \le 0$$

proving that $u^- \equiv 0$, i.e. that $u \ge 0$ in Ω. Therefore by part (i) we have that $u > 0$ or $u \equiv 0$ in Ω. This completes the proof of part (iii).

(iv) By part (ii) we have that $u_i \in C^{1+\beta}(\bar{\Omega})$, and by part (i), we have that $u_2 > 0$ in Ω and that $\frac{\partial u_2}{\partial v} < 0$ at that part of $\partial\Omega$ where $u_2 = 0$. Therefore there exists $c > 1$ such that $u_1 < cu_2$ in Ω. Consider the problem

$$\begin{cases} -\Delta_p v = -\Delta_p u_2 & \text{in } \Omega \\ v = u_2 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

Then u_1 and cu_2 are sub and supersolutions, respectively, of (2.3). Thus, the method of sub and supersolutions (see, for example, Theorem 4.14 of Diaz [8]) yields the existence of a solution $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ to (2.3), with $u_1 \le v \le cu_2$, which must be nonnegative.

We claim that (2.3) has a unique nonnegative solution in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Suppose we have two such solutions v_1 and v_2. Then parts (i) and (ii) imply that $v_1/v_2,\ v_2/v_1 \in L^\infty(\Omega)$. Then following a proof similar to Lemma 3.4 proved in the next chapter, we have that $v_1 = cv_2$ for some constant c. Since $v_1 = v_2$ on $\partial\Omega$ we have proved the claim.

Therefore $v = u_2$ and we have $u_1 \le u_2$, completing the proof of the theorem. □

Theorem 2.11. *There exists a map $J : D^{1,p} \to (D^{1,p})^*$. such that J, J^{-1} are contin-*

uous, and

$$\|Ju\| = \|u\|, \ \langle Ju, u\rangle = \|u\|^2 \quad \forall\, u \in D^{1,p}.$$

Proof: This follows from Proposition 8 of Browder [5]. □

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^N$ be a domain and let $X \subset L^{1,p}(\Omega)$ be a linear function space equipped with norm $\|u\|_X = \|\nabla u\|_{p,\Omega} = \left(\int_\Omega |\nabla u|^p\right)^{1/p}$ such that with this norm X is a reflexive Banach space. Let X^* be the dual space with norm $\|\cdot\|_{X^*}$ on X^* and with $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. Define the operator $J : X \to X^*$ by*

$$\langle J(u), v\rangle_X = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v$$

for any $u, v \in X$. Then the operator J is a homeomorphism between X and X^*. In particular, given $f \in X^*$, the equation $J(u) = f$ has a unique solution $u_f \in X$ satisfying

$$\|u_f\|_X \leq \|f\|_{X^*}^{1/(p-1)}.$$

Proof: This follows from Theorem 2.1 of Drabek and Simander [11]. □

Theorem 2.13. *(i) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the Dirichlet problem*

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $f \in L^{p^{*\prime}}(\Omega)$ and $p^{*\prime} = \frac{p^*}{p^*-1} = \frac{Np}{Np-N-p}$. Then (2.4) has a unique weak solution $u_f \in W_0^{1,p}(\Omega)$, i.e.

$$\int_\Omega |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla v = \int_\Omega fv$$

for any $v \in W_0^{1,p}(\Omega)$ *(or equivalently for any* $v \in C_0^\infty(\Omega)$*), where* $W_0^{1,p}(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *with respect to the norm* $\|\nabla \cdot\|_{p,\Omega}$*. In addition, we have* $\|\nabla u_f\|_{p,\Omega} \leq C\|f\|_{p^{*\prime}}^{1/(p-1)}$.

(ii) Let $f \in L^{p^{*\prime}}(\mathbb{R}^N)$*. Then there is a unique weak solution* $u_f \in D^{1,p}(\mathbb{R}^N)$ *to*

$$-\Delta_p u_f = f,$$

i.e.

$$\int |\nabla u_f|^{p-2} \nabla u_f \cdot \nabla v = \int f v$$

for all $v \in D^{1,p}(\mathbb{R}^N)$.

Proof: (i) This follows from Theorem 3.1 of Drabek and Simander [11]. For the reader's convenience, we include the entire proof here.

Let $X = W_0^{1,p}(\Omega)$ and let $f \in X^*$. It is well known that the space X equipped with the norm $\|\nabla \cdot\|_{p,\Omega}$ satisfies the conditions of Theorem 2.12. Then we have $u \in X$ is a weak solution to (2.4) if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \langle f, v \rangle_X$$

for all $v \in X$. This equation is uniquely solvable for any $f \in X^*$ by Theorem 2.12. By the Sobolev Embedding Theorem any $f \in L^{p^{*\prime}}(\Omega)$ can be identified with an $f \in X^*$ satisfying $\langle f, v \rangle_X = \int_\Omega f v$ for any $v \in X$, proving the first part of the theorem. For the estimates on $\|\nabla u_f\|_{p,\Omega}$, we again apply Theorem 2.12 and note that

$$\begin{aligned}\|f\|_{X^*} &= \sup_{\|v\|_X=1} |\langle f, v\rangle_X| \\ &\leq \sup_{\|v\|_X=1} \|f\|_{p^{*\prime}}\|v\|_{p^*} \\ &\leq C \sup_{\|v\|_X=1} \|f\|_{p^{*\prime}}\|v\|_X \\ &= C\|f\|_{p^{*\prime}}\end{aligned}$$

proving the first part of the theorem.

(ii) This follows from Theorem 4.1 of Drabek and Simander [11]. Note that in the notation of that paper, $\hat{H}_0^{1,p}(\mathbb{R}^N)$ is the same space as $D^{1,p}(\mathbb{R}^N)$ in the notation of this paper, both being the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int |\nabla u|^p\right)^{1/p}.$$

□

Definition 2.14. Let $\Omega \subset \mathbb{R}^N$ be an open set.

(i) A function $u \in H^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ is said to be p-harmonic in Ω if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

(ii) A lower-semicontinuous function $u : \Omega \to (-\infty, \infty]$ is called p-superharmonic if u is not identically infinite in each component of Ω, and if for all open $D \subset\subset \Omega$ and all $h \in C(\bar{D})$, p-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D.

Theorem 2.15. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and let u be a bounded, nonnegative, p-superharmonic function in Ω such that*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_\Omega \phi \, d\mu$$

for some nonnegative Radon measure μ on Ω and all $\phi \in C_0^\infty(\Omega)$. Define

$$W_{1,p}^\mu(x,r) = \int_0^r \left(\frac{\mu(B_t(x))}{t^{N-p}} \right)^{q-1} \frac{dt}{t}.$$

If $B_{3r}(a) \subset \Omega$, then there exists constants A_1, A_2 and A_3 such that

$$A_1 W_{1,p}^\mu(a,r) \le u(a) \le A_2 \inf_{x \in B_r(a)} u(x) + A_3 W_{1,p}^\mu(a, 2r).$$

Proof: This is Theorem 1.6 of Kilpelainen and Maly [17] for the case $A(x,h) = |h|^{p-2}h$. (See also Theorem 3.1 of [16]) $\square$

CHAPTER 3

EIGENVALUE PROBLEM

Existence Results

Before presenting our main theorem, we first need to study the following eigenvalue problem in $\mathbb{R}^N$:

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u,$$

where a non-trivial solution $u \in D^{1,p}$ is called an eigenfunction provided

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla \phi = \lambda \int a(x)|u|^{p-2}u\phi \tag{3.1}$$

for all $\phi \in C_0^\infty$, and by completeness for all $\phi \in D^{1,p}$. In this case λ is called an eigenvalue.

For the next theorem, we need the concept of genus. Let V be the completion of C_0^∞ with respect to the norm

$$\|u\|_V^p = \int |\nabla u|^p + \int \frac{|u|^p}{(1+|x|)^p}.$$

Let $G = \{u \in V \mid \int a(x)|u|^p = 1\}$, and define

$$\Gamma_k = \{A \subset G \mid A \text{ is symmetric, compact, and } \gamma(A) \geq k\},$$

where $\gamma(A)$ is the genus of A, i.e. the smallest integer k such that there exists an odd continuous map from A to $\mathbb{R}^k \backslash \{0\}$.

Theorem 3.1. *The eigenvalue problem above has a sequence of solutions (λ_k, Φ_k) with $\int a(x)|\Phi_k|^p = 1$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. Furthermore,*

$$\lambda_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \int |\nabla u|^p.$$

As such, the solutions (λ_k, Φ_k) are called minimax eigenvalues and eigenvectors.

Note: For all $u \in D^{1,p}$ we have $\left\{ \frac{u}{(\int a(x)|u|^p)^{1/p}}, -\frac{u}{(\int a(x)|u|^p)^{1/p}} \right\} \in \Gamma_1$, so

$$\lambda_1 = \inf_{u \in D^{1,p}} \frac{\int |\nabla u|^p}{\int a(x)|u|^p}.$$

Proof: Define $I(u) = \frac{1}{p} \int |\nabla u|^p$ and $\Psi(u) = \frac{1}{p} \int a(x)|u|^p$. Clearly I is well-defined on V. Furthermore, I is bounded below on G because

$$\int a(x)|u|^p \leq C\|a\|_{N/p} \|\nabla u\|_p^p$$

by Lemma 2.2. Before proceeding, we need the following lemma:

Lemma 3.2. *The functional I satisfies the Palais-Smale condition on G, i.e. for*

$\{u_n\} \subset G$, if $I(u_n)$ is bounded and $I'(u_n) \to 0$, then $\{u_n\}$ has a convergent subsequence in V.

Proof: For the convenience of the reader, we reproduce the proof of Allegretto and Huang [3].

We first claim that $I'(u_n) \to 0$ and $\{I(u_n)\}$ bounded with $\{u_n\} \subset G$ implies that

$$I'(u_n) - \frac{\langle I'(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle}\Psi'(u_n) \to 0.$$

Let J be as in Theorem 2.11, so that $\|Ju\| = \|u\| = \|J^{-1}u\|$ and $\langle Ju, u\rangle = \|u\|^2$ for all $u \in D^{1,p}$. By definition, $I'(u_n) \to 0$ implies that

$$J^{-1}(I'(u_n)) - \frac{\langle \Psi'(u_n), J^{-1}I'(u_n)\rangle}{\|\Psi'(u_n)\|^2}J^{-1}(\Psi'(u_n)) \to 0,$$

and since J is continuous that

$$I'(u_n) - \frac{\langle \Psi'(u_n), J^{-1}I'(u_n)\rangle}{\|\Psi'(u_n)\|^2}\Psi'(u_n) \to 0.$$

Now, since $\{I(u_n)\} = \{\frac{1}{p}\int |\nabla u_n|^p\}$ is bounded, then by Hardy's inequality (Lemma 2.3), we can conclude that $\{u_n\}$ is bounded in V. Therefore

$$\langle I'(u_n), u_n\rangle - \frac{\langle \Psi'(u_n), J^{-1}I'(u_n)\rangle}{\|\Psi'(u_n)\|^2}\langle \Psi'(u_n), u_n\rangle \to 0.$$

Now, $\langle \Psi'(u_n), u_n \rangle = \int a(x)|u_n|^{p-2}u_n u_n = \int a(x)|u_n|^p = 1$, so that

$$\frac{\langle I'(u_n), u_n \rangle}{\langle \Psi'(u_n), u_n \rangle} - \frac{\langle \Psi'(u_n), J^{-1}I'(u_n) \rangle}{\|\Psi'(u_n)\|^2} \to 0,$$

and hence

$$I'(u_n) - \frac{\langle I'(u_n), u_n \rangle}{\langle \Psi'(u_n), u_n \rangle} \Psi'(u_n) \to 0,$$

proving the claim.

Next, we proved above that $\{u_n\}$ is bounded in V. Therefore, passing to a subsequence if necessary, there exists $u_0 \in V$ such that $u_n \rightharpoonup u_0$ in V.

We claim that $u_0 \not\equiv 0$. First note that on any bounded domain Ω, we have

$$\int_\Omega a(x)|u_0|^p = \lim_{n\to\infty} \int_\Omega a(x)|u_n|^p,$$

by the Sobolev Embedding Theorem (Theorem 2.1). Suppose $u_0 \equiv 0$. Then $\int_\Omega a(x)|u_n|^p \to 0$ for all bounded domains Ω. Choose Ω so that for all n sufficiently large we have

$$C\|a\chi_{\mathbb{R}^N \setminus \Omega}\|_{N/p}\|\nabla u_n\|_p^p < \frac{1}{4},$$

where χ denotes the usual characteristic function and C is the constant in Lemma 2.2.

Then we can choose n large enough so that

$$\begin{aligned}\int a(x)|u_n|^p &= \int_\Omega a(x)|u_n|^p + \int_{\mathbb{R}^N\setminus\Omega} a(x)|u_n|^p \\ &\leq \int_\Omega a(x)|u_n|^p + C\|a\chi_{\mathbb{R}^N\setminus\Omega}\|_{N/p}\|\nabla u_n\|_p^p \\ &\leq \frac{1}{4}+\frac{1}{4}=\frac{1}{2}\end{aligned}$$

contradicting the fact that $u_n \in G$ implies that $\int a(x)|u_n|^p = p > 1$. Therefore $u_0 \not\equiv 0$.

Now, by the assumption

$$I'(u_n) - \frac{\langle I'(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle}\Psi'(u_n) \to 0$$

we have that, for any $\phi \in C_0^\infty$,

$$\int |\nabla u_n|^{p-2}\nabla u_n \nabla\phi = c_n \int a(x)|u_n|^{p-2}u_n\phi + o(1) \tag{3.2}$$

where

$$\begin{aligned} c_n &= \frac{\langle I'(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle} \\ &= \frac{\int |\nabla u_n|^{p-2}\nabla u_n \nabla u_n}{\int a(x)|u_n|^{p-2}u_n u_n} \\ &= \int |\nabla u_n|^p = \frac{1}{p}I(u_n)\end{aligned}$$

because $u_n \in G$.

Take $\phi = u_n - u_m$ in (3.2) (twice). Then, if $\Omega = B_r(0)$ is a ball of radius r, we have

$$\begin{aligned}
&\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m) \\
&= \int a(x)(c_n|u_n|^{p-2}u_n - c_m|u_m|^{p-2}u_m)(u_n - u_m) + o(1) \\
&= \int_\Omega a(x)c_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \\
&+ \int_{\mathbb{R}^N\setminus\Omega} a(x)c_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \\
&+(c_n - c_m)\int a(x)|u_m|^{p-2}u_m(u_n - u_m) + o(1) = A_{n,m} + B_{n,m} + C_{n,m} + o(1).
\end{aligned}$$

We have

$$\begin{aligned}
B_{n,m} &= \int_{\mathbb{R}^N\setminus\Omega} a(x)c_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \\
&= \int_{\mathbb{R}^N\setminus\Omega} a(x)c_n(|u_n|^p + |u_m|^p - (|u_n|^{p-2} + |u_m|^{p-2})u_n u_m) \\
&\leq Cc_n \int_{\mathbb{R}^N\setminus\Omega} a(x)(|u_n|^p + |u_m|^p) \\
&\leq C'c_n \|a\chi_{\mathbb{R}^N\setminus\Omega}\|_{N/p}(\|\nabla u_n\|_p^p + \|\nabla u_m\|_p^p)
\end{aligned}$$

which approaches 0 (because $\|a\chi_{\mathbb{R}^N\setminus\Omega}\|_{N/p} \to 0$ as $r \to \infty$ independent of n and m, and $\{I(u_n)\}$ is bounded.)

Now, for any fixed Ω, we have (passing to a subsequence if necessary)

$$A_{n,m} = \int_\Omega a(x)c_n(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m) \to 0$$

as $n, m \to \infty$, since $u_n \to u_0$ in $L^p(\Omega)$ (because $D^{1,p}(\Omega)$ is compactly contained in

$L^q(\Omega)$ for all $p \leq q < p^*$). Finally, we have

$$\begin{aligned}\left|\int a(x)|u_m|^{p-2}u_m(u_n - u_m)\right| &\leq \int a(x)|u_m|^{p-2}|u_m u_n| + \int a(x)|u_m|^p \\ &\leq 2\int a(x)|u_n|^p + 2\int a(x)|u_m|^p\end{aligned}$$

so $\{\int a(x)|u_n|^p\}$ bounded implies that $\{\int a(x)|u_m|^{p-2}u_m(u_n - u_m)\}$ is bounded. Since $\{c_n\} = \{\frac{1}{p}I(u_n)\}$ is bounded, we have, passing to a subsequence if necessary, $c_n - c_m \to 0$ as $n, m \to \infty$. Therefore $A_{n,m} + B_{n,m} + C_{n,m} \to 0$ as $n, m \to \infty$.

On the other hand, note that for any $a, b \in \mathbb{R}^N$,

$$|a - b|^p \leq c((|a|^{p-2}a - |b|^{p-2}b)(a - b))^{s/2}(|a|^p + |b|^p)^{1-s/2},$$

where $s = p$ if $1 < p < 2$ and $s = 2$ if $p \geq 2$ (which follows from Lemma 4.2 of Lindqvist [19]). Therefore we have

$$\begin{aligned}|\nabla u_n - \nabla u_m|^p &\leq c((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m))^{s/2} \\ &\quad \cdot (|\nabla u_n|^p + |\nabla u_m|^p)^{1-s/2}\end{aligned}$$

By applying Holder's Inequality, using $\frac{1}{p_1} + \frac{1}{p_2} = 1$ with $p_1 = \frac{1}{s/2}$ and $p_2 = \frac{1}{1-s/2}$, we obtain

$$\begin{aligned}\int |\nabla u_n - \nabla u_m|^p &\leq c\left(\int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)\nabla(u_n - u_m)\right)^{s/2} \\ &\quad \cdot \left(\int |\nabla u_n|^p + \int |\nabla u_m|^p\right)^{1-s/2} \\ &\leq c(A_{n,m} + B_{n,m} + C_{n,m} + o(1))^{s/2} \\ &\quad \cdot \left(\int |\nabla u_n|^p + \int |\nabla u_m|^p\right)^{1-s/2}\end{aligned}$$

Since $\left\{\int |\nabla u_n|^p\right\}$ is bounded, we can conclude that $\nabla u_n \to \nabla u_0$ in L^p. Combining this with Hardy's Inequality (Lemma 2.3) gives $u_n \to u_0$ in V. This completes the proof of the lemma. □

Theorem 3.1 now follows from Ljusternik-Schnirelmann theory, see for example Costa [6]. In particular, for any integer $k > 0$, $\lambda_k = \inf_{A\in\Gamma_j} \sup_{u\in A} pI(u)$ is a critical value of I restricted to G. Thus, there exists $A_k \in \Gamma_k$ and $\Phi_k \in A_k$ such that $\lambda_k = pI(\Phi_k) = \sup_{u\in A_k} pI(u)$ and (λ_k, Φ_k) is a solution to the eigenvalue problem (3.1). Moreover, $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. This completes the proof of Theorem 3.1. □

Properties of Eigenfunctions and the First Eigenvalue

We next investigate some properties of the first eigenvalue λ_1 and its eigenfunctions.

Theorem 3.3. *There exists a first eigenfunction Φ_1 such that $\Phi_1 > 0$ on $\mathbb{R}^N$.*

Proof: We first show that we can choose Φ_1 such that $\Phi_1 \geq 0$ on $\mathbb{R}^N$. For $u \in V$, write $u = u^+ - u^-$ where $u^+ > 0, u^- \geq 0$. Then $\nabla u = \nabla u^+ - \nabla u^-$ and $\nabla|u| = \nabla u^+ + \nabla u^-$.

Then we have

$$\int |\nabla u|^p = \int |\nabla u^+|^p + \int |\nabla u^-|^p = \int |\nabla |u||^p.$$

Therefore $|u| \in V$, and hence if Φ_1 minimizes $I(u)$ then so does $|\Phi_1|$, so $(\lambda_1, |\Phi_1|)$ is a solution to (3.1).

So choose Φ_1 such that $\Phi_1 \geq 0$ on $\mathbb{R}^N$. We next show that in fact $\Phi_1 > 0$ on $\mathbb{R}^N$. Suppose $\Phi_1(x_0) = 0$. For a ball B around x_0, clearly $\Phi_1 \geq 0$ on B. Then Φ_1 is a weak solution of the problem

$$\begin{aligned} -\Delta_p u &= \lambda a(x)|\Phi_1|^{p-2}\Phi_1 \text{ in } B, \\ u &\geq 0 \text{ on } \partial B. \end{aligned}$$

Then an application of Theorem 2.10 part (iii) implies that $\Phi_1 \equiv 0$ in B. Since B can by any ball, we have that $\Phi_1 \equiv 0$ on $\mathbb{R}^N$, a contradiction. This completes the proof of the theorem. □

Lemma 3.4. *Suppose $u, v \in C^1 \cap D^{1,p}$, $u, v > 0$ on $\mathbb{R}^N$, $\frac{u}{v}, \frac{v}{u} \in L^\infty$, and let*

$$I(u,v) = \left(-\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\right) - \left(-\Delta_p v, \frac{u^p - v^p}{v^{p-1}}\right).$$

Then $\frac{u^p - v^p}{u^{p-1}}, \frac{u^p - v^p}{v^{p-1}} \in D^{1,p}$, $I(u,v) \geq 0$ and $I(u,v) = 0$ if and only if there exists $\alpha > 0$ such that $u = \alpha v$.

Proof: We follow logic similar to Anane [4]. First, since $u, v \in C^1 \cap D^{1,p}$, $\frac{u}{v}, \frac{v}{u} \in L^\infty$

and

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \nabla u + (p-1)\left(\frac{v}{u}\right)^p \nabla u - p\left(\frac{v}{u}\right)^{p-1} \nabla v,$$

we have that $\frac{u^p - v^p}{u^{p-1}} \in D^{1,p}$, and similarly for $\frac{u^p - v^p}{u^{p-1}}$. Now, we have

$$\begin{aligned}
&\left(-\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\right) \\
&\quad = \int |\nabla u|^{p-2} \nabla u \cdot \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) \\
&\quad = \int \left(-p\left(\frac{v}{u}\right)^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla v + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right) |\nabla u|^p\right) \\
&\quad = \int \left(p\left(\frac{v}{u}\right)^{p-1} |\nabla u|^{p-2} (|\nabla u||\nabla v| - \nabla u \cdot \nabla v)\right. \\
&\quad \left. + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right) |\nabla u|^p - p\left(\frac{v}{u}\right)^{p-1} |\nabla u|^{p-1} |\nabla v|\right)
\end{aligned}$$

We obtain a similar result interchanging u and v, thus we have

$$I(u, v) = \int F\left(\frac{v}{u}, \nabla u, \nabla v\right) + \int G\left(\frac{v}{u}, |\nabla u|, |\nabla v|\right)$$

where F and G are defined by

$$F(t, R, S) = p(t^{p-1}|R|^{p-2} + t^{1-p}|S|^{p-2})(|R||S| - R \cdot S)$$

$$G(t, r, s) = (1 + (p-1)t^p)r^p + (1 + (p-1)t^{-p})s^p - pt^{p-1}r^{p-1}s - pt^{1-p}s^{p-1}r$$

for all $t > 0$, $R, S \in \mathbb{R}^N$, and $r, s \geq 0$. Clearly $F \geq 0$, and $F(t, R, S) = 0$ if and only if $|R||S| = R \cdot S$. We will show that $G \geq 0$ as well. First note that $G(t, 0, s) \geq 0$ and that $G(t, 0, s) = 0$ implies $s = 0$. If $r \neq 0$, let $z = (s/tr)^p$. Then we have

$$G(t, r, s) = r^p(t^p f(z) + g(z)),$$

where $f(z) = z^p - pz + p - 1$ and $g(z) = (p-1)z^p - pz^{p-1} + 1$. A simple calculus exercise shows that $f(z), g(z) \geq 0$ for $z \geq 0$, proving that $I(u, v) \geq 0$.

Furthermore, the same calculus exercise shows that (for $z \geq 0$) $f(z), g(z) = 0$ if and only if $z = 1$. Therefore, in the case $r \neq 0$, we have that $G(t, r, s) = 0$ if and only if $s = tr$. Consequently, if $I(u, v) = 0$ then $F(v/u, \nabla u, \nabla v), G(v/u, |\nabla u|, |\nabla v|) = 0$ on all of $\mathbb{R}^N$, which is equivalent to $|\nabla u||\nabla v| = \nabla u \cdot \nabla v$ and $u|\nabla v| = v|\nabla u|$. This is equivalent to $(u\nabla v - v\nabla u)^2 = 0$, which implies that $u = \alpha v$, completing the proof of the lemma. □

Theorem 3.5. *(i) λ_1 is simple, i.e. the positive eigenfunction corresponding to λ_1 is unique up to a constant multiple.*

(ii) λ_1 is unique, i.e. if $v \geq 0$ is an eigenfunction associated with an eigenvalue λ with $\int a(x)|v|^p = 1$, then $\lambda = \lambda_1$.

(iii) If $\mu > \lambda_1$ is an eigenvalue with eigenfunction v, then v must change signs in $\mathbb{R}^N$.

Proof: Let $u > 0$ and $v \geq 0$ be the eigenfunctions associated with λ_1 and λ respectively, chosen such that $\int a(x)|u|^p = \int a(x)|v|^p = 1$. Then, as in the last part of the proof of Theorem 3.3, we must have $v > 0$. By Theorem 2.9 we have

$u, v \in C^1(\mathbb{R}^N)$. Therefore we can apply Lemma 3.4 to get

$$\begin{aligned}
&\left(-\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\right) - \left(-\Delta_p v, \frac{u^p - v^p}{v^{p-1}}\right) \\
&= \int \lambda_1 a(x) u^{p-1} \frac{u^p - v^p}{u^{p-1}} - \int \lambda a(x) v^{p-1} \frac{u^p - v^p}{v^{p-1}} \\
&= (\lambda_1 - \lambda) \int a(x)(u^p - v^p) = 0,
\end{aligned}$$

which implies that $u = \alpha v$ for some $\alpha > 0$. However, by our assumption that $\int a(x)|u|^p = \int a(x)|v|^p$, we then have that $u = v$, and hence $\lambda_1 = \lambda$. This completes the proof of all parts of the theorem. □

Lemma 3.6. *Let Φ_1 denote a first eigenfunction of (3.1) satisfying $\Phi_1 > 0$. Then $\Phi_1 \in L^r$ for all $p^* \leq r < \infty$.*

Proof: For this proof we apply a method similar to that used in Appendix B of Struwe [25]. Let $u = \Phi_1$, and let $s, M \geq 0$. Let $\phi = \phi_{s,M} = u \min(u^{ps}, M^p)$. Keep in mind that $u > 0$ and $u \in D^{1,p} \subset L^{p^*}$, with $p^* = \frac{pN}{N-p}$.

We first wish to show that if $u \in L^{p(s+1)}$ then $u^{s+1} \in D^{1,p}$. To simplify notation, we denote the set $\{x \in \mathbb{R}^N \mid (u(x))^s \leq M\}$ by $\{u^s \leq M\}$. We have

$$\begin{aligned}
\int |\nabla(u \min(u^s, M))|^p &= \int |\nabla u \min(u^s, M) + u \chi_{u^s \leq M} s u^{s-1} \nabla u|^p \\
&\leq 2^p \left[\int |\nabla u|^p \min(u^{ps}, M^p) + \int_{u^s \leq M} s^p u^{sp} |\nabla u|^p\right]
\end{aligned}$$

Now, use u and ϕ in (3.1). We get

$$\int |\nabla u|^{p-2}\nabla u \cdot \left(\nabla u \min(u^{ps}, M^p) + u\chi_{u^s \le M} ps u^{ps-1}\nabla u\right)$$
$$= \lambda_1 \int a(x)u^{p-2}u^2 \min(u^{ps}, M^p)$$

so that

$$\int |\nabla u|^p \min(u^{ps}, M^p) + ps \int_{u^s \le M} u^{ps}|\nabla u|^p = \lambda_1 \int a(x)u^p \min(u^{ps}, M^p).$$

Therefore there exists a constant C depending only on s, p, and $\|a\|_\infty$ such that

$$\int |\nabla(u \min(u^s, M))|^p \le C \int u^p \min(u^{ps}, M^p)$$

Letting $M \to \infty$ and using the fact that $u \in L^{p(s+1)}$, we have that $u^{s+1} \in D^{1,p} \subset L^{p^*}$, i.e. that $u \in L^{p^*(s+1)}$. Now, let $s_0 = \frac{p}{N-p}$. Then $p(s_0+1) = \frac{Np}{N-p} = p^*$, so $u \in L^{p^*} = L^{p(s_0+1)}$. Therefore the above demonstrates that $u \in L^{p^*(s_0+1)}$. Now we can iteratively let $p(s_i+1) = p^*(s_{i-1}+1)$, i.e. that $s_i + 1 = (s_{i-1}+1)\frac{N}{N-p}$ and obtain $u \in L^r$ for all $p^* \le r < \infty$. Note that here we used the fact that $f \in L^{p_1} \cap L^{p_2}$ implies $f \in L^{p_3}$ for all $p_1 \le p_3 \le p_2$, which follows from an application of Holder's inequality. This completes the proof of the lemma. □

Theorem 3.7. *Let Φ_1 denote a first eigenfunction of (3.1) satisfying $\Phi_1 > 0$. Then $\Phi_1 \in D^{1,p} \cap L^\gamma_{a(x)}$ for all $\gamma \ge p$.*

Proof: We have by Holder's Inequality

$$\begin{aligned}\int a(x)\Phi_1^\gamma &\le \left(\int a(x)^{\frac{N}{p}}\right)^{\frac{p}{N}} \left(\int \Phi_1^{\frac{\gamma N}{N-p}}\right)^{\frac{N-p}{N}} \\ &= \|a\|_{N/p} \|\Phi_1\|^\gamma_{\gamma N/(N-p)} < \infty\end{aligned}$$

by Lemma 3.6, because $\frac{\gamma N}{N-p} \ge \frac{pN}{N-p} = p^*$. This proves the theorem. $\square$

CHAPTER 4

SCALAR PROBLEM

In this chapter we let $\lambda_1 = \lambda_1(a,p)$ be the first eigenvalue of Theorem 3.1.

Main Theorem 4.1. *Consider*

$$-\Delta_p u = a(x)\lambda|u|^{p-2}u - a(x)g(u) \quad x \in \mathbb{R}^N \tag{4.1}$$

where

(A_0) $g : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous,*

(A_1) $\lim_{s\to 0^+} \frac{g(s)}{s^{p-1}} = 0$,

(A_2) $0 < \liminf_{s\to\infty} \frac{g(s)}{s^{\gamma-1}} \leq \limsup_{s\to\infty} \frac{g(s)}{s^{\gamma-1}} < \infty$ *with* $\gamma > p$,

(A_3) $\frac{g(s)}{s^{p-1}}$ *is nondecreasing,*

(B_1) $0 < a(x) \in L^{N/p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

(B_2) $1 < p < N$ *and* $\lambda > \lambda_1$.

Then (4.1) has a solution $u_0 > 0$ *in* $\mathbb{R}^N$ *which satisfies,*

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p}{p-1}}} \text{ for } |x| \text{ large.}$$

If we change conditions (A_3) *and* (B_1) *to be*

(A_3') $\frac{g(s)}{s^{p-1}}$ *is increasing,*

(B_1') $0 < a(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and for all* $x \in \mathbb{R}^N$ *we have*

$$|x|^{\frac{N}{Q(p-1)}} \|a\|_{L^P(\mathbb{R}^N \setminus B_{|x|}(0))} \leq C$$

for some constant C *and some* $P > \frac{N}{p}$, *with* $\frac{1}{P} + \frac{1}{Q} = 1$.

Then we can conclude in addition that u_0 *is the unique positive solution and that*

$$u_0(x) = \frac{d(x)}{|x|^{\frac{N-p}{p-1}}} \text{ for } |x| \text{ large,} \tag{4.2}$$

where $c \leq d(x) \leq C$ *for all* $x \in \mathbb{R}^N$ *and some constants* $c, C > 0$.

Existence of Solution

Lemma 4.2. *For any* $\epsilon > 0$ *there exists constants* $C_1 = C_1(\epsilon)$ *and* $C_2 = C_2(\epsilon)$ *such that*

$$-\epsilon(s^+)^{p-1} + C_1(s^+)^{\gamma-1} \leq g(s) \leq \epsilon(s^+)^{p-1} + C_2(s^+)^{\gamma-1},$$

$$-\epsilon(s^+)^p + C_1(s^+)^\gamma \leq G(s) \leq \epsilon(s^+)^p + C_2(s^+)^\gamma,$$

where $G(s) = \int_0^s g(t)dt$.

Proof: This follows from our conditions on g. □

We say that $u \in D^{1,p}$ is a (weak) solution to (4.1) if

$$\int |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda \int a(x)|u|^{p-2}uv + \int a(x)g(u)v = 0 \tag{4.3}$$

holds for all $v \in D^{1,p} \cap L^\gamma_{a(x)}$. Note that the condition $v \in L^\gamma_{a(x)}$ arises from the fact that

$$\int a(x)u^{\gamma-1}v \le \left(\int a(x)u^{\gamma}\right)^{\frac{\gamma}{\gamma-1}} \left(\int a(x)v^{\gamma}\right)^{\frac{1}{\gamma}},$$

and in our construction of a weak solution below via minimization we determine that $u \in L^{\gamma}_{a(x)}$. In addition, we do not require $v \in L^{p}_{a(x)}$, because $D^{1,p} \subset L^{p}_{a(x)}$ by Lemma 2.2.

Theorem 4.3. *If $u \in D^{1,p}$ is a positive weak solution to (4.1) then $\lambda > \lambda_1$.*

Proof: By Theorem 3.1 we have that $\lambda_1 \le \frac{\int |\nabla u|^p}{\int a(x)|u|^p}$ for all $u \in D^{1,p}$. Therefore, for our positive solution u we have

$$\begin{aligned} \lambda_1 \int a(x)u^p &\le \int |\nabla u|^p \\ &= \lambda \int a(x)u^p - \int a(x)g(u)u \end{aligned}$$

so that

$$(\lambda_1 - \lambda)\int a(x)u^p \le -\int a(x)g(u)u < 0$$

because $a(x) > 0$ and $g(u), u > 0$. Therefore $\lambda > \lambda_1$, completing the proof. □

We consider the functionals

$$J : D^{1,p} \to \mathbb{R} \cup \{\infty\},\ J(u) = \int a(x)G(u)$$

and

$$I : D^{1,p} \to \mathbb{R} \cup \{\infty\}$$

$$I(u) = \begin{cases} \frac{1}{p}\int |\nabla u|^p - \frac{\lambda}{p}\int a(x)(u^+)^p + J(u) & J(u) < \infty \\ \infty & J(u) = \infty \end{cases}$$

Our goal is to find a solution to (4.1) by minimizing I.

Lemma 4.4. *(a) I is coercive, i.e. $I(u) \to \infty$ as $\|u\| \to \infty$.*

(b) I is a weakly lower semi-continuous functional.

Proof: (a) We generalize the methods of Du and Ma [12].

Assume not. Then there exists $\{u_n\} \subset D^{1,p}$ such that $\{I(u_n)\}$ is bounded above and $\|u_n\| \to \infty$. Let $d_n = \left(\int a(x)(u_n^+)^p\right)^{1/p}$. Then $\|u_n\| \to \infty$ implies that $d_n \to \infty$. ($I(u_n) \geq \frac{1}{p}\|u_n\|^p - \frac{\lambda}{p}d_n^p$ and $\|u_n\| \to \infty$, so if $\{d_n\}$ is bounded then $I(u_n) \to \infty$, contradicting the assumption that $\{I(u_n)\}$ is bounded above.)

Set $\bar{u}_n = \frac{u_n}{d_n}$. Then $\int a(x)(\bar{u}_n^+)^p = 1$ and

$$\begin{aligned} I(u_n) &= \frac{1}{p}\int |\nabla u_n|^p - \frac{\lambda}{p}\int a(x)(u_n^+)^p + \int a(x)G(u_n) \\ &= \frac{d_n^p}{p}\int |\nabla \bar{u}_n|^p - \frac{\lambda}{p}d_n^p\int a(x)(\bar{u}_n^+)^p + \int a(x)G(d_n\bar{u}_n) \end{aligned}$$

so that

$$\frac{pI(u_n)}{d_n^p} = \int |\nabla \bar{u}_n|^p - \lambda + \frac{p}{d_n^p}\int a(x)G(d_n\bar{u}_n) \geq \int |\nabla \bar{u}_n|^p - \lambda$$

Therefore, since $\frac{I(u_n)}{d_n^p} \to 0$, we have that $\{\|\bar{u}_n\|\}$ is bounded. This implies, passing to a subsequence if necessary, that $\bar{u}_n \rightharpoonup \bar{u}$ in $D^{1,p}$, $\bar{u}_n \to \bar{u}$ a.e. in $\mathbb{R}^N$, and $\bar{u}_n \to \bar{u}$ in $L^p_{a(x)}$ (by Lemma 2.6). In addition, we have

$$\begin{aligned}\frac{pI(u_n)}{d_n^p} &= \int |\nabla \bar{u}_n|^p - \lambda + \frac{p}{d_n^p}\int a(x)G(d_n\bar{u}_n)\\ &\geq \int |\nabla \bar{u}_n|^p - \lambda - \frac{\epsilon p}{d_n^p}\int a(x)(d_n\bar{u}_n^+)^p + \frac{C_1 p}{d_n^p}\int a(x)(d_n\bar{u}_n^+)^\gamma\\ &\geq -\lambda - \epsilon p + C_1 p d_n^{\gamma-p}\int a(x)(\bar{u}_n^+)^\gamma\end{aligned}$$

by Lemma 4.2. Therefore, $\gamma > p$ and $d_n \to \infty$ imply that $\int a(x)(\bar{u}_n^+)^\gamma \to 0$. Hence, by Fatou's lemma we have

$$\int a(x)(\bar{u}^+)^\gamma = 0.$$

Therefore, since $a(x) > 0$ on $\mathbb{R}^N$, we have that $\bar{u} \leq 0$ on $\mathbb{R}^N$. However this contradicts the fact that $\bar{u}_n \to \bar{u}$ in $L^p_{a(x)}$ and $\int a(x)(\bar{u}_n^+)^p = 1$ for all n. This proves part (a) of the lemma.

(b) Assume $u_n \rightharpoonup u$ in $D^{1,p}$. Then Lemma 2.6 implies

$$\int a(x)u_n^p \to \int a(x)u^p.$$

Furthermore, since $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$, we have by Lebesgue Dominated Convergence Theorem that

$$\int a(x)|u_n^+|^p \to \int a(x)|u^+|^p,$$

and by Fatou's Lemma that

$$\int a(x)G(u) \le \liminf_{n\to\infty} \int a(x)G(u_n).$$

Finally, by Lemma 2.5, $\|\cdot\|^p$ is weakly l.s.c., so that $u_n \rightharpoonup u$ in $D^{1,p}$ implies $\frac{1}{p}\|u\|^p \le \liminf_{n\to\infty} \frac{1}{p}\|u_n\|^p$. Therefore $I(u) \le \liminf_{n\to\infty} I(u_n)$, hence is weakly l.s.c. This proves the lemma. □

Theorem 4.5. *(Solution to the Minimization Problem) The minimization problem*

$$\inf I(u), \;\; u \in D^{1,p}$$

(i.e. find $u \in D^{1,p}$ for which $\inf I(u)$ *is achieved) has a solution $u_0 \in D^{1,p}$ with*

$$\inf_{u\in D^{1,p}} I(u) = I(u_0) \tag{4.4}$$

Proof: This follows directly from Lemma 4.4 and Lemma 2.5. □

Theorem 4.6. *Suppose $\bar{u}$ is a minimum point of I, i.e. a solution to (4.4). Assume that* $\{t_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$. *Then, if $v \in D^{1,p} \cap L^{\gamma}_{a(x)}$, we have*

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(\bar{u}+t_n v) - I(\bar{u})}{t_n} \\ &= \int |\nabla \bar{u}|^{p-2}\nabla \bar{u}\cdot\nabla v - \lambda\int a(x)(\bar{u}^+)^{p-1}v + \int a(x)g(\bar{u})v. \end{aligned}$$

Proof: It is enough to show that for $v \in D^{1,p} \cap L^{\gamma}_{a(x)}$, we have

$$\lim_{n\to\infty} \frac{J(\bar{u}+t_n v) - J(\bar{u})}{t_n} = \int a(x)g(\bar{u})v.$$

(This then proves the second equality above, and both then equal 0 due to the fact that $\bar{u}$ is a minimum for I and that v can be replaced by $-v$ above.)

To prove the above equation we need to prove that

$$\int a(x)\left(\frac{1}{t_n}\int_{\bar{u}}^{\bar{u}+t_n v} g(s)ds\right) dx \to \int a(x)g(\bar{u})v \; dx.$$

Define $F_n(x) = \frac{1}{t_n}\int_{\bar{u}}^{\bar{u}+t_n v} g(s)ds$. Since g is continuous, $F_n(x) \to g(\bar{u}(x))v(x)$ for a.e. $x \in \mathbb{R}^N$. In addition, by Lemma 4.2, we have (for some $0 \le \bar{t}_n \le t_n$ and assuming without loss of generality that $t_n \le 1$)

$$\begin{aligned}
|F_n(x)| &\le \frac{1}{t_n}(t_n|v|)g(\bar{u}+\bar{t}_n v) \\
&\le \frac{1}{t_n}(t_n|v|)(\epsilon(\bar{u}^+ + v^+)^{p-1} + C_2(\bar{u}^+ + v^+)^{\gamma-1}) \\
&\le \epsilon C((\bar{u}^+)^{p-1}|v| + (v^+)^p) + C((\bar{u}^+)^{\gamma-1}|v| + (v^+)^{\gamma})
\end{aligned}$$

Therefore, for any domain $\Omega \subset \mathbb{R}^N$,

$$\begin{aligned}
\left|\int_\Omega a(x)F_n(x)\right| \le\; & \epsilon C\left(\int_\Omega a(x)|\bar{u}^+|^p\right)^{\frac{p-1}{p}} \cdot \left(\int_\Omega a(x)|v|^p\right)^{\frac{1}{p}} + \epsilon C\int_\Omega a(x)|v|^p \\
&+ C\left(\int_\Omega a(x)|\bar{u}^+|^{\gamma}\right)^{\frac{\gamma-1}{\gamma}} \cdot \left(\int_\Omega a(x)|v|^{\gamma}\right)^{\frac{1}{\gamma}} + C\int_\Omega a(x)|v|^{\gamma}
\end{aligned}$$

Now, we have $v \in D^{1,p} \cap L^{\gamma}_{a(x)} \subset L^{p}_{a(x)} \cap L^{\gamma}_{a(x)}$, $\bar{u} \subset D^{1,p} \subset L^{p}_{a(x)}$ and $\int a(x)|\bar{u}|^{\gamma} < \infty$ (because $\bar{u}$ minimizing I implies $\int a(x)G(\bar{u}) < \infty$.) The theorem now follows from an application of Vitali's Convergence Theorem, Theorem 2.4. □

From this point on we will always assume that $\lambda > \lambda_1$.

Corollary 4.7. *A solution u_0 to the minimization problem (4.4) is a weak solution to the problem*

$$-\Delta_p u = \lambda a(x)(u^+)^{p-1} - a(x)g(u), \;\; x \in \mathbb{R}^N, \tag{4.5}$$

i.e.

$$\int |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v = \int (\lambda a(x)(u_0^+)^{p-1} - a(x)g(u_0))v \;\; \forall\, v \in L^{\gamma}_{a(x)} \cap D^{1,p}. \tag{4.6}$$

In addition, $I(u_0) < 0$, so that u_0 is non-trivial.

Proof: The first part follows immediately from Theorem 4.6. For the second part, let $\Phi_1 \geq 0$ be a first eigenfunction (as in Theorem 3.3), normalized so that $\|\Phi_1\| = 1$. Then we have

$$\begin{aligned} I(t\Phi_1) &= \frac{1}{p}t^p \int |\nabla \Phi_1|^p - \frac{\lambda t^p}{p}\int a(x)\Phi_1^p + \int a(x)G(t\Phi_1) \\ &\leq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)t^p + \int a(x)\left(\epsilon t^p \Phi_1^p + C_2 t^{\gamma}\Phi_1^{\gamma}\right) \\ &= \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1} + \frac{p\epsilon}{\lambda_1}\right)t^p + C_2 t^{\gamma}\int a(x)\Phi_1^{\gamma}. \end{aligned}$$

The result now follows from having $\lambda > \lambda_1$ by choosing ϵ and t sufficiently small and using the facts that $\gamma > p$ and $\Phi_1 \in L^{\gamma}_{a(x)}$, as proved in Theorem 3.7. □

Lemma 4.8.

$$u_0(x) \geq 0 \ \ \forall\ x \in \mathbb{R}^N.$$

Proof: Since G is a function of u^+ and $\int |\nabla u_0| \geq \int |\nabla u_0^+|$ we have that $I(u_0) \geq I(u_0^+)$. Since u_0 minimizes $I(u)$ we in fact have $I(u_0) = I(u_0^+)$, and thus we can assume that $u_0(x) \geq 0$. □

For the remainder of this Chapter, we let u_0 be any nontrivial solution to (4.6) satisfying $u_0 \geq 0$, rather than assuming it is the solution that we arrived at via minimization.

Theorem 4.9.

$$u_0 \in L^\infty \cap L^\gamma_{a(x)} \cap C^{1,\alpha}_{\text{loc}} \ \textit{and} \ \lim_{|x|\to\infty} u_0(x) = 0.$$

Proof: u_0 is a weak solution to the variational inequality

$$-\Delta_p u \geq \lambda a(x) \chi_{\{x \ | \ u_0(x)>0\}} |u|^{p-2}u, \ x \in \mathbb{R}^N.$$

Therefore, since $u_0 \in L^{p^*}$, an application of Corollary 2.8 part (ii) (with $\gamma = p^*$ and moving the center of the cube $K(3\rho)$ to any $x \in \mathbb{R}^N$) yields that $\sup u_0^+ \leq C' \|u_0^+\|_{p^*} \leq C\|u_0^+\|$ and $\lim_{|x|\to\infty} u_0^+(x) = 0$ for some constant $C = C(\lambda, \|a\|_\infty)$. Therefore

$$\sup u_0^+ \leq \tilde{C}, \tag{4.7}$$

where $\tilde{C} = \tilde{C}(\lambda, \|a\|_\infty, \|u_0^+\|)$. Therefore, since $u_0(x) \geq 0$ by Lemma 4.8, we have

$u_0 \in L^\infty$ and $\lim_{|x|\to\infty} u_0(x) = 0$.

Next, since $u_0 \in L^\infty \cap L^p_{a(x)}$ and $\gamma > p$, we have $u_0 \in L^\gamma_{a(x)}$.

Finally, we have $u_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ by Theorem 2.9. $\square$

Lemma 4.10. *Set* $f(x) = \lambda a(x)(u_0^+(x))^{p-1} - a(x)g(u_0(x))$. *Then* $f(x) \geq 0$ *for all* $x \in \mathbb{R}^N$.

Proof: We apply similar methods to Costa, Drabek and Tehrani [7].

Let $S = \max\{s \mid \frac{g(s)}{s^{p-1}} = \lambda\}$. Then, by assumption (A_3) in our conditions on g, we have that $a(x)(\lambda(u_0^+(x))^{p-1} - g(u_0(x))) < 0$ if and only if $u_0(x) > S$. Now define $v = (u_0 - S)^+$. Then since $0 \leq v \leq u_0^+ \leq |u_0|$ and $u_0 \in L^\gamma_{a(x)} \cap D^{1,p}$, v is an admissible test function in (4.6). Therefore, if $\{u_0 > S\}$ is nonempty then

$$
\begin{aligned}
0 &\leq \int_{\{u_0 > S\}} |\nabla u_0|^p \\
&= \int |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla v \\
&= \int a(x)(\lambda(u_0^+)^{p-1} - g(u_0))v \\
&= \int_{\{u_0 > S\}} a(x)(\lambda(u_0^+)^{p-1} - g(u_0))(u_0 - S)^+ < 0,
\end{aligned}
$$

a contradiction, so that $u_0 \leq S$ in $\mathbb{R}^N$, proving the lemma. $\square$

At this point we can prove that $u_0 > 0$ by applying the Maximum Principle of Theorem 2.10. However we employ a different method, which in addition provides estimates for the behavior of u_0 at infinity.

Lemma 4.11. *Given $\epsilon > 0$, set*

$$V_\epsilon = \{x \mid u_0(x) > \epsilon a(x)^{\frac{N-p}{p^2}},\ f(x) > \epsilon a(x)(u_0^+(x))^{p-1}\}.$$

Then there exists positive constants ϵ_0, L_0 and $R_1 \geq 1$ such that, for all $0 < \epsilon \leq \epsilon_0$ we have

$$\|a\chi_\Omega\|_{N/p} \geq L_0,$$

where $\Omega = V_\epsilon \cap B_{R_1}(0) = V_\epsilon \cap \{x \mid |x| \leq R_1\}$. Here our constants ϵ_0 and L_0 may depend on $\lambda, p, \|a\|_{N/p}$ and $\|u_0\|$.

Proof: Again we apply similar methods to Costa, Drabek and Tehrani [7]. Let $\epsilon > 0$. To simplify notation, we write $V = V_\epsilon$. Then, letting $v = u_0$ in (4.6), we have

$$\begin{aligned}\|u_0\|^p &= \int \lambda a(x)(u_0^+)^{p-1}u_0 - \int a(x)g(u_0)u_0 \\ &= \int_V \lambda a(x)(u_0^+)^{p-1}u_0 - \int_V a(x)g(u_0)u_0 + \int_{\mathbb{R}^N\setminus V} f(x)u_0 \\ &\leq \int_V \lambda a(x)(u_0^+)^{p-1}u_0 + \int_{\mathbb{R}^N\setminus V} f(x)u_0 \\ &\leq \lambda C\|a\chi_V\|_{N/p}\|u_0\|^p + \int_{\mathbb{R}^N\setminus V} f(x)u_0.\end{aligned}$$

Consider the decomposition $\mathbb{R}^N\setminus V = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, where

$$A_1 = \{x \mid u_0(x) \leq \epsilon a(x)^{\frac{N-p}{p^2}}\},$$

$$A_2 = \{x \mid f(x) \leq \epsilon a(x)(u_0^+(x))^{p-1},\ u_0(x) > \epsilon a(x)^{\frac{N-p}{p^2}}\}.$$

Then we obtain

$$\begin{aligned}
\int_{A_1} f(x)u_0 &= \lambda \int_{A_1} a(x)(u_0^+)^p - \int_{A_1} a(x)g(u_0)u_0 \\
&\leq \lambda\epsilon^p \int_{A_1} a(x)\left(a(x)^{\frac{N-p}{p^2}}\right)^p \\
&= \lambda\epsilon^p \int_{A_1} a(x)^{N/p} \\
&= \lambda\epsilon^p \|a\|_{N/p}^{N/p}.
\end{aligned}$$

Furthermore,

$$\int_{A_2} f(x)u_0 \leq \epsilon \int_{A_2} a(x)|u_0|^p \leq \epsilon C\|a\|_{N/p}\|u_0\|^p.$$

Therefore, combining the above estimates, we obtain

$$\|u_0\|^p \leq \lambda C\|a\chi_V\|_{N/p}\|u_0\|^p + \lambda\epsilon^p\|a\|_{N/p}^{N/p} + \epsilon C\|u_0\|^p\|a\|_{N/p}.$$

Therefore we can find $\epsilon_0 > 0$ (depending only on λ, $\|u_0\|$ and $\|a\|_{N/p}$) such that, for $\epsilon \leq \epsilon_0$,

$$\lambda\epsilon^p\|a\|_{N/p}^{N/p} + \epsilon C\|u_0\|^p\|a\|_{N/p} \leq \frac{1}{2}\|u_0\|^p,$$

and hence

$$\|a\chi_V\|_{N/p} \geq \frac{1}{2\lambda C}.$$

Next, we let $L_0 = \frac{1}{4\lambda C}$. Since $a \in L^{N/p}(\mathbb{R}^N)$ there exists $R_1 \geq 1$ such that

$$\|a\chi_{\mathbb{R}^N\setminus B_{R_1}(0)}\|_{N/p} < L_0.$$

Therefore, considering that $V_{\epsilon_0} \subset V_\epsilon$ for $0 < \epsilon < \epsilon_0$, it follows that

$$\|a\chi_{V_\epsilon \cap B_{R_1}(0)}\|_{N/p} \geq L_0,$$

completing the proof of the lemma. $\square$

The reader should note that the above proof works without the pointwise estimate of Lemma 4.8, because we have $g(s)s \geq 0$ for all $s \in \mathbb{R}$.

We are finally ready to study the behavior of u_0 at infinity, proving in the process that $u_0 > 0$.

Theorem 4.12. *There exists $C > 0$ such that a solution $u_0 \geq 0$ of (4.6) is a positive weak solution to problem (4.1) and satisfies*

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p}{p-1}}} \text{ for } |x| \text{ large.}$$

Proof: We have that u_0 is a weak solution to (4.5) in $\mathbb{R}^N$. Therefore, using the notation of Lemma 4.11 and letting $V = V_{\epsilon_0}$, we have by the definition of V that

$$-\Delta_p u_0 = \lambda a(x)(u_0^+)^{p-1} - a(x)g(u_0) = f(x) \geq \epsilon_0^p (a(x))^{\frac{Np-N+p}{p^2}} \text{ on } V \cap B_{R_1}(0).$$

For $R > R_1$, consider $z = z_R$, the solution to the Dirichlet problem

$$\begin{cases} -\Delta_p z = \epsilon_0^p (a(x))^{\frac{Np-N+p}{p^2}} \chi_{V \cap B_{R_1}(0)} & \text{in } B_R(0), \\ z = 0 & \text{on } \partial B_R(0). \end{cases}$$

The solution z exists by Theorem 2.13, is continuous (and hence bounded) by Theorem 2.9, and is p-superharmonic by part (iv) of Theorem 2.10. Since $f(x) \geq \epsilon_0^p(a(x))^{\frac{Np-N+p}{p^2}}$ in $V \cap B_{R_1}(0)$ and $f(x) \geq 0$ in $\mathbb{R}^N$ (by Lemma 4.10), we have

$$-\Delta_p u_0 \geq -\Delta_p z \text{ in } B_R(0).$$

Furthermore, since $u_0 \geq 0$ in $\mathbb{R}^N$, we have that

$$u_0 \geq z \text{ on } \partial B_R(0).$$

Therefore, by the Weak Comparison Principle of Theorem 2.10, we conclude that

$$u_0 \geq z \text{ in } B_R(0).$$

Now, choose $R \geq 24R_1$. Then we have, for $x \in B_{R/24}(0)$,

$$B_{R_1}(0) \subset B_{R/12}(x) \subset B_{R/6}(x), \text{ and } B_{R/2}(x) \subset B_R(0).$$

So, since $-\Delta_p z = \epsilon_0^p(a(x))^{\frac{Np-N+p}{Np}} \chi_{V \cap B_{R_1}(0)}$ in $B_{R/2}(x) \subset B_R(0)$, we can apply Theorem 2.15 to $\mu(\Omega) = \int_\Omega \epsilon_0^p(a(x))^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)} dx$ and get

$$
\begin{aligned}
z(x) &\geq A_1 \int_0^{R/6} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} \epsilon_0^p(a(y))^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\geq A_1 \int_{R/12}^{R/6} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} \epsilon_0^p(a(y))^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&\geq A_1 \int_{R/12}^{R/6} \left(\frac{1}{t^{N-p}} \int_{B_{R_1}(0)} \epsilon_0^p(a(y))^{\frac{Np-N+p}{p^2}} \chi_{V\cap B_{R_1}(0)} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= A_1 \int_{R/12}^{R/6} \left(\frac{1}{t^{N-p}} \int_{V\cap B_{R_1}(0)} \epsilon_0^p(a(y))^{\frac{Np-N+p}{p^2}} dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\
&= A_1 \int_{R/12}^{R/6} \left(\int_{V\cap B_{R_1}(0)} \epsilon_0^p(a(y))^{\frac{Np-N+p}{p^2}} dy \right)^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-p}{p-1}} \frac{dt}{t} \\
&= C\epsilon_0^{\frac{p}{p-1}} \left(\frac{1}{R}\right)^{\frac{N-p}{p-1}} \left(\int_{V\cap B_{R_1}(0)} a^{\frac{Np-N+p}{p^2}} \right)^{\frac{1}{p-1}}
\end{aligned}
$$

for $|x| \leq \frac{R}{24}$. Now, by Theorem 4.11, we have

$$
L_0^{N/p} \leq \int_{V\cap B_{R_1}(0)} a^{N/p} \leq \|a\|_\infty^\alpha \int_{V\cap B_{R_1}(0)} a^{\frac{Np-N+p}{p^2}}
$$

where $\alpha = \frac{N}{p} - \frac{Np-N+p}{p^2} = \frac{N-p}{p^2}$. Therefore, taking $|x| = \frac{R}{24}$, and using the fact that $R \geq 24R_1$ is arbitrary, we have that there exists $C_1 = C_1(\epsilon_0, \|a\|_\infty, L_0)$ such that

$$
u_0(x) \geq \frac{C_1}{|x|^{\frac{N-p}{p-1}}}, \text{ for } |x| \geq R_1.
$$

Furthermore, choosing $R = 24R_1$, we have that there exists $C_2 = C_2(\epsilon_0, \|a\|_\infty, L_0)$ such that

$$
u_0(x) \geq C_2, \text{ for } |x| \leq \frac{R}{24} = R_1.
$$

Therefore $u_0 > 0$ and there exists $C > 0$ such that

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p}{p-1}}}$$

for $|x|$ sufficiently large. This completes the proof. □

Uniqueness and Sharp Estimate at Infinity

We finish this chapter by addressing the uniqueness and the sharp estimate at infinity for u_0.

In order to prove the sharp estimate at infinity for u_0, we need the following lemma.

Lemma 4.13. *Suppose $0 \leq h \in L^1 \cap L^\infty$ and for all $x \in \mathbb{R}^N$ we have*

$$|x|^{\frac{N}{Q(p-1)}} \|h\|_{L^P(\mathbb{R}^N \setminus B_{|x|}(0))} \leq C \tag{4.8}$$

for some constant C and some $P > \frac{N}{p}$, with $\frac{1}{P} + \frac{1}{Q} = 1$. Then there exists a unique weak solution w to

$$-\Delta_p w = h$$

with $w \in D^{1,p} \cap C^1 \cap L^\infty$, $\lim_{|x| \to \infty} w(x) = 0$, and there exists a constant $d > 0$ such that

$$w(x) \leq \frac{d}{|x|^{\frac{N-p}{p-1}}} \quad \forall\, x \in \mathbb{R}^N.$$

Proof: We use techniques developed in Lemma 4 of Allegretto and Odiobala [2].

The solution $w \in D^{1,p}$ exists and is unique by Theorem 2.13 and is p-superharmonic on bounded domains by part (iv) of Theorem 2.10. In addition, $w \in C^1 \cap L^\infty$ and $\lim_{|x|\to\infty} w(x) = 0$ by an application of Theorem 2.9 and Corollary 2.8.

Now, let $r > 0$. Then by Theorem 2.15 we have $A_2, A_3 > 0$ such that

$$w(x) \leq A_2 \inf_{a\in B(x,r)} w(a) + A_3 \int_0^{2r} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} h(y)dy \right)^{\frac{1}{p-1}} \frac{dt}{t}$$

Letting $r \to \infty$ and using the fact that $\lim_{|x|\to\infty} w(x) = 0$ we get

$$\begin{aligned} w(x) &\leq 0 + A_3 \int_0^{\infty} \left(\frac{1}{t^{N-p}} \int_{B_t(x)} h(y)dy \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &= A_3 \int_0^\infty \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-p}{p-1}+1} dt \\ &= A_3 \int_0^\infty \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt \\ &= A_3 \int_0^{|x|/2} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt + A_3 \int_{|x|/2}^\infty \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt. \end{aligned}$$

For the second term on the right we have

$$\begin{aligned} \int_{|x|/2}^\infty \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt &\leq \|h\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p-1}} \int_{|x|/2}^\infty \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt \\ &= C\|h\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p-1}} |x|^{\frac{1-N}{p-1}+1} \\ &= C\|h\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p-1}} \left(\frac{1}{|x|}\right)^{\frac{N-p}{p-1}}. \end{aligned}$$

Now, we also have, with $h_x(y) = h(y+x)$ and $\frac{1}{P} + \frac{1}{Q} = 1$,

$$\begin{aligned}
\|h\|_{L^1(B_t(x))} &= \int_{B_t(x)} h(y)dy \\
&= \int_{B_t(0)} h_x(y)dy \\
&= c_1 \int_{B_1(0)} h_x(ty)t^N dy \\
&\leq c_2 t^N \left(\int_{B_1(0)} (h_x(ty))^P dy \right)^{1/P} \\
&= c_3 t^N \left(\int_{B_t(0)} (h_x(y))^P t^{-N} dy \right)^{1/P} \\
&= c_3 t^{N/Q} \|h\|_{L^P(B_t(x))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{|x|/2} \|h\|_{L^1(B_t(x))}^{\frac{1}{p-1}} \left(\frac{1}{t}\right)^{\frac{N-1}{p-1}} dt &= c_3^{\frac{1}{p-1}} \int_0^{|x|/2} \|h\|_{L^P(B_t(x))}^{\frac{1}{p-1}} t^{\frac{1-N}{p-1} + \frac{N}{Q(p-1)}} dt \\
&\leq c_4 \|h\|_{L^P(B_{|x|/2}(x))}^{\frac{1}{p-1}} \left(\frac{|x|}{2}\right)^{\frac{p-N}{p-1} + \frac{N}{Q(p-1)}} \\
&\leq c_5 \|h\|_{L^P(\mathbb{R}^N \setminus B_{|x|/2}(0))}^{\frac{1}{p-1}} \left(\frac{1}{|x|}\right)^{\frac{N-p}{p-1}} |x|^{\frac{N}{Q(p-1)}}
\end{aligned}$$

where we used the fact that $P > \frac{N}{p}$ implies that $\frac{p-N}{p-1} + \frac{N}{Q(p-1)} > 0$. The lemma now follows from our condition (4.8) on h. □

Now, by our conditions on g we have that there exists an $S > 0$ such that

$$S = \sup_{s \in \mathbb{R}^+} \lambda s^{p-1} - g(s).$$

Therefore $a(x)(\lambda u_0^{p-1}(x) - g(u_0(x))) \leq Sa(x)$, and so the sharp estimate (4.2) now follows from Theorem 4.12 and Lemma 4.13, where we apply condition (B_1') of Theorem 4.1.

Finally, suppose we have two positive solutions $u = u_0$ and v to (4.1). Then, since our results following Lemma 4.8 hold for any nonnegative solution to (4.6), we get sharp estimates for both u and v at infinity, proving that $\frac{u}{v}, \frac{v}{u} \in L^\infty$. In addition, we have $u, v \in C^1$ by Theorem 4.9. Therefore, by Lemma 3.4, we can apply (4.3) to our admissible test functions $\frac{u^p - v^p}{u^{p-1}}$ and $\frac{u^p - v^p}{v^{p-1}}$ to get

$$\begin{aligned}
0 &\leq I(u,v) \\
&= \lambda \int a(x) u^{p-1} \frac{u^p - v^p}{u^{p-1}} - \int a(x) g(u) \frac{u^p - v^p}{u^{p-1}} \\
&\quad - \lambda \int a(x) v^{p-1} \frac{u^p - v^p}{v^{p-1}} + \int a(x) g(v) \frac{u^p - v^p}{v^{p-1}} \\
&= \int a(x) \left(\frac{g(v)}{v^{p-1}} - \frac{g(u)}{u^{p-1}} \right) (u^p - v^p).
\end{aligned}$$

Therefore, by condition (A_3') on g (that $\frac{g(s)}{s^{p-1}}$ is increasing) we have that $0 \leq I(u,v) < 0$ if u and v are not identical, a contradiction. Therefore $u \equiv v$, and hence we have uniqueness for positive solutions to (4.1).

CHAPTER 5

SYSTEM PROBLEM

In this chapter we let $\lambda_1(p) = \lambda_1(a, p)$ be the first eigenvalues of Theorem 3.1.

Main Theorem 5.1. *Consider*

$$\begin{cases} -\Delta_{p_1} u = a_1(x)(\mu_1 |u|^{p_1-2}u - g_1(u)) + F_u(x,u,v) \\ -\Delta_{p_2} v = a_2(x)(\mu_2 |v|^{p_2-2}v - g_2(v)) + F_v(x,u,v) \end{cases} \tag{5.1}$$

where for $i = 1, 2$,

(A_0) $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous,*

(A_1) $\lim_{s\to 0^+} \frac{g_i(s)}{s^{p_i-1}} = 0$,

(A_2) $0 < \liminf_{s\to\infty} \frac{g_i(s)}{s^{\gamma_i-1}} \le \limsup_{s\to\infty} \frac{g_i(s)}{s^{\gamma_i-1}} < \infty$ *with* $\gamma_i > p_i$,

(A_3) $\frac{g_i(s)}{s^{p_i-1}}$ *is nondecreasing,*

(B_0) $F(x,s,t) : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is given by* $F(x,s,t) = b(x)s^{\frac{p_1}{m}}t^{\frac{p_2(m-1)}{m}}$ *for some* $1 < m < \infty$,

(B_1) $0 < b(x) = \Gamma(x)(a_1(x))^{\frac{1}{m}}(a_2(x))^{\frac{m-1}{m}}$ *for all* $x \in \mathbb{R}^N$ *and some* $\Gamma \in L^\infty(\mathbb{R}^N)$,

(C_1) $0 < a_i(x) \in L^{N/p_i}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

(C_2) $1 < p_i < N$, $m \le p_1$, $\frac{m}{m-1} \le p_2$ *and* $\mu_i > \lambda_1(p_i)$.

Then (5.1) has a solution (u_0, v_0) *with* $u_0, v_0 > 0$ *in* $\mathbb{R}^N$. *In addition,*

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p_1}{p_1-1}}} \text{ for } |x| \text{ large,}$$

and

$$v_0(x) \geq \frac{C}{|x|^{\frac{N-p_2}{p_2-1}}} \text{ for } |x| \text{ large.}$$

Existence of Solution

Since we are varying p in this chapter, we define

$$\|u\|_{1,p} = \left(\int |\nabla u|^p\right)^{1/p}$$

Lemma 5.2. *(i) For any* $\epsilon > 0$ *there exists constants* $C_1 = C_1(\epsilon)$ *and* $C_2 = C_2(\epsilon)$ *such that*

$$-\epsilon(s^+)^{p_i-1} + C_1(s^+)^{\gamma_i-1} \leq g_i(s) \leq \epsilon(s^+)^{p_i-1} + C_2(s^+)^{\gamma_i-1},$$

$$-\epsilon(s^+)^{p_i} + C_1(s^+)^{\gamma_i} \leq G_i(s) \leq \epsilon(s^+)^{p_i} + C_2(s^+)^{\gamma_i},$$

where $G_i(s) = \int_0^s g_i(t)dt$.

(ii) There exists $C_0 > 0$ *such that*

$$\int |F(x,u,v)|, \int |F_u(x,u,v)u|, \int |F_v(x,u,v)v|$$
$$\leq C_0 \left(\int a_1(x)(u^+)^{p_1}\right)^{\frac{1}{m}} \left(\int a_2(x)(v^+)^{p_2}\right)^{\frac{m-1}{m}}.$$

Proof: This follows from our conditions on g_i and F (and Holder's Inequality). □

We consider the functionals

$$J : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R} \cup \{\infty\},\ J(u,v) = \int a_1(x)G_1(u) + \int a_2(x)G_2(v)$$

$$K : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R}^+,\ K(u,v) = \int F(x,u,v)$$

and $I : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R} \cup \{\infty\}$ where $I(u,v) = \infty$ if $J(u,v) = \infty$ and

$$\begin{aligned} I(u,v) &= \frac{1}{p_1}\int |\nabla u|^{p_1} - \frac{\mu_1}{p_1}\int a_1(x)(u^+)^{p_1} + \frac{1}{p_2}\int |\nabla v|^{p_2} - \frac{\mu_2}{p_2}\int a_2(x)(v^+)^{p_2} \\ &\quad + J(u,v) - K(u,v) \end{aligned}$$

if $J(u,v) < \infty$. Our goal is to find a solution to (5.1) by minimizing I.

Lemma 5.3. *(a) I is coercive, i.e. $I(u,v) \to \infty$ as $\|u\|_{1,p_1} + \|v\|_{1,p_2} \to \infty$.*

(b) I is a weakly lower semi-continuous functional.

Proof: (a) Assume not. Then there exists $\{(u_n, v_n)\} \subset D^{1,p_1} \times D^{1,p_2}$ such that $\{I(u_n, v_n)\}$ is bounded above and $\|u_n\|_{1,p_1} + \|v_n\|_{1,p_2} \to \infty$.

Let $d_n = \left(\int a_1(x)(u_n^+)^{p_1}\right)^{1/p_1}$ and $e_n = \left(\int a_2(x)(v_n^+)^{p_2}\right)^{1/p_2}$. Now, without loss of generality and passing to a subsequence if necessary, we can assume that $\|u_n\|_{1,p_1} \to \infty$.

First we show that $\|u_n\|_{1,p_1} \to \infty$ implies that $d_n \to \infty$ or $e_n \to \infty$. We have, applying Lemma 5.2, that

$$\begin{aligned} I(u_n, v_n) &\geq \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1} + \frac{1}{p_2}\|v_n\|_{1,p_2}^{p_2} - \frac{\mu_2}{p_2}e_n^{p_2} - \int |F(x, u_n, v_n)| \\ &\geq \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1} - \frac{\mu_2}{p_2}e_n^{p_2} - C_0 d_n^{\frac{p_1}{m}} e_n^{\frac{p_2(m-1)}{m}} \end{aligned}$$

and $\|u_n\|_{1,p_1} \to \infty$, so if $\{d_n\}$ and $\{e_n\}$ are bounded then $I(u_n, v_n) \to \infty$, contradicting the assumption that $\{I(u_n, v_n)\}$ is bounded above. Therefore, passing to a subsequence if necessary, $d_n \to \infty$ or $e_n \to \infty$, as desired. If we have that $\|v_n\|_{1,p_2}$ is bounded, then because $e_n^{p_2} \leq C\|a_2\|_{N/p_2}\|v_n\|_{1,p_2}^{p_2}$ we also have that $\{e_n\}$ is bounded. Therefore either $d_n \to \infty$ and $\|u_n\|_{1,p_1} \to \infty$, or we have $e_n \to \infty$ and (passing to a subsequence if necessary) $\|v_n\|_{1,p_2} \to \infty$, or both cases occur simultaneously. Hence, without loss of generality, we can assume that $\|u_n\|_{1,p_1} \to \infty$, $d_n \to \infty$ and $\limsup_{n\to\infty} \frac{e_n^{p_2}}{d_n^{p_1}} < \infty$.

Next, set $\bar{u}_n = \frac{u_n}{d_n}$ and $\bar{v}_n = \frac{v_n}{e_n}$. Then

$$\int a_1(x)(\bar{u}_n^+)^{p_1} = \int a_2(x)(\bar{v}_n^+)^{p_2} = 1$$

and

$$
\begin{aligned}
I(u_n, v_n) &= \frac{1}{p_1}\int |\nabla u_n|^{p_1} - \frac{\mu_1}{p_1}\int a_1(x)(u_n^+)^{p_1} + \int a_1(x)G_1(u_n) \\
&\quad + \frac{1}{p_2}\int |\nabla v_n|^{p_2} - \frac{\mu_2}{p_2}\int a_2(x)(v_n^+)^{p_2} + \int a_2(x)G_2(v_n) \\
&\quad - \int F(x, u_n, v_n) \\
&= \frac{d_n^{p_1}}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1}\int a_1(x)(\bar{u}_n^+)^{p_1} + \int a_1(x)G_1(d_n\bar{u}_n) \\
&\quad + \frac{e_n^{p_2}}{p_2}\int |\nabla \bar{v}_n|^{p_2} - \frac{\mu_2}{p_2}e_n^{p_2}\int a_2(x)(\bar{v}_n^+)^{p_2} + \int a_2(x)G_2(e_n\bar{v}_n) \\
&\quad - \int F(x, u_n, v_n)
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{I(u_n, v_n)}{d_n^{p_1}} &= \frac{1}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1} + \frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n) \\
&\quad + \frac{e_n^{p_2}}{p_2 d_n^{p_1}}\int |\nabla \bar{v}_n|^{p_2} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} + \frac{1}{d_n^{p_1}}\int a_2(x)G_2(e_n\bar{v}_n) \\
&\quad - \frac{1}{d_n^{p_1}}\int F(x, u_n, v_n) \\
&\geq \frac{1}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - \frac{1}{d_n^{p_1}}\int |F(x, u_n, v_n)| \\
&\geq \frac{1}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - C_0\frac{d_n^{\frac{p_1}{m}} e_n^{\frac{p_2(m-1)}{m}}}{d_n^{p_1}} \\
&= \frac{1}{p_1}\|\bar{u}_n\|_{1,p_1}^{p_1} - \frac{\mu_1}{p_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}}.
\end{aligned}
$$

Therefore, since $\frac{I(u_n,v_n)}{d_n^{p_1}} \to 0$ and $\limsup_{n\to\infty}\frac{e_n^{p_2}}{d_n^{p_1}} < \infty$, we have that $\{\|\bar{u}_n\|_{1,p_1}\}$ is bounded. This implies, passing to a subsequence if necessary, that $\bar{u}_n \rightharpoonup \bar{u}$ in D^{1,p_1}, $\bar{u}_n \to \bar{u}$ a.e. in $\mathbb{R}^N$, and $\bar{u}_n \to \bar{u}$ in $L^{p_1}_{a_1(x)}$ (by Lemma 2.6). In addition, applying

Lemma 5.2 we have

$$\begin{aligned}
\frac{I(u_n,v_n)}{d_n^{p_1}} &= \frac{1}{p_1}\int|\nabla\bar{u}_n|^{p_1} - \frac{\mu_1}{p_1} + \frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n) \\
&\quad + \frac{e_n^{p_2}}{p_2 d_n^{p_1}}\int|\nabla\bar{v}_n|^{p_2} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} + \frac{1}{d_n^{p_1}}\int a_2(x)G_2(e_n\bar{v}_n) \\
&\quad - \frac{1}{d_n^{p_1}}\int F(x,u_n,v_n) \\
&\geq -\frac{\mu_1}{p_1} + \frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n) - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - \frac{1}{d_n^{p_1}}\int|F(x,u_n,v_n)| \\
&\geq -\frac{\mu_1}{p_1} - \frac{\epsilon}{d_n^{p_1}}\int a_1(x)(d_n\bar{u}_n^+)^{p_1} + \frac{C_1}{d_n^{p_1}}\int a_1(x)(d_n\bar{u}_n^+)^{\gamma_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} \\
&\quad - C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}} \\
&= -\frac{\mu_1}{p_1} - \epsilon + C_1 d_n^{\gamma_1-p_1}\int a_1(x)(\bar{u}_n^+)^{\gamma_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}}.
\end{aligned}$$

Therefore, $\gamma_1 > p_1$ and $d_n \to \infty$ imply that $\int a_1(x)(\bar{u}_n^+)^{\gamma_1} \to 0$. Hence, by Fatou's lemma we have

$$\int a_1(x)(\bar{u}^+)^{\gamma_1} = 0.$$

Therefore, since $a_1(x) > 0$ on $\mathbb{R}^N$, we have that $\bar{u} \leq 0$ on $\mathbb{R}^N$. However this contradicts the fact that $\bar{u}_n \to \bar{u}$ in $L^{p_1}_{a_1(x)}$ and $\int a_1(x)(\bar{u}_n^+)^{p_1} = 1$ for all n. This proves part (a) of the lemma.

(b) Assume $u_n \rightharpoonup u$ in D^{1,p_1} and $v_n \rightharpoonup v$ in D^{1,p_2}. Then Lemma 2.6 implies

$$\int a_1(x)u_n^{p_1} \to \int a_1(x)u^{p_1} \text{ and } \int a_2(x)v_n^{p_2} \to \int a_2(x)v^{p_2}.$$

Furthermore, since $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$, we have by Lebesgue

Dominated Convergence Theorem that

$$\int a_1(x)|u_n^+|^{p_1} \to \int a_1(x)|u^+|^{p_1}, \int a_2(x)|v_n^+|^{p_2} \to \int a_2(x)|v^+|^{p_2},$$

and

$$\int F(x, u_n, v_n) \to \int F(x, u, v).$$

It follows by Fatou's Lemma that

$$\int a_1(x)G_1(u) \leq \liminf_{n\to\infty} \int a_1(x)G_1(u_n) \text{ and } \int a_2(x)G_2(v) \leq \liminf_{n\to\infty} \int a_2(x)G_2(v_n).$$

Finally, $\|\cdot\|^p$ is weakly l.s.c. by Lemma 2.5, so that $(u_n, v_n) \rightharpoonup (u, v)$ in $D^{1,p_1} \times D^{1,p_2}$ implies $\frac{1}{p_1}\|u\|_{1,p_1}^{p_1} + \frac{1}{p_2}\|v\|_{1,p_2}^{p_2} \leq \liminf_{n\to\infty} \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} + \frac{1}{p_2}\|v_n\|_{1,p_2}^{p_2}$. Therefore $I(u, v) \leq \liminf_{n\to\infty} I(u_n, v_n)$, hence is weakly l.s.c. This proves the lemma. □

Theorem 5.4. *(Solution to the Minimization Problem) The minimization problem*

$$\inf I(u, v), \ (u, v) \in D^{1,p_1} \times D^{1,p_2}$$

(i.e. find $(u, v) \in D^{1,p_1} \times D^{1,p_2}$ for which $\inf I(u, v)$ is achieved) has a solution $(u_0, v_0) \in D^{1,p_1} \times D^{1,p_2}$ with

$$\inf_{(u,v)\in D^{1,p_1}\times D^{1,p_2}} I(u, v) = I(u_0, v_0). \tag{5.2}$$

Proof: This follows directly from Lemma 5.3 and Lemma 2.5. □

Theorem 5.5. *Suppose $(\bar{u}, \bar{v})$ is a minimum point of I, i.e. a solution to (5.2). Assume that $\{t_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$. Then, if $w \in D^{1,p_1} \cap L^{\gamma_1}_{a_1(x)}$ and $z \in D^{1,p_2} \cap L^{\gamma_2}_{a_2(x)}$, we have*

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(\bar{u} + t_n w, \bar{v}) - I(\bar{u}, \bar{v})}{t_n} \\ &= \int |\nabla \bar{u}|^{p_1-2} \nabla \bar{u} \cdot \nabla w - \mu_1 \int a_1(x)(\bar{u}^+)^{p_1-1} w \\ &\quad + \int a_1(x) g_1(\bar{u}) w - \int F_u(x, \bar{u}, \bar{v}) w \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(\bar{u}, \bar{v} + t_n z) - I(\bar{u}, \bar{v})}{t_n} \\ &= \int |\nabla \bar{v}|^{p_2-2} \nabla \bar{v} \cdot \nabla z - \mu_2 \int a_2(x)(\bar{v}^+)^{p_2-1} z \\ &\quad + \int a_2(x) g_2(\bar{v}) z - \int F_v(x, \bar{u}, \bar{v}) z \end{aligned}$$

Proof: In view of Lemma 5.2, the proof follows exactly as in the scalar case, Theorem 4.6. □

Corollary 5.6. *A solution (u_0, v_0) to the minimization problem (5.2) is a weak solution to the problem*

$$\begin{aligned} -\Delta_{p_1}(u_0) &= \mu_1 a_1(x)(u_0^+)^{p_1-1} - a_1(x) g_1(u_0) + F_u(x, u_0, v_0) \\ -\Delta_{p_2}(v_0) &= \mu_2 a_2(x)(v_0^+)^{p_2-1} - a_2(x) g_2(v_0) + F_v(x, u_0, v_0), \end{aligned} \tag{5.3}$$

in $\mathbb{R}^N$, *i.e.*

$$\int |\nabla u_0|^{p_1-2}\nabla u_0 \cdot \nabla w = \int (\mu_1 a_1(x)(u_0^+)^{p_1-1} - a_1(x)g_1(u_0))w + \int F_u(x,u_0,v_0)w,$$
$$\int |\nabla v_0|^{p_2-2}\nabla v_0 \cdot \nabla z = \int (\mu_2 a_2(x)(v_0^+)^{p_2-1} - a_2(x)g_2(v_0))z + \int F_v(x,u_0,v_0)z,$$
$$\forall\ w \in L^{\gamma_1}_{a_1(x)} \cap D^{1,p_1}, \quad \forall\ z \in L^{\gamma_2}_{a_2(x)} \cap D^{1,p_2}. \tag{5.4}$$

In addition, $I(u,v) < 0$ *so that* (u_0, v_0) *is non-trivial. Finally, we have that both* u_0 *and* v_0 *are each non-trivial.*

Proof: The first part follows immediately from Theorem 5.5. For the second part, we provide a proof that works even without the condition that $F \geq 0$. Let $\Phi_1, \Psi_1 \geq 0$ be first eigenfunctions associated to the first eigenvalues $\lambda_1(p_1), \lambda_1(p_2)$, respectively, normalized so that $\|\Phi_1\|_{1,p_1} = \|\Psi_1\|_{1,p_2} = 1$. Assume without loss of generality that $p_1 \leq p_2$. Then we have, applying Lemma 5.2,

$$\begin{aligned}
I(t\Phi_1, t^2\Psi_1) &= \frac{1}{p_1}t^{p_1} - \frac{\mu_1 t^{p_1}}{p_1}\int a_1(x)\Phi_1^{p_1} + \int a_1(x)G_1(t\Phi_1) + \frac{1}{p_2}t^{2p_2} \\
&\quad - \frac{\mu_2 t^{2p_2}}{p_2}\int a_2(x)\Psi_1^{p_2} + \int a_2(x)G_2(t^2\Psi_1) - \int F(x, t\Phi_1, t^2\Psi_1) \\
&\leq \frac{1}{p_1}\left(1 - \frac{\mu_1}{\lambda_1(p_1)}\right)t^{p_1} + \int a_1(x)\left(\epsilon t^{p_1}\Phi_1^{p_1} + C_2 t^{\gamma_1}\Phi_1^{\gamma_1}\right) \\
&\quad + \frac{1}{p_2}\left(1 - \frac{\mu_2}{\lambda_1(p_2)}\right)t^{2p_2} + \int a_2(x)\left(\epsilon t^{2p_2}\Psi_1^{p_2} + C_2 t^{2\gamma_2}\Psi_1^{\gamma_2}\right) \\
&\quad + C_0 t^{\frac{p_1}{m} + \frac{2p_2(m-1)}{m}}\left(\int a_1(x)\Phi_1^{p_1}\right)^{\frac{1}{m}}\left(\int a_2(x)\Psi_1^{p_2}\right)^{\frac{m-1}{m}} \\
&= \frac{1}{p_1}\left(1 - \frac{\mu_1}{\lambda_1(p_1)} + \frac{p_1\epsilon}{\lambda_1(p_1)}\right)t^{p_1} + C_2 t^{\gamma_1}\int a_1(x)\Phi_1^{\gamma_1} \\
&\quad + \frac{1}{p_2}\left(1 - \frac{\mu_2}{\lambda_1(p_2)} + \frac{p_2\epsilon}{\lambda_1(p_2)}\right)t^{2p_2} + C_2 t^{2\gamma_2}\int a_2(x)\Psi_1^{\gamma_2} \\
&\quad + C_0 t^{\frac{p_1}{m} + \frac{2p_2(m-1)}{m}}\left(\frac{1}{\lambda_1(p_1)}\right)^{\frac{1}{m}}\left(\frac{1}{\lambda_1(p_2)}\right)^{\frac{m-1}{m}}.
\end{aligned}$$

Now, we have $\gamma_i > p_i$. In addition, since $1 < p_1 \leq p_2$ we have

$$p_1 = \frac{p_1}{m} + \frac{p_1(m-1)}{m} < \frac{p_1}{m} + \frac{2p_2(m-1)}{m}.$$

The result now follows in this case from having $\mu_i > \lambda_1(p_i)$, by choosing ϵ and t sufficiently small, and using the fact that $\Phi_1 \in L^{\gamma_1}_{a_1(x)}, \Psi_1 \in L^{\gamma_2}_{a_2(x)}$ as proved in Theorem 3.7. Finally, if $u_0 \equiv 0$, a similar proof as above shows that $I(t\Phi_1, v_0) < I(u_0, v_0)$ for t sufficiently small, contradicting the fact that (u_0, v_0) minimizes I. Therefore u_0 is non-trivial. The proof that v_0 is non-trivial is the same. □

Properties of Solution

Lemma 5.7.

$$u_0(x) \geq 0,\ v_0(x) \geq 0 \ \ \forall\, x \in \mathbb{R}^N.$$

Proof: Since F is a function of u^+ and v^+, the proof is identical to that of the scalar case, Lemma 4.8. □

For the remainder of this chapter, we let (u_0, v_0) be any nontrivial solution to (5.4) satisfying $u_0, v_0 \geq 0$, rather than assuming it is the solution that we arrived at via minimization.

Theorem 5.8.

$$u_0 \in L^\infty \cap L^{\gamma_1}_{a_1(x)} \cap C^{1,\alpha}_{\text{loc}} \ \textit{and} \ \lim_{|x|\to\infty} u_0(x) = 0,$$

$$v_0 \in L^\infty \cap L^{\gamma_2}_{a_2(x)} \cap C^{1,\alpha}_{\text{loc}} \text{ and } \lim_{|x|\to\infty} v_0(x) = 0.$$

Proof: First note that

$$F_u(x,u,v) = \frac{p_1}{m} b(x)(u^+)^{\frac{p_1}{m}-1}(v^+)^{\frac{p_2(m-1)}{m}}$$

and

$$F_v(x,u,v) = \frac{p_2(m-1)}{m} b(x)(u^+)^{\frac{p_1}{m}}(v^+)^{\frac{p_2(m-1)}{m}-1}.$$

Our proof mimics that of Appendix 7 in [14].

Let $y \in \mathbb{R}^N$ and $R > 0$. For $x \in B_2(0)$ and any function h defined on $B_{2R}(y)$ we define $\hat{h}(x) = h(Rx+y)$, i.e.

$$\hat{h}\left(\frac{x-y}{R}\right) = h(x).$$

Let $u = \hat{u_0}$ and $v = \hat{v_0}$. Then we have (with $z = Rx + y$)

$$\begin{aligned}
&\int |\nabla u(x)|^{p_1-2}\nabla u(x)\cdot\nabla\hat{w}(x)dx \\
&\quad = R^{p_1}\int |\nabla u_0(Rx+y)|^{p_1-2}\nabla u_0(Rx+y)\cdot\nabla w(Rx+y)dx \\
&\quad = R^{p_1}\int |\nabla u_0(z)|^{p_1}\nabla u_0(z)\cdot\nabla w(z)R^{-N}dz \\
&\quad \le R^{p_1}\int \left(\mu_1 a_1(z)(u_0^+(z))^{p_1-1} + \frac{p_1}{m} b(z)(u_0^+(z))^{\frac{p_1}{m}-1}(v_0^+(z))^{\frac{p_1(m-1)}{m}}\right) w(z)R^{-N}dz \\
&\quad = R^{p_1}\int \left(\mu_1 \hat{a}_1(x)(\hat{u_0}^+(x))^{p_1-1} + \frac{p_1}{m}\hat{b}(x)(\hat{u_0}^+(x))^{\frac{p_1}{m}-1}(\hat{v_0}^+(x))^{\frac{p_1(m-1)}{m}}\right)\hat{w}(x)dx,
\end{aligned}$$

so that (u, v) are weak solutions to the variational inequalities

$$-\Delta_{p_1} u \leq \mu_1 R^{p_1} \hat{a}_1(x)|u|^{p_1-2}u^+ + \frac{p_1}{m}\hat{b}(x)R^{p_1}|u|^{\frac{p_1}{m}-2}u^+|v|^{\frac{p_2(m-1)}{m}} \tag{5.5}$$

and (similar to above)

$$-\Delta_{p_2} v \leq \mu_2 R^{p_2} \hat{a}_2(x)|v|^{p_2-2}v^+ + \frac{p_2(m-1)}{m}\hat{b}(x)R^{p_2}|u|^{\frac{p_1}{m}}|v|^{\frac{p_2(m-1)}{m}-2}v^+. \tag{5.6}$$

Hereafter we let K_0, K_1, K, K', K'' denote any constants depending only on p_1, p_2, N, μ_1, μ_2, m, $\|a_1\|_\infty$, $\|a_2\|_\infty$ and $\|b\|_\infty$.

Without loss of generality we can assume that $p_1 \geq p_2$. Let $c = \frac{N}{N-p_2} > 1$. Then for any ball $B \subset B_2(0)$ we have

$$\text{for any } w \in W_0^{1,p_1}(B),\ \|w\|_{L^{cp_1}(B)} \leq K_0\|w\|_{L^{p_1^*}(B)} \leq K_1\|\nabla w\|_{L^{p_1}(B)}$$

$$\text{for any } w \in W_0^{1,p_2}(B),\ \|w\|_{L^{cp_2}(B)} = \|w\|_{L^{p_2^*}(B)} \leq K_1\|\nabla w\|_{L^{p_2}(B)}.$$

We construct the following sequences:

$$p_{i,k} = p_i c^k \text{ for } i = 1, 2 \text{ and } k \geq 0,$$

$$m_k = p_1(c^k - 1),\ t_k = p_2(c^k - 1),$$

$$\rho_0 = 2,\ \rho_k = 2 - \tfrac{1}{\sigma}\textstyle\sum_{j=0}^{k-1} c^{-\frac{j}{p_1'}},\ \text{for any } k \geq 1,$$

where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ and $\sigma = \sum_{j=0}^{\infty} c^{-\frac{j}{p_1'}}$.

Denote $D_k = B_{\rho_k}(0)$. Consider $\eta \in C_0^\infty(\mathbb{R}^N)$ defined such that $0 \leq \eta \leq 1$, $\eta = 1$

on D_{k+1}, η is supported in D_k, and

$$|\nabla\eta(t)| \leq Kc^{\frac{k}{p_1'}}, \text{ for all } t \in D_k. \tag{5.7}$$

Let $|u|^{m_k}u\eta^{p_1}$ be a test function in (5.5), obtaining $I_1 + I_2 \leq I_3 + I_4$ where

$$I_1 = (1+m_k)\int_{D_k}\eta^{p_1}|u|^{m_k}|\nabla u|^{p_1},$$

$$I_2 = p_1\int_{D_k}\eta^{p_1-1}\nabla\eta\cdot\nabla u|\nabla u|^{p_1-2}|u|^{m_k}u,$$

$$I_3 = \mu_1 R^{p_1}\int_{D_k}\hat{a}_1(x)\eta^{p_1}|u^+|^{p_1+m_k},$$

$$I_4 = \tfrac{p_1}{m}R^{p_1}\int_{D_k}\hat{b}(x)|u^+|^{\frac{p_1}{m}+m_k}|v^+|^{\frac{p_2(m-1)}{m}}\eta^{p_1}.$$

Next define

$$E_k = \max\{\|u^{c^k}\|^{p_1}_{L^{p_1}(D_k)}, \|v^{c^k}\|^{p_2}_{L^{p_2}(D_k)}\}.$$

Then we have

$$|I_3| \leq \mu_1 R^{p_1}\|a_1\|_\infty\int_{D_k}\eta^{p_1}|u|^{p_1c^k} \leq R^{p_1}KE_k.$$

Similarly, applying Holder's inequality and noting that

$$\frac{p_1/m+m_k}{p_1c^k} + \frac{p_2(m-1)/m}{p_2c^k} = \frac{1}{c^k}\left(\frac{1}{m}+\frac{m-1}{m}\right) + \frac{p_1(c^k-1)}{p_1c^k} = 1,$$

we have

$$|I_4| \leq \frac{p_1}{m}R^{p_1}\|b\|_\infty\left(\int_{D_k}|u|^{p_1c^k}\right)^{\frac{p_1/m+m_k}{p_1c^k}}\left(\int_{D_k}|v|^{p_2c^k}\right)^{\frac{p_2(m-1)/m}{p_2c^k}} \leq R^{p_1}KE_k.$$

On the other hand, since $(1+m_k) = (p_1-1)(c^k-1)+c^k$, we have for any $s>0$,

$$\begin{aligned}
|I_2| &\le \int_{D_k} p_1\eta^{p_1-1}|\nabla u|^{p_1-1}|\nabla\eta||u|^{1+m_k} \\
&= p_1\int_{D_k} c^{\frac{k}{p_1'}}\eta^{p_1-1}|\nabla u|^{p_1-1}|u|^{(p_1-1)(c^k-1)}c^{-\frac{k}{p_1'}}|\nabla\eta||u|^{c^k} \\
&\le p_1\cdot\frac{1}{p_1'}\left(sc^{\frac{k}{p_1'}}\right)^{p_1'}\int\eta^{(p_1-1)p_1'}|\nabla u|^{(p_1-1)p_1'}|u|^{p_1'(p_1-1)(c^k-1)} \\
&\quad + p_1\cdot\frac{1}{p_1}\left(\frac{c^{-\frac{k}{p_1'}}}{s}\right)^{p_1}\int|\nabla\eta|^{p_1}|u|^{p_1c^k} \\
&= \frac{p_1 s^{p_1'}c^k}{p_1'}\int_{D_k}\eta^{p_1}|\nabla u|^{p_1}|u|^{m_k} + \frac{c^{-\frac{kp_1}{p_1'}}}{s^{p_1}}\int_{D_k}|\nabla\eta|^{p_1}|u|^{p_1c^k},
\end{aligned}$$

where we used Young's Inequality $AB\le\frac{1}{p_1'}(sA)^{p_1'}+\frac{1}{p_1}\left(\frac{B}{s}\right)^{p_1}$. Since

$$c^k = \frac{m_k}{p_1}+1\le 1+m_k,$$

applying (5.7) for $s=\left(\frac{p_1'}{2p_1}\right)^{1/p_1'}$ (i.e. $2p_1s^{p_1'}\le p_1'$), we obtain

$$|I_2|\le\frac{1}{2}|I_1|+KE_k.$$

Therefore $|I_1|\le|I_2|+|I_3|+|I_4|\le\frac{1}{2}|I_1|+KE_k+2R^{p_1}KE_k$, so that

$$|I_1|\le(1+R^{p_1})KE_k.$$

Now, by Sobolev Embedding, we have

$$\|\eta u^{c^k}\|^{p_1}_{L^{cp_1}(D_k)}\le K_1\|\nabla(\eta u^{c^k})\|^{p_1}_{L^{p_1}(D_k)}\le K(I_5+I_6)$$

where

$$I_5 = \int_{D_k} |\nabla \eta|^{p_1} |u|^{p_1 c^k} \leq K c^{k(p_1-1)} E_k,$$

and

$$\begin{aligned} I_6 &= c^{kp_1} \int_{D_k} \eta^{p_1} |\nabla u|^{p_1} |u|^{p_1(c^k-1)} \\ &= c^{k(p_1-1)} c^k \int_{D_k} \eta^{p_1} |\nabla u|^{p_1} |u|^{m_k} \\ &\leq c^{k(p_1-1)} (1+m_k) \int_{D_k} \eta^{p_1} |\nabla u|^{p_1} |u|^{m_k} \\ &= c^{k(p_1-1)} I_1. \end{aligned}$$

Combining the above results, we get

$$\|\eta u^{c^k}\|^{p_1}_{L^{cp_1}(D_k)} \leq (1+R^{p_1}) K c^{k(p_1-1)} E_k. \tag{5.8}$$

Similarly, letting $|v|^{t_k} v \eta^{p_2}$ be a test function in (5.6), we obtain (because $p_2 \leq p_1$)

$$\|\eta v^{c^k}\|^{p_2}_{L^{cp_2}(D_k)} \leq (1+R^{p_1}) K c^{k(p_1-1)} E_k. \tag{5.9}$$

Setting $\Theta_k = E_k^{\frac{1}{p_{1,k}}}$, then we obtain by (5.8) and (5.9) and the fact that $\eta \equiv 1$ on

D_{k+1},

$$
\begin{aligned}
\Theta_{k+1} &= \left(\max\{\|u^{c^{k+1}}\|^{p_1}_{L^{p_1}(D_{k+1})}, \|v^{c^{k+1}}\|^{p_2}_{L^{p_2}(D_{k+1})}\}\right)^{\frac{1}{p_{1,k+1}}} \\
&= \left(\max\{\|u^{c^k}\|^{cp_1}_{L^{cp_1}(D_{k+1})}, \|v^{c^k}\|^{cp_2}_{L^{cp_2}(D_{k+1})}\}\right)^{\frac{1}{p_{1,k}}\cdot\frac{1}{c}} \\
&= \left(\max\{\|u^{c^k}\|^{p_1}_{L^{cp_1}(D_{k+1})}, \|v^{c^k}\|^{p_2}_{L^{cp_2}(D_{k+1})}\}\right)^{\frac{1}{p_{1,k}}} \\
&\leq \left(\max\{\|\eta u^{c^k}\|^{p_1}_{L^{cp_1}(D_k)}, \|\eta v^{c^k}\|^{p_2}_{L^{cp_2}(D_k)}\}\right)^{\frac{1}{p_{1,k}}} \\
&\leq [(1+R^{p_1})K]^{\frac{1}{p_{1,k}}} c^{\frac{k(p_1-1)}{p_{1,k}}} \Theta_k
\end{aligned}
$$

for all $k \geq 0$. Therefore,

$$
\|u\|_{L^{p_{1,k}}(D_k)} = \left(\|u^{c^k}\|^{p_1}_{L^{p_1}(D_k)}\right)^{\frac{1}{p_{1,k}}} \leq \Theta_k \leq [(1+R^{p_1})K]^{\sum_{j=0}^{\infty}\frac{1}{p_{1,j}}} c^{\sum_{j=0}^{\infty}\frac{j(p_1-1)}{p_{1,j}}} \Theta_0,
$$

where

$$
\sum_{j=0}^{\infty} \frac{1}{p_{1,j}} = \sum_{j=0}^{\infty} \frac{1}{p_1 c^j} = \frac{1}{p_1} \cdot \frac{c}{c-1} = \frac{N}{p_1 p_2},
$$

and (because $c > 1$)

$$
\sum_{j=0}^{\infty} \frac{j(p_1-1)}{p_{1,j}} = \frac{p_1-1}{p_1} \sum_{j=0}^{\infty} \frac{j}{c^j} = K < \infty.
$$

We therefore obtain

$$
\begin{aligned}
\|u\|_{L^\infty(B_1)} &\leq \limsup_{k\to+\infty} \|u\|_{L^{p_{1,k}}(D_k)} \\
&\leq K'(1+R^{p_1})^{\frac{N}{p_1 p_2}} \Theta_0 \\
&= K'(1+R^{p_1})^{\frac{N}{p_1 p_2}} \max\left\{\|u\|_{L^{p_1}(B_2)}, \|v\|^{\frac{p_2}{p_1}}_{L^{p_2}(B_2)}\right\}.
\end{aligned}
$$

Similarly, define $\Psi_k = E_k^{\frac{1}{p_{2,k}}}$, and we get from (5.8) and (5.9),

$$\Psi_{k+1} \leq [(1+R^{p_1})K]^{\frac{1}{p_{2,k}}} c^{\frac{k(p_1-1)}{p_{2,k}}} \Psi_k, \text{ for all } k \geq 0,$$

which implies

$$\|v\|_{L^\infty(B_1)} \leq K'(1+R^{p_1})^{\frac{N}{p_2^2}} \max\{\|u\|_{L^{p_1}(B_2)}^{\frac{p_1}{p_2}}, \|v\|_{L^{p_2}(B_2)}\}.$$

By the embeddings

$$L^{p_i^*}(B_2) \subset L^{p_i}(B_2), \ i = 1, 2,$$

we obtain that

$$\|u\|_{L^\infty(B_1)} \leq K''(1+R^{p_1})^{\frac{N}{p_1 p_2}} \max\{\|u\|_{L^{p_1^*}(B_2)}, \|v\|_{L^{p_2^*}(B_2)}^{\frac{p_2}{p_1}}\},$$

and

$$\|v\|_{L^\infty(B_1)} \leq K''(1+R^{p_1})^{\frac{N}{p_2^2}} \max\{\|u\|_{L^{p_1^*}(B_2)}^{\frac{p_1}{p_2}}, \|v\|_{L^{p_2^*}(B_2)}\}.$$

Finally, changing variables back to (u_0, v_0), we get

$$\|u_0\|_{L^\infty(B_R(y))} \leq K(1+R^{p_1})^{\frac{N}{p_1 p_2}} \max\{R^{\frac{p_1-N}{p_1}} \|u_0\|_{L^{p_1^*}(B_{2R}(y))}, R^{\frac{p_2-N}{p_1}} \|v_0\|_{L^{p_2^*}(B_{2R}(y))}^{\frac{p_2}{p_1}}\},$$

and

$$\|v_0\|_{L^\infty(B_R(y))} \leq K(1+R^{p_1})^{\frac{N}{p_2^2}} \max\{R^{\frac{p_1-N}{p_2}} \|u_0\|_{L^{p_1^*}(B_{2R}(y))}^{\frac{p_1}{p_2}}, R^{\frac{p_2-N}{p_2}} \|v_0\|_{L^{p_2^*}(B_{2R}(y))}\}.$$

Therefore we have $u_0, v_0 \in L^\infty$ and $\lim_{|x|\to\infty} u_0(x) = 0$, $\lim_{|x|\to\infty} v_0(x) = 0$.

Next, since $u_0 \in L^\infty \cap L^{p_1}_{a_1(x)}$ and $\gamma_1 > p_1$, we have $u_0 \in L^{\gamma_1}_{a_1(x)}$. Similarly, $v_0 \in L^{\gamma_2}_{a_2(x)}$.

Finally, we have $u_0, v_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ by Theorem 2.9. □

Lemma 5.9. *Set*

$$f_1(x) = \mu_1 a_1(x)(u_0^+(x))^{p_1-1} - a_1(x)g_1(u_0(x)) + F_u(x, u_0, v_0),$$

$$f_2(x) = \mu_2 a_2(x)(v_0^+(x))^{p_2-1} - a_2(x)g_2(v_0(x)) + F_v(x, u_0, v_0).$$

Then there exists $R_0 > 0$ such that $f_1(x), f_2(x) \geq 0$ for all $x \in \mathbb{R}^N \backslash B_{R_0}(0)$, and such that

$$\Omega = \{x \in \mathbb{R}^N \mid u_0(x) > S_1\} \cup \{x \in \mathbb{R}^N \mid v_0(x) > S_2\} \subset \bar{\Omega} \subset B_{R_0}(0)$$

where $S_i = \max\{s \mid g_i(s) = \mu_i s^{p_i-1}\}$, $i = 1.2$.

Proof: Fix $x \in \mathbb{R}^N$ and consider

$$\frac{f_1(x)}{(u_0^+(x))^{p_1-1}} = a_1(x)\left(\mu_1 - \frac{g_1(u_0(x))}{(u_0^+(x))^{p_1-1}}\right) + \frac{F_u(x, u_0(x), v_0(x))}{(u_0^+(x))^{p_1-1}}.$$

Since $s^{-p_1+1}F_u(x, s, v_0(x)) = \frac{p_1}{m} b(x) s^{p_1\left(\frac{1}{m}-1\right)} (v_0^+(x))^{\frac{p_2(m-1)}{m}}$ is a decreasing function of s and $-\frac{g_1(s)}{s^{p_1-1}}$ is non-increasing by our conditions on g_1 and F, we have that there exists $T_1(x) > 0$ such that $f_1(x) < 0$ if and only if $u_0(x) > T_1(x)$. Furthermore, since $s^{-p_1+1}F_u(x, s, v_0(x)) \geq 0$, if we let $S_1 = \max\{s \mid g_1(s) = \mu_1 s^{p_1-1}\}$ then we have $T_1(x) \geq S_1 > 0$ for all $x \in \mathbb{R}^N$. Since $\lim_{|x|\to\infty} u_0(x) = 0$ the result now follows for $f_1(x)$ by a sufficiently large choice of R_0. The proof for $f_2(x)$ is the same. □

Lemma 5.10. *$u_0(x) > 0$ and $v_0(x) > 0$ for all $x \in \mathbb{R}^N$.*

Proof: We know from Lemma 5.6 that u_0 is not identically zero, so let $\Omega = \{x \in \mathbb{R}^N \mid u_0(x) = 0\} \neq \mathbb{R}^N$ and assume that Ω is nonempty. Note that by Lemma 5.7 we have $u_0 \geq 0$, and since u_0 is continuous by Lemma 5.8 we have that Ω is closed. Let $x_0 \in \partial\Omega$, so that $u_0(x_0) = 0$. Then by continuity of u_0, we can find a $\delta > 0$ such that

$$\max_{x \in B_\delta(x_0)} u_0(x) \leq \frac{S_1}{2} < S_1,$$

where S_1 is as in the proof of Lemma 5.9. Therefore u_0 is a weak solution to the system

$$\begin{aligned} -\Delta_p u &= f_1(x) \geq 0 \text{ in } B_\delta(x_0), \\ u &\geq 0 \text{ on } \partial B_\delta(x_0). \end{aligned}$$

Then an application of the maximum principle (Theorem 2.10 part (iii)) implies that $u_0 \equiv 0$ in $B_\delta(x_0)$, contradicting the fact that $x_0 \in \partial\Omega$. Therefore $u_0 > 0$ on $\mathbb{R}^N$. The proof for v_0 is the same. $\square$

We now provide estimates for the behavior of u_0 and v_0 at infinity.

Lemma 5.11. *(i) Given $\epsilon > 0$, set*

$$V_\epsilon = \{x \mid u_0(x) > \epsilon a_1(x)^{\frac{N-p_1}{p_1^2}},\ f_1(x) > \epsilon a_1(x)(u_0^+(x))^{p_1-1}\}.$$

Then there exists positive constants ϵ_0, L_0 and $R_1 > R_0$ (where R_0 is as in Lemma 5.9) *such that, for all* $0 < \epsilon \leq \epsilon_0$ *we have*

$$\|a_1 \chi_{\Omega_1}\|_{N/p_1} \geq L_0,$$

where $\Omega_1 = V_\epsilon \cap B_{R_1}(0)$.

(ii) Given $\epsilon > 0$, *set*

$$W_\epsilon = \{x \mid v_0(x) > \epsilon a_2(x)^{\frac{N-p_2}{p_2^2}},\ f_2(x) > \epsilon a_2(x)(v_0^+(x))^{p_2-1}\}.$$

Then there exists positive constants ϵ_0, L_0 *and* $R_1 > R_0$ *such that, for all* $0 < \epsilon \leq \epsilon_0$ *we have*

$$\|a_2 \chi_{\Omega_2}\|_{N/p_2} \geq L_0,$$

where $\Omega_2 = W_\epsilon \cap B_{R_1}(0)$.

In both parts our constants ϵ_0 *and* L_0 *may depend on* m, p_1, p_2, μ_1, μ_2, $\|u_0\|_{1,p_1}$, $\|v_0\|_{1,p_2}$, $\|a_1\|_{N/p_1}$ *and* $\|a_2\|_{N/p_2}$.

Proof: Again we apply similar methods to Costa, Drabek and Tehrani [7].

Let $\epsilon > 0$. To simplify notation, we write $V = V_\epsilon$. Then, letting $w = u_0$ in (5.4) and applying Lemma 5.2, we have

$$
\begin{aligned}
\|u_0\|_{1,p_1}^{p_1} &= \int \mu_1 a_1(x)(u_0^+)^{p_1-1}u_0 - \int a_1(x)g_1(u_0)u_0 + \int F_u(x,u_0,v_0)u_0 \\
&= \int_V \mu_1 a_1(x)(u_0^+)^{p_1-1}u_0 - \int_V a_1(x)g_1(u_0)u_0 + \int_V F_u(x,u_0,v_0)u_0 \\
&\quad + \int_{\mathbb{R}^N\setminus V} f_1(x)u_0 \\
&\leq \int_V \mu_1 a_1(x)(u_0^+)^{p_1-1}u_0 + \int_{\mathbb{R}^N\setminus V} f_1(x)u_0 \\
&\quad + C_4'\|a_1\chi_V\|_{N/p_1}^{1/m}\|u_0\|_{1,p_1}^{p_1/m}\|a_2\chi_V\|_{N/p_2}^{p_2(m-1)/m}\|v_0\|_{1,p_2}^{p_2(m-1)/m} \\
&\leq C\mu_1\|a_1\chi_V\|_{N/p_1}\|u_0\|_{1,p_1}^{p_1} + \int_{\mathbb{R}^N\setminus V} f_1(x)u_0 \\
&\quad + C_4'\|a_1\chi_V\|_{N/p_1}^{1/m}\|u_0\|_{1,p_1}^{p_1/m}\|a_2\chi_V\|_{N/p_2}^{p_2(m-1)/m}\|v_0\|_{1,p_2}^{p_2(m-1)/m}
\end{aligned}
$$

Consider the decomposition $\mathbb{R}^N\setminus V = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, where

$$A_1 = \{x \mid u_0(x) \leq \epsilon a_1(x)^{\frac{N-p_1}{p_1^2}}\},$$

$$A_2 = \{x \mid f_1(x) \leq \epsilon a_1(x)(u_0^+(x))^{p_1-1},\ u_0(x) > \epsilon a_1(x)^{\frac{N-p_1}{p_1^2}}\}.$$

Then we obtain, again applying Lemma 5.2,

$$\begin{aligned}
\int_{A_1} f_1(x)u_0 &= \mu_1 \int_{A_1} a_1(x)(u_0^+)^{p_1} - \int_{A_1} a_1(x)g_1(u_0)u_0 + \int_{A_1} F_u(x,u_0,v_0)u_0 \\
&\le \mu_1 \epsilon^{p_1} \int_{A_1} a_1(x)\left(a_1(x)^{\frac{N-p_1}{p_1^2}}\right)^{p_1} \\
&\quad + C_4'' \epsilon^{p_1/m} \left(\int_{A_1} a_1(x)\left(a_1(x)^{\frac{N-p_1}{p_1^2}}\right)^{p_1}\right)^{1/m} \|a_2\|_{N/p_2}^{(m-1)/m} \|v_0\|_{1,p_2}^{p_2(m-1)/m} \\
&= \mu_1 \epsilon^{p_1} \int_{A_1} a_1(x)^{N/p_1} \\
&\quad + C_4'' \epsilon^{p_1/m} \left(\int_{A_1} a_1(x)^{N/p_1}\right)^{1/m} \|a_2\|_{N/p_2}^{(m-1)/m} \|v_0\|_{1,p_2}^{p_2(m-1)/m} \\
&= \mu_1 \epsilon^{p_1} \|a_1\|_{N/p_1}^{N/p_1} + C_4'' \epsilon^{p_1/m} \|a_1\|_{N/p_1}^{N/(mp_1)} \|a_2\|_{N/p_2}^{(m-1)/m} \|v_0\|_{1,p_2}^{p_2(m-1)/m}
\end{aligned}$$

Furthermore,

$$\int_{A_2} f_1(x)u_0 \le \epsilon \int_{A_2} a_1(x)|u_0|^{p_1} \le C\epsilon \|a_1\|_{N/p_1} \|u_0\|_{1,p_1}^{p_1}.$$

Therefore, combining the above estimates, we obtain

$$\begin{aligned}
\|u_0\|_{1,p_1}^{p_1} &\le C\mu_1 \|a_1\chi_V\|_{N/p_1} \|u_0\|_{1,p_1}^{p_1} \\
&\quad + C_4' \|a_1\chi_V\|_{N/p_1}^{1/m} \|u_0\|_{1,p_1}^{p_1/m} \|a_2\chi_V\|_{N/p_2}^{p_2(m-1)/m} \|v_0\|_{1,p_2}^{p_2(m-1)/m} \\
&\quad + \mu_1 \epsilon^{p_1} \|a_1\|_{N/p_1}^{N/p_1} + C_4'' \epsilon^{p_1/m} \|a_1\|_{N/p_1}^{N/(mp_1)} \|a_2\|_{N/p_2}^{(m-1)/m} \|v_0\|_{1,p_2}^{p_2(m-1)/m} \\
&\quad + C\epsilon \|a_1\|_{N/p_1} \|u_0\|_{1,p_1}^{p_1}.
\end{aligned}$$

Now, we can find $\epsilon_0 > 0$ (depending on μ_1, $\|u_0\|_{1,p_1}$, $\|v_0\|_{1,p_2}$, $\|a_1\|_{N/p_1}$ and $\|a_2\|_{N/p_2}$)

such that

$$\frac{1}{2}\|u_0\|_{1,p_1}^{p_1} \geq \mu_1\epsilon^{p_1}\|a_1\|_{N/p_1}^{N/p_1} + C_4''\epsilon^{p_1/m}\|a_1\|_{N/p_1}^{N/(mp_1)}\|a_2\|_{N/p_2}^{(m-1)/m}\|v_0\|_{1,p_2}^{p_2(m-1)/m}$$
$$+ C\epsilon\|a_1\|_{N/p_1}\|u_0\|_{1,p_1}^{p_1}$$

for all $\epsilon \leq \epsilon_0$. Therefore

$$\frac{1}{2}\|u_0\|_{1,p_1}^{p_1} \leq C\mu_1\|a_1\chi_V\|_{N/p_1}\|u_0\|_{1,p_1}^{p_1}$$
$$+ C_4'\|a_1\chi_V\|_{N/p_1}^{1/m}\|u_0\|_{1,p_1}^{p_1/m}\|a_2\chi_V\|_{N/p_2}^{p_2(m-1)/m}\|v_0\|_{1,p_2}^{p_2(m-1)/m},$$

so that there exists constants $C_5, C_6 > 0$ such that $X^m + C_5X - C_6 \geq 0$, where $X = \|a_1\chi_V\|_{N/p_1}^{1/m}$. Hence, there exists a constant L_0 depending on the constants listed in the statement of the lemma such that, for $0 < \epsilon \leq \epsilon_0$,

$$\|a_1\chi_V\|_{N/p_1} \geq 2L_0.$$

Since $a_1 \in L^{N/p_1}(\mathbb{R}^N)$ there exists $R_1 > R_0$ such that

$$\|a_1\chi_{\mathbb{R}^N\setminus B_{R_1}(0)}\|_{N/p_1} < L_0.$$

Therefore, considering that $V_{\epsilon_0} \subset V_\epsilon$ for $0 < \epsilon < \epsilon_0$, it follows that

$$\|a_1\chi_{V_\epsilon\cap B_{R_1}(0)}\|_{N/p_1} \geq L_0,$$

completing the proof of part (i) the lemma. The proof of part (ii) is identical. □

We are finally ready to study the behaviors of u_0 and v_0 at infinity.

Theorem 5.12. *There exists $C > 0$ such that a solution (u_0, v_0) of the minimization problem (5.2) is a positive solution to problem (5.1) and satisfies*

$$u_0(x) \geq C|x|^{-\frac{N-p_1}{p_1-1}} \text{ for } |x| \text{ large.}$$
$$v_0(x) \geq C|x|^{-\frac{N-p_2}{p_2-1}} \text{ for } |x| \text{ large.}$$

Proof: We recall that (u_0, v_0) is a solution to (5.3) in $\mathbb{R}^N$. Therefore, using the notation of Lemma 5.11 and letting $V = V_{\epsilon_0}$ and $W = W_{\epsilon_0}$, we have by the definition of V and W that

$$\begin{aligned} -\Delta_{p_1} u_0 &= \mu_1 a_1(x)(u_0^+)^{p_1-1} - a_1(x)g_1(u_0) + F_u(x, u_0, v_0) \\ &= f_1(x) \geq \epsilon_0^{p_1}(a_1(x))^{\frac{Np_1-N+p_1}{p_1^2}} \quad \text{on } V \cap B_{R_1(0)} \end{aligned}$$

and

$$\begin{aligned} -\Delta_{p_2} v_0 &= \mu_2 a_2(x)(v_0^+)^{p_2-1} - a_2(x)g_2(v_0) + F_v(x, u_0, v_0) \\ &= f_2(x) \geq \epsilon_0^{p_2}(a_2(x))^{\frac{Np_2-N+p_2}{p_2^2}} \quad \text{on } W \cap B_{R_1(0)}. \end{aligned}$$

For $R \geq 24R_1 > 24R_0$, consider $z = z_R$, the solution to the Dirichlet problem

$$\begin{cases} -\Delta_{p_1} z = \epsilon_0^{p_1}(a_1(x))^{\frac{Np_1-N+p_1}{p_1^2}} \chi_{V \cap B_{R_1}(0)} & \text{in } B_R(0), \\ z = 0 & \text{on } \partial B_R(0). \end{cases}$$

The solution z exists by Theorem 2.13, is continuous (and hence bounded) by Theorem 2.9, and is p-superharmonic by part (iv) of Theorem 2.10. Let $\Omega_0 = \{x \in \mathbb{R}^N \mid u_0(x) > S_1\}$, where $S_1 = \max\{s \mid g_1(s) = \mu_1 s^{p_1-1}\}$ as in Lemma 5.9. Then $\Omega_0 \subset \overline{\Omega_0} \subset B_{R_0}(0) \subset B_R(0)$ is bounded and $f_1(x) \geq 0$ for all $x \in \mathbb{R}^N \backslash \Omega_0$. Let $\Omega_1 = B_R(0) \backslash \Omega_0$. Then by Theorem 5.8 we have that Ω_1 is a bounded domain of class $C^{1+\alpha}$, $0 < \alpha < 1$, and that $\partial\Omega_1 = \partial B_R(0) \cup \Gamma$ with $u_0 \equiv S_1$ on Γ.

Now, let $K(\rho)$ be the cube centered at 0 with sides of length ρ. Then $\Gamma \subset K(R_0) \subset K(2R_0) \subset B_R(0)$, so we can apply Corollary 2.8 (where in the notation of that result we use $\gamma = p_1^*$) and Sobolev Embedding to get

$$\begin{aligned} \max_{x\in\Gamma} z(x) &\leq \max_{x\in K(R_0)} z(x) \\ &\leq C R_0^{-\frac{N}{p_1^*}} \|z\|_{L^{p_1^*}(K(2R_0))} \\ &\leq C' R_0^{-\frac{N}{p_1^*}} \|\nabla z\|_{L^{p_1}(K(2R_0))} \\ &\leq C' R_0^{-\frac{N}{p_1^*}} \|\nabla z\|_{L^{p_1}(B_R(0))}. \end{aligned}$$

We can then apply Corollary 2.13 to get

$$\max_{x\in\Gamma} z(x) \leq C'' R_0^{-\frac{N}{p_1^*}} \left\| \epsilon_0^{p_1} (a_1(x))^{\frac{Np_1 - N + p_1}{p_1^2}} \chi_{V\cap B_{R_1}(0)} \right\|_{p_1^{*\prime}}^{1/(p_1-1)}.$$

Therefore there exists a constant $C_1 = C_1(\epsilon_0, a_1, p_1, N, R_0, R_1)$ independent of R such that $C_1 = \max_\Gamma z$. Let $\epsilon_1 = \min(1, \frac{S_1}{C_1})$, so that $S_1 \geq \max_\Gamma \epsilon_1 z$. Note that S_1 depends only on g_1, μ_1 and p_1. We have that $z_1 = \epsilon_1 z$ is the solution to the Dirichlet problem

$$\begin{cases} -\Delta_{p_1} w = \epsilon_1^{p_1-1} \epsilon_0^{p_1} (a_1(x))^{\frac{Np_1-N+p_1}{p_1^2}} \chi_{V \cap B_{R_1}(0)} & \text{in } B_R(0), \\ w = 0 & \text{on } \partial B_R(0). \end{cases}$$

Since $f_1(x) \geq \epsilon_0^{p_1} (a_1(x))^{\frac{Np_1-N+p_1}{p_1^2}}$ in $V \cap B_{R_1}(0)$, $\epsilon_1 \leq 1$ and $f_1(x) \geq 0$ in Ω_1, we have

$$-\Delta_{p_1} u_0 \geq -\Delta_{p_1} z_1 \text{ in } \Omega_1.$$

Furthermore, since $u_0 \geq 0$ in $\mathbb{R}^N$ by Lemma 5.7 and $u_0 \geq S_1 \geq z_1$ on Γ, we have that

$$u_0 \geq z_1 \text{ on } \partial\Omega_1.$$

Therefore, by the Weak Comparison Principle of Theorem 2.10, we conclude that

$$u_0 \geq z_1 \text{ in } \Omega_1.$$

The remainder of the proof for u_0 then follows exactly as in the scalar case, Theorem 4.12. The proof for v_0 is the same. □

Remark 5.13. We could generalize our condition that $F(x, s, t) = b(x) s^{\frac{p_1}{m}} t^{\frac{p_2(m-1)}{m}}$, instead requiring:

(1) $|F_s(x, s, t)| \leq C_1 (a_1(x))^{\frac{1}{m}} (a_2(x))^{\frac{m-1}{m}} s^{\frac{p_1}{m}-1} t^{\frac{p_2(m-1)}{m}}$,

(2) $|F_t(x, s, t)| \leq C_2 (a_1(x))^{\frac{1}{m}} (a_2(x))^{\frac{m-1}{m}} s^{\frac{p_1}{m}} t^{\frac{p_2(m-1)}{m}-1}$,

(3) $s^{-p_1+1} F_s(x, s, t)$ is a decreasing function of s,

(4) $t^{-p_2+1}F_t(x, s, t)$ is a decreasing function of t,

(5) $F_s(x, s, t)$ and $F_t(x, s, t)$ are continuous.

For the sake of simplicity, we focused in this chapter on the main example of such a function, that of expressing $F(x, s, t)$ as powers of s and t.

CHAPTER 6

SYSTEM PROBLEM WITH HARVESTING

In this chapter we let $\lambda_1(p) = \lambda_1(a, p)$ be the first eigenvalues of Theorem 3.1.

Main Theorem 6.1. *Consider*

$$\begin{cases} -\Delta_{p_1} u = a_1(x)(\mu_1 |u|^{p_1-2}u - g_1(u)) + F_u(x,u,v) - \nu_1 h_1(x) \\ -\Delta_{p_2} v = a_2(x)(\mu_2 |v|^{p_2-2}v - g_2(v)) + F_v(x,u,v) - \nu_2 h_2(x) \end{cases} \tag{6.1}$$

where for $i = 1, 2$,

(A_0) $g_i : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous,*

(A_1) $\lim_{s\to 0^+} \frac{g_i(s)}{s^{p_i-1}} = 0$,

(A_2) $0 < \liminf_{s\to\infty} \frac{g_i(s)}{s^{\gamma_i-1}} \le \limsup_{s\to\infty} \frac{g_i(s)}{s^{\gamma_i-1}} < \infty$ *with* $\gamma_i > p_i$,

(A_3) $\frac{g_i(s)}{s^{p_i-1}}$ *is nondecreasing,*

(B_0) $F(x,s,t) : \mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is given by* $F(x,s,t) = b(x) s^{\frac{p_1}{m}} t^{\frac{p_2(m-1)}{m}}$ *for some* $1 < m < \infty$,

(B_1) $0 < b(x) = \Gamma(x)(a_1(x))^{\frac{1}{m}}(a_2(x))^{\frac{m-1}{m}}$ *for all* $x \in \mathbb{R}^N$ *and some* $\Gamma \in L^\infty(\mathbb{R}^N)$,

(C_1) $0 < h_i(x) \in L^{q_i}_{\sigma_i}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *where* $\sigma_i(x) = (1+|x|)^{q_i}$ *and* $\frac{1}{p_i} + \frac{1}{q_i} = 1$,

(C_2) $0 < a_i(x) \in L^{N/p_i}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

(C_3) $1 < p_i < N$, $m \le p_1$, $\frac{m}{m-1} \le p_2$, $0 < \nu_i \le \delta_1$ *and* $\mu_i > \lambda_1(p_i)$.

Then for δ_1 *sufficiently small, (6.1) has a nontrivial solution* (u_1, v_1) *in* $\mathbb{R}^N$. *In* *addition, suppose we also have*

(C_4) *$h_i \in L^1(\mathbb{R}^N)$ and for all $x \in \mathbb{R}^N$ we have*

$$|x|^{N/Q}\|h\|_{L^P(\mathbb{R}^N\setminus B_{|x|}(0))} \leq C$$

for some constant C and some $P > \frac{N}{2}$, with $\frac{1}{P}+\frac{1}{Q}=1$.

Then, for $p_1 = p_2 = 2 = m$ there is a solution with $u_1, v_1 > 0$ in $\mathbb{R}^N$ and satisfying

$$u_1(x) \geq \frac{C}{|x|^{N-2}} \text{ for } |x| \text{ large,}$$

and

$$v_1(x) \geq \frac{C}{|x|^{N-2}} \text{ for } |x| \text{ large.}$$

Existence of Solution

Since we are varying p in this chapter, we define

$$\|u\|_{1,p} = \left(\int |\nabla u|^p\right)^{1/p}$$

Lemma 6.2. *(i) For any $\epsilon > 0$ there exists constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ such that*

$$-\epsilon(s^+)^{p_i-1} + C_1(s^+)^{\gamma_i-1} \leq g_i(s) \leq \epsilon(s^+)^{p_i-1} + C_2(s^+)^{\gamma_i-1},$$

$$-\epsilon(s^+)^{p_i} + C_1(s^+)^{\gamma_i} \leq G_i(s) \leq \epsilon(s^+)^{p_i} + C_2(s^+)^{\gamma_i},$$

where $G_i(s) = \int_0^s g_i(t)dt$.

(ii) There exists $C_0 > 0$ *such that*

$$\int |F(x,u,v)|, \int |F_u(x,u,v)u|, \int |F_v(x,u,v)v|$$
$$\leq C_0 \left(\int a_1(x)(u^+)^{p_1}\right)^{\frac{1}{m}} \left(\int a_2(x)(v^+)^{p_2}\right)^{\frac{m-1}{m}}.$$

Proof: This follows from our conditions on g_i and F (and Holder's Inequality). □

We consider the functionals

$$J : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R} \cup \{\infty\},\ J(u,v) = \int a_1(x)G_1(u) + \int a_2(x)G_2(v)$$
$$K : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R}^+,\ K(u,v) = \int F(x,u,v)$$

and $I = I_{\nu_1,\nu_2} : D^{1,p_1} \times D^{1,p_2} \to \mathbb{R} \cup \{\infty\}$ where $I(u,v) = \infty$ if $J(u,v) = \infty$ and

$$\begin{aligned} I(u,v) &= \frac{1}{p_1}\int |\nabla u|^{p_1} - \frac{\mu_1}{p_1}\int a_1(x)(u^+)^{p_1} + \nu_1 \int h_1(x)u \\ &+ \frac{1}{p_2}\int |\nabla v|^{p_2} - \frac{\mu_2}{p_2}\int a_2(x)(v^+)^{p_2} + \nu_2 \int h_2(x)v + J(u,v) - K(u,v) \end{aligned}$$

if $J(u,v) < \infty$. Our goal is to find a solution to (6.1) by minimizing I.

We note that $h_i \in L^{q_i}_{\sigma_i}$ $(\frac{1}{p_i} + \frac{1}{q_i} = 1)$ implies by Holder's and Hardy's Inequalities (Lemma 2.3) that

$$\begin{aligned} \left|\int h_1(x)u(x)\right| &= \left|\int h_1(x)(1+|x|)\frac{u(x)}{1+|x|}\right| \\ &\leq \left(\int [h_1(x)(1+|x|)]^{q_1}\right)^{1/q_1} \left(\int \frac{u^{p_1}(x)}{(1+|x|)^{p_1}}\right)^{1/p_1} \\ &\leq C_{N,p_1}\|h_1\|_{q_1,\sigma_1}\|u\|_{1,p_1}, \end{aligned}$$

and similarly that

$$\left|\int h_2(x)v(x)\right| \leq C_{N,p_2}\|h_2\|_{q_2,\sigma_2}\|v\|_{1,p_2}.$$

Lemma 6.3. *(a) There exists $\delta > 0$ such that I is uniformly coercive for $0 < \nu_1, \nu_2 \leq \delta$, i.e. for all $M > 0$ there exists $R_0 > 0$ such that for $0 < \nu_1, \nu_2 \leq \delta$,*

$$I(u,v) \geq M \quad \forall \ \|u\|_{1,p_1} + \|v\|_{1,p_2} \geq R_0.$$

(b) I is a weakly lower semi-continuous functional.

Proof: (a) Assume not. Then there exists

$$\{(u_n, v_n)\} \subset D^{1,p_1} \times D^{1,p_2} \text{ and } \{\nu_{1,n}, \nu_{2,n}\} \subset (0, \delta]$$

such that $\{I(u_n, v_n)\}$ is bounded above and $\|u_n\|_{1,p_1} + \|v_n\|_{1,p_2} \to \infty$. Let $d_n = \left(\int a_1(x)(u_n^+)^{p_1}\right)^{1/p_1}$ and $e_n = \left(\int a_2(x)(v_n^+)^{p_2}\right)^{1/p_2}$. Now, without loss of generality and passing to a subsequence if necessary, we can assume that $\|u_n\|_{1,p_1} \to \infty$. First we show that $\|u_n\|_{1,p_1} \to \infty$ implies that $d_n \to \infty$ or $e_n \to \infty$. We have, applying Lemma 6.2, that

$$\begin{aligned} I(u_n, v_n) \geq\ & \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1} + \frac{1}{p_2}\|v_n\|_{1,p_2}^{p_2} - \frac{\mu_2}{p_2}e_n^{p_2} - \int |F(x, u_n, v_n)| \\ & + \nu_{1,n}\int h_1(x)u_n + \nu_{2,n}\int h_2(x)v_n \\ \geq\ & \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1} + \frac{1}{p_2}\|v_n\|_{1,p_2}^{p_2} - \frac{\mu_2}{p_2}e_n^{p_2} \\ & - C_0 d_n^{\frac{p_1}{m}} e_n^{\frac{p_2(m-1)}{m}} - C\delta(\|h_1\|_{q_1,\sigma_1}\|u_n\|_{1,p_1} + \|h_2\|_{q_2,\sigma_2}\|v_n\|_{1,p_2}). \end{aligned}$$

Now, $\|u_n\|_{1,p_1} \to \infty$ and $p_1, p_2 > 1$, so if $\{d_n\}$ and $\{e_n\}$ are bounded then $I(u_n, v_n) \to \infty$, contradicting the assumption that $\{I(u_n, v_n)\}$ is bounded above. Therefore, passing to a subsequence if necessary, $d_n \to \infty$ or $e_n \to \infty$, as desired. If we have that $\|v_n\|_{1,p_2}$ is bounded, then because $e_n^{p_2} \le C\|a_2\|_{N/p_2}\|v_n\|_{1,p_2}^{p_2}$ we also have that $\{e_n\}$ is bounded. Therefore either $d_n \to \infty$ and $\|u_n\|_{1,p_1} \to \infty$, or we have $e_n \to \infty$ and (passing to a subsequence if necessary) $\|v_n\|_{1,p_2} \to \infty$, or both cases occur simultaneously. Hence, without loss of generality, we can assume that $\|u_n\|_{1,p_1} \to \infty$, $d_n \to \infty$ and $\limsup_{n\to\infty} \frac{e_n^{p_2}}{d_n^{p_1}} < \infty$.

Next, set $\bar{u}_n = \frac{u_n}{d_n}$ and $\bar{v}_n = \frac{v_n}{e_n}$. Then

$$\int a_1(x)(\bar{u}_n^+)^{p_1} = \int a_2(x)(\bar{v}_n^+)^{p_2} = 1$$

and

$$\begin{aligned}
I(u_n, v_n) &= \frac{1}{p_1}\int |\nabla u_n|^{p_1} - \frac{\mu_1}{p_1}\int a_1(x)(u_n^+)^{p_1} + \int a_1(x)G_1(u_n) \\
&\quad + \frac{1}{p_2}\int |\nabla v_n|^{p_2} - \frac{\mu_2}{p_2}\int a_2(x)(v_n^+)^{p_2} + \int a_2(x)G_2(v_n) \\
&\quad - \int F(x, u_n, v_n) + \nu_{1,n}\int h_1(x)u_n + \nu_{2,n}\int h_2(x)v_n \\
&= \frac{d_n^{p_1}}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1}d_n^{p_1}\int a_1(x)(\bar{u}_n^+)^{p_1} + \int a_1(x)G_1(d_n\bar{u}_n) \\
&\quad + \frac{e_n^{p_2}}{p_2}\int |\nabla \bar{v}_n|^{p_2} - \frac{\mu_2}{p_2}e_n^{p_2}\int a_2(x)(\bar{v}_n^+)^{p_2} + \int a_2(x)G_2(e_n\bar{v}_n) \\
&\quad - \int F(x, u_n, v_n) + \nu_{1,n}\int h_1(x)u_n + \nu_{2,n}\int h_2(x)v_n
\end{aligned}$$

so that

$$
\begin{aligned}
\frac{I(u_n,v_n)}{d_n^{p_1}} &= \frac{1}{p_1}\int|\nabla\bar{u}_n|^{p_1}-\frac{\mu_1}{p_1}+\frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n)\\
&\quad+\frac{e_n^{p_2}}{p_2d_n^{p_1}}\int|\nabla\bar{v}_n|^{p_2}-\frac{\mu_2e_n^{p_2}}{p_2d_n^{p_1}}+\frac{1}{d_n^{p_1}}\int a_2(x)G_2(e_n\bar{v}_n)\\
&\quad-\frac{1}{d_n^{p_1}}\int F(x,u_n,v_n)+\frac{\nu_{1,n}}{d_n^{p_1-1}}\int h_1(x)\bar{u}_n+\frac{\nu_{2,n}e_n}{d_n^{p_1}}\int h_2(x)\bar{v}_n\\
&\geq \frac{1}{p_1}\int|\nabla\bar{u}_n|^{p}-\frac{\mu_1}{p_1}-\frac{\mu_2e_n^{p_2}}{p_2d_n^{p_1}}-\frac{1}{d_n^{p_1}}\int|F(x,u_n,v_n)|\\
&\quad+\frac{\nu_{1,n}}{d_n^{p_1-1}}\int h_1(x)\bar{u}_n+\frac{\nu_{2,n}e_n}{d_n^{p_1}}\int h_2(x)\bar{v}_n\\
&\geq \frac{1}{p_1}\int|\nabla\bar{u}_n|^{p_1}+\frac{1}{p_2}\int|\nabla\bar{v}_n|^{p_2}-\frac{\mu_1}{p_1}-\frac{\mu_2e_n^{p_2}}{p_2d_n^{p_1}}-C_0\frac{d_n^{\frac{p_1}{m}}e_n^{\frac{p_2(m-1)}{m}}}{d_n^{p_1}}\\
&\quad+\frac{\nu_{1,n}}{d_n^{p_1-1}}\int h_1(x)\bar{u}_n+\frac{\nu_{2,n}e_n}{d_n^{p_1}}\int h_2(x)\bar{v}_n\\
&= \frac{1}{p_1}\|\bar{u}_n\|_{1,p_1}^{p_1}+\frac{1}{p_2}\|\bar{v}_n\|_{1,p_2}^{p_2}-\frac{\mu_1}{p_1}-\frac{\mu_2e_n^{p_2}}{p_2d_n^{p_1}}-C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}}\\
&\quad-C\delta\left(\frac{1}{d_n^{p_1-1}}\|h_1\|_{q_1,\sigma_1}\|\bar{u}_n\|_{1,p_1}+\frac{e_n}{d_n^{p_1}}\|h_2\|_{q_2,\sigma_2}\|\bar{v}_n\|_{1,p_2}\right).
\end{aligned}
$$

Now, $d_n\to\infty$ and $\limsup_{n\to\infty}\frac{e_n^{p_2}}{d_n^{p_1}}<\infty$ imply that $\limsup_{n\to\infty}\frac{e_n}{d_n^{p_1}}<\infty$. Therefore, since $\frac{I(u_n,v_n)}{d_n^{p_1}}\to 0$, $1<p_1,p_2$ and $\limsup_{n\to\infty}\frac{e_n^{p_2}}{d_n^{p_1}}<\infty$, we have that $\{\|\bar{u}_n\|_{1,p_1}\}$ and $\{\|\bar{v}_n\|_{1,p_2}\}$ are bounded. This implies, passing to a subsequence if necessary, that $\bar{u}_n\rightharpoonup\bar{u}$ in D^{1,p_1}, $\bar{u}_n\to\bar{u}$ a.e. in $\mathbb{R}^N$, and $\bar{u}_n\to\bar{u}$ in $L^{p_1}_{a_1(x)}$ (by Lemma 2.6). In addition, applying Lemma 6.2 we have

$$\begin{aligned}
\frac{I(u_n,v_n)}{d_n^{p_1}} &= \frac{1}{p_1}\int |\nabla \bar{u}_n|^{p_1} - \frac{\mu_1}{p_1} + \frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n) \\
&\quad + \frac{e_n^{p_2}}{p_2 d_n^{p_1}}\int |\nabla \bar{v}_n|^{p_2} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} + \frac{1}{d_n^{p_1}}\int a_2(x)G_2(e_n\bar{v}_n) \\
&\quad - \frac{1}{d_n^{p_1}}\int F(x,u_n,v_n) + \frac{\nu_{1,n}}{d_n^{p_1-1}}\int h_1(x)\bar{u}_n + \frac{\nu_{2,n}e_n}{d_n^{p_1}}\int h_2(x)\bar{v}_n \\
&\geq -\frac{\mu_1}{p_1} + \frac{1}{d_n^{p_1}}\int a_1(x)G_1(d_n\bar{u}_n) - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - \frac{1}{d_n^{p_1}}\int |F(x,u_n,v_n)| \\
&\quad - C\delta\left(\frac{1}{d_n^{p_1-1}}\|h_1\|_{q_1,\sigma_1}\|\bar{u}_n\|_{1,p_1} + \frac{e_n}{d_n^{p_1}}\|h_2\|_{q_2,\sigma_2}\|\bar{v}_n\|_{1,p_2}\right) \\
&\geq -\frac{\mu_1}{p_1} - \frac{\epsilon}{d_n^{p_1}}\int a_1(x)(d_n\bar{u}_n^+)^{p_1} + \frac{C_1}{d_n^{p_1}}\int a_1(x)(d_n\bar{u}_n^+)^{\gamma_1} \\
&\quad - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}} \\
&\quad - C\delta\left(\frac{1}{d_n^{p_1-1}}\|h_1\|_{q_1,\sigma_1}\|\bar{u}_n\|_{1,p_1} + \frac{e_n}{d_n^{p_1}}\|h_2\|_{q_2,\sigma_2}\|\bar{v}_n\|_{1,p_2}\right) \\
&= -\frac{\mu_1}{p_1} - \epsilon + C_1 d_n^{\gamma_1-p_1}\int a_1(x)(\bar{u}_n^+)^{\gamma_1} - \frac{\mu_2 e_n^{p_2}}{p_2 d_n^{p_1}} - C_0\left(\frac{e_n^{p_2}}{d_n^{p_1}}\right)^{\frac{m-1}{m}} \\
&\quad - C\delta\left(\frac{1}{d_n^{p_1-1}}\|h_1\|_{q_1,\sigma_1}\|\bar{u}_n\|_{1,p_1} + \frac{e_n}{d_n^{p_1}}\|h_2\|_{q_2,\sigma_2}\|\bar{v}_n\|_{1,p_2}\right).
\end{aligned}$$

Therefore, $\gamma_1 > p_1 > 1$ and $d_n \to \infty$ imply that $\int a_1(x)(\bar{u}_n^+)^{\gamma_1} \to 0$. Hence, by Fatou's lemma we have

$$\int a_1(x)(\bar{u}^+)^{\gamma_1} = 0.$$

Therefore, since $a_1(x) > 0$ on $\mathbb{R}^N$, we have that $\bar{u} \leq 0$ on $\mathbb{R}^N$. However this contradicts the fact that $\bar{u}_n \to \bar{u}$ in $L^{p_1}_{a_1(x)}$ and $\int a_1(x)(\bar{u}_n^+)^{p_1} = 1$ for all n. This proves part (a) of the lemma.

(b) Assume $u_n \rightharpoonup u$ in D^{1,p_1} and $v_n \rightharpoonup v$ in D^{1,p_2}. Then Lemma 2.6 and our estimates on $|\int h_1(x)u|$ and $|\int h_2(x)v|$ imply

$$\int a_1(x)u_n^{p_1} \to \int a_1(x)u^{p_1} \text{ and } \int a_2(x)v_n^{p_2} \to \int a_2(x)v^{p_2},$$

$$\int h_1(x)u_n \to \int h_1(x)u \text{ and } \int h_2(x)v_n \to \int h_2(x)v.$$

Furthermore, since $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$, we have by Lebesgue Dominated Convergence Theorem that

$$\int a_1(x)|u_n^+|^{p_1} \to \int a_1(x)|u^+|^{p_1}, \int a_2(x)|v_n^+|^{p_2} \to \int a_2(x)|v^+|^{p_2},$$

and

$$\int F(x, u_n, v_n) \to \int F(x, u, v).$$

It follows by Fatou's Lemma that

$$\int a_1(x)G_1(u) \leq \liminf_{n\to\infty} \int a_1(x)G_1(u_n) \text{ and } \int a_2(x)G_2(v) \leq \liminf_{n\to\infty} \int a_2(x)G_2(v_n).$$

Finally, $\|\cdot\|^p$ is weakly l.s.c. by Lemma 2.5, so that $(u_n, v_n) \rightharpoonup (u, v)$ in $D^{1,p_1} \times D^{1,p_2}$ implies $\frac{1}{p_1}\|u\|_{1,p_1}^{p_1} + \frac{1}{p_2}\|v\|_{1,p_2}^{p_2} \leq \liminf_{n\to\infty} \frac{1}{p_1}\|u_n\|_{1,p_1}^{p_1} + \frac{1}{p_2}\|v_n\|_{1,p_2}^{p_2}$. Therefore $I(u,v) \leq \liminf_{n\to\infty} I(u_n, v_n)$, hence is weakly l.s.c. This proves the lemma. □

Theorem 6.4. *(Solution to the Minimization Problem) The minimization problem*

$$\inf I(u, v), \ (u, v) \in D^{1,p_1} \times D^{1,p_2}$$

(i.e. find $(u, v) \in D^{1,p_1} \times D^{1,p_2}$ for which $\inf I(u, v)$ is achieved) has a solution

$(u_1, v_1) \in D^{1,p_1} \times D^{1,p_2}$ *with*

$$\inf_{(u,v)\in D^{1,p_1}\times D^{1,p_2}} I(u,v) = I(u_1, v_1). \tag{6.2}$$

Proof: This follows directly from Lemma 6.3 and Lemma 2.5. □

Theorem 6.5. *Suppose* $(\bar{u}, \bar{v})$ *is a minimum point of* I, *i.e. a solution to (6.2). Assume that* $\{t_n\} \in \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$. *Then, if* $w \in D^{1,p_1} \cap L^{\gamma_1}_{a_1(x)}$ *and* $z \in D^{1,p_2} \cap L^{\gamma_2}_{a_2(x)}$, *we have*

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(\bar{u} + t_n w, \bar{v}) - I(\bar{u}, \bar{v})}{t_n} \\ &= \int |\nabla \bar{u}|^{p_1-2} \nabla \bar{u} \cdot \nabla w - \mu_1 \int a_1(x)(\bar{u}^+)^{p_1-1} w + \int a_1(x) g_1(\bar{u}) w \\ &\quad - \int F_u(x, \bar{u}, \bar{v}) w + \nu_1 \int h_1(x) w \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{I(\bar{u}, \bar{v} + t_n z) - I(\bar{u}, \bar{v})}{t_n} \\ &= \int |\nabla \bar{v}|^{p_2-2} \nabla \bar{v} \cdot \nabla z - \mu_2 \int a_2(x)(\bar{v}^+)^{p_2-1} z + \int a_2(x) g_2(\bar{v}) z \\ &\quad - \int F_v(x, \bar{u}, \bar{v}) z + \nu_2 \int h_2(x) z \end{aligned}$$

Proof: In view of Lemma 6.2, the proof follows exactly as in the scalar case, Theorem 4.6. □

Corollary 6.6. *A solution* (u_1, v_1) *to the minimization problem (6.2) is a weak so-*

lution to the problem

$$
\begin{aligned}
-\Delta_{p_1}(u_1) &= \mu_1 a_1(x)(u_1^+)^{p_1-1} - a_1(x)g_1(u_1) + F_u(x,u_1,v_1) - \nu_1 h_1(x)\\
-\Delta_{p_2}(v_1) &= \mu_2 a_2(x)(v_1^+)^{p_2-1} - a_2(x)g_2(v_1) + F_v(x,u_1,v_1) - \nu_2 h_2(x),
\end{aligned}
\tag{6.3}
$$

in $\mathbb{R}^N$*, i.e.*

$$
\begin{aligned}
&\int |\nabla u_1|^{p_1-2}\nabla u_1 \cdot \nabla w = \int (\mu_1 a_1(x)(u_1^+)^{p_1-1} - a_1(x)g_1(u_1))w\\
&+ \int F_u(x,u_1,v_1)w - \nu_1 \int h_1(x)w,\\
&\int |\nabla v_1|^{p_2-2}\nabla v_1 \cdot \nabla z = \int (\mu_2 a_2(x)(v_1^+)^{p_2-1} - a_2(x)g_2(v_1))z\\
&+ \int F_v(x,u_1,v_1)z - \nu_2 \int h_2(x)z,\\
&\ \forall\ w \in L^{\gamma_1}_{a_1(x)} \cap D^{1,p_1}, \quad \forall\ z \in L^{\gamma_2}_{a_2(x)} \cap D^{1,p_2}.
\end{aligned}
\tag{6.4}
$$

In addition, there exists $0 < \delta_0 \le \delta$ *and* $\beta_0 > 0$*,* β_0 *independent of* ν_i *and* h_i*, such that*

$$
\inf_{(u,v)\in D^{1,p_1}\times D^{1,p_2}} I(u,v) \le -\beta_0 \quad \forall\ 0 \le \nu_1, \nu_2 \le \delta_0,
$$

so that (u_1, v_1) *is non-trivial. Finally, there exists* $b_0, B_0 > 0$ *independent of* ν_1, ν_2 *such that*

$$
b_0 \le \|u_1\|_{1,p_1} + \|v_1\|_{1,p_2} \le B_0 \quad \forall\ 0 < \nu_1, \nu_2 \le \delta_0.
$$

Proof: The first part follows immediately from Theorem 6.5. For the second part, let $\Phi_1, \Psi_1 \ge 0$ be first eigenfunctions associated to the first eigenvalues $\lambda_1(p_1), \lambda_1(p_2)$, respectively, normalized so that $\|\Phi_1\|_{1,p_1} = \|\Psi_1\|_{1,p_2} = 1$. Assume without loss of generality that $p_1 \le p_2$. Then we have, applying Lemma 6.2,

$$\begin{aligned}
I(t\Phi_1, t^2\Psi_1) &= \frac{1}{p_1}t^{p_1} - \frac{\mu_1 t^{p_1}}{p_1}\int a_1(x)\Phi_1^{p_1} + \int a_1(x)G_1(t\Phi_1) + \frac{1}{p_2}t^{2p_2} \\
&\quad - \frac{\mu_2 t^{2p_2}}{p_2}\int a_2(x)\Psi_1^{p_2} + \int a_2(x)G_2(t^2\Psi_1) - \int F(x, t\Phi_1, t^2\Psi_1) \\
&\quad + \nu_1 t\int h_1(x)\Phi_1 + \nu_2 t^2 \int h(x)\Psi_1 \\
&\le \frac{1}{p_1}\left(1 - \frac{\mu_1}{\lambda_1(p_1)}\right)t^{p_1} + \int a_1(x)\left(\epsilon t^{p_1}\Phi_1^{p_1} + C_2 t^{\gamma_1}\Phi_1^{\gamma_1}\right) \\
&\quad + \frac{1}{p_2}\left(1 - \frac{\mu_2}{\lambda_1(p_2)}\right)t^{2p_2} + \int a_2(x)\left(\epsilon t^{2p_2}\Psi_1^{p_2} + C_2 t^{2\gamma_2}\Psi_1^{\gamma_2}\right) \\
&\quad + C_0 t^{\frac{p_1}{m} + \frac{2p_2(m-1)}{m}}\left(\int a_1(x)\Phi_1^{p_1}\right)^{\frac{1}{m}}\left(\int a_2(x)\Psi_1^{p_2}\right)^{\frac{m-1}{m}} \\
&\quad + \nu_1 t C_{N,p_1}\|h_1\|_{q_1,\sigma_1}\|\Phi_1\|_{1,p_1} + \nu_2 t^2 C_{N,p_2}\|h_2\|_{q_2,\sigma_2}\|\Psi_1\|_{1,p_2} \\
&= \frac{1}{p_1}\left(1 - \frac{\mu_1}{\lambda_1(p_1)} + \frac{p_1\epsilon}{\lambda_1(p_1)}\right)t^{p_1} + C_2 t^{\gamma_1}\int a_1(x)\Phi_1^{\gamma_1} \\
&\quad + \frac{1}{p_2}\left(1 - \frac{\mu_2}{\lambda_1(p_2)} + \frac{p_2\epsilon}{\lambda_1(p_2)}\right)t^{2p_2} + C_2 t^{2\gamma_2}\int a_2(x)\Psi_1^{\gamma_2} \\
&\quad + \nu_1 t C_{N,p_1}\|h_1\|_{q_1,\sigma_1} + \nu_2 t^2 C_{N,p_2}\|h_2\|_{q_2,\sigma_2} \\
&\quad + C_0 t^{\frac{p_1}{m} + \frac{2p_2(m-1)}{m}}\left(\frac{1}{\lambda_1(p_1)}\right)^{\frac{1}{m}}\left(\frac{1}{\lambda_1(p_2)}\right)^{\frac{m-1}{m}}.
\end{aligned}$$

Now, we have $\gamma_i > p_i > 1$. In addition, since $1 < p_1 \le p_2$ we have

$$p_1 = \frac{p_1}{m} + \frac{p_1(m-1)}{m} < \frac{p_1}{m} + \frac{2p_2(m-1)}{m}.$$

Then there exists $\beta_0 > 0$ such that

$$\begin{aligned} -2\beta_0 &= \frac{1}{p_1}\left(1 - \frac{\mu_1}{\lambda_1(p_1)} + \frac{p_1\epsilon}{\lambda_1(p_1)}\right) t^{p_1} + C_2 t^{\gamma_1} \int a_1(x)\Phi_1^{\gamma_1} \\ &+ \frac{1}{p_2}\left(1 - \frac{\mu_2}{\lambda_1(p_2)} + \frac{p_2\epsilon}{\lambda_1(p_2)}\right) t^{2p_2} + C_2 t^{2\gamma_2} \int a_2(x)\Psi_1^{\gamma_2} \\ &+ C_0 t^{\frac{p_1}{m} + \frac{2p_2(m-1)}{m}} \left(\frac{1}{\lambda_1(p_1)}\right)^{\frac{1}{m}} \left(\frac{1}{\lambda_1(p_2)}\right)^{\frac{m-1}{m}} \end{aligned}$$

which follows from having $\mu_i > \lambda_1(p_i)$, by choosing ϵ and t sufficiently small, and using the fact that $\Phi_1 \in L^{\gamma_1}_{a_1(x)}$, $\Psi_1 \in L^{\gamma_2}_{a_2(x)}$ as proved in Theorem 3.7. Then choose δ_0 small so that

$$\nu_1 t C_{N,p_1} \|h_1\|_{q_1,\sigma_1} + \nu_2 t^2 C_{N,p_2} \|h_2\|_{q_2,\sigma_2} \leq \beta_0$$

for $\nu_1, \nu_2 \leq \delta_0$. This proves the estimate on I.

Finally, for the last part we have

$$\begin{aligned} I(u,v) &\geq \frac{1}{p_1}\int |\nabla u|^{p_1} - \frac{\mu_1}{p_1}\int a_1(x)(u^+)^{p_1} + \nu_1 \int h_1(x)u \\ &+ \frac{1}{p_2}\int |\nabla v|^{p_2} - \frac{\mu_2}{p_2}\int a_2(x)(v^+)^{p_2} + \nu_2 \int h_2(x)v - \int F(x,u,v) \\ &\geq \frac{1}{p_1}(1 - C\mu_1\|a_1\|_{N/p_1})\|u\|^{p_1}_{1,p_1} - C_{N,p_1}\nu_1\|h_1\|_{q_1,\sigma_1}\|u\|_{1,p_1} \\ &+ \frac{1}{p_2}(1 - C\mu_2\|a_2\|_{N/p_2})\|v\|^{p_2}_{1,p_2} - C_{N,p_2}\nu_2\|h_2\|_{q_2,\sigma_2}\|v\|_{1,p_2} \\ &- C_0(C\|a_1\|_{N/p_1}\|u\|^{p_1}_{1,p_1})^{1/m}(C\|a_2\|_{N/p_2}\|v\|^{p_2}_{1,p_2})^{(m-1)/m}. \end{aligned}$$

From this we get that

$$\liminf_{\|u\|_{1,p_1} + \|v\|_{1,p_2} \to 0} I(u,v) \geq 0.$$

Since $I(u_1, v_1) \leq -\beta_0 < 0$, we have that there exists $b_0 > 0$ such that

$$b_0 \leq \|u_1\|_{1,p_1} + \|v_1\|_{1,p_2} \ \ \forall\ 0 < \nu_1, \nu_2 \leq \delta_0.$$

Since, in addition, I is uniformly coercive, there exists $B_0 > 0$ such that

$$b_0 \leq \|u_1\|_{1,p_1} + \|v_1\|_{1,p_2} \leq B_0,$$

completing the proof of the corollary. □

Properties of Solution

Theorem 6.7.

$$u_1 \in L^\infty \cap L^{\gamma_1}_{a_1(x)} \cap C^{1,\alpha}_{\text{loc}} \textit{ and } \lim_{|x|\to\infty} u_1(x) = 0,$$

$$v_1 \in L^\infty \cap L^{\gamma_2}_{a_2(x)} \cap C^{1,\alpha}_{\text{loc}} \textit{ and } \lim_{|x|\to\infty} v_1(x) = 0.$$

Proof: This follows exactly as in Theorem 5.8. □

At this point, we need linearity of the operator Δ_p, so we must assume that $p_1 = p_2 = 2 = m$. Let (u_2, v_2) satisfy $-\Delta u_2 = \nu_1 h_1(x)$ and $-\Delta v_2 = \nu_2 h_2(x)$, whose existence follows from Lemma 4.13 and by our conditions on h_1 and h_2. In addition, by that same lemma, we have a constant $d > 0$ such that

$$u_2(x) \leq \frac{\nu_1 d}{|x|^{N-2}} \text{ for all } x \in \mathbb{R}^N$$

and

$$v_2(x) \leq \frac{\nu_2 d}{|x|^{N-2}} \text{ for all } x \in \mathbb{R}^N.$$

Lemma 6.8. *Let $u_0 = u_1 + u_2$ and $v_0 = v_1 + v_2$. Then (u_0, v_0) is a solution to (5.1) and $u_0, v_0 \geq 0$.*

Proof: We have that

$$\begin{aligned} -\Delta(u_0) &= -\Delta(u_1 + u_2) \\ &= -\Delta(u_1) - \Delta(u_2) \\ &= \mu_1 a_1(x)(u_1^+)^{p_1-1} - a_1(x)g_1(u_1) + F_u(x, u_1, v_1) - \nu_1 h_1(x) + \nu_1 h_1(x) \\ &= \mu_1 a_1(x)(u_1^+)^{p_1-1} - a_1(x)g_1(u_1) + F_u(x, u_1, v_1). \end{aligned}$$

Now let $w = u_0^-$. Then since $u_2 > 0$ (by the maximum principle) we have $0 \leq w \leq u_1^- \leq |u_1|$, and so $u_1 \in L^{\gamma_1}_{a_1(x)} \cap D^{1,p_1}$ implies that w is an admissible test function in the above equation, which gives

$$-\int |\nabla u_0^-|^2 = -\int \nabla u_0 \cdot \nabla w = \int (\mu_1 a_1(x)(u_1^+)^{p_1-1} - a_1(x)g_1(u_1) + F_u(x, u_1, v_1))u_0^-.$$

So, since $u_0 < 0$ implies that $u_1 < 0$, and since u_1^+, g_1, and F_u are nonzero only when $u_1 > 0$, we have that $\int |\nabla u_0^-|^2 = 0$, i.e. that $u_0 \geq 0$. Similarly, $v_0 \geq 0$, proving the lemma. □

Therefore, our Main Theorem 6.1 follows from choosing $0 < \nu_1, \nu_2 \leq \delta_1$ with $\delta_1 \leq \delta_0$ sufficiently small, and using our estimates at infinity from the System Problem, Main Theorem 5.1 (recalling that for any nonnegative solution (u_0, v_0) to (5.1) we have that $u_0 > 0$, $v_0 > 0$ and these estimates hold):

$$u_0(x) \geq \frac{C}{|x|^{\frac{N-p_1}{p_1-1}}} \text{ and } v_0(x) \geq \frac{C}{|x|^{\frac{N-p_2}{p_2-1}}} \text{ for } |x| \text{ large.}$$

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