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EM Wave Interaction with a Bounded Plasma Column Supporting an Electron Beam

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EM WAVE INTERACTION WITH A BOUNDED PLASMA COLUMN SUPPORTING AN ELECTRON BEAM

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ABSTRACT

In this thesis, the disruption of an incoming external transverse electromagnetic wave by an inhomogeneous plasma with energetic electron beam is examined. The plasma-beam characteristics are motivated by theory and experiment. Wave-plasma interactions and wave propagation and reflection in and from a plasma medium is studied. Physical sources such as the plate current density are considered. The inhomogeneity of the plasma slab supporting the energetic electron beam is partially built into the supported fields. The wave-plasma-beam interaction is examined over a wide parameter space. Absorption or reflection of electromagnetic waves can be achieved by changing the plasma number density, collision frequency, beam number density, and the Gaussian nature of the beam and slab thickness. Under appropriate condition in the presence of an energetic electron beam supported by the plasma slab, the externally generated wave incident on and exciting the slab can resonate with the beam. Although insignificant for the parameters studied, this becomes apparent when the operation frequency (both the wave and the beam) approaches the electron plasma frequency. Initial studies conducted have not exhausted all possible parameter space scenarios and physics mechanisms. Based on the results obtained, more involved investigations are warranted.
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CHAPTER 1. INTRODUCTION

Electromagnetic (EM) wave propagation in plasma has received tremendous attention due to widespread application in plasma physics and engineering, radio wave propagation, plasma diagnostics with microwave and plasma stealth technology and air chemistry [1] [2].

Plasma is one of the four fundamental states of matter, which can be formed by ionization processes. Typically, a gas discharge is driven by a voltage or a current source, when the source is turned off the plasma disappears completely in a fraction of a second. An important property of plasma is its tendency to act as an electrical conductor. When a plasma in an equilibrium state is disturbed by an external field, an oscillation occurs due to the collective particle motions. The frequency of oscillation is called the plasma frequency. In order to have a damped oscillation, the electron-neutral collision frequency has to be smaller than the plasma frequency. When collision frequency is greater than or equal to the frequency of the incident wave, there is a high degree of attenuation. When the collision frequency is less than that of the incident wave frequency, the attenuation is less [2] [3] [4] [5].

It is well known that an incident electromagnetic wave propagates in a plasma only when the incident wave frequency is greater than the plasma frequency. When the frequency of the electromagnetic wave is below the plasma frequency, the wave exhibits continuous multiple reflections as it propagates into the plasma medium away from its stimulus. Such a wave is said to be an evanescent wave. The medium is unable to support the wave. The wave is incrementally reflected in the direction of propagation yields, for the lossless plasma, total wave reflection at the source or plasma-non-plasma interface [4] [6] [7] [8] [9] [10].

The scattering of an incident electromagnetic wave is remarkably affected by the plasma density, the collision frequency and incident wave frequency [5]. By changing the parameters of
the plasma such as number density, plasma frequency, and collision frequency, the plasma can be regarded as reflector, absorber, and transmitter. For a fixed electron density, the degree of reflection and transmission of an incident wave is highly affected by the incident wave frequency [11]. The attenuation coefficient of plasma is directly related to plasma number density [12]. Therefore, the total absorbed power increases with increasing number density until the plasma reaches its oscillation. For highly collisional and non-uniform plasma, there is a transmission of electromagnetic waves even beyond the critical density [13]. For highly collisional plasma the lower density is, the higher transmission. The higher the density, the lower the transmission. High density but less collisional plasma reflects and absorbs microwaves. Collision effects ultimately reduce the amplitude of the reflected wave [14] [15] [16] [17] [18]. Plasma electron density and collision frequency play a significant role for maximum electromagnetic wave absorption. The degree of attenuation of electromagnetic waves is determined by the plasma thickness. For a plasma frequency \( f_{pe} = 8 \text{GHz} \), electromagnetic wave frequency \( f = 16 \text{GHz} \), and electron collision frequency \( \nu = 2\omega \), attenuation is not linear and incident waves are highly absorbed, oscillation is observed when slab thickness is small [19].

A tenuous plasma in earth’s atmosphere from sea level to 270,000 ft. (100km) can be modeled as a cold collisional plasma. Such a plasma acts as a broadband absorber from VHF to S-band [1]. The amount of radiation detected is a function of the plasma density as well as gradients in the plasma.

A numerical study of the interaction of electromagnetic waves with a plasma cylinder that has a Gaussian distribution of electron density was studied. The transmission coefficient for a scattered field increases with plasma frequency and electron density [20].
The attenuation characteristics of the EM wave in a radar absorbing structure which is made of plasma and radar absorbing material (RAM) is affected not only by the characteristics of the RAMs but also width of the plasma layer, the electron density of the plasma and the collision frequency of electrons and neutrals [21].

A plasma antenna is a radio frequency antenna that can easily be switched on and off unlike a traditional wire antenna. These antennas have a tendency to transmit radio signals when they are on, and letting signals pass through when they are off. Such an antenna is efficient with low noise and has potential application in military within the range between 3 MHz and 30 MHz [22].

Electronic attack on computers and computer systems from electromagnetic signatures is a concern. One way of protecting them is by reconfiguring the internal parts of computers so that the incident electric field is smoothened [23]. A plasma microwave absorptive material can be used as a protective device by making a plasma-microwave absorptive material to protect a device from electromagnetic attack.

Application motivated, it is of interest to examine the interaction of an electromagnetic wave incident on a constricted plasma column in a guided structure. Others have studied wave-plasma interactions. Researchers have studied and implemented microwave interaction with a plasma slab with applications to microwave steering. The microwaves used in their experiments are typically X-band (8GHz-12GHz) [18] [19] [24] [25] [26] [27].

The main purpose of this research is to examine the disruptive properties of an inhomogeneous plasma slab supporting a y-directed energetic electron beam, that is Gaussian with respect to x, to a z-directed incident wave. The plasma slab is contained in a parallel-plate waveguide. The plasma with energetic electron beam is modeled as three interpenetrating fluids: a cool drift-free thermal electron fluid, a cool drift-free thermal ion fluid, and an energetic electron
beam fluid. The latter is energetic enough that collisions are statistically unfavorable. The energetic electron beam is cold and is modeled as a source current density in the y-direction. The plasma slab containing the energetic electron beam is Gaussian in x and uniform in y and z. The slab is unbounded in x. The cold plasma characteristics are built into the dielectric properties of the slab. Maxwell's equations appropriately coupled to the cold fluid equations are analytically solved for four regions in the waveguide and simulated for three regions where the first and the last regions are free space. The middle region(s), or slab(s), contains the plasma with beam. In the theoretical analysis, the waveguide is assumed to support the complete set of transverse electromagnetic, TEM, transverse electric, TE, and transverse magnetic, TM modes. A transverse electromagnetic wave generated in region 1 with no variation in x propagating towards the plasma with beam region is to be disrupted. Numerically, the disrupted properties of the plasma are examined assuming the overmoded waveguide supports $\text{TE}_0$ (TEM) and $\text{TE}_1$ and $\text{TM}_1$ modes. From simulation, the characteristics of the disruptive wave are examined based on changes in plasma density, collision frequency, the Gaussian nature of the plasma density, beam current density and the depth of the plasma slab, and the frequency of the source wave.

Other researchers have studied wave-plasma interactions and wave propagation and reflection in and from a plasma medium. In their work, the plasma is treated as a dielectric medium with the attenuation property of the plasma built into the conductivity term. In this paper, the disruption of an incoming external wave by an inhomogeneous plasma with energetic electron beam is examined. The plasma-beam characteristics are motivated by theory and experiments [28]. The electron beam is due to secondary electron emission at the cathode electrode sourced by the discharge current. The electron beam is due to the external discharge current. It is anticipated that
the electron beam will have a significant influence in the disruption process. This work develops a detailed theory and investigates some of the disruption mechanisms numerically.

Chapter 1 provides a simple discussion on basic plasma physics and offers a mathematical motivation to this work. In Chapter II, the governing equations are solved with a source beam present. In Chapters III, IV, V, and VI, field solutions are determined using perturbation type technique for the x-variation and a Green function technique for the source terms. Source terms that involve Gaussian inhomogeneity are not considered here. Finally, in Chapters VII and VIII, boundary conditions and simulation results are discussed and the concluding remarks are found in Chapter IX. The relative time average transmitted power is numerically simulated for over a wide parameter space. Simulation is compared to a simple homogeneous unbounded, three-medium geometry where the finite in z slab region supports a uniform beam current density. Good agreement is shown.

1.1 Basic Plasma Discharge Theory

A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior. Collective nature implies that the internally generated and the externally applied electromagnetic fields can influence the nature of the plasma medium. One of the most important properties of plasma is its tendency to remain nearly electrically neutral. That is, in any given macroscopic volume element, the electron and the ion density are nearly equal.

Because the ion to electron mass ratio is large, the electrons tend to respond faster than the ions in an electromagnetic field. Assuming the plasma is in a thermal dynamic equilibrium, the electron thermal speed is significantly higher than the ion thermal speed. Consequently, the initial space charge buildup, on an isolated electrode, is mainly due to negative charge. In equilibrium, sheath formation results around the electrode ideally preventing the space charge fields from
penetrating the plasma as a whole. Sheath formation also results about the electrodes in a DC and a pulse power plasma discharge.

It has been suggested by Anderson and Schill [28] that a glow discharge at a moderate pressure can be modeled as a dynamic, nonlinear, multi-fluid of charged particle species. In their work, a cold plasma fluid model was imposed. The model describes the initial stages of electron channeling characterizing the dynamic behavior of the background electrons after beam interpenetration. The plasma is modeled as three statistically independent, interpenetrating, charged particle fluids - a cool thermal ion fluid, a cool thermal electron fluid, and an energetic electron beam fluid.

Paschen effects and collision statistics play significant roles in plasma formation. The architecture of a steady-state DC discharge tube and a single-signed, pulsed power discharge tube is similar if the pulse width of the discharge voltage is much longer than the plasma formation time of the discharge. The potential difference between any two points in the negative glow and/or the positive column regions of a DC discharge is very small. This implies that most of the potential difference of the discharge electrodes in the DC plasma discharge is across the sheath region. Since the architecture of the steady-state DC discharge and the pulsed power glow discharge are similar, the potential difference of the pulsed power discharge is distributed mainly in the electrode sheath region. Further, similar to the architecture of the DC discharge, most of the potential of the discharge lies across the cathode sheath. This particular sheath region is responsible for generating the high-energy beam electrons. Positive ions drifting into the cathode sheath region fall through a large potential field gaining significant kinetic energy upon impact with the electrode. Low energy secondary electron emission results. The secondary electrons fall through the large, cathode
sheath potential composing the energetic electron beam contained in the plasma discharge (glow and positive column regions).

1.2 Mathematical Motivation

When a phenomena is in resonance with a system, large amounts of energy can be transferred between the phenomena and the system. Depending on the nature of the system, the energy in the phenomena can grow at the expense of energy in the system yielding phenomena energy amplification. Under appropriate conditions, the energy of the system can grow at the expense of the phenomena yielding phenomena energy damping. Consequently, at or near resonance more power is reradiated or absorbed by the system. The following analysis mathematically motivates how to handle a wave-like system in resonance without the need for a complete solution.

Consider the system is not in resonance. Assume the system has vanishing boundary conditions at its endpoints. Let the system potential be oscillatory in nature characterized by the following one dimensional wave equation with source excitation

$$\frac{d^2}{dx^2} y + a^2 y = A_e \cos bx + A_o \sin bx$$  \hspace{1cm} (1-1)

subject to the boundary conditions

$$y(x = 0) = 0$$  \hspace{1cm} (1-2a)

$$y(x = d) = 0$$  \hspace{1cm} (1.2b)

The natural response (homogeneous equation) of Eq. (1-1) is characterized by

$$\frac{d^2}{dx^2} y + a^2 y = 0$$  \hspace{1cm} (1-3)

yielding a solution of the form
\[ y_h(x) = y_{h1} e^{i \alpha x} + y_{h2} e^{-j \alpha x} \]  

(1-4a)

Using the method of undetermined coefficients, let

\[ y_f(x) = B_1 \cos bx + B_2 \sin bx \]  

(1.4b)

Substituting Eq. (1.4b) into Eq. (1-1) yields,

\[-b^2(B_1 \cos bx + B_2 \sin bx) + a^2(B_1 \cos bx + B_2 \sin bx) = A_e \cos bx + A_o \sin bx \]

Comparing terms, we have

\[ B_1 = \frac{A_e}{a^2 - b^2} \]  

(1-5a)

\[ B_2 = \frac{A_o}{a^2 - b^2} \]  

(1-5b)

provided \(a^2 - b^2 \neq 0\). Therefore, the general solution is

\[ y(x) = y_{h1} e^{i \alpha x} + y_{h2} e^{-j \alpha x} + \frac{1}{a^2 - b^2} (A_e \cos bx + A_o \sin bx) \]  

(1-6)

Applying boundary condition Eq. (1-2a) yields

\[ y_{h1} = -(y_{h2} + \frac{A_e}{a^2 - b^2}) \]  

(1-7)

Upon substituting Eq. (1-7) into Eq. (1-6) one obtains

\[ y(x) = -j 2 y_{h2} \sin ax + \frac{1}{a^2 - b^2} [(-e^{j ax} \cos bx)A_e + A_o \sin bx]. \]  

(1-8)

Applying the boundary condition Eq. (1.2b), Eq. (1-8) gives rise to

\[ y_{h2} = \frac{1}{(a^2 - b^2)j 2 \sin ad} [( -e^{j ad} + \cos bd)A_e + A_o \sin bd] \]  

(1-9)

Consequently, Eq. (1-8) can be re-expressed as

8
\[ y(x) = \{(\cos ad - \cos bd) \sin ax \\
+ \sin ad (\cos bx - \cos ax)\} \frac{A_e}{(a^2-b^2) \sin ad} \\
+ \{\sin bd \sin ax + \sin ad \sin bx\} \frac{-A_o}{(a^2-b^2) \sin ad} \]  

where \( a^2-b^2 \neq 0 \) and \( \sin(ad) \neq 0 \). It can be observed that Eq. (1-10) is proportional to \([\sin ad]^{-1}\). Thus, when the system is not in resonance the natural solutions appear to be part of the total solutions. If the system resonates, \( \sin ad \to 0 \) as \( a \to \frac{n\pi}{d} \). This is a resonance condition for the natural system.

Consider Eq. (1-1) assuming that no source exists \((A_e = A_o = 0)\). The complete solution subject to boundary conditions yield

\[ y(x) = A_{01} e^{jax} + A_{02} e^{-jax} \]  

(1-11)

Applying boundary conditions Eq. (1.2a, b) to Eq. (1-11) yields

\[ y(x) = \tilde{A}_{01} \sin(a_n x) \]  

(1-12)

where \( a = \frac{n\pi}{d} = a_n \) for \( n = 1,2,\ldots \) and \( \tilde{A}_{01} = j2A_{01} \).

Now let \( a = \frac{n\pi}{d} \triangleq a_n \) in Eq. (1-1), to determine how the system responds to a general source assuming the natural response is in resonance with the system. That is, \( \sin ad = 0 \).

Consider

\[ \frac{d^2}{dx^2} y + a_n^2 y = A_e \cos bx + A_o \sin bx \]  

(1-13)

subject to the boundary conditions Eq. (1.2a, b). Comparing Eqs. (1-1) and (1-4a) one can, from analogy with Eq. (1-10), deduce the general solution to be
\[ y(x) = y_{h1} e^{j a_n x} + y_{h2} e^{-j a_n x} + \frac{A_e}{a_n^2 - b^2} \cos bx + \frac{A_o}{a_n^2 - b^2} \sin bx \]  

(1-14)

where \(a_n = \frac{n\pi}{d}\) for \(n = 1, 2, \ldots\). Applying the boundary condition Eq. (1-2a) yields the constraint

\[ y_{h1} = -(y_{h2} + \frac{A_e}{a_n^2 - b^2}) \]  

(1-15)

Consequently, Eq. (1-14) can be written as

\[ y(x) = -j2 y_{h2} \sin a_n x - \left(\frac{A_e}{a_n^2 - b^2}\right) e^{j a_n x} + \frac{A_e}{a_n^2 - b^2} \cos bx + \frac{A_o}{a_n^2 - b^2} \sin bx \]

Imposing the boundary condition Eq. (1.2b) requires

\[ y(x = d) = -j2 y_{h2} \sin a_n d - \left(\frac{A_e}{a_n^2 - b^2}\right) e^{j a_n d} + \frac{A_e}{a_n^2 - b^2} \cos bd + \frac{A_o}{a_n^2 - b^2} \sin bd = 0. \]

Since \(\sin a_n d = 0\), \(y_{h2}\) is arbitrary. Let \(Y_{hn} = -j2 y_{h2}\). Further, since \(A_e \) and \(A_o \) cannot be zero, we require

\[ \cos n\pi - \cos bd = 0 \]  

(1-16a)

\[ \sin bd = 0 \]  

(1.16b)

provided \(a_n^2 - b^2 \neq 0\). Therefore, \(b = \frac{m\pi}{d} = b_m\) where \(m = 0, 1, 2, \ldots\) and \(\cos n\pi - \cos m\pi = 0\). For the latter condition to be satisfied, \((m, n)\) are either both even or both odd where \(m \neq n\). Thus, the solution when \(m \neq n\) under the condition that the natural response is in resonance with the system is given by

\[ y(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ Y_{hn} \sin(a_n x) + \bar{\Gamma}_{mn} \frac{A_{em}}{a_n^2 - b_m^2} [\cos(a_n x) - \cos(b_m x)] 
+ \delta_{nm} \frac{A_{om}}{a_n^2 - b_m^2} \sin(b_m x) \right] \]  

(1-17)

where \(\delta_{nm}\) is the Kronecker delta function having the properties
\[
\begin{align*}
\delta_{nm} &= \begin{cases} 
0 & \text{for } n \neq m \\
1 & \text{for } n = m
\end{cases} \quad (1.18a) \\
\bar{\delta}_{nm} &= 1 - \delta_{nm} \quad (1.18b) \\
\bar{\Gamma}_{mn} &= 0.5\bar{\delta}_{nm}[1 + (-1)^{n+m}] \quad (1.18c)
\end{align*}
\]

If the condition in Eq. (1-16a) is not satisfied, the even source term does not generate a wave solution to Eq. (1-13) and, as a consequence of linearity, is omitted. This is built into the solution through the term \(0.5[1 + (-1)^{n+m}]\). Physically, certain conditions (boundary conditions) must be satisfied for a wave to fit in a box. If these conditions are not met, the box cannot support the energy associated with this “wave” condition (evanescent wave). Consequently, on time average, the energy is reflected back to the source on the line connecting the source to the box. Typically, only a discrete number of waves can fit into the box and resonate in the box. For the condition when \(m \neq n\), we denote the type of waves (phenomena) supported by the system as “harmonic” waves (phenomena).

It is observed from Eq. (1-17) that the harmonic modes of the source are coupled to the resonant natural modes of the system. This implies that if an external wave is generated and that wave is supported by the system, by way of the forcing function, energy transfer can occur to other modes supported by the system. This is more transparent in a time average power calculation where the modes separate based on an orthogonality condition for ideal systems.

Now, consider the case when \(m = n\) implying \(a_n = b_m\). This type of wave (or phenomenon) resonant with the system is denoted as a “fundamental” wave (phenomenon). It is anticipated that the fundamental wave (phenomenon) is unstable growing without bounds in an ideal situation. Typically, a realistic solution can be attained by adding small loss effects.
The forced solution, by method of undetermined coefficients with repeated root is guessed as

\[ y_f(x) = x(B_1 \cos bx + B_2 \sin bx). \]

Substituting in Eq. (1-13) yields

\[-xb^2(B_1 \cos bx + B_2 \sin bx) + 2b(-B_1 \sin bx + B_2 \cos bx) + a^2x(B_1 \cos bx + B_2 \sin bx) = A_e \cos bx + A_o \sin bx\]

Comparing the coefficients of \( x \sin bx \), \( x \cos bx \), \( \sin bx \), and \( \cos bx \) the following constraints must be imposed for Eq. (1-13) to have a solution subject to boundary conditions:

\[ a = \pm b \]  \hspace{1cm} (1-19a)

\[ B_2 = \pm \frac{A_e}{2b} \]  \hspace{1cm} (1.19b)

\[ B_1 = \mp \frac{A_o}{2b} \]  \hspace{1cm} (1.19c)

Then, the complete solution when \( a = b \), is given by

\[ y(x) = y_{h1} e^{jx} + y_{h2} e^{-jx} + \frac{x}{2a} [-A_o \cos(ax) + A_e \sin(ax)] \]  \hspace{1cm} (1-20)

Applying the boundary condition Eq. (1-2a) and substituting the results in Eq. (1-20) yields

\[ y(x) = -j2y_{h2} \sin(ax) + \frac{x}{2a} [-A_o \cos(ax) + A_e \sin(ax)] \]  \hspace{1cm} (1-21)

Our goal is to examine how the source can drive the system at its fundamental mode. Therefore, let \( a = a_n = \frac{n\pi}{d} \). Applying the second boundary condition Eq. (1.2b) requires

\[ -j2y_{h2} \sin(ad) + \frac{d}{2a_n} [-A_o \cos(ad) + A_e \sin(ad)] = 0 \]
or equivalently, \( A_0 \cos a_n d = A_0 \cos n\pi = 0 \). This implies that although an odd source function can exist, it cannot satisfy the physics modeled by the governing equation Eq. (1-13). Therefore, the system cannot support the phenomenon. For the wave equation, odd function source terms cannot drive propagating waves supported by the system. We therefore force \( A_0 = 0 \). Forcing \( A_0 = 0 \) does not imply that a physical source term with odd symmetry cannot exist. It simply implies that a wave solution cannot be generated by the source. Typical “waves” that have this nature are evanescent waves supported by the system. The main thrust in the thesis is to transfer energy among source waves resonating with the system interacting with the natural, source-free, resonant mode supported by the system. Therefore, forcing \( A_0 = 0 \) implies that we are only omitting the component of the source driving evanescent waves.

Consequently, we have,

\[
y(x) = Y_0 \sin a_n x + \frac{A_e}{2a_n} (x \sin a_n x) \tag{1-22a}
\]

\[
y(x) = Y_0 \sin a_n x + \frac{A_e}{b_n} (x \sin b_n x) \tag{1.22b}
\]

where \( b_n = a_n = \frac{n\pi}{d} \). Note that \( Y_0 \) is a free variable and can be set to zero without loss in generality. The eigen solution, \( a_n = \frac{n\pi}{d} \) is the source free solution. As anticipated, Eq. (1.22a, b) grows with \( x \). The forcing function resonates identically with the natural response of the system. The system or phenomena characterized absorbs to absorb energy supplied by the source.
CHAPTER 2. PLASMA AND BEAM CHARACTERISTICS

The loading effects of the plasma and the energetic electron beam on the wave are determined. The energetic electron beam and plasma are created by a set of isolated biased parallel plates one each embedded in the parallel plate wave guide. The biased plates generate a plasma slab within the region between \( z = l_1 \) and \( z = l_3 \). The region is divided into two slab sub-regions (regions 2 and 3) each with a plasma density that is uniform in \( z \). The plasma and energetic beam contributions in region 2 is different than that in region 3. Even so, the Gaussian nature of the beam in the \( x \)-direction is the same.

Figure 2-1 A parallel plate wave guide, where regions 1 and 4 are free space and region 2 and 3 are the plasma with beam.
The momentum transport fluid equations characterizing the energetic electron beam and the cool, drift-free electron and ion fluids composing the cool plasma in the $i^{th}$ region ($i = 2, 3$) are respectively given by

\begin{align*}
n_b(\vec{r}(t), t) m_b \frac{d}{dt} \vec{v}_b(t) \\
&= n_b(\vec{r}(t), t) q_b \left[ \vec{E}_p(\vec{r}(t), t) + \vec{E}_b(\vec{r}(t), t) + \vec{E}_s(\vec{r}(t), t) \right] \\
&- m_b n_b(\vec{r}(t), t) \nu_{cb} \vec{v}_b(t)
\end{align*}

(2.1a)

\begin{align*}
n_e(\vec{r}(t), t) m_e \frac{d}{dt} \vec{v}_e(t) \\
&= n_e(\vec{r}(t), t) q_e \left[ \vec{E}_b(\vec{r}(t), t) + \vec{E}_s(\vec{r}(t), t) \right] \\
&- m_e n_e(\vec{r}(t), t) \nu_{ce} \vec{v}_e(t)
\end{align*}

(2.1b)

\begin{align*}
n_i(\vec{r}(t), t) m_i \frac{d}{dt} \vec{v}_i(t) \\
&= n_i(\vec{r}(t), t) q_i \left[ \vec{E}_b(\vec{r}(t), t) + \vec{E}_s(\vec{r}(t), t) \right] \\
&- m_i n_i(\vec{r}(t), t) \nu_{ci} \vec{v}_i(t)
\end{align*}

(2.1c)

where respectively $\vec{v}_k$, $n_k$, $q_k$, and $\nu_{ck}$ are the velocity, number density, charge-neutral collision frequency with or for the $k^{th}$ species. Here, $k = b, e, i$ represent energetic beam electron, plasma electron, and plasma ion respectively. It is assumed that $\nu_{cb} = \nu_{ce} = \nu_c$. The external fields $E_a$ where $a = p, s, b$, represent the electric field of the plate, source signal, and beam respectively. External magnetic fields are small since the problem is non-relativistic and hence neglected. Plate field effects, $\vec{E}_p$ in Eqs. (2.1b) and (2.1c) are neglected since the discharge plate voltage is mainly contained in the sheath region of plates concentrated near the electrode surface. The sheath region tends to shield the plasma glow discharge and positive plasma column from the discharge electrodes inhibiting the plate field from penetrating into the plasma regions.
Initially, the formation of plasma without the presence of the beam electrons,

\[ n_{pl}(x, t = t_0^+) \approx n_{el}(x, t = t_0^+) \approx n_{i0}(x) \]

where \( n_{pl}, n_{el} \) and \( n_{i0} \) are number densities of plasma, electrons and ions respectively. Note, subscripts \( p \) and \( p \) respectively imply plate and plasma.

It is assumed that the frequency of the stimulus field to be scattered is high enough that the sluggish heavy ions are approximately stationary. The ions in effect cannot respond to the incident wave or the supported waves. Therefore, there is no current density contribution due to ions and the ion number density is approximately constant with time.

Define the frequency/time and wavenumber/space Fourier transform pairs respectively by,

\[ \mathcal{F}\{f(x, t)\} = \int_{t=-\infty}^{\infty} f(x, t)e^{-j\omega t} dt = F(x, \omega) \]  
(2.2a)

\[ f(x, t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(x, \omega)e^{j\omega t} d\omega = \mathcal{F}^{-1}\{F(x, \omega)\} \]  
(2.2b)

\[ \mathcal{F}\{f(x, t)\} = \int_{t=-\infty}^{\infty} f(\beta, t)e^{j\beta t} dt = F(\beta, t) \]  
(2.2c)

\[ f(x, t) = \frac{1}{2\pi} \int_{\beta=-\infty}^{\infty} F(\beta, t)e^{-j\beta t} d\beta = \mathcal{F}^{-1}\{F(\beta, t)\} \]  
(2.2d)

In the frequency domain, the cool electron plasma fluid equation can be expressed as

\[ j\omega m_e \tilde{v}_e(\omega) = q_e [\tilde{E}_b(\vec{r}, \omega) + \tilde{E}_s(\vec{r}, \omega)] - m_e v_c \tilde{v}_e(\omega) \]  
(2-3)

yielding the spectral cool electron plasma fluid velocity
\[ \tilde{v}_e(\omega) = \left( \frac{q_e/m_e}{j\omega + \nu_c} \right) \left[ \tilde{E}_b(\tilde{r}, \omega) + \tilde{E}_s(\tilde{r}, \omega) \right] \]  
(2-4)

The plasma electron velocity in the time domain is

\[ \tilde{v}_e(t) = \frac{q_e}{2\pi m_e} \int_{\omega = -\infty}^{\infty} \left[ \tilde{E}_b(\tilde{r}, \omega) + \tilde{E}_s(\tilde{r}, \omega) \right] e^{j \omega t} d\omega \]  
(2-5)

yielding the plasma current density in the \( i^{th} \) region, \( \tilde{J}_{pi} \),

\[ \tilde{J}_{pi}(\tilde{r}, t) = q n_{ei}(\tilde{r}, t) \tilde{v}_e(t) \]  
(2-6a)

The beam properties (energetic electron beam velocity and density) are assumed to be dictated by the discharge voltage and discharge current obtained from the pulsed power experiments [28], as depicted in . The embedded plate fields are mainly distributed within a narrow sheath region between the plate electrodes and the plasma in a DC-like pulsed power discharge. Such a case occurs when the plasma formation time is small compared to discharge time. Since the plate field contribution is isolated and already built into the properties of the energetic beam and the presence of the cool plasma, the plate field contributions are no longer considered. In this regard, one can imagine that an energetic electron beam is injected into an existing plasma medium.

As a consequence of Coulomb effects and momentum, the number density of the cool plasma electrons adjusts in value as the energetic electron beam density varies in time

\[ n_e(x, t) \approx n_{i0}(x) - n_{bp}(x, t) \geq 0 \]  
(2.6b)

If \( n_b(x, t) > n_{i0}(x) \), then \( n_e(x, t) = 0 \). The inequality is valid since the distribution of each charge species over space is the same, \( e^{-ax^2} \) and because both the number and number density are always positive or zero.
Figure 2-2 Plots of discharge voltage (kV) and discharge current (A) vs. time (μs). Adapted from [29].

Figure 2-3 Plots of discharge current: the plot with markers is for the sampled points, and the line plot is its corresponding curve fit.
In lowest order, the total discharge current is approximately equal to the beam current. That is, \( I_d(t) = I_{bp}(t) \). The energetic beam and the cool electron plasma current densities are uniform in \( y \) and \( z \) and Gaussian in \( x \). For the parallel plate geometry, the total discharge current is the sum of the current densities in regions 2 and 3 passing normal through the plate conductor in these regions. Let \( D \) be the approximate beam width with respect to \( x \) such that \( e^{-ax^2} \to 0 \) implying \( D \gg \frac{1}{\sqrt{a}} \). Consequently, the energetic electron beam current or equivalently the discharge current is

\[
I_d(t) \approx \sum_{i=2}^{3} \int_{s_i} \int_{l_i} J_{bp_i}(x, t) \cdot ds_i = \sum_{i=2}^{3} \int_{-D/2}^{D/2} \int_{l_i-1}^{l_i} J_{bp_i}(x, t) dx dz
\]

\[
= \left[ J_{bp_{y20}}(t) \Delta l_2 \int_{-D/2}^{D/2} e^{-ax^2} dx + J_{bp_{y30}}(t) \Delta l_3 \int_{-D/2}^{D/2} e^{-ax^2} dx \right]
\]

(2-7)

where \( \Delta l_i = l_i - l_{i-1} \).

Allowing \( D \) to approach infinity, \( \int_{-D/2}^{D/2} e^{-ax^2} dx \approx \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\pi}{\sqrt{a}} \). Consequently,

\[
I_d(t) \approx \frac{\pi}{\sqrt{a}} [J_{bp_{y20}}(t) \Delta l_2 + J_{bp_{y30}}(t) \Delta l_3].
\]

(2-8)

As suggested by Eq. (2-7), \( I_d(t) \) is the sum of the discharge currents in regions 2 and 3. The distributions of the total current in the individual slabs can be written as \( I_{d2}(t) = \xi I_d(t) \) and \( I_{d3}(t) = (1 - \xi) I_d(t) \) that add up to the total discharge current, where the constraint \( \xi \) is \( 0 < \xi < 1 \) is a free parameter based on physics and geometry.

From Eq. (2-8) the discharge current in regions, \( i = 2, 3 \), can be written as
\[ I_{di}(t) = J_{byi0}(t)\Delta l_i \sqrt{\frac{\pi}{\alpha}} \]  

(2-9)

The beam current density in the \(i^{th}\) region in terms of the discharge current is given by

\[ J_{byi}(x, t) = 0.5 \left( 1 - (-1)^i(1 - 2\xi) \right) \sqrt{\frac{\alpha I_d(t)}{\pi \Delta l_i}} e^{-\alpha x^2} \]  

(2-10)

In the frequency domain, using Eq. (2-10)

\[ \tilde{J}_{bpi}(x, \omega) = \hat{y} \left[ \frac{1}{2} \left( 1 - (-1)^i(1 - 2\xi) \right) \right] \sqrt{\frac{\alpha I_d(t)}{\pi \Delta l_i}} e^{-\alpha x^2} e^{-j\omega t} dt \]  

(2-11)

In the \(\beta_x-\omega\) domain, the beam current density is written as,

\[ \tilde{J}_{bpi}(\beta_x, \omega) = \hat{y} \left[ \frac{1}{2} \left( 1 - (-1)^i(1 - 2\xi) \right) \right] \sqrt{\frac{\alpha}{\pi \Delta l_i}} \int_{-\infty}^{\infty} I_d(t)e^{-j\omega t} dt \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-j\beta_x x} dx \]

where

\[ \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-j\beta_x x} dx = \frac{\sqrt{\pi}}{\sqrt{\alpha}} e^{-\beta_x^2/4\alpha} \]  

(2-12)

Therefore,

\[ \tilde{J}_{bpi}(\beta_x, \omega) = \hat{y} \left[ \frac{1}{2} \left( 1 - (-1)^i(1 - 2\xi) \right) \right] \frac{1}{\Delta l_i} e^{-\beta_x^2/4\alpha} \int_{-\infty}^{\infty} I_d(t)e^{-j\omega t} dt \]  

(2-13)

It is anticipated, as motivated by Section 1.2, that this beam current density is the source term responsible, in part, for disrupting the incoming wave propagating towards the plasma. The discharge current \(I_d\) is determined using a curve fitting technique, from the experiment performed by Andersen and Schill [28]. Refer to Figure 2-2. The resulting integral will be evaluated using Matlab (see Appendix B).
Given the beam current density due to the plates in terms of the discharge current Eq. (2-13), one can determine the discharge electron beam density and the velocity. The beam electron charge falls through the potential well of the sheath region with approximate sheath potential difference given by the discharge voltage. Assuming the initial energy of the secondary electron at the cathode is zero, from the conservation of energy we have

\[ v_{bp}(t) = \sqrt{\frac{2q|V_d(t)|}{m_e}}. \]  

(2-14)

In view of Eq. (2-14), one can observe that the beam velocity, \( v_{bp} \), is a function of time because the discharge voltage, \( V_d(t) \) is a function of time. Since regions 2 and 3 in the problem set up are in parallel configuration with respect to the electrodes, the beam velocities in both regions are equal as the discharge voltage is the same in both regions. Thus, \( \tilde{v}_{bp}(t) = \tilde{v}_{bp2}(t) = \tilde{v}_{bp3}(t) \). The energetic electron beam current density is given by \( \tilde{J}_{pl}(\bar{r}, t) = qn_{bp1}(\bar{r}, t)v_{bp}(t) \). From this relation given Eqs. (2-10) and (2-14) the beam number density, \( n_{bp1} \), in region \( i = 2,3 \) is

\[ n_{bp1}(x, t) = 0.5\left(1 - (-1)^i(1 - 2\xi)\right) \frac{\sqrt{\pi}}{q|\Delta l_i} \frac{m_e}{2|q|V_d(t)} e^{-\alpha x^2} \]  

(2-15)

A curve fitting technique for both the discharge voltage, \( V_d(t) \), and discharge current, \( I_{di}(t) \), is used to determine the number density, beam velocity, and current density in regions \( i = 2,3 \) due to plate source.

Now we are interested in the loading effects of the plasma onto the wave leading to dissipation and/or phase changes. Such effects can be built into Maxwell’s equations using the
point form of Ohm’s law. Consequently, substituting Eqs. (2-4), (2.6b), and (2-15) in (2-6a) the y-directed current density in the $i^{th}$ region is

$$\vec{J}_{pl}(x, t) = \tilde{f}_{i}(x, t) \int_{-\infty}^{\infty} \frac{\tilde{E}_{i}(x, \omega)}{j\omega + \nu_c} e^{j\omega t} d\omega$$

(2.16)

where

$$\tilde{f}_{i}(x, t) = \frac{\varepsilon_0}{2\pi} \omega_{pi}^2(x) - f_{i}(x, t)$$

(2.17a)

$$f_{i}(x, t) = \frac{q^2}{2\pi m_e} 0.5 \left( 1 - (-1)^i(1 - 2\xi) \right) \frac{[\alpha l_q(t)}{|q|\Delta l_i} \sqrt{\frac{m_e}{2|q|V_d(t)}} e^{-ax^2}$$

(2.17b)

$$\omega_{pi}^2(x) = \frac{q^2}{m_e} n_{io}e^{-ax^2}$$

(2.17c)

$$\omega_{pe}^2 = \frac{q^2}{m_e} n_{oi}$$

(2.17d)

where $\tilde{E}_{i}(\vec{r}, \omega) = \tilde{E}_{bi}(\vec{r}, \omega) + \tilde{E}_{si}(\vec{r}, \omega)$ is the total vector electric field, which is the sum of the signal field to be disrupted, $\tilde{E}_{si}$, and the field generated by the energetic electron beam, $\tilde{E}_{bi}$, in region $i$. Further, $\omega_{pi}(x)$ is the initial plasma frequency, $n_{io}(x) = n_{oi}e^{-ax^2}$ is the initial number density of the plasma, and $n_{oi}$ is the initial, quasi-neutral ion number density at $x=0$ in region $i$. Subscript $p_{ii}$ on the initial plasma frequency [$\omega_{pi}(x)$] and subscript $i0i$ on the initial plasma number density [$n_{io}(x)$] are respectively plasma-initial- $i^{th}$ medium and ion-initial- $i^{th}$ medium.

In the space-frequency domain, Eq. (2-16) becomes
\[
\tilde{J}_{pl}(x, \omega) = \int_{t=-\infty}^{\infty} \left[ \tilde{f}_i(x, t) \int_{\tilde{\omega}=-\infty}^{\infty} \frac{\tilde{E}_i(x, \tilde{\omega})}{j\tilde{\omega} + \nu_c} e^{j\tilde{\omega}t} d\tilde{\omega} \right] e^{-j\omega t} dt 
= \int_{\tilde{\omega}=-\infty}^{\infty} \frac{\tilde{E}_i(x, \tilde{\omega})}{j\tilde{\omega} + \nu_c} \tilde{f}_i(x, \omega - \tilde{\omega}) d\tilde{\omega} 
= \frac{\tilde{E}_i(x, \omega)}{j\omega + \nu_c} \bigotimes \tilde{f}_i(x, \omega)
\] (2-18)

By analogy, the total beam current density with the contribution of the plate generated energetic beam in region \(i\) can be written as

\[
\tilde{J}_{bi}(x, \omega) = \frac{\tilde{E}_i(x, \omega)}{j\omega + \nu_c} \bigotimes f_i(x, \omega) + \tilde{J}_{bpi}(x, \omega) 
\] (2-19)

where \(\tilde{J}_{bpi}(x, \omega) = \tilde{\gamma} \sqrt{\frac{\pi}{\alpha}} e^{-\alpha x^2} \int_{-\infty}^{\infty} I_d(t)e^{-j\omega t} dt\).

The resultant current density is the sum of the energetic electron beam current density and the plasma current density. Thus,

\[
\tilde{J}_{rei}(x, \omega) = \tilde{J}_{pl}(x, \omega) + \tilde{J}_{bpi}(x, \omega) = \int_{\tilde{\omega}=-\infty}^{\infty} \frac{\tilde{E}(x, \tilde{\omega})}{j\tilde{\omega} + \nu_c} \tilde{f}_i(x, \omega - \tilde{\omega}) + \]

\[
f_i(x, \omega - \tilde{\omega}) d\tilde{\omega} + \tilde{J}_{bpi}(x, \omega)
\] (2-20)

In the frequency domain, Eq. (2-17a) becomes

\[
\tilde{f}_i(x, \omega) + f_i(x, \omega) = \varepsilon_0 \omega_{pl}^2(x) [\delta(\omega)] \]

(2-21)

Using Eqs. (2-20) and (2-21), the total current density can be written as,

\[
\tilde{J}_{rei}(\vec{r}, \omega) = \frac{\varepsilon_0 \omega_{pl}^2(x)}{j\omega + \nu_c} \tilde{E}(\vec{r}, \omega) + \tilde{J}_{bpi}(x, \omega)
\] (2-22a)

From the point form of Ohms law in the spectral domain,

\[
\tilde{J}_{rei}(\vec{r}, \omega) = \sigma_{ti}(\vec{r}, \omega) \tilde{E}(\vec{r}, \omega) + \tilde{J}_{bpi}(x, \omega)
\] (2.22b)
where the total conductivity of the plasma with beam contribution in the $i^{th}$ region, $\sigma_{Ti}$, is given by

$$\sigma_{Ti} = \sigma_{pi}(\vec{r}, \omega) + \sigma_{bi}(\vec{r}, \omega) = \frac{\varepsilon_0 \omega^2 \beta_i(x)}{j\omega + v_c}$$ (2-23)

$\vec{E}_i(\vec{r}, \omega) = \vec{E}_{si}(\vec{r}, \omega) + \vec{E}_{bi}(\vec{r}, \omega)$ is the sum of the external signal field and the fields generated by the energetic electron beam contributions.

### 2.1 Plasma in Waveguide

Consider a parallel plate waveguide with plates located in the $y = 0$ and $y = d$ planes as illustrated in . The region internal to the waveguide is segregated into four regions. Regions 1 and 4 are void matter, regions 2 and 3 contain a cool quasi-neutral plasma with an energetic electron beam ($\gamma$). The electron beam in lowest order is modeled as a y-directed current density with constant beam velocity. The beam current density within each region is independent of $y$ and $z$ and is Gaussian in $x$. The source wave in region 1 propagating in the +z-direction is to be disrupted ideally transmitting no energy to region 4.

Maxwell’s curl equations in the $i^{th}$ region are given by

$$\vec{\nabla} \times \vec{E}_i(x, y, z, t) = -\frac{\partial}{\partial t} \vec{B}_i(x, y, z, t)$$ (2.24a)

$$\vec{\nabla} \times \vec{H}_i(x, y, z, t) = \vec{J}_i(x, y, z, t) + \frac{\partial}{\partial t} \vec{D}_i(x, y, z, t)$$ (2.24b)

where $\vec{J}_i = \vec{J}_{pi} + \vec{J}_{bi}$ and $\vec{E}_i$ and $\vec{H}_i$ are due to the beam and source signal effects. The plate field contributions responsible for the energetic electron beam is contained in $\vec{J}_{bi}$ through $\vec{J}_{bpi}$ as indicated in Eq. (2-19). Explicitly, $\vec{H}_i = \vec{H}_{bi} + \vec{H}_{si}$ and $\vec{E}_i = \vec{E}_{bi} + \vec{E}_{si}$.

Assume that the plasma medium is nonmagnetic. Regions 1 and 4 in the parallel plate waveguide are empty. Regions 2 and 3 contain the plasma with energetic beam existing in free
space. Therefore, the permittivity in all four regions is \( \varepsilon_0 \). The constitutive relations for the fields in the \( i^{th} \) region where \( i = 1,2,3, \) and 4 are \( \vec{D}_i = \varepsilon_0 \vec{E}_i, \vec{B}_i = \mu_0 \vec{H}_i \). For a time harmonic form of solution \( e^{j\omega t} \), Eqs. (2.24a) and (2.24b) become, with the functional dependence on \( \omega \) implied,

\[
\nabla \times \vec{E}_i(x,y,z) = -j\omega \mu_0 \vec{H}_i(x,y,z) \\
\n\nabla \times \vec{H}_i(x,y,z) = j\omega \varepsilon_0 \vec{E}_i(x,y,z) + \vec{J}_i(x,y,z)
\]

where

\[
\vec{J}_i = \vec{J}_{pi} + \vec{J}_{bi}
\]

\[
\vec{J}_{pi} = \vec{J}_{pb} + \vec{J}_{ps},
\]

\[
\vec{J}_{bi} = \vec{J}_{bpi} + \vec{J}_{bsi} + \vec{J}_{bbi},
\]

where \( \vec{J}_{pr} = \sigma_{pr} \vec{E}_r \) and \( \vec{J}_{br} = \sigma_{br} \vec{E}_r \), for \( r = s, b \). The current density in the \( i^{th} \) region, \( \vec{J}_i \), is the resultant current density due to the plasma and the beam. The plasma current density, \( \vec{J}_{pi} \), is due to the properties of the plasma, the beam current density is due to the plate effect, \( \vec{J}_{bpi} \), and the current density due to external signal and beam generated waves, \( \vec{J}_{bsi} + \vec{J}_{bbi} \). Also, the magnetic and electric fields in the \( i^{th} \) medium, \( \vec{H}_i = \vec{H}_{bi} + \vec{H}_{si}, \vec{E}_i = \vec{E}_{bi} + \vec{E}_{si} \), are due to the beam and source signal effects.

From Eqs. (2.25b) and (2.26a),

\[
\nabla \times \vec{H}_i(x,y,z) = j\omega \varepsilon_0 \vec{E}_i(x,y,z) + \sigma_{pi} \vec{E}_i(x,y,z) + \sigma_{bi} \vec{E}_i(x,y,z) + \vec{J}_{bpi}(x,y,z)
\]

Then, Eq. (2.27) can be written as

\[
\nabla \times \vec{H}_i(x,y,z) = j\omega \varepsilon_0 \left( 1 + \frac{\sigma_{pi}(x,\omega) + \sigma_{bi}(x,\omega)}{j\omega \varepsilon_0} \right) \vec{E}_i(x,y,z) + \vec{J}_{bpi}(x,y,z)
\]
where from Eq.(2-23) $\sigma_{pl} + \sigma_{bi} = \frac{\varepsilon_0 \omega_{pl}^2(x)}{j \omega + v_c}$ and $\omega_{pl}^2(x) = \omega_{pe}^2 e^{-\alpha x^2}$. The electron plasma frequency, $\omega_{pe}$, is provided in Eq. (2.17d).

Define the effective permittivity of the medium in region $i = 2,3$ by,

$$
\varepsilon_i(x, \omega) = \varepsilon_0 \left[ 1 + \frac{\sigma_{pl}(x, \omega) + \sigma_{bi}(x, \omega)}{j \omega \varepsilon_0} \right] \tag{2.28a}
$$

or, equivalently

$$
\varepsilon_i(x, \omega) = \varepsilon_0 \left[ 1 + \frac{\sigma_{pl}(\omega) + \sigma_{bi}(\omega)}{j \omega \varepsilon_0} e^{-\alpha x^2} \right] \tag{2.28b}
$$

The spectral dispersion loading effects of the medium without medium inhomogeneity is separated from that with inhomogeneity. The physics of the latter and former effects allows one to treat these on different orders or with different emphasis. Adding and subtracting the coefficient of the Gaussian function, Eq. (2.28b) can be re-written as

$$
\varepsilon_i(x, \omega) = \varepsilon_{effi}(\omega) + \frac{\sigma_{Tl}(x, \omega)}{j \omega \varepsilon_0} \tag{2-29a}
$$

where

$$
\varepsilon_{effi}(\omega) = \varepsilon_0 \left[ 1 + \frac{\sigma_{pl}(\omega) + \sigma_{bi}(\omega)}{j \omega \varepsilon_0} \right] = \varepsilon_0 \left[ 1 - \frac{\omega_{pe}^2}{\omega (\omega - j \nu_c)} \right] \tag{2.29b}
$$

$$
\sigma_{Tl}(x, \omega) = \bar{\sigma}_{Tl}(\omega) (e^{-\alpha x^2} - 1) \tag{2.29c}
$$

$$
\bar{\sigma}_{Tl}(\omega) = \bar{\sigma}_{pl}(\omega) + \bar{\sigma}_{bi}(\omega) = \frac{\varepsilon_0 \omega_{pe}^2}{j \omega + v_c} \tag{2.29d}
$$

In view of Eq. (2-29a), one can verify that if there is no variation in $x$, the effective permittivity becomes that of a frequency dispersive homogeneous plasma. Therefore, Eqs. (2-25a) and (2.25b) become,
\[ \vec{\nabla} \times \vec{E}_i(x, y, z) = -j \omega \mu_0 \vec{H}_i(x, y, z) \]  
\[ \vec{\nabla} \times \vec{H}_i(x, y, z) = j \omega \varepsilon_i(x, \omega) \vec{E}_i(x, y, z) + \vec{j}_{bp}(x, y, z) \]  

From Eqs. (2-30) and (2-31), we have the following coupled first order differential equations,

\[
\frac{\partial}{\partial y} E_{xl}(x, y, z) - \frac{\partial}{\partial z} E_{yl}(x, y, z) = -j \omega \mu_0 H_{xl}(x, y, z) 
\]  
\[
\frac{\partial}{\partial z} E_{xl}(x, y, z) - \frac{\partial}{\partial x} E_{zl}(x, y, z) = -j \omega \mu_0 H_{yl}(x, y, z) 
\]  
\[
\frac{\partial}{\partial x} E_{yl}(x, y, z) - \frac{\partial}{\partial y} E_{zl}(x, y, z) = -j \omega \mu_0 H_{zl}(x, y, z) 
\]  
\[
\frac{\partial}{\partial y} H_{zl}(x, y, z) - \frac{\partial}{\partial z} H_{yl}(x, y, z) 
\]  
\[
= J_{bp\text{x}l}(x, y, z) + j \omega \varepsilon_i(x) E_{xl}(x, y, z) 
\]  
\[
\frac{\partial}{\partial z} H_{xl}(x, y, z) - \frac{\partial}{\partial x} H_{zl}(x, y, z) 
\]  
\[
= J_{bp\text{y}l}(x, y, z) + j \omega \varepsilon_i(x) E_{yl}(x, y, z) 
\]  
\[
\frac{\partial}{\partial x} H_{yl}(x, y, z) - \frac{\partial}{\partial y} H_{xl}(x, y, z) 
\]  
\[
= J_{bp\text{z}l}(x, y, z) + j \omega \varepsilon_i(x) E_{zl}(x, y, z) 
\]

where subscript \( i \) represents the \( i^{th} \) medium in the parallel plate waveguide. It is implied that the field amplitudes are functions of \( \omega \).

The transverse field components can be expressed in terms of the longitudinal field components and the source current density. It is assumed that the \( x \)- and \( z \)-components of the energetic electron beam current density are negligible as compared to the energetic beam generated perpendicular to the plate. Thus, substituting \( H_{yl} \) from Eq. (2-33) into Eq. (2-35), and \( H_{xl} \) from Eq. (2-32) into Eq. (2-36), the transverse electric fields are decoupled as,
\[
\frac{\partial^2}{\partial x^2} + \omega^2 \mu_0 \varepsilon_i(x) E_{x_1}(x, y, z) = -j\omega \mu_0 \frac{\partial}{\partial y} H_{z_1}(x, y, z) + \frac{\partial^2}{\partial x \partial z} E_{z_1}(x, y, z)
\] (2-38)

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_i(x) E_{y_1}(x, y, z)
\]

\[
= \frac{\partial^2}{\partial y \partial z} E_{z_1}(x, y, z) + j\omega \mu_0 \frac{\partial}{\partial x} H_{z_1}(x, y, z)
\] (2-39)

\[+ j\omega \mu_0 j_{by_1}(x, y, z)\]

In a similar fashion applying Eq. (2-36) into Eq. (2-32), and Eq. (2-35) into Eq. (2-33) yields respectively,

\[
\frac{\partial^2}{\partial x \partial z} H_{x_1}(x, y, z)
\]

\[
= j\omega \varepsilon_i(x) \frac{\partial}{\partial y} E_{z_1}(x, y, z) + \frac{\partial}{\partial z} j_{by_1}(x, y, z)
\] (2-40)

\[+ \frac{\partial}{\partial z \partial x} H_{z_1}(x, y, z)\]

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_i(x) H_{y_1}(x, y, z) = \frac{\partial^2}{\partial y \partial z} H_{z_1}(x, y, z) - j\omega \varepsilon_i(x) \frac{\partial}{\partial x} E_{z_1}(x, y, z)
\] (2-41)

With the aid of Eqs. (2-32) and (2-33), Eq. (2-37) simplifies as

\[
\frac{\partial}{\partial x} \left[ \frac{\partial}{\partial z} E_{x_1}(x, y, z) - \frac{\partial}{\partial x} E_{z_1}(x, y, z) \right] - \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial y} E_{z_1}(x, y, z) - \frac{\partial}{\partial z} E_{y_1}(x, y, z) \right]
\]

\[= \omega^2 \mu_0 \varepsilon_i(x) E_{z_1}(x, y, z)\]

applying Gauss electric law yields the inhomogeneous wave equation for the longitudinal electric field

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_i(x) \right] E_{z_1}(x, y, z) = \frac{\partial}{\partial z} \left( \frac{\rho}{\varepsilon_0} \right)
\] (2-42)
Similarly, substituting Eqs. (2-35) and (2-36) into Eq. (2-34) for \( E_{x_i} \) and \( E_{y_i} \) respectively and multiplying through by \( j \omega \varepsilon_i(x) \) yields,

\[
\varepsilon_i(x) \frac{\partial}{\partial x} \left\{ \frac{1}{\varepsilon_i(x)} \left[ \frac{\partial}{\partial z} H_{x_i}(x, y, z) - \frac{\partial}{\partial x} H_{z_i}(x, y, z) - J_{ybp_i}(x, y, z) \right] \right\} - \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial y} H_{z_i}(x, y, z) - \frac{\partial}{\partial z} H_{y_i}(x, y, z) \right] = \omega^2 \mu_0 \varepsilon_i(x) H_{z_i}(x, y, z).
\]

This can be re-written as

\[
\varepsilon_i(x) \left[ \frac{-1}{\varepsilon_i^2(x)} \frac{\partial \varepsilon_i(x)}{\partial x} \right] \left[ \frac{\partial}{\partial z} H_{x_i}(x, y, z) - \frac{\partial}{\partial x} H_{z_i}(x, y, z) - J_{ybp_i}(x, y, z) \right] + \frac{\partial^2}{\partial x \partial z} H_{x_i}(x, y, z)
\]

\[
- \frac{\partial^2}{\partial x^2} H_{z_i}(x, y, z) - \frac{\partial}{\partial x} J_{ybp_i}(x, y, z) - \frac{\partial^2}{\partial y^2} H_{z_i}(x, y, z) + \frac{\partial^2}{\partial y \partial z} H_{y_i}(x, y, z)
\]

\[= \omega^2 \mu_0 \varepsilon_i(x) H_{z_i}(x, y, z)\]

Combining terms together and rearranging,

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\varepsilon_i(x)} \frac{\partial \varepsilon_i(x)}{\partial x} \frac{\partial}{\partial x} + \omega^2 \mu \varepsilon_i(x) \right] H_{z_i}(x, y, z)
\]

\[= - \frac{1}{\varepsilon_i(x)} \frac{\partial \varepsilon_i(x)}{\partial x} \left[ \frac{\partial}{\partial z} H_{x_i}(x, y, z) - J_{ybp_i}(x, y, z) \right] + \frac{\partial^2}{\partial x \partial z} H_{x_i}(x, y, z)
\]

\[+ \frac{\partial^2}{\partial y \partial z} H_{y_i}(x, y, z) - \frac{\partial}{\partial x} J_{ybp_i}(x, y, z)
\]

\[= \frac{\partial}{\partial z} \left[ - \frac{1}{\varepsilon_i(x)} \frac{\partial \varepsilon_i(x)}{\partial x} H_{x_i}(x, y, z) + \vec{\nabla} \cdot \vec{H}_i(x, y, z) - \frac{\partial}{\partial z} H_{z_i}(x, y, z) \right]
\]

\[+ \frac{1}{\varepsilon_i(x)} \frac{\partial \varepsilon_i(x)}{\partial x} J_{ybp_i}(x, y, z) - \frac{\partial}{\partial x} J_{ybp_i}(x, y, z)
\]

Therefore, applying Gauss magnetic law to the above equation, yields
\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_i(x) \right] H_{zi}(x,y,z) \\
= - \frac{1}{\epsilon_i(x)} \frac{\partial \epsilon_i(x)}{\partial x} \left[ \frac{\partial}{\partial z} H_{xi}(x,y,z) - \frac{\partial}{\partial x} H_{zi}(x,y,z) \right] \\
+ \frac{1}{\epsilon_i(x)} \frac{\partial \epsilon_i(x)}{\partial x} J_{ybp}(x,y,z) - \frac{\partial}{\partial x} J_{ybp}(x,y,z)
\]

Using Eq. (2-36), this can be written as

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_i(x) \right] H_{zi}(x,y,z) \\
= - \frac{1}{\epsilon_i(x)} \frac{\partial \epsilon_i(x)}{\partial x} \left[ J_{ybp}(x,y,z) + j \omega \epsilon_i(x) E_{yi}(x,y,z) \right] \\
+ \frac{1}{\epsilon_i(x)} \frac{\partial \epsilon_i(x)}{\partial x} J_{ybp}(x,y,z) - \frac{\partial}{\partial x} J_{ybp}(x,y,z) \\
= - \frac{\partial}{\partial x} J_{ybp}(x,y,z) - j \omega \frac{\partial \epsilon_i(x)}{\partial x} E_{yi}(x,y,z)
\]

Therefore, the inhomogeneous wave equation for the magnetic field takes the form

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_i(x) \right] H_{zi}(x,y,z) \\
= - \frac{\partial}{\partial x} J_{ybp}(x,y,z) - j \omega \frac{\partial \epsilon_i(x)}{\partial x} E_{yi}(x,y,z)
\] (2-43)

Using Eq. (2.28b), Eq. (2-43) becomes

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_i(x) \right] H_{zi}(x,y,z) \\
= - \frac{\partial}{\partial x} J_{ybp}(x,y,z) \tag{2-44} \\
+ 2 \alpha x \left( \bar{\sigma}_{pl}(\omega) + \bar{\sigma}_{bi}(\omega) \right) e^{-\alpha x^2} E_{yi}(x,y,z)
\]
Then, multiplying Eq. (2-44) through by \( \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_i(x) \) and using Eq. (2-39)

\[
\begin{align*}
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_i(x) \right] \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_i(x) \right] H_{zi}(x, y, z) &= - \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu \varepsilon_i(x) \right] \frac{\partial}{\partial x} J_{ybpl}(x, y, z) \\
&+ 2\alpha x \left( \tilde{\varepsilon}_{pl}(\omega) + \tilde{\varepsilon}_{bpl}(\omega) \right) e^{-\alpha x^2} \left[ \frac{\partial^2}{\partial y \partial z} E_{zi}(x, y, z) + j\omega \mu_0 \frac{\partial}{\partial x} H_{zi}(x, y, z) \right] \\
&+ j\omega \mu_0 J_{ybpl}(x, y, z)
\end{align*}
\]

This implies the TE and TM modes do not exist in the medium separately. For \( \alpha \gg 1 \), the TE and TM modes are weakly coupled. Therefore, in lowest order, we neglect the last term in Eq. (2-44) and assumed the existence of uncoupled TE and TM modes in the source free sense. Therefore, Eq. (2-44) simplifies to

\[
\begin{align*}
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_i(x) \right] H_{zi}(x, y, z) &\approx - \frac{\partial}{\partial x} J_{ybpl}(x, y, z) \\
\end{align*}
\]  

(2-45)

where it is implied that \( H_{zi}(x, y, z) = H_{zi}(x, y, z, \omega), J_{ybpl}(x, y, z) = J_{ybpl}(x, y, z, \omega) \) and \( \varepsilon_i(x) = \varepsilon_i(x, \omega) \), where \( \varepsilon_i(x, \omega) \) is defined in Eq. (2-29a).

Taking the forward Fourier transform of Eq. (2-45) making use of Eq. (2-2a) (Refer Appendix A) results in

\[
\begin{align*}
\left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \beta_x^2 + \frac{\omega^2 \mu_0}{2\pi} \varepsilon_i(\beta_x) \bigoplus \beta_x \right] H_{zi}(\beta_x, y, z) &= j\beta_x J_{ybpl}(\beta_x, y, z) \\
\end{align*}
\]  

(2-46)

In the same manner, the Fourier transform of Eqs. (2-38) through (2-42) are respectively,
\[
\begin{align*}
\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2 \mu_0}{2\pi} \epsilon_i(\beta_x) \odot \beta_x \right] E_{x_l}(\beta_x, y, z) \\
= -j\omega \mu_0 \frac{\partial}{\partial y} H_{z_l}(\beta_x, y, z) - j\beta_x \frac{\partial}{\partial z} E_{z_l}(\beta_x, y, z) \\
\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2 \mu_0}{2\pi} \epsilon_i(\beta_x) \odot \beta_x \right] E_{y_l}(\beta_x, y, z) \\
= \frac{\partial^2}{\partial y \partial z} E_{z_l}(\beta_x, y, z) + \omega \mu_0 \beta_x H_{z_l}(\beta_x, y, z) + j \omega \mu_0 j_{y_{bpl}}(\beta_x, y, z) \\
\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2 \mu_0}{2\pi} \epsilon_i(\beta_x) \odot \beta_x \right] H_{x_l}(\beta_x, y, z) \\
= \frac{j\omega}{2\pi} \frac{\partial}{\partial y} \left[ \epsilon_i(\beta_x) \odot \beta_x E_{z_l}(\beta_x, y, z) \right] + \frac{\partial}{\partial z} j_{y_{bpl}}(\beta_x, y, z) \\
- j\beta_x \frac{\partial}{\partial z} H_{z_l}(\beta_x, y, z) \\
\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2 \mu_0}{2\pi} \epsilon_i(\beta_x) \odot \beta_x \right] H_{y_l}(\beta_x, y, z) \\
= \frac{\partial^2}{\partial y \partial z} H_{z_l}(\beta_x, y, z) - \omega \frac{\partial}{\partial z} \left[ \epsilon_i(\beta_x) \odot \beta_x \{ \beta_x E_{z_l}(\beta_x, y, z) \} \right] \\
\left[ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} - \beta_x^2 + \frac{\omega^2 \mu_0}{2\pi} \epsilon_i(\beta_x) \odot \beta_x \right] E_{z_l}(\beta_x, y, z) = 0
\end{align*}
\]

Where the convolution is defined as

\[
g(\beta_x) \odot \beta_x f(\beta_x) = \int_{-\infty}^{\infty} g(\beta_x) f(\beta_x - \beta_x) d\beta_x = \int_{-\infty}^{\infty} f(\beta_x) g(\beta_x - \beta_x) d\beta_x
\]

In regions 2 and 3, the current density is independent of z. Consequently, from the continuity equation charge density is independent of z. Therefore, the spatial change in the charge density with respect to z, corresponding to the right-hand side of Eqs. (2-42) and (2-51) is zero.
In the phase domain, the permittivity of the medium in region $i$, from Eq. (2-29a), is

$$\varepsilon_i(\beta_x, \omega) = 2\pi \varepsilon_{effi}(\omega) \delta(\beta_x) + \frac{\sigma_{T}(\beta_x, \omega)}{j\omega}$$

(2-53a)

where $\varepsilon_{effi}(\omega)$, the effective permittivity, is defined in Eq. (2.29b). The transformed conductivity of Eq. (2.29c) is

$$\sigma_{T}(\beta_x, \omega) = \left(\tilde{\sigma}_{pi}(\omega) + \tilde{\sigma}_{bi}(\omega)\right) \left[\frac{\sqrt{\pi}}{4a} e^{-\frac{-\beta_x^2}{4a}} - 2\pi \delta(\beta_x)\right]$$

(2.53b)

where the integral definition, $\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)} dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{(b^2-4ac)/4a}$ is used to compute Eq. (2.53b).

With the aid of Eq. (2-53a), and Eqs. (2-46) through (2-51) are rewritten as,

$$\left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \beta_x^2 + \omega^2 \mu_0 \varepsilon_{effi}(\omega) \right] H_{zi}(\beta_x, y, z)$$

$$= -\frac{\omega \mu_0}{j2\pi} \sigma_{T}(\beta_x, \omega) \Theta \beta_x H_{zi}(\beta_x, y, z) + j\beta_x J_{ybp}(\beta_x, y, z)$$

(2-54)

$$\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{effi}(\omega) \right] E_{xi}(\beta_x, y, z)$$

$$= -\frac{\omega \mu_0}{j2\pi} \sigma_{T}(\beta_x, \omega) \Theta \beta_x E_{xi}(\beta_x, y, z) - j\omega \mu_0 \frac{\partial}{\partial y} H_{zi}(\beta_x, y, z)$$

$$- j\beta_x \frac{\partial}{\partial z} E_{zi}(\beta_x, y, z)$$

(2-55)

$$\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{effi}(\omega) \right] E_{yi}(\beta_x, y, z)$$

$$= -\frac{\omega \mu_0}{j2\pi} \sigma_{T}(\beta_x, \omega) \Theta \beta_x E_{yi}(\beta_x, y, z) + \frac{\partial^2}{\partial y \partial z} E_{zi}(\beta_x, y, z)$$

$$+ \omega \mu_0 \beta_x H_{zi}(\beta_x, y, z) + j\omega \mu_0 J_{ybp}(\beta_x, y, z)$$

(2-56)
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] H_{xi}(\beta_x, y, z) = - \frac{\omega \mu_0}{j2\pi} \sigma_{T1}(\beta_x, \omega) \odot \beta_x H_{xi}(\beta_x, y, z, \omega)
\]
\[+ j\frac{\omega}{2\pi} \frac{\partial}{\partial y} \left[ \varepsilon_i(\beta_x) \odot \beta_x E_{zi}(\beta_x, y, z) \right] + \frac{\partial}{\partial z} I_{y_{p}}(\beta_x, y, z) \]
\[ - j\beta_x \frac{\partial}{\partial z} H_{zi}(\beta_x, y, z) \]
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] H_{yi}(\beta_x, y, z) = - \frac{\omega \mu_0}{j2\pi} \sigma_{T1}(\beta_x, \omega) \odot \beta_x H_{yi}(\beta_x, y, z) + \frac{\partial^2}{\partial y \partial z} H_{zi}(\beta_x, y, z) \]
\[ - \frac{\omega}{2\pi} \left[ \varepsilon_i(\beta_x) \odot \beta_x \{ \beta_x E_{zi}(\beta_x, y, z) \} \right] \]
\[
\left[ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} - \beta_x^2 + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] E_{zi}(\beta_x, y, z) = - \frac{\omega \mu_0}{j2\pi} \sigma_{T1}(\beta_x, \omega) \odot \beta_x E_{zi}(\beta_x, y, z) \]

Equations (2-54) through (2-59) characterize each of the four regions internal to the parallel plate waveguide with a medium that is inhomogeneous with x for two of the regions. Waves propagate in the \( \pm z \)-direction at normal incidence to the planar interfaces separating the regions. The terms on the right-hand side of Eqs. (2-54) through (2-59) are treated as combinations of source terms and effective source terms driving the fields in each region.

Making use of linearity, each field component is defined in terms of the sum of three sub-fields of the same type. Let \( \bar{F}_{kth} \) represent the homogeneous or natural, source-free, field (solution). Define \( \bar{F}_{klf} \) to represent the force field (solution) due to physical external sources such as the y-directed beam current density. Further, let \( \bar{F}_{klf} \) represent the correction of the force field.
(solution). This field results from effective force field source terms due to the inhomogeneous nature of the medium. Specifically, these source terms are associated with the convolutions of the modeled total conductivity of the medium with the resultant field component. The resultant field is therefore written as

$$F_{kl}(\beta_x, y, z) = \bar{F}_{kh}(\beta_x, y, z) + \bar{F}_{kf}(\beta_x, y, z) + \bar{F}_{kf}(\beta_x, y, z)$$ (2-60a)

where \( F = E \) or \( H \) and \( k = x, y, \) or \( z. \) The subscript \( f \) implies the force field solution due to either the physical or effective source terms. The subscript \( h \) implies the homogeneous solution.

Using Eq. (2-53a), the following convolution terms on the right-hand side of Eqs. (2-57) and (2-58) can be written as

$$\varepsilon_i(\beta_x) \otimes^\beta_x F_{kl}(\beta_x, y, z) = 2\pi\varepsilon_{eff}(\omega)F_{kl}(\beta_x, y, z) + \frac{\sigma_{Ti}(\beta_x, \omega)}{j\omega} \otimes^\beta_x F_{kl}(\beta_x, y, z)$$ (2.60b)

Only in the effective source terms, the inhomogeneous nature of the medium is weak. This implies that \( \alpha \) in the Gaussian distribution of Eq. (2.28a, b), Eq. (2.29a-d) and Eq. (2.53b) is small or approaches zero. The total conductivity of the medium \( \sigma_{Ti} \) is therefore small implying the field correction term \( \bar{F}_{ki} \) is small or approaches zero. Consequently, \( \sigma_{Ti} \otimes^\beta_x \bar{F}_{ki} \) is assumed to be negligibly small compared to the remaining contributions. Therefore,

$$\sigma_{Ti}(\beta_x, \omega) \otimes^\beta_x \left[ \bar{F}_{kih}(\beta_x, y, z) + \bar{F}_{kif}(\beta_x, y, z) + \bar{F}_{kf}(\beta_x, y, z) \right] \approx \sigma_{Ti}(\beta_x, \omega) \otimes^\beta_x \left[ \bar{F}_{kih}(\beta_x, y, z) + \bar{F}_{kif}(\beta_x, y, z) \right]$$ (2.60c)

The governing systems of equations characterizing the homogeneous, forced and corrected effective fields are, respectively

*Homogenous (natural) Equations*
\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{E}_{x_0}(\beta_x, y, z) = 0 \quad (2.61a) \]

\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{E}_{y_0}(\beta_x, y, z) = 0 \quad (2.61b) \]

\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{H}_{x_0}(\beta_x, y, z) = 0 \quad (2.61c) \]

\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{H}_{y_0}(\beta_x, y, z) = 0 \quad (2.61d) \]

\[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 \bar{H}_{z_0}(\beta_x, y, z) = 0 \quad (2.61e) \]

\[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 \bar{E}_{z_0}(\beta_x, y, z) = 0 \quad (2.61f) \]

The transverse field equations Eqs. (2.61a-d) are redundant since they are automatically built into the decoupled expressions Eqs. (2.61e, f). These expressions will no longer be considered.

**Forced Equations due to physical sources**

\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{E}_{x_{if}} = -j \omega \mu_0 \frac{\partial H_{z_{if}}}{\partial y} - j \beta_x \frac{\partial}{\partial z} E_{z_{if}} \quad (2.62a) \]

\[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \bar{E}_{y_{if}}(\beta_x, y, z) = \frac{\partial^2}{\partial y \partial z} E_{z_{if}}(\beta_x, y, z) + \omega \mu_0 \beta_x H_{z_{if}}(\beta_x, y, z) + j \omega \mu_0 J_{y_{bp}}(\beta_x, y, z) \quad (2.62b) \]
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] \tilde{H}_{xif}(\beta_x, y, z) = \frac{\partial}{\partial z} J_\text{ybp} (\beta_x, y, z) - j \beta_x \frac{\partial}{\partial z} H_z \text{(2.62c)}
\]

\[
+ j \omega \varepsilon_{eff}(\omega) \frac{\partial}{\partial y} E_z (\beta_x, y, z)
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] \tilde{H}_{yif}(\beta_x, y, z)
\]

\[
= \frac{\partial^2}{\partial y \partial z} H_z (\beta_x, y, z) - \varepsilon_{eff}(\omega) \omega \beta_x E_z (\beta_x, y, z) \text{(2.62d)}
\]

\[
\left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \right] \tilde{H}_{zif}(\beta_x, y, z) = j \beta_x J_\text{ybp} (\beta_x, y, z) \text{(2.62e)}
\]

**Forced governing equations due to effective sources**

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] \tilde{E}_{xi}(\beta_x, y, z) = - \frac{\omega \mu_0 \sigma_{Ti}(\beta_x)}{j 2\pi} \odot \beta_x \tilde{E}_{xi}(\beta_x, y, z) \text{(2-63a)}
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] \tilde{E}_{yi}(\beta_x, y, z) = - \frac{\omega \mu_0}{j 2\pi} \sigma_{Ti}(\beta_x, \omega) \odot \beta_x \tilde{E}_{yi}(\beta_x, y, z) \text{(2.63b)}
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) \right] \tilde{H}_{xi}(\beta_x, y, z)
\]

\[
= - \frac{\omega \mu_0 \sigma_{Ti}(\beta_x)}{j 2\pi} \odot \beta_x \tilde{H}_{xi}(\beta_x, y, z) \text{(2.63c)}
\]

\[
+ \frac{1}{2\pi} \left[ \sigma_{Ti}(\beta_x, \omega) \odot \beta_x \frac{\partial}{\partial y} E_z(\beta_x, y, z) \right]
\]
\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) \tilde{H}_{yi}(\beta_x, y, z) = -\frac{\omega \mu_0}{j2\pi} \sigma_{Ti}(\beta_x, \omega) \otimes \beta_x \tilde{H}_{yi}(\beta_x, y, z)
\]

(2.63d)

\[
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) - \beta_x^2 \tilde{H}_{zi}(\beta_x, y, z)
\]

(2.63e)

\[
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) - \beta_x^2 \tilde{E}_{zi}(\beta_x, y, z)
\]

(2.63f)

To simplify Eqs. (2-63a) through (2.63f), we redefine \( F_{kl} \) as

\[
F_{kl}(\beta_x, y, z) = \frac{1}{2\pi} \sigma_{Ti}(\beta_x, \omega) \otimes \beta_x \tilde{F}_{kl}(\beta_x, y, z)
\]

Consequently, Eqs. (2-63a) through (2.63d) simplify as

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) \tilde{E}_{xli}(\beta_x, y, z) = j\omega \mu_0 \tilde{E}_{xli}(\beta_x, y, z)
\]

(2-64a)

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) \tilde{E}_{yli}(\beta_x, y, z) = j\omega \mu_0 \tilde{E}_{yli}(\beta_x, y, z)
\]

(2-64b)

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) \tilde{H}_{xli}(\beta_x, y, z) = j\omega \mu_0 \tilde{H}_{xli}(\beta_x, y, z) + \frac{\partial}{\partial y} E_{zli}(\beta_x, y, z)
\]

(2-64c)

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 e_{eff}(\omega) \tilde{H}_{yli}(\beta_x, y, z) = j\omega \mu_0 \tilde{H}_{yli}(\beta_x, y, z) + j\beta_x E_{zli}(\beta_x, y, z)
\]

(2-64d)
The homogeneous solution will be solved in Chapter 3 for each mode. Chapter 4 we evaluate the y-variation of the inhomogeneous wave equation. Using a Green’s function technique, the z-variation is handled in Chapter 5. Chapter 6 yields the solution of the transverse fields. Boundary conditions are solved in Chapter 7.
CHAPTER 3. HOMOGENEOUS WAVE EQUATION

The inhomogeneous wave equations are characterized by Eqs. (2-54) and (2-59). In the absence of the source terms on the right-hand side of the equations, transverse electric and transverse magnetic modes are supported by the guide. It is desired to determine the natural modes (eigenfunctions and eigenvalues) supported by the system. Since the natural modes are orthogonal, an expansion technique in terms of these modes may be applied to the general source functions.

The homogeneous wave equation fits the following form

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 \right] \tilde{F}_{zh}(\beta_x, y, z) = 0 \]  

(3-1)

where \( F = E \) or \( H \). Based on the method of separation of variables,

\[ \tilde{F}_{zh}(\beta_x, y, z) = \tilde{F}_{zh1}(\beta_x, z)e^{-j\beta_y y} + \tilde{F}_{zh2}(\beta_x, z)e^{j\beta_y y} \]

where the functional dependence of the fields with respect to \( \omega \) is implied.

The ideal parallel-plate waveguide supports discrete standing waves normal to the plate surface. Boundary conditions give rise to the nature of these standing waves to fit between the plates. For both the TE and TM modes, the tangential electric field must vanish on the surface of each plate internal to the guide. Consequently, \( \tilde{F}_{zh} \) must be proportional to \( \sin(\beta_y y) \) or \( \cos(\beta_y y) \).

The boundary condition in the electric field requires the tangential field to be proportional to \( \sin(\beta_y y) \). This is required since this field must vanish at \( y = 0^+ \) and \( y = d^- \) simultaneously. This requires

\[ \beta_y = \frac{n\pi}{d} \equiv \beta_{yn}, \text{ for } n = 0, 1, 2, ... \]  

(3-2)
where \( n = 0 \) for the \( TM_0 \) mode yields a trivial solution and \( n = 0 \) for the \( TE_0 \) mode is non-trivial. Based on time average, the concept of modes typically implies that energy is contained in the \( n^{th} \) set of fields. In other words, the \( n^{th} \) set of fields are orthogonal to the \( m^{th} \) set of fields.

Equation (3-1) can be re-expressed as a source-free one-dimensional, wave equation of the form

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{zin}(\beta_x) \right] \vec{F}_{zth}(\beta_x, z) = 0
\]

where

\[
\beta_{zin}(\beta_x) = \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 - \beta_y^2
\]

valid for all modes of both the \( TE_n \) and \( TM_n \) types.

The \( TE_0 \) and \( TM_0 \) modes are treated as a special case in Appendix G. There it is explicitly shown that the \( TM_0 \) mode does not exist. For the special case when \( \beta_x \to 0 \), the \( TE_0 \) mode becomes TEM in nature.
CHAPTER 4. INHOMOGENEOUS WAVE EQUATION Y-VARIATION

In this section, the y-variation of the inhomogeneous wave equation is solved for both the TE and TM modes. Only the fundamental and harmonic waves will be examined.

The ideal, empty, parallel-plate waveguide can only support a discrete set of standing waves in the y-direction. On time average, the energy contained in one set of fields comprising the wave is totally contained in this set. There is no coupling among field sets. Such sets of fields are called modes. Modes, similar to eigenfunctions, exhibit the property of function orthogonality. Since the empty waveguide structure can only support a discrete set of modes, one can argue that the general source can be expanded in terms of a complete set of modes or eigenfunctions supported by the system. Only these m-discrete source fields subject to boundary conditions lead to wave solutions supported by the system.

Consequently, the y-directed current density can be expanded in terms of a discrete series of even and odd eigenfunctions of the parallel-plate waveguide. The eigenvalues are to be explicitly determined by satisfying boundary conditions. In the $\beta_x$ space, the current density (more correctly the spatial spectral current density) in the y-direction for the $i^{th}$ medium is in general written as

$$J_y(\beta_x, y, z, \omega) = \sum_{m=0}^{\infty} \{J_{elbm}(\beta_x, z) \cos(b_m y) + J_{olbm}(\beta_x, z) \sin(b_m y)\}$$

(4-1)

where $b_m$ is unknown but discrete in $m$ and needs to be determined. The index $m$ is reserved for the source modes. The frequency dependence in the series’ coefficients is implied.
4.1 TE Mode Resonance

4.1.1 Natural Response (Homogeneous Solution)

Using the method of separation of variables, (see Section Chapter 3) the solution to Eq. (2-61a) becomes

\[
\tilde{H}_{zilh}(\beta_x, y, z) = \tilde{H}_{zilh1}(\beta_x, z) e^{j\beta_y y} + \tilde{H}_{zilh2}(\beta_x, z) e^{-j\beta_y y}
\]  

(4-2)

where \( \tilde{H}_{zilh1}, \tilde{H}_{zilh2}, \) and \( \beta_y \) are to be determined from boundary conditions.

Based on the source free condition in Section Chapter 3, \( \beta_y = \beta_{yn} \) where \( n = 0, 1, 2, \ldots \). The index \( n \) will be reserved for the natural response of the system for the \( n^{th} \) mode. If the \( m^{th} \) set of source fields (\( m^{th} \) source mode) resonates with the \( n^{th} \) set of natural response fields of the system where \( m \neq n \), source harmonic resonance or harmonic resonance, for short, occurs. If \( m = n \) the source mode and the same natural response mode of the system resonates. This resonance interaction is denoted by as fundamental resonance. Here fundamental and harmonic are loosely used to describe the direct modal number relation between the system response and the beam characteristics.

4.1.2 Forced Response “Harmonic Resonance” (Particular Solution, \( m \neq n \))

Consider Eq. (2.62e). The source term is a function of \( \cos(b_m y) \) and \( \sin(b_m y) \) as expressed in Eq. (4-1). Using the method of undetermined coefficients, the guessed force solution is

\[
\tilde{H}_{zilf}(\beta_x, y, z) = \tilde{H}_{zilfm}(\beta_x, z) \cos(b_m y) + \tilde{H}_{zilfm}(\beta_x, z) \sin(b_m y)
\]  

(4-3)

where \( b_m \) is a particular eigenvalue of the system. Substituting in Eq. (2.62e) and invoking orthogonality yields

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\[
\left[ \frac{\partial^2}{\partial y^2} + \left( \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \epsilon_{eff} - \beta_x^2 \right) \right] [\overline{H}_{zefm}(\beta_x, z) \cos(b_m y)
+ \overline{H}_{zoifm}(\beta_x, z) \sin(b_m y)]
= j\beta_x [J_{eibm}(\beta_x, z) \cos(b_m y) + J_{oibm}(\beta_x, z) \sin(b_m y)]
\]

Performing the operation on y and once again invoking orthogonality Eq. (4-4) yields

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \epsilon_{eff} - \beta_x^2 - b_m^2 \right] \overline{H}_{zefm}(\beta_x, z) = j\beta_x J_{eibm}(\beta_x, z) \quad (4-5)
\]
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \epsilon_{eff} - \beta_x^2 - b_m^2 \right] \overline{H}_{zoifm}(\beta_x, z) = j\beta_x J_{oibm}(\beta_x, z) \quad (4-6)
\]

Consequently,

\[
\overline{H}_{zif}(\beta_x, y, z) = \overline{H}_{zefm}(\beta_x, z) \cos(b_m y) + \overline{H}_{zoifm}(\beta_x, z) \sin(b_m y)
\]

where $\overline{H}_{zefm}(\beta_x, z)$ and $\overline{H}_{zoifm}(\beta_x, z)$ are determined by Eqs. (4-5) and (4-6) respectively subject to boundary conditions with z.

4.1.3 The Correction Term

Consider Eq. (2.63e). Since the governing equation is linear, the corrected field is subdivided into two components. Let $\overline{H}_{zifh}$ represent the field driven by the effective source term resulting from the homogeneous fields interacting with the medium’s inhomogeneity. In a similar manner let $\overline{H}_{ziff}$ represent the field driven by the effective source term due to the field generated by the physical source terms interacting with the medium’s inhomogeneity. Therefore,

\[
\overline{H}_{zif}(\beta_x, y, z) = \overline{H}_{zifh}(\beta_x, y, z) + \overline{H}_{ziff}(\beta_x, y, z) \quad (4-7)
\]
Based on the method of superposition, substituting Eq. (4-7) into Eq. (2.63e) yields

\[
\begin{bmatrix}
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} \beta_x^2 \\
\end{bmatrix}
\bar{H}_{zifh}(\beta_x, y, z)
= - \frac{\omega \mu_0}{j2\pi} \left[ \sigma_{\tau l}(\beta_x) \odot \beta_x \bar{H}_{zih}(\beta_x, y, z) \right] 
\tag{4-8a}
\]

\[
\begin{bmatrix}
\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} \beta_x^2 \\
\end{bmatrix}
\bar{H}_{ziff}(\beta_x, y, z)
= - \frac{\omega \mu_0}{j2\pi} \left[ \sigma_{\tau l}(\beta_x) \odot \beta_x \bar{H}_{zih}(\beta_x, y, z) \right] 
\tag{4.8b}
\]

It is noted that \( \bar{H}_{zih} \) satisfying Eq. (2.61e) has the same form of solution as the homogeneous form of Eq. (2.63e). Since \( \bar{H}_{zih} \) is the solution to the left-hand side of the Eq. (2.63e), a repeated form of solution is anticipated. Let,

\[
\bar{H}_{zifh} = \gamma(\bar{H}_{zifh1} e^{j\beta_y y} + \bar{H}_{zifh2} e^{-j\beta_y y}) 
\tag{4-9}
\]

With the aid of Eq. (4-9), Eq. (4-8a) becomes

\[
j2\beta_y \left( \bar{H}_{zifh1} e^{j\beta_y y} - \bar{H}_{zifh2} e^{-j\beta_y y} \right) - \beta_x^2 y \left[ \bar{H}_{zifh1} e^{j\beta_y y} + \bar{H}_{zifh2} e^{-j\beta_y y} \right]
+ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} \beta_x^2 \right] \left[ \gamma(\bar{H}_{zifh1} e^{j\beta_y y} + \bar{H}_{zifh2} e^{-j\beta_y y}) \right]
\]

\[
= - \frac{\omega \mu_0}{j2\pi} \left[ \sigma_{\tau l}(\beta_x) \odot \beta_x \bar{H}_{zih1} \right] e^{j\beta_y y} - \frac{\omega \mu_0}{j2\pi} \left[ \sigma_{\tau l}(\beta_x) \odot \beta_x \bar{H}_{zih2} \right] e^{-j\beta_y y}
\]

Comparing terms, we get

\[
\bar{H}_{zifh1}(\beta_x, z) = \frac{\omega \mu_0}{4\pi \beta_y} \left[ \sigma_{\tau l}(\beta_x) \odot \beta_x \bar{H}_{zih1}(\beta_x, z) \right] 
\tag{4-10a}
\]
\[
\overline{H}_{zifh2}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_y} [\sigma_{Tl}(\beta_x) \odot^{\beta_x} \overline{H}_{zih2}(\beta_x, z)]
\]  \hspace{1cm} (4.10b)

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] \overline{H}_{zifhp}(\beta_x, z) = 0
\] \hspace{1cm} (4.10c,d)

where \( p = 1,2 \). The last expression is the homogeneous solution of the relation. This has already been accounted for. If we did consider the contribution, all homogeneous coefficients would be added together yielding an equivalent pair of constants in \( y \) that would be constrained by the same boundary conditions in \( y \). Thus,

\[
\overline{H}_{ziff} = y \frac{\omega \mu_0}{4\pi \beta_y} \left\{ \left[ \sigma_{Tl}(\beta_x) \odot^{\beta_x} \overline{H}_{zih1} \right] e^{j\beta_y y} - \left[ \sigma_{Tl}(\beta_x) \odot^{\beta_x} \overline{H}_{zih2} \right] e^{-j\beta_y y} \right\}
\] \hspace{1cm} (4-11)

Now consider Eq. (4.8b). Let

\[
\overline{H}_{ziff} = \overline{H}_{ziff} \cos(b_m y) + \overline{H}_{ziff} \sin(b_m y)
\] \hspace{1cm} (4-12)

Substituting Eq. (4-12) into Eq. (4.8b) rearranging and comparing terms based on the method of substitution yields

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - b_m^2 \right] \overline{H}_{ziff}(\beta_x, y, z)
\] \hspace{1cm} (4-13)

\[
= -\frac{\omega \mu_0}{j2\pi} \left[ \sigma_{Tl}(\beta_x) \odot^{\beta_x} \overline{H}_{ziff}(\beta_x, y, z) \right]
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - b_m^2 \right] \overline{H}_{ziff}(\beta_x, y, z)
\] \hspace{1cm} (4-14)

\[
= -\frac{\omega \mu_0}{j2\pi} \left[ \sigma_{Tl}(\beta_x) \odot^{\beta_x} \overline{H}_{ziff}(\beta_x, y, z) \right]
\]

Combining Eqs. (4-2), (4-3), (4-7), (4-9), (4-12) and (2-60a) yields
\[ H_{z1}(\beta_x, y, z) = \tilde{H}_{z1h1}(\beta_x, z)e^{j\beta_y y} + \tilde{H}_{z1h2}(\beta_x, z)e^{-j\beta_y y} \]
\[ + \tilde{H}_{z1fm}(\beta_x, z)\cos(b_m y) + \tilde{H}_{z1ofm}(\beta_x, z)\sin(b_m y) \]
\[ + H(\tilde{H}_{z1h1}(\beta_x, z)e^{j\beta_y y} + \tilde{H}_{z1h2}(\beta_x, z)e^{-j\beta_y y}) \]
\[ + \tilde{H}_{z1ffm}(\beta_x, z)\cos(b_m y) + \tilde{H}_{z1ofm}(\beta_x, z)\sin(b_m y) \]

based on the constraints

\[ \frac{\partial^2}{\partial z^2} + \omega^2\mu_0\varepsilon_{eff} - \beta_x^2 - b_m^2 \tilde{H}_{z1h1}(\beta_x, z) = j\beta_x \tau_{kibm}(\beta_x, z) \] (4-16)

\[ \tilde{H}_{z1hp}(\beta_x, z) = (-1)^p+1 \frac{\omega\mu_0}{4\pi\beta_y} [\sigma_t(\beta_x) \otimes \beta_x \tilde{H}_{z1hp}(\beta_x, z)] \] (4-17)

\[ \frac{\partial^2}{\partial z^2} + \omega^2\mu_0\varepsilon_{eff} - \beta_x^2 - b_m^2 \tilde{H}_{z1ffm}(\beta_x, z) = -\frac{\omega\mu_0}{j2\pi} [\sigma_t(\beta_x) \otimes \beta_x \tilde{H}_{z1ffm}] \] (4-18)

where \( k = o, e \) and \( p = 1, 2 \).

Boundary condition requires the tangential component of the electric field to vanish at a perfect conductor, for the TE mode, \( E_{xi} \) is tangential to the plate surface. With the aid of Eq. (4-15), Eq. (2-47) for the TE case becomes
\[
\left[ \frac{\partial^2}{\partial z^2} + \frac{\omega^2 \mu_0}{2\pi} \varepsilon_{t}(\beta_x, \omega, \odot \beta_x) \right] E_{xi}(\beta_x, y, z) \\
= -j \omega \mu_0 (j \beta_y + \frac{\omega \mu_0}{4 \pi \beta_y} [\sigma_{Ti}(\beta_x, \odot \beta_x)] \bar{H}_{zih1}(\beta_x, z) e^{j \beta_y y} \\
- (j \beta_y + \frac{\omega \mu_0}{4 \pi \beta_y} [\sigma_{Ti}(\beta_x, \odot \beta_x)] \bar{H}_{zih2}(\beta_x, z) e^{-j \beta_y y} \\
+ j \beta_y \frac{\omega \mu_0}{4 \pi \beta_y} [\sigma_{Ti}(\beta_x, \odot \beta_x)] (\bar{H}_{zih1}(\beta_x, z) e^{j \beta_y y} \\
+ \bar{H}_{zih2}(\beta_x, z) e^{-j \beta_y y}) \\
- b_m \{ \bar{H}_{zeimf}(\beta_x, z) + \bar{H}_{zoeim}(\beta_x, z) \} \sin(b_m y) \\
+ b_m \{ \bar{H}_{zeimf}(\beta_x, z) + \bar{H}_{zoeim}(\beta_x, z) \} \cos(b_m y) \\
\] (4-19)

For consistency in solutions, \( E_{xi} \) must have the same form of solution in terms of the \( y \)-variation of \( H_{zi} \). Thus, let

\[
E_{xi}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_{xi}(\beta_x, y, z) e^{-j \beta_x x} d\beta_x \\
(4-20)
\]

\[
E_{xi}(\beta_x, y, z) = \left[ \bar{E}_{zih1}(\beta_x, z) + \frac{1}{j \beta_y} \bar{E}_{zihf1}(\beta_x, z) \right] e^{j \beta_y y} \\
+ \left[ \bar{E}_{zih2}(\beta_x, z) - \frac{1}{j \beta_y} \bar{E}_{zihf2}(\beta_x, z) \right] e^{-j \beta_y y} \\
+ y \bar{E}_{zihf1}(\beta_x, z) e^{j \beta_y y} + y \bar{E}_{zihf2}(\beta_x, z) e^{-j \beta_y y} \\
+ [\bar{E}_{zeimf}(\beta_x, z) + \bar{E}_{zoeim}(\beta_x, z)] \cos(b_m y) + [\bar{E}_{zeimf}(\beta_x, z) + \bar{E}_{zoeim}(\beta_x, z)] \sin(b_m y) \\
(4-21)
\]

where \( \bar{E}_{zihf} \) is proportional to \( \bar{E}_{zih} \) through a convolution where \( p = 1, 2 \).

For the TE mode, the boundary condition is \( E_{xi}(x, y = 0^+, z) = 0 \) for all \( x \) and \( z \). Thus, based on Eqs. (4-20) and (4-21)
\[
\left[ \bar{\mathcal{E}}_{xih2}(\beta_x, z) - \frac{1}{j\beta_y} \bar{\mathcal{E}}_{xifh2}(\beta_x, z) \right]
\]

\[
= - \left[ \bar{\mathcal{E}}_{xih1}(\beta_x, z) + \frac{1}{j\beta_y} \bar{\mathcal{E}}_{xifh1}(\beta_x, z) \right] - [\bar{\mathcal{E}}_{xeif}(\beta_x, z)]
\]

\[
+ \bar{\mathcal{E}}_{xeiff}(\beta_x, z)
\]

By analogy the boundary condition at \( y = d^- \), \( E_x(x, y = d^-, z) = 0 \), requires

\[
\left[ \bar{\mathcal{E}}_{xih1}(\beta_x, z) + \frac{1}{j\beta_y} \bar{\mathcal{E}}_{xifh1}(\beta_x, z) \right] e^{j\beta_y d}
\]

\[
+ \left[ \bar{\mathcal{E}}_{xih2}(\beta_x, z) - \frac{1}{j\beta_y} \bar{\mathcal{E}}_{xifh2}(\beta_x, z) \right] e^{-j\beta_y d}
\]

\[
+ d\bar{\mathcal{E}}_{xifh1}(\beta_x, z)e^{j\beta_y d} + d\bar{\mathcal{E}}_{xih2}(\beta_x, z)e^{-j\beta_y d} + [\bar{\mathcal{E}}_{xeif}(\beta_x, z)]
\]

\[
+ \bar{\mathcal{E}}_{xeiff}(\beta_x, z) \cos(b_m d) + [\bar{\mathcal{E}}_{xoif}(\beta_x, z)]
\]

\[
+ d\left[ \bar{\mathcal{E}}_{xifh1}(\beta_x, z) e^{j\beta_y d} + \bar{\mathcal{E}}_{xih2}(\beta_x, z) e^{-j\beta_y d} \right] = 0
\]

Substituting Eq. (4-22) into (4-23) and rearranging yields

\[
j2 \left[ \bar{\mathcal{E}}_{xih1}(\beta_x, z) + \frac{1}{j\beta_y} \bar{\mathcal{E}}_{xifh1}(\beta_x, z) \right] \sin(\beta_y d)
\]

\[
- \left[ \bar{\mathcal{E}}_{xeif}(\beta_x, z) + \bar{\mathcal{E}}_{xeiff}(\beta_x, z) \right] e^{-j\beta_y d} - \cos(b_m d)
\]

\[
+ \left[ \bar{\mathcal{E}}_{xoif}(\beta_x, z) + \bar{\mathcal{E}}_{xoiff}(\beta_x, z) \right] \sin(b_m d)
\]

\[
+ d\left[ \bar{\mathcal{E}}_{xifh1}(\beta_x, z) e^{j\beta_y d} + \bar{\mathcal{E}}_{xih2}(\beta_x, z) e^{-j\beta_y d} \right] = 0
\]

This condition can only be satisfied for harmonic wave interactions \((m \neq n)\) when

\[
\beta_y = \frac{m\pi}{d} \triangleq \beta_m
\]

\[
b_m = \frac{m\pi}{d} \triangleq \beta_m
\]
\[ \tilde{E}_{xifh1} = \tilde{E}_{xifh2} = 0 \]  

(4-26)

under the conditions \( m \) and \( n \) are both integers and \( m + n \) equals and even number. Further \( \tilde{E}_{xifh1} \) and \( \tilde{E}_{xifh2} \) must be set equal to zero since the source effects leading to these field contributions are not supported by the waveguide as propagating waves. If \( m \) and \( n \) are not both even or not both odd, then \( \tilde{E}_{xif} \) and \( \tilde{E}_{xeiff} \) must be set to zero since, the source contribution that lead to these fields are not supported as propagating wave in the waveguide.

With the aid of Eq. (2.25b)

\[ \tilde{E}_{xih2}(\beta_x, z) = -\tilde{E}_{xih1}(\beta_x, z) - [\tilde{E}_{xif} (\beta_x, z) + \tilde{E}_{xeiff} (\beta_x, z)] \]  

(4-27)

Therefore, Eq. (4-21) can be rewritten as

\[
E_{x}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \{j2\tilde{E}_{xihn1}(\beta_x, z) \sin(\beta_{yn}y) \\
- \bar{\Gamma}_{mn}[\tilde{E}_{xeimf}(\beta_x, z) + \tilde{E}_{xeimff}(\beta_x, z)]e^{-j\beta_{yn}y} \\
- \cos(\beta_{ym}y) \\
+ \delta_{nm}[\tilde{E}_{xoimf}(\beta_x, z) + \tilde{E}_{xoimff}(\beta_x, z)] \sin(b_my) \}
\]

(4-28)

where \( \delta_{mn} \), \( \delta'_{nm} \), and \( \Gamma_{mn} \) are defined by Eqs. (1.8a-c) respectively. Both \( \tilde{E}_{xifh1} \) and \( \tilde{E}_{xifh2} \) are omitted since boundary conditions cannot be satisfied for these source contributions.

As anticipated in Section 1.2, the boundary condition can be solved for any \( \beta_y \neq \beta_{yn} \). Energy in these waves is of little value since the adjoining empty waveguide can only support
eigensolutions. Even so, it may be possible that coupling may result at a \( z = \) constant interface between the empty and filled waveguide. Our attention is focused around harmonic and fundamental wave/beam-plasma interaction.

In Eq. (2-47) it is observed that \( E_{xi} \propto \frac{\partial}{\partial y} H_{zi} \). Substituting Eqs. (4-15) and (4-28) into (2-47) and comparing terms on both sides we have, \( \tilde{E}_{xihp} \propto (-1)^{p+1} j \beta_{yn} \tilde{H}_{zihp} \), \( \tilde{E}_{xihp} \propto (-1)^{p+1} j \beta_{y} \tilde{H}_{zinhp} \), \( \tilde{H}_{zinhp} = 0 \), \( \tilde{E}_{xelf} \propto b_{m} \tilde{H}_{zofil}, \tilde{E}_{xofil} \propto -b_{m} \tilde{H}_{zelf}, \tilde{E}_{xelf} \propto b_{m} \tilde{H}_{zofil} \), \( \tilde{E}_{xofil} \propto -b_{m} \tilde{H}_{zelf} \), where \( \beta_{y} = \beta_{yn} = \frac{n \pi}{d}, b_{m} = \beta_{ym} = \frac{m \pi}{d} \) and \( p = 1,2 \) and \( k = e, o \).

The homogeneous terms and forcing terms are exclusively associated with the subscript \( n \) and \( m \) respectively. Consequently, Eq. (4-27) with the corresponding substitutions \( \tilde{E}_{xihp} = (-1)^{p+1} j \beta_{yn} \tilde{H}_{zihnp} \) becomes

\[
\tilde{H}_{zihn2}(\beta_{x}, z) = \tilde{H}_{zihn1}(\beta_{x}, z) + \delta_{0n} \frac{\beta_{ym}}{j \beta_{yn}} [\tilde{H}_{zoimf}(\beta_{x}, z) + \tilde{H}_{zoimf}(\beta_{x}, z)] \quad (4-29)
\]

\[
\tilde{H}_{zinfh1}(\beta_{x}, z) = \tilde{H}_{zinfh2}(\beta_{x}, z) = 0 \quad (4-30)
\]

where \( \delta_{mn} \) is defined in Eq. (1.18b).

Using the connecting relations and Eqs. (4-29) and (4-30), Eq. (4-15) can be rewritten as,
\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ 2H_{zihn1}(\beta_x, z) \cos(\beta_{yn}y) \right. \\
+ \delta_{nm} \left( \bar{H}_{zeimf}(\beta_x, z) + \bar{H}_{zeimff}(\beta_x, z) \right) \cos(\beta_{ym}y) \\
+ \bar{\Gamma}_{mn} \left( \bar{H}_{zoimf}(\beta_x, z) + \bar{H}_{zoimff}(\beta_x, z) \right) \left[ \frac{\beta_{ym}}{j\beta_{yn}} e^{-j\beta_{yn}y} \right. \\
\left. + \sin(\beta_{ym}y) \right] \right\} \] (4-31a)

In summary, Harmonic Resonance

\[ H_{zi}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{zi}(\beta_x, y, z) e^{-j\beta_{x}x} d\beta_x \]

where

\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ 2H_{zihn1}(\beta_x, z) \cos(\beta_{yn}y) \right. \\
+ \delta_{nm} \left( \bar{H}_{zeimf}(\beta_x, z) + \bar{H}_{zeimff}(\beta_x, z) \right) \cos(\beta_{ym}y) \\
+ \bar{\Gamma}_{mn} \left( \bar{H}_{zoimf}(\beta_x, z) + \bar{H}_{zoimff}(\beta_x, z) \right) \left[ \frac{\beta_{ym}}{j\beta_{yn}} e^{-j\beta_{yn}y} \right. \\
\left. + \sin(\beta_{ym}y) \right] \right\} \] (4-31a)

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff1} - \beta_x^2 - \beta_{ym}^2 \right] H_{zikmff}(\beta_x, \beta_{ym}, z) = j\beta_{x}l_{kibm}(\beta_x, \beta_{ym}, z) \] (4-31b)
Comparing terms yield

\[
\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_{ym}^2 \bar{H}_{zilmf}(\beta_x, z) = -\frac{\omega \mu_0}{j\pi} \left[ \sigma_{T1}(\beta_x) \odot \beta_x \bar{H}_{zilmf}(\beta_x, z) \right]
\] (4-31c)

\[J_{ylb}(\beta_x, y, z) = \sum_{m=0}^{\infty} \left[ J_{elbm}(\beta_x, z) \cos(\beta_{ym} y) + J_{oibm}(\beta_x, z) \sin(\beta_{ym} y) \right]
\] (4-31d)

where \( k = e, o \). \( \bar{H}_{zilm1} \) and \( \bar{H}_{zilm2} \) are unknowns.

4.1.4 Forced response, “Fundamental Resonance” (Particular Solution; \( m = n \))

In this case, fundamental resonance between the forced response and the natural response of the medium is considered. Equation (2.62e) is evaluated with \( m = n \). If \( b_n = \beta_{yn} \) (fundamental resonance), repeated solutions exist. Hence, using the method of undetermined coefficients, let

\[
\bar{H}_{zif} = y(\bar{H}_{zef}(\beta_x, z) \cos(b_n y) + \bar{H}_{zoi}(\beta_x, z) \sin(b_n y))
\] (4-32a)

\[
\frac{\partial}{\partial y} \bar{H}_{zif}(\beta_x, z) = \bar{H}_{zef}(\beta_x, z) \cos(b_n y) + \bar{H}_{zoi}(\beta_x, z) \sin(b_n y) - b_n y (\bar{H}_{zef}(\beta_x, z) \sin(b_n y) - \bar{H}_{zoi}(\beta_x, z) \cos(b_n y))
\] (4.32b)

\[
\frac{\partial^2}{\partial y^2} \bar{H}_{zif}(\beta_x, z) = -b_n^2 y (\bar{H}_{zef}(\beta_x, z) \cos(b_n y) + \bar{H}_{zoi}(\beta_x, z) \sin(b_n y)) - 2b_n \bar{H}_{zef}(\beta_x, z) \sin(b_n y) - \bar{H}_{zoi}(\beta_x, z) \cos(b_n y))
\] (4.32c)

Substituting Eqs. (4-32a) and (4.32c) into Eq. (2.62e) yields,

\[
-b_n^2 y (\bar{H}_{zef}(\beta_x, z) \cos(b_n y) + \bar{H}_{zoi}(\beta_x, z) \sin(b_n y))
\]

\[
-2b_n \left[ \bar{H}_{zef}(\beta_x, z) \sin(b_n y) - \bar{H}_{zoi}(\beta_x, z) \cos(b_n y) \right]
\]

\[
+ y \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \right] \left[ \bar{H}_{zef}(\beta_x, z) \cos(b_n y) + \bar{H}_{zoi}(\beta_x, z) \sin(b_n y) \right]
\]

\[= j \beta_x \left[ J_{elbm}(\beta_x, z) \cos(b_n y) + J_{oibm}(\beta_x, z) \sin(b_n y) \right].
\]

Comparing terms yield
\[ \tilde{H}_{zeif}(\beta_x, z) = -\frac{j\beta_x}{2b_n} J_{olbn}(\beta_x, z) \]  
(4.33a)

\[ \tilde{H}_{zoif}(\beta_x, z) = \frac{j\beta_x}{2b_n} J_{elbn}(\beta_x, z) \]  
(4.33b)

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - b_n^2 \right] \tilde{H}_{zeif}(\beta_x, z) = 0 \]  
(4.33c)

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - b_n^2 \right] \tilde{H}_{zoif}(\beta_x, z) = 0 \]  
(4.33d)

If \( b_n = \beta_{yn} \), then the last two equations, (4.33c) and (4.33d) are identical to the natural solution of Eq. (2.62e) or equivalently Eq. (2.61e). Therefore,

\[ \tilde{H}_{zif} = y \left[ \tilde{H}_{zeif}(\beta_x, z) \cos(b_n y) + \tilde{H}_{zoif}(\beta_x, z) \sin(b_n y) \right] \]  
(4.34)

Now consider Eq. (2.63e). Let

\[ \tilde{H}_{zif} = \tilde{H}_{zifh} + \tilde{H}_{ziff} \]  
(4.35)

Then substituting Eq. (4.35) into Eq. (2.63e) with \( m = n \) and based on the physics of the problem

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \right] \tilde{H}_{zifh}(\beta_x, y, z) = 0 \]  
(4.36a)

\[ = -\frac{\omega \mu_0}{j2\pi} \sigma_{Ti}(\beta_x) \bigodot \tilde{H}_{zifh}(\beta_x, y, z) \]

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \right] \tilde{H}_{ziff}(\beta_x, y, z) = 0 \]  
(4.36b)

\[ = -\frac{\omega \mu_0}{j2\pi} \sigma_{Ti}(\beta_x) \bigodot \tilde{H}_{ziff}(\beta_x, y, z) \]

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] \tilde{H}_{zifhp}(\beta_x, z) = 0 \]  
(4.36c)

where \( p = 1, 2 \).
Equations (4-8a) and (4-36a) are identical and have the same form of source term as the homogeneous solution. Since this solution does not result in a propagating solution in the “Harmonic Resonance” case, the effects may be treated here without concern that its contributions are counted twice. Therefore, by analogy, the solution to Eq. (4-36a) is

\[
\overline{H}_{zif}(\beta_x, y, z) = y\overline{H}_{zifh1}(\beta_x, z)e^{j\beta_2 y} + y\overline{H}_{zifh2}(\beta_x, z)e^{-j\beta_2 y}
\]  \hspace{1cm} (4.37a)

\[
\overline{H}_{zifh1}(\beta_x, z) = \frac{\omega \mu_0}{4\pi \beta_y} \sigma_{T1}(\beta_x) \odot \beta_x \overline{H}_{zilh1}(\beta_x, z)
\]  \hspace{1cm} (4.37b)

\[
\overline{H}_{zifh2}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_y} \sigma_{T1}(\beta_x) \odot \beta_x \overline{H}_{zilh2}(\beta_x, z)
\]  \hspace{1cm} (4.37c)

Now consider Eq. (4.36b) with the aid of Eq. (4.34). For the sake of simplicity, let

\[
K_k = -\frac{\omega \mu_0}{j2\pi} \left[ \sigma_{T1}(\beta_x) \odot \beta_x \overline{H}_{zilf}(\beta_x, z) \right] = K_k(\beta_x, z)
\]  \hspace{1cm} (4.38)

where \( k = e, o \). Therefore, Eq. (4-36a) can be re-written as

\[
\left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \right] \overline{H}_{ziff} = K_e y \cos(\beta_y y) + K_0 y \sin(\beta_y y)
\]  \hspace{1cm} (4.39)

To solve Eq. (4.39) the method of undetermined coefficients is used. Let

\[
\overline{H}_{ziff} = (A_1 y^2 + A_2 y + A_3) \cos(\beta_y y) + (B_1 y^2 + B_2 y + B_3) \sin(\beta_y y)
\]  \hspace{1cm} (4.40a)

where \( A_q = A_q(\beta_x, z) \) and \( B_q = B_q(\beta_x, z) \), \( q = 1, 2, 3 \). Then,

\[
\frac{\partial}{\partial y} \overline{H}_{ziff} = (2A_1 y + A_2) \cos(\beta_y y) + (2B_1 y + B_2) \sin(\beta_y y)
\]  \hspace{1cm} (4.40b)

\[
-\beta_y (A_1 y^2 + A_2 y + A_3) \sin(\beta_y y)
\]  \hspace{1cm} (4.40b)

\[
+\beta_y (B_1 y^2 + B_2 y + B_3) \cos(\beta_y y)
\]  \hspace{1cm} (4.40b)
\[ \frac{\partial^2}{\partial y^2} \bar{H}_{zff} = 2A_1 \cos(\beta_y y) + 2B_1 \sin(\beta_y y) - 2\beta_y (2A_1 y + A_2) \sin(\beta_y y) \]  
\[ + 2\beta_y (2B_1 y + B_2) \cos(\beta_y y) \]  
\[ - \beta_y^2 (A_1 y^2 + A_2 y + A_3) \cos(\beta_y y) \]  
\[ - \beta_y^2 (B_1 y^2 + B_2 y + B_3) \sin(\beta_y y) \]  

Substituting Eqs. (4.40a) and (4.40c) into Eq. (4.39) yields

\[
2A_1 \cos(\beta_y y) + 2B_1 \sin(\beta_y y) - 2\beta_y (2A_1 y + A_2) \sin(\beta_y y) + 2\beta_y (2B_1 y + B_2) \cos(\beta_y y) - \beta_y^2 (A_1 y^2 + A_2 y + A_3) \cos(\beta_y y) - \beta_y^2 (B_1 y^2 + B_2 y + B_3) \sin(\beta_y y) + \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff \parallel} - \beta_x^2 \right] \left[ (A_1 y^2 + A_2 y + A_3) \cos(\beta_y y) + (B_1 y^2 + B_2 y + B_3) \sin(\beta_y y) \right] = K_e y \cos(\beta_y y) + K_o y \sin(\beta_y y)
\]

Comparing terms on both sides of the above equation, based on the concept of linearly independent set, the following equations are obtained

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff \parallel} - \beta_x^2 - \beta_y^2 \right] A_1 = 0 \]  
\[ (4.41a) \]  
\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff \parallel} - \beta_x^2 - \beta_y^2 \right] B_1 = 0 \]  
\[ (4.41b) \]  
\[ 4B_1 \beta_y + \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff \parallel} - \beta_x^2 - \beta_y^2 \right] A_2 = K_e \]  
\[ (4.41c) \]  
\[ -4A_1 \beta_y + \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff \parallel} - \beta_x^2 - \beta_y^2 \right] B_2 = K_o \]  
\[ (4.41d) \]
Since the fundamental resonance case is only being considered in this section, $\beta_y = \beta_{yn} = \frac{n\pi}{d}$. Here $\beta_{yn}$ is the eigenvalue solution for homogeneous case subject to the boundary condition. Therefore, since

$$\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] F_j = 0$$

where $F = A_j$ or $B_j$ and $j = 1, 2, 3$. Then, Eqs. (4.41a) – (4.41f) imply

$$B_1 = \frac{K_e}{4\beta_y}$$  \hspace{1cm} (4.42a)

$$A_1 = -\frac{K_o}{4\beta_y}$$  \hspace{1cm} (4.42b)

$$B_2 = -\frac{A_1}{\beta_y} = \frac{K_o}{4\beta_y^2}$$  \hspace{1cm} (4.42c)

$$A_2 = \frac{B_1}{\beta_y} = \frac{K_e}{4\beta_y^2}$$  \hspace{1cm} (4.42d)

where $A_3$ and $B_3$ are free parameters, so let $A_3 = B_3 = 0$. Based on Eqs. (4.42a)-(4.42d), the even and odd sinusoidal function solutions of $\vec{H}_{ziff}$ must coexist for the solution to be meaningful. Therefore,

$$\vec{H}_{ziff} = \vec{H}_{zeiff} + \vec{H}_{zoiff}$$

$$\vec{H}_{ziff} = (A_1 y^2 + A_2 y) \cos(\beta_y y) + (B_1 y^2 + B_2 y) \sin(\beta_y y)$$  \hspace{1cm} (4.43)
Consequently, for $\vec{H}_{ziff}$ to be a wave solution for fundamental resonance, all boundary conditions directly or indirectly associated with $\vec{H}_{ziff}$ must be satisfied simultaneously. Therefore, the solution to Eq. (4-43) becomes

$$\vec{H}_{ziff} = \left( -\frac{K_o}{4\beta_y}y^2 + \frac{K_e}{4\beta_y^2}y \right) \cos(\beta_y y) + \left( \frac{K_e}{4\beta_y}y^2 + \frac{K_o}{4\beta_y^2}y \right) \sin(\beta_y y) \quad (4.44)$$

Therefore, Eq. (2-60a) becomes

$$H_zl(\beta_x, y, z) = \vec{H}_{zih1}e^{j\beta_y y} + \vec{H}_{zih2}e^{-j\beta_y y} + y\vec{H}_{zeif} \cos(\beta_y y)$$
$$+ y\vec{H}_{zoif} \sin(\beta_y y) + \vec{H}_{zlhf1}y e^{j\beta_y y} + \vec{H}_{zlhf2}y e^{-j\beta_y y}$$
$$+ \left( -\frac{K_o}{4\beta_y}y^2 + \frac{K_e}{4\beta_y^2}y \right) \cos(\beta_y y) \quad (4.45)$$
$$+ \left( \frac{K_e}{4\beta_y}y^2 + \frac{K_o}{4\beta_y^2}y \right) \sin(\beta_y y)$$

where

$$\vec{H}_{zeif}(\beta_x, z) = -\frac{j\beta_x}{2\beta_y}J_{1\beta yn}(\beta_x, z) \quad (4.46a)$$
$$\vec{H}_{zoif}(\beta_x, z) = \frac{j\beta_x}{2\beta_y}J_{1\beta yn}(\beta_x, z) \quad (4.46b)$$
$$\vec{H}_{zilhp}(\beta_x, z) = (-1)^{p+1} \frac{\omega \mu_0}{4\pi \beta_y} \sigma_{T1}(\beta_x) \otimes \beta_x \vec{H}_{zihp}(\beta_x, z) \quad (4.46c, d)$$
$$K_k(\beta_x, z) = -\frac{\omega \mu_0}{j2\pi} \left[ \sigma_{T1}(\beta_x) \otimes \beta_x \vec{H}_{zklf}(\beta_x, z) \right] \quad (4.46e, f)$$

where $k = e, o$ all the coefficients are independent of $y$. The above equations are subject to the following constraints.
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] \tilde{H}_{zkif}(\beta_x, z) = 0 \quad (4.46g,h)
\]
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] K_k(\beta_x, z) = 0 \quad (4.46i,j)
\]
\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] \tilde{H}_{zif hp}(\beta_x, z) = 0 \quad (4.46k,l)
\]

where \(k = e, o\) and \(p = 1, 2\).

Boundary conditions need to be addressed on the \(x\)-component of the electric field \(E_{xli}\), as it is the only electric field component tangential to the parallel plates at \(y = 0\) and \(y = d\). Considering only the TE mode contribution on the right-hand side of Eq. (2-47), the first term on the right hand side can be expanded using Eq. (4-45) to yield
\[-j\omega \mu_0 \frac{\partial}{\partial y} H_{z1}(\beta_x, y, z) \]

\[= -j\omega \mu_0 \left\{ j\beta_y H_{zih1}(\beta_x, z) e^{j\beta_y y} - j\beta_y H_{zih2}(\beta_x, z) e^{-j\beta_y y} + \beta_y \left[ \frac{1}{\beta_y} \cos(\beta_y y) - y\sin(\beta_y y) \right] H_{2el} (\beta_x, z) \right. \]

\[+ \beta_y \left[ \frac{1}{\beta_y} \sin(\beta_y y) + y\cos(\beta_y y) \right] H_{2ol} (\beta_x, z) + j\beta_y \left[ \frac{1}{j\beta_y} + y \right] e^{j\beta_y y} H_{zifh1}(\beta_x, z) \]

\[- j\beta_y \left[ - \frac{1}{j\beta_y} + y \right] e^{-j\beta_y y} H_{zifh2}(\beta_x, z) + \beta_y \left[ \left( - \frac{K_o}{2\beta_y^2} y + \frac{K_e}{4\beta_y^3} \right) \cos(\beta_y y) \right. \]

\[\left. - \left( - \frac{K_o}{4\beta_y^2} y^2 + \frac{K_e}{4\beta_y^3} y \right) \sin(\beta_y y) \right] + \beta_y \left[ \left( \frac{K_e}{2\beta_y^2} y + \frac{K_o}{4\beta_y^3} \right) \sin(\beta_y y) + \left( \frac{K_e}{4\beta_y^2} y^2 + \frac{K_o}{4\beta_y^3} y \right) \cos(\beta_y y) \right\} \]

For functional consistency, the electric field has the form
\[ E_{xi}(\beta_x, y, z) = E_{xih1}(\beta_x, z)e^{i\beta_y y} + E_{xih2}(\beta_x, z)e^{-i\beta_y y} \]
\[ + \tilde{E}_{xelf}(\beta_x, z)[\cos(\beta_y y) - \beta_y y\sin(\beta_y y)] \]
\[ + \tilde{E}_{xolf}(\beta_x, z)[\sin(\beta_y y) + \beta_y y\cos(\beta_y y)] \]
\[ + [e^{i\beta_y y} + j\beta_y ye^{i\beta_y y}]\tilde{E}_{xih1}(\beta_x, z) \]
\[ + [e^{-i\beta_y y} - j\beta_y ye^{-i\beta_y y}]\tilde{E}_{xih2}(\beta_x, z) \]
\[ + (\tilde{E}_{A1} + \tilde{E}_{A2})\cos(\beta_y y) + (\tilde{E}_{B1} + \tilde{E}_{B2})\sin(\beta_y y) \]
\[ + (\tilde{E}_{A3} \gamma^2 + \tilde{E}_{A4} y)\sin(\beta_y y) + (\tilde{E}_{B3} \gamma^2 + \tilde{E}_{B4} y)\cos(\beta_y y) \]

Boundary conditions on all eight terms containing \( \tilde{E}_{Aj} \) and \( \tilde{E}_{Bj} \) \( j = 1, 2, 3, 4 \) must be satisfied simultaneously for \( \tilde{H}_{z_{iff}} \) source terms to contribute to the wave propagating solution for the fundamental resonance case.

Comparing Eq. (4-47) with Eq. (4-48a), the electric field amplitude are proportionally related to \( z \)-component of the magnetic field as

\[ \tilde{E}_{xihp} \propto (-1)^{p+1} j\beta_y \tilde{H}_{z_{ihp}} \]
\[ \tilde{E}_{xifhp} \propto \tilde{H}_{z_{iffp}} \]
\[ \tilde{E}_{xkif} \propto \tilde{H}_{z_{kif}} \]
\[ \tilde{E}_{xolf} \propto \frac{j\beta_x}{2\beta_y n} J_{eibn} \]
\[ \tilde{E}_{xelf} \propto -\frac{j\beta_x}{2\beta_y n} J_{oibn} \]
\[ \tilde{E}_{xifhp} \propto (-1)^{p+1} \frac{\omega \mu_0}{4\pi \beta_y n} \sigma_T(\beta_x) \odot \beta_x \tilde{H}_{z_{ihp}} \]
\[ \tilde{E}_{A1} \propto -\frac{K_o}{2\beta_y n} \]
\[\vec{E}_{A2} \propto \frac{K_e}{4\beta y_n}\]
\[\vec{E}_{A3} \propto \frac{K_0}{4}\]
\[\vec{E}_{A4} \propto -\frac{K_e}{4\beta y_n}\]
\[\vec{E}_{B1} \propto \frac{K_e}{2\beta y_n}\]
\[\vec{E}_{B2} \propto \frac{K_0}{4\beta y_n}\]
\[\vec{E}_{B3} \propto \frac{K_e}{4}\]
\[\vec{E}_{B4} \propto \frac{K_0}{4\beta y_n}\]

\[\vec{H}_{self} = -\frac{j\beta_x}{2\beta y_n}J_{otbn}\]
\[\vec{H}_{gol} = \frac{j\beta_x}{2\beta y_n}J_{elbm}\]

where \(k = e, o\) and \(p = 1,2\).

The TE boundary condition at \(y = 0\) requires \(E_{xl}(x, y = 0^+, z) = 0\) for all \(x\) and \(z\). In the \(\beta_x\) space this implies that

\[\vec{E}_{xih1}(\beta_x, z) + \vec{E}_{xih2}(\beta_x, z) + \vec{E}_{xelf}(\beta_x, z) + \vec{E}_{xih1}(\beta_x, z) + \vec{E}_{xih2}(\beta_x, z) + \vec{E}_{A2}(\beta_x, z) = 0\]  \(\text{(4-49)}\)

Further at \(y = d\), \(E_{xl}(x, y = d, z) = 0\) for all \(x\) and \(z\). Therefore, in \(\beta_x\) space,
\[ E_{xih1}(\beta_x, z)e^{j\beta_y d} + E_{xih2}(\beta_x, z)e^{-j\beta_y d} \]

\[ + \tilde{E}_{xelf}(\beta_x, z)[ \cos(\beta_y d) - \beta_y d\sin(\beta_y d)] \]

\[ + \tilde{E}_{xolf}(\beta_x, z)[\sin(\beta_y d) + \beta_y d\cos(\beta_y d)] \]

\[ + [1 + j\beta_y d]e^{j\beta_y d} \tilde{E}_{xih1}(\beta_x, z) \]

\[ + [1 - j\beta_y d]e^{-j\beta_y d} \tilde{E}_{xih2}(\beta_x, z) + (\tilde{E}_{A1}d + \tilde{E}_{A2})\cos(\beta_y d) \]

\[ + (\tilde{E}_{B1}d + \tilde{E}_{B2})\sin(\beta_y d) + (\tilde{E}_{A3}d^2 + \tilde{E}_{A4}d)\sin(\beta_y d) \]

\[ + (\tilde{E}_{B3}d^2 + \tilde{E}_{B4}d)\cos(\beta_y d) = 0 \]

Solving for \( \tilde{E}_{xih2} \) in Eq. (4-49) and substituting into Eq. (4-50) yields

\[ \tilde{E}_{xih1}(\beta_x, z)(j2\sin(\beta_y d)) \]

\[ + \tilde{E}_{xelf}(\beta_x, z)[ \cos(\beta_y d) - \beta_y d\sin(\beta_y d) - e^{-j\beta_y d}] \]

\[ + \tilde{E}_{xolf}(\beta_x, z)[\sin(\beta_y d) + \beta_y d\cos(\beta_y d)] + [1 + j\beta_y d \]

\[ - e^{-j2\beta_y d}]e^{j\beta_y d} \tilde{E}_{xih1}(\beta_x, z) + [-j\beta_y d]e^{-j\beta_y d} \tilde{E}_{xih2}(\beta_x, z) \]

\[ + (\tilde{E}_{A1}d + \tilde{E}_{A2})\cos(\beta_y d) + (\tilde{E}_{B1}d + \tilde{E}_{B2})\sin(\beta_y d) \]

\[ + (\tilde{E}_{A3}d^2 + \tilde{E}_{A4}d)\sin(\beta_y d) + (\tilde{E}_{B3}d^2 + \tilde{E}_{B4}d)\cos(\beta_y d) \]

\[ - \tilde{E}_{A2}e^{j\beta_y d} = 0 \]

Since \( \beta_y = \frac{n\pi}{d} = \beta_{yn} \), Eq. (4-51) simplifies to

\[ n\pi(-1)^n\tilde{E}_{xolf}(\beta_x, z) + jn\pi(-1)^n[\tilde{E}_{xih1}(\beta_x, z) - \tilde{E}_{xih2}(\beta_x, z)] \]

\[ + \tilde{E}_{A1}d(-1)^n + (\tilde{E}_{B3}d^2 + \tilde{E}_{B4}d)(-1)^n = 0 \]

where the relationship between \( E_x \) and \( H_z \) are given by Eq. (4.48b). It appears that Eq. (4-52) cannot be satisfied uniquely for all \( z \). Consequently, \( \tilde{E}_{xolf}, \tilde{E}_{xihp}, \tilde{E}_{xihf} \) where \( p = 1,2 \) do not
contribute to the propagating wave solution and need to be omitted. Further, this implies that \( \mathcal{H}_{zolf} \), \( \mathcal{H}_{ziff} \), and \( \mathcal{H}_{zifp} \) must be omitted for consistency. Hence Eq. (4-49) becomes

\[
\mathcal{E}_{xih2}(\beta_x, z) = -\mathcal{E}_{xih1}(\beta_x, z) - \mathcal{E}_{xief}(\beta_x, z)
\]

(4-53)

Using the field relationship in Eq. (4.48a), Eq. (4-53) becomes

\[
\mathcal{H}_{zih2}(\beta_x, z) = \mathcal{H}_{zih1}(\beta_x, z) + \frac{1}{j\beta_y} \mathcal{H}_{zief}(\beta_x, z)
\]

(4-54)

The \( z \)-component of the magnetic field satisfying boundary conditions at the plate surface for fundamental resonance is given by

\[
H_{z}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ 2\mathcal{H}_{zihn1}(\beta_x, z) \cos(\beta_{yn}y) + \delta_{nm} \left[ \frac{1}{j\beta_{yn}} e^{-j\beta_{yn}y} + y \cos(\beta_{yn}y) \right] \mathcal{H}_{zief}(\beta_x, z) \right\}
\]

(4-55)

Because the eigenfunctions exhibit functional orthogonality, the coefficients of the eigenfunctions are in general different for each \( n \). Therefore, an odd subscript \( n \) index letter has been added to the amplitude symbol. The double summation and the use of the kronecker delta are used to emphasize that these conditions stem from the fundamental resonance condition. The term \( H_{zio} \) takes into consideration the \( TE_0 \) mode which is treated separately as a special case. In the limit as \( \beta_x \to 0 \), the TEM mode is recovered.

Summary: Fundamental Resonance (\( m = n \))

In summary, the governing TE relations leading to wave propagation in the parallel-plate wave guide based on the fundamental resonance condition are
\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left\{ 2\tilde{H}_{zinh1}(\beta_x, z) \cos(\beta_yny) \right. \]
\[ + \delta_{nm} \left[ \frac{1}{j\beta_yn} e^{-j\beta_yny} + y \cos(\beta_yny) \right] \tilde{H}_{zeinf}(\beta_x, z) \right\} \]
\[ + H_{zio}(\beta_x, z) \]

subject to the following conditions

\[ \left[ \frac{\partial^2}{\partial z^2} + \beta_{zln}^2(\beta_x) \right] \tilde{H}_{zinh1}(\beta_x, z) = 0 \] \hspace{1cm} \text{(4.56a)}

\[ \left[ \frac{\partial^2}{\partial z^2} + \beta_{zln}^2(\beta_x) \right] \tilde{H}_{zeinf}(\beta_x, z) = 0 \] \hspace{1cm} \text{(4.56b)}

\[ \tilde{H}_{zeinf}(\beta_x, z) = -\frac{j\beta_x}{2\beta_yn} J_{0ibn}(\beta_x, z) \] \hspace{1cm} \text{(4.57)}

\[ \beta_{zln}^2(\beta_x) \triangleq \omega^2 \mu_0 \varepsilon_{effi}(\omega) - \beta_x^2 - \beta_{yn}^2 \] \hspace{1cm} \text{(4.58)}

4.2 TM contributions

We now consider the TM contributions as given by Eq. (2.61f) and Eq. (2.63f) Recall the general TM wave equation, Eq. (2.59), has no physical source term contribution. Consequently,

\[ E_{zi}(\beta_x, y, z) = \tilde{E}_{zih}(\beta_x, y, z) + \tilde{E}_{zlf}(\beta_x, y, z) \] \hspace{1cm} \text{(4-59)}

By analogy to Eq. (4.2), Eq. (2.61f) yields,

\[ \tilde{E}_{zih}(\beta_x, y, z) = \tilde{E}_{zih1}(\beta_x, z) e^{j\beta_yny} + \tilde{E}_{zih2}(\beta_x, z) e^{-j\beta_yny} \] \hspace{1cm} \text{(4-60a)}

where

\[ \left[ \frac{\partial^2}{\partial z^2} + \beta_{zi}^2 \right] \tilde{E}_{zihp}(\beta_x, z) = 0 \] \hspace{1cm} \text{(4-60b)}

\[ \beta_{zi}^2 = \omega^2 \mu_0 \varepsilon_{effi}(\omega) - \beta_x^2 - \beta_{yn}^2 \] \hspace{1cm} \text{(4-60c)}
and \( p = 1,2 \).

Equation (4-60a) has the same form as Eq. (4-2) for the TE mode. Therefore, by analogy to Eqs. (4-9), (4-10a) and (4-10b) one can write

\[
\vec{E}_{zif} \equiv \vec{E}_{zifh}(x, y, z) = y(\vec{E}_{zifh1}(\beta_x, z)e^{j\beta_yy} + \vec{E}_{zifh2}(\beta_x, z)e^{-j\beta_yy})
\]  

(4-61)

where

\[
\vec{E}_{zifhp}(\beta_x, \beta_y, z) = (-1)^{p+1} \frac{\omega \mu_0}{4\pi \beta_y} [\sigma_{Tl}(\beta_x) \otimes \beta_x \vec{E}_{zihp}(\beta_x, \beta_y, z)]
\]  

(4-62)

subject to

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 - \beta_y^2 \right] \vec{E}_{zifhp}(\beta_x, z) = 0
\]  

(4-63)

for \( p = 1,2 \). With the aid of Eqs. (4-60a) and (4-61), Eq. (4-59) becomes,

\[
E_{zl}(\beta_x, y, z) = \vec{E}_{zih1}(\beta_x, z)e^{j\beta_yy} + \vec{E}_{zih2}(\beta_x, z)e^{-j\beta_yy} + y[\vec{E}_{zifh1}(\beta_x, z)e^{j\beta_yy}
\]

\[
+ \vec{E}_{zifh2}(\beta_x, z)e^{-j\beta_yy}]
\]  

(4-64)

Both the x- and z-components of the electric field at the surface of the parallel plates must vanish for all \( x \) and \( z \). Boundary conditions require

\[
E_{xl}(x, y = 0, z) = 0 \quad \text{and} \quad E_{xl}(x, y = d, z) = 0
\]

\[
E_{zl}(x, y = 0, z) = 0 \quad \text{and} \quad E_{zl}(x, y = d, z) = 0
\]

Satisfying the boundary condition at \( y = 0 \) leads to the result

\[
\vec{E}_{zih1}(\beta_x, \beta_y, z) = -\vec{E}_{zih2}(\beta_x, \beta_y, z)
\]  

(4-65a)

Satisfying the boundary condition at \( y = d \) and using Eq. (4-65a) yields
\[ E_{z_1}(\beta_x, y = d, z) = j2E_{z_1h1}(\beta_x, z) \sin(\beta_y d) + d[E_{z_1h1}(\beta_x, z)e^{j\beta_y d} + E_{z_1h2}(\beta_x, z)e^{-j\beta_y d}] = 0 \] (4-65b)

It is desired to examine the coupling of the source contributions to the resonant natural response.

The resonant natural response requires

\[ \beta_y = \beta_{yn} = \frac{\pi}{d} \] (4-66a)

resulting in

\[ \beta_{zi}^2 = \frac{\omega^2}{\mu_0}\varepsilon_{eff}(\omega) - \beta_x^2 - \beta_{yn}^2 \triangleq \beta_{zin}^2 \] (4.66b)

Therefore, boundary condition at \( y = d \) yields the constraint

\[ \bar{E}_{z1f1}e^{j\beta_{yn} d} = -\bar{E}_{z1f2}e^{-j\beta_{yn} d} \]

implying

\[ \bar{E}_{z1f1}(\beta_x, z) = -\bar{E}_{z1f2}(\beta_x, z) \] (4-67)

But, from Eq. (4-62) and Eq. (4-65a), \( \bar{E}_{z1f1} = \bar{E}_{z1f2} \) which is a contradiction with the boundary condition. Thus, \( \bar{E}_{z1fhp}, p = 1,2 \) does not contribute to the wave solution and therefore must be omitted. Therefore,

\[ E_{z_1}(\beta_x, y, z) = \sum_{n=1}^{\infty} j2\bar{E}_{z1hn1}(\beta_x, z)\sin(\beta_{yn} y) \] (4-68)

where \( \beta_y = \beta_{yn} = \frac{n\pi}{d} \).
The tangential component of the electric field $E_{xi}$, is related to the z-component as dictated by Eq. (2-47) suppressing TE mode contributions. This relation suggests that $E_x$ and $E_z$ vary with respect to $y$ in the same manner.
CHAPTER 5. INHOMOGENEOUS WAVE EQUATION, Z -VARIATION

5.1  z —variation, Fundamental Resonance-TE mode

Equations (4-56a) and (4.56b) are the one-dimensional wave equations. Eq. (4-56a) yields the solution

\[
\tilde{H}_{zih1}(\beta_x, \beta_{zin}, z) = \tilde{H}_{zih1}^+(\beta_x, \beta_{zin}) e^{-j\beta_{zin}z} + \tilde{H}_{zih1}^-(\beta_x, \beta_{zin}) e^{j\beta_{zin}z} \tag{5-1}
\]

Equation (4-56b) with the aid of Eq. (4-57) can be re-expressed as

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{zin}^2 \right] J_{oibn}(\beta_x, z) = 0 \tag{5-2}
\]

We can express \( J_{oibn} \) in terms of propagating and counter propagating waves with \( z \) as

\[
J_{oibn}(\beta_x, z) = J_{oibn}^+(\beta_x) e^{-j\beta_{zin}z} + J_{oibn}^-(\beta_x) e^{j\beta_{zin}z} \tag{5-3}
\]

But, \( J_{oibn}(\beta_x, z) \) is known. If \( J_{oibn} \) does not fit this form, then \( H_{zel} \) is not a propagating solution of \( H_{z1}(\beta_x, y, z) \) and must be omitted. In this research, \( J_{oibn} \) is independent of \( z \). Therefore,

\[
\beta_{zin}^2(\beta_x) = 0 = \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 - \beta_{yn}^2 \tag{5-4}
\]

satisfies the wave equation, Eq. (5-2) and the wave solution, Eq. (5-3). Equation (5-4) is substituted in Eq. (5-3) which in turn is substituted in Eq. (4-57). Here the sum of the two constants \( J_{oibn}^+(\beta_x) \) and \( J_{oibn}^-(\beta_x) \) is replaced by a single constant. For generality, we will retain the \( z \)-functional dependence property in \( J_{oibn} \). Excluding the \( TE_0 \) mode contribution, the resultant equation and Eq. (5-1) is then substituted in Eq. (4-55) to yield...
\[ H_{zi}(\beta_x, y, z) - H_{zi0}(\beta_x, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ 2[\bar{H}_{zihn1}^+(\beta_x) e^{-j\beta_{zin}z} + \bar{H}_{zihn1}^-(\beta_x) e^{j\beta_{zin}z}] \cos(\beta_{yn}y) - \left( e^{-j\beta_{yn}y} \frac{-j\beta_x}{\beta_{yn}} + y \cos(\beta_{yn}y) \right) \frac{j\beta_x}{2\beta_{yn}} J_{oibn}(\beta_x, z) \delta_{mn} \right] \]

where \( \beta_{xin}^2 = \beta_{xn}^2 = \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_{yn}^2 \) for \( J_{oibn}(\beta_x, z) = J_{oibn}(\beta_x) \). To emphasize the constraint in Eq. (5-5) one may use the Dirac delta function as a function of \( \beta_x \). Including the \( TE_0 \) mode contribution, the resultant longitudinal magnetic field is

\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ 2[\bar{H}_{zihn1}^+(\beta_x) e^{-j\beta_{zin}z} + \bar{H}_{zihn1}^-(\beta_x) e^{j\beta_{zin}z}] \cos(\beta_{yn}y) - \left( e^{-j\beta_{yn}y} \frac{-j\beta_x}{\beta_{yn}} + y \cos(\beta_{yn}y) \right) \frac{j\beta_x}{2\beta_{yn}} J_{oibn}(\beta_x, z) \delta_{mn} \right] + H_{zi0}(\beta_x, z) \]

where \( \beta_{zin}^2 \) is defined by Eq. (4-58) and, based on orthogonality, \( H_{zi0}(\beta_x, z) \) is representative of the \( TE_0 \) with boundary conditions satisfied for this special case. Refer to Appendix FG for the full treatment of this special case. For a current density independent of \( z \),

\[ J_{oibn}(\beta_x, z) = J_{oibn}(\beta_x) = J_{oibn}(\beta_x) \delta(\beta_x \pm \beta_{xn}) \]

The \( \pm \) sign in the Dirac delta function allows for both positive and negative propagating waves in the \( x \)-direction. It is noted that \( \beta_{xn} \) is real. For generality purpose only, we retain the \( z \)-variation in \( J_{oibn} \). Further \( \bar{E}_{xifh1} \) and \( \bar{E}_{xifh2} \) or equivalently \( \bar{H}_{zifh1} \) and \( \bar{H}_{zifh2} \) do not contribute to the above wave solution since boundary condition cannot be satisfied with this contribution.
5.2 \( z \) -Variation, Harmonic Resonance-TE mode

The \( z \)-component of the magnetic field is given by Eq. (4-31a) subject to Eqs. (4-31b-d). The natural (homogeneous) solution to the wave equation is given by Eq. (4-2).

For the \((m \neq n)\) case, we use the Green’s function technique to evaluate Eqs. (4-31b, c). We will make use of the free space Green’s function based on the Sommerfeld radiation condition and then satisfy boundary conditions with the aid of the homogeneous solution.

Consider Eq. (4-31b) where \( \beta_{zim}^2(\beta_x) = \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 - \beta_{ym}^2 \). Refer to Appendix B for the mathematical construction of the Green’s function with inclusion of boundary conditions. Treating the complicated source as the sum of many point sources, the Green’s function equation is

\[
\frac{d^2}{dz^2} + \beta_{zim}^2(\beta_x) \left[ G(\beta_x, \beta_{ym}, z; \bar{z}) \right] = \delta(z - \bar{z}) \quad (5-7)
\]

where excluding boundary conditions, the resultant field is

\[
\tilde{H}_{zimf}(\beta_x, \bar{z}) = j \beta_x \int_{l_{i-1}}^{l_i} J_{etbm}(\beta_x, \beta_{ym}, z) G(\beta_x, \beta_{ym}, z; \bar{z}) \, dz \quad (5-8)
\]

Here \( l_{i-1} < z < l_i \), \( l_{i-1} < \bar{z} < l_i \) for \( i = 2,3 \). Because boundary conditions are to be handled separately with the homogeneous solution contributions, the limits on \( z \) and \( \bar{z} \) are extended to \( \pm \infty \) in obtaining both the Green’s function from Eq. (5-7) and the field from Eq. (5-8). Consequently, the free space green function with Sommerfeld radiation conditions satisfied is given by

\[
G(\beta_x, \beta_{ym}, z; \bar{z}) = j \frac{e^{-j \beta_{zim}(\beta_x)|z - \bar{z}|}}{2\beta_{zim}} \quad (5-9)
\]

Therefore,
\[
\tilde{H}_{zeimf}(\beta_x, \tilde{z}) = \frac{-\beta_x}{2\beta_{zim}(\beta_x)} \int_{-\infty}^{\infty} J_{elm}(\beta_x, z)e^{-j\beta_{zim}|z-\tilde{z}|}dz
\] (5-10)

For convenience, the role of \(z\) and \(\tilde{z}\) is interchanged. The source contributions driving the longitudinal magnetic field as dictated by Eqs. (4-31b-d) yield the following contributions respectively,

\[
\tilde{H}_{zeimf}(\beta_x, z) = \frac{-\beta_x}{2\beta_{zim}} \int_{-\infty}^{\infty} J_{elm}(\beta_x, \tilde{z})e^{-j\beta_{zim}|z-\tilde{z}|}d\tilde{z}
\] (5-11)

\[
\tilde{H}_{zoimf}(\beta_x, z) = \frac{-\beta_x}{2\beta_{zim}} \int_{-\infty}^{\infty} J_{oim}(\beta_x, \tilde{z})e^{-j\beta_{zim}|z-\tilde{z}|}d\tilde{z}
\] (5-12)

\[
\tilde{H}_{zeimf}(\beta_x, z) = -\frac{\omega\mu_0}{4\pi\beta_{zim}} \int_{-\infty}^{\infty} [\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zeimf}(\beta_x, \tilde{z})]e^{-j\beta_{zim}|z-\tilde{z}|}d\tilde{z}
\] (5-13)

\[
\tilde{H}_{zoimf}(\beta_x, z) = -\frac{\omega\mu_0}{4\pi\beta_{zim}} \int_{-\infty}^{\infty} [\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zoimf}(\beta_x, \tilde{z})]e^{-j\beta_{zim}|z-\tilde{z}|}d\tilde{z}
\] (5-14)

**Summary: Harmonic Resonance**

\[
H_{zi}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{zi}(\beta_x, y, z)e^{-j\beta_x x}d\beta_x
\]
\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left \{ 2\tilde{H}_{zinh}(\beta_x, z) \cos(\beta_y y) ight \} 
+ \delta_{nm} \left( \tilde{H}_{zeimf}(\beta_x, z) \right) 
+ \tilde{H}_{zeimff}(\beta_x, z) \cos(\beta_y m y) \]
\[ + \tilde{H}_{zoimf}(\beta_x, z) \left[ \frac{\beta_y m y}{j\beta_y n} e^{-j\beta_y m y} + \sin(\beta_y m y) \right] \]
\[ = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left \{ 2\tilde{H}_{zinh}(\beta_x, z) \cos(\beta_y y) ight \} 
+ \delta_{nm} \left( \tilde{H}_{zeimf}(\beta_x, z) \right) 
+ \tilde{H}_{zeimff}(\beta_x, z) \cos(\beta_y m y) \]
\[ + \tilde{H}_{zoimf}(\beta_x, z) \left[ \frac{\beta_y m y}{j\beta_y n} e^{-j\beta_y m y} + \sin(\beta_y m y) \right] \]

\[ \tilde{H}_{zeimf}(\beta_x, z) = \frac{-\beta_x}{2\beta_{zim}} \int_{-\infty}^{\infty} f_{ebm}(\beta_x, \bar{z}) e^{-j\beta_{zim}|\bar{z}-z|} d\bar{z} \]  
\[ (5.11) \]

\[ \tilde{H}_{zoimf}(\beta_x, z) = \frac{-\beta_x}{2\beta_{zim}} \int_{-\infty}^{\infty} f_{oibm}(\beta_x, \bar{z}) e^{-j\beta_{zim}|\bar{z}-z|} d\bar{z} \]  
\[ (5.12) \]

\[ \tilde{H}_{zeimff}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_{zim}} \int_{-\infty}^{\infty} \left[ \sigma_T(\beta_x) \otimes \beta_x \tilde{H}_{zeimf}(\beta_x, \bar{z}) e^{-j\beta_{zim}|\bar{z}-z|} \right] d\bar{z} \]  
\[ (5.13) \]

\[ \tilde{H}_{zoimff}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_{zim}} \int_{-\infty}^{\infty} \left[ \sigma_T(\beta_x) \otimes \beta_x \tilde{H}_{zoimf}(\beta_x, \bar{z}) e^{-j\beta_{zim}|\bar{z}-z|} \right] d\bar{z} \]  
\[ (5.14) \]

\[ \varepsilon_i(\beta_x, \omega) = 2\pi \varepsilon_{effi}(\omega) \delta(\beta_x) + \frac{\sigma_T(\beta_x, \omega)}{j\omega} \]

\[ \sigma_T(\beta_x, \omega) = \left( \tilde{\sigma}_{pl}(\omega) + \tilde{\sigma}_{bi}(\omega) \right) \left[ \frac{\pi}{\epsilon} e^{\frac{\beta_x^2}{4\epsilon \alpha}} - 2\pi \delta(\beta_x) \right] \]

\[ \varepsilon_{effi}(\omega) = \varepsilon_0 \left[ 1 + \frac{\tilde{\sigma}_{pl}(\omega) + \tilde{\sigma}_{bi}(\omega)}{j\omega \varepsilon_0} \right] \]

\[ \beta_{ym} = \frac{m\pi}{d}, \quad \beta_{yn} = \frac{n\pi}{d} \]  
\[ (5-16a,b) \]
\[ \beta_{zim}^2(\beta_x) = \omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 - \beta_{ym}^2 \] (5-17)

where \( \delta_{mn}, \bar{\delta}_{nm}, \) and \( \bar{\Gamma}_{mn} \) are defined by Eqs. (1.18a-c) respectively.

5.3 TM mode

Satisfying boundary conditions with respect to \( y \) yielded the solution depicted by Eq. (4-67) and (4-68) subject to Eq. (4.60b). Solving the one dimensional homogeneous wave equation given by Eq. (4.60b) yields

\[ E_{zl}(\beta_x, y, z) = \sum_{n=1}^{\infty} j2 \left[ \bar{E}_{zlh1}^+(\beta_x) e^{-j\beta_{zin}z} + \bar{E}_{zlh1}^- (\beta_x)e^{j\beta_{zin}z} \right] \sin(\beta_{yn}y) \] (5-18)

where \( \beta_{zin} = \beta_{zin}(\beta_x) \) and \( \beta_{yn} \) are defined by Eqs. (4.66b) and (4-66a) respectively.

The longitudinal field component for propagating waves in the \( \beta_x \) phase space is complete.
CHAPTER 6. TRANSVERSE FIELDS

The transverse fields are determined in this section knowing the form of solutions for the longitudinal fields and the source current density as given by Eqs. (2-47) through (2-50), equivalently in other forms Eqs. (2.55) through (2.58), or the combination of Eqs. (2.62a-d) and Eqs. (2.63a-d). Each of these expressions has similar forms. A TE and TM mode can be treated separately if one can distinguish or discriminate the portion of the source current density driving the TE mode and that portion driving the TM mode. In Maxwell’s equations, the fields and source terms presented are resultant fields and resultant source terms. Physical meaning is placed to field sets known as modes based on functional orthogonality due to power considerations. If one is unable ‘a priori’ to discriminate the source current density based on the mode concept (functional orthogonality) then the total current density must be related to the total field (TE and TM and if appropriate TEM) of that vector component. This has been correctly performed on the transverse fields. Consequently, the source current density on its own right couples the TE and TM fields. That is, it couples the TE and TM field of the empty parallel-plate waveguide.

Consider the x-component of the magnetic field characterized by Eqs. (2.62c) and (2.63c).
The z-component of the magnetic field in Eqs. (2.62c) and (2.63c) contain terms of the form
\[ \int_{-\infty}^{\infty} h(z)e^{j\beta_z|z-\bar{z}|} d\bar{z} \]
these terms and their variation with respect to z can be written as
\[ \int_{-\infty}^{\infty} h(\bar{z})e^{-j\beta_z|\bar{z}-z|} d\bar{z} = \int_{-\infty}^{z} h(\bar{z})e^{j\beta_z(z-\bar{z})} d\bar{z} + \int_{z}^{\infty} h(\bar{z})e^{-j\beta_z(z-\bar{z})} d\bar{z} \]

and
\[
\frac{\partial}{\partial z} \int \limits_{-\infty}^{\infty} h(\tilde{z}) e^{-j\beta_x|\tilde{z} - z|} d\tilde{z} = -j\beta_z \int \limits_{-\infty}^{z} h(\tilde{z}) e^{j\beta_x(\tilde{z} - z)} d\tilde{z} + h(z) + j\beta_z \int \limits_{z}^{\infty} h(\tilde{z}) e^{-j\beta_x(\tilde{z} - z)} d\tilde{z} - h(z)
\]

\[
= -j\beta_z \left[ \int \limits_{-\infty}^{z} h(\tilde{z}) e^{j\beta_x(\tilde{z} - z)} d\tilde{z} - \int \limits_{z}^{\infty} h(\tilde{z}) e^{-j\beta_x(\tilde{z} - z)} d\tilde{z} \right]
\]

Using these forms and substituting Eqs. (4-1), (5.6a), (5.11)-(5.16), and (5.18) into right hand side of Eq. (2.62c) yields

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} (\omega) \right] \bar{H}_{xi}(\beta_x, y, z)
\]

\[
= -j\beta_x \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ j2\beta_{zin} \left[ -\bar{H}_zn^{+}_1(\beta_x) e^{-j\beta_{zin}z} + \bar{H}_zn^{+}_{thn}(\beta_x) e^{j\beta_{zin}z} \right] \cos(\beta_{yn}y)
\right.
\]

\[
+ \delta_{nm} \left[ \frac{j\beta_x}{2} \left[ \int \limits_{-\infty}^{z} J_{elbm}(\beta_x, \tilde{z}) e^{j\beta_{zin}(\tilde{z} - z)} d\tilde{z} - \int \limits_{z}^{\infty} J_{elbm}(\beta_x, \tilde{z}) e^{-j\beta_{zin}(\tilde{z} - z)} d\tilde{z} \right] \right]
\]

\[
+ \frac{j\omega \mu_0}{4\pi} \left[ \int \limits_{-\infty}^{z} [\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zemf}(\beta_x, \tilde{z})] e^{j\beta_{zin}(\tilde{z} - z)} d\tilde{z} \right.
\]

\[
- \int \limits_{z}^{\infty} [\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zemf}(\beta_x, \tilde{z})] e^{-j\beta_{zin}(\tilde{z} - z)} d\tilde{z} \left\} \cos(\beta_{yn}y) \right\} +
\]
Therefore, for the form of equation (6.1) yields

\[
\left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} \right] \mathcal{H}_{xlh}(\beta_x, y, z) = h_1(\beta_x, y, \beta_{zim}) e^{j\beta_{zim}z} + h_1(\beta_x, y, \beta_{zim}) e^{j\beta_{zim}z} + h_2(\beta_x, y, \beta_{zim}) e^{-j\beta_{zim}z} + h_2(\beta_x, y, \beta_{zim}) e^{-j\beta_{zim}z}
\]

The last term is a placeholder for the TE₀ mode which is fully treated with in appendix G.

Making use of the method of superposition, all source terms of the form \( e^{\pm j\beta_{zim}z} \) are treated separately from the terms of the form \( F_1(z) = \int_{-\infty}^{z} f_1(\tilde{z}; z) \, d\tilde{z} \) and \( F_2(z) = \int_{-\infty}^{z} f_2(\tilde{z}; z) \, d\tilde{z} \) where

\[
f_1(\tilde{z}; z) = f_1(\tilde{z}) e^{\pm j\beta_{zim}z} \quad \text{and} \quad f_2(\tilde{z}; z) = f_2(\tilde{z}) e^{\pm j\beta_{zim}z}
\]

Let \( h_1(\beta_x, y, \beta_{zim}) \) and \( h_2(\beta_x, y, \beta_{zim}) \) be the coefficients associated with \( e^{\pm j\beta_{zim}z} \). Therefore, for the form of equation, Eq. (6.1) yields

\[
\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \tilde{E}_{mn} \left[ \int_{-\infty}^{z} j \beta_x \mathcal{H}_{oibm}(\beta_x, \tilde{z}) e^{j\beta_{zim}(\tilde{z}-z)} d\tilde{z} - \int_{z}^{\infty} j \beta_x \mathcal{H}_{oibm}(\beta_x, \tilde{z}) e^{-j\beta_{zim}(\tilde{z}-z)} d\tilde{z} \right]
\]
\[ \bar{H}_{xih}(\beta_x, y, z) = \frac{1}{\omega^2 \mu_0 \varepsilon_{eff} - \beta_{zim}^2} [h_1(\beta_x, y, \beta_{zim}) e^{j \beta_{zim} z} + h_2(\beta_x, y, \beta_{zim}) e^{-j \beta_{zim} z}] \]

\[ + \frac{1}{\omega^2 \mu_0 \varepsilon_{eff} - \beta_{zim}^2} [h_1(\beta_x, y, \beta_{zim}) e^{j \beta_{zim} z} + h_2(\beta_x, y, \beta_{zim}) e^{-j \beta_{zim} z}] \]

For the forms \( F_1(z) = \int_{-\infty}^{z} f_1(\tilde{z}; z) d\tilde{z}, \) \( F_2(z) = \int_{z}^{\infty} f_2(\tilde{z}; z) d\tilde{z} \) and the non-integral form \( F_0(\beta_x, y, z), \) we will employ the Green’s function approach (see Appendix B). Fitting to the form of Eqs. (4-31), and (5-7)-(5-10) we have

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{eff} \right] \bar{H}_{xif}(\beta_x, y, z) = F_1(\beta_x, y, z) + F_2(\beta_x, y, z) + F_0(\beta_x, y, z) \]  

(6-3)

Omitting finite boundary condition contributions,

\[ \bar{H}_{xif}(\beta_x, y, z) = \int_{-\infty}^{z} \int_{-\infty}^{0} f_1(\tilde{z}; z) d\tilde{z} + \int_{z}^{\infty} f_2(\tilde{z}; z) d\tilde{z} \]

\[ G_F(\beta_x, \beta_{ym}, \tilde{z}; z) d\tilde{z} \]

\[ + \int_{z}^{\infty} \int_{-\infty}^{0} f_1(\tilde{z}; z) d\tilde{z} + \int_{z}^{\infty} f_2(\tilde{z}; z) d\tilde{z} \]

\[ G_F(\beta_x, \beta_{ym}, \tilde{z}; z) d\tilde{z} \]

\[ + \int_{-\infty}^{z} F_0(\tilde{z}) G_F(\beta_x, \beta_{ym}, \tilde{z}; z) d\tilde{z} \]

\[ + \int_{z}^{\infty} F_0(\tilde{z}) G_F(\beta_x, \beta_{ym}, \tilde{z}; z) d\tilde{z} \]

(6-4)

where

\[ \left[ \frac{d^2}{d\tilde{z}^2} + \omega^2 \mu_0 \varepsilon_{eff} \right] G_F(\beta_x, \beta_{ym}, \tilde{z}; z) = \delta(\tilde{z} - z) \]

(6-5)

yielding the free space Green’s function
\[ G_F(\beta_x, \beta_{ym}, z; \bar{z}) = e^{\frac{-j\omega \|\bar{\mu}_0\|_{eff}|z-\bar{z}|}{2\omega \sqrt{\mu_0 \varepsilon_{eff}}}}. \]  

(6-6)

We now develop a notation to concisely write and therefore simplify the Greens function notation. Define

\[ \mathcal{G}_z(f(\beta_x, y), \beta_{zim}) = \frac{j}{2\beta_{zim}} \int_{-\infty}^{\infty} f(\beta_x, y, \bar{z}) e^{-j\beta_{zim}|z-\bar{z}|} d\bar{z} \]

\[ = \frac{j}{2\beta_{zim}} \left[ \int_{-\infty}^{z} f(\beta_x, y, \bar{z}) e^{j\beta_{zim}(z-\bar{z})} d\bar{z} \right. \]

\[ + \left. \int_{z}^{\infty} f(\beta_x, y, \bar{z}) e^{-j\beta_{zim}(\bar{z}-z)} d\bar{z} \right] \]

(6-7)

\[ \triangleq \mathcal{G}_<z(f(\beta_x, y), \beta_{zim}) + \mathcal{G}_>z(f(\beta_x, y), \beta_{zim}) \]

where the subscript \(< z\) implies \(-\infty < \bar{z} < z\), and \(> z\) implies \(z < \bar{z} < \infty\). Therefore,

\[ \mathcal{G}_<z(f(\beta_x, y), \beta_0) = \begin{cases} 
\frac{j}{2\beta_0} \int_{-\infty}^{z} f(\beta_x, y, \bar{z}) e^{j\beta_0(\bar{z}-z)} d\bar{z} \\
\frac{j}{2\beta_0} \int_{z}^{\infty} f(\beta_x, y, \bar{z}) e^{-j\beta_0(\bar{z}-z)} d\bar{z} 
\end{cases} \]  

(6-8)

It is noted that the function \(f(\beta_x, y)\) in the argument of \(\mathcal{G}\) is, in the integrand, a function of the parameter being integrated over \(f(\beta_x, y, \bar{z})\). Other useful variations of this notation can be found in Appendix F.

Using the short hand notations Eqs. (6-7) and (6-8) and other variations tabulated in Appendix F, the longitudinal field components with the aid of Appendix G [Eqs. G-25a-c] can be written as

\[ \bar{H}_{z1}(\beta_x, y, z) = \bar{H}_{z1}(\beta_x, y, z) + \bar{H}_{z1}(\beta_x, z) \]  

(6-9a)
\[ H_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left[ 2[H_{zithn1}(\beta_x)e^{-j\beta zin^2} + H_{zithn1}(\beta_x)e^{j\beta zin^2}] \cos(\beta_{yn}y) \right. \\
+ \left. \bar{\Gamma}_{mn} \left[ j\beta_x G_z(J_{oibm}, \beta_{zim}) + j \frac{\omega \mu_0}{2\pi} G_z(\sigma_T(\beta_x) \odot \beta_x \bar{H}_{zimf}(\beta_x), \beta_{zim}) \right] \frac{\beta_{yn}}{j} e^{-j\beta_{yn}y} \right. \\
+ \left. \sin(\beta_{ym}y) \right] \\
+ \left. \delta_{nm} \left[ j\beta_x G_z(J_{eibm}, \beta_{zim}) + j \frac{\omega \mu_0}{2\pi} G_z(\sigma_T(\beta_x) \odot \beta_x \bar{H}_{zimf}(\beta_x), \beta_{zim}) \right] \cos(\beta_{yn}y) \right. \\
- \left. \frac{\beta_x \delta_{mn}}{2\beta_{yn}} \left[ e^{-j\beta_{yn}y} + y \cos(\beta_{yn}y) \right] J_{oibm}(\beta_x, z) \right. \\
+ \left. \frac{\beta_x}{\omega \mu_0} \left[ \bar{E}_{yih0}^{+}(\beta_x) e^{-j\beta_{zio}z} + \bar{E}_{yih0}^{-}(\beta_x) e^{j\beta_{zio}z} \right] \right. \\
+ \left. j\beta_x G_z(J_{eib0}, \beta_{zio}) + j \frac{\beta_x}{2\pi} G_z(\sigma_T(\beta_x) \odot \beta_x \bar{E}_{yif0}(\beta_x), \beta_{zio}) \right] \sin(\beta_{yn}y) \right) \\
(6.9b) \]

The longitudinal field for TM\(_n\) mode can be expressed as

\[ E_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} j2[H_{zithn1}(\beta_x)e^{-j\beta zin^2} + H_{zithn1}(\beta_x)e^{j\beta zin^2}] \sin(\beta_{yn}y) \]

(6-10)
In short hand notation, \( \tilde{H}_{x1} \) satisfying Eq. (6.1) with the aid of Eqs. (G-24a-c) can be written as

\[
\tilde{H}_{x1}(\beta_x, y, z) = -j\beta_x \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{j2\beta_{zin}}{\beta_x^2 + \beta_y^2} \left[ \tilde{H}_{zim1}^+(\beta_x) e^{-j\beta_{zin} z} \right. \right.
\]

\[
+ \tilde{H}_{zim1}^-(\beta_x) e^{j\beta_{zin} z} \cos(\beta_y y) \left[ \Gamma_{mn} \frac{\beta_x \beta_{zim}}{2\pi} \left[ G_z(G_{<z}(\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zimf}(\beta_x, \beta_{zim}), \beta_{0i})) - G_z(G_{>z}(\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zimf}(\beta_x, \beta_{zim}), \beta_{0i})) \right] \right. \left[ \frac{\beta_y m}{\beta_y n} e^{-j\beta_{yn} y} + \sin(\beta_y y) \right] \right.
\]

\[
+ \delta_{nm} \left[ \beta_x \beta_{zim} \left[ G_z(G_{<z}(\sigma_{eibm}(\beta_x, \beta_{zim}), \beta_{0i})) - G_z(G_{>z}(\sigma_{eibm}(\beta_x, \beta_{zim}), \beta_{0i})) \right] \right. \left[ G_z(G_{<z}(\sigma_{T1}(\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zimf}(\beta_x, \beta_{zim}), \beta_{0i}))) - G_z(G_{>z}(\sigma_{T1}(\sigma_{T1}(\beta_x) \odot \beta_x \tilde{H}_{zimf}(\beta_x, \beta_{zim}), \beta_{0i}))) \right] \cos(\beta_y y) \right.
\]

\[
- j\beta_x \delta_{nm} \left[ \frac{e^{-j\beta_{yn} y}}{\beta_y n} + y \cos(\beta_y y) \right] G_z \left( \frac{\partial}{\partial z} J_{oibm}(\beta_x, z), \beta_{0i}) \right) \}
\]
\[-\sum_{n=1}^{\infty} \frac{2\omega \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e_{eff}(\omega) \left[ E_{zih}^+(\beta_x) e^{-j\beta_{zio}z} + E_{zih}^-(\beta_x) e^{j\beta_{zio}z} \right] \cos(\beta_{yn}y) \]

\[+ \sum_{m=1}^{\infty} \left[ G_z \left( \frac{\partial}{\partial z} I_{ebm}, \beta_{0i} \right) \cos(\beta_{yn}y) + G_z \left( \frac{\partial}{\partial z} I_{obm}, \beta_{0i} \right) \sin(\beta_{yn}y) \right] \]

\[-\frac{\beta_{zio}(\beta_x)}{\omega \mu_0} \left[ E_{yih0}^+(\beta_x) e^{-j\beta_{zio}z} - E_{yih0}^-(\beta_x) e^{j\beta_{zio}z} \right] \]

\[- j\beta_{zio}(\beta_x) \left[ G_{<z}(I_{eib0}(\beta_x, \bar{z}), \beta_{z0}) - G_{>z}(I_{eib0}(\beta_x, \bar{z}), \beta_{z0}) \right] \]

\[- j\frac{\beta_{zio}(\beta_x)}{2\pi} \left[ G_{<z}(\sigma_T(\beta_x) \odot^\beta_x E_{yif0}(\beta_x, \bar{z}), \beta_{zio}) - G_{>z}(\sigma_T(\beta_x) \odot^\beta_x E_{yif0}(\beta_x, \bar{z}), \beta_{zio}) \right] \]

where $I_{kibm} = I_{kibm}(\beta_x, \bar{z})$ and $k = e, o$.

Following the same procedure as for $\bar{H}_{xl}$, Eq. (2.63c) can be solved for the correction term $\bar{H}_{xl}$ resulting from inhomogeneous medium contribution in the transverse field equations. The corrected field may be written as follows using the short hand notation of Appendix F.
\[
\bar{H}_{xi}(\beta_x, y, z) = \omega \mu_0 \beta_x \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{j^2 \beta_{zim}^2}{(\beta_x^2 + \beta_y^2)^2} [-\bar{H}_{z.im} e^{j \beta_{zim}^2 z} + \bar{H}_{z.im} e^{j \beta_{zim}^2 z}] \cos(\beta_{ym} y) \\
+ \bar{G}_{zim} \left[ \beta_x \beta_{zim} [G_{<z}(\Gamma_{obm}, \beta_{zim}), \beta_{0i}) - G_{>z}(\Gamma_{obm}, \beta_{zim}), \beta_{0i})] \right] + \frac{\omega \mu_0 \beta_{zim}^2}{2\pi} [G_{<z}^2 (\sigma_{Tz}^2(\beta_x) \otimes \beta_x \bar{H}_{z.im}^2, \beta_{zim}), \beta_{0i}) - G_{>z}^2 (\sigma_{Tz}^2(\beta_x) \otimes \beta_x \bar{H}_{z.im}^2, \beta_{zim}), \beta_{0i})] [\frac{\beta_{ym}}{j \beta_{yn}} e^{j \beta_{ym} y} \\
+ \sin(\beta_{ym} y)] \\
+ \bar{\delta}_{nm} \left[ \beta_x \beta_{zim} [G_{<z}(\Gamma_{ebm}, \beta_{zim}), \beta_{0i}) - G_{>z}(\Gamma_{ebm}, \beta_{zim}), \beta_{0i})] \right] + \frac{\omega \mu_0 \beta_{zim}^2}{2\pi} [G_{<z}^2 (\sigma_{Tz}^2(\beta_x) \otimes \beta_x \bar{H}_{z.im}^2, \beta_{zim}), \beta_{0i}) - G_{>z}^2 (\sigma_{Tz}^2(\beta_x) \otimes \beta_x \bar{H}_{z.im}^2, \beta_{zim}), \beta_{0i})] \cos(\beta_{ym} y) \\
- \frac{j \beta_x \delta_{nm}}{2 \beta_{yn}} [e^{j \beta_{ym} y} + \gamma \cos(\beta_{ym} y)] G_{z}^2 \left( \frac{\partial}{\partial z} \Gamma_{ebm}(\beta_x), \beta_{0i}) \right) \right\} 
\]

(6-12a)
\[
+ \sum_{n=1}^{\infty} j^2 \beta_n \left[ \frac{1}{\beta_x^2 + \beta_y^2} - \frac{\omega^2 \mu_0 \varepsilon_{eff}(\omega)}{\beta_x^2 + \beta_y^2} \right] \left[ E^+_{zln1}(\beta_x) e^{-j \beta_z n z} \right] \\
+ \frac{E^-_{zln1}(\beta_x) e^{j \beta_z n z}}{\cos(\beta_y n)} \\
+ j \omega \mu_0 \sum_{m=1}^{\infty} \left[ G^2 \left( \frac{\partial}{\partial z} J_{ebm}(\beta_x), \beta_{0i} \right) \cos(\beta_{ym} n) \right] \\
+ G^2 \left( \frac{\partial}{\partial z} J_{obm}(\beta_x), \beta_{0i} \right) \sin(\beta_{ym} n) \right]
\]

where

\[
\bar{H}_{x1}(\beta_x, y, z) = \frac{\sigma_{TX}(\beta_x)}{2\pi} \otimes \beta_x \bar{H}_{x1}(\beta_x, y, z) + \bar{H}_{x10}(\beta_x, z) \\
\bar{H}_{x10}(\beta_x, z) = \frac{-j \beta_{z10}}{2\pi} \left[ G_{<z} \left( \sigma_{TX}(\beta_x) \otimes \beta_x \bar{E}_{y10}(\beta_x, z), \beta_{z10} \right) \\
- G_{>z} \left( \sigma_{TX}(\beta_x) \otimes \beta_x \bar{E}_{y10}(\beta_x, z), \beta_{z10} \right) \right]
\]

Now consider Eqs. (2-62a) and (2-63a). Following the same procedure as, for \( \bar{H}_{x1} \) and \( \bar{H}_{x1} \) the solution for \( E_{x1} \) and \( \bar{E}_{x1} \) can be written as
\[ E_{xi}(\beta_x, y, z) = -j\omega \mu_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{-2\beta_{yn}}{\beta_x^2 + \beta_{yn}^2} \left[ H_{zihn1}^+(\beta_x) e^{-j\beta_{zin} z} \right] \right. \\
+ H_{zihn1}^-(\beta_x) e^{j\beta_{zin} z} \left. \right) \sin(\beta_{yn} y) \]

\[ + \Gamma_{mn} \left[ j\beta_x G_z(G_z(j0ibm, \beta_{zim}), \beta_{0i}) \right] \]

\[ + j\omega \mu_0 \frac{1}{2\pi} g_z(G_z(\sigma_{Ti}(\beta_x) \right) \]

\[ \bigotimes_{\beta_x} H_{zioimf}(\beta_x, \beta_{zim}, \beta_{0i}) \Big] \left[ -\beta_{ym} e^{-j\beta_{yn} y} \right. \]

\[ + \beta_{ym} \cos(\beta_{ym} y) \Big] \]

\[ + \delta_{nm} \left[ j\beta_x G_z(G_z(j0ibm, \beta_{zim}), \beta_{0i}) \right] \]

\[ + j\omega \mu_0 \frac{1}{2\pi} g_z(G_z(\sigma_{Ti}(\beta_x) \right) \]

\[ \bigotimes_{\beta_x} H_{zieimf}(\beta_x, \beta_{zim}, \beta_{0i}) \Big] \left[ -\beta_{ym} \sin(\beta_{ym} y) \right] \]

\[ - \frac{j\beta_x \delta_{nm}}{2\beta_{yn}} \left[ -e^{-j\beta_{yn} y} + \cos(\beta_{yn} y) \right. \]

\[ - y\beta_{yn} \sin(\beta_{yn} y) G_z(j0ibm(\beta_x, z), \beta_{0i}) \right\} \]

\[ - j\beta_x \sum_{n=1}^{\infty} \frac{2\beta_{zin}}{\beta_x^2 + \beta_{yn}^2} \left[ E_{zihn1}^+(\beta_x) e^{-j\beta_{zin} z} \right. \]

\[ \left. - E_{zihn1}^-(\beta_x) e^{j\beta_{zin} z} \right] \sin(\beta_{yn} y) \]

\[ \text{and} \]
\[
\vec{E}_{xi}(\beta_x, y, z) = \omega^2 \mu_0^2 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{-2\beta_{yn}}{(\beta_x^2 + \beta_{yn}^2)^2} \left[ \tilde{H}_{zithn1}^+(\beta_x)e^{-j\beta_{zn}z} \right] \right. \\
+ \left. \tilde{H}_{zithn1}^-(\beta_x)e^{j\beta_{zn}z} \right\} \sin(\beta_{yn}y) \\
+ \Gamma_{mn} \left[ j\beta_x G_2^2(G_{\beta}(U_{ibm}(\beta_x), \beta_{zim}), \beta_{oi}) \right] \left[ -\beta_{ym}e^{-j\beta_{yn}y} \right. \\
+ \left. \beta_{ym} \cos(\beta_{ym}y) \right] \\
+ \delta_{nm} \left[ j\beta_x G_2^2(G_{\beta}(U_{ibm}(\beta_x), \beta_{zim}), \beta_{oi}) \right] \left[ -\beta_{ym} \sin(\beta_{ym}y) \right] \\
- \frac{j\beta_x \delta_{nm}}{2\beta_{yn}} \left[ -e^{-j\beta_{yn}y} + \cos(\beta_{yn}y) \right. \\
- \left. y\beta_{yn} \sin(\beta_{yn}y) \right] \left[ G_2^2(U_{ibm}(\beta_x, z), \beta_{oi}) \right] \\
+ \omega \mu_0 \beta_x \sum_{n=1}^{\infty} \left. \frac{2\beta_{zn}}{(\beta_x^2 + \beta_{zn}^2)^2} \left[ \tilde{E}_{zithn1}^+(\beta_x)e^{-j\beta_{zn}z} \right] \right. \\
- \left. \tilde{E}_{zithn1}^-(\beta_x)e^{j\beta_{zn}z} \right\} \sin(\beta_{yn}y) \\
\]

where

\[
\vec{E}_{xt}(\beta_x, y, z) = \frac{\sigma_{Ti}(\beta_x)}{2\pi} \odot_{\beta_x} \tilde{\vec{E}}_{xt}(\beta_x, y, z)
\]
The $TE_0$ mode is independent of the $x$- component of the electric field and therefore does not contribute to the solution for $E_{xi}$.

In the same spirit as $\tilde{E}_{xi}$ and $\tilde{H}_{xi}$, using Eqs. (2.62b), and Eqs. (G-23a-c), (2.63b) $\tilde{E}_{yi}$ and $\bar{E}_{yi}$ can be written as

$$
\tilde{E}_{yi}(\beta_x, y, z) = \omega \mu_0 \beta_x \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{2}{\kappa^2 + \beta_y^2} \left[ \tilde{H}_{zithn1}^+(\beta_x) e^{-j \beta_{zin} z} + \tilde{H}_{zithn1}^-(\beta_x) e^{j \beta_{zin} z} \right] \cos(\beta_{yn} y) 
+ \tilde{G}_{yi} \left[ j \beta_x G_z(\sigma_{Ti}(\beta_x) \otimes \beta_x \tilde{H}_{zithn1}^+(\beta_x, \beta_{zi0}), \beta_{ti0}) \right] \cos(\beta_{yn} y) \right\} 
+ \bar{E}_{yi} + \tilde{E}_{yi}(\beta_x, y, z)\left\{ e^{-j \beta_{zio} z} + e^{j \beta_{zio} z} \right\} 
+ j \omega \mu_0 G_z(\sigma_{Ti}(\beta_x) \otimes \beta_x \bar{E}_{yi} + \tilde{E}_{yi}(\beta_x, \beta_{zi0}) \right\} 
+ j \omega \mu_0 G_z(\sigma_{Ti}(\beta_x) \otimes \beta_x \bar{E}_{yi}(\beta_x, \beta_{zi0}) 
+ j \omega \mu_0 G_z(\sigma_{Ti}(\beta_x) \otimes \beta_x \bar{E}_{yi}(\beta_x, \beta_{zi0}) 
$$

(6-15)
\begin{align}
&+ \sum_{n=1}^{\infty} 2\beta_{yn}\beta_{zn} \left[ E_{zn1}^{+}(\beta_x) e^{-j\beta_{zn}z} - E_{zn1}^{-}(\beta_x) e^{j\beta_{zn}z} \right] \cos(\beta_{yn}y) \\
&+ j\omega_0 \sum_{m=1}^{\infty} \left[ G_z(J_{eibm}(\beta_x), \beta_{0i}) \cos(\beta_{ym}y) + G_z(J_{oibm}(\beta_x), \beta_{0i}) \sin(\beta_{ym}y) \right]
\end{align}

\bar{E}_{yi}(\beta_x, y, z) = j\omega^2 \mu_0^2 \beta_x \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{2}{(\beta_x^2 + \beta_{yn}^2)^2} \left[ H_{zn1}^{+}(\beta_x) e^{-j\beta_{zn}z} + H_{zn1}^{-}(\beta_x) e^{j\beta_{zn}z} \right] \right\}

\begin{align}
&+ \bar{E}_{yi0}(\beta_x, z) + \sum_{n=0}^{\infty} \frac{2\beta_{yn}\beta_{zn}}{\beta_x^2 + \beta_{yn}^2} \left[ E_{zn1}^{+}(\beta_x) e^{-j\beta_{zn}z} - E_{zn1}^{-}(\beta_x) e^{j\beta_{zn}z} \right] \cos(\beta_{yn}y) \\
&- \omega^2 \mu_0^2 \sum_{m=1}^{\infty} \left[ G_z^2(J_{eibm}(\beta_x), \beta_{0i}) \cos(\beta_{ym}y) + G_z^2(J_{oibm}(\beta_x), \beta_{0i}) \sin(\beta_{ym}y) \right]
\end{align}

where
\[\bar{E}_{yi}(\beta_x, y, z) = \frac{\sigma_{Ti}(\beta_x)}{2\pi} \odot_{\beta_x} \bar{E}_{yi}(\beta_x, y, z) + \bar{E}_{yi0}(\beta_x, z) \tag{6.16b}\]

\[\bar{E}_{yi0}(\beta_x, z) = \frac{j\omega\mu_0}{2\pi} \left[ G_2(\sigma_{Ti}(\beta_x) \odot_{\beta_x} \bar{E}_{yi0}(\beta_x), \beta_{zi0}(\beta_x)) \right] \tag{6.16b}\]

Now consider Eqs. (2.62d) and (2.63d). Solving for the \(y\)-component of the magnetic field and its correction associated with medium inhomogeneities Eqs. (2.62d) and (2.63d) yields respectively

\[
\bar{H}_{yi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( \frac{2j\beta_{yn}\beta_{zin}}{\beta_x^2 + \beta_{yn}^2} [\bar{H}_{zithn1}(\beta_x)e^{-j\beta_{zin}z} \right.

\left. - \bar{H}_{zithn1}(\beta_x)e^{j\beta_{zin}z}] \sin(\beta_{yn}y) \right.

\left. + \bar{H}_{mn} \beta_x \beta_{zin} [G_2(G_{<z}(J_{oibm}, \beta_{zin}), \beta_{0i}) \right.

\left. - G_2(G_{>z}(J_{oibm}, \beta_{zin}), \beta_{0i}) \right] \right)

\left. + \frac{\omega\mu_0\beta_{zin}}{2\pi} \left[ G_2(G_{<z}(\sigma_{Ti}(\beta_x) \odot_{\beta_x} \bar{H}_{zoimf}, \beta_{zin}), \beta_{0i}) \right. \right.

\left. - G_2(G_{>z}(\sigma_{Ti}(\beta_x) \odot_{\beta_x} \bar{H}_{zoimf}, \beta_{zin}), \beta_{0i}) \right] \left[ -\beta_{ym}e^{-j\beta_{yn}y} \right. \right.

\left. + \beta_{ym} \cos(\beta_{ym}y) \right) \right\} + \left(6-17\right)
\[ \bar{H}_y(l, \beta_x, y, z) = j \omega \mu_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{2j \beta_y \beta_{zim}}{(\beta_x^2 + \beta_{yn}^2)^2} \left[ \bar{H}_{zim}^+ (\beta_x) e^{-j \beta_{zim} z} - \bar{H}_{zim}^- (\beta_x) e^{j \beta_{zim} z} \right] \sin(\beta_{yn} y) + \tilde{\Gamma}_{mn} \beta_x \beta_{zim} [G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi}) - G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi})] \right\} \]

\[ + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \tilde{\delta}_{nm} \left[ \beta_x \beta_{zim} [G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi}) - G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi})]\right] \]

and

\[ \begin{align*}
\bar{H}_y(l, \beta_x, y, z) &= j \omega \mu_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \frac{2j \beta_y \beta_{zim}}{(\beta_x^2 + \beta_{yn}^2)^2} \left[ \bar{H}_{zim}^+ (\beta_x) e^{-j \beta_{zim} z} - \bar{H}_{zim}^- (\beta_x) e^{j \beta_{zim} z} \right] \sin(\beta_{yn} y) + \tilde{\Gamma}_{mn} \beta_x \beta_{zim} [G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi}) - G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi})] \right\} \\
&+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \tilde{\delta}_{nm} \left[ \beta_x \beta_{zim} [G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi}) - G_{z < \bar{z}}(\sigma_{Tl}(\beta_x) \odot \beta_x \bar{H}_{zim} f(\beta_x, \beta_{zim}), \beta_{oi})]\right] \]
\end{align*} \tag{6.18a} \]
\[ + j \omega \mu_0 \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \left( \delta_{nm} \left[ \beta_x \beta_{zim} [G^2_z(G_{<\bar{z}}(j \epsilon_{ibm}, \beta_{zim}), \beta_{0i}) \right] - G^2_z(G_{>\bar{z}}(j \epsilon_{ibm}, \beta_{zim}), \beta_{0i}) \right] + \omega \mu_0 \beta_{zim} \frac{2 \pi}{2 \pi} \left[ \sigma_{Ti}(\beta_x) \otimes \beta_x \bar{H}_{zimf}(\beta_x, \beta_{zim}), \beta_{0i}) \right] \right) \left[ - \beta_{ym} \sin(\beta_{ym}y) \right] \] 

\[ = \frac{\beta_x \delta_{nm}}{2 \beta_{yn}} \left[ - e^{-j \beta_{yn}y} + \cos(\beta_{yn}y) - y \beta_{yn} \sin(\beta_{yn}y) \right] G^2_z \left( \frac{\partial}{\partial \bar{z}} I_{ibm}(\beta_x, \bar{z}), \beta_{0i}) \right) \] 

\[ = j \omega^2 \mu_0 \sum_{n=1}^{\infty} \left( 2j \beta_x e_{eff}^2(\omega) \left[ E^+_{zim1}(\beta_x) e^{-j \beta_{zim}^2} + E^-_{zim1}(\beta_x) e^{j \beta_{zim}^2} \right] \sin(\beta_{yn}y) \right) \] 

\[ - 2 \beta_x \sum_{n=1}^{\infty} \frac{1}{\beta_x^2 + \beta_{yn}^2} \left[ E^+_{zim1}(\beta_x) e^{-j \beta_{zim}^2} + E^-_{zim1}(\beta_x) e^{j \beta_{zim}^2} \right] \sin(\beta_{yn}y) \] 

where

\[ \bar{H}(\beta_x, y, z) = \frac{\sigma_{Ti}(\beta_x)}{2 \pi} \otimes \beta_x \bar{H}(\beta_x, y, z) \] 

(6.18b)

The TE\(_0\) mode is independent of the y-component of the magnetic field and therefore does not contribute to the solution for \(H_{yi}\).

The fundamental resonance contributions in the double summation associated with \(\delta_{nm}\) in Eqs. (6.9) - (6.18) are constrained to the physics of the problem when satisfying boundary conditions. Refer to the text in Section 5.1 about Eqs. (5.2) and (5.3).

The connecting equations to the field solutions in the TE\(_0\) mode are found in Appendix G. They are presented below for convenience.

\[ \bar{E}_{yil0}(\beta_x, z) = \bar{E}^+_{yil0}(\beta_x) e^{-j \beta_{zil0}z} + \bar{E}^-_{yil0}(\beta_x) e^{j \beta_{zil0}z} \] 

(G-19)

\[ \bar{E}_{yil0}(\beta_x, z) = \frac{-\omega \mu_0}{2 \beta_{zil0}} \int_{-\infty}^{\infty} \bar{E}_{iib0}(\beta_x, \bar{z}) e^{-j \beta_{zil0}|\bar{z} - z|} d\bar{z} = j \omega \mu_0 G_z(j \epsilon_{ilb0}(\beta_x), \beta_{zil0}) \] 

(G-20)

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CHAPTER 7. THEORETICAL MODEL AND BOUNDARY CONDITIONS

The general theory is now tailored to a particular model subject to a specific set of conditions. The parallel-plate waveguide is divided into four regions. Regions 1, 2, 3, and 4 lie between $-\infty$ and $l_1$, $l_1$ and $l_2$, $l_2$ and $l_3$, and $l_3$ and $\infty$ respectively. Regions 2 and 3 contain the cool plasma with energetic beam exhibiting a Gaussian-like distribution with respect to $x$. The plasma density is different in the two regions characterizing the inhomogeneous nature of the constricted plasma discharge with $z$. Within regions 2 and 3 the plasma with energetic beam is assumed to be uniform with $z$. This further implies that the energetic electron beam density is assumed to be constant with $y$ and $z$. Reradiation effects coupled to the beam have been neglected in this model. The inhomogeneous nature of the plasma with the beam has been retained in the longitudinal fields. The correction term to the transverse fields are neglected. These terms are sourced by the inhomogeneous nature of the medium by way of the convolution operations displayed on the right-hand side of Eqs. (2.63a-f).

Since the current density is a source term and source terms have been treated using a Greens function technique it stands to reason that this source term appears in the shorthand function notation $\mathcal{G}$. Therefore, let any function $f$ be independent of $z$. For generality sake let the wave number be $\beta_{zim}$ as given by Eqs. (3.4), (4-58), (5-17), and (4-66b). Then,

$$\mathcal{G}_e(f, \beta_{zim}) = \frac{ff}{2\beta_{zim}} \left[ \int_{-\infty}^{Z} e^{j\beta_{zim}(z-z')} d\bar{z} + \int_{Z}^{\infty} e^{-j\beta_{zim}(z-z')} d\bar{z} \right]$$

(7-1)

Since the loading effects of the plasma with the beam have been modeled as an effective permittivity $\varepsilon_{eff}$ in a homogeneous medium given by Eq. (2-29b). As indicated in Eqs. (2-53a,b), the inhomogeneous nature of the plasma is characterized by a nonuniform conductivity.
contribution to the permittivity of the slab. Expressing the wavenumber Eq. (3-4) in the $z$-direction in polar form yields

$$\beta_{zim} = \pm \left[ \psi_r + j \psi_j \right]^{1/2} = \pm \left[ \psi_r^2 + \psi_j^2 \right]^{1/4} e^{j \phi / 2}$$  \hspace{1cm} (7-2)

where

$$\Phi = \tan^{-1} \left( \frac{\psi_j}{\psi_r} \right)$$ \hspace{1cm} (7-3a)

$$\psi_r = \omega^2 \mu_0 \varepsilon_0 \left( \frac{\omega^2 + \nu_c^2 - \omega_{pel}^2}{\omega^2 + \nu_c^2} \right) - \beta_x^2 - \beta_y^2$$ \hspace{1cm} (7-3b)

$$\psi_j = -\omega \mu_0 \left( \frac{\nu_c \omega_{pel}^2}{\omega^2 + \nu_c^2} \right)$$ \hspace{1cm} (7-3c)

The direction of propagation has already been correctly chosen based on physics in the wave solution. Therefore, the upper sign of Eq. (7-2) is chosen. When

$$\omega^2 \mu_0 \varepsilon_0 (\omega^2 + \nu_c^2) > \omega^2 \mu_0 \varepsilon_0 \omega_{pel}^2 + (\beta_x^2 + \beta_y^2)(\omega^2 + \nu_c^2)$$

then $-90^\circ < \Phi < 0$. This leads to the physical case for the wave attenuation as it propagates on the $+z$ direction. Since the medium is passive one can only expect wave attenuation in the direction of propagation. Therefore, it can be shown that

$$\int_{-\infty}^{z} e^{j \beta_{zim}(z-z')}d\tilde{z} = \frac{1}{j \beta_{zim}}$$ \hspace{1cm} (7-4a)

$$\int_{z}^{\infty} e^{-j \beta_{zim}(z-z')}d\tilde{z} = \frac{1}{j \beta_{zim}}$$ \hspace{1cm} (7.4b)

Consequently, for $f$ to be a function independent of $z$. 

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\[ G_z(f, \beta_{zim}) = \frac{f}{\beta_{zim}^2} \]  

(7-5)

\[ G_{<z}(f, \beta_{zim}) = G_{>z}(f, \beta_{zim}) = \frac{f}{2\beta_{zim}^2} \]  

(7-6)

By analogy,

\[ G_z(f, \beta_{0i}) = \frac{f}{\beta_{0i}^2} \]  

(7-7)

\[ G_{<z}(f, \beta_{0i}) = G_{>z}(f, \beta_{0i}) = \frac{f}{2\beta_{0i}^2} \]  

(7-8)

where \( \beta_{0i} = \omega \sqrt{\mu_0 \varepsilon_{eff} \omega} \). These results may be extended to the nested integration operations yielding

\[ G_z( G_{<z}(f, \beta_{zim}), \beta_{0i}) - G_z( G_{>z}(f, \beta_{zim}), \beta_{0i}) = 0 \]  

(7-9)

\[ G_z( G_z(f, \beta_{zim}), \beta_{0}) = \frac{1}{\beta_{zim}^2} \frac{f}{\beta_{0i}^2} \]  

(7-10)

\[ G^2_z( G_{<z}(f, \beta_{zim}), \beta_{0i}) = \frac{1}{\beta_{0i}^2} \frac{f}{\beta_{zim}^2} \]  

(7-11)

\[ G^2_z( G_{<z}(f, \beta_{zim}), \beta_{0}) - G^2_z( G_{>z}(f, \beta_{zim}), \beta_{0i}) = 0 \]  

(7-12)

where under the constraints of this model \( f \) is \( f_{kibm}(\beta_x) \) or \( \sigma_{Ti}(\beta_x) \odot \tilde{\eta}_{zimf}(\beta_x) \) and \( k = e, o \).

The spatial uniformity of the current density in the \( y \) direction implies all \( m \) terms in the series expansion except \( m = 0 \) vanishes. Further this implies by the symmetry of the current density for \( m = 0 \), the odd symmetric component of the current density is zero. Consequently, the beam current density as given by Eq. (2-13) can be written as
\[
\tilde{J}_{bpl}(\beta_x, \omega) = \hat{y} \left[ 0.5 \left( 1 - (-1)^i (1 - 2\xi) \right) \right] \frac{1}{\Delta l_i} e^{-\beta_x^2 \frac{t^2}{4\alpha}} \int_{-\infty}^{\infty} I_d(t)e^{-j\omega t} dt
\]

\[= \sum_{m=0}^{\infty} \left[ J_{eibm}(\beta_x, \omega) \cos(\beta_y m y) + J_{oibm}(\beta_x, \omega) \sin (\beta_y m y) \right] \]

\[J_{eibm}(\beta_x, \omega) = 0 \text{ for all } m \neq 0 \tag{7.14a}\]

\[J_{oibm}(\beta_x, \omega) = 0 \text{ for all } m \tag{7.14b}\]

With the neglect of correction terms for the transverse fields, all convolution contributions are neglected except for one. For the resonance response, the inhomogeneous nature of the mediums with respect to \(x\) is coupled through this term to the field solutions for the homogeneous medium. With the aid of Eqs. (5-11), (5-12), (7-1), (7-5), (2-53b), (2-13) and (4-1), the convolution contribution is

\[
\sigma_T(\beta_x, \omega) \odot^\beta_x H_{zklmf}(\beta_x, z) = \sigma_T(\beta_x) \odot^\beta_x \left[ j\beta_x G_z(\beta_x, \omega) \right]
\]

\[= \sigma_T(\beta_x) \odot^\beta_x \left[ \frac{j\beta_x}{\beta_{zim}(\beta_x)} I_{kibm}(\beta_x, \omega) \right] \]

\[= T_i \frac{j}{\Delta l_i} \int_{-\infty}^{\infty} I_d(t)e^{-j\omega t} dt \left[ \tilde{\sigma}_{pl}(\omega) + \tilde{\sigma}_{bi}(\omega) \right] \left[ \frac{\pi}{\sqrt{\alpha}} e^{-\beta_x^2 \frac{t^2}{4\alpha}} \right] \]

\[= 2\pi\delta(\beta_x) \odot^\beta_x \left[ \frac{\beta_x}{\beta_{zim}(\beta_x)} e^{-\beta_x^2 \frac{t^2}{4\alpha}} \right] \]

where \(T_i = 0.5 \left( 1 - (-1)^i (1 - 2\xi) \right)\). Only the \(m = 0\) term in the current density expansion exists. Therefore, \(\beta_{zim}^2(\beta_x) = \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2\) where \(\varepsilon_{eff}\) is independent of \(\beta_x\). Performing the convolution operation and rearranging yields
\[ \sigma_{T_i}(\beta_x) \odot \beta_x \bar{H}_{z kim}(\beta_x, z) \]

\[ = \left[ \frac{j}{\Delta l_i} \bar{T}_i \bar{T}_i(\omega) \int_{-\infty}^{\infty} I_d(t) e^{-j\omega t} dt \right] \left[ -\frac{2\pi \beta_x}{\beta_{zim}^2} e^{-\frac{\beta_x^2}{4\alpha}} \right] \]

\[ + \left[ \frac{\bar{T}_x}{\alpha} e^{-\frac{\beta_x^2}{4\alpha}} \int_{-\infty}^{\infty} \frac{\beta_x}{\beta_{zim}^2} e^{-(\beta_x^2-\beta_x\bar{\beta}_x)/2\alpha} d\beta_x \right]_{m=0} \]

where \( \beta_{0i} = \omega \sqrt{\mu_0 \varepsilon_{eff}} = \omega \sqrt{\mu_0 \varepsilon_0} \left[ 1 + \frac{\sigma_{T_i}(\omega)}{\omega \varepsilon_0} \right]^{1/4} \left[ \cos \frac{\Phi_i}{2} - j \sin \frac{\Phi_i}{2} \right] \) and \( \bar{T}_i(\omega) \) is given by Eq. (2.29d). Note \( \beta_{0i} = \beta_{zim} \) when \( \beta_x \) and \( \beta_{ym} \) in Eqs. (7.2) and (7.3a-b) are set equal to zero.

For a current density independent of \( y \) and \( z \) (\( m=0 \)), the fields in region \( i = 2, 3 \) containing the plasma and the beam contributions are

\[ \bar{H}_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} 2[\bar{H}^+_{zihn1}(\beta_x)e^{-j\beta_{xin}z} + \bar{H}^-_{zihn1}(\beta_x)e^{j\beta_{xin}z}] \cos(\beta_{yn}y) \]

\[ + \frac{j\omega \mu_0}{2\pi \beta_{zim}^2} [\sigma_{T_i}(\beta_x) \odot \beta_x \frac{j\beta_x}{\beta_{zim}^2} J_{eib0}(\beta_x)] \]

\[ + \frac{\beta_x}{\omega \mu_0} [\bar{E}^+_{yih}(\beta_x)e^{-j\beta_{zio}z} + \bar{E}^-_{yih}(\beta_x)e^{j\beta_{zio}z}] + \frac{j\beta_x}{\beta_{zim}^2} J_{eib0}(\beta_x) \]

\[ - \frac{\omega \mu_0 \beta_x}{2\pi \beta_{zim}^2} [\sigma_{T_i}(\beta_x) \odot \beta_x \frac{1}{\beta_{zim}^2} J_{eib0}(\beta_x)] \]

\[ E_{zi}(\beta_x, y, z) = \sum_{n=1}^{\infty} j2[\bar{E}^+_{zihn1}(\beta_x)e^{-j\beta_{xin}z} + \bar{E}^-_{zihn1}(\beta_x)e^{j\beta_{xin}z}] \sin(\beta_{yn}y) \]

\[ J_{yib} = J_{eib0}(\beta_x, \omega) = \left[ 0.5 \left( 1 - (-1)^i(1 - 2\xi) \right) \right] \frac{1}{\Delta l_i} e^{-\frac{\beta_x^2}{4\alpha}} \int_{-\infty}^{\infty} I_d(t) e^{-j\omega t} dt \]
\[
\bar{H}_{xi}(\beta_x, y, z) = -j \beta_x \sum_{n=1}^{\infty} \frac{j2 \beta_{zin}}{\beta_x^2 + \beta_{yn}^2} \left[ -\bar{H}_{zih11}(\beta_x) e^{-j \beta_{zin} z} + \bar{H}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \cos(\beta_{yn} y) \\
- 2\omega \sum_{n=1}^{\infty} \frac{\beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e_{eff}(\omega) \left[ \bar{E}_{zih11}^+(\beta_x) e^{-j \beta_{zin} z} + \bar{E}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \cos(\beta_{yn} y) \\
- \frac{\beta_{zio}}{\omega \mu_0} \left[ \bar{E}_{yih}(\beta_x) e^{-j \beta_{zio} z} - \bar{E}_{yih}^-(\beta_x) e^{j \beta_{zio} z} \right]
\]

\[
\bar{E}_{xi}(\beta_x, y, z) = j \omega \mu_0 \sum_{n=1}^{\infty} \frac{2 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{H}_{zih11}(\beta_x) e^{-j \beta_{zin} z} + \bar{H}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \sin(\beta_{yn} y) \\
- j \beta_y \sum_{n=1}^{\infty} \frac{2 \beta_{zin}}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{E}_{zih11}^+(\beta_x) e^{-j \beta_{zin} z} - \bar{E}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \sin(\beta_{yn} y)
\]

\[
\bar{E}_{yi}(\beta_x, y, z) = \left\{ \frac{j \omega \mu_0}{\beta_{zio}^2} + \frac{j \omega \mu_0 \beta_x^2}{\beta_{zio}^2 + \beta_{yn}^2} \right\} \bar{J}_{eib0}(\beta_x) + \frac{j \omega^2 \mu_0^2 \beta_x}{2 \pi \beta_{zio}^2} \left[ \sigma_{T1}(\beta_x) \otimes \beta_x \frac{j \beta_y}{\beta_{zio}^2} \bar{J}_{eib0}(\beta_x) \right] \\
- \frac{\omega^2 \mu_0^2}{2 \pi \beta_{zio}^2} \left[ \sigma_{T1}(\beta_x) \otimes \beta_x \frac{1}{\beta_{zio}^2} \bar{J}_{eib0}(\beta_x) \right] \\
+ \omega \mu_0 \beta_x \sum_{n=1}^{\infty} \frac{2}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{H}_{zih11}(\beta_x) e^{-j \beta_{zin} z} + \bar{H}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \cos(\beta_{yn} y) \\
+ \sum_{n=1}^{\infty} \frac{2 \beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{E}_{zih11}^+(\beta_x) e^{-j \beta_{zin} z} - \bar{E}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \cos(\beta_{yn} y) \\
+ \bar{E}_{yih}^+(\beta_x) e^{-j \beta_{zio} z} + \bar{E}_{yih}^-(\beta_x) e^{j \beta_{zio} z}
\]

\[
\bar{H}_{yi}(\beta_x, y, z) = \sum_{n=1}^{\infty} \left\{ \frac{2 j \beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{H}_{zih11}(\beta_x) e^{-j \beta_{zin} z} - \bar{H}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \sin(\beta_{yn} y) \\
- \omega \sum_{n=1}^{\infty} \frac{j2 \beta_x \epsilon_{eff}^2}{\beta_x^2 + \beta_{yn}^2} \left[ \bar{E}_{zih11}(\beta_x) e^{-j \beta_{zin} z} + \bar{E}_{zih11}^-(\beta_x) e^{j \beta_{zin} z} \right] \sin(\beta_{yn} y) \right\}
\]

where \( \beta_{yn} = \frac{n \pi}{d} \)
\[
\beta_{zin} = \left[\omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2 - \beta_{yn}^2\right]^{1/2}
\]

\[
\beta_{zi0} = \left[\omega^2 \mu_0 \varepsilon_{eff}(\omega) - \beta_x^2\right]^{1/2}
\]

\[
\sigma_{ti}(\beta_x, \omega) = \left(\bar{\sigma}_{pl}(\omega) + \bar{\sigma}_{bl}(\omega)\right)\left[\sqrt{\pi} e^{\frac{\beta_x^2}{4\alpha}} - 2\pi \delta(\beta_x)\right]
\]

For simplicity in notation, let the TE mode constant of integration, \(2\tilde{H}_{z i h n1}\), and the TM mode amplitudes, \(2\tilde{E}_{z i h n1}\), be denoted by \(A_{in}^{\pm}(\beta_x)\) and \(B_{in}^{\pm}(\beta_x)\) respectively. The fundamental dependence of \(A_{in}^{\pm}\) and \(B_{in}^{\pm}\) on \(\beta_x\) will be suppressed for simplicity in notation only and is implied. The functional dependence of \(\tilde{E}_{z i h n}^{\pm}\) on \(\beta_x\) will be retained due to unique notation to represent both the \(TE_0\) and TEM modes. Then, making use of the tilde fields components above as the components of the corresponding electric and magnetic fields in region \(i\) for \(i = 2, 3\). The transverse fields with subscript ‘Ti’ and longitudinal fields with subscript ‘Li’ in region \(i\) for \(i = 2, 3\) are given below.
\[
\tilde{E}_{T1}(\beta_x, y, z) = \hat{x} \sum_{n=1}^{\infty} \left[ j \omega \mu_0 \beta_{yn} \frac{1}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn} z} A^+_n + j \omega \mu_0 \beta_{yn} \frac{1}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{zn} z} A^-_n \right] \\
+ \frac{-j \beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn} z} B^+_n + \frac{j \beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{zn} z} B^-_n \sin(\beta_{yn} y) \\
+ \hat{y} \left[ \left( \frac{j \omega \mu_0}{\beta_x^2 + \beta_{yn}^2} + j \omega \mu_0 \frac{\beta_x^2}{\beta_x^2 + \beta_{yn}^2} \right) J_{elbo}(\beta_x) \right] \\
+ \frac{j \omega^2 \mu_0^2 \beta_x}{2\pi \beta_{0i}^2 \beta_{zi0}^2} \left[ \sigma_{T1}(\beta_x) \circ \beta_x \frac{j \beta_x}{\beta_x^2 + \beta_{zi0}^2} J_{elbo}(\beta_x) \right] \\
- \frac{\omega^2 \mu_0^2}{2\pi \beta_{zi0}^2} \left[ \sigma_{T1}(\beta_x) \circ \frac{1}{\beta_x^2 + \beta_{zi0}^2} J_{elbo}(\beta_x) \right] + e^{-j \beta_{zi0} z} \tilde{E}^+_y E_{y1h}(\beta_x) \\
+ e^{j \beta_{zi0} z} \tilde{E}^-_y E_{y1h}(\beta_x) \\
+ \sum_{n=1}^{\infty} \left[ \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn} z} A^+_n + \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{zn} z} A^-_n \right] \\
+ \frac{\beta_{yn} \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn} z} B^+_n - \frac{\beta_{yn} \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{zn} z} B^-_n \cos(\beta_{yn} y) \right] \]

(7-17)
\[ H_{Ti}(\beta_x, y, z) = \hat{2} \left\{ -\frac{\beta_{z_0}}{\omega \mu_0} [e^{-j\beta_{z_0} z} \tilde{E}_{yih}^+ (\beta_x) - e^{j\beta_{z_0} z} \tilde{E}_{yih}^- (\beta_x)] \right. \\
\left. + \sum_{n=1}^\infty \frac{-\beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j\beta_{zn}^2 A_{in}^+} + \frac{\beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{zn}^2 A_{in}^-} \right. \\
\left. + \frac{-\omega \beta_{yn} \varepsilon_{eff} (\omega)}{\beta_x^2 + \beta_{yn}^2} e^{-j\beta_{zn}^2 B_{in}^+} \left[ \cos (\beta_{yn} y) \right] \right\} \\
\left( 7-18 \right) \\
\left. + \hat{y} \sum_{n=1}^\infty \frac{j \beta_y \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j\beta_{zn}^2 A_{in}^+} - \frac{j \beta_y \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{zn}^2 A_{in}^-} \\
- \frac{j \omega \beta_x \varepsilon_{eff}}{\beta_x^2 + \beta_{yn}^2} e^{-j\beta_{zn}^2 B_{in}^+} \left[ \sin (\beta_{yn} y) \right] \right]\] \\

and

\[ H_{Li}(\beta_x, y, z) = \hat{2} H_{zi}(\beta_x, y, z) \\
= \hat{2} \left\{ \sum_{n=1}^\infty [e^{-j\beta_{zn}^2 A_{in}^+} + e^{j\beta_{zn}^2 A_{in}^-}] \cos (\beta_{yn} y) \right. \\
\left. + \frac{j \omega_0}{2\pi \beta_{z_0}^2} \left[ \sigma_{Ti}(\beta_x) \bigotimes \beta_x \frac{j \beta_x}{\beta_{z_0}^2} J_{ebo}(\beta_x) \right] \right. \\
\left. + \frac{\beta_x}{\omega_0} \left[ \tilde{E}_{yih}^+ (\beta_x) e^{-j\beta_{z_0} z} + \tilde{E}_{yih}^- (\beta_x) e^{j\beta_{z_0} z} \right] + \frac{j \beta_x}{\beta_{z_0}^2} J_{ebo}(\beta_x) \right. \\
\left. - \frac{\omega_0 \beta_x}{2\pi \beta_{z_0}^2} \left[ \sigma_{Ti}(\beta_x) \bigotimes \beta_x \frac{1}{\beta_{z_0}^2} J_{ebo}(\beta_x) \right] \right\} \right\} \\

\[ \tilde{E}_{Li}(\beta_x, y, z) = \hat{2} E_{zi}(\beta_x, y, z) = \hat{2} \sum_{n=1}^\infty j \left[ e^{-j\beta_{zn}^2 B_{in}^+} + e^{j\beta_{zn}^2 B_{in}^-} \right] \sin (\beta_{yn} y) \]
The TE and TM modes in regions 1 and 4 are to be determined from the summarized expressions of Eqs. (7.17) and (7.18). All current and plasma contributions are equal to zero. We should recover the empty waveguide fields with the spatial variation in x. In general, for the $i^{th}$ empty waveguide:

$$
\tilde{E}_{Tl}(\beta_x, y, z) = \hat{x} \sum_{n=1}^{\infty} \left\{ \frac{j \omega}{\beta_x^2 + \beta_y^2} e^{-j \beta_{xin} z} A_{in}^+ + \frac{j \omega \mu_0}{\beta_x^2 + \beta_y^2} e^{j \beta_{xin} z} A_{in}^- 
+ \frac{-j \beta_x \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} B_{in}^+ + \frac{j \beta_x \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} B_{in}^- \right\} \sin(\beta_{yn} y)
+ \hat{y} \left\{ e^{-j \beta_{zio} z} \tilde{E}_{yinh}(\beta_x) + e^{j \beta_{zio} z} \tilde{E}_{yinh}(\beta_x) \right\}
+ \sum_{n=1}^{\infty} \left\{ \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_y^2} e^{-j \beta_{xin} z} A_{in}^+ + \frac{\omega \mu_0 \beta_y}{\beta_x^2 + \beta_y^2} e^{j \beta_{xin} z} A_{in}^- 
+ \frac{-\beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} B_{in}^+ - \frac{-\beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} B_{in}^- \right\} \cos(\beta_{yn} y) \right\}
+ \hat{z} \left\{-\frac{\beta_{zio}}{\omega \mu_0} \left[ e^{-j \beta_{zio} z} \tilde{E}_{yinh}(\beta_x) - e^{j \beta_{zio} z} \tilde{E}_{yinh}(\beta_x) \right] \right\}
+ \sum_{n=1}^{\infty} \left\{ \frac{-\beta_x \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} A_{in}^+ + \frac{\beta_x \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} A_{in}^- 
+ \frac{-\omega \beta_{yn} \varepsilon_{eff}(\omega)}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} B_{in}^+ + \frac{-\omega \beta_{yn} \varepsilon_{eff}(\omega)}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} B_{in}^- \right\} \cos(\beta_{yn} y) \right\}
+ \hat{y} \sum_{n=1}^{\infty} \left\{ \frac{j \beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} A_{in}^+ - \frac{j \beta_{yn} \beta_{zin}}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} A_{in}^- 
+ \frac{-j \omega \beta_x \varepsilon_{eff i}}{\beta_x^2 + \beta_y^2} e^{-j \beta_{zin} z} B_{in}^+ + \frac{-j \omega \beta_x \varepsilon_{eff i}}{\beta_x^2 + \beta_y^2} e^{j \beta_{zin} z} B_{in}^- \right\} \sin(\beta_{yn} y) \right\} \right\}
\( \vec{H}_{li}(\beta_x, y, z) \)

\[
\begin{aligned}
= \hat{\beta} \left( \sum_{n=1}^{\infty} \left[ e^{-j\beta_{zin}^2 A_{in}^+} + e^{j\beta_{zin}^2 A_{in}^-} \right] \cos(\beta_{yn} y) \\
+ \frac{\beta_x}{\omega \mu_0} \left[ \vec{E}_{y1h}^+(\beta_x) e^{-j\beta_{zo}^2} + \vec{E}_{y1h}^-(\beta_x) e^{j\beta_{zo}^2} \right] \right)
\end{aligned}
\]

\( \vec{E}_{li}(\beta_x, y, z) = \hat{\beta} \sum_{n=1}^{\infty} \left[ e^{-j\beta_{zin}^2 B_{in}^+} + e^{j\beta_{zin}^2 B_{in}^-} \right] \sin(\beta_{yn} y) \)

A TEM incident field generated from the external source (antenna), and reflected fields in due to the multi-medium loading effects of the cascaded regions exist in region 1. Plasma and the electron beam in region 2 and 3 exist in region 1. Therefore, in region 1 the transverse fields are

\( \vec{E}_{r1}(\beta_x, y, z) = \hat{\beta} \sum_{n=1}^{\infty} \left[ j\omega \mu_0 \beta_{yn} e^{j\beta_{zin}^2 A_{1n}^-} \right. \\
+ \frac{j\beta_x \beta_{zo}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{zo}^2 B_{1n}^-} \left. \right] \sin(\beta_{yn} y) \\
+ \hat{y} \left\{ 2\pi \delta(\beta_x) e^{-j\beta_{zo}^2 E_{y1h}^+(\beta_x)} + e^{j\beta_{zo}^2 E_{y1h}^-(\beta_x)} \right\} + j\sum_{n=1}^{\infty} \left( \omega \mu_0 \beta_x \right. \left. \beta_{zn} \beta_{zo} \right) e^{j\beta_{zn}^2 A_{1n}^-} \\
+ \frac{\beta_{yn} \beta_{zo}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{zo}^2 B_{1n}^-} \right] \cos(\beta_{yn} y) \right) \}

(7-19)
\[ \tilde{H}_{T1}(\beta_x, y, z) = \hat{x} \left\{ -\frac{\beta_{z10}}{\omega \mu_0} 2\pi \delta(\beta_x) e^{-j\beta_{z10}z} \tilde{E}_{y1ha}^+(\beta_x) + \frac{\beta_{z10}}{\omega \mu_0} e^{j\beta_{z10}z} \tilde{E}_{y1h}^-(\beta_x) \right. \\
+ \sum_{n=1}^{\infty} \left[ \frac{\beta_x \beta_{z1n}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{z1n}z} A_{1n}^- \right] - \frac{\omega \varepsilon_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{j\beta_{z1n}z} B_{1n}^- \cos(\beta_{yn} y) \right\} \] (7-20)

\[ \tilde{H}_{L1}(\beta_x, y, z) = \hat{x} \left\{ \sum_{n=1}^{\infty} e^{j\beta_{z1n}z} A_{1n}^- \cos(\beta_{yn} y) \right. \\
+ \frac{\beta_x}{\omega \mu_0} \left[ 2\pi \delta(\beta_x) e^{-j\beta_{z10}z} \tilde{E}_{y1ha}^+(\beta_x) + \tilde{E}_{y1h}^-(\beta_x) e^{j\beta_{z10}z} \right] \right\} \\
\tilde{E}_{L1}(\beta_x, y, z) = \hat{x} \sum_{n=1}^{\infty} j e^{j\beta_{z1n}z} B_{1n}^- \sin(\beta_{yn} y) \]

where \( \beta_{yn} = \frac{n\pi}{d} \)

\[ \beta_{z1n}^2 = \omega^2 \mu_0 \varepsilon_0 - (\beta_x^2 + \beta_{yn}^2) \]

Region 4 is the region in the waveguide where only transmitted fields exist. These transmitted fields are due to the loading effects of the source beam current density and the external source in region 1.
\[ \tilde{E}_{T4}(\beta_x, y, z) = \hat{x} \sum_{n=1}^{\infty} \left[ \frac{j \omega \mu_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 A_{4n}^+} - \frac{j \beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 B_{4n}} \right] \sin(\beta_{yn} y) + \hat{y} \left( e^{-j \beta_{zn} z} \tilde{E}_{y4h}(\beta_x) \right) \]

\[ + \sum_{n=1}^{\infty} \left[ \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 A_{4n}^+} + \frac{\beta_{yn} \beta_{4n}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 B_{4n}^+} \right] \cos(\beta_{yn} y) \]  

\[ \tilde{H}_{T4}(\beta_x, y, z) = \hat{x} \left\{ -\frac{\beta_{40}}{\omega \mu_0} e^{-j \beta_{zn}^2 E_{y4h}(\beta_x)} + \sum_{n=1}^{\infty} \left[ -\frac{\beta_x \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 A_{4n}^+} - \frac{\omega \varepsilon_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 B_{4n}^+} \right] \cos(\beta_{yn} y) \right\} + \hat{y} \sum_{n=1}^{\infty} \left[ \frac{j \beta_{yn} \beta_{zn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 A_{4n}^+} + \frac{j \omega \varepsilon_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{zn}^2 B_{4n}^+} \right] \sin(\beta_{yn} y) \]  

(7-21)

and the longitudinal fields are

\[ \tilde{H}_{LA}(\beta_x, y, z) = \hat{z} \sum_{n=1}^{\infty} e^{-j \beta_{zn}^2 A_{4n}^+} \cos(\beta_{yn} y) + \frac{\beta_x}{\omega \mu_0} E_{y4h}(\beta_x) e^{-j \beta_{zn}^2 z} \]

\[ \tilde{E}_{LA}(\beta_x, y, z) = \hat{z} \sum_{n=1}^{\infty} j e^{-j \beta_{zn}^2 A_{4n}^+} \sin(\beta_{yn} y) \]
where \( \beta_{z4n}^2 = \omega^2 \mu_0 \varepsilon_0 - (\beta_x^2 + \beta_{yn}^2) \)

7.1.1 Boundary Conditions

The field solutions in each of the four regions are summarized subject to boundary conditions. In region 1, an undesired wave is generated by an external source propagating in the \( z \)-direction towards the constricted plasma. It is desired that the incident beam be attenuated by the plasma or coupled to other modes supported by the structure. In this section boundary conditions are defined and satisfied assuming normal incidence. Due to the property that a discrete set of standing waves resonate with \( y \), functional orthogonality in \( y \) is applied to separate the modes of the system.

Regions 1 and 4 do not contain an ionized gas. The incident (superscript +) positive propagating wave in region 1 is due to the presence of the external wave to be disrupted. The negative propagating waves in region 1 are a consequence of reflection of the incident wave at the boundary between mediums 1 and 2 plus the contribution of waves transmitted from region 2 to region 1 as the consequence of active and passive medium loading effects of regions 2, 3 and 4. The inhomogeneous nature of the adjoining medium (region 2), through boundary conditions can drive an inhomogeneous wave in region 1. Region 1 can support waves with amplitudes that are dependent and independent of \( x \). To determine expressions for those fields with field amplitudes having \( x \) (or \( \beta_x \)) variation in region 1, all source contributions are forced to be zero in Eqs. (2.62a-f). For the case that the fields are independent of \( x \), the field amplitudes are further constrained to be an appropriate constant, subject to boundary conditions, times \( 2\pi \delta(\beta_x) \). Taking the inverse Fourier transform of these field components is equivalent to forcing \( \beta_x = 0 \) in these field components. By analogy, the field solutions in region 4 have the same form but differ in the
constants of integration. In region 4 Sommerfeld radiation condition excludes the existence of negative propagating fields in the region.

The boundary conditions are to be satisfied at the \(z=\text{constant} \) planes separating adjoint regions. Wave propagation at normal incidence is assumed. Region 1 and 4 are free space and regions 2 and 3 contain the plasma with energetic electron beam. Thus, we have three interfaces,

1. Interface (1) - (2) \( z = l_1 \)
2. Interface (2) - (3) \( z = l_2 \)
3. Interface (3) - (4) \( z = l_3 \)

The \( z = l_i \) interface does not support surface charges or surface currents. Therefore, the boundary conditions require the tangential fields in the adjacent regions to the \( z = l_i \) interfaces to be continuous across the boundary interface for all \( x \) and \( y \) within the parallel-plate waveguide.

\[
\begin{align*}
\tilde{E}_{x1}(\beta_x, y, z = l_i^-) &= \tilde{E}_{x(i+1)}(\beta_x, y, z = l_i^+) \quad (7.23a) \\
\tilde{E}_{y1}(\beta_x, y, z = l_i^-) &= \tilde{E}_{y(i+1)}(\beta_x, y, z = l_i^+) \quad (7.23b) \\
\tilde{H}_{x1}(\beta_x, y, z = l_i^-) &= \tilde{H}_{x(i+1)}(\beta_x, y, z = l_i^+) \quad (7.23c) \\
\tilde{H}_{y1}(\beta_x, y, z = l_i^-) &= \tilde{H}_{y(i+1)}(\beta_x, y, z = l_i^+) \quad (7.23d)
\end{align*}
\]

where \( i = 1,2,3 \). Thus, satisfying Eqs. (7.23a-d) and enforcing the orthogonality condition with regards to \( y \), we have the following constraints at each interface

1. Interface (1) - (2): \( [z = l_1] \)
\[
\tilde{E}_{x1}(\beta_x, y, z = l_1^-) = \tilde{E}_{x2}(\beta_x, y, z = l_1^+)
\]
\[
\frac{j \omega \mu_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x1n} l_1} A_{1n}^- + \frac{j \beta_x \beta_{z1n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z1n} l_1} B_{1n}^- \\
= \frac{j \omega \mu_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{x2n} l_1} A_{2n}^+ + \frac{j \omega \mu_0 \beta_{yn}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x2n} l_1} A_{2n}^- \\
+ \frac{-j \beta_x \beta_{x2n}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{x2n} l_1} B_{2n}^+ + \frac{j \beta_x \beta_{x2n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x2n} l_1} B_{2n}^-
\]
(7.24a)

\[
e^{-j \beta_{x10} l_2} 2 \pi \delta(\beta_x) \tilde{E}_{y1h}^+ + e^{j \beta_{x10} l_2} \tilde{E}_{y1h}^- = K_2 + e^{-j \beta_{x20} l_2} \tilde{E}_{y2h}^+ + e^{j \beta_{x20} l_2} \tilde{E}_{y2h}^- 
\]
(7.24b)

\[
\frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x1n} l_1} A_{1n}^- + \frac{j \beta_{yn} \beta_{z1n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z1n} l_1} B_{1n}^- \\
= \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{x2n} l_1} A_{2n}^+ + \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x2n} l_1} A_{2n}^- \\
+ \frac{\beta_{yn} \beta_{x2n}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{x2n} l_1} B_{2n}^+ - \frac{\beta_{yn} \beta_{x2n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{x2n} l_1} B_{2n}^-
\]
(7.24c)

\[
\tilde{H}_{x1}(\beta_x, y, z = l_1^-) = \tilde{H}_{x2}(\beta_x, y, z = l_1^+) 
\]
(7.24d)

\[
-\frac{\beta_{x10}}{\omega \mu_0} 2 \pi \delta(\beta_x) e^{-j \beta_{x10} l_2} \tilde{E}_{y1h}^+ + \frac{\beta_{x10}}{\omega \mu_0} e^{j \beta_{x10} l_2} \tilde{E}_{y1h}^- \\
= -\frac{\beta_{x20}}{\omega \mu_0} [e^{-j \beta_{x20} l_2} \tilde{E}_{y2h}^+ - e^{j \beta_{x20} l_2} \tilde{E}_{y2h}^-]
\]
(7.24e)
where

\[ K_i = \left( \frac{j\omega\mu_0}{\beta_{zi0}} + \frac{j\omega\mu_0\beta_x^2}{\beta_{zi0}^2} \right) J_{elb0}(\beta_x) + \frac{j\omega^2\mu_0^2\beta_x}{2\pi\beta_{zi0}^2} \left[ \sigma_{Ti}(\beta_x) \otimes \beta_x \frac{j\beta_x}{\beta_{zi0}} J_{elb0}(\beta_x) \right] \]

\[ - \frac{\omega^2\mu_0^2}{2\pi\beta_{zi0}^2} \left[ \sigma_{Ti}(\beta_x) \otimes \beta_x \frac{1}{\beta_{zi0}} J_{elb0}(\beta_x) \right] \]  

(7.25)

where \( i = 2, 3 \) and \( n = 1, 2, \ldots \).

2. Interface (2) - (3): [\( z = l_2 \)]

\[ \mathcal{E}_{x2}(\beta_x, y, z = l_2^-) = \mathcal{E}_{x3}(\beta_x, y, z = l_2^+) \]
\[
\frac{j\omega_0\beta_yn}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_2 l_2} A_2^+ + \frac{j\omega_0\beta_yn}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_2 l_2} A_2^- + \frac{-j\beta_x \beta_z n}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_2 l_2} B_{2n}^-
\]
\[
+ \frac{j\beta_x \beta_z n}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_2 l_2} B_{2n}^-
\]
\[
= \frac{j\omega_0\beta_yn}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_3 l_2} A_3^+ + \frac{j\omega_0\beta_yn}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_3 l_2} A_3^-
\]
\[
+ \frac{-j\beta_x \beta_z n}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_3 l_2} B_{3n}^+ + \frac{j\beta_x \beta_z n}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_3 l_2} B_{3n}^-
\]
\[
E_{y2}(\beta_x, y, z = l_2^-) = E_{y3}(\beta_x, y, z = l_2^+)
\]
\[
K_2 + e^{-j\beta_x z_2 l_2} \bar{E}_{y2h} + e^{j\beta_x z_2 l_2} \bar{E}_{y2h} = K_3 + e^{-j\beta_x z_3 l_2} \bar{E}_{y3h} + e^{j\beta_x z_3 l_2} \bar{E}_{y3h}
\]
\[
\frac{\omega_0\beta_x}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_2 l_2} A_2^+ + \frac{\omega_0\beta_x}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_2 l_2} A_2^- + \frac{\beta_y \beta_z n}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_2 l_2} B_{2n}^-
\]
\[
- \frac{\beta_y \beta_z n}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_2 l_2} B_{2n}^-
\]
\[
= \frac{\omega_0\beta_x}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_3 l_2} A_3^+ + \frac{\omega_0\beta_x}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_3 l_2} A_3^-
\]
\[
+ \frac{\beta_y \beta_z n}{\beta_x^2 + \beta_y^2} e^{-j\beta_x z_3 l_2} B_{3n}^+ - \frac{\beta_y \beta_z n}{\beta_x^2 + \beta_y^2} e^{j\beta_x z_3 l_2} B_{3n}^-
\]
\[
(7.26a)
\]
\[
\bar{H}_{x2}(\beta_x, y, z = l_2^-) = \bar{H}_{x3}(\beta_x, y, z = l_2^+)
\]
\[
- \frac{\beta_z n}{\omega_0} \left[ e^{-j\beta_x z_2 l_2} \bar{E}_{y2h}^+ - e^{j\beta_x z_2 l_2} \bar{E}_{y2h}^- \right]
\]
\[
= - \frac{\beta_z n}{\omega_0} \left[ e^{-j\beta_x z_3 l_2} \bar{E}_{y3h}^+ - e^{j\beta_x z_3 l_2} \bar{E}_{y3h}^- \right]
\]
\[
(7.26b)
\]
\[
(7.26c)
\]
\[
(7.26d)
\]
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\[
-\beta_x \beta_{z2n} e^{-j \beta_{z2n} l_2} A_{2n}^+ + \frac{\beta_x \beta_{z2n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z2n} l_2} A_{2n}^-
\]
\[
+ \frac{-\omega \beta_{yn} \varepsilon_{eff2}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z2n} l_2} B_{2n}^+
\]
\[
+ \frac{-\omega \beta_{yn} \varepsilon_{eff2}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z2n} l_2} B_{2n}^-
\]
\[
= -\beta_x \beta_{z3n} e^{-j \beta_{z3n} l_2} A_{3n}^+ + \frac{\beta_x \beta_{z3n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z3n} l_2} A_{3n}^-
\]
\[
+ \frac{-\omega \beta_{yn} \varepsilon_{eff3}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z3n} l_2} B_{3n}^+
\]
\[
+ \frac{-\omega \beta_{yn} \varepsilon_{eff3}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z3n} l_2} B_{3n}^-
\]

\[
\bar{H}_{y2}(\beta_x, y, z = l_2^z) = \bar{H}_{y3}(\beta_x, y, z = l_2^z)
\]
\[
\frac{j \beta_{yn} \beta_{z2n}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z2n} l_2} A_{2n}^+ - \frac{j \beta_{yn} \beta_{z2n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z2n} l_2} A_{2n}^- - \frac{j \omega \beta_y \varepsilon_{eff2}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z2n} l_2} B_{2n}^+
\]
\[
- \frac{j \omega \beta_y \varepsilon_{eff2}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z2n} l_2} B_{2n}^-
\]
\[
= \frac{j \beta_{yn} \beta_{z3n}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z3n} l_2} A_{3n}^+ - \frac{j \beta_{yn} \beta_{z3n}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z3n} l_2} A_{3n}^- - \frac{j \omega \beta_y \varepsilon_{eff3}}{\beta_x^2 + \beta_{yn}^2} e^{-j \beta_{z3n} l_2} B_{3n}^+ - \frac{j \omega \beta_y \varepsilon_{eff3}}{\beta_x^2 + \beta_{yn}^2} e^{j \beta_{z3n} l_2} B_{3n}^-
\]

\[3. \text{ Interface (3)-(4): } [z = l_3]\]
\[
\bar{E}_{x3}(\beta_x, y, z = l_3^z) = \bar{E}_{x4}(\beta_x, y, z = l_3^z)
\]
\[
\frac{j \omega_0 \beta_n}{\beta_x^2 + \beta_y^2} e^{-j \beta z_3 n l_3} A_{3n}^+ + \frac{j \omega_0 \beta_n}{\beta_x^2 + \beta_y^2} e^{j \beta z_3 n l_3} A_{3n}^- + \frac{-j \beta_x \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{3n} l_3} B_{3n}^+
\]

\[
+ \frac{j \beta_x \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{j \beta z_{3n} l_3} B_{3n}^-
\]

(7.27a)

\[
= \frac{j \omega_0 \beta_n}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} A_{4n}^+ - \frac{j \beta_x \beta z_{4n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} B_{4n}^+
\]

\[
\tilde{E}_{y3}(\beta_x, y, z = l_3^+) = \tilde{E}_{y4}(\beta_x, y, z = l_3^+)
\]

\[
K_3 + e^{-j \beta z_{30} l_3} \tilde{E}_{y3h}^+ + e^{j \beta z_{30} l_3} \tilde{E}_{y3h}^- = e^{-j \beta z_{40} l_3} \tilde{E}_{y4h}^+
\]

(7.27b)

\[
\frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{3n} l_3} A_{3n}^+ + \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_y^2} e^{j \beta z_{3n} l_3} A_{3n}^- + \frac{\beta y \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{3n} l_3} B_{3n}^+
\]

\[
- \frac{\beta y \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{j \beta z_{3n} l_3} B_{3n}^-
\]

\[
= \frac{\omega \mu_0 \beta_x}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} A_{4n}^+ + \frac{\beta y \beta z_{4n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} B_{4n}^+
\]

(7.27c)

\[
\tilde{H}_{x3}(\beta_x, y, z = l_3^+) = \tilde{H}_{x4}(\beta_x, y, z = l_3^+)
\]

\[
- \frac{\beta z_{30}}{\omega \mu_0} \left[ e^{-j \beta z_{30} l_3} \tilde{E}_{y3h}^+ - e^{j \beta z_{30} l_3} \tilde{E}_{y3h}^- \right] = - \frac{\beta z_{40}}{\omega \mu_0} \tilde{E}_{y4h}^+ e^{-j \beta z_{40} l_3}
\]

(7.27d)

\[
- \frac{\beta x \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{3n} l_3} A_{3n}^+ + \frac{\beta x \beta z_{3n}}{\beta_x^2 + \beta_y^2} e^{j \beta z_{3n} l_3} A_{3n}^-
\]

\[
- \frac{\omega \beta y e_{eff}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{3n} l_3} B_{3n}^+ - \frac{\omega \beta y e_{eff}}{\beta_x^2 + \beta_y^2} e^{j \beta z_{3n} l_3} B_{3n}^-
\]

\[
= - \frac{\beta x \beta z_{4n}}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} A_{4n}^+ - \frac{\omega \varepsilon_0 \beta y}{\beta_x^2 + \beta_y^2} e^{-j \beta z_{4n} l_3} B_{4n}^+
\]

(7.27e)
\[ H_{y3}(\beta_x, y, z = l_3^z) = H_{y4}(\beta_x, y, z = l_3^z) \]
\[
j \beta \gamma_0 \beta \gamma_0 e^{-j\beta \gamma_0 l_3^z} A_{3n}^+ - \frac{j \beta \gamma_0 \beta \gamma_0 e^{j\beta \gamma_0 l_3^z}}{\beta_x^2 + \beta_y^2} A_{3n}^- - \frac{j \omega \beta \gamma_0 e_{eff}^3}{\beta_x^2 + \beta_y^2} e^{-j\beta \gamma_0 l_3^z} B_{3n}^+ \]
\[- \frac{j \omega \beta \gamma_0 e_{eff}^3}{\beta_x^2 + \beta_y^2} e^{j\beta \gamma_0 l_3^z} B_{3n}^- \]
\[
\frac{j \beta \gamma_0 \beta \gamma_0 e_{eff}^3}{\beta_x^2 + \beta_y^2} e^{-j\beta \gamma_0 l_3^z} A_{4n}^+ - \frac{j \omega \gamma_0 e^{-j\beta \gamma_0 l_3^z} B_{4n}^+}{\beta_x^2 + \beta_y^2} \]

(7.27f)

Now each of the above Eqs. (7.26a-f) and (7.27a-f) can be respectively written in matrix form as

\[
\mathcal{A}_{12} x + \mathcal{B}_{12} y = \mathcal{B}_{12} y \quad (7.28a) \\
\mathcal{B}_{23} y + \mathcal{C}_{23} z = \mathcal{C}_{23} z \quad (7.28b) \\
\mathcal{C}_{34} z + \mathcal{D}_{34} x = \mathcal{D}_{34} x \quad (7.28c)
\]

where \( \mathcal{A}_{12} \) and \( \mathcal{B}_{12}, \mathcal{B}_{23} \) and \( \mathcal{C}_{23} \), and \( \mathcal{C}_{34} \) and \( \mathcal{D}_{34} \) are 6×6 square matrices and \( \mathcal{B}_{12}, \mathcal{B}_{23}, \) and \( \mathcal{B}_{34} \) are columns of constants of size 1×6 determined from Eqs. (7.24a-f), (7.26a-f) and (7.27a-f) respectively, and

\[
x = \begin{pmatrix} A_{1n}^- \\ B_{1n}^- \\ A_{10}^+ \\ A_{4n}^+ \\ B_{4n}^+ \\ A_{40} \end{pmatrix}, \quad y = \begin{pmatrix} A_{2n}^+ \\ B_{2n}^+ \\ A_{20} \\ B_{20} \end{pmatrix}, \quad z = \begin{pmatrix} A_{3n}^+ \\ B_{3n}^+ \\ A_{30} \\ B_{30} \end{pmatrix}
\]

where \( A_{10}^\pm = \bar{E}_{yin}^\pm \).

Solving Eqs. (7.28a) and (7.28b) for \( z \) and for \( y \) respectively, we have
\[
z = C_{34}^{-1}D_{34}x - C_{34}^{-1}b_{34} \quad (7.29a)
\]
\[
y = B_{23}^{-1}C_{23}z - B_{23}^{-1}b_{23} \quad (7.29b)
\]

Substituting Eq. (7.29a) into Eq. (7.29b)

\[
y = B_{23}^{-1}C_{23}C_{34}^{-1}D_{34}x - B_{23}^{-1}C_{23}C_{34}^{-1}b_{34} - B_{23}^{-1}b_{23} \quad (7.30)
\]

Combining Eqs. (7-28a) and (7-30), collecting like terms together and rearranging and solving for \(x\) yields

\[
x = \left( B_{12}B_{23}^{-1}C_{23}C_{34}^{-1}D_{34} - A_{12} \right)^{-1} \left( B_{12}B_{23}^{-1}C_{23}C_{34}^{-1}b_{34} + B_{12}B_{23}^{-1}b_{23} + b_{12} \right) \quad (7-31)
\]

Equation (7-31) is the solution for the systems of equations obtained from the boundary conditions where the vector \(x\) contains the information on the reflected fields in region 1 and the transmitted fields in region 4.

### 7.2 Simplified Model for Simulation and Boundary Conditions

In this section, the theoretical model is simplified by assuming the beam current density is uniformly distributed over regions 2 and 3 with respect to both \(y\) and \(z\) unlike as defined in previous chapters. Further, correction terms in the transverse fields due to medium inhomogeneities are neglected. The goal of this effort is to identify scattering, absorption, mode conversion and/or reflection mechanisms under the assumptions and neglects imposed. It is desired to minimize the transmission of a high energy electromagnetic wave impinging on the plasma from region 1. For the purpose of notation in this section and the remainder of the thesis, original plasma regions 2 and 3 will now be denoted as region 2. The original region 4 will be region 3. For generality sake within the constraints imposed the subscript \(i\) will still be employed as either the region index number or represent a property of the ion specie. Only the \(TE_0\) (TEM special case), \(TE_1\) and \(TM_1\)
modes are assumed to exist. For the two-interface, three medium case problem. Eq. (7-31) simplifies to

\[(B_{12}B_{23}^{-1}C_{23} - A_{12})x = B_{12}B_{23}^{-1}b_{23} + b_{12} \quad (7-32)\]

where

\[x = (B_{12}B_{23}^{-1}C_{23} - A_{12})^{-1}(B_{12}B_{23}^{-1}b_{23} + b_{12}) \quad (7-33)\]

7.2.1 Power Calculations

The time average power flow entering and leaving region 2 by means of regions 1 and 3, is determined using the Poynting vector,

\[\mathcal{P} = \frac{1}{2} Re \left\{ \oint E \times H^* \cdot d\vec{s} \right\} \quad (7-34)\]

where \(d\vec{s} = \hat{z} \, dx \, dy\) and ‘*’ represents complex conjugate, and \(\vec{E}\) and \(\vec{H}\) are the total fields in the space - frequency domain. Defining the cross-product,

\[\vec{E} \times \vec{H}^* = \hat{x}[E_y H_x^* - E_x H_y^*] + \hat{y}[E_z H_x^* - E_x H_z^*] + \hat{z}[E_x H_y^* - E_y H_x^*],\]

yielding

\[\vec{E} \times \vec{H}^* \cdot d\vec{s} = [E_x H_y^* - E_y H_x^*] \, dx \, dy\]

Therefore,

\[\mathcal{P} = \frac{1}{2} Re \int_0^d \int_{-\infty}^{\infty} [E_x H_y^* - E_y H_x^*] \, dx \, dy \quad (7-35)\]

where the electric field and magnetic field are in the space-frequency domain and their amplitudes are determined by solving Eq. (7-31). For the wave propagation to occur the wave number in the
$z$- direction must be positive real. From Eq. (3-4), this places a restriction on the value of $\beta_x$ namely $|\beta_x| < \omega \sqrt{\mu_0 \varepsilon_0}$ for $\beta_{yn} = 0$.

Consequently, using Eq. (7-35), the time average power incident towards, $P_{1a}$, and transmitted through, $P_{t3}$, the plasma beam are given by

$$P_{1a} = \int_{x=x_1}^{x_1+\Delta x} \frac{1}{2} \Re \left\{ \int_0^d -\tilde{E}_{y1a} \tilde{H}_{x1a}^* dy \right\} dx = \frac{d\Delta x}{2} \frac{\varepsilon_0}{\mu_0} = .00135d\Delta x$$

(7-36)

where the amplitude of the incident wave, $\tilde{E}_{y1a}^+(\beta_x) = \tilde{E}_{y1a0}^+ \delta(\beta_x)$, and $\tilde{E}_{y1a0}^+$ is a free non-zero parameter set equal to 1. This is used in the normalization and $\Delta x$ is determined from the experiment [29]. From the analysis, it is observed that all amplitudes except for the $TE_0$, modes vanish, therefore the power transmitted in region 3 is given by

$$P_{t3} = \int_{x=x_1}^{x_1+\Delta x} \frac{d}{2} \left( \int_{-\infty}^{\infty} \tilde{E}_{y3h} e^{-j\beta_x x} d\beta_x \right) \left( \int_{-\infty}^{\infty} \frac{\beta_{x30}}{\omega \mu_0} \tilde{E}_{y3h} e^{-j\beta_x x} d\beta_x \right)$$

(7-37)

As it can be observed the power transmitted is related to plate separation $d$, of the plates, the thickness of the plasma slab and the amplitudes of the transmitted fields that are coupled with the source and the external source wave.

Equations (7-36) and (7-37) are numerically solved using Matlab and analyzed to study what parameters of the plasma disrupt the incident wave. Since the main objective of this study is to minimize the power transfer for different parameter space such as, plasma density, $n_{0i}$, electron-neutral collision frequency, $\nu$, beam current density, $J_{yib}$, Gaussian nature of the beam, $\alpha$ and beam thicknesses along $x$ and $z$, we only analyze the fraction of power transmitted and plot the ratio $(P_{t3}/P_{1a})$ versus frequency.
The value of $\Delta x$ in the above equations is adapted from [29], pages 108-109. The 90/10 rule is implemented to determine a better approximation for the length along $x$ for the power calculation. From Figure 7.2-2, it is observed that 90% of the glow is in the range of 0.6 inches to -0.4 inches, thus the $\Delta x \approx 1 \text{ inch} = .0254 \text{ meter}$.

Figure 7.2-1 Plot showing line placement used for analysis (zoomed in and labels placed for this paper). Adapted from [29].

Figure 7.2-2 Plot of relative light intensity of the glow discharge vs. position (inches) The plot is intensity plot along the vertical lines. Adapted from [29].
CHAPTER 8. NUMERICAL CHECKS AND RESULTS

In general, numerical studies examine the ability of a plasma discharge in the shape of a slab bounded by a parallel-plate waveguide to disrupt an incoming, y-polarized, TEM electromagnetic wave propagating normal to the slab along the waveguide in the z direction. The waveguide consists of two perfectly conducting, infinite-in-extent, parallel plates with one plate in the y=0 plane and the second in the y=d plane. The slab is $z_0$ thick. The plasma slab supports not only a drift-free, cool, thermal plasma but also an energetic electron beam with drift velocity in the y direction. The sluggish ions are assumed to be stationary in space. Consequently, the theory is not valid at low frequencies where ion motion is more prevalent. The plasma and beam are constricted or pinched in the x coordinate and therefore, are represented by a one-dimensional Gaussian distribution in x. Plasma loss effects are built in the electron-neutral collision frequency. The beam current density is uniform in both y and z within the slab.

Equations (7-36) and (7-37) are simulated and discussed for different values of parameter space in this section. Special conditions are checked against a simpler model. The independent analysis used as a check may be found in Appendix H and is based on a y-polarized, TEM plane wave propagating at normal incidence to a lossy dielectric slab with beam current density. For clarity, the region of space in the simpler model is unbounded in all three dimensions. The TEM wave supported by the parallel-plate waveguide and the wave in the unbounded three medium space, with slab supported current density, have the exact same form and boundary conditions in the direction of incidence. Good agreement is shown. Matlab codes of the theoretical model presented in this thesis demonstrate how the power of an incident wave is disrupted by a (pulsed) constricted, discharge plasma supporting an energetic electron beam. The power transmitted through the constricted plasma with beam relative to the desired incident power to be disrupted is
explored by changing the plasma density, beam thickness, plasma frequency, incident wave frequency, collision frequency, and Gaussian coefficient. Only the following modes are taken into consideration in the code: $TE_0$ ($TEM$), $TE_1$, and $TM_1$. Simplifying the beam and the incident field configurations, neglecting wave coupling to the current density, and neglecting plasma radiation effects, only the $TE_0$ mode is significant.

8.1 Numerical Checks

Numerical Checks for the validation of the code with the hand calculation above is given below.

A simple three-medium (to be referred to as the “simple model”) theory presented in Appendix 0 has been developed assuming the slab medium is homogeneous in $x$. The code written for the general inhomogeneous in $x$ slab geometry is compared to the independent theory in the limit when the medium approaches the homogeneous case. The inhomogeneous nature of the medium for the general theory is removed, for comparison purposes, by forcing in the limit $\beta_x$ to approach zero. This constraint automatically removes the Gaussian nature of the beam and plasma without requiring $\alpha$ in the argument of the Gaussian distribution to be equal to zero. Simulations are examined over the frequency range between 1 Hz to 1 THz typically in frequency steps of 100 MHz.

Neglect the beam current contribution and assume that the electron-to-neutral collision frequency is zero. The magnitude of the system transmission coefficient for various plasma densities over a broad frequency range is examined. Refer to Figure 8.1-2a. The markers in the figure are the simple model (Refer to Appendix H) results and the solid lines are the results obtained from the theory presented in this thesis with appropriate constraints imposed. As depicted in Figures 8.1-1a, b, both theories appear to match identically over a wide frequency range. It
must be understood that only one data point (1 Hz) in the low frequency range is evaluated for frequencies between 1 Hz and 100 MHz due to the 100 MHz step size in frequency imposed. As anticipated at a particular frequency, it is observed that as the magnitude of the plasma density increases, the magnitude of the transmission coefficient of the system decreases. The plasma frequency of each of the three cases in ascending order of plasma density are: \( N = 2.5 \times 10^{17} m^{-3}, N = 5 \times 10^{17} m^{-3} \), and \( N = 10^{18} m^{-3} \). As the operating frequency of the externally generated wave increases beyond the plasma frequency, the plasma slab becomes more transparent to the wave implying that the wave passes freely through the plasma. It is interesting to note beyond the plasma frequency, that the magnitude of the transmission coefficient does not approach one monotonically. Refer to Fig. 8.1-1b. At the high frequency end of the spectrum plotted, some of the transmission coefficient curves exhibits aperiodic oscillation. Although not explicitly observed in the figure, the simple model theory and the theory developed in this thesis agree identically even at the high frequencies. The plasma is treated as an effective permittivity. The effective permittivity contains the following plasma parameters: plasma density, electron mass, plasma frequency and electron-to-neutral collision frequency contributions.
Figure 8.1-1a Plots for the magnitude of the transmission coefficient of the three-medium system vs. frequency for different values of plasma number density over 1 Hz - 1 THz range.
Figure 8.1-1b Plots for the magnitude of the transmission coefficient of the three-medium system vs. frequency for different values of plasma number density over 100 MHz-1 THz range.

Figure 8.1-2a, b (a) displays plots for the magnitude of the transmission coefficient of the three-medium system vs. frequency for different values of plasma number density. The slab thickness $z_0 = 0.03$ m; the beam current density and the electron-neutral collision frequency are both zero. The plotted ‘Markers’ are discrete points obtained from an independent simple model theory (Appendix H). The line plots represent the simulation of the theory developed in this thesis. The frequency range starts at 1 Hz and the frequency step size is 100 MHz. The electron plasma frequencies for the associated plasma densities, $10^{18}m^{-3}$, $5 \times 10^{17}m^{-3}$, and $2.5 \times 10^{17}m^{-3}$ are respectively 9 GHz, 6 GHz, and 4 GHz. The line plots for simple model theory and the line plots for the thesis analysis are depicted in (b) and exhibit identical aperiodic oscillations in the magnitude of the transmission coefficient curve near the 10 GHz end of the spectrum. The symbols X and Y in the label notes refer to the plasma frequency and the respective transmission coefficient points on the curves.

Now consider a fixed plasma number density ($N = 10^{18}m^{-3}$) with a highly electron–neutral collisional plasma ($\nu = 0.01 \text{ GHz}, \nu = 1 \text{ GHz}, \nu = 100 \text{ GHz}$). Further, let the beam current density be zero. Refer to Figure 8.1-5. Again, the markers plotted are discrete points from the simple model theory. The theory developed in this thesis and the simple model theory agree identically. Allow the operating frequencies to be well below the plasma frequency. As inferred from Figure 8.1-5, the higher the collision frequency, the larger the transmission coefficient. When the operating frequency is above plasma frequency, the inverse is observed. It must be understood that only one data point (1 Hz) in the low frequency range is evaluated for frequencies between 1 Hz and 100 MHz due to the 100 MHz step size in frequency imposed.
Figure 8.1-3a Plots for the magnitude of the transmission coefficient of the three-medium system vs. frequency for a range of collision frequencies over the spectral frequency range from 1 Hz to 1 THz.

Figure 8.1-4b Plots for the magnitude of the transmission coefficient of the three-medium system vs. frequency for a range of collision frequencies over the spectral frequency range from 10 GHz to 1 THz.
Figure 8.1-5a, b The magnitude of the transmission coefficient of the three-medium system vs. frequency for a range of collision frequencies over the spectral frequency range from (a) 1 Hz to 1 THz and (b) 10 GHz to 1 THz are plotted. The plotted ‘Markers’ are discrete points obtained from the simple model theory. The line plots represent the simulation of the theory developed in this thesis. The frequency range starts at 1 Hz and the frequency step size is 100 MHz. The electron plasma frequency is 9 GHz. The electron-neutral collision frequencies used are \( \nu = 0.01 \text{ GHz}, \nu = 1 \text{ GHz}, \nu = 100 \text{ GHz} \). The beam current density, plasma number density, and the plasma width are respectively, 0 A/m\(^2\), \(10^{18} \text{ m}^{-3}\), and 0.03 m. Number density and collision frequency are chosen according to values found in literature [9]. The symbols X and Y in the label note refer to the plasma frequency and the respective transmission coefficient points on the curves.

The consequences of three different plasma-beam slab widths are explored. For each slab width, three different cases are considered in the presence of a y-directed beam current density. The first case displayed in Fig. 8.1-3a, the beam current is fix among the three different beam-plasma slab widths considered. Consequently, the spatially independent beam current density in the slab is given by \(J_0 = 100/z_0 \text{ A/m}^2\) throughout the volume of the slab. Therefore, for following beam-plasma thicknesses \(z_0 = 0.03 \text{ m}, 0.003 \text{ m}, \) and \(0.0003 \text{ m}\) the respective beam current densities are: \(J_0 = 3.33 \times 10^3 \text{ A/m}^2, 3.33 \times 10^4 \text{ A/m}^2, \) and \(3.33 \times 10^5 \text{ A/m}^2\). For the second case, Fig. 8.1-3b, the beam current density is the same constant value for each of the three beam-plasma slab widths considered; \(J_0 = 3.33 \times 10^3 \text{ A/m}^2\). For the third case, refer to Fig. 8.1-3c, the beam current density is identically zero for each of the slab thicknesses. For each case, the plasma density and the electron-to-neutral collision frequency are, respectively, \(N = 10^{18} \text{ m}^{-3}\) and \(\nu = 1 \text{ GHz}\). The beam-plasma thickness values, given above, were chosen consistent with experimental observations of a laboratory pulsed power discharge [29]. For the first case associated with Fig. 8.1-3a, it is observed that as the beam thickness increases, the transmission coefficient of the system decreases. For frequencies larger than the plasma frequency near 10 GHz, the transmission coefficient curve for the larger beam thickness exhibits oscillation. Both the simple model theory and the simulated theory developed in this thesis appear to agree. Because Figs. 8.1-3a through c appear identical, the relative power transmitted for the same plasma-beam slab thickness is insensitive to the beam current density based on the values explored. The wave attenuation effect could be due to the background plasma primarily operating in evanescent mode or dissipation mode. In evanescent mode, there is no dissipation of energy. Instead, the energy is continuously
reflected back towards the source as the wave propagates further into the slab medium. As above, one data point (1 Hz) in the low frequency range is evaluated for frequencies between 1 and 100 MHz due to the 100 MHz step size in frequency imposed.

Figure 8.1-6a Over a wide spectrum, the magnitude of the transmission coefficient is displayed for three different values of beam (and hence plasma) thicknesses. The beam current is the same for each of the three plasma slab thicknesses.
Figure 8.1-7b Over a wide spectrum, the magnitude of the transmission coefficient is displayed for three different values of beam (and hence plasma) thicknesses. The beam current density, $J_0 = 3.33 \times 10^3 \, A/m^2$ is the same for each of three slab thicknesses.

![Graph showing transmission coefficient vs frequency for different beam thicknesses.](image)

**Figure 8.1-8c** Over a wide spectrum, the magnitude of the transmission coefficient is displayed for three different values of beam (and hence plasma) thicknesses. The beam current density is zero for each of the three slab thicknesses.

**Figure 8.1-9a-c** Over a wide spectrum, the magnitude of the transmission coefficient is displayed for three different values of beam (and hence plasma) thicknesses. (a) The beam current is the same for each of the three plasma slab thicknesses. Consequently, the beam current density amplitude is given by $J_0 = 100/z_0 \, A/m^2$ where $z_0$ is the beam thickness. (b) The beam current density, $J_0 = 3.33 \times 10^3 \, A/m^2$ is the same for each of three slab thicknesses. (c) The beam current density is zero for each of the three slab thicknesses. For each case, the collision frequency and the plasma density are $\nu = 1 \, GHz$ and $N = 10^{18} \, m^{-3}$ respectively. The plotted ‘Markers’ are discrete points obtained from an independent simple model theory. The line plots represent the simulation of the theory developed in this thesis. The frequency range starts at 1 Hz and the frequency step size is 100 MHz. The electron plasma frequency is 9 GHz.
It is of interest to examine the transmission coefficient effects on the incident wave for various values of the beam current density ranging over seven orders of magnitude. Refer to Figure 8.1-13a-c. The following parameters are fixed: plasma density \( N = 10^{18} \text{ m}^{-3} \), beam-plasma slab thickness \( z_0 = 0.03 \text{ m} \), collision frequency \( \nu = 1 \text{ GHz} \). In the presence of the beam at a particular frequency (Refer to Fig. 8.1-4c), an increase in the magnitude of the beam current density results in a decrease in the transmission coefficient. This is observed for all frequencies evaluated above and below the plasma frequency. Even so, for the parameter space constraints imposed, the current density must be large before a significant difference is observed in the transmission coefficient curves. Both the simple model theory and the simulated theory developed in this thesis appear to agree. But, the simple model theory provided in Appendix H further suggests that there are some beam current densities in which the transmission coefficient is zero at a particular frequency. For convenience, Eq. (H-6) is rewritten in the following form

\[
\frac{J_{y02}}{E^{+}_{y01} e^{-j\beta_1 l_1}} = \frac{2\eta_2 \omega \varepsilon_2}{j[\eta_2(1 - \cos (\beta_2 \Delta l)) - j\eta_1 \sin (\beta_2 \Delta l)]}
\]  

(8-1)

The complex nature of the ratio indicates that the transmission coefficient is zero when the current density is out of phase with the incident field at the \( z = l_1 \) boundary interface.
Figure 8.1-10a Transmission coefficient magnitude curves for beam current density amplitudes $3.3 \times 10^1$, $6.7 \times 10^2$, and $1.3 \times 10^3$ A/m$^2$.

Figure 8.1-11b Transmission coefficient magnitude curves for beam current density amplitudes $3.3 \times 10^0$, $3.3 \times 10^5$, and $1.7 \times 10^5$ A/m$^2$. 
Figure 8.1-12c Transmission coefficient magnitude curves for beam current density amplitudes 6.7×10^{-1}, 6.7×10^{8}, and 1.7×10^{8} A/m².

Figure 8.1-13a-c Transmission coefficient magnitude curves for beam current density amplitudes (a) 3.3×10^1, 6.7×10^2, and 1.3×10^3 A/m², (b) 3.3×10^0, 3.3×10^5, and 1.7×10^5 A/m², and (c) 6.7×10^{-1}, 6.7×10^8, and 1.7×10^8 A/m². The plotted ‘Markers’ are discrete points obtained from an independent simple model theory. The line plots represent the simulation of the theory developed in this thesis. The frequency range starts at 1 Hz and the frequency step size is 100 MHz. The following parameters are fixed: plasma density (N = 10^{18} m^{-3}), beam and plasma slab thickness (z₀ = 0.03 m), and the collision frequency (ν = 1 GHz). The electron plasma frequency in each case is 9 GHz. The current densities are chosen in such a manner to demonstrate a significant change in the magnitude of the power transmission coefficient curves. The symbols X and Y in the label note refer to the plasma frequency and the respective transmission coefficient points on the curves.
8.2 Numerical Results

The following six parameters are varied to explore the disruption of an incoming wave: plasma number density, electron-neutral collision frequency, Gaussian coefficient for one-dimensional Gaussian variations of the plasma-beam slab with respect to x, beam current density, beam-plasma slab thickness along the z-direction, and the power flow area of the slab. The power flow area is defined as that area in which the normal component of the power flux density (Poynting Vector) is averaged over to calculate the power transmitted to or reflected from the region the wave impinges upon. In all cases considered, the normal vector to the power flow area is in the +z-direction. The +z-directed power flow area is spanned by the space parameters x and y. Numerical values for the extent of x and y are based on the beam width as measured in experiment $x_0 = 0.0254 \, m$ from recombination/de-excitation light intensity curves (Refer to Fig. 7.2-2b) and the plate separation $d = 0.05 \, m$ respectively. In experiment, the beam width of 2.54 cm is a worst case scenario measured at the cathode surface. Typically, it is half of this value in the plasma column region of the discharge. One half the beam width corresponds to the distance from the peak intensity location to the 10% of the maximum beam intensity location. The Gaussian distribution is given by $e^{-\alpha x^2}$. Define $\alpha$ as the Gaussian coefficient. The Gaussian coefficient, $\alpha$, for the constricted plasma observed in experiment is $\sim14,280$ (for 2.54 cm beam width). In this section, we are concerned with the time average power transmitted into region 3 relative to the time average power of the incident wave to be disrupted impinging on the plasma-beam slab surface. Unless otherwise stated, the distance in the x-direction of the power flow area is twice the distance from the peak intensity location to the 10% of the maximum beam-plasma intensity location.
The discharge plasma slab supports the cool thermal plasma and the energetic electron beam. Both the plasma and the beam are Gaussian, with respect to x, modeling the constricted discharge observed in experiment. Refer to Figs. 7.2-1 and 7.2-2. Examine Eqs. (7.17), (7.18), and the two longitudinal equations \( \vec{H}_{Li} \) and \( \vec{E}_{Li} \) for \( i=2,3 \) that immediately follow Eq. (7.18). When the beam current density is zero, the Gaussian contribution of both the beam and plasma vanish identically in these relations. The Gaussian contribution of the plasma exists in the convolution terms associated with the conductivity of the medium. These convolution terms are convolved with the energetic electron beam current density. Consequently, when the beam current density is zero, the Gaussian nature of the slab vanishes implying that the value of the Gaussian coefficient, \( \alpha \), has no effect on the fields supported by the medium. Consequently, in the cases below, no reference is made to the Gaussian coefficient value when the beam current density is zero.

For all cases considered, the range of frequencies in the simulations is typically from 1 Hz -1 THz starting at 1 Hz with a frequency step size of 100 MHz. Consequently, the first seven orders of magnitude in the frequency spectrum is based on the properties of the fields at 1 Hz and 100 MHz. When explicitly investigating tendencies in this frequency range, the characteristics at 1 Hz and 100 MHz are typical. It should be realized that the theory developed in this thesis is not necessarily valid at the low frequencies since we assumed the sluggish heavy ions to be stationary. In reality, this assumption is not valid at the low frequencies. Unless otherwise stated with this in mind, the field and power characteristics at the low frequencies are “somewhat” representative of the characteristics at the 1 Hz and 100 MHz points within the validity of the analysis.

Case 1. Beam current density, \( (J_{e2b0} = 0) \), Electron-neutral collision frequency, \( (\nu = 0) \), Beam-plasma slab thickness \( (z_0=0.03 \, m) \), and Extent of power flow area \( (-x_0 < x < +x_0), (0 < y < 0.05 \, m) \) where \( x_0= 0.0125 \, m \), and Number density, \( N \) varies.
For case 1, the relative time average power transmitted into region 3 is examined for the following three plasma number densities: \( N = 2.5 \times 10^{17} \text{m}^{-3} \), \( N = 5 \times 10^{17} \text{m}^{-3} \), and \( N = 10^{18} \text{m}^{-3} \). In Fig. 8.2-1a, it is observed for a constant frequency, increasing the density of plasma decreases the relative time average power transmitted. In other words, only high frequency waves penetrate the plasma medium. The plasma acts as a high pass filter. Beyond 5 GHz, the power transmitted into the third region no longer monotonically approaches one. As illustrated in Figs. 8.1-1a, b, this result is predicted at least in form by the simple model theory.

Figure 8.2-1a The relative, time average, spectral power transmitted is evaluated at the following three plasma number densities: \( N = 2.5 \times 10^{17} \text{m}^{-3} \), \( N = 5 \times 10^{17} \text{m}^{-3} \), and \( N = 10^{18} \text{m}^{-3} \) over the frequency range 1 Hz – 1 THz.
The relative, time average, spectral power transmitted is evaluated at the following three plasma number densities: \( N = 2.5 \times 10^{17} \, \text{m}^{-3} \), \( N = 5 \times 10^{17} \, \text{m}^{-3} \), and \( N = 10^{18} \, \text{m}^{-3} \) over the frequency range 1 GHz – 1 THz.

The frequency range starts at 1 Hz and the frequency step size is 100 MHz for (a). In (b), the step size in frequency has been adjusted for finer detail. The following plasma number densities: \( N = 2.5 \times 10^{17} \, \text{m}^{-3} \), \( N = 5 \times 10^{17} \, \text{m}^{-3} \), and \( N = 10^{18} \, \text{m}^{-3} \) respectively correspond to the following plasma frequencies: 9 GHz, 6 GHz, and 4 GHz. Similar to Figs. 8.1-1a, b, aperiodic oscillations are observed around the 10 GHz side of the frequency spectrum.
Case 2. Beam current density, \( J_{e2b0} = 0 \), Number density \( (N = 10^{18} \, m^{-3}) \), Beam-plasma slab thickness of \( (z_0 = 0.03 \, m) \), Extent of power flow area \((-x_0 < x < +x_0), (0 < y < 0.05 \, m)\) where \( x_0 = 0.0125 \, m \), and Collision frequency, \( \nu \) varies.

Consider an operating frequency below the electron plasma frequency. Refer to Figs. 8.2-2a, b. As the electron-neutral collision frequency increases, the relative time-averaged transmitted power increases. For those frequencies above the plasma frequency, the inverse in tendency is observed. These observations are similar to the homogeneous plasma case in the previous section.

Figure 8.2-4a Plots the relative power transmitted for the following three electron-neutral collision frequency: \( \nu = 0.01 \, GHz, \nu = 1 \, GHz, \nu = 100 \, GHz \) over the frequency range 1 Hz to 1 THz.
Figure 8.2-5b Plots the relative power transmitted for the following three electron-neutral collision frequency: \( \nu = 0.01 \, GHz, \nu = 1 \, GHz, \nu = 100 \, GHz \) over the frequency range 1 GHz to 1THz.

Figure 8.2-6a,b  Plots the relative power transmitted for the following three electron-neutral collision frequency: \( \nu = 0.01 \, GHz, \nu = 1 \, GHz, \nu = 100 \, GHz \) over the frequency range (a) 1 Hz to 1 THz and (b) 1 GHz to 1THz. The following parameters are fixed: beam current density \( (J_{e2b0} = 0) \), plasma slab thickness \( (z_0=0.03 \, m) \), plasma number density, and \( (N = 10^{18} \, m^{-3}) \). The extent of the power flow area is \( (-x_0 < x < +x_0), (0 < y < 0.05 \, m) \) where \( x_0 = 0.0125 \, m \). The plasma frequency is 9 GHz. The results obtained in this case are similar to those in Fig. 8.1-2. The symbols X and Y in the label note refer to the plasma frequency and the respective relative time average power transmitted.
Case 3. Beam current density \( J_0 = 3.33 \times 10^3 \, A/m^2 \), Electron-neutral collision frequency \( \nu = 1 \, GHz \), Number density \( N = 10^{18} \, m^{-3} \), Beam/plasma slab thickness \( z_0 = 0.03 \, m \), Power flow area is base off of the 10% of peak points with x, and Gaussian coefficient \( \alpha \) varies.

When evaluating the power in these cases, the x interval of the power flow area is based on the 10% points of the Gaussian maximum. In Figs. 8.2-3a-c, the relative power transmitted is examined for three different Gaussian coefficients, namely, 14276, 280, and \( 10^{-5} \, m^{-2} \). Respectively, the corresponding beam widths in the x direction between the 10% points of the Gaussian maximum \( \{ Beam \, Width = 2 \sqrt{-\frac{\ln 0.1}{\alpha}} \} \) are 0.0254, 0.18 m, and 960 m. In principle, as the Gaussian coefficient approaches zero, the inhomogeneous nature of the plasma slab with beam should become more homogeneous in nature. This is observed for \( \alpha = 10^{-5} \, m^{-2} \). The tendencies displayed for the remaining two Gaussian coefficients examined seems to give rise to an increase in relative power transmitted in a bandwidth about a center frequency (3 GHz) that is below the plasma frequency (9 GHz) as observed in Figs. 8.1-4b,c. Further, the larger the Gaussian coefficient, the larger the relative power transmitted at this center frequency. It is noted in Fig. 8.1-4c beyond 10 GHz, the relative power transmitted does not increase monotonically to one as the frequency is increased.

![Graph](image-url)

Figure 8.2-7a The relative transmitted power for three different Gaussian coefficients \( \alpha = 14276, 280, \) and \( 10^{-5} \, m^{-2} \) over the frequency range 1 Hz - 1 THz.
Figure 8.2-8b The relative transmitted power for three different Gaussian coefficients $\alpha = 14276$, 280, and $10^{-5} \, m^{-2}$ over the frequency range 1 Hz - 10 GHz.

Figure 8.2-9c The relative transmitted power for three different Gaussian coefficients $\alpha = 14276$, 280, and $10^{-5} \, m^{-2}$ over the frequency range 10 GHz - 1 THz.
Figure 8.2-10a-c The relative transmitted power for three different Gaussian coefficients $\alpha = 14276$, $280$, and $10^{-5} \text{ m}^{-2}$ over the frequency range (a) 1 Hz - 1 THz, (b) 1 Hz - 10 GHz, and (c) 10 GHz - 1 THz. The following parameters are fixed: beam thickness ($z_0 = 0.03 \text{ m}$), electron-neutral collision frequency, ($\nu = 1 \text{ GHz}$), beam current density, ($J_0 = 3.33 \times 10^3 \frac{A}{m^2}$) and plasma number density, ($N = 10^{18} \text{ m}^{-3}$). The plasma frequency is 9 GHz. The step size in frequency has been adjusted for finer detail in (a)-(c). For frequencies below 100 MHz, the ratio of the power transmitted to the incident power is near zero. The x interval of the power flow area of integration is based on the 10% points of the Gaussian maximum.

Case 4. Beam current density ($J_0 = 3.33 \times 10^3 \frac{A}{m^2}$), Electron-neutral collision frequency ($\nu = 1 \text{ GHz}$), Plasma number density ($N = 10^{18} \text{ m}^{-3}$), Gaussian coefficient ($\alpha = 14276 \text{ m}^{-2}$), and the plasma slab thickness, $z_0$, varies.

Consider a particular operating frequency in Figures 8.2-4a, b. As the slab thickness increases from 0.03 cm to 3 cm, the relative transmitted power decreases. But if the slab thickness is too large, the monotonic nature of the curve changes for frequencies greater than the electron plasma frequency (9 GHz). Although not explicitly shown, this oscillation effect may be a consequence of constructive and destructive interference effects.
Figure 8.2-11a The relative transmitted spectral power is examined for three different beam widths (3 cm, 3 mm, and 0.3 mm) over the frequency range of 1 Hz to 100 MHz.

Figure 8.2-12b The relative transmitted spectral power is examined for three different beam widths (3 cm, 3 mm, and 0.3 mm) over the frequency range of 100 MHz to 1 THz.
Figure 8.2-13a, b The relative transmitted spectral power is examined for three different beam widths (3 cm, 3 mm, and 0.3 mm) over the frequency range of (a) 1 Hz to 100 MHz and (b) 100 MHz to 1 THz. At a particular operating frequency, increasing the thickness of the slab decreases the relative power transmitted. The following parameters are fixed: beam current density ($J_0 = 3.33 \times 10^3 \text{ A/m}^2$), plasma density ($N = 10^{18} \text{ m}^{-3}$), collision frequency ($\nu = 1 \text{ GHz}$), and Gaussian coefficient ($\alpha = 14276 \text{ m}^{-2}$). The frequency step size is (a) fine enough to resolve the tendencies between 1 Hz and 100 MHz and (b) 100 MHz. The plasma frequency is 9 GHz.

Case 5. Plasma density ($N = 10^{18} \text{ m}^{-3}$), Beam and plasma slab thickness ($z_0 = 0.03 \text{ m}$), Electron-neutral collision frequency ($\nu = 1 \text{ GHz}$), and Gaussian coefficient ($\alpha = 14276 \text{ m}^{-2}$), Beam current density, $J_{yib0}$, varies.

In Figs. 8.2-5a-c, the relative power transmitted is explored for the following three fixed current densities: 3.3, $1.7 \times 10^5$, and $3.3 \times 10^5 \text{ A/m}^2$. As observed in Fig. 8.2-5a, the relative time averaged power increases significantly for frequencies greater than the electron plasma frequency. The relative time average transmitted spectral power seems to be insensitive to a five order of magnitude change in the beam current density over the spectral range between 1 Hz and 1 THz. Consider any frequency sufficiently far from the 9 GHz value below the plasma frequency. Although very small for the current densities studied, an increase in current density leads to a decrease in the transmitted power. Here, small is on the order of three to five orders of magnitude small compared to one. For frequencies far above the plasma frequency, the inverse is valid. That is, an increase in current density leads to an increase in relative power transmitted. At a bandwidth of frequencies about 9 GHz, the transmitted power curve changes its shape significantly. It is conjectured that the beam may be in resonance with the wave to be disrupted in the slab.
The relative transmitted spectral power is examined for the following three beam current densities: $3.3\times10^5$, $1.7\times10^5\frac{A}{m^2}$, and $3.3\times10^5\frac{A}{m^2}$ over the spectral range $1\text{ Hz} - 1\text{ THz}$.

The relative transmitted spectral power is examined for the following three beam current densities: $3.3\times10^5$, $1.7\times10^5\frac{A}{m^2}$, and $3.3\times10^5\frac{A}{m^2}$ over the spectral range $1\text{ Hz} - 1\text{ GHz}$. 

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Figure 8.2-16c The relative transmitted spectral power is examined for the following three beam current densities: 3.3, 1.7×10^5, and 3.3×10^5 \( \frac{A}{m^2} \) over the spectral range 6.156 GHz to 2.166 GHz.

Figure 8.2-17a-c The relative transmitted spectral power is examined for the following three beam current densities: 3.3, 1.7×10^5, and 3.3×10^5 \( \frac{A}{m^2} \) over the spectral range (a) 1 Hz – 1 THz, (b) 1 Hz – 1 GHz, and (c) 6.156 GHz to 2.166 GHz. The following parameters are fixed: plasma density \( (N = 10^{18} m^{-3}) \), beam and plasma slab thickness \( (z_0 = 0.03 m) \), collision frequency \( (\nu = 1 GHz) \), and Gaussian coefficient \( (\alpha = 14276 m^{-2}) \). The frequency step size is (a) fine enough to resolve the tendencies between 1 Hz and 1 THz, (b) 1 Hz and 1 GHz, and (c) fine enough to resolve the resonant tendencies between 6.156 GHz to 2.166 GHz. The plasma frequency is 9 GHz.

Case 6. Current density \( (J_0 = 3.33 \times 10^3 A/m^2) \), Plasma number density \( (N = 10^{18} m^{-3}) \), beam and slab thickness \( (z_0 = 0.03 m) \), electron-neutral collision frequency \( (\nu = 1 GHz) \), and Gaussian coefficient \( (\alpha = 14276 m^{-2}) \), x- interval in the power flow area varies
The differences in the time average transmitted spectral power as a consequence of choosing the flow area based on: (i) the 10% of the maximum light intensity of the constricted plasma (full width 0.0254 m), (ii) one-tenth of the ten percent width (full width 0.00254 m), (iii) 10 times the ten percent width (full width 0.254 m) and (iv) 100 times the ten percent width (full width 2.54 m) are depicted in Figs. 8.2-6a-d. The latter distance was arbitrarily chosen to be well beyond the bounds of the Gaussian beam-plasma width. The purpose of such an investigation characterizes the “stopping or capture region” of the constricted plasma. Because of the finite nature of the constricted plasma, there will be a certain dimension in x in which the presence of the plasma with beam no longer influences the disruption properties of the slab. Remember, the incident electromagnetic energy incident on the slab is a plane-like, TEM wave with phase and amplitude independent of x. Therefore, its amplitude is constant over a plane that extends from minus infinity to plus infinity with respect to x. For frequencies below about 100 MHz there is no significant change in relative transmitted power. At around the plasma frequency (9 GHz), the relative power transmitted changes abruptly from zero to near one within an order of magnitude of the plasma frequency. Examining three of the four x-extents in the power flow area as illustrated in Figs. 8.2-6b-d, no significant differences seem to exist on both sides of the plasma frequency. This was not anticipated and may requires further research in the future.
Figure 8.2-18a Exhibits the relative transmitted spectral power for four different extent in $x$ to be 2.54 mm, 2.54 cm (the 10% of the maximum light intensity of the constricted plasma), 25.4 cm and 2.54 m for the flow area over the spectrum between 1 Hz and 1 THz.

Figure 8.2-19b Exhibits the relative transmitted spectral power for four different extent in $x$ to be 2.54 cm for the flow area over the spectrum between 1 Hz and 1 THz.
Figure 8.2-20c Exhibits the relative transmitted spectral power for four different extents in x to be 2.54 mm for the flow area over the spectrum between 1 Hz and 1 THz.

Figure 8.2-21d Exhibits the relative transmitted spectral power for four different extents in x to be 25.4 cm for the flow area over the spectrum between 1 Hz and 1 THz.
Figure 8.2-22e Exhibits the relative transmitted spectral power for four different extents in x to be 2.54 m for the flow area over the spectrum between 1 Hz and 1 THz.

Figure 8.2-23a-e Exhibits the relative transmitted spectral power for four different extents in x for the flow area over the spectrum between 1 Hz and 1 THz. In (a), an overall view of the spectral domain is exhibits the power flow area for the x extent to be 2.54 mm, 2.54 cm (the 10% of the maximum light intensity of the constricted plasma), 25.4 cm and 2.54 m. Examining (b) for 2.54 cm extent in x, (c) for 2.54 mm in extent in x, (d) 25.4 cm and (e) 2.54 m it is observed that the three curves are identical. For frequencies below 9 GHz, the transmitted power is nearly zero. The following parameters are fixed: plasma density ($N = 10^{18} m^{-3}$), beam and plasma slab thickness ($z_0 = 0.03 m$), collision frequency ($\nu = 1 GHz$), and Gaussian coefficient ($\alpha = 14276 m^{-2}$). The plasma frequency is 9 GHz. At 9 GHz, the beam appears to resonate with the external wave to be disrupted.
CHAPTER 9. CONCLUSION AND EXTENSIONS TO RESEARCH

9.1 Research Summary

Base on a fundamental and harmonic resonance, a general theory has been developed to potentially characterize the interaction of an external TEM wave propagating normal to a plasma discharge in a slab configuration supporting an energetic electron beam in a parallel-plate waveguide. The plasma with energetic electron beam is Gaussian with $x$ to model the constrictive nature of the plasma discharge observed in experiments. The theory incorporates empirical data from experiment to model the constricted discharge geometry, the discharge current, and the discharge voltage. For generality, the contribution of all $\text{TE}_n$ and $\text{TM}_m$ modes for a medium inhomogeneous with respect to $x$ supported by the parallel-plate waveguide are incorporated in the theory. The plasma characteristics are built into an effective permittivity whereas the energetic electron beam is treated as a source through the current density in Maxwell’s equations. The beam properties are directly linked to the discharge voltage and discharge current based on similarities between the pulsed power discharge and the DC discharge [28]. The inhomogeneous contribution of the plasma with beam has not been fully explored in this effort but has been built into the theory and treated as a small effect as far as the transverse fields are concerned. The theoretical study portion of this theory yields the coupled resonant-harmonic field solutions in the $i^{\text{th}}$ medium. Boundary conditions are applied to a four-medium case where two plasma slabs with different plasma-beam characteristics characterize in part what is observed in experiment. In this case, the correction terms to the transverse fields are neglected. Typically, these terms are sourced by the inhomogeneous nature of the medium by way of the convolution operations. The inhomogeneous nature of the plasma has been retained in the longitudinal fields. Further, the plasma and energetic beam have been assumed to be uniform with respect to $y$ and $z$. 

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Coherent and incoherent re-radiation effects of the energetic electron beam have been neglected. That is, wave interaction on the electron beam and its resulting field generation have been neglected. Plasma radiation effects have been neglected as well. It is anticipated that significant mode coupling and desired external wave disruption will result when beam re-radiation effects are included.

Simulations are performed for the three-medium case where a plasma discharge slab separates the regions of the waveguide void of medium. For a three-medium single plasma slab system, the interaction of an external incident transverse electromagnetic wave with a plasma discharge is considered. It is understood that the plasma discharge is a cool thermal plasma supporting an energetics electron beam. Under appropriate conditions, the energetic electron beam supported by the plasma discharge couples and resonates with the external field contributions. Power transmission is highly affected by plasma density, collision frequency, beam current density, Gaussian coefficient of the beam, and beam-plasma slab thickness.

From simulation, a dense but collisional plasma transmits less time average power compared to less dense and highly collisional plasmas. In the latter case, the medium acts more transparent to the external wave to be disrupted. As anticipated waves with frequencies below the plasma frequency are more attenuated or evanescent as compared to high frequency waves. In effect, the plasma acts like a high pass filter except when the fields tend to resonate with the internal energetic electron beam.
With all parameters fixed, the plasma thickness along the direction of propagation has a significant effect on power absorption and/or transmission. From simulations, it was observed that as the plasma thickness increases, the amount of power transmitted by the system decreases.

9.2 Extensions to the Research

Recall that fundamental (n) and harmonic (m) are loosely used to describe the direct modal number relations between the system’s natural spatial resonance response (n) and the spatial resonance response of the sources internally driving the system (m). It has been shown that many of the source contributions present at the harmonic resonance in this effort do not couple with the fundamental resonance (natural response) of the system. Even so, resonance was demonstrated between the externally generated energetic electron beam and the externally generated wave to be disrupted. It is anticipated that incorporating re-radiation effects allows for mode coupling mechanisms for dispersing the undesired, externally generated, incident wave. It is suggested that re-radiation effects be built into the theory exploring the mode coupling mechanisms that may exist.

Initial results obtained from numerical simulations have not exhausted all parameter space scenarios and physics mechanisms. This effort has shown that oddities (e.g., negative power) exist in simulation and hence the theoretical analysis developed. Special care around the plasma frequency and possibly its harmonics need to be examined. It also stands to reason that increasing the magnitude of the beam current density beyond some threshold will exhibit an increase in power transmitted. This has not been shown at this point. Although built into the theory, the consequences of phase differences between the external field to be disrupted relative to the independent, externally-generated, beam current density has not been explored in simulations.
These and other appropriate studies exploring consequences of the full parameter space are recommended.

Wave-plasma-beam interaction was explored in a parallel-plate waveguide. This waveguide was chosen because of simplicity in satisfying boundary conditions. A more realistic waveguide geometry for application purposes is to extend the analysis to the rectangular and possibly the cylindrical waveguide geometries.

In pulsed power experiments, the constricted beam-plasma configuration was observed from de-excitation and recombination effects. Based on light intensity studies, the discharge appeared somewhat cylindrical in geometry. The light intensity appeared to be non-uniform in radius. It would be prudent to extend the theory to model a more realistic plasma wire in a cylindrical geometry as compared to the planar slab configuration as examined in this effort. Diffraction and scattering effects may be significantly different due to the geometrical finiteness, geometrical curvature, and nonuniformity in plasma/energetic electron beam density of the discharge.

The inhomogeneity and coupling effects have only been initially explored in this work. Higher order coupling and more complex energetic beam configurations may allow for other interesting physics that has not be uncovered in this work. For example, inhomogeneity effects of the transverse fields add a new contribution to the fields that could play more than a perturbation role in the overall field and power flow analyses. In the formulation of the general theory, the contribution of medium homogeneities was not originally assumed small until we explored the disruption properties of the medium numerically.
Finally, theoretical studies provide predictions under well-defined constraints that must be challenged and verified with experiment. Such experiments will require a unique way to measure various coupled modes in the waveguide of choice in order to fully characterize and corroborate theory.
APPENDIX A: INTEGRALS INVOLVING DERIVATIVES IN THE PHASE DOMAIN

To express the following integral in the phase domain,

\[
\int_{-\infty}^{\infty} \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon(x, \omega) \right\} H_{zi} e^{i\beta_s x} \, dx
\]

where \( H_{zi} = H_{zi}(x, y, z, \omega) \).

Then performing term by term integration for those that have variations or changes in \( x \), we have,

First term in the integrand is,

\[
\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} H_{zi} e^{i\beta_s x} \, dx
\]

\[
= \int_{-\infty}^{\infty} e^{i\beta_s x} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} H_{zi} \right) \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left[ e^{i\beta_s x} \frac{\partial}{\partial x} H_{zi} \right] \, dx - \int_{-\infty}^{\infty} \frac{\partial H_{zi}}{\partial x} j\beta_s e^{i\beta_s x} \, dx
\]

\[
= - e^{i\beta_s x} \frac{\partial}{\partial x} H_{zi} \bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (H_{zi} j\beta_s e^{i\beta_s x}) \, dx + \int_{-\infty}^{\infty} -\beta_s^2 H_{zi} e^{i\beta_s x} \, dx
\]

\[
= - e^{i\beta_s x} \left( \frac{\partial H_{zi}}{\partial x} - j\beta_s H_{zi} \right) \bigg|_{-\infty}^{\infty} - \beta_s^2 \int_{-\infty}^{\infty} H_{zi} e^{i\beta_s x} \, dx
\]

The first term in the last expression must vanish due to the radiation condition or that \( H_{zi} \) and its derivative vanishes at \( \pm \infty \). The only exception is the TEM mode. In this case, the field components are independent of \( x \) implying \( \beta_s = 0 \). Therefore,

\[
\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} H_{zi} e^{i\beta_s x} \, dx = -\beta_s^2 \int_{-\infty}^{\infty} H_{zi} e^{i\beta_s x} \, dx = -\beta_s^2 H_{zi}(\beta_s, y, z)
\]

The last term in the integrand,

\[
\int_{-\infty}^{\infty} \varepsilon(x, \omega) H_{zi} e^{i\beta_s x} \, dx
\]

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Writing the integrands in terms of their inverse transform equivalent,

\[
\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{i\beta_x} \int \frac{1}{2\pi} H_{z1}(\beta_x, y, z, \omega) e^{-i\beta_x} \right] e^{i\beta_x} \, dx \\
= \frac{1}{4\pi^2} \int_{\beta_x=-\infty}^{\infty} \int_{\beta_x=-\infty}^{\infty} e(\beta_x, \omega) H_{z1}(\beta_x, y, z, \omega) \left[ \int_{-\infty}^{\infty} e^{i\beta_x} \right] \, d\beta_x \, d\beta_x \\
= \frac{1}{4\pi^2} \int_{\beta_x=-\infty}^{\infty} \int_{\beta_x=-\infty}^{\infty} e(\beta_x, \omega) H_{z1}(\beta_x, y, z, \omega) 2\pi \delta(\beta_x - \beta_x - \beta_x) \, d\beta_x \, d\beta_x \\
= \frac{1}{2\pi} \int_{\beta_x=-\infty}^{\infty} e(\beta_x - \beta_x, \omega) H_{z1}(\beta_x, y, z, \omega) \, d\beta_x = \frac{1}{2\pi} e(\beta_x, \omega) \bigcirc H_{z1}(\beta_x, y, z, \omega)
\]

Therefore, in the phase domain, Eq. (2-45) becomes

\[
\left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \beta_x^2 + \frac{\omega^2 \mu_0}{2\pi} e(\beta_x, \omega) \bigcirc \beta_x \right] H_{z1}(\beta_x, y, z, \omega) = -jI_{yib}(\beta_x, y, z, \omega)
\]

In general, if \( \tilde{F}(x, y, z, \omega; t) = \tilde{F}(x, y, \beta_z, \omega) e^{i\omega t} e^{-i\beta_x z} \), which is the +z-propagating wave.

The following is a list of useful transforms:

1. \( \int_{-\infty}^{\infty} \tilde{F}(x, y, z, \omega) e^{i\beta_x} \, dx = \tilde{F}(\beta_x, y, z, \omega) \)
2. \( \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \tilde{F}(x, y, z, \omega) e^{i\beta_x} \, dx = -j\beta_x \tilde{F}(\beta_x, y, z, \omega) \)
3. \( \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} \tilde{F}(x, y, z, \omega) e^{i\beta_x} \, dx = -\beta_x^2 \tilde{F}(\beta_x, y, z, \omega) \)
4. \( \int_{-\infty}^{\infty} [e(\omega)] \, \tilde{F}(x, y, z, \omega) e^{i\beta_x} \, dx = \frac{1}{2\pi} e(\beta_x, \omega) \bigcirc \tilde{F}(\beta_x, y, z, \omega) \)
5. \( \int_{-\infty}^{\infty} [e(\omega)] \frac{\partial}{\partial x} \tilde{F}(x, y, z, \omega) e^{i\beta_x} \, dx = \frac{1}{2\pi} e(\beta_x, \omega) \bigcirc [-j\beta_x \tilde{F}(\beta_x, y, z, \omega)] \)
APPENDIX B: GREEN’S FUNCTION TECHNIQUE

Consider the differential equation,

\[
\left( \frac{d^2}{dz^2} + \beta^2 \right) H_{zn}(\beta_x, \beta_y, z) = j\beta_x J_{yn}(\beta_x, \beta_y, z)
\]

where \( \beta^2_z = \omega^2 \mu \varepsilon - \beta^2_x - \beta^2_y \). This inhomogeneous differential equation is solved using the Green’s function technique. Let \( G(z; \bar{z}) \) be the Green’s function, then multiplying this equation through by \( G(z; \bar{z}) \) and integrating with respect to \( \bar{z} \) we get,

\[
\int G(z; \bar{z}) \left( \frac{d^2}{dz^2} + \beta^2 \right) H_{zn}(\beta_x, \beta_y, z) \, d\bar{z} = \int \beta_x J_{yn}(\beta_x, \beta_y, z) G(z; \bar{z}) \, d\bar{z}
\]

But,

\[
G \frac{d^2 H_{zn}}{dz^2} = \frac{d}{dz} \left( G \frac{dH_{zn}}{dz} \right) - \frac{dH_{zn}}{dz} \frac{dG}{dz} = \frac{d}{dz} \left( G \frac{dH_{zn}}{dz} \right) - \left\{ \frac{d}{dz} \left( \frac{dH_{zn}}{dz} \right) - H_{zn} \frac{d^2 G}{dz^2} \right\}
\]

Integrating this expression with respect to \( z \) and rearranging we have,

\[
\int G \frac{d^2 H_{zn}}{dz^2} \, dz = \int \left( \frac{d}{dz} \left( G \frac{dH_{zn}}{dz} \right) - \frac{d}{dz} \left( H_{zn} \frac{dG}{dz} \right) \right) \, dz - \int H_{zn} \frac{d^2 G}{dz^2} \, dz
\]

Substituting this into the differential equation and integrating over the interval \([l_{i-1}, l_i], i = 2,3\) the above becomes,

\[
\int_{l_{i-1}}^{l_i} G \frac{d^2 H_{zn}}{dz^2} \, dz = \left[ G \frac{dH_{zn}}{dz} - \frac{dG}{dz} \right]_{l_{i-1}}^{l_i} + \int_{l_{i-1}}^{l_i} H_{zn} \frac{d^2 G}{dz^2} \, dz
\]

Then the differential equation becomes
\[
\int_{l_{i-1}}^{l_i} H_{zni} \left( \frac{d^2 G}{dz^2} + \beta_z^2 G \right) dz + \left[ G \frac{dH_{zni}}{dz} - H_{zni} \frac{dG}{dz} \right]_{l_{i-1}}^{l_i} = \int_{l_{i-1}}^{l_i} \beta_x J_{yny}(\beta_x, \beta_{yn}, z) G(z; \bar{z}) \, dz
\]

If \( \left[ \frac{d^2}{dz^2} + \beta_{zn}^2 \right] G(\beta_x, \beta_{yn}, z; \bar{z}) = \delta(z - \bar{z}) \), the magnetic field is activated for at \( z = \bar{z} \), where \( l_{i-1} < z, \bar{z} < l_i \). By method of superposition, we now consider only the source effects by extending the medium boundary \( l_{i-1} \) and \( l_i \) to \(-\infty\) and \(+\infty\) respectively and solve for the free space Green’s function. In effect, we are solving the particular (forced) solution of the wave equation. Joining with the source-free (natural or homogeneous) solution, boundary conditions at \( l_{i-1} \) and \( l_i \) will be satisfied yielding a single solution characterizing wave propagation in the slab region. The free space Green’s function is used to solve the above differential equation. Since the Green’s function is the response to a sheet current source, when applied in the equation the magnetic field is activated and gives the sum of all field response contributions due to the general source.

\[
H_{zni}(\beta_x, \beta_{yn}, \bar{z}) = \int_{-\infty}^{\infty} \beta_x J_{yny}(\beta_x, \beta_{yn}, z) G(z; \bar{z}) \, dz - \left[ G \frac{dH_{zni}}{dz} - H_{zni} \frac{dG}{dz} \right]_{z=-\infty}^{z=\infty}
\]

where

\[
G(\bar{z}; z) = \int_{-\infty}^{\infty} e^{-j\beta_{zn}|z-\bar{z}|} \frac{z^{\alpha-1}}{(2\beta_{zn})^\alpha} \, dz
\]

is the solution to the free space Green’s function.
APPENDIX C: DELTA FUNCTION PROPERTIES

\[ F(x, \omega) = \int_{-\infty}^{\infty} f(x, t)e^{-j\omega t} \, dt = \mathcal{F}\{f(x, t)\} \]

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x, \omega)e^{j\omega t} \, d\omega = \mathcal{F}^{-1}\{F(x, \omega)\} \]

Thus,

\[ F(x, \omega) = \int_{t=\infty}^{0} \left[ \frac{1}{2\pi} \int_{\tilde{\omega}=\infty}^{0} F(x, \tilde{\omega})e^{j\tilde{\omega} t} \, d\tilde{\omega} \right] e^{-j\omega t} \, dt \]

\[ = \int_{\tilde{\omega}=\infty}^{0} F(x, \tilde{\omega}) \left[ \frac{1}{2\pi} \int_{t=\infty}^{0} e^{-j(\omega-\tilde{\omega})t} \, dt \right] d\tilde{\omega} = \int_{\tilde{\omega}=\infty}^{0} F(x, \tilde{\omega}) \delta(\omega - \tilde{\omega}) \, d\tilde{\omega} \]

\[ = F(x, \omega) \]

\[ f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{\tilde{\omega}=\infty}^{0} f(x, \tilde{\omega})e^{-j\omega t} \, d\tilde{\omega} \right] e^{j\omega t} \, d\omega \]

\[ = \int_{t=\infty}^{0} f(x, \tilde{t}) \left[ \frac{1}{2\pi} \int_{t=\infty}^{0} e^{-j\omega(t-\tilde{t})} \, d\omega \right] d\tilde{t} \]

\[ = \int_{\tilde{\omega}=\infty}^{0} f(x, \tilde{t}) \delta(t - \tilde{t}) \, d\tilde{t} = f(x, t) \]

Therefore,

\[ \int_{t=\infty}^{0} e^{-j(t-\tilde{t})\omega} \, d\omega = 2\pi \delta(t - \tilde{t}) = 2\pi \delta(\tilde{t} - t) = 2\pi \mathcal{F}^{-1}\{e^{-j\omega t}\} \]

\[ \int_{t=\infty}^{0} e^{-j(\omega-\tilde{\omega})t} \, dt = 2\pi \delta(\omega - \tilde{\omega}) = 2\pi \delta(\tilde{\omega} - \omega) = \mathcal{F}\{e^{j\tilde{\omega}t}\} \]

By analogy the spatial Fourier transform,
\[ F(\beta_x) = \int_{-\infty}^{\infty} f(x)e^{j\beta_x x} \, dx = \mathcal{F}\{f(x)\} \]

\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\beta_x)e^{-j\beta_x x} \, d\omega = \mathcal{F}^{-1}\{F(\beta_x)\} \]

\[ \int_{\chi=-\infty}^{\infty} e^{-j(\chi-x)\beta_x} \, d\beta_x = 2\pi \delta(\chi - \tilde{x}) = 2\pi \delta(\tilde{x} - x) = 2\pi \mathcal{F}^{-1}\{e^{j\beta_x \tilde{x}}\} \]

\[ \int_{\chi=-\infty}^{\infty} e^{-j(\beta_x-\tilde{\beta}_x)x} \, dx = 2\pi \delta(\beta_x - \tilde{\beta}_x) = 2\pi \delta(\tilde{\beta}_x - \beta_x) = \mathcal{F}\{e^{-j\tilde{\beta}_x x}\} \]
APPENDIX D: FLOW CHART

A flow chart below shows how the program is simulated to determine the output. The input parameters are plasma thickness, plate separation, incident wave frequency. Based on the values of those parameter space the amplitudes of the transmitted fields are calculated and finally a plot of the power transmitted is determined.

![Flow chart for the simulation](image-url)

The parameters \( n_0, \rho, d, a \) are plasma density, incident wave frequency, thickness in the \( y \) and \( z \) respectively.

Figure D-1 Flow chart for the simulation
APPENDIX E: MATLAB CODE

%% This matlab program solves a system of equation of the form A*x=b
%% where the coefficient matrix A has its entries obtained from the boundary
%% conditions in the problem set up( Refer Pages 106-111). Our problem set up consists of a
%% parallel plate
%% waveguide with three regions where the first(region 1) and the last
%% region (3) are free spaces and region (2) is with source beam
%% current density .The column of constants, b02 and b01 are constituents of the loss and source
terms in regions 2 and 1 resp.
clc
close all

syms f Bx t N

% Parameter space to be played with
x0=.000254;% Beam thickness in the x-direction
z0=3*10^-2;%plasma thickness in the z-direction
d0=.05;% plate separation
a00=10^0;
a=a00*.25;% gaussian coefficient
s0=100;
noi0=N;% number density of plasma
nu=10^8;% collision frequency
Jyb0=0;% amplitude of beam current density

w=2*pi*f;% angular freq
s=inf;% limit of integration for the transforms
n=1;% integer value for the lowest mode
l1=0;% first interface location
d=d0; % distance of separation of the plates
Byn=@(n)n*pi/d;% eigen values from the boundary conditions
Byn=Byn(n);
e0=8.854*10^(-12);% permittivity of free space
mu0=4*pi*10^(-7);% permeability of free space
q=1.6*10^(-19);% charge of an electron
me=9.1*10^(-31);% mass of an electron

wp_e=1/(2*pi)*sqrt(q^2*noi0/(me*e0))
sigma_i=e0*wp_e^2/(1i*w+nu);
Di0= (Bx^2+Byn(n)^2);
% Effective permittivities of the mediums
Eeff1=e0; % region 1
Eeff2=e0*(1- wp_e^2./(w.*(w -1i*nu))));% effective permittivity of the plasma(region 2)
Eeff3=e0; % region 3

% The wave numbers in the three regions for n <> 0
Bz1n= sqrt(w^2*mu0*Eeff1-Di0 );
\[ B_{z2n} = \sqrt{w^2 \mu_0 E_{eff2} - D_i0}; \]
\[ B_{z3n} = \sqrt{w^2 \mu_0 E_{eff3} - D_i0}; \]

\% Wave numbers in each region for all \( n \neq 0 \)

\% Wave numbers for \( n = 0 \)
\[ B_{z10} = \sqrt{w^2 \mu_0 E_{eff1} - B_x^2}; \]
\[ B_{z20} = \sqrt{w^2 \mu_0 E_{eff2} - B_x^2}; \]
\[ B_{z30} = \sqrt{w^2 \mu_0 E_{eff3} - B_x^2}; \]

\% Wave numbers in each region, \( n = 0 \).
\[ B_{z1n} = w^2 \mu_0 e_0 - Byn(n)^2; \]
\% wave number of the incident wave (\( n \) not equal to zero)
\[ B_{z10a} = w^2 \mu_0 e_0; \]
\% wave number for \( n = 0 \)
\[ B_{0i} = \sqrt{w^2 \mu_0 e_0}; \]

\% /////////////////////////////////////////////////////
\% Region 1
\% \( z = l1; \)
\n\[ a_{11e} = (i\cdot w\cdot \mu_0 \cdot Byn \cdot \exp(i\cdot Bz1n^*l1)); \]
\[ a_{112m} = (i\cdot Bx \cdot Bz1n \cdot \exp(i\cdot Bz1n^*l1)); \]
\[ a_{121e} = (Bx \cdot Bz1n \cdot \exp(i\cdot Bz1n^*l1)); \]
\[ a_{122m} = (-w \cdot e_0 \cdot Byn \cdot \exp(i\cdot Bz1n^*l1)); \]
\[ a_{131e} = (-i\cdot Byn \cdot Bz1n \cdot \exp(i\cdot Bz1n^*l1)); \]
\[ a_{132m} = (-i\cdot w \cdot e_0 \cdot Bx \cdot \exp(i\cdot Bz1n^*l1)); \]

\% Define the coefficient matrix, \( A_{12} \) associated to the systems of equations in region 1
\[ A_{12} = \begin{bmatrix} a_{11e} & a_{112m} & 0 & 0 \\ a_{121e} & a_{122m} & 0 & 0 \\ a_{131e} & a_{132m} & 0 & 0 \\ a_{141e} & a_{142m} & 0 & 0 \end{bmatrix}; \]
\% the column of constants are defined
\[ B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \]

\% /////////////////////////////////////////////////////
\% Region 2 (\( z = l1; \))
\n\[ a_{21e} = (i\cdot w\cdot \mu_0 \cdot Byn \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{212e} = (i\cdot w\cdot \mu_0 \cdot Byn \cdot \exp(i\cdot Bz2n^*l1)); \]
\[ a_{213m} = (-i\cdot Bx \cdot Bz2n \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{214m} = (i\cdot Bx \cdot Bz2n \cdot \exp(i\cdot Bz2n^*l1)); \]
\[ a_{221e} = (w \cdot \mu_0 \cdot Bx \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{222e} = (w \cdot \mu_0 \cdot Bx \cdot \exp(i\cdot Bz2n^*l1)); \]
\[ a_{223m} = (-Byn \cdot Bz2n \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{224m} = (-Byn \cdot Bz2n \cdot \exp(i\cdot Bz2n^*l1)); \]
\[ a_{231e} = (-Bx \cdot Bz2n \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{232e} = (Bx \cdot Bz2n \cdot \exp(i\cdot Bz2n^*l1)); \]
\[ a_{233m} = (-w \cdot Byn \cdot Eeff2 \cdot \exp(-i\cdot Bz2n^*l1)); \]
\[ a_{234m} = (-w \cdot Byn \cdot Eeff2 \cdot \exp(i\cdot Bz2n^*l1)); \]
a241e = (1i*Byn*Bz2n*exp(-1i*Bz2n*l1));
a242e=(-1i*Byn*Bz2n*exp(1i*Bz2n*l1));
a243m=(-1i*w*Bx*Eeff2*exp(-1i*Bz2n*l1));
a244m=(-1i*w*Bx*Eeff2*exp(1i*Bz2n*l1));

% the coefficient matrix in region 2, for z=l1
B12=[a211e a212e a213m a214m ;a221e a222e a223m a224m ;a231e a232e a233m a234m ;a241e a242e a243m a244m ];

%% z=l2=l1+a
% Region 2

b211e=(1i*w*mu0*Byn*exp(-1i*Bz2n*(l1+z0))); b212e=(1i*w*mu0*Byn*exp(1i*Bz2n*(l1+z0))); b213m=(-1i*Bx*Bz2n*exp(-1i*Bz2n*(l1+z0))); b214m=(1i*Bx*Bz2n*exp(1i*Bz2n*(l1+z0)));

b221e=(w*mu0*Bx*exp(-1i*Bz2n*(l1+z0))); b222e=(w*mu0*Bx*exp(1i*Bz2n*(l1+z0))); b223m=(Byn*Bz2n*exp(-1i*Bz2n*(l1+z0))); b224m=-(Byn*Bz2n*exp(1i*Bz2n*(l1+z0)));

b231e=(-Bx*Bz2n*exp(-1i*Bz2n*(l1+z0))); b232e=(Bx*Bz2n*exp(1i*Bz2n*(l1+z0))); b233m=-(w*Byn*Eeff2*exp(-1i*Bz2n*(l1+z0))); b234m=-(w*Byn*Eeff2*exp(1i*Bz2n*(l1+z0)));

b241e=(1i*Byn*Bz2n*exp(-1i*Bz2n*(l1+z0))); b242e=(-1i*Byn*Bz2n*exp(1i*Bz2n*(l1+z0))); b243m=(-1i*w*Bx*Eeff2*exp(-1i*Bz2n*(l1+z0))); b244m=(-1i*w*Bx*Eeff2*exp(1i*Bz2n*(l1+z0)));

% The coefficient matrix in region 2, z=l2=l1+a
B23=[b211e b212e b213m b214m ;b221e b222e b223m b224m ;b231e b232e b233m b234m ;b241e b242e b243m b244m ];

% Region 3

a311e=(1i*w*mu0*Byn*exp(-1i*Bz3n*(l1+z0))); a312m=(-1i*Bx*Bz3n*exp(-1i*Bz3n*(l1+z0)));

a321e=w*mu0*Bx*exp(-1i*Bz3n*(l1+z0)); a322m=Byn*Bz3n*exp(-1i*Bz3n*(l1+z0));
a331e=-Bx*Bz3n*exp(-i*1i*Bz3n*(l1+z0));
a332m=-w*e0*Byn*exp(-i*1i*Bz3n*(l1+z0));
a341e=i*Byn*Bz3n*exp(-i*1i*Bz3n*(l1+z0));
a342m=-i*i*w*e0*Bx*exp(-i*1i*Bz3n*(l1+z0));

% Coefficient matrix region 3
A23=[0 0 a311e a312m ; 0 0 a321e a322m ; 0 0 a331e a332m ;0 0 a341e a342m];
% Compute the following
AA=B12*((B23)\A23);
A=AA-A12;
u=A\B0;
% Solution for the TE1 and TM1 mode
x1n=u(1,1)
y1n=u(2,1)
x3n=u(3,1)
y3n=u(4,1)

%% coefficients corresponding to the system of equations in the coupling (n=0), for
% reflected wave in region 1
a1110= exp(1i*Bz10*l1);
a1210= Bz10/(w*mu0)*exp(1i*Bz10*l1);
% waves in region 2
%% for 1-2 interface
a2110= exp(-1i*Bz20*l1);
a2120= exp(1i*Bz20*l1);
a2210= (-Bz20/(w*mu0))*exp(-1i*Bz20*l1);
a2220= (Bz20/(w*mu0))*exp(1i*Bz20*l1);

%% for 2-3 interface
b2110= exp(-1i*Bz20*(l1+z0));
b2120= exp(1i*Bz20*(l1+z0));
b2210= -Bz20/(w*mu0)*exp(-1i*Bz20*(l1+z0));
b2220= Bz20/(w*mu0)*exp(1i*Bz20*(l1+z0));
% transmitted waves in region 3
a3120= exp(-i*Bz30*(l1+z0));
a3220= -Bz30/(w*mu0)*exp(-1i*Bz30*(l1+z0));
%% Curve Fit-Discharge current,Id0
% General model Gauss1:%% Curve Fit-Discharge current,Id0
% General model Gauss1:
% Coefficients (with 95% confidence bounds):
a1 = 10^6;
b1 = 10^-6;
c1 = .0000000001 ;
\[ f = @\(x\) a1*exp(-(x-b1)/c1)^2 \]

% Goodness of fit:
% SSE: 3.625
% R-square: 0.8622
% Adjusted R-square: 0.8484
% RMSE: 0.4257

I=f(t)*exp(-i*w*t);
% frequency transform of discharge current
Id0= int(I,t,s,s);
% Cases to be considered
Id0m=matlabFunction(Id0)

%% beam current density (Refer Eq. 2.14)

Je2b0=Jyb0*(1/z0)*exp(-Bx^2/(4*a))*Id0;

% Je2b0m=matlabFunction(Je2b0);
% Curve fit I00
% General model Exp2:
% Coefficients (with 95% confidence bounds):
a0 = 0.7884 ;
b0 = 3.033e-08 ;
c0 = -0.2351 ;
d0 = -9.539e-05 ;
I00 = a0*exp(b0*w) + c0*exp(d0*w);
% Goodness of fit:
% SSE: 3.719e-06
% R-square: 0.9984
% Adjusted R-square: 0.998
% RMSE: 0.000498

%% Curve fit I01
% General model Fourier6:
% Coefficients (with 95% confidence bounds):
a01 = 1.016e-15 ;
a11 = 7.308e-16 ;
b11 = -1.934e-15 ;
a21 = -1.221e-15 ;
b21 = -2.558e-16 ;
a31 = 9.341e-16 ;
b31 = -3.049e-16 ;
a41 = -9.601e-16 ;
b41 = 3.806e-16 ;
a51 = -1.474e-15 ;
\[ b51 = 1.694 \times 10^{-15}; \]
\[ a61 = 1.744 \times 10^{-15}; \]
\[ b61 = 1.748 \times 10^{-16}; \]
\[ w1 = 9.141 \times 10^{-5}; \]
\[ I01 = a01 + a11 \cos(w*w1) + b11 \sin(w*w1) + a21 \cos(2*w*w1) + b21 \sin(2*w*w1) + a31 \cos(3*w*w1) + b31 \sin(3*w*w1) + a41 \cos(4*w*w1) + b41 \sin(4*w*w1) + a51 \cos(5*w*w1) + b51 \sin(5*w*w1) + a61 \cos(6*w*w1) + b61 \sin(6*w*w1); \]

% Goodness of fit:
%   SSE: 1.24e-29
%   R-square: 0.9266
%   Adjusted R-square: 0.7359
%   RMSE: 1.575e-15

%% Curve fit I02
% Linear model Poly2:

%   where x is normalized by mean 9.803e+04 and std 5.015e+04
% Coefficients (with 95% confidence bounds):
%   p1 = 2.137e-07;
%   p2 = 8.614e-07;
%   p3 = -0.7927;
\[ I02 = p1*w^2 + p2*w + p3; \]
% Goodness of fit:
%   SSE: 5.554e-15
%   R-square: 0.9996
%   Adjusted R-square: 0.9996
%   RMSE: 1.863e-08

%% Curve fit I3
% General model Fourier7:
% Coefficients (with 95% confidence bounds):
%   a03 = 1.276e-19;
%   a13 = 5.935e-18;
%   b13 = -4.44e-19;
%   a23 = -4.516e-20;
%   b23 = 1.166e-18;
%   a33 = -6.037e-18;
%   b33 = -1.203e-18;
%   a43 = -1.479e-18;
%   b43 = 2.282e-18;
%   a53 = -3.437e-19;
%   b53 = -9.008e-18;
%   a63 = -7.492e-18;
%   b63 = 3.887e-18;
%   a73 = 7.513e-18;
%   b73 = -1.783e-18;
\[ w3 = 6.268 \times 10^{-5} \]

\[ I03 = a03 + a13 \cos(w \cdot w3) + b13 \sin(w \cdot w3) + a23 \cos(2 \cdot w \cdot w3) + b23 \sin(2 \cdot w \cdot w3) + a33 \cos(3 \cdot w \cdot w3) + b33 \sin(3 \cdot w \cdot w3) + a63 \cos(6 \cdot w \cdot w3) + b63 \sin(6 \cdot w \cdot w3) + a73 \cos(7 \cdot w \cdot w3) + b73 \sin(7 \cdot w \cdot w3) \]

% Goodness of fit:
% SSE: \(1.598 \times 10^{-34}\)
% R-square: 0.9043
% Adjusted R-square: 0.4258
% RMSE: 7.297 \times 10^{-18}
%
% Curve fit I04
% Linear model Poly2:
% where x is normalized by mean 9.803e+04 and std 5.015e+04
% Coefficients (with 95% confidence bounds):
% \[ p_{14} = -2.251 \times 10^{-08} ; \]
% \[ p_{24} = -8.691 \times 10^{-08} ; \]
% \[ p_{34} = -0.07927 ; \]

\[ I04 = p_{14} \cdot w^2 + p_{24} \cdot w + p_{34} \]

% Goodness of fit:
% SSE: \(1.705 \times 10^{-17}\)
% R-square: 0.9999
% Adjusted R-square: 0.9999
% RMSE: 1.032 \times 10^{-09}

\[ I_{Je2b0} = Jyb0 \cdot (I00 + Bx/(2 \cdot a) \cdot I01 + Bx^2/(8 \cdot a^2) \cdot I02 + Bx^3/(48 \cdot a^3) \cdot I03); \]
\[ I_{He2b0} = Jyb0 \cdot (I01 + Bx/(2 \cdot a) \cdot I02 + Bx^2/(8 \cdot a^2) \cdot I03 + Bx^3/(48 \cdot a^3) \cdot I04); \]

\[ I_{convH} = 1i \cdot \sigma_i \cdot I_{d0}/z_0 \cdot (\sqrt{\pi/a} \cdot \exp(-Bx^2/(4 \cdot a)) \cdot I_{He2b0} - 2 \pi \cdot \exp(-Bx^2/(4 \cdot a)) \cdot (w^2 \cdot mu_0 \cdot E_{eff2} - Bx^2)); \]
\[ I_{convJ} = 1i \cdot \sigma_i \cdot I_{d0}/z_0 \cdot (\sqrt{\pi/a} \cdot \exp(-Bx^2/(4 \cdot a)) \cdot I_{Je2b0} - 2 \pi \cdot \exp(-Bx^2/(4 \cdot a)) \cdot (w^2 \cdot mu_0 \cdot E_{eff2} - Bx^2)); \]
\[ I_{001} = (1i \cdot w^2 \cdot mu_0/(Bz_2^0) + (1i \cdot w^2 \cdot mu_0 \cdot Bx^2)/(B0i^2 \cdot Bz_2^0)) \cdot I_{Je2b0}; \]

\[ K = I_{001} - (w^2 \cdot mu_0^2)/(2 \cdot pi \cdot Bz_2^0) \cdot I_{convJ} + (1i \cdot w^2 \cdot mu_0^2 \cdot Bx^2)/(2 \cdot pi \cdot B0i^2 \cdot Bz_2^0 \cdot 2^2) \cdot I_{convH}; \]

\[ K = 0 \]

% Define the coefficient matrix form of the system of equations from the boundary conditions(n=0)
% the systems of equation to be solved is of the form
% \[ A01 \cdot x + A02 \cdot y = b \]
% where \( x = [x01, x02] = [A10, A30]^+ \)
% thus the system of equations is equivalent to the matrix equation
% \[ A \cdot x = b \]
% where \( A = A01 - A02 \cdot \text{inv}(A11) \cdot A12 \) and \( b0 = b + A02 \cdot \text{inv}(A11) \cdot b \)

\[ A01 = [a1100; a1210 0]; \]
\[ A02 = [a2100 a2120; a2210 a2220]; \]
\[ B01 = [b2110 b2120; b2210 b2220]; \]
\[ B02 = [0 a3120; 0 a3220]; \]

% Define the column of constants
b01=[exp(-i*B0i*l1);-Bz10/(w*mu0)*exp(-i*B0i*l1)];
b02=[K;0];
% Solve for the unknowns in the Matrix equations.
B_inv=inv(B01);
A=A01-A02*B_inv*B02;
b0=b02 - A02*B_inv*b02-b01;

% solve for the unknown field amplitudes
x_01=inv(A)*(b02- A02*B_inv*b02)
x_02= -inv(A)*b01
x=x_01+x_02;

% Compute the transmitted Power
% The power transmitted is
F_inh=real(.5*d)*x_01(2,1)^2*conj(Bz30)/(w*mu0));
F_inhn=matlabFunction(F_inh)
F_hom=matlabFunction((.5*d*x0)*real(conj(Bz30))/(w*mu0)*(x_01(2,1)*conj(x_02(2,1))+x_02(2,1)*conj(x_01(2,1))));
F_hom1=matlabFunction((.5*d)*real(conj(Bz30))/(w*mu0)*((x_01(2,1)*conj(x_02(2,1))+x0*x_02(2,1)*conj(x_02(2,1))));
% The the power flowing in
P_in=.00135*d*x0;

x02m=matlabFunction(x_02(2,1))
% Plot the output power vs. Frequency
hold on
m0=10 ;
for N=10^18;
for f=(1:10^7:m0*10^9)
   Bx0=2*pi*f*sqrt(e0*mu0); % define limits of integration
   if F_inh ~=0
      funt= @(Bx)F_inhn(Bx,N,f);
      Qt=integral(funt,-Bx0,Bx0);
      p1= semilogx(f,((Qt+F_hom(0,N,f))/P_in),'r-*'); % Plot of the total power vs. frequency
      set(gca, 'XScale', 'log');
      grid on
   else
      p1=semilogx(f,(F_hom(0,N,f))/P_in,'r-*'); % Plot of the total power vs. frequency
      set(gca, 'XScale', 'log');
      grid on
   end
end
title('Plot, Power ratio(Pt/Pi) vs. Frequency');
xlabel('Frequency(Hz)');
ylabel('Pt/Pi');
end
for \( N = 5 \times 10^{18} \);  
for \( f = (1:10^7:m0*10^9) \)  
\[
Bx0 = 2\pi f \sqrt{e0*mu0}; \quad \% \text{define limits of integration}
\]

if \( F_{\text{inh}} \approx 0 \)
    \[
    \text{funt} = @(Bx)F_{\text{inhn}}(Bx,N,f);
    \text{Qt} = \text{integral(funt,-Bx0,Bx0)};
    \text{p2} = \text{semilogx}(f,((\text{Qt}+F_{\text{hom}}(0,N,f))/P_{\text{in}}),'b*-'); \quad \% \text{Plot of the total power vs. frequency}
    \text{set(gca,'XScale','log');}
    \text{grid on}
    \]
else
    \[
    \text{p2} = \text{semilogx}(f,(F_{\text{hom}}(0,N,f))/P_{\text{in}},'b*-'); \quad \% \text{Plot of the total power vs. frequency}
    \text{set(gca,'XScale','log');}
    \text{grid on}
    \]
end

end

for \( N = 10^{19} \);  
for \( f = (1:10^7:m0*10^9) \)  
\[
Bx0 = 2\pi f \sqrt{e0*mu0}; \quad \% \text{define limits of integration}
\]

if \( F_{\text{inh}} \approx 0 \)
    \[
    \text{funt} = @(Bx)F_{\text{inhn}}(Bx,N,f);
    \text{Qt} = \text{integral(funt,-Bx0,Bx0)};
    \text{p3} = \text{semilogx}(f,((\text{Qt}+F_{\text{hom}}(0,N,f))/P_{\text{in}}),'k*-'); \quad \% \text{Plot of the total power vs. frequency}
    \text{set(gca,'XScale','log');}
    \text{grid on}
    \]
else
    \[
    \text{p3} = \text{semilogx}(f,(F_{\text{hom}}(0,N,f))/P_{\text{in}},'k*-'); \quad \% \text{Plot of the total power vs. frequency}
    \text{set(gca,'XScale','log');}
    \text{grid on}
    \]
end

end

legend([p1 p2 p3],['N=' num2str(10^18,'%10.1e\n')],['N=' num2str(5*10^18,'%10.1e\n')],['N=' num2str(10*10^18,'%10.1e\n')])
% legend([p10 p20 p30],[N= num2str(10^18,'%10.1e\n')],[N= num2str(5*10^18,'%10.1e\n')],[N= num2str(10*10^18,'%10.1e\n')])
APPENDIX F: SHORT HAND NOTATIONS

The short hand notations below have been written based on extending \( l_{i-1} \) to \(-\infty\) and \( l_i \) to \(+\infty\). The short hand notations are valid for the finite region by replacing \(-\infty\) by \( l_{i-1} \) and \(+\infty\) by \( l_i \) with the understanding that the argument of the integrand pertains the \( i^{th} \) medium. Refer to Figure 2-1 for region demarcations relative to boundary interfaces separating the regions.

\[
G_{>z}(G_{>\tilde{z}}(f, \beta_{zim}), \beta_0), \beta_0) = G_{>z}^2(G_{>\tilde{z}}(f, \beta_{zim}), \beta_0) \quad (G-1)
\]

\[
= \left( \frac{j}{2\beta_0} \right)^2 \frac{j}{2\beta_{zim}} \int_z^\infty \left( \int_{\tilde{z}}^\infty f(\tilde{z}) e^{-j\beta_{zim}(\tilde{z}-\tilde{z})} d\tilde{z} \right) e^{-j\beta_0(\tilde{z}-z)} d\tilde{z} \quad (G-2)
\]

\[
G_z(G_z(f, \beta_{zim}), \beta_0) \quad (G-1)
\]

\[
= \frac{j}{2\beta_0} \int_z^\infty \left\{ \frac{j}{2\beta_{zim}} \int_{-\infty}^\infty f(\tilde{z}) e^{j\beta_{zim}(\tilde{z}-z)} d\tilde{z} \right\} \quad (G-1)
\]
\[ G_{>z}(G_{\bar{z}}, f, \beta_{\text{zim}}, \beta_0) \]  
\[ = \frac{j}{2\beta_0} \int_{-z}^{\infty} \left\{ \frac{j}{2\beta_{\text{zim}}} \int_{-\infty}^{\bar{z}} f(\bar{z}) e^{j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} \delta_{\bar{z}>z} \]  
\[ + \frac{j}{2\beta_{\text{zim}}} \int_{-z}^{\infty} f(\bar{z}) e^{-j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} e^{-j\beta_0(\bar{z}-z)} d\bar{z} \]

\[ G_{>z}(G_{\bar{z}}(af, \beta_{\text{zim}}), \beta_0) \]  
\[ = \frac{j}{2\beta_0} \int_{-z}^{\infty} \left\{ \frac{j}{2\beta_{\text{zim}}} \int_{-\infty}^{\bar{z}} af(\bar{z}) e^{j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} \delta_{\bar{z}>z} \]  
\[ + \frac{j}{2\beta_{\text{zim}}} \int_{-z}^{\infty} af(\bar{z}) e^{-j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} e^{-j\beta_0(\bar{z}-z)} d\bar{z} \]

\[ G_{>z}(G_{\bar{z}}(af, \beta_{\text{zim}}), \beta_0) \]  
\[ = \frac{j a}{2\beta_0} \int_{-z}^{\infty} \left\{ \frac{j}{2\beta_{\text{zim}}} \int_{-\infty}^{\bar{z}} f(\bar{z}) e^{j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} \delta_{\bar{z}>z} \]  
\[ + \frac{j}{2\beta_{\text{zim}}} \int_{-z}^{\infty} f(\bar{z}) e^{-j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} e^{-j\beta_0(\bar{z}-z)} d\bar{z} \]

\[ = a G_{>z}(G_{\bar{z}}(f, \beta_{\text{zim}}), \beta_0) \]

where \( a \) is a constant.

\[ G_{<\bar{z}}(G_{<\bar{z}}(f, \beta_{\text{zim}}), \beta_0) \]  
\[ = \left\{ \frac{j}{2\beta_0} \frac{j}{2\beta_{\text{zim}}} \int_{-\infty}^{2} \int_{-\infty}^{\bar{z}} f(\bar{z}) e^{j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} e^{j\beta_0(\bar{z}-z)} d\bar{z} \]  
\[ + \left\{ \frac{j}{2\beta_0} \frac{j}{2\beta_{\text{zim}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\bar{z}} f(\bar{z}) e^{-j\beta_{\text{zim}}(\bar{z}-2)} d\bar{z} \right\} e^{-j\beta_0(\bar{z}-z)} d\bar{z} \]
\[ G_{z}(G_{z}(G_{\bar{z}}(f, \beta_{zim}), \beta_{0}), \beta_{0}) = G_{z}^{2}(G_{\bar{z}}(f, \beta_{zim}), \beta_{0}) \]  
\[ (g) \]

\[ G_{z}^{3}(f, \beta_{0}) = G_{z}^{2}(G_{\bar{z}}(f, \beta_{0}), \beta_{0}) = G_{z}(G_{z}(G_{\bar{z}}(f, \beta_{0}), \beta_{0}), \beta_{0}) \]  
\[ (G-7) \]

\[ \frac{\partial}{\partial z} G_{z}(f, \beta_{zim}) = \frac{\partial}{\partial z} G_{<z}(f, \beta_{zim}) + \frac{\partial}{\partial z} G_{>z}(f, \beta_{zim}) \]  
\[ (G-8) \]

\[ \frac{\partial}{\partial z} G_{<z}(f, \beta_{zim}) = \frac{j}{2\beta_{zim}} f(z) - j\beta_{zim} G_{<z}(f, \beta_{zim}) \]  
\[ (G-9) \]

\[ \frac{\partial}{\partial z} G_{>z}(f, \beta_{zim}) = -\frac{j}{2\beta_{zim}} f(z) + j\beta_{zim} G_{>z}(f, \beta_{zim}) \]  
\[ (G-10) \]
\[
\frac{\partial}{\partial z} G_z(f, \beta_{zim}) = -j\beta_{zim}[G_{<z}(f, \beta_{zim}) - G_{>z}(f, \beta_{zim})]
\]  

(G-11)

The error that results from extending the boundary limits \(l_{i-1}\) and \(l_i\) to \(-\infty\) and \(+\infty\) can be obtained by looking at \(G_z(f, \beta_{zim})\) where \(f\) (like the current density) is independent of \(z\). Expanding to its long hand form yields

\[
G_z(f, \beta_{zim}) = \frac{jf}{2\beta_{zim}} \left[ \int_{l_{i-1}}^{z} e^{j\beta_{zim}(z-2)} \, d\bar{z} + \int_{z}^{l_i} e^{-j\beta_{zim}(2-z)} \, d\bar{z} \right]
\]  

(G-12)

where

\[
\beta_{zim} = |\beta_{zim}| \left[ \cos \frac{\phi}{2} - j \sin \frac{\phi}{2} \right]
\]

Integrating yields

\[
G_z(f, \beta_{zim}) = \frac{f}{\beta_{zim}^2} \left[ 1 - \text{Error} \right]
\]  

(G-13)

where

\[
\text{Error} = \frac{1}{2} e^{-j|\beta_{zim}|(z-l_{i-1})}\cos \frac{\phi}{2} e^{-|\beta_{zim}|(z-l_{i-1})\sin \frac{\phi}{2}}
\]

\[
+ \frac{1}{2} e^{-j|\beta_{zim}|(l_{i}-z)}\cos \frac{\phi}{2} e^{-|\beta_{zim}|(l_{i}-z)\sin \frac{\phi}{2}}
\]

(G-14)

The magnitude of the error is

\[
|\text{Error}| = \frac{1}{2} \left[ e^{-2|\beta_{zim}|(z-l_{i-1})}\sin \frac{\phi}{2} + e^{-2|\beta_{zim}|(l_{i}-z)}\sin \frac{\phi}{2} \right]
\]

\[
+ 2 \cos \left[ (l_{i} + l_{i-1} + z)|\beta_{zim}| \cos \frac{\phi}{2} \right] e^{-|\beta_{zim}|(l_{i}-l_{i-1})\sin \frac{\phi}{2}} \right]^{1/2}
\]

(G-15)

To simplify the analysis, let \(l_{i-1} = 0\). Consequently, the slab thickness \(\Delta l = l_i - l_{i-1} = l_i\). Define \(\bar{l}_i = l_i/\lambda\) and \(\bar{z} = z/l_i\). Then,

\[
|\beta_{zim}|[z - l_{i-1}] = 2\pi \bar{l}_i \bar{z}
\]

and
\[ |\beta_{zim}|[l_i - z] = 2\pi \tilde{l}_i[1 - \tilde{z}] \]

Therefore,

\[
|Error| = \frac{1}{2} \left[ e^{-4\pi \tilde{l}_i \tilde{z} \sin\frac{\Phi}{2}} + e^{-4\pi \tilde{l}_i [1 - \tilde{z}] \sin\frac{\Phi}{2}} \right] 
+ 2 \cos \left[ 2\pi \tilde{l}_i [1 - 2\tilde{z}] \cos\frac{\Phi}{2} \right] e^{-2\pi \tilde{l}_i \sin\frac{\Phi}{2}} \right]^{1/2}
\]

This expression is plotted versus \( \tilde{z} \) for three cases when the slab thickness is 0.1, 1, 10 times the operating wave length. In each case, \(|\Phi| = 0.0.2,0.4,0.6,0.8,1.2, and 1.4 \) radians. For \(|\Phi| = 0 \) case, the median does not attenuate the wave. It is observed that the magnitude of the error exhibits even symmetry about \( \tilde{z} = 0.5 \). Further, when the slab thickness is small compared to a wavelength, \( \Delta \tilde{l} < 0 \), the magnitude of the error approaches 1 for all \( |\Phi| \) and for all \( \tilde{z} \) in the slab. When \( \Delta \tilde{l} = 1 \) as \( |\Phi| \) increase, the endpoints of the slab approach 0.5. Further, the error at each point throughout the slab is smaller than at the slab ends. This excludes the \( |\Phi| = 0 \) case. Under a high frequency approximation where the slab thickness is large compared to wavelength, the error at the ends of the slab equals 0.5 and the error approaches zero at \( \tilde{z} = 0.5 \). In all three cases the error as a whole decreases as \( |\Phi| \) increases. Although extending the limits of integration to \( \pm \infty \) resulted in a more convenient analytical solution for \( G \), the simplification becomes more valid in a high frequency approximation with the slab ends a \( \frac{1}{2} \) error value. These arguments do not hold for the \( |\Phi| = 0 \) case.
Figure F-1 The plots in (a), (b), and (c) are plots of the magnitudes of the Error vs. the normalized slab thickness.

The code for the above plots is given below.

```matlab
% Maximum error in Asmelash's thesis for extending the limits of the G function to + and - infinity
clear all
close all
clc

myfun = @(psitilde,phi) exp(-2.*psitilde.*sin(abs(phi/2)))+...
    2*cot(abs(phi/2)).*sin(psitilde.*cos(abs(phi/2)))...
    .*exp(-psitilde.*sin(abs(phi/2)))-1; % function

psitildeV=[];
phiV=[];
psitilde0=10; % initial point

for phi=-0.0000001:-0.0001:-pi/2;
    phi;
    fun=@(psitilde) myfun(psitilde,phi);
    psitilde = fzero(fun,psitilde0);
    psitildeV=[psitildeV,psitilde];
    phiV=[phiV,phi];
```
end
plot(phiV,psitildeV)
xlabel('phi')
ylabel('psitilde')
title('psitilde at dError/dz=0')

% Special Case phi=90,pi/2 or phi/2=45,pi/4
myfun = @(psitilde,phi) exp(-2.*psitilde.*sin(abs(phi/2)))+...%
    2*cot(abs(phi/2)).*sin(psitilde.*cos(abs(phi/2)))...%
    .*exp(-psitilde.*sin(abs(phi/2)))-1; % function
fun=@(psitilde) myfun(psitilde,pi/4);
psitildeSC = fzero(fun,psitilde0)

% Magnitude of error vs z in the ith slab.
% NOTE: Normalization: li-1=0  z=ztilde*li  litilde=li/lambda
figure
hold on
litilde=10;
for phi=0:.2:pi/2; % redefine
    GError=[];
    zz=[];
    for ztilde=0:.001:1; % Ratio of position in slab to slab thickness (li)
      % ratio of slab thickness to wavelength
      psi1=2*pi*litilde*ztilde;
      psi2=2*pi*litilde*(1-ztilde);
      Error1=0.5*exp(-j*psi1.*cos(phi/2)).*exp(-psi1.*sin(abs(phi/2)));
      Error2=0.5*exp(-j*psi2.*cos(phi/2)).*exp(-psi2.*sin(abs(phi/2)));
      GError=[GError,abs(Error1+Error2)];
      zz=[zz,ztilde];
    end
    plot(zz,GError)
end
xlabel('ztilde')
ylabel('GError')
title(['GError vs ztilde; Ratio of slab thickness to wavelength = ', num2str(litilde)])
legend('phi=0','phi=-0.2','phi=-0.4','phi=-0.6','phi=-0.8','phi=-1.0','phi=-1.2','phi=-1.4','phi=-1.57')
grid

The convolution approximation assumes the following error:
The integral in Eq. (7.16) is
\[
\int_{-\infty}^{\infty} \frac{\beta_x}{\beta_0 - \beta_x^2} e^{-\frac{(\beta_x^2 - \beta_0 \beta_x)}{2 \alpha}} d\beta_x = \int_{-\infty}^{\infty} \frac{\beta_x}{\beta_0 - \beta_x^2} e^{-\frac{\beta_x^2}{2 \alpha}} e^{-\frac{\beta_0 \beta_x}{2 \alpha}} d\beta_x.
\]
For the sake of simplicity in the numerical approximation, we approximate \( e^{-\frac{\beta_0 \beta_x}{2 \alpha}} \) in its Taylor polynomial of degree 3 at \( \beta_x = 0 \)

\[
f = e^{y_1 \cdot y_2} \approx 1 + y_1 \cdot y_2 + \frac{(y_1 \cdot y_2)^2}{2!} + \frac{(y_1 \cdot y_2)^3}{3!}
\]

where \( y_1 = \frac{\beta_x}{2 \alpha} \) and \( y_2 = \beta_x \). The approximation used involves expansion of exponential with a Gaussian coefficient, \( \alpha \). The error for different values of Gaussian coefficients are given below.

Figure F-2 Plots showing error in the convolution approximation.

As it can be seen from the plots in Figure F-2, the error in the approximation is minimum when the value of \( \alpha \) around 0.25. Thus, the code is valid only for values of the Gaussian factor close enough to this value.
APPENDIX G: SPECIAL CASE: $TE_0$ MODE

With $\beta_y = 0$ (no $y$-variation), the $E_y(x, z)$, $H_x(x, z)$ and $H_z(x, z)$ field components are coupled. With all variations with respect to $y$ suppressed and $J_x$ and $J_z$ set equal to zero, Maxwell’s equation in component form yields

\[
\frac{\partial}{\partial z} E_y(x, z) = j\omega\mu_0 H_x(x, z) \tag{G-1}
\]

\[
\frac{\partial}{\partial z} E_x(x, z) - \frac{\partial}{\partial x} E_z(x, z) = -j\omega\mu_0 H_y(x, z) \tag{G-2}
\]

\[
\frac{\partial}{\partial x} E_y(x, z) = -j\omega\mu_0 H_z(x, z) \tag{G-3}
\]

\[
-\frac{\partial}{\partial z} H_y(x, z) = j\omega\varepsilon(x) E_x(x, z) \tag{G-4}
\]

\[
\frac{\partial}{\partial z} H_x(x, z) - \frac{\partial}{\partial x} H_z(x, z) = J_y(x, z) + j\omega\varepsilon(x) E_y(x, z) \tag{G-5}
\]

\[
\frac{\partial}{\partial x} H_y(x, z) = j\omega\varepsilon(x) E_z(x, z) \tag{G-6}
\]

\[
\varepsilon(x, \omega) = \varepsilon_{eff}(\omega) + \frac{\sigma_{Ti}(x, \omega)}{j\omega} \tag{G-7}
\]

Since $E_x$ and $E_z$ is independent of $y$ and $E_x(x, z) = 0$ and $E_z(x, z) = 0$ for all $x$ and $z$ when $y = 0$ and $y = d$, then $E_x(x, z)$ and $E_z(x, z)$ must be zero for all $y$. Consequently, the $TM_0$ mode for all $\beta_x$ is not supported by the waveguide. The only field components characterizing wave propagation are $H_z$, $E_y$, and $H_x$ composing the $TE_0$ mode.

The coupled equations in the $i^{th}$ region are written as

\[
\frac{\partial}{\partial z} E_{yi}(x, z) = j\omega\mu_0 H_{xi}(x, z) \tag{G-8}
\]
\[
\frac{\partial}{\partial x} E_{yi}(x, z) = -j\omega \mu_0 H_{zi}(x, z) \tag{G-9}
\]
\[
\frac{\partial}{\partial z} H_{xi}(x, z) - \frac{\partial}{\partial x} H_{zi}(x, y, z) = jy_i(x, z) + j\omega \varepsilon(x) E_{yi}(x, z) \tag{G-10}
\]

Decoupling in terms of \( E_y \) yields the inhomogeneous equation

\[
\left[ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \omega^2 \varepsilon(x) \right] E_{yi}(x, z) = j\omega \mu_0 J_{yi}(x, z) \tag{G-11}
\]

Therefore, for all \( \beta_x \) when \( \beta_y = 0 \), \( J_y(x, z) \) solely drives \( E_y \), \( H_x \), and \( H_z \) and therefore is distinguishable under these conditions. Furthermore, when \( \beta_y = 0 \), \( E_y \) is also distinguishable associated with only the \( TE_0 \) and TEM modes. TEM only occurs when \( \beta_x = 0 \). The \( TE_0 \) is the TEM mode in the limit as \( \beta_x \) approaches zero.

Since both \( E_y \) and \( J_y \) for \( \beta_y = 0 \) can only belong to one mode type for all \( \beta_x \), the wave equation is decoupled in terms of \( E_y \) (instead of \( H_z \)) allowing for a more general solution without the approximation \( \frac{\partial \varepsilon(x)}{\partial x} \) being negligible. We could not use this type of logic for \( n \neq 0 \) since the transverse components of the fields compounded by the presence of the total current density belong to both TE and TM modes. One cannot distinguish what contributions or fractions of field is a consequence of the TE and TM modes. Therefore, Eq. (G-11) in the phase domain becomes

\[
\left[ \frac{\partial^2}{\partial z^2} - \beta_x^2 + \frac{\omega^2 \mu_0}{2\pi} \varepsilon(\beta_x) \right] E_{yi0}(\beta_x, z) = j\omega \mu_0 J_{yi0}(\beta_x, z) \tag{G-12}
\]

Writing the solution to Eq. (G-12) as

\[
E_{yi0}(\beta_x, z) = \tilde{E}_{yi0}(\beta_x, z) + \tilde{E}_{yi0}(\beta_x, z) + \tilde{E}_{yi0}(\beta_x, z) + \tilde{E}_{yi0}(\beta_x, z) \tag{G-13}
\]

Substituting Eq. (G-13) into Eq. (G-12), separating fields and writing the correction terms due to medium inhomogeneity (convolution terms) in terms of \( \tilde{E}_{zh} \) and \( \tilde{E}_{zf} \), yields
\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{z0}^2(\beta_x) \right] \bar{E}_{yih0}(\beta_x, z) = 0 \quad (G-14)
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{z0}^2(\beta_x) \right] \bar{E}_{yifo}(\beta_x, z) = j\omega \mu_0 j_{el0}(\beta_x, z) \quad (G-15)
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{z0}^2(\beta_x) \right] \bar{E}_{yfh0}(\beta_x, z) = -\frac{\omega \mu_0}{j2\pi} \sigma_{\gamma l}(\beta_x) \oint \bar{E}_{yih0}(\beta_x, z) \quad (G-16)
\]

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta_{z0}^2(\beta_x) \right] \bar{E}_{yf0}(\beta_x, z) = -\frac{\omega \mu_0}{j2\pi} \sigma_{\gamma l}(\beta_x) \oint \bar{E}_{yf0}(\beta_x, z) \quad (G-17)
\]

where

\[
\beta_{z0}^2(\beta_x) = \omega^2 \mu_0 \varepsilon_{eff} - \beta_x^2 \quad (G-18)
\]

Because the fields are independent of y and the electric field component is perpendicular to the electrode surfaces, boundary conditions on y do not play a role in the solution. The solutions to Eqs. (G-14) - (G-17) become

\[
\bar{E}_{yih0}(\beta_x, z) = \bar{E}_{yih0}(\beta_x) e^{-j\beta_{z0}^2} + \bar{E}_{yih0}(\beta_x) e^{j\beta_{z0}^2} \quad (G-19)
\]

\[
\bar{E}_{yifo}(\beta_x, z) = \frac{-\omega \mu_0}{2\beta_{z0}} \int_{-\infty}^{\infty} j_{el0}(\beta_x, \tilde{z}) e^{-j\beta_{z0}^2|\tilde{z}|} d\tilde{z} \quad (G-20)
\]

\[
\bar{E}_{yfh0}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_{z0}} \int_{-\infty}^{\infty} [\sigma_{\gamma l}(\beta_x) \oint \bar{E}_{yih0}(\beta_x, \tilde{z})] e^{-j\beta_{z0}^2|\tilde{z}|} d\tilde{z} \quad (G-21)
\]

\[
\bar{E}_{yf0}(\beta_x, z) = -\frac{\omega \mu_0}{4\pi \beta_{z0}} \int_{-\infty}^{\infty} [\sigma_{\gamma l}(\beta_x) \oint \bar{E}_{yf0}(\beta_x, \tilde{z})] e^{-j\beta_{z0}^2|\tilde{z}|} d\tilde{z} \quad (G-22)
\]

Therefore, using the short hand integral notations in

\[
E_y(\beta_x, z) = \bar{E}_{y0}(\beta_x, z) + \bar{E}_{y0}(\beta_x, z) \quad (G-23a)
\]
where

\[ \bar{E}_{yi0}(\beta_x, z) = \bar{E}_{yi0}^+(\beta_x) e^{-j\beta_{z10} z} + \bar{E}_{yi0}^-(\beta_x) e^{j\beta_{z10} z} \]  
\[ + j\omega_0 \mu_0 G_z(\mathcal{I}_{ei0}(\beta_x, \bar{z}), \beta_{z10}) \]
\[ + j\frac{\omega_0 \mu_0}{2\pi} G_z(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x), \beta_{z10}) \]

\[ \bar{E}_{yi0}(\beta_x, z) = j\frac{\omega_0 \mu_0}{2\pi} G_z(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x), \beta_{z10}) \]  

From Faraday’s law, the magnetic field components can be determined using the integral forms in

\[ H_{xi}(\beta_x, z) = \bar{H}_{xi0}(\beta_x, z) + \bar{H}_{xi0}(\beta_x, z) \]  

where

\[ \bar{H}_{xi0}(\beta_x, z) = -\frac{\beta_{z10}}{\omega_0 \mu_0} \left[ \bar{E}_{yi0}^+(\beta_x) e^{-j\beta_{z10} z} - \bar{E}_{yi0}^-(\beta_x) e^{j\beta_{z10} z} \right] \]
\[ + j\beta_{z10} [G_{>z}(\mathcal{I}_{ei0}(\beta_x, \bar{z}), \beta_{z10}) - G_{<z}(\mathcal{I}_{ei0}(\beta_x, \bar{z}), \beta_{z10})] \]
\[ + j\frac{\beta_{z10}}{2\pi} \left[ G_{>z}(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x, \bar{z}), \beta_{z10}) \right] \]
\[ - G_{<z}(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x, \bar{z}), \beta_{z10}) \]  

\[ \bar{H}_{xi0}(\beta_x, z) = \frac{j\beta_{z10}}{2\pi} \left[ G_{>z}(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x, \bar{z}), \beta_{z10}) \right] \]
\[ - G_{<z}(\sigma_{T1}(\beta_x) \odot \beta_x \bar{E}_{yi0}(\beta_x, \bar{z}), \beta_{z10}) \]  

\[ H_{zi}(\beta_x, z) = \frac{\beta_x}{\omega_0 \mu_0} E_{yi}(\beta_x, z) = \bar{H}_{zi0}(\beta_x, z) + \bar{H}_{zi0}(\beta_x, z) \]  

where
\[
\vec{H}_{z0}(\beta_x, z) = \frac{\beta_x}{\omega \mu_0} \left[ \vec{E}_{yth0}^+(\beta_x) e^{-j \beta z_0 z} + \vec{E}_{yth0}^-(\beta_x) e^{j \beta z_0 z} \right] 
\]
\[
+ j \beta_x G_z (J_{i\beta}(-\beta_x, z), \beta z_0) 
\]
\[
+ \frac{j \beta_x}{2 \pi} G_z (\sigma_{Ti}(\beta_x) \circ \beta_x \vec{E}_{yf0}(\beta_x, z), \beta z_0) 
\]
\[
\vec{H}_{z0}(\beta_x, z) = \frac{j \beta_x}{2 \pi} G_z (\sigma_{Ti}(\beta_x) \circ \beta_x \vec{E}_{yf0}(\beta_x, z), \beta z_0) 
\]
\[
\text{where} \\
\vec{E}_{yth0}(\beta_x, z) = \vec{E}_{yth0}^+(\beta_x) e^{-j \beta z_0 z} + \vec{E}_{yth0}^-(\beta_x) e^{j \beta z_0 z} 
\]
\[
\vec{E}_{yf0}(\beta_x, z) = j \omega \mu_0 G_z (J_{i\beta}(-\beta_x, z), \beta z_0) 
\]
For the special case where \( \beta_y = 0 \) and \( \beta_x = 0 \), \( H_{z0}(\beta_x, z) \) vanishes yielding the TEM mode condition supporting the field components \( E_y(\beta_x, z) \) and \( H_x(\beta_x, z) \). This case can be realized by defining the source and field amplitudes and \( \sigma_{Ti} \) to be proportional to \( \delta(\beta_x) \).
APPENDIX H: NUMERICAL CHECK- THREE MEDIUM PROBLEM WITH BEAM: TEM MODE

An independent analysis is developed to validate the Matlab code characterizing the more complex and general theory in this thesis. Validating the code, in part, validates the analysis under appropriate constraints of the independent analysis. Consider a \( y \)-polarized, TEM plane wave propagating in the \( z \) direction at normal incidence to a lossy dielectric slab with a spatially uniform beam current density. For clarity, the space in this effort is unbounded in all three dimensions. The TEM wave supported by the parallel-plate waveguide and the wave in the unbounded three medium space, with slab supported current density, have the exact same form and boundary conditions in the direction of incidence. Further, the presence of the parallel-plates does not constrain the fields or beam in the bounded system and hence are comparable to the unbounded medium problem. The mediums outside of the slab in both the bounded and unbounded problems are free space. The unbounded medium problem is assumed to be homogenous in \( x \). Consequently, the fields are independent of \( x \). The Matlab code characterizing the general theory developed in this thesis is based on a general, inhomogeneous in \( x \), slab geometry. To attain the homogeneous condition in the spatial phase domain, one takes the limit that \( \beta_x \) approaches zero in the Matlab code. The code written for the general, inhomogeneous in \( x \), geometry is compared to the independent theory in the limit when the medium approaches the homogeneous case.

Consequently, the bounded and unbounded medium problems are the similar; the sources have the same form of solution, \( e^{j\omega t} \); the beam current and the incident wave are driven by different independent sources; and boundary conditions on the fields and current density are the same. The unbounded three medium case is theoretically developed in this section.

Respectively, assuming a \( e^{j\omega t} \) time harmonic form of solution, the \( y \)-polarized fields in Regions 1 and 3 with an incident wave in region 1 propagating in the +\( z \) direction are given by
Region 1

\[ E_{y1} = E_{y01}^+ e^{-j\beta_1 z} + E_{y01}^- e^{j\beta_1 z} \]

\[ H_{x1} = \frac{-\beta_2}{\omega \mu_0} \left( E_{y1}^+ e^{-j\beta_2 z} - E_{y1}^- e^{j\beta_2 z} \right) = \frac{-1}{\eta_1} \left( E_{y01}^+ e^{-j\beta_1 z} - E_{y01}^- e^{j\beta_1 z} \right) \]

where

\[ \beta_1 = \omega \sqrt{\varepsilon_1 \mu_1} \quad \text{and} \quad \eta_1 = \frac{\mu_1}{\sqrt{\varepsilon_1}}. \]

Region 3

\[ E_{y3} = E_{y03}^+ e^{-j\beta_3 z} \]

\[ H_{x3} = \frac{-\beta_3}{\omega \mu_0} \left( E_{y03}^+ e^{-j\beta_3 z} \right) = \frac{-1}{\eta_3} \left( E_{y03}^+ e^{-j\beta_3 z} \right) \]

where

\[ \beta_3 = \omega \sqrt{\varepsilon_3 \mu_3} \quad \text{and} \quad \eta_3 = \frac{\mu_3}{\sqrt{\varepsilon_3}}. \]

The fields and current density in Region 2 must satisfy the following set of Maxwell Equations

\[ \frac{\partial}{\partial z} E_{y2} = j \omega \mu_2 H_{x2} \]

\[ \frac{\partial}{\partial z} H_{x2} = J_{y2} + j \omega \varepsilon_2 E_{y2} \]

where \( \varepsilon_2 = \varepsilon_0 \left[ 1 - \frac{\omega_{pe}^2}{\omega (\omega - j\nu_c)} \right] \). The current density is assumed to be spatially uniform. Therefore, \( J_{y2} = J_{y02} \). Decoupling the above equations yields

\[ \left[ \frac{\partial^2}{\partial z^2} + \omega^2 \mu_2 \varepsilon_2 \right] E_{y2} = j \omega \mu_2 J_{y02} \]

Then, solving the homogeneous solution yields,

\[ E_{yh2} = E_{y02}^+ e^{-j\beta_2 z} + E_{y02}^- e^{j\beta_2 z} \]
where $\beta_2 = \omega \sqrt{\varepsilon_2 \mu_2}$. In general, $\varepsilon_2$ is complex. This allows one to incorporate evanescent and loss effects in the region. This implies that $\beta_2$ and $\eta_2$ are complex. The forced solution, $E_{yf2}$ using the method of undetermined coefficients is

$$E_{yf2} = \frac{jf_{y02}}{\omega \varepsilon_2}$$

Consequently, the general solution becomes

$$E_{y2} = E_{y02}^+ e^{-j\beta_2 z} + E_{y02}^- e^{j\beta_2 z} + \frac{jf_{y02}}{\omega \varepsilon_2}$$

The magnetic field in the x-direction is

$$H_{x2} = \frac{1}{j \omega \mu_2} \frac{\partial}{\partial z} E_{y2} = -\frac{j \beta_2}{j \omega \mu_2} (E_{y02}^+ e^{-j\beta_2 z} - E_{y02}^- e^{j\beta_2 z}) = -\frac{1}{\eta_2} (E_{y02}^+ e^{-j\beta_2 z} - E_{y02}^- e^{j\beta_2 z})$$

where $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$.

Boundary conditions require the transverse electric and magnetic fields to be continuous. Therefore,

$$E_{yi}(z = l_i^-) = E_{y(i+1)}(z = l_i^+)$$

$$H_{xi}(z = l_i^-) = H_{x(i+1)}(z = l_i^+)$$

where $i = 1, 2$.

For $z = l_1$

$$E_{y01}^+ e^{-j\beta_1 l_1} + E_{y01}^- e^{j\beta_1 l_1} = E_{y02}^+ e^{-j\beta_2 l_1} + E_{y02}^- e^{j\beta_2 l_1} + \frac{jf_{y02}}{\omega \varepsilon_2} \quad (H-1)$$

$$-\frac{1}{\eta_1} (E_{y01}^+ e^{-j\beta_1 l_1} - E_{y01}^- e^{j\beta_1 l_1}) = -\frac{1}{\eta_2} (E_{y02}^+ e^{-j\beta_2 l_1} - E_{y02}^- e^{j\beta_2 l_1}) \quad (H-2)$$

For $z = l_2$

$$E_{y02}^+ e^{-j\beta_2 l_2} + E_{y02}^- e^{j\beta_2 l_2} + \frac{jf_{y02}}{\omega \varepsilon_2} = E_{y03}^+ e^{-j\beta_3 l_2} \quad (H-3)$$
\[
\frac{-1}{\eta_2} \left( E_{y02}^+ e^{-j\beta_2 l_2} - E_{y02}^- e^{j\beta_2 l_2} \right) = \frac{-1}{\eta_3} \left( E_{y03}^+ e^{-j\beta_3 l_2} \right)
\]  
\hspace{1cm} \text{(H-4)}

Define the thickness of the slab as, \( \Delta l = l_2 - l_1 \) where \( l_2 > l_1 \).

Adding and subtracting Eqs. (H-3) and (H-4) and solving for the forward and the backward wave amplitudes in region 2, also subtracting Eq. (H-1) from Eq. (H-2) the fields in region 1 and 3 are coupled as

\[
E_{y01}^+ e^{-j\beta_1 l_1} = \frac{1}{4} \left[ \left( 1 + \frac{\eta_1}{\eta_2} \right) \left( 1 + \frac{\eta_2}{\eta_3} \right) e^{j\beta_2 \Delta l} 
+ \left( 1 - \frac{\eta_1}{\eta_2} \right) \left( 1 - \frac{\eta_2}{\eta_3} \right) e^{-j\beta_2 \Delta l} \right] E_{y03}^+ e^{-j\beta_3 l_2} 
+ j \left[ 1 - \cos (\beta_2 \Delta l) - j \frac{\eta_1}{\eta_2} \sin (\beta_2 \Delta l) \right] \frac{J_{y02}}{2 \omega \varepsilon_2}
\]  
\hspace{1cm} \text{(H-5)}

Equation (H-5) can be re-written as

\[
\frac{E_{y03}^+ e^{-j\beta_3 l_2}}{E_{y01}^+ e^{-j\beta_1 l_1}} = \mathcal{T}_{31} \left( E_{y01}^+, J_{y02}, \omega \right)
\]

\[
= \frac{1 - j \left[ 1 - \cos (\beta_2 \Delta l) - j \frac{\eta_1}{\eta_2} \sin (\beta_2 \Delta l) \right]}{\frac{1}{4} \left[ \left( 1 + \frac{\eta_1}{\eta_2} \right) \left( 1 + \frac{\eta_2}{\eta_3} \right) e^{j\beta_2 \Delta l} + \left( 1 - \frac{\eta_1}{\eta_2} \right) \left( 1 - \frac{\eta_2}{\eta_3} \right) e^{-j\beta_2 \Delta l} \right] \frac{J_{y02}}{2 \omega \varepsilon_2} E_{y01}^+ e^{-j\beta_1 l_1}}
\]

\[
= \frac{4 \eta_2 \eta_3 - j [\eta_2 \eta_3 (1 - \cos (\beta_2 \Delta l)) - j \eta_1 \eta_3 \sin (\beta_2 \Delta l)] \frac{2J_{y02}}{\omega \varepsilon_2 E_{y01}^+ e^{-j\beta_1 l_1}}}{[(\eta_2 + \eta_1) (\eta_3 + \eta_2) e^{j\beta_2 \Delta l} + (\eta_2 - \eta_1) (\eta_3 - \eta_2) e^{-j\beta_2 \Delta l}]}
\]

where \( \beta_q = \omega \sqrt{\varepsilon_q \mu_q} \) for \( q = 1, 2 \). Let \( K = \frac{2J_{y02}}{\omega \varepsilon_2 E_{y01}^+ e^{-j\beta_1 l_1}} \), where \( J_{y02} \) and \( E_{y01}^+ \) are external, independently driven, sources. For \( \mathcal{T}_{31} = 0 \),

\[
K = \frac{4 \eta_2}{j [\eta_2 (1 - \cos (\beta_2 \Delta l)) - j \eta_1 \sin (\beta_2 \Delta l)]}
\]  
\hspace{1cm} \text{(H-6)}

Equation (H-6) gives rise to the resonant condition between both independent sources yielding no transmission. Although, \( J_{y02} \) is fixed in amplitude, \( E_{y01}^+ \) can be chosen to satisfy the no
transmission condition. Here, $T_{31}(E_{y01}^+, J_{y02}, \omega)$ is the transmission coefficient of the medium system with beam in region 2. Let us redefine $T_{31}$ into two parts. One part being the three-medium system without beam $\tilde{T}_{31}$ and one part being the contribution of the transmission coefficient due to the presence of the beam, $\tilde{T}_{31}$, where

$$T_{31} = \tilde{T}_{31} + \tilde{T}_{31}(E_{y01}^+, J_{y02}, \omega)$$

where

$$\tilde{T}_{31} = \frac{4\eta_2\eta_3}{[(\eta_2 + \eta_1)(\eta_3 + \eta_2)e^{j\beta_2\Delta l} + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-j\beta_2\Delta l}]}$$

$$\tilde{T}_{31} = \tilde{T}_{31} - \frac{J_{y02}}{\omega\varepsilon_2 E_{y01}^+ e^{-j\beta_1 l_1}}$$

$$\tilde{T}_{31} = \frac{-2j[\eta_2\eta_3(1 - \cos(\beta_2\Delta l)) - j\eta_1\eta_3 \sin(\beta_2\Delta l)]}{[(\eta_2 + \eta_1)(\eta_3 + \eta_2)e^{j\beta_2\Delta l} + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-j\beta_2\Delta l}]}$$

Therefore,

$$E_{y03}^+ e^{-j\beta_3 l_2} = \tilde{T}_{31} E_{y01}^+ e^{-j\beta_1 l_1} + \tilde{T}_{31} \frac{J_{y02}}{\omega\varepsilon_2}$$

$$E_{y03}^+ = -\eta_3 H_{x03}^+$$

As a result, the transmitted power is

$$P_3^+(z = l_2^+) = \frac{1}{2} \text{Re} \left[ \left\{ E_{y03}^+ e^{-j\beta_3 l_2} \right\} \left\{ \frac{E_{y03}^+}{\eta_3} e^{-j\beta_3 l_2} \right\}^* \right]$$

Since $\eta_3$ is real,

$$P_3^+(z = l_2^+) = \frac{|E_{y03}^+|^2}{2\eta_3} = \frac{|E_{y03}^+ e^{-j\beta_3 l_2}|^2}{2\eta_3} = \frac{\tilde{T}_{31} E_{y01}^+ e^{-j\beta_1 l_1} + \tilde{T}_{31} \frac{J_{y02}}{\omega\varepsilon_2}}{2\eta_3}$$
Consequently,

\[ P_3^+(z = l_2^+) = \frac{1}{2\eta_3} |\tilde{T}_{31}|^2 |E_{y01}^+|^2 + \frac{1}{2\eta_3} |\tilde{T}_{31}|^2 |J_{y02}|^2 + \frac{1}{\eta_3} \text{Re} \left\{ \tilde{T}_{31}\tilde{T}_{31}^*E_{y01}^+e^{-j\beta_1 l_1}J_{y02}^+ \right\} \]

\[ = \frac{\eta_1}{\eta_3} |\tilde{T}_{31}|^2 |E_{y01}^+|^2 + \frac{1}{2\eta_3} |\tilde{T}_{31}|^2 |J_{y02}|^2 + \frac{1}{\eta_3} \text{Re} \left\{ \tilde{T}_{31}\tilde{T}_{31}^*E_{y01}^+e^{-j\beta_1 l_1}J_{y02}^+ \right\} \]

The following points are emphasized:

1. \( E_{y01}^+ \) and \( J_{y02} \) are driven by two different independent sources. In this analysis and in the theory developed in this thesis, \( E_{y01}^+ \) does NOT drive or alter \( J_{y02} \). But, it can resonate with \( J_{y02} \) under certain conditions. In resonance, the \( E_{y01}^+ \) wave can be disrupted by the current. To study the effect of resonance given \( J_{y02} \) one has to vary \( E_{y01}^+ \) (magnitude and phase) to satisfy the condition given by Eq. (H-6).

2. The transmission coefficient of the system is nonlinear. That is \( T_{31} = T_{31}(E_{y01}^+, J_{y02}, \omega) \). Under special conditions \( T_{31} = 0 \).

3. Regions 1 and 3 are lossless. Consequently, one can talk about an incident and reflected power in region 1 and a transmitted power in region 3. Consider the transmitted power at \( z = l_2^+ \) in region 3 relative to the incident power in region 1 at \( z = l_1^- \). The power transmitted in region 3 is due to: a fraction of the incident power in region 1, the presence of the beam current in region 2, and the resonance (cross terms resulting from constructive /destructive interference effects) between the beam fields and the incident field in region 1 to be disrupted.
BIBLIOGRAPHY


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