May 2017

Investigation of Determinacy for Games of Variable Length

Emi Ikeda
University of Nevada, Las Vegas

INVESTIGATION OF DETERMINACY FOR GAMES OF VARIABLE LENGTH

Emi Ikeda

Bachelor of Science - Mathematical Sciences
Yamaguchi University, Japan
2004

Master of Science - Mathematical Sciences
Yamaguchi University, Japan
2006

A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
May 2017

UNLV | GRADUATE COLLEGE

Dissertation Approval

The Graduate College
The University of Nevada, Las Vegas

March 31, 2017

This dissertation prepared by

Emi Ikeda

entitled

Investigation of Determinacy for Games of Variable Length

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy - Mathematical Sciences
Department of Mathematical Sciences

Derrick DuBose, Ph.D.
Examination Committee Chair

Douglas Burke, Ph.D.
Examination Committee Member

Peter Shiue, Ph.D.
Examination Committee Member

Pushkin Kachroo, Ph.D.
Graduate College Faculty Representative

Kathryn Hausbeck Korgan, Ph.D.
Graduate College Interim Dean

Abstract

Many well-known determinacy results calibrate determinacy strength in terms of large cardinals (e.g., a measurable cardinal) or a "large cardinal type" property (e.g., zero sharp exists). Some of the other results are of the form that subsets of reals of a certain complexity will satisfy a well-known property when a certain amount of determinacy holds. The standard game tree considered in the study of determinacy involves games in which all moves are from omega and all plays have length omega (i.e. the game tree is $\omega^{<\omega}$ and the body of the game tree is ω^ω). There are also many well-known results on the game trees $\omega^{<\alpha}$ for α countable (all moves from omega and all paths are of fixed length α). However, one can easily construct a nondetermined open game on a game tree T, in which all moves are from ω, but some paths of T have length omega while the others of length $\omega + 1$.

Many determinacy results consider games on a fixed game tree with each path having the same length. In this dissertation, we investigate the determinacy of games on game trees with variable length paths. Especially, we investigate two types of such game trees, which we named Type 1 and Type 2. The length of each path in a Type 1 tree is determined by its first ω moves. A Type 2 tree is generalization of a Type 1 tree. In other words, a Type 1 tree is a special case of a Type 2 tree. We shall consider collections C of such game trees,

that will be defined from particular parameters ranging over certain sets. A $Tree_1$ collection will be a collection of Type 1 trees. A $Tree_2$ collection will be a collection of Type 2 trees. Given a $Tree_1$ (respectively, $Tree_2$) collection C and a fixed complexity (e.g., open, Borel, $\boldsymbol{\Sigma}^1_1$), we calibrate the strength of the determinacy of games with that complexity on all trees in the collection C in terms of well-known determinacy.

Acknowledgments

I would like to take this time to thank the many people that have helped me achieving my goal of acquiring my Ph.D. First, I would like to thank my advisor Dr. Derrick DuBose for his help and guidance throughout my time here at UNLV and my dissertation process. He has taken many of days and long nights to help me gain a better understanding of Set Theory. I would also like to especially thank Dr. Douglas Burke for taking time out of his schedule to help me through my whole time at UNLV. He has been like a co-advisor to me. He has taken time out of his day to meet with me every week to help me study and present me with a lot of ideas to help me in understanding Set Theory. He has helped me solve many of the problems I had concerning my studies. He guided me to the kind of game trees to look for in my dissertation. He has been a great influence for me. I would remise if I did not thank him for his guidance and patience. Without these two I do not know if I could had made it to the position I am in today. I would like to thank my other committee members Dr. Peter Shiue and Graduate College Representative Dr. Pushkin Kachroo. They have generously taken time out of their schedules to listen to my comprehensive exam and dissertation defense. I would also like to thank the Mathematical Sciences Department at UNLV for affording me the opportunity to study and achieve my dream of obtaining my

Ph.D.

My original area of focus was topology while I was working on my Master's. I was planning on continuing to study Topology when I came to UNLV. Sadly, I was not able to find a professor to work with me in topology. When I was taking MAT 701 and MAT 702, Dr. DuBose suggested to me to study Set Theory. I learned Set Theory from scratch. He gave me the opportunity to pursue a new field of study and he invited me to the Set Theory Seminars. For that I am grateful for the opportunity to learn this new and exciting field of which I have been studying for the last few years. I would also like to thank a Ph.D. student Josh Reagan and a Master's student Katherine Yost for studying with me in the Set Theory Student Workshop here on campus. Finally, I would like to give a special acknowledgment to my family and friends for their loving support throughout the years of my studies as a Ph.D. student.

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Chapter 1

Preliminaries and Introduction

The determinacy of games has been an active area of study in set theory. In this dissertation, we will focus on two-player perfect information games on a certain type of "long trees", all of which have heights greater than (or equal to) ω. Our goal to this dissertation will be the classification of certain long games. Before we start discussing games, we will review standard definitions and well-known theorems.

In this chapter, we will review the basic concepts of games and set of notations for this dissertation. In section 1.1, we will review some notations for a product space and sequences. In section 1.2, we will define trees and games. In section 1.3, we will define complexities. We will use the product topology taking each set as a discrete space. Thus defining open sets, we will use "finiteness". Then we will define the Borel, projective, and difference hierarchies. In section 1.4, we will review several well-known determinacy results for games on trees $\omega^{<\omega}$ and $X^{<\omega}$ for any nonempty set X. Then in section 1.5, we will start the introduction to this dissertation and introduce some new concepts and notations, particular to this dissertation.

We will use the following notation 1.0.1 throughout the paper.

Notation 1.0.1. *We use $\dashv$ to signify that this is the end of the statement of definition, theorem, proposition, lemma, corollary, observation and notation.* $\dashv$

By using notation 1.0.1, it is easier to distinguish the end of a statement. While we use "$\dashv$" symbol to identify the end of a theorem, we will use this "$\square$" symbol to identify the end of a proof.

For the material in this dissertation, the following books and publication are standard references:

- Martin (2017 draft). Borel and Projective Games (unpublished).

  http://www.math.ucla.edu/~dam/booketc/thebook.pdf.

  The main reference for this dissertation is Martin's unpublished book. The 2017 draft does not include Chapter 5. The cited page numbers and theorems for Chapter 5 are from an older draft.

- Jech (2003). Descriptive Set Theory, the Third Millennium Edition, Revised and Expanded. Springer, 2003.

- Kechris (2010). Classical Descriptive Set Theory, Graduate Texts in Mathematics: vol. 156. Springer-Verlag.

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- Neeman (2004). The Determinacy of Long Games: de Gruyter Series in Logic and Its Applications, vol. 7 Berlin, Germany: de Gruyter GmbH, Walter.

- Steel (1988). Long Games. In: Kechris A.S., Martin D.A., Steel J.R. (eds) Cabal Seminar 81-85. Lecture Notes in Mathematics, vol 1333. Springer, Berlin, Heidelberg https://link.springer.com/chapter/10.1007%2FBFb0084970

These and the additional references are listed under the References on page 418.

1.1 General notations for a product space

In this section, we will review some standard notations for sequences.

Definition 1.1.1. *(Definition of countable, denumerable and uncountable sets)*

A set X is finite if there is a bijection between X and some finite subset of the set of natural numbers. A set X is denumerable if there is a bijection between X and the set of all natural numbers. A set X is countable if it is either finite or denumerable. A set X is uncountable if it is not countable, i.e., it is infinite and not denumerable. $\dashv$

Definition 1.1.2. *ω is the least countable ordinal and ω_1 is the least uncountable ordinal.* $\dashv$

Suppose X and Y are nonempty sets. X^Y is a set of functions from Y into X and thus it is called a function space. In particular, we will consider the case that Y is an ordinal number α. Then $X^\alpha = \{f \,|\, f : \alpha \to X\}$. Since the domain of each function in X^α is an ordinal α, by letting $x_\beta = f(\beta)$ for each $\beta \in \alpha$, each function f can be identified with a sequence of length α. Thus each element of X^α is a sequence $\langle x_0, x_1, ..., x_\beta, ...\rangle$ where $\beta \in \alpha$ and each $x_\beta \in X$. Each x_β is called the β-th entry of the sequence. Hence X^α is the α Cartesian product of X, i.e., $X \times X \times \cdots$ multiplied α times. Recall $\{0,1\}^\omega = 2^\omega$ is called the *Cantor space* and ω^ω is called the *Baire space*. We also use $\mathcal{N}$ to represent the Baire space. We define $X^{<\alpha}$ and $X^{\leq\alpha}$ by $X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$ and $X^{\leq\alpha} = \bigcup_{\beta\leq\alpha} X^\beta$.

Notation 1.1.3. *The length of a sequence p is the domain of p and is denoted by $lh(p)$.* $\dashv$

Note that for any sequence p, there exists a unique ordinal α such that

$$p = \langle p_0, p_1, ..., p_i, ...\rangle, i \in \alpha.$$

Thus p can be identified with the function $\{\langle i, p_i\rangle \,|i \in \alpha\}$. Hence the domain of a sequence p is the domain of the corresponding function, i.e., $lh(p) = \alpha$.

Definition 1.1.4. *(Definition of a concatenation)*

Suppose $f = \langle f(0), f(1), ...\rangle$ *and* $g = \langle g(0), g(1), ...\rangle$ *are sequences. Then a concatenation of* f *and* g*, denoted by* $f^\frown g$ *is defined to mean* $\langle f(0), f(1), ..., g(0), g(1), ...\rangle$*. i.e., if* $\alpha = dom(f)$ *and* $\beta = dom(g)$*, then* $dom(f^\frown g) = \alpha + \beta$ *and*

$$f^\frown g\,(\gamma) = \begin{cases} f\,(\gamma) & \text{if } \gamma < \alpha, \\ g\,(\delta) & \text{if } \alpha \leq \gamma < \beta, \text{ where } \gamma = \alpha + \delta. \end{cases}$$

⊣

Notation 1.1.5. *(Definition of* $\restriction$ *for a sequence)*

Suppose x *is a sequence of length* α*. Then for any* $\beta \leq \alpha$*, define* $x \restriction \beta$ *to be the sequence of length* β *such that* $(x \restriction \beta)(\gamma) = x(\gamma)$ *for* $\gamma \in \beta$*, i.e.,* $x \restriction \beta$ *and* x *have the same* γ*th component for any* $\gamma \in \beta$*. If* $\beta > \alpha$*, then we define* $x \restriction \beta$ *to be* x*.* ⊣

For a function f and a set A, the restriction of f to A, $f \restriction A$,

$$f \restriction A = f \restriction (A \cap dom(f)).$$

Since any sequence x can be identified as a function, we can obtain $x \restriction \beta = x \restriction (\beta \cap dom(x))$. Since the domain of x is the length of x, if $\beta > lh(x)$, then $\beta \cap dom(x) = \beta \cap lh(x) = lh(x)$. Thus we obtain $x \restriction \beta = x$ as in notation 1.1.5.

$$x \restriction \beta = x \restriction (\beta \cap lh(x)) = x \restriction lh(x) = x.$$

Definition 1.1.6. *(Definition of an initial segment and an extension of a sequence)*

If $s = t \restriction \alpha$ *for some ordinal* α*, then we say* s *is an initial segment of* t *and* t *is an extension* *of s (possibly $s = t$). If $s = t \restriction \alpha$ for some ordinal α and $s \neq t$, then we say s is a proper initial segment of t and t is a proper extension of s.* $\dashv$

1.2 Definition of a game

In this section, we will give standard definitions related to game trees. We regularly refer to a "game tree" as a "tree". Then we will give standard definitions related to a game on a tree. By a "game", we mean a "two-player perfect information game."

1.2.1 Definitions related to a game tree

Definition 1.2.1. *(Definition of a game tree)*

T is a game tree if T satisfies the following 4 properties.

1. *T is a set of sequences.*

2. *T is closed under initial segments, i.e., if $t \in T$ then $t \upharpoonright \alpha \in T$ for all $\alpha \in lh(t)$.*

3. *If $s \in T$ and $lh(s)$ is a limit ordinal, then there exists $t \in T$ such that $s \subsetneq t$.*

(Property 3 is a convention that we need to fix to avoid confusion. This assumption implies that there is no path of limit length which is a position. A path and a position are defined in definition 1.2.3 below.)

When we say a "tree", we mean a "game tree". Note that every tree contains the empty sequence. $\dashv$

Note that ω^ω is not a tree since it is not closed under initial segments. The typical example of a game tree is $\omega^{<\omega}$. If the game tree is not specified, we assume this tree.

Definition 1.2.2. *(Definition of the height of a tree)*

Suppose T is a tree. The height of T, denoted by $ht(T)$ is defined by

$$ht\,(T) = \sup_{p \in T} (lh\,(p)) \qquad \dashv$$

Definition 1.2.3. *(Definition of a position, a move, a play and a path)*

Suppose T is a tree. Each $p \in T$ is called a position. For any $p \in T$, define

$$M_p = \{m \mid p^\frown \langle m \rangle \in T\}.$$

Then a move at p in T is an a such that $a \in M_p$, i.e., $p^\frown \langle a \rangle \in T$. a is called a move if there exists a position $p \in T$ such that $a \in M_p$. A play is a sequence x in which every proper initial segment of x is in T and for any move a, $x^\frown \langle a \rangle$ is not in T. Each play is also called a branch or a path through the tree T. $\dashv$

Note that property 3 in definition 1.2.1 affects of a definition of a play. Suppose every proper initial segment of x is in T and no proper extension of x is in T. If the length of x is a successor ordinal and $x \in T$, then x is a play in T. If the length of x is a limit ordinal, then x is a play in T but $x \notin T$.

Definition 1.2.4. *(Definition of the body of a tree)*

Suppose T is a tree. The body of a tree is the set of all plays in T and is denoted by $[T]$. $\dashv$

If $x \in [T] \backslash T$, then the length of x is a limit ordinal. If $x \in T \cap [T]$, then the length of x is a successor ordinal.

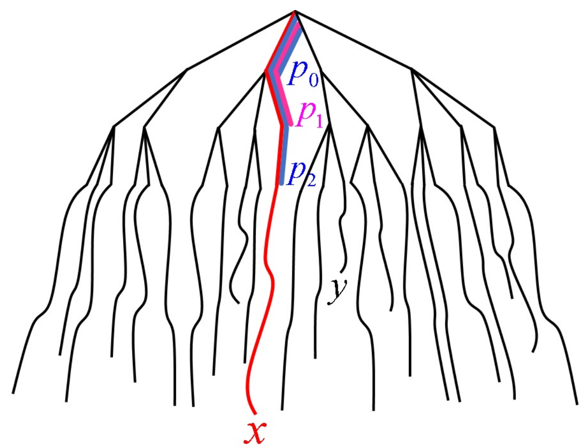

Figure 1.2.1: Illustration of $p \in T$ and $x \in [T]$.

Definition 1.2.5. *(Definition of a well-founded tree)*

Suppose T is a tree. If T has no infinite branches, T is called well-founded. Otherwise, T is called ill-founded. ⊣

Definition 1.2.6. *(Definition of the rank of a well-founded tree)*

Suppose T is a well-founded tree. Then $[T] \subseteq T$. Define the rank of T recursively.

$$rank_T : T \to \omega$$
$$p \mapsto \begin{cases} 0 & \text{if } p \in [T]\,, \\ \sup\{rank_T(p^\frown\langle k\rangle) + 1 \mid p^\frown\langle k\rangle \in T\} & \text{if } p \in T \setminus [T]\,. \end{cases}$$ ⊣

1.2.2 Definitions related to a game on a tree

Definition 1.2.7. *(Definition of a two-player perfect information game $G(A;T)$)*

Define a two-player perfect information game $G(A;T)$ as follows (see Figure 1.2.2):

1. There are two players, usually called player I and player II.

2. Player I and player II alternatively play moves as follows: Suppose $p \in T$.

(a) If $lh(p)$ is even (e.g., $lh(p) = 0$), then I plays a move a such that $a \in M_p$, i.e., $p^\frown\langle a\rangle \in T$.

(b) If $lh(p)$ is odd, then player II plays a move a such that $a \in M_p$.

3. Each player has complete knowledge of the previous moves of the way that has been played, i.e., when a player makes a move a_p at a position $p \in T$, then the player knows p, therefore knows all moves previous to a_p.

4. A play of the game is exactly a play on the tree, i.e., $f \in [T]$.

5. f is a win for player I if and only if $f \in A$, respectively, f is a win for player II if and only if $f \notin A$. A is called the payoff set for player I. $[T]\backslash A$ is called the payoff set for player II.

We denote such a game by $G(A;T)$. We will also use $G(A,[T])$ for the notation (sometimes it is easier to use $[T]$ rather than T for notational issues with cross products). ⊣

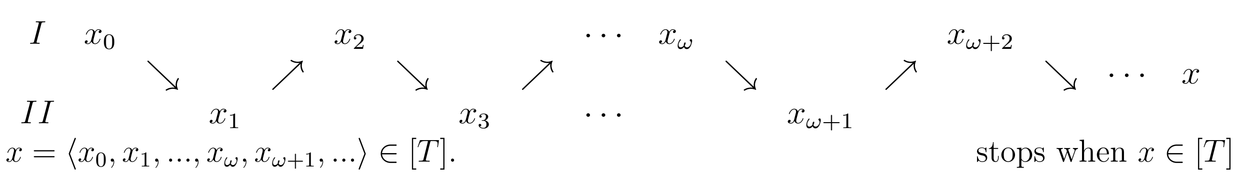

Figure 1.2.2: Illustration of a play $x \in [T]$ for $lh(x) > \omega$.

The notation $G(A;T)$ is not universal. In Jech (2003, p. 627), he uses G_A with fixed tree $\omega^{<\omega}$. In Moschovakis (2009, p. 218), he uses $G_X(A)$ for a fixed tree $X^{<\omega}$. In Kechris (2010,

p. 137), he uses $G(X, A)$ or $G(A)$ for a fixed tree $X^{<\omega}$. Since we will be considering games on different trees, we will use Martin's notation $G(A; T)$ by Martin (2017 draft, p. 5). We will sometimes use $G(A; [T])$ since we can uniquely obtain T from $[T]$.

From now on, we will only consider two-player perfect information games.

Definition 1.2.8. *(Definition of a strategy)*

Suppose T is a tree. Recall $M_p = \{m \mid p^\frown\langle m\rangle \in T\}$ for each $p \in T$. A strategy s for player I is a function such that

$$s : \{p \in T \backslash [T] \ \mid lh\,(p) \ \textit{ is even}\} \to \bigcup_{p\in T} M_p$$

and $s(p) \in M_p$.

Similarly, a strategy s for player II is a function such that

$$s : \{p \in T \backslash [T] \ \mid lh\,(p) \ \textit{ is odd}\} \to \bigcup_{p\in T} M_p$$

and $s(p) \in M_p$.

s is a strategy on the tree T if s is a strategy for player I or player II. ⊣

Definition 1.2.9. *(Definition of being according to a strategy)*

Suppose T is a tree and s is a strategy on T. For any $f \in T \cup [T]$, f is according to s if and only if for any β such that $f \restriction \beta \in dom(s)$, $f(\beta) = s(f \restriction \beta)$. ⊣

Note that each strategy s gives rise to the following tree $\mathcal{T}_s = \mathcal{T}(T, s)$.

Notation 1.2.10. *Suppose T is a tree and s is a strategy on T. Define*

$$\mathcal{T}_s = \{p \in T \mid \ p \textit{ is according to } s\}.$$ ⊣

Definition 1.2.11. *(Definition of a winning strategy for a game $G(A;T)$)*

Suppose T is a tree and $A \subseteq [T]$ is a payoff set for player I. A strategy s is a winning strategy for player I for $G(A;T)$ if for any $f \in [T]$ according to s, $f \in A$, i.e., $[\mathcal{T}_s] \subseteq A$. Similarly, a strategy s is a winning strategy for player II for $G(A,T)$ if for any $f \in [T]$ according to s, $f \notin A$, i.e., $[\mathcal{T}_s] \subseteq [T]\backslash A$. ⊣

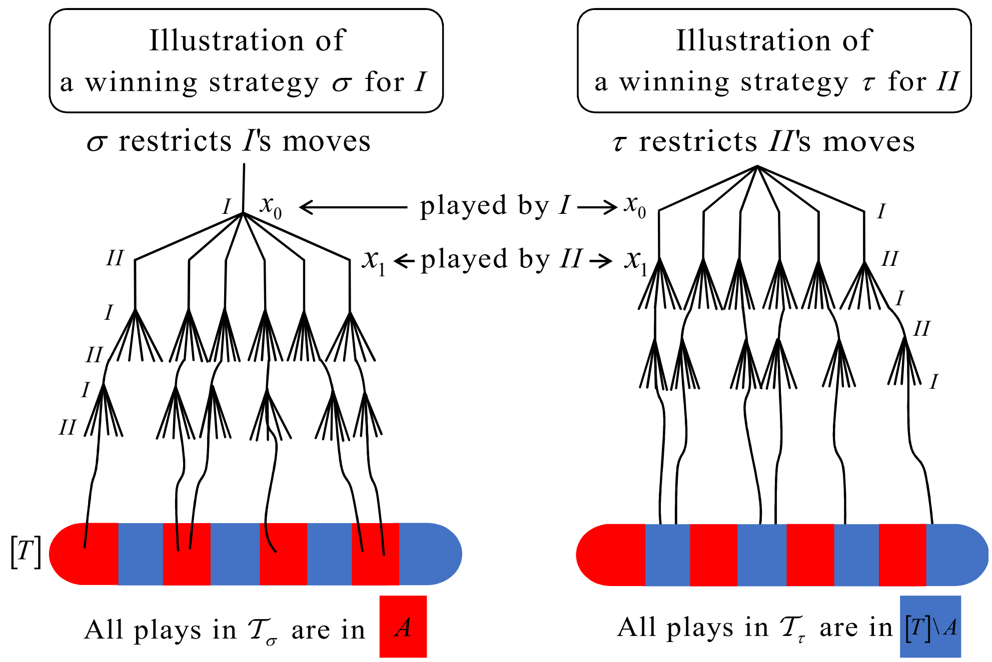

Figure 1.2.3: Illustration of winning strategies.

Definition 1.2.12. *(Definition of a game being determined)*

Suppose T is a tree and $A \subseteq [T]$ is a payoff set for player I. We say the game $G(A;T)$ is determined if and only if player I or player II has a winning strategy, i.e., there exists a strategy s on T such that $[\mathcal{T}_s] \subseteq A$ and s is a strategy of player I, or $[\mathcal{T}_s] \subseteq [T]\backslash A$ and s is a strategy for player II. ⊣

Notice that determinacy corresponds to the existence of a subtree $\mathcal{T}_\sigma$ or $\mathcal{T}_\tau$ of T as illustrated in Figure 1.2.3.

1.3 Definition of complexities

In this section, we will review standard complexities on subsets of $[T]$. We will first define open sets in a space, from which we will define Borel hierarchy, projective hierarchy, and the difference hierarchy on $\mathbf{\Pi}^1_1$ sets.

Notation 1.3.1. *(Definition of a complexity)*

In this dissertation, whenever we mention a "complexity" in chapters 2 and 3, we mean the complexities defined in this section, i.e., Borel, projective and difference hierarchy, unless specified. More precisely, the definition of a complexity in this dissertation is the following: Suppose we have Ξ such that for each tree T, $\Xi \upharpoonright [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then we say Ξ is a complexity. ⊣

1.3.1 Definitions related to topologies

First, we will review the definition of topologies.

Definition 1.3.2. *(Definition of a topology)*

Suppose X is a set. A topology on a set X is a collection τ of subsets of X such that:

1. *$\emptyset$, $X \in \tau$,*
2. *Any union of elements in τ is in τ,*
3. *The finite intersection of elements of τ is in τ.*

A set X with a topology τ, (X, τ) is called a topological space. The elements of τ are called open sets in X. ⊣

A basis for a topology (X, τ) is $\mathcal{B} \subseteq \tau$ in which every open set $A \in \tau$ can be written as unions of elements of $\mathcal{B}$.

Definition 1.3.3. *(Definition of a basis)*

Suppose X is a set. A basis of a topology (X, τ) is a collection $\mathcal{B}$ of subsets in X such that

1. *For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.*

2. *If there exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.* ⊣

If $\mathcal{B}$ satisfies both of the conditions 1 and 2, then there is a unique topology on X for which $\mathcal{B}$ is a basis. It is called the *topology generated by $\mathcal{B}$*.

We will consider the product topology $\prod_{i \in I} X_i$ with each X_i discrete. We will review the product topology and the discrete topology.

Definition 1.3.4. *(Definition of the product topology)*

Suppose X_i are sets and τ_i is a topology for X_i for $i \in I$. Consider the Cartesian product $\prod_{i \in I} X_i$. The basic open sets of $\prod_{i \in I} X_i$ are sets of the form $\prod_{i \in I} U_i$ where each U_i is an open set in X_i and $U_i \neq X_i$ for finitely many $i \in I$ (that is, "finiteness"). ⊣

Definition 1.3.5. *(Definition of the discrete topology)*

Suppose X is a set. The discrete topology on X is defined by setting every subset of X to be an open set in X. ⊣

1.3.2 Open sets

We will review open sets in product topology $\prod_{i\in I} X_i$ with each X_i discrete. Then we will define open sets over a tree T by using "finiteness".

Observation 1.3.6. *Suppose X_i for $i \in I$ are nonempty sets with the discrete topology. Then for every $x_i \in X_i$, $\{x_i\}$ is an open set in X_i. Consider the product topology for $\prod_{i\in I} X_i$. Note that every $x \in \prod_{i\in I} X_i$ is a sequence $x = \langle x_i \,|i \in I\rangle$ with $x_i \in X_i$. The basic open sets are of the form*

$$O\left(\{\langle i, x_i\rangle\,|i \in E\}\right) = \left\{ f \in \prod_{i\in I} X_i \middle| f \supseteq \{\langle i, x_i\rangle\,|i \in E\} \right\}$$

for some finite $E \subseteq I$ and some $x_i \in X_i$ for $i \in I$. ⊣

For a tree T, we will define open sets over T in a way similar to our definition of open sets in the product topology, by using "finiteness". Once we define open sets over $[T]$, we can naturally define the Borel and projective sets on $[T]$.

Definition 1.3.7. *Suppose T is a tree. Define $Finite(T)$ by*

$$Finite(T) = \{q|q \text{ is finite } \wedge \exists p \in T(q \subseteq p)\}$$

The basic open sets in $[T]$ are the $O(q)$'s for $q \in Finite(T)$ where

$$O(q) = \{h \in [T]|h \supseteq q\}.$$ ⊣

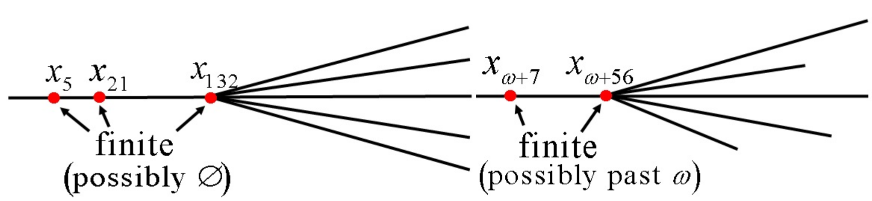

Figure 1.3.1: Illustration of q and $O(q)$.

Proposition 1.3.8. *Suppose T is a tree. The set of open sets defined in definition 1.3.7 form a basis for a topology on $[T]$.* $\dashv$

1.3.3 Borel hierarchy

Sets are classified in hierarchies according to the complexity. The collection of Borel sets on a set $[T]$ are the smallest collection containing all open sets and closed under complements and countable unions. We will denote the class of Borel sets over $[T]$ by $\mathbf{B} \upharpoonright [T]$. Borel sets are defined by the smallest σ-algebra containing all open sets. We will review the definitions of algebra and σ-algebra.

Definition 1.3.9. *(Definition of an algebra and σ-algebra)*
An algebra of sets is a collection $\mathcal{S}$ of subsets of a given set S such that

1. $S \in \mathcal{S}$,
2. *if* $X \in \mathcal{S}$ *and* $Y \in \mathcal{S}$ *then* $X \cup Y \in \mathcal{S}$,
3. *if* $X \in \mathcal{S}$ *then* $S \backslash X \in \mathcal{S}$.

Note that $\mathcal{S}$ is also closed under finite intersections. A σ-algebra is additionally closed under countable unions (and countable intersections):

4. *If* $X_n \in \mathcal{S}$ *for all* $n \in \omega$, *then* $\bigcup_{n\in\omega} X_n \in \mathcal{S}$. $\dashv$

Now, we define the Borel sets over a tree $[T]$. First, we will define the restriction notation over classes.

Notation 1.3.10. *(Moschovakis, 2009, p. 27)*

Suppose X is a space and $\mathcal{A}$ is an arbitrary collection sets. Then define $\mathcal{A} \upharpoonright X$ by

$$\mathcal{A} \upharpoonright X = \{A \subseteq X \,|A \in \mathcal{A}\}\,.$$

If the space is clear from the context, we will omit it. ⊣

Definition 1.3.11. *(Definition of the Borel sets over $[T]$)*

Suppose T is a tree. A set $B \subseteq [T]$ is Borel if it belongs to the smallest σ-algebra of subsets of $[T]$ that contains all open sets of $[T]$. We will use $\mathbf{B} \upharpoonright [T]$ to represent the collection of Borel sets over $[T]$. ⊣

We will review the definition of the Borel Hierarchy. The notation of the $\mathbf{\Sigma}$'s, $\mathbf{\Pi}$'s and $\mathbf{\Delta}$'s were introduced by Addison (1959). For more details, see Moschovakis (2009, p. 48) and Jech (2003, p. 153).

Definition 1.3.12. *(Hierarchy of Borel sets for $[T]$)(Notation by Addison, 1959)*

Suppose T is a tree. For any $1 \leq \alpha \in \omega_1$,

$$\begin{aligned}
\mathbf{\Sigma}^0_1 \upharpoonright [T] &= \text{the collection of all open sets on } [T],\\
\mathbf{\Pi}^0_1 \upharpoonright [T] &= \text{the collection of all closed sets on } [T],\\
\mathbf{\Sigma}^0_\alpha \upharpoonright [T] &= \text{the collection of all sets } A = \textstyle\bigcup_{n\in\omega} A_n, \text{where each } A_n \in \mathbf{\Pi}^0_{\beta_n} \upharpoonright [T] \text{ for some } \beta_n \in \alpha,\\
\mathbf{\Pi}^0_\alpha \upharpoonright [T] &= \text{the collection of all complements of sets in } \mathbf{\Sigma}^0_\alpha \upharpoonright [T],\\
&= \text{the collection of all sets } A = \textstyle\bigcap_{n\in\omega} A_n, \text{where each } A_n \in \mathbf{\Sigma}^0_{\beta_n} \upharpoonright [T] \text{ for some } \beta_n \in \alpha,\\
\mathbf{\Delta}^0_\alpha \upharpoonright [T] &= \mathbf{\Sigma}^0_\alpha \upharpoonright [T] \cap \mathbf{\Pi}^0_\alpha \upharpoonright [T].
\end{aligned}$$

⊣

Note that

$$\mathbf{B} = \bigcup_{\alpha\in\omega_1} \mathbf{\Sigma}^0_\alpha \upharpoonright [T] = \bigcup_{\alpha\in ON} \mathbf{\Sigma}^0_\alpha \upharpoonright [T] = \bigcup_{\alpha\in ON} \mathbf{\Pi}^0_\alpha \upharpoonright [T] = \bigcup_{\alpha\in\omega_1} \mathbf{\Pi}^0_\alpha \upharpoonright [T].$$

where ON represents the class of all ordinal numbers.

Proposition 1.3.13. *(Martin, 2017 draft, p. 7, Lemma 1.1.1 for X^ω)*

Suppose each collection in Figure 1.3.2 is defined over X^ω. For any $\alpha \in \omega_1$, we have the following inclusions.

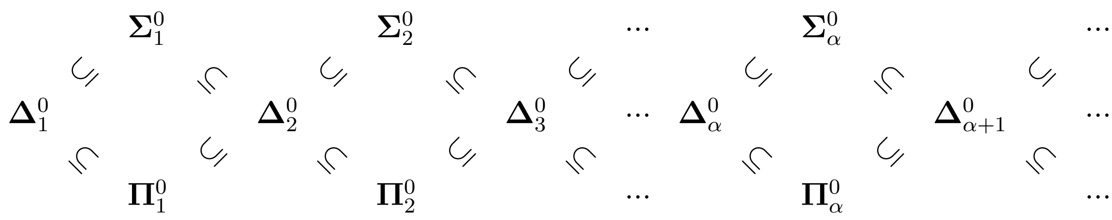

Figure 1.3.2: Diagram of Borel hierarchy for X^ω. $\dashv$

To show the diagram above, one needs to show $\mathbf{\Sigma}^0_1 \restriction X^\omega \subseteq \mathbf{\Sigma}^0_2 \restriction X^\omega$. For a countable X, one can use separability to get this. However, for even uncountable X, $\mathbf{\Sigma}^0_1 \restriction X^\omega \subseteq \mathbf{\Sigma}^0_2 \restriction X^\omega$ holds as shown in Martin (2017 draft).

In Martin (2017 draft), for $T = X^{<\omega}$, to show $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$, he uses that

$$O_n = \bigcup \{[T_p] \,|p \in T \wedge lh\,(p) = n \wedge [T_p] \subseteq A\}$$

is clopen where

$$T_p = \{q \in T \,|q \subseteq p \vee p \subseteq q\}.$$

It is routine to adjust the above argument to get $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$ for game trees of countable height. Thus, the diagram in figure 1.3.2 is true for any tree with countable height.

In general, one must be careful whether the above diagram holds for other game trees T. Dr. Burke communicated that $\mathbf{\Sigma}^0_1 \restriction 2^{\omega_1} \not\subseteq \mathbf{\Sigma}^0_2 \restriction 2^{\omega_1}$ so that the above diagram is false when $T = 2^{<\omega_1}$.

Assume that $T = 2^{<\omega_1}$. Let

$$O = \{f \in [T] | \exists \beta \in \omega_1 \text{ such that } f(\beta) = 0\}.$$

Then $O \in \mathbf{\Sigma}^0_1 \restriction [T]$. Notice that $[T] \backslash O = \{f_1\}$ where $f_1 : \omega_1 \to \{0, 1\}$ is the constant function $f(\alpha) = 1$ for any $\alpha \in \omega_1$.

We show that $O \notin \mathbf{\Sigma}^0_2 \restriction [T]$. Suppose, for a contradiction, $O \in \mathbf{\Sigma}^0_2 \restriction [T]$. Then there exists $\langle C_n | n \in \omega \rangle$ such that each $C_n \in \mathbf{\Pi}^0_1 \restriction [T]$ and $O = \bigcup_{n \in \omega} C_n$. Then $[T] \backslash O = \bigcap_{n \in \omega} O_n$ where each O_n is a complement of C_n. Thus each O_n is open. Hence each $O_n = \bigcup_{m \in \omega} B^m_n$ where each B^m_n a basic open neighborhood. Then for each B^m_n, there is $q^m_n \in Finite(T)$ such that $B^m_n = O(q^m_n)$ (see notations for definition 1.3.7). Hence

$$\{f_1\} = [T] \backslash O = \bigcap_{n \in \omega} O_n = \bigcap_{n \in \omega} \bigcup_{m \in \omega} O(q^m_n).$$

Thus, for any $n \in \omega$, there exists m_n such that $f_1 \in O(q^{m_n}_n)$, i.e., $f_1 \supseteq q^{m_n}_n$. Hence each $q^{m_n}_n$ is a sequence of countable length such that every entry is 1. Define

$$\begin{aligned} \pi : \ & \omega \to \omega \\ & n \to \mu i \in \omega \, (f_1 \in O(q^i_n)) \end{aligned}$$ [1]

Then we have

$$f_1 \in \bigcap_{n \in \omega} O\left(q^{\pi(n)}_n\right) \subseteq \bigcap_{n \in \omega} \bigcup_{m \in \omega} O(q^m_n) = \bigcap_{n \in \omega} O_n = [T] \backslash O.$$

Let $r = \sup_{n \in \omega} dom(q^{\pi(n)}_n)$. Then $r \in \omega_1$ since ω_1 is regular.[2] Let $f \in 2^{\omega_1}$ such that $f \restriction r$ is a sequence with every entry 1 and $f(r) = 0$. Then for every $n \in \omega$, $f \supseteq q^{\pi(n)}_n$. Thus

$$f \in \bigcap_{n \in \omega} O\left(q^{\pi(n)}_n\right) \subseteq [T] \backslash O = \{f_1\}$$

[1] μ represents "the least".
[2] An infinite cardinal α is regular if cofinality of α is α.

Since $f \neq f_1$, this is a contradiction. Hence $O \notin \mathbf{\Sigma}^0_2 \upharpoonright [T]$.

1.3.4 Projective hierarchy

The collection of Borel sets of reals is closed under countable unions and intersections and closed under complements, but it is not closed under continuous images. The image of a Borel set by a continuous function need not be a Borel set (Jech, 2003, p. 142).

Beyond the Borel hierarchy, we have the projective hierarchy. The $\mathbf{\Sigma}^1_1 \upharpoonright [T]$ sets are obtained from taking projections of a closed subset of $[T] \times \mathcal{N}$ along the Baire space. The $\mathbf{\Pi}^1_1 \upharpoonright [T]$ sets are the complement of $\mathbf{\Sigma}^1_1 \upharpoonright [T]$ sets. In general, for $1 \leq n < \omega$, the $\mathbf{\Sigma}^1_{n+1} \upharpoonright ([T] \times \mathcal{N}^n)$ sets are obtained from taking projections of $\mathbf{\Pi}^1_n \upharpoonright ([T] \times \mathcal{N}^{n+1})$ sets along the Baire space. In this section, we will review basic definitions associated with projective hierarchy. We will denote the class of projective sets over $[T]$ by $\mathbf{P} \upharpoonright [T]$.

Definition 1.3.14. *(Definition of the projection of S along Y)(Moschovakis, 2009, p. 19) The projection of a set $S \subseteq X \times Y$ along Y (into X) is the set*

$$P_S = \{x \in X \,|\exists y \in Y \,(\langle x, y \rangle \in S)\}\,. \qquad \dashv$$

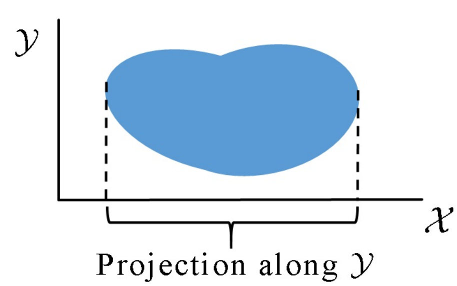

Figure 1.3.3: Illustration of a projection along $\mathcal{Y}$.

Suslin first discovered that there are $\mathbf{\Sigma}^1_1$ sets which are not Borel. Together with Lusin, they established most of the basic properties of analytic sets (as cited in Moschovakis, 2009, p. 2). Projective sets were introduced by Lusin in 1925 and independently by Sierpinski in 1925. (as cited in Moschovakis, 2009, p. 47). See more historic details in Moschovakis (2009, p. 2, p. 47).

Definition 1.3.15. *(Hierarchy of projective sets over $[T]$)(Lusin, 1925* [3]*)*

Suppose T is a tree. Define $\mathbf{\Sigma}^1_0 \upharpoonright [T] = \mathbf{\Sigma}^0_1 \upharpoonright [T]$ and $\mathbf{\Pi}^1_0 = \mathbf{\Pi}^0_1 \upharpoonright [T]$. For each $n \in \omega$ and $i \in \omega$, inductively define

$\mathbf{\Sigma}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)$= *the collection of the projections along $\mathcal{N}$ of the $\mathbf{\Pi}^1_n\upharpoonright([T]\times\mathcal{N}^{i+1})$ sets,*

$\mathbf{\Pi}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)$= *the collection of complements of the $\mathbf{\Sigma}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)$ sets,*

$\mathbf{\Delta}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)=\mathbf{\Sigma}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)\cap\mathbf{\Pi}^1_{n+1}\upharpoonright([T]\times\mathcal{N}^i)$.

Denote that the collection of projective sets over $[T]$ by $\mathbf{P} \upharpoonright [T]$. ⊣

Thus, for example, for any $A \subseteq [T]$, A is $\mathbf{\Sigma}^1_1 \upharpoonright [T]$ if and only if A is the projection of a closed set of $[T] \times \mathcal{N}$ along $\mathcal{N}$ and the collection of projective sets $\mathbf{P} \upharpoonright [T]$ is

$$\mathbf{P} \upharpoonright [T] = \bigcup\nolimits_{n\in\omega} \mathbf{\Sigma}^1_n \upharpoonright [T] = \bigcup\nolimits_{n\in\omega} \mathbf{\Pi}^1_n \upharpoonright [T].$$

$\mathbf{B} \upharpoonright [T] \subseteq \mathbf{\Delta}^1_1 \upharpoonright [T]$ is obtained from the following well-known proposition.

Proposition 1.3.16. *(Sierpinski, 1928* [4]*)*

$\mathbf{\Sigma}^1_n \upharpoonright [T]$ and $\mathbf{\Pi}^1_n \upharpoonright [T]$ are closed under countable unions and countable intersections. ⊣

There is a proof for the cases $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ in Jech (2003, pp. 142-143). See lemma 2.3.22 and lemma 2.5.13 for proofs of proposition 1.3.16.

[3] as cited in Moschovakis (2009, p. 29).
[4] as cited in Moschovakis (2009, p. 47).

Theorem 1.3.17. *(Suslin [5])*

Suppose T is a countable tree. Every $\mathbf{\Sigma}^1_1 \restriction [T]$ whose complement is also $\mathbf{\Sigma}^1_1 \restriction [T]$ is a Borel set. Thus $\mathbf{\Delta}^1_1 \restriction [T] = \mathbf{B} \restriction [T]$. ⊣

Definition 1.3.18. *(Definition of an open-separated union)(Martin, 1990; Martin, 2017 draft, p.80)*

Suppose T is a tree. $A \subseteq [T]$ is the open separated union of $\{B_j \subseteq [T] | j \in J\}$ where each $B_j \subseteq [T]$, if

1. $A = \bigcup_{j \in J} B_j$

2. *there are disjoint open sets D_j, $j \in J$ such that $B_j \subseteq D_j$ for each $j \in J$* ⊣

Definition 1.3.19. *(Definition of a quasi-Borel set)(Martin, 1990; Martin, 2017 draft, p.80)*

Suppose T is a tree. The quasi-Borel subsets of $[T]$ form the smallest class of subsets of $[T]$ containing all open sets and closed under the operations:

1. *complementation*

2. *countable union*

3. *open-separated union*

We will denote the collection of quasi-Borel sets on $[T]$ by $\mathbf{qB} \restriction [T]$. ⊣

By closure under complementation (1) and countable union (2) of quasi-Borel sets, $\mathbf{B} \restriction [T] \subseteq \mathbf{qB} \restriction [T]$ for any tree T.

[5] as cited in Jech (2003, p. 145, Theorem 11.10).

Theorem 1.3.20. *(Martin, 1990, p281 Remarks (1))*

Suppose T is tree. If T is countable, the quasi-Borel subsets of $[T]$ are Borel subsets of $[T]$. Thus $\mathbf{qB} \restriction [T] = \mathbf{B} \restriction [T]$ for a countable tree T. $\dashv$

If T is uncountable, not all the quasi-Borel subsets of $[T]$ are Borel. For example, let $T = \{\langle a \rangle^\frown p | p \in \omega^{<\omega} \wedge \alpha \in \omega_1\}$. For each $\alpha \in \omega_1$, fix $B_\alpha \in (\mathbf{\Pi}^0_\alpha \restriction \omega^\omega) \setminus (\mathbf{\Sigma}^0_\alpha \restriction \omega^\omega)$. Define $A = \{\langle a \rangle^\frown y | y \in B_\alpha\}$. Then A is quasi-Borel but not Borel (Martin, 2017 draft, p. 83, Remark(a)).

Suslin's theorem 1.3.17 generalizes:

Theorem 1.3.21. *(Hansell, 1973 [6])*

For any tree T, $\mathbf{\Delta}^1_1 \restriction [T] = \mathbf{qB} \restriction [T]$. $\dashv$

This is shown in Martin (1990, p. 281, Theorem 1).

1.3.5 Difference hierarchy

The difference kernel was discussed by Hausdorff (as cited in Welch, 1996, p. 1).

Definition 1.3.22. *(Definition of the difference kernel)(Hausdorff, 1944 [7])*

Denote the difference kernel of $\vec{A} = \langle A_\beta \,|\beta \in \alpha \rangle$ by $dk(\vec{A})$ and define

$$dk(\vec{A}) = \{x \in [T] \,|\mu\beta \,(x \notin A_\beta \vee \beta = \alpha) \text{ is odd}\}. \quad \dashv$$

Definition 1.3.23. *Suppose Λ is a class of subsets of $[T]$ and Λ is closed under countable intersections. Suppose $\alpha \in \omega_1$. Define*

$$\alpha\text{-}\Lambda \restriction [T] = \left\{ A \subseteq [T] \,\middle|\, \exists \vec{A} = \langle A_\beta \,|\beta \in \alpha \rangle \left(\text{each } A_\beta \in \Lambda \text{ and } A = dk(\vec{A}) \right) \right\}. \quad \dashv$$

[6] as cited in Martin (1990); Martin (2017 draft, p. 84, Theorem 2.2.3).

[7] as cited in Welch (1996, p. 1).

Since Λ is closed under countable intersections, without loss of generality, we can assume each $A_\beta \supseteq A_\gamma$ for any $\beta < \gamma$. Note that

$$1\text{-}\Lambda = \Lambda$$

$$2\text{-}\Lambda = \{A \subseteq [T] \,|\, \exists A_0, A_1 \in \Lambda (A = A_0 \backslash \underbrace{A_1}_{1\text{-}\Lambda})\}$$

$$3\text{-}\Lambda = \{A \subseteq [T] \,|\, \exists A_0, A_1, A_2 \in \Lambda (A = A_0 \backslash \underbrace{(A_1 \backslash A_2)}_{2\text{-}\Lambda})\}$$

$$4\text{-}\Lambda = \{A \subseteq [T] \,|\, \exists A_0, A_1, A_2, A_3 \in \Lambda (A = A_0 \backslash \underbrace{(A_1 \backslash (A_2 \backslash A_3))}_{3\text{-}\Lambda})\}$$

$$\vdots$$

In general, for any finite n,

$$n\text{-}\Lambda = \{A \subseteq [T] \,|\, \exists A_0, A_1, ..., A_{n-1} \in \Lambda (A = A_0 \backslash \underbrace{(A_1 \backslash (A_2 \backslash (A_3 \backslash (\cdots (A_{n-2} \backslash A_{n-1}))))))}_{(n-1)\text{-}\Lambda})\}$$

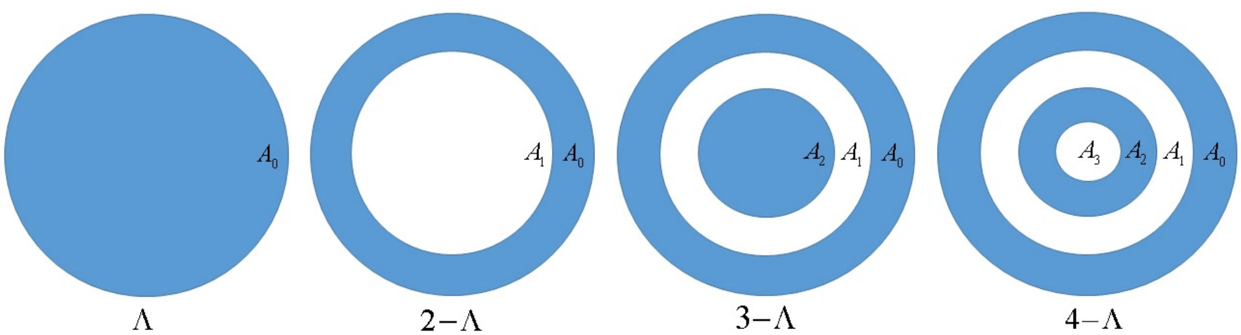

Figure 1.3.4: Illustration of difference kernel.

Consider 2-Λ. Then

$$\underbrace{A_0}_{\Lambda\restriction[T]} \backslash \underbrace{A_1}_{\Lambda\restriction[T]} = \underbrace{A_0}_{\Lambda\restriction[T]} \cap \underbrace{(X^\omega \backslash A_1)}_{co\text{-}\Lambda\restriction[T]} = \underbrace{(X^\omega \backslash A_1)}_{co\text{-}\Lambda\restriction[T]} \backslash \underbrace{(X^\omega \backslash A_0)}_{co\text{-}\Lambda\restriction[T]}.$$

Thus

$$2\text{-}\Lambda \restriction [T] = (\Lambda \wedge co\text{-}\Lambda) \restriction [T] = 2\text{-}(co\text{-}\Lambda) \restriction [T]$$

where the notation $\wedge$ is defined in notation 1.5.5 on page 43. In particular,

$$2\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega = \left(\mathbf{\Sigma}_1^1 \wedge \mathbf{\Pi}_1^1\right) \restriction X^\omega = 2\text{-}\mathbf{\Sigma}_1^1 \restriction X^\omega.$$

This gives us

$$co\text{-}2\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega = \left\{A \subseteq X^\omega \,\middle|\, X^\omega \backslash A \in 2\text{-}\mathbf{\Pi}_1^1\right\} = \left(\mathbf{\Sigma}_1^1 \vee \mathbf{\Pi}_1^1\right) \restriction X^\omega$$

where notation $\vee$ is defined in notation 1.5.5 on page 43. We also have $\mathbf{\Sigma}_1^1 \restriction X^\omega \subseteq 2\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega$ since for any $E \in \mathbf{\Sigma}_1^1 \restriction X^\omega$,

$$E = \underbrace{X^\omega}_{\mathbf{\Pi}_1^1 \restriction X^\omega} \backslash \underbrace{(X^\omega \backslash E)}_{\mathbf{\Pi}_1^1 \restriction X^\omega} \in 2\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega.$$

The following classes are also well-known and are presented in Martin (2017 draft). Note that Martin (2017 draft) does not include Chapter 5. The page numbers listed below under Chapter 5 are from an older draft.

Definition 1.3.24. *(Martin, 2017 draft, p. 24, Chapter 5, p. 203)*

$$\mathbf{Diff}\left(\mathbf{\Pi}_1^1 \restriction [T]\right) = \bigcup\nolimits_{\alpha \in \omega_1} \alpha\text{-}\mathbf{\Pi}_1^1 \restriction [T] \qquad \dashv$$

Definition 1.3.25. *(Martin, 2017 draft, p.275, Chapter 5 Section 5.4)*

Define $\mathbf{\Sigma}_1^0\left(\mathbf{\Pi}_1^1\right)$ to be the collection of all countable unions of Boolean combinations of sets belonging to $\mathbf{\Pi}_1^1$ sets. $\dashv$

Lemma 1.3.26. *(Martin, 2017 draft, p. 276, Chapter 5 Lemma 5.4.1)*

Suppose T is a tree and let $A \subseteq [T]$. Then $A \in \mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1) \restriction [T]$ if and only if A is a countable union of differences of $\mathbf{\Pi}^1_1$ sets. $\dashv$

Thus for any $\gamma \leq \beta$, we have

$$\gamma - \mathbf{\Pi}^1_1 \restriction [T] \subseteq \beta - \mathbf{\Pi}^1_1 \restriction [T] \subseteq \mathbf{Diff}\left(\mathbf{\Pi}^1_1 \restriction [T]\right) \subseteq \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right) \restriction [T].$$

1.4 Well-known determinacy results

In this section, we will list some well-known determinacy results. In section 1.4.1, we will list some well-known determinacy results from ZFC. In section 1.4.2, we will list some well-known determinacy results from large cardinal properties. The list of well-known determinacy results are also on page 367 Appendix D.

Definition 1.4.1. *(Axioms of Zermelo-Fraenkel (ZF) and ZFC)(Jech, 2003)*

1. *Axiom of Extensionality.*

   If X and Y have the same elements, then $X = Y$.

2. *Axiom of Pairing.*

   For any a and b, there exists a set $\{a, b\}$ that contains exactly a and b.

3. *Axiom Schema of Separation (Comprehension).*

   If P is a property (with parameter p), then for any X and p, there exists a set $Y = \{u \in X | P(u, p)\}$ that contains all those $u \in X$ that have property P.

4. *Axiom of Union.*

   For any X, there exists a set $Y = \bigcup X$, the union of all elements of X.

5. *Axiom of Power Set.*

   For any X, there exists a set $Y = \wp(X)$, the set of all subsets of X.

6. *Axiom of Infinity.*

   There exists an infinite set.

7. *Axiom Schema of Replacement.*

   If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x)|x \in X\}$.

8. *Axiom of Regularity (Foundation).*

   Every nonempty set has an $\in$-minimal element.

9. *Axiom of Choice.*

   Every family of nonempty sets has a choice function.

The theory with axioms 1-8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of choice; ZF-P denotes the theory with ZF without the Power Set Axiom. $\dashv$

1.4.1 Determinacy results from ZFC

Theorem 1.4.2 through theorem 1.4.9 are theorems of ZFC.

Theorem 1.4.2. *(Gale and Stewart, 1953)*

Suppose T is a tree. If T is well-founded, then for any $A \subseteq [T]$, $G(A;T)$ is determined. $\dashv$

Theorem 1.4.3. *(AC)(Gale and Stewart, 1953)(as cited in Moschovakis, 2009, p. 222, 6A.6)*

There exists $A \subseteq \omega^\omega$ such that $G(A;\omega^{<\omega})$ is not determined. $\dashv$

Definition 1.4.4. *(Definition of an open game)*

Suppose T is a tree. Suppose $A \subseteq [T]$. If A is an open set, we call $G(A;T)$ is called an open game. Similarly for the other complexities. $\dashv$

Notation 1.4.5. *Suppose T is a tree. We denote all open games on T are determined by $Det(\mathbf{\Sigma}^0_1 \restriction [T])$. In this case, we say $\mathbf{\Sigma}^0_1$ determinacy on T holds. Similarly for the other complexities.* ⊣

Theorem 1.4.6. *(Gale and Stewart, 1953)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{\Sigma}^0_1 \restriction [T])$ and $Det(\mathbf{\Pi}^0_1 \restriction [T])$. ⊣

Theorem 1.4.7. *(Wolfe, 1955)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{\Sigma}^0_2 \restriction [T])$. ⊣

Theorem 1.4.8. *(Martin, 1975; Martin, 1990)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{B} \restriction [T])$. ⊣

Theorem 1.4.9. *(Martin, 1990)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{qB} \restriction [T])$. ⊣

1.4.2 Determinacy results from large cardinals

An uncountable cardinal κ is a measurable cardinal if there is a κ-complete nonprincipal ultrafilter on κ. We will review definitions of filters related to the definition of a measurable cardinal.

Definition 1.4.10. *(Definitions of a filter, a principal filter, an ultrafilter and a κ-complete filter)*

A filter on a nonempty set S is a collection F of subsets of S such that

1. *$S \in F$ and $\emptyset \neq F$,*

2. if $X \in F$ and $Y \in F$, then $X \cap Y \in F$,

3. if $X, Y \subseteq S$, $X \in F$ and $X \subseteq Y$, then $Y \in F$.

Let X_0 be a nonempty subset of S. The filter $F = \{X \subseteq S | X \supseteq X_0\}$ is a principal filter.

A filter U on S is an ultrafilter if for every $X \subseteq S$, either $X \in U$ or $S \backslash X \in U$.

If κ is a regular uncountable cardinal and F is a filter on S, then F is called κ-complete if F is closed under intersection of less than κ sets, i.e., for any $\{X_\alpha \in F | \alpha \in \gamma\}$ with $\gamma \in \kappa$, $\bigcap_{\alpha \in \gamma} X_\alpha \in F$. ⊣

Definition 1.4.11. *(Definition of a measurable cardinal)*

An uncountable cardinal κ is measurable if there is a κ-complete nonprincipal ultrafilter U on κ. ⊣

1.4.2.1 List of results related to the existence of measurable cardinals

The following are results obtained from the existence of a measurable cardinal.

Theorem 1.4.12. *(Martin, 1970)*

If there is a measurable cardinal, then $Det(\mathbf{\Pi}^1_1 \upharpoonright \omega^\omega)$. ⊣

Theorem 1.4.13. *(Martin, 1970)(as cited in Martin, 2017 draft, p.187, Theorem 4.1.6)*

Let T be a game tree. Assume there is a measurable cardinal larger than $|T|$. Then $Det(\mathbf{\Pi}^1_1 \upharpoonright [T])$. ⊣

Theorem 1.4.14. *(Martin, 1990, p. 287, Theorem 3)*

If there is a measurable cardinal, then $Det(\omega^2\text{-}\mathbf{\Pi}^1_1 \upharpoonright \omega^\omega)$. ⊣

Martin proved the above result in 1970's. In the 1980's he proved the following generalization which uses quasi-Borel determinacy.

Theorem 1.4.15. *(Martin, 1990, p. 292, Theorem 4)*

If there is a measurable cardinal, then Det $\Delta((\omega^2+1)\text{-}\mathbf{\Pi}^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.16. *(Martin, 2017 draft, p.241, Chapter 5 Theorem 5.2.32)*

Let α *be a countable ordinal and* $T = X^{<\omega}$*. If the class of measurable cardinals greater than* $|T|$ *has order type* $\geq \alpha$*, then Det* $\Delta((\omega^2 \cdot \alpha + 1)\text{-}\mathbf{\Pi}^1_1 \restriction [T])$. ⊣

Martin's student John Simms proved the following in his dissertation.

Theorem 1.4.17. *(Simms 1979* [8]*)*

Let $T = X^{<\omega}$*. If there is a measurable limit of measurable cardinals that is larger than* $|T|$*, then* $Det(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1) \restriction [T])$. ⊣

1.4.2.2 Projective Determinacy

In general, to obtain each level of projective determinacy $Det(\mathbf{\Pi}^1_{n+1} \restriction \omega^\omega)$, we will need the existence of n Woodin cardinals. We will review the definition of an elementary embedding and define a Woodin cardinal.

Definition 1.4.18. *(Definition of an elementary embedding and a critical point)*

Suppose $\mathcal{M} = (M, E)$ *and* $\mathcal{N} = (N, F)$ *are models of set theory. An elementary embedding of* $\mathcal{M}$ *into* $\mathcal{N}$ *is a function* $j : M \to N$ *such that for any formula* $\varphi(v_1, ..., v_n)$ *of the language of set theory and for any* $a_1, ..., a_n \in M$*,*

[8] as cited in Martin (2017 draft, p. 281, Chapter 5 Theorem 5.4.5).

$$\mathcal{M} \models \varphi [a_1, ..., a_n] \Leftrightarrow \mathcal{N} \models \varphi [j(a_1), ..., j(a_n)]$$

Suppose M and N are both transitive and $j : M \to N$ is an elementary embedding. Then an ordinal $\kappa \in M$ is the critical point if κ is the least such that $j(\kappa) \neq \kappa$. ⊣

If α is an ordinal, then $j(\alpha)$ is an ordinal and if $\alpha < \beta$, then $j(\alpha) < j(\beta)$ so that $\alpha \leq j(\alpha)$. Thus we can replace $j(\kappa) \neq \kappa$ by $j(\kappa) > \kappa$. Also, for any $n \in \omega$, $j(n) = n$ and thus $j(\omega) = \omega$. Hence $\kappa > \omega$.

Theorem 1.4.19. *(Jech, 2003, p. 287)*
If there exists a measurable cardinal, then there exists a nontrivial elementary embedding of the universe. Conversely, if $j : V \to M$ is a nontrivial elementary embedding, then there exists a measurable cardinal. ⊣

Definition 1.4.20. *(Definition of the cumulative hierarchy V_α of sets)*
Inductively, for each ordinal α, define a set V_α by :

1. $V_0 = \emptyset$;

2. $V_{\alpha+1} = \wp(V_\alpha)$;

3. $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ *if λ is a limit ordinal.*

Define the class $V = \bigcup_{\alpha \in ON} V_\alpha$. ⊣

Definition 1.4.21. *(Definition of a Woodin cardinal)*
A cardinal δ is a Woodin cardinal if for all $A \subseteq V_\delta$ there are arbitrary large $\kappa < \delta$ such that

for all $\lambda < \delta$ there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $A \cap V_\lambda = j(A) \cap V_\lambda$. ⊣

Each level of projective determinacy $Det(\mathbf{\Pi}^1_{n+1} \upharpoonright \omega^\omega)$ is obtained from the existence of a measurable cardinal above n Woodin cardinals.

Theorem 1.4.22. *(Martin and Steel, 1985)*

For $n \in \omega$, if there exist n Woodin cardinals with a measurable cardinal above them, then $Det(\mathbf{\Pi}^1_{n+1} \upharpoonright \omega^\omega)$. ⊣

Projective determinacy $Det(\mathbf{P} \upharpoonright \omega^\omega)$ is obtained from the existence of infinitely many Woodin cardinals.

Theorem 1.4.23. *(Martin and Steel, 1985)*

Suppose there are infinitely many Woodin cardinals. Then $Det(\mathbf{P} \upharpoonright \omega^\omega)$. ⊣

1.4.2.3 Lightface results related to the existence of $0^\#$

We will observe theorems of difference hierarchy of lightface version. Recall that definition of Π^1_1 is obtained from a recursive relation.

We will review the definition of $0^\#$. The theory of $0^\#$ is provided in Jech (2003, p. 313, chapter 18). First, we will review the definition of Gödel's constructible universe L.

Definition 1.4.24. *(Definition of the Gödel's constructible universe L)(Martin, 2017 draft)*

Gödel's constructible universe L and hierarchy of constructible sets are defined as follows:

1. $L_0 = \emptyset$

2. *$L_{\alpha+1}$ is the collection of all subsets of L_α that are first order definable over L_α from elements of L_α. In other words, a set x belongs to $L_{\alpha+1}$ if and only if there is a formula $\varphi(v_0, ..., v_n)$ of the language of set theory and there are elements $y_1, ..., y_n$ of L_α such that*

$$x = \{y_0 \in L_\alpha \,|(L_\alpha; \in) \vDash \varphi\,[y_0, y_1..., y_n]\}\,.$$

3. *If α is a limit ordinal, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.*

4. *$L = \bigcup_{\alpha\in ON} L_\alpha$.* ⊣

By Gödel, L is a transitive class model of ZFC (as cited in Martin, 2017 draft; Kunen, 2006). See Kunen (2006) for details.

We will review the definition of a class of indiscernibles.

Definition 1.4.25. *(Definition of a class of indiscernibles)(Martin, 2017 draft)*

A class U is a class of indiscernibles for a transitive class M if

1. *$U \subseteq ON \cap M$;*

2. *if $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ are elements of U and $\varphi(v_1, ..., v_n)$ is a formula of the language of set theory, then*

$$M \vDash \varphi\,[\alpha_1, ..., \alpha_n] \leftrightarrow M \vDash \varphi\,[\beta_1, ..., \beta_n]$$ ⊣

We will review the definition of $0^\#$.

Definition 1.4.26. *(Definition of $0^\#$)(Martin, 2017 draft)*

Fix some recursive bijection $\varphi \mapsto n_\varphi$ from the set of formulas of the language of set theory *whose free variables are among* $v_1, v_2, ..$ *to the set* ω. *If there is a closed unbounded subset* C *of* ω_1 *such that* C *is a set of indiscernibles for* L_{ω_1}, *then* $0^{\#}$ *is*

$$\left\{ n_{\varphi(v_1,...,v_n)} \,|L_{\omega_1} \vDash \varphi\left[\alpha_1, ..., \alpha_n\right] \right\}$$

where $\alpha_1 < \cdots < \alpha_n$ *are elements of* C. *In such case, we say that* $0^{\#}$ *exists.* ⊣

It is well-known that if a measurable cardinal exists, then $0^{\#}$ exists (Martin, 2017 draft, p. 173, Corollary 3.4.9).

1.4.2.4 List of results related to the existence of $0^{\#}$

Initially, Martin proved the following lightface result of theorem 1.4.12 with a weaker hypothesis than an existence of a measurable cardinal.

Theorem 1.4.27. *(Martin, 1970; Martin, 2017 draft, p.2.9, Theorem 4.4.3)*

If $0^{\#}$ *exists, then* $Det(\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.28. *(Friedman, 1971* [9]*)*

If $0^{\#}$ *exists, then* $Det(3\text{-}\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.29. *(Martin, early 1970's* [10]*)*

If $0^{\#}$ *exists, then* $Det(\bigcup_{\beta\in\omega^2} \beta\text{-}\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.30. *(Martin, 1975)*

$Det(3\text{-}\Pi^1_1 \restriction \omega^\omega)$ *implies* $0^{\#}$ *exists.* ⊣

[9]as cited in DuBose (1990, p. 504).
[10]as cited in DuBose (1990, p. 512).

Theorem 1.4.31. *(Harrington, 1978* [11]*)*

$Det(\Pi^1_1 \restriction \omega^\omega)$ implies $0^\#$ exists. $\dashv$

All together, we have the following.

Theorem 1.4.32. *(Martin and Harrington)*

$Det(\Pi^1_1 \restriction \omega^\omega)$ if and only if $0^\#$ exists if and only if $Det(\bigcup_{\beta\in\omega^2} \beta\text{-}\Pi^1_1 \restriction \omega^\omega)$. $\dashv$

Martin commented that no direct proof of $Det(\Pi^1_1 \restriction \omega^\omega)$ if and only if $Det(\bigcup_{\beta\in\omega^2} \beta\text{-}\Pi^1_1 \restriction \omega^\omega)$ is known witout going through the existence of $0^\#$ (Martin, 2017 draft, p. 253, Chapter 5 Remark (a)). One can find more details regarding to these theorems in DuBose (1990, p. 512) and Martin (2017 draft, p. 253, under Remarks after Chapter 5 theorem 5.3.10).

[11] as cited in DuBose (1990, p. 512); Martin (2017 draft, p. 209).

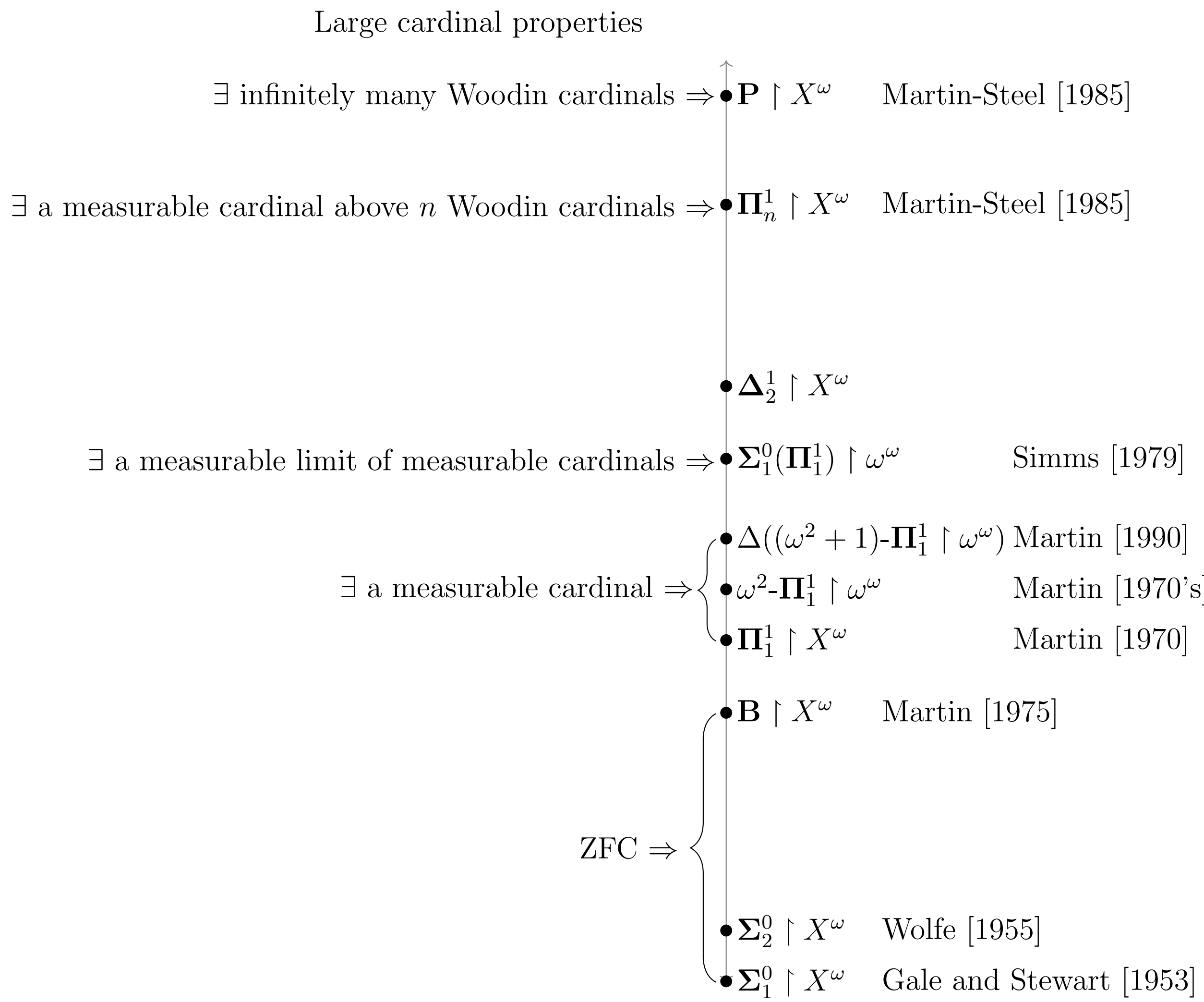

Figure 1.4.1: Illustration of well-known boldface determinacy results. The "Det" before each class is suppressed/excluded.

In the above Figure 1.4.1, $A \Rightarrow B$ abbreviates A implies the determinacy of B. Similarly,

$$A \Rightarrow \begin{cases} B \\ C \\ D \end{cases}$$

abbreviates A implies the determinacy of the classes listed, i.e., A implies the determinacy of B, A implies the determinacy of C, and A implies the determinacy of D.

1.5 Introduction to this dissertation

In this dissertation, we will focus on a certain type of "long trees". Typically, these trees will have height greater than ω. Our goal to this dissertation is to classify determinacy involving certain long games. Also, we will work on determinacy comparison in a different way as usual determinacy comparison. In section 1.5.2, we will explain the difference. The big picture for some of the determinacy results in this dissertation is shown in Appendix A on page 355.

1.5.1 Motivation to study long trees

As we have seen in section 1.4, games on trees $\omega^{<\omega}$ and $X^{<\omega}$ for a nonempty X, have been extensively studied. Every path in the trees $X^{<\omega}$ has length ω. Next, we will consider trees with height greater than ω.

First, let's consider two contrasting examples about open games.

Proposition 1.5.1. *ZF-P(folklore)*

Suppose $n \in \omega$. Then for odd n such that $1 \leq n < \omega$,

$$Det\left(\mathbf{\Sigma}^0_n \upharpoonright \omega^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \upharpoonright \omega^{\omega+n-1}\right).$$

(See more details in theorem 2.4.5). $\dashv$

Thus in ZF, we have determinacy of open games for the tree $\omega^{\omega+n-1}$ for any odd $n \in \omega$ by theorem 1.4.8.

For our contrasting example, consider $A \subseteq \omega^\omega$ such that $G\left(A; \omega^{<\omega}\right)$ is not determined. Such a set A exists by theorem 1.4.3. By theorem 1.4.8, this A is not Borel. Define a tree T

as follows.

$$h \in [T] \Leftrightarrow \begin{cases} h \in \omega^{\omega} & \text{if } h \restriction \omega \notin A, \\ h \in \omega^{\omega+1} & \text{if } h \restriction \omega \in A. \end{cases}$$

Then $ht(T) = \omega + 1$. Define $B = \{h \in [T] \,|lh\,(h) = \omega + 1\}$. We can see that $B \in \mathbf{\Sigma}_1^0 \restriction [T]$ since

$$B = \bigcup_{n \in \omega} O\,(\langle \omega, n \rangle).$$

Suppose $G\,(B;T)$ is determined. Then there is a winning strategy s^* for player I^* or player II^* for $G\,(B;T)$. If s^* is player I^*'s winning strategy for $G\,(B;T)$, then define $s = s^* \restriction \omega^{<\omega}$. Then any play f according to s is according to s^* and playing one more move $\langle a \rangle$ according to s^* gives us that $f^\frown\langle a \rangle \in B$. Thus $f \in A$ so s is a winning strategy for player I for $G\,(A;\omega^{<\omega})$. If s^* is player II^*'s winning strategy for $G\,(B;T)$, then define $s = s^*$. Then any play f according to s is according to s^*. Thus $f \in [T] \backslash B$. Hence $f \notin A$ so s is a winning strategy for player II for $G\,(A;\omega^{<\omega})$. Thus $G\,(A;\omega^{<\omega})$ is determined, a contradiction. Hence $G\,(B;T)$ is not determined.

Therefore, there is an open nondetermined game on a long tree (just one more move is added to some plays in $\omega^{<\omega}$). Note that above game has variable length. Thus we can conclude that there is a long tree T such that $Det(\mathbf{\Sigma}_1^0 \restriction [T])$ fails and such T is above. Hence we have the following proposition.

Proposition 1.5.2. *There exists a game tree T with $ht(T) = \omega + 1$ such that*

$$\neg Det\left(\mathbf{\Sigma}_1^0 \restriction [T]\right). \qquad \dashv$$

A natural project is to characterize such long T for which open determinacy holds (or fails). The trees that we will be considering will have variable length as in example above. We will calibrate the determinacy strength on these trees, by trying to obtain such determinacy that is equivalent to well-known determinacy results (as founded in section 1.4).

In this dissertation, we will define two types of trees, Type 1 trees and Type 2 trees. Type 1 trees are special case of Type 2 trees and the tree mentioned above is a special case of a Type 1 tree. Each play in these trees possibly has variable length. We will determine which games are determined on these long trees by comparing determinacy of games on these long trees and determinacy of games on the usual tree $X^{<\omega}$. In order to talk about determinacy of games on long trees, we will introduce some new concepts.

1.5.2 Difference from usual determinacy results

Each of determinacy results in section 1.4 refer to determinacy results on a fixed tree. Often we consider determinacy of games on a fixed a tree T. For example, $Det(\mathbf{\Sigma}^0_1 \upharpoonright [T])$ refers to any open set $A \subseteq [T]$, $G(A;T)$ is determined. Thus every game in $\{G\,(A;T)\,|A \in \mathbf{\Sigma}^0_1 \upharpoonright [T]\}$ is determined for a fixed tree T.

When we compare determinacy, often we are comparing the determinacy of certain games on a fixed game tree T_1 to determinacy of certain games on a fixed game tree T_2 (possibly $T_1 = T_2$). For example, we have

$$Det\left(\mathbf{\Sigma}^0_n \upharpoonright \omega^\omega\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \upharpoonright \omega^{\omega+n}\right)$$

for any $n \in \omega$. For $Det\,(\mathbf{\Sigma}^0_n \upharpoonright \omega^\omega)$, we are considering the tree $\omega^{<\omega}$ and for $Det\,(\mathbf{\Delta}^0_1 \upharpoonright \omega^{\omega+n})$, we are considering the tree $\omega^{\leq\omega+n}$.

In this dissertation, instead of fixing one tree and considering certain games on the one fixed tree, we will consider collections of trees, a $Tree_1$ collection (respectively, a $Tree_2$

collection), corresponding to Type 1 trees (respectively, Type 2 trees). Then for each tree T in a $Tree$ collection and $A \subseteq [T]$, we have the game $G(A;T)$. Thus when we say "games on a $Tree$ collection", we mean

$$\bigcup_{T \in Tree} \{G(A;T) \,|A \subseteq [T]\}.$$

Similar to how we defines open games on a fixed tree T, we will define open games on a $Tree$ collection. When we say "open games on a $Tree$ collection", we mean

$$\bigcup_{T \in Tree} \left\{G(A;T) \,\middle|A \in \mathbf{\Sigma}_1^0 \upharpoonright [T]\right\}.$$

As a notation, we use $\mathcal{G}$ for games on a $Tree$ collection. We will denote open games on a $Tree$ collection by

$$\mathcal{G}(\mathbf{\Sigma}_1^0; Tree) = \bigcup_{T \in Tree} \left\{G(A;T) \,\middle|A \in \mathbf{\Sigma}_1^0 \upharpoonright [T]\right\}.$$

Similarly, we shall define other complexities of games on a $Tree$ collection. With respect to determinacy, we usually write $Det(\mathbf{\Sigma}_1^0 \upharpoonright [T])$ for $\mathbf{\Sigma}_1^0$ determinacy on the game tree T. Similarly, $\mathbf{\Sigma}_1^0$ determinacy on a $Tree$ collection will be expressed by $Det\ \mathcal{G}(\mathbf{\Sigma}_1^0; Tree)$. See definition 2.2.7 and definition 3.2.10 for precise definition.

1.5.3 Notations for this dissertation

We will define notations particular to this dissertation. Some of the notations that we will see often in this dissertation are listed in Appendix B : List of Symbols (page 359). Notations and definitions are also listed on page 371 Appendix E.

Consider the game tree $X^{\leq \omega+n}$. We shall identify the body of the tree $[X^{\leq \omega+n}] = X^{\omega+n}$

with the product $X^\omega \times X^n$. Let $x = \langle x_0, x_1, ...\rangle \in X^\omega$ and $g = \langle g_0, g_1, ..., g_{n-1}\rangle \in X^n$. Then

$$x^\frown g = \langle x_0, x_1, ..., g_0, g_1, ..., g_{n-1}\rangle \in X^{\omega+n}.$$

Thus we can think $X^{\leq n}$ as a "tail tree" and games on $X^{\leq \omega+n}$ as games on $X^{<\omega}$ composed with "tail games", i.e., games on $X^{\leq n}$. Using the idea of a tail game, we will apply the cross product notation with X^n being replaced by arbitrary tail games.

Notation 1.5.3. *(Abuse of product notation)*

Suppose T, T_1, T_2 are trees and satisfies the following properties.

1. *every path of $[T_1]$ has length α for a fixed α,*
2. *for any $\langle f, g\rangle \in [T_1] \times [T_2]$, $f^\frown g \in [T]$ and*
3. *for any $h \in [T]$, $\langle h \restriction \alpha, h \restriction [\alpha, lh(h))\rangle \in [T_1] \times [T_2]$.*

Then to simplify notation, we abuse the cross product notation and express

$$[T] = [T_1] \times [T_2].$$

Caution :

We will use the actual cross product in some places. Readers should identify them from the context. ⊣

Notation 1.5.4. *(Abuse of inverse image notation)*

Suppose f is a function from A to B. If $b \in B$ is a singleton, we suppress $\{\}$ for $f^{-1}(\{b\})$, i.e., we write $f^{-1}(b)$ to mean $f^{-1}(\{b\})$. ⊣

Notation 1.5.5. *Define*

$$(\Lambda \wedge \Xi) \restriction [T] = \{A \,|\exists B \in \Lambda \restriction [T] \,\exists C \in \Xi \restriction [T] \,(A = B \cap C)\}.$$

Similarly, define

$$(\Lambda \vee \Xi) \restriction [T] = \{A \,|\exists B \in \Lambda \restriction [T] \,\exists C \in \Xi \restriction [T] \,(A = B \cup C)\}. \qquad \dashv$$

Notation 1.5.6. *Define*

$$co\text{-}\Lambda \restriction [T] = \{A \subseteq [T] \,|[T] \setminus A \in \Lambda\}.$$

Define

$$\Delta(\Lambda) \restriction [T] = \Lambda \restriction [T] \cap (co\text{-}\Lambda) \restriction [T]. \qquad \dashv$$

Definition 1.5.7. $f : X_1 \to X_2$ *is* Λ*-measurable if for any open* $O \subseteq X_2$, $f^{-1}(O) \in \Lambda \restriction X_1$.$\dashv$

Notation 1.5.8. *Define* $\Gamma(Y, \Lambda) = \{\Psi : X^\omega \to Y \,|\Psi$ *is* Λ*-measurable*$\}$.

Note that $\mathbf{\Sigma}^0_1$*-measurable is continuous.* $\dashv$

Observation 1.5.9. *Suppose* Ξ *is any complexity. Then*

$$\Gamma(\omega, \Xi) = \Gamma(\omega, \Delta(\Xi)) = \Gamma(\omega, co\text{-}\Xi).$$

In particular, for any $\gamma \in \omega_1$ *and* $n \in \omega$, $\Gamma(\omega, \mathbf{\Sigma}^0_\gamma) = \Gamma(\omega, \mathbf{\Delta}^0_\gamma) = \Gamma(\omega, \mathbf{\Pi}^0_\gamma)$ *and* $\Gamma(\omega, \mathbf{\Sigma}^1_n) = \Gamma(\omega, \mathbf{\Delta}^1_n) = \Gamma(\omega, \mathbf{\Pi}^1_n)$. $\dashv$

Proof.

Show $\Gamma(\omega, \Xi) \subseteq \Gamma(\omega, \Delta(\Xi))$. Suppose $\Psi \in \Gamma(\omega, \Xi)$. Then Ψ is a function from X^ω into ω such that for every open set $O \subseteq \omega$, $\Psi^{-1}(O) \in \Xi \restriction X^\omega$. Fix $O \in \mathbf{\Sigma}^0_1 \restriction X^\omega$. Since we are using the discrete topology on ω, $O = \bigcup_{n\in O}\{n\}$ and each singleton $\{n\}$ is clopen. Consider $\omega\backslash O = \bigcup_{n\in\omega\backslash O}\{n\}$. Then $\omega\backslash O$ is open. Thus $\Psi^{-1}(\omega\backslash O) \in \Xi \restriction X^\omega$. Since $\Psi^{-1}(\omega\backslash O) = X^\omega\backslash\Psi^{-1}(O)$, $\Psi^{-1}(O) \in co\text{-}\Xi \restriction X^\omega$. Thus $\Psi^{-1}(O) \in \Xi \restriction X^\omega \cap co\text{-}\Xi \restriction X^\omega = \Delta(\Xi) \restriction X^\omega$. Therefore, $\Psi \in \Gamma(\omega, \Delta(\Xi))$. Since $\Gamma(\omega, \Delta(\Xi)) \subseteq \Gamma(\omega, \Xi)$, we have $\Gamma(\omega, \Delta(\Xi)) = \Gamma(\omega, \Xi)$. Similarly, we have $\Gamma(\omega, \Delta(\Xi)) = \Gamma(\omega, co\text{-}\Xi)$. $\square$

Notation 1.5.10. *Let FIN be the collection of nonempty finite sets and CTB be the collection of nonempty countable sets.* $\dashv$

Notation 1.5.11. *Let WF be the set of nonempty well-founded trees. Let $CWF \subseteq WF$ be the set of nonempty well founded trees such that each move is from some countable set. Similarly, let $FWF \subseteq CWF$ be the set of nonempty well-founded trees such that each move is from some finite set.* $\dashv$

Chapter 2

Type 1 Tree : $T_{X,Y}^{\Psi,B}$

In this chapter, we will consider games on a certain type of long trees, called a "Type 1 tree".

In section 2.1, we will define a "Type 1" tree. Paths of a Type 1 tree typically will have variable lengths. Some paths will have length ω while other paths will have length greater than ω and less than $\omega + \omega$. The first ω-moves of each play will come from a nonempty set X. Any move made at a position of infinite length will come from a nonempty set Y. Note that Y could be just X. The length of each play will be determined by a function Ψ and a subset B of X^ω. B will be used to determine whether any play ends at length ω. If a play doesn't end at length ω, we will use the function Ψ to decide the length of the play. We will also provide a separate characterization of Type 1 trees called (X, Y)-*TEP*-$[\omega, \omega + \omega)$ (TEP is an abbreviation for "tail exchange property").

In section 2.2, we will define a certain collection of Type 1 trees, called a "$Tree_1$ collection". Each $Tree_1$ collection will have four parameters, a fixed nonempty set X, a collection of nonempty sets Y, a collection of functions and a collection of subsets of X^ω. Then, we will use standard complexities on trees ($\mathbf{\Sigma}_1^0 \upharpoonright [T], \mathbf{\Pi}_1^0 \upharpoonright [T]$) as in section 1.3 to define complexities on each $Tree_1$ collection. (Recall section 1.5.2.)

In sections 2.3 through 2.6, we will observe the determinacy strength on games on $Tree_1$ collections. In section 2.3, by shifting, we will compare the determinacy of $\mathbf{\Sigma}_\alpha^0$ (respectively, $\mathbf{\Sigma}_n^1$) games on a particular $Tree_1$ collection and $\mathbf{\Pi}_\alpha^0$ (respectively, $\mathbf{\Pi}_n^1$) games on the same $Tree_1$ collection, for $\alpha \in \omega_1$ and $n \in \omega$. In sections 2.4 through 2.6, we will compare the determinacy strength of games on a certain $Tree_1$ collection and standard determinacy of games on $X^{<\omega}$:

- In section 2.4, we will use the determinacy of a fixed complexity of games on a certain $Tree_1$ collection to obtain the determinacy of a certain complexity of games on $X^{<\omega}$.
- In section 2.5, we will obtain the determinacy of Borel and projective games on particular $Tree_1$ collections from the determinacy of a fixed complexity of games on $X^{<\omega}$. Some of these results will be converses to results in section 2.4.
- In section 2.6, we will conclude this chapter with the resulting determinacy equivalences from the earlier determinacy results between games on $X^{<\omega}$ and games on a $Tree_1$ collection.

Lastly, in section 2.7, we will generalize a Type 1 tree to an α-Type 1 tree for a limit ordinal α.

2.1 Definition of a Type 1 tree

In this section, we will give a definition of a Type 1 tree. We will also provide a separate characterization of Type 1 trees called (X, Y)-*TEP*-$[\omega, \omega + \omega)$ (TEP is an abbreviation for "tail exchange property"). Throughout this chapter, we will assume the following notation 2.1.1.

Notation 2.1.1. *We will assume the following notational conventions throughout chapter 2:*

- *X and Y will always denote nonempty sets.*
- *B will always denote a subset of X^ω.*
- *Ψ will always denote a function from X^ω into ω.* $\dashv$

Definition 2.1.2. *(Definition of a Type 1 tree)*

Suppose X and Y are nonempty sets. Let B be a subset of X^ω and let Ψ be a function from X^ω into ω. For any $h \in X^\omega \times Y^{<\omega}$, define $[T^{\Psi,B}_{X,Y}]$ by :

$$h \in \left[T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times Y^{\Psi(h\restriction\omega)+1} & \text{if } h \restriction \omega \in B. \end{cases}$$

A tree T is a Type 1 tree if and only if $T = T^{\Psi,B}_{X,Y}$ for some nonempty sets X and Y, a function Ψ from X^ω into ω and a subset B of X^ω. (Possibly $X = Y$ and also B could be the empty set.) $\dashv$

As in definition 2.1.2, fix X, Y, a function Ψ from X^ω to ω and $B \subseteq X^\omega$. Then for any $h \in [T^{\Psi,B}_{X,Y}]$,

1. $h = h \restriction \omega \in X^{\omega} \backslash B$, or $h \restriction \omega \in B$ and $h \restriction [\omega, lh(h)) \in Y^{\Psi(h\restriction\omega)+1}$. [1] We will call $h \restriction [\omega, lh(h))$, the "tail of h". Hence the tail of h is in $Y^{\Psi(h\restriction\omega)+1}$. Thus $lh(h) = \omega$ if and only if $h \in X^{\omega} \backslash B$, $lh(h) > \omega$ if and only if $h \in B$.

2. If $lh(h) > \omega$, then the length of h is determined by Ψ and $h \restriction \omega$. The length of h is $\omega + \Psi(h \restriction \omega) + 1$. Thus, the length of a long play h depends on which $\Psi^{-1}(n)$ contains h as an element.[2]

Therefore, $[T_{X,Y}^{\Psi,B}] = \bigcup_{n\in\omega} [(B \cap \Psi^{-1}(n)) \times Y^{n+1}] \dot\cup (X^{\omega} \backslash B)$. [3] In particular, when $B = \emptyset$, $[T_{X,Y}^{\Psi,\emptyset}] = X^{\omega}$ and when $B = X^{\omega}$ and Ψ is a constant function at $n \in \omega$, then $[T_{X,Y}^{\Psi,X^{\omega}}] = X^{\omega} \times Y^{n+1}$.

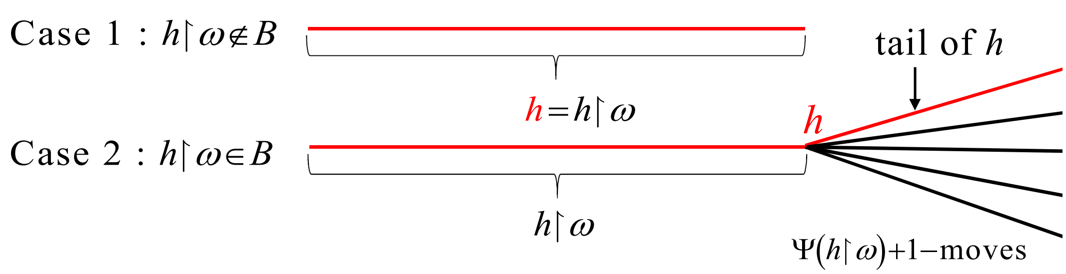

Figure 2.1.1: Illustration of paths $h \in [T]$ for a Type 1 tree $T = T_{X,Y}^{\Psi,B}$ for $B \neq \emptyset$.

Next, we provide an alternate description of Type 2 trees. In definition 2.1.3 below, we will define a property "(X,Y)-*TEP*-$[\omega, \omega + \omega)$ property" for a tree T.

Definition 2.1.3. *(Definition of the* (X,Y)*-TEP-*$[\omega, \omega + \omega)$ *property)*

Suppose X *and* Y *are nonempty sets. Let* T *be a tree.* T *satisfies* (X,Y)*-TEP-*$[\omega, \omega + \omega)$

[1] 1 is added to make sure that the plays with length ω are exactly the ones that are not in B (e.g., if $h \restriction \omega \in B$ and $\Psi(h \restriction \omega) = 0$, then $lh(h) = \omega + 1$).

[2] Recall notation 1.5.4. Abuse of notation : we suppress $\{\}$ for $\Psi^{-1}(\{n\})$, i.e., we write $\Psi^{-1}(n)$ to mean $\Psi^{-1}(\{n\})$. $\Psi^{-1}(n)$ does not mean the inverse image of $\{0, 1, ..., n-1\}$ here.

[3] Recall notation 1.5.3: abuse of product notation. The dot "·" above the union symbol represents the disjoint union.

property if for all $y \in [T]$, y satisfies the following four properties:

1. $y \upharpoonright \omega \in X^{\omega}$.

2. $lh\,(y) \in [\omega, \omega+\omega)$.

3. *If $lh\,(y) > \omega$, then each move of the tail of h is from Y.*

4. *If $lh\,(y) > \omega$, then there exists a unique $n \in \omega \backslash \{0\}$ such that*

$$\forall g \in Y^{n} \;\; (y \upharpoonright \omega)^\frown g \in [T] \quad (\textit{tail exchange property}) \qquad \dashv$$

TEP abbreviates the "tail exchange property". For any nonempty sets X, Y, a function $\Psi : X^{\omega} \to \omega$ and $B \subseteq X^{\omega}$, $T^{\Psi,B}_{X,Y}$ satisfies (X,Y)-*TEP*-$[\omega, \omega+\omega)$ property.

Observation 2.1.4. *Every Type 1 tree satisfies (X,Y)-TEP-$[\omega, \omega+\omega)$ property. Conversely, for any (X,Y)-TEP-$[\omega, \omega+\omega)$ tree T, there exist $\Psi : X^{\omega} \to \omega$ and a unique $B \subseteq X^{\omega}$ such that $T = T^{\Psi,B}_{X,Y}$. In fact, $B = \{h \upharpoonright \omega \,|h \in [T] \wedge lh\,(h) > \omega\}$ and $\Psi \upharpoonright B$ is unique.* $\dashv$

2.2 Definition of a $Tree_1$ collection and a collection of games on a $Tree_1$ collection with complexity Ξ

In this section, we will first define a $Tree_1$ collection of Type 1 trees. Then we will define games on a $Tree_1$ collection. We will be considering a collection of trees $T_{X,Y}^{\Psi,B}$ in which Y varies over Θ, Ψ varies over Γ and B varies over Λ, while X is fixed. Thus, each $Tree_1$ collection will be defined from X, Θ, Γ and Λ. We will denote a $Tree_1$ collection by $Tree_1(X, \Theta, \Gamma, \Lambda)$ constructed from some X, Θ, Γ and Λ. Throughout the rest of this chapter, we will assume notation 2.2.1 below. Then, in definition 2.2.7, we will define games on a $Tree_1$ collection with complexity Ξ (e.g., $\mathbf{\Sigma}_1^0$, $\mathbf{\Sigma}_2^0, ...$). We will also make some observations concerning $Tree_1$ collections and games on $Tree_1$ collections.

Notation 2.2.1. *We will assume the following notation throughout the chapter 2:*

- Θ*, respectively,* Θ_i *will always denote a nonempty collection of nonempty sets.*
- Γ*, respectively,* Γ_i *will always denote a nonempty collection of functions from* X^ω *into* ω*.*
- Λ*, respectively,* Λ_i *will always denote a nonempty collection of subsets of* X^ω*.* $\dashv$

We next define a collection of Type 1 trees constructed from X, Θ, Γ and Λ.

Definition 2.2.2. *(Definition of a* $Tree_1$ *collection)*

Fix a nonempty set X*. Let* $\Theta \neq \emptyset$ *be any collection of nonempty sets. Suppose* $\Lambda \neq \emptyset$ *is any collection of subsets of* X^ω *and* $\Gamma \neq \emptyset$ *is a collection of functions from* X^ω *to* ω*. Define*

$$Tree_1(X, \Theta, \Gamma, \Lambda) = \left\{ T_{X,Y}^{\Psi,B} \mid Y \in \Theta, \Psi \in \Gamma, B \in \Lambda \right\}.$$

A collection is a $Tree_1$ *collection if and only if it is* $Tree_1(X, \Theta, \Gamma, \Lambda)$ *for some nonempty set* X*, a nonempty collection* Θ *of* Y*'s, a nonempty collection* Γ *of functions from* X^ω *into* ω *and a nonempty collection* Λ *of subsets of* X^ω*.*

We sometimes let $\mathcal{T}_1$ *be a* $Tree_1$ *collection when we wish to suppress* X, Θ, Γ *and* Λ*, i.e.,* $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. ⊣

For example, $Tree_1(X, \Theta, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Pi}_1^0 \restriction X^\omega)$ [4] is a collection of trees such that each tree is constructed by a set $Y \in \Theta$, a continuous function Ψ from X^ω into ω and $B \in \mathbf{\Pi}_1^0 \restriction X^\omega$.

Notation 2.2.3. *When dealing with singletons for any of the last three components of* $Tree_1(X, \Theta, \Gamma, \Lambda)$*, we will suppress* $\{\}$*, i.e., if* Θ *is a singleton* $\{Y\}$*,* $Tree_1(X, Y, \Gamma, \Lambda)$ *abbreviates* $Tree_1(X, \{Y\}, \Gamma, \Lambda)$*. Similarly, if* Γ *is a singleton* $\{f\}$*,* $Tree_1(X, \Theta, f, \Lambda)$ *abbreviates* $Tree_1(X, \Theta, \{f\}, \Lambda)$ *and if* Λ *is a singleton* $\{B\}$*,* $Tree_1(X, \Theta, \Gamma, B)$ *abbreviates* $Tree_1(X, \Theta, \Gamma, \{B\})$. ⊣

Observation 2.2.4. *Fix a nonempty set* X*. Suppose* $\Theta, \Theta_1, \Theta_2$ *are collections of sets;* $\Gamma, \Gamma_1, \Gamma_2$ *are collections of functions from* X^ω *into* ω*; and* $\Lambda, \Lambda_1, \Lambda_2$ *are collections of subsets of* X^ω*. If* $\Theta_1 \subseteq \Theta_2$*, then*

$$Tree_1(X, \Theta_1, \Gamma, \Lambda) \subseteq Tree_1(X, \Theta_2, \Gamma, \Lambda).$$

Similarly, if $\Gamma_1 \subseteq \Gamma_2$*, then*

$$Tree_1(X, \Theta, \Gamma_1, \Lambda) \subseteq Tree_1(X, \Theta, \Gamma_2, \Lambda),$$

and if $\Lambda_1 \subseteq \Lambda_2$*, then*

$$Tree_1(X, \Theta, \Gamma, \Lambda_1) \subseteq Tree_1(X, \Theta, \Gamma, \Lambda_2).$$

[4] Recall for notation 1.5.8, $\Gamma(\omega, \mathbf{\Delta}_1^0)$ is a set of continuous functions from X^ω into ω.

Thus $Tree_1$ is an increasing operation on last three components. $\dashv$

Now we will consider the set of functions $\Gamma(\omega, \Xi)$ for some complexity Ξ over X^ω. Since we are using the discrete topology on ω, we have $\Gamma(\omega, \Xi) = \Gamma(\omega, co\text{-}\Xi) = \Gamma(\omega, \mathbf{\Delta}(\Xi))$ by observation 1.5.9 on page 43. Thus we have observation 2.2.5. (Recall that for example, if Ξ is $\mathbf{\Sigma}^0_\gamma$, then co-Ξ is $\mathbf{\Pi}^0_\gamma$ and $\mathbf{\Delta}(\Xi)$ is $\mathbf{\Delta}^0_\gamma$.)

Observation 2.2.5. *Let Θ be a collection of sets and $\Lambda \subseteq X^\omega$. Suppose we have Ξ such that $\Xi \upharpoonright X^\omega$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $Tree_1\left(X, \Theta, \Gamma(\omega, \Xi), \Lambda\right)$
- $Tree_1\left(X, \Theta, \Gamma(\omega, co\text{-}\Xi), \Lambda\right)$
- $Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda\right)$ $\dashv$

Now, we consider games on a $Tree_1$ collection. Corresponding to each tree T in a $Tree_1$ collection and $A \subseteq [T]$, we have the game $G(A; T)$.

Definition 2.2.6. *(Definition of "games on a $Tree_1$ collection")*

Let $Tree_1$ collection $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$ for some X, Θ, Γ and Λ. Define "games on the $Tree_1$ collection $\mathcal{T}_1$" by

$$\bigcup_{T \in \mathcal{T}_1} \left\{ G\left(A; T\right) | A \subseteq [T] \right\} \qquad \dashv$$

If Ξ is a complexity, we define Ξ games on a $Tree_1$ collection as follows.

Definition 2.2.7. *(Definition of Ξ games on a $Tree_1$ collection)*

Let $Tree_1$ collection $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in$

$Tree_1$, $\Xi \restriction [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ games on a $Tree_1$ collection $\mathcal{T}_1$ by

$$\mathcal{G}(\Xi; \mathcal{T}_1) = \bigcup_{T \in \mathcal{T}_1} \{G(A;T) \,|A \in \Xi \restriction [T]\}.$$

As a notation, we will use $\mathcal{G}$ for a collection of games on a $Tree_1$ collection. ⊣

For example, open games on a $Tree_1$ collection $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$ is

$$\mathcal{G}(\mathbf{\Sigma}^0_1; \mathcal{T}_1) = \bigcup_{T \in \mathcal{T}_1} \left\{G(A;T) \,\middle|\, A \in \mathbf{\Sigma}^0_1 \restriction [T]\right\}.$$

Though often Ξ will be a standard classes (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$), note that $G(\Xi;T)$ is defined as long as we have defined $\Xi \restriction [T] \subseteq \wp([T])$.

Definition 2.2.8. *(Definition of Ξ determinacy on a $Tree_1$ collection)*

Let $Tree_1$ collection $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in \mathcal{T}_1$, $\Xi \restriction [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ determinacy on the $Tree_1$ collection $\mathcal{T}_1$ by

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_1),$$

i.e., for any $T \in \mathcal{T}_1$ and $A \in \Xi \restriction [T]$, every game $G(A;T)$ is determined. ⊣

Next, we will make three observations about games on Type 1 trees.

Observation 2.2.9. *Suppose X is a nonempty set, Θ is a collection of sets, Γ is a collection of functions from X^ω into ω, Λ is a collection of subsets of X^ω. Let $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. Suppose we have Ξ_1, Ξ_2 such that for each $T \in \mathcal{T}_1$, $\Xi_1 \restriction [T] \subseteq \wp([T])$ and $\Xi_2 \restriction [T] \subseteq \wp([T])$ are defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). If for any $T \in \mathcal{T}_1$, $\Xi_1 \restriction [T] \subseteq \Xi_2 \restriction [T]$, then*

$$\mathcal{G}(\Xi_1; \mathcal{T}_1) \subseteq \mathcal{G}(\Xi_2; \mathcal{T}_1).$$

Thus $\mathcal{G}$ is an increasing operation on the first component. $\dashv$

Since the three $Tree_1$ collections in observation 2.2.5 are equal, we have the following observation.

Observation 2.2.10. *Let Θ be a collection of sets and $\Lambda \subseteq X^\omega$. Suppose we have Ξ such that $\Xi \restriction X^\omega \subseteq \wp(X^\omega)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Suppose we have Ξ_1 such that for each $T \in Tree_1(X, \Theta, \Gamma(\omega, \Xi), \Lambda)$, $\Xi_1 \restriction [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, \Xi), \Lambda))$
- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, co\text{-}\Xi), \Lambda))$
- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda))$ $\dashv$

2.3 Equivalence between $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ determinacy on a $Tree_1$ collection and equivalence between $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ determinacy on a $Tree_1$ collection

In this section, for a countable α, we will show that the determinacy of $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ games on certain $Tree_1$ collections of Type 1 trees are equivalent. We will also obtain the determinacy equivalence of $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ games on certain $Tree_1$ collections for a finite n. The main theorem of this section is theorem 2.3.1.

Theorem 2.3.1. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Then for any X and Θ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_1\right) \tag{2.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_1\right) \tag{2.2}$$

for $\mathcal{T}_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ [5] where:

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* $\dashv$

We will prove this theorem on page 80.

For each Type 1 tree T, we will define a corresponding "Shift tree" $Sft(T)$ in definition 2.3.3. Then in definition 2.3.7, we will define a function "Shift" which takes a subset of $[T]$ for a Type 1 tree T to a particular subset of $[Sft(T)]$. In lemma 2.3.9 and lemma 2.3.15, we will find the complexity of $Shift(A)$ for A being Borel (respectively, a projective set) on $[T]$

[5] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

for Type 1 trees in a specific $Tree_1$ collection. For each Type 1 tree, there is a natural Shift tree which is also a Type 1 tree. In order to define a Shift tree for each Type 1 tree $T = T^{\Psi,B}_{X,Y}$, we define B^+ and Ψ^+ from $B \subseteq X^\omega$ and a function Ψ from X^ω into ω which satisfy "shift" relation. B^+ corresponds to adding a "dummy" copy of X to the front of B: for any $a \in X$, $\langle a \rangle^\frown f \in B^+$ if and only if $f \in B$. Hence there is a "shift" relation between B and B^+. Ψ^+ is a function on X^ω into ω and for any $f \in X^\omega$ and for any $a \in X$, $\Psi^+(\langle a \rangle^\frown f) = \Psi(f) + 1$. Hence there is a "shift" relation between Ψ and Ψ^+.

Definition 2.3.2. *(Definition of B^+ and Ψ^+)*

Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$. Then $B \subseteq X^\omega$ and $\Psi : X^\omega \to \omega$. Define

1. $B^+ = X \times B \subseteq X^\omega$, [6]

2. $\Psi^+ : X^\omega \to \omega$ *such that* $\Psi^+(f) = \Psi(f \restriction [1, \omega)) + 1$. ⊣

By using B^+ and Ψ^+, we will define a Shift tree as follows.

Definition 2.3.3. *(Definition of a Shift tree $Sft(T)$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Define a Shift tree $Sft(T)$ by

$$Sft(T) = T^{\Psi^+,B^+}_{X,Y}.$$ ⊣

[6] Recall abuse of notation 1.5.3 on page 42.

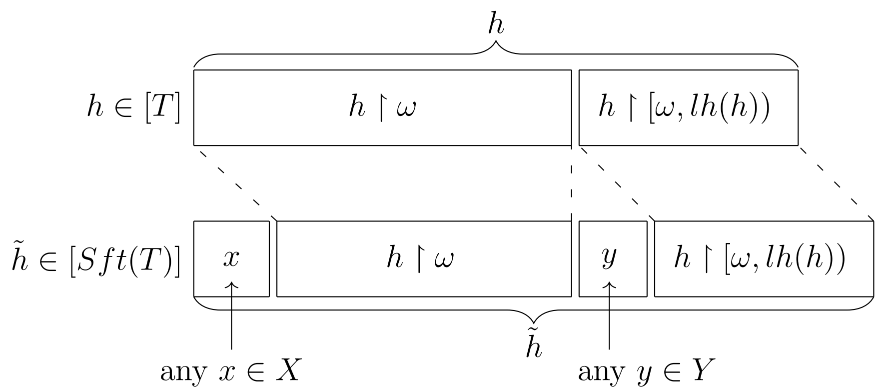

Figure 2.3.1: Illustration of $\tilde{h} \in [Sft(T)]$ with $lh(\tilde{h}) > \omega$.

Observation 2.3.4. *For any Type 1 tree* $T = T^{\Psi,B}_{X,Y}$*,* $B = \emptyset$ *if and only if* $T = Sft(T)$. $\dashv$

Proof.

Notice that $B = \emptyset$ if and only if $B^+ = \emptyset$.

($\Rightarrow$) For any Type 1 tree $T = T^{\Psi,B}_{X,Y}$ with $B = \emptyset$, $T = X^{<\omega} = Sft(T)$.

($\Leftarrow$) Suppose $T = Sft(T)$. Show $B = B^+ = \emptyset$.

First, show $B = B^+$. Suppose $B \neq B^+$ for a contradiction. Then there exists $f \in B \backslash B^+$ or $f \in B^+ \backslash B$. Suppose $f \in B \backslash B^+$. Pick an arbitrary $h \in [T]$ such that $h \restriction \omega = f$. Since $h \restriction \omega = f \in B$, $lh(h) > \omega$. By assumption, we have $T = Sft(T)$ so that $h \in [Sft(T)]$. Since $h \restriction \omega = f \notin B^+$, $lh(h) = \omega$, a contradiction. Similarly for the case $f \in B^+ \backslash B$. Therefore, we have $B = B^+$ and thus $B = X \times B$.

Show $B = \emptyset$ or $B = X^\omega$. Suppose $B \neq \emptyset$. Since $B = X \times B$, $B = X \times (X \times B) = X \times X \times B$. Inductively, we have that each component of elements of B is from X. Hence $B = X^\omega$.

Show $B \neq X^\omega$. For a contradiction, assume $B = X^\omega$. Let $\hat{f} \in X^\omega = B = B^+$ be the sequence of all 0's. Notice that since $\hat{f}$ is the sequence of all 0's, $\hat{f} \restriction [1, \omega) = \hat{f}$. Pick an arbitrary $h \in [Sft(T)]$ such that $h \restriction \omega = \hat{f}$. Then

$$lh(h) = \Psi^+(h \restriction \omega) + 1 = \Psi(h \restriction [1, \omega)) + 2 = \Psi(\hat{f} \restriction [1, \omega)) + 2 = \Psi(\hat{f}) + 2.$$

Since $T = Sft(T)$, $h \in [T]$. Thus

$$lh(h) = \Psi(h \restriction \omega) + 1 = \Psi(\hat{f}) + 1.$$

Hence, $\Psi(\hat{f}) = \Psi(\hat{f}) + 1$, a contradiction.

Therefore, $B = B^+ = \emptyset$. $\square$

Notice that for each Type 1 tree $T = T^{\Psi,B}_{X,Y}$ and for each $h \in [Sft(T)]$, there is a unique $f \in [T]$ such that $h(i+1) = f(i)$ for every $i \in lh(f)$ (e.g., $h(1) = f(0), h(2) = f(1), ..., h(\omega + 1) = f(\omega), h(\omega + 2) = f(\omega + 1), ...$ for $h \restriction \omega \in B$).

Proposition 2.3.5. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Then for every $h \in [Sft(T)]$,*

$$\begin{cases} h \restriction [1, \omega) \in [T] & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1, \omega)^\frown h \restriction [\omega + 1, lh(h)) \in [T] & \text{if } h \restriction \omega \in B^+. \end{cases}$$ $\dashv$

Proof.

Pick an arbitrary $h \in [Sft(T)]$. Notice that $h \restriction \omega \in B^+$ if and only if $h \restriction [1, \omega) \in B$. Also, $\Psi^+(h \restriction \omega) = \Psi^+(h(0)^\frown h \restriction [1, \omega)) = \Psi(h \restriction [1, \omega)) + 1$. Thus

$$h \in Sft\,[T] \leftrightarrow \begin{cases} h \in X^{\omega} & \text{if } h \upharpoonright \omega \notin B^{+}, \\ h \in X^{\omega} \times Y^{\Psi^{+}(h\upharpoonright\omega)+1} & \text{if } h \upharpoonright \omega \in B^{+}. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^{\omega} & \text{if } h \upharpoonright [1,\omega) \notin B, \\ h \in X^{\omega} \times Y^{\Psi(h\upharpoonright[1,\omega))+2} & \text{if } h \upharpoonright [1,\omega) \in B. \end{cases}$$

$$\leftrightarrow \begin{cases} h \upharpoonright [1,\omega) \in X^{\omega} & \text{if } h \upharpoonright [1,\omega) \notin B, \\ h \upharpoonright [1,\omega)^\frown h \upharpoonright [\omega+1, lh\,(h)) \in X^{\omega} \times Y^{\Psi(h\upharpoonright[1,\omega))+1} & \text{if } h \upharpoonright [1,\omega) \in B. \end{cases}$$

$$\leftrightarrow \begin{cases} h \upharpoonright [1,\omega) \in [T] & \text{if } h \upharpoonright \omega \notin B^{+}, \\ h \upharpoonright [1,\omega)^\frown h \upharpoonright [\omega+1, lh\,(h)) \in [T] & \text{if } h \upharpoonright \omega \in B^{+}. \end{cases} \qquad \square$$

Proposition 2.3.5 give us a natural erasing function e from $[Sft(T)]$ into $[T]$.

Definition 2.3.6. *(Definition of the erasing function $e : [Sft(T)] \to [T]$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Define the erasing function e from $[Sft(T)]$ into $[T]$ by

$$e : \quad [Sft\,(T)] \to [T]$$

$$h \mapsto \begin{cases} h \upharpoonright [1,\omega) & \textit{if } h \upharpoonright \omega \notin B^{+}, \\ h \upharpoonright [1,\omega)^\frown h \upharpoonright [\omega+1, lh\,(h)) & \textit{if } h \upharpoonright \omega \in B^{+}. \end{cases} \qquad \dashv$$

Now, we define a function "Shift" which maps subsets A of $[T]$ for a Type 1 tree T to a particular subset of $[Sft(T)]$.

Definition 2.3.7. *(Definition of Shift)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. *Define*

$$Shift: \quad \wp([T]) \to \wp([Sft(T)])$$
$$A \mapsto \{h \in [Sft(T)] \mid e(h) \in [T] \setminus A\}. \qquad \dashv$$

Theorem 2.3.8. *For any Type 1 tree* T, *the determinacy of* $G(Shift(A); Sft(T))$ *implies the determinacy of* $G(A, T)$. $\dashv$

Proof.

Pick an arbitrary Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Assume $G(Shift(A); Sft(T))$ is determined. Then I^* or II^* has a winning strategy s^* for $G(Shift(A); Sft(T))$. Show that $G(A; T)$ is determined.

Case I : s^* is a I^*'s winning strategy for $G(Shift(A); Sft(T))$. Define a strategy s for II for $G(A; T)$ as follows: Suppose $a_0 = s^*(\emptyset)$.

For $p \in T$ such that p is finite and $\langle a_0 \rangle^\frown p \in dom(s^*)$ or

p is infinite and $\langle a_0 \rangle^\frown p \restriction \omega^\frown \langle a_\omega \rangle^\frown p \restriction [\omega, lh(p)) \in dom(s^*)$ where $a_\omega = s^*\left(\langle a_0 \rangle^\frown p \restriction \omega\right)$,

$p \in dom(s)$ and

$$s(p) = \begin{cases} s^*\left(\langle a_0 \rangle^\frown p\right) & \text{if } p \text{ is finite,} \\ s^*\left(\langle a_0 \rangle^\frown p \restriction \omega^\frown \langle a_\omega \rangle^\frown p \restriction [\omega, lh(p))\right) & \text{if } p \text{ is infinite.} \end{cases}$$

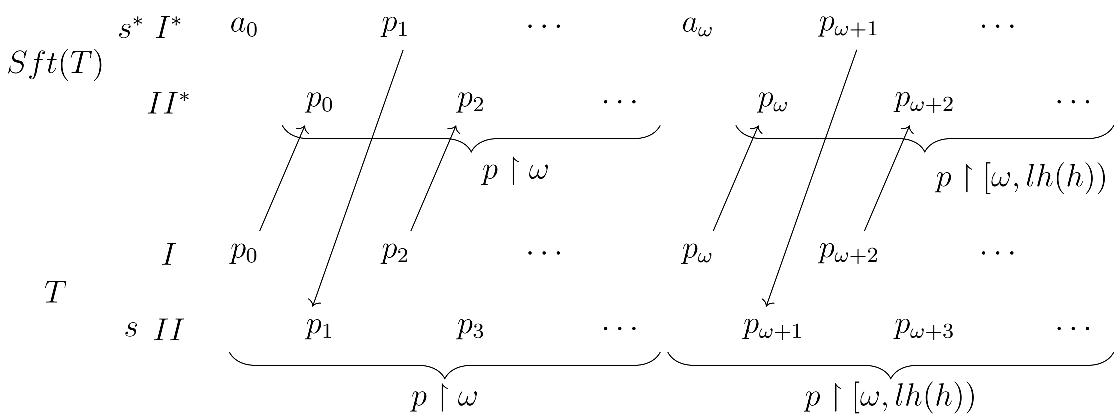

Figure 2.3.2: Illustration of $p \in T$, $lh(p) > \omega$ according to II's strategy s.

Show s is a w.s. for II for $G(A;T)$. Pick an arbitrary $x \in [T]$ according to s.

Subcase 1 : $x \restriction \omega \notin B$.

Then $x = x \restriction \omega$ and $s^*(\emptyset)^\frown x \notin B^+$. Thus $s^*(\emptyset)^\frown x \in [Sft(T)]$ and it is according to s^*. Hence $s^*(\emptyset)^\frown x \in Shift(A)$ and thus $x = e(s^*(\emptyset)^\frown x) \notin A$. [7]

Subcase 2 : $x \restriction \omega \in B$.

Then $s^*(\emptyset)^\frown x \restriction \omega \in B^+$ and $\Psi^+(s^*(\emptyset)^\frown x \restriction \omega) = \Psi(x \restriction \omega) + 1 = lh(x)$. Let

$$h = s^*(\emptyset)^\frown (x \restriction \omega)^\frown s^*\big(s^*(\emptyset)^\frown x \restriction \omega\big)^\frown x \restriction [\omega, lh(x)) \,.$$

Then $lh(h) = lh(x) + 1$, $h \in [Sft(T)]$ and h is according to s^*. Thus $h \in Shift(A)$. Hence $x = e(h) \notin A$. Therefore, s is a w.s. for II for $G(A;T)$.

Case II : s^* is a II^*'s winning strategy for $G(Shift(A); Sft(T))$.

Define a strategy s for I for $G(A;T)$ as follows: Suppose $a_0 \in X$ and $a_\omega \in Y$ are arbitrary. For $p \in T$ such that p is finite and $\langle a_0 \rangle^\frown p \in dom(s^*)$ or

[7] Recall definition 2.3.6 for the erasing function e.

p is infinite and $\langle a_0\rangle^\frown p\restriction\omega^\frown\langle a_\omega\rangle^\frown p\restriction[\omega, lh\,(p)) \in dom(s^*)$, $p \in dom(s)$ and

$$s\,(p) = \begin{cases} s^*\,(\langle a_0\rangle^\frown p) & \text{if } p \text{ is finite,} \\ s^*\,(\langle a_0\rangle^\frown p\restriction\omega^\frown\langle a_\omega\rangle^\frown p\restriction[\omega, lh\,(p))) & \text{if } p \text{ is infinite.} \end{cases}$$

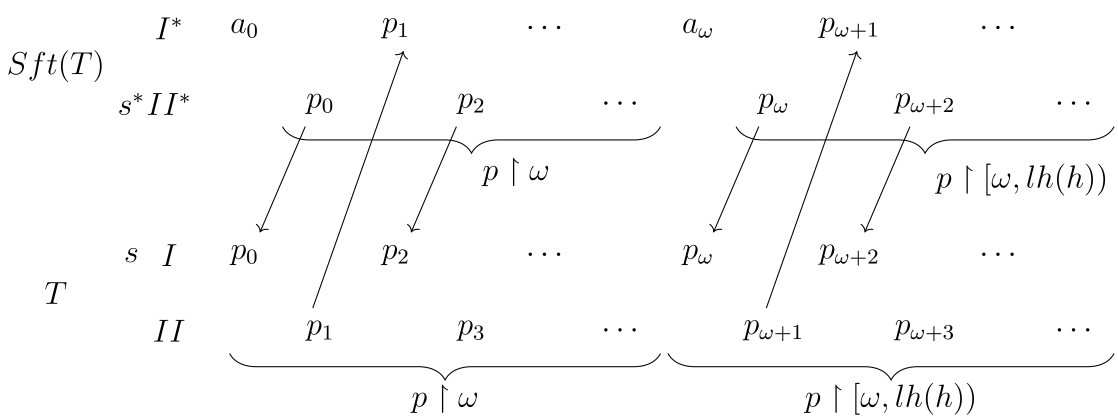

Figure 2.3.3: Illustration of $p \in T$, $lh(p) > \omega$ according to I's strategy s.

Show s is a w.s. for I for $G(A;T)$. Pick an arbitrary $x \in [T]$ according to s. Let $a_0 \in X$ and $a_\omega \in Y$ be arbitrary.

Subcase 1 : $x\restriction\omega \notin B$.

Then $x = x\restriction\omega$ and $\langle a_0\rangle^\frown x \notin B^+$. Thus $\langle a_0\rangle^\frown x \in [Sft(T)]$ and it is according to s^*. Hence $\langle a_0\rangle^\frown x \notin Shift(A)$ and thus $x = e(\langle a_0\rangle^\frown x) \in A$.

Subcase 2 : $x\restriction\omega \in B$.

Then $\langle a_0\rangle^\frown x\restriction\omega \in B^+$ and $\Psi^+(\langle a_0\rangle^\frown x\restriction\omega) = \Psi(x\restriction\omega) + 1 = lh(x)$. Let

$$h = \langle a_0\rangle^\frown(x\restriction\omega)^\frown\langle a_\omega\rangle^\frown x\restriction[\omega, lh(x))\,.$$

Then $lh(h) = lh(x) + 1$, $h \in [Sft(T)]$ and h is according to s^*. Thus $h \notin Shift(A)$. Hence $x = e(h) \in A$. Therefore, s is a w.s. for I for $G(A;T)$.

By cases I and II, $G(A;T)$ is determined. $\square$

We shall eventually use theorem 2.3.8, to prove theorem 2.3.1.

Theorem 2.3.1. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Then for any X and Θ,*

$$Det\ \mathcal{G}\left(\boldsymbol{\Sigma}^0_\alpha; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\boldsymbol{\Pi}^0_\alpha; \mathcal{T}_1\right) \tag{2.1}$$

$$Det\ \mathcal{G}\left(\boldsymbol{\Sigma}^1_n; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\boldsymbol{\Pi}^1_n; \mathcal{T}_1\right) \tag{2.2}$$

for $\mathcal{T}_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ [8] where:

- $\mathcal{C} \in \{\boldsymbol{\Sigma}^0_\gamma, \boldsymbol{\Pi}^0_\gamma, \boldsymbol{\Delta}^0_\gamma, \mathbf{B}, \boldsymbol{\Sigma}^1_m, \boldsymbol{\Pi}^1_m, \boldsymbol{\Delta}^1_m, \boldsymbol{\Sigma}^1_m \wedge \boldsymbol{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\boldsymbol{\Sigma}^0_\beta, \boldsymbol{\Pi}^0_\beta, \boldsymbol{\Delta}^0_\beta, \mathbf{B}, \boldsymbol{\Sigma}^1_m, \boldsymbol{\Pi}^1_m, \boldsymbol{\Delta}^1_m, \boldsymbol{\Sigma}^1_m \wedge \boldsymbol{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* $\dashv$

For the equivalences in theorem 2.3.1, we won't be obtaining the determinacy of a game $G(A;T)$ from the same tree T (except for the case when $B = \emptyset$, recall observation 2.3.4). We will instead use two trees T and $Sft(T)$ in the same $Tree_1$ collection.

Before we prove theorem 2.3.1, we will find the complexity of $Shift(A)$ for Borel sets A on $[T]$ in lemma 2.3.9. In lemma 2.3.15, we will find the complexity of $Shift(A)$ for projective sets A on $[T]$. In sublemma 2.3.19 and sublemma 2.3.20, we will find the complexity of B^+ for each $B \in \mathcal{B}$ where $\mathcal{B}$ is as in theorem 2.3.1. In sublemma 2.3.19, we will find the complexity of B^+ when B is a Borel set, and in sublemma 2.3.20, we will find the complexity of B^+ when B is a projective set. In sublemma 2.3.21, we will find the complexity of Ψ^+ for each $\Psi \in \Gamma(\omega, \mathcal{C})$ where $\mathcal{C}$ is as in theorem 2.3.1.

First, we compute the complexity of $Shift(A)$ for Borel set A on $[T]$.

Lemma 2.3.9. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Then, for any $\alpha \in \omega_1$:*

[8] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

1. *If* $A \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$.

2. *If* $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Pi}^0_\alpha \upharpoonright [Sft(T)]$. $\dashv$

We will prove lemma 2.3.9 by using sublemma 2.3.14 below. Given $S \subseteq [T]$, we define $S^+ \subseteq [Sft(T)]$ as follows.

Definition 2.3.10. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *Given* $S \subseteq [T]$, *define*

$$S^+ = \{h \in [Sft\,(T)]\,| e\,(h) \in S\}\,. \qquad \dashv$$

Observation 2.3.11. *If* $A \subseteq [T]$, *then*

$$([T]\setminus A)^+ = \{h \in [Sft\,(T)]\,| e\,(h) \in [T]\setminus A\} = Shift\,(A)\,. \qquad \dashv$$

In definition 2.3.2, for $B \subseteq X^\omega$, we defined $B^+ = X \times B$. The following observation shows that the + notation in definition 2.3.10 is a consistent notation with definition 2.3.2 over X^ω.

Observation 2.3.12. *Recall that* X^ω *is the special case of Type 1 trees* $T = T^{\Psi,B}_{X,Y}$ *with* $B = \emptyset$. *By observation 2.3.4,* $T = Sft(T) = X^{<\omega}$. *Thus for* $S \subseteq X^\omega$,

$$S^+ = \{h \in X^\omega\,| e(h) \in S\} = X \times S.$$

Thus the definition of + that appear in definitions 2.3.2 and 2.3.10 are the same for subsets of $T = X^{<\omega}$. $\dashv$

Sublemma 2.3.13. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *For any* $S \subseteq [T]$,

$$[Sft\,(T)]\setminus S^+ = ([T]\setminus S)^+. \qquad \dashv$$

Proof.

Fix $S \subseteq [T]$. Then

$$\begin{aligned}([Sft(T)]) \setminus S^{+} &= \{h \in [Sft(T)] \,|\, e(h) \notin S\} \\ &= \{h \in [Sft(T)] \,|\, e(h) \in [T] \setminus S\} \\ &= ([T] \setminus S)^{+}. \end{aligned}$$

□

Using sublemma 2.3.13, we obtain the following.

Sublemma 2.3.14. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *For any* $\alpha \in \omega_1 \setminus \{0\}$ *and for any* $S \subseteq [T]$:

1. *If* $S \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, *then* $S^{+} \in \mathbf{\Sigma}^0_\alpha \restriction [Sft(T)]$.
2. *If* $S \in \mathbf{\Pi}^0_\alpha \restriction [T]$, *then* $S^{+} \in \mathbf{\Pi}^0_\alpha \restriction [Sft(T)]$. ⊣

Proof.

We prove both (1) and (2) simultaneously by induction on α. (2) follows from (1) and sublemma 2.3.13. The case for $S \in \mathbf{\Sigma}^0_1 \restriction [T]$ is obtained from sublemma 2.3.18 below.

Base case : $\alpha = 1$.

Show that if $S \in \mathbf{\Sigma}^0_1 \restriction [T]$, then $S^{+} \in \mathbf{\Sigma}^0_1 \restriction [Sft(T)]$. We shall obtain this as a special case of sublemma 2.3.18 below with $k = 0$.

Show that if $S \in \mathbf{\Pi}^0_1 \restriction [T]$, then $S^{+} \in \mathbf{\Pi}^0_1 \restriction [Sft(T)]$. $\cdots (*)$

Suppose $S \in \mathbf{\Pi}^0_1 \restriction [T]$. Then $[T] \setminus S \in \mathbf{\Sigma}^0_1 \restriction [T]$. Since we have already shown (1) for $\alpha = 1$, we have:

$$([T] \setminus S)^{+} \in \mathbf{\Sigma}^0_1 \restriction [Sft(T)] .$$

By sublemma 2.3.13,

$$[Sft(T)] \setminus S^{+} = ([T] \setminus S)^{+}.$$

Thus $S^+ \in \mathbf{\Pi}^0_1 \upharpoonright [Sft\,(T)]$.

Induction step : As an induction hypothesis, assume that for any $\beta \in \alpha$, if $S \in \mathbf{\Sigma}^0_\beta \upharpoonright [T]$, then $S^+ \in \mathbf{\Sigma}^0_\beta \upharpoonright [Sft(T)]$ and if $S \in \mathbf{\Pi}^0_\beta \upharpoonright [T]$, then $S^+ \in \mathbf{\Pi}^0_\beta \upharpoonright [Sft(T)]$.

Assume $S \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Show $S^+ \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$.

Since $S \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$, there exists $\langle A_n \,|n \in \omega\rangle$ such that each $A_n \in \mathbf{\Pi}^0_{\beta_n} \upharpoonright [T]$, $\beta_n \in \alpha$ and $S = \bigcup_{n\in\omega} A_n$. Then by induction hypothesis, each $A^+_n \in \mathbf{\Pi}^0_{\beta_n} \upharpoonright [Sft(T)]$.

$$\begin{aligned} S^+ &= \{h \in [Sft\,(T)]\,|e\,(h) \in S\} \\ &= \left\{ h \in [Sft\,(T)] \,\middle|\, e\,(h) \in \bigcup_{n\in\omega} A_n \right\} \\ &= \bigcup_{n\in\omega} \{h \in [Sft\,(T)]\,|e\,(h) \in A_n\} \\ &= \bigcup_{n\in\omega} \underbrace{A^+_n}_{\mathbf{\Pi}^0_{\beta_n}\upharpoonright[Sft(T)]} \\ &\in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft\,(T)]\,. \end{aligned}$$

Show that if $S \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$, then $S^+ \in \mathbf{\Pi}^0_\alpha \upharpoonright [Sft\,(T)]$. We repeat the same proof of $(*)$ on page 65.

Suppose $S \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$. Then $[T] \setminus S \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Since we have already shown (1) for the case α, we have:

$$([T] \setminus S)^+ \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft\,(T)]\,.$$

By sublemma 2.3.13,

$$[Sft\,(T)] \setminus S^+ = ([T] \setminus S)^+.$$

Thus $S^+ \in \mathbf{\Pi}^0_\alpha \upharpoonright [Sft\,(T)]$. $\square$

Lemma 2.3.9 is obtained immediately from sublemma 2.3.14. Recall lemma 2.3.9.

Lemma 2.3.9. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *Then, for any* $\alpha \in \omega_1$*:*

1. *If* $A \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$.

2. *If* $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Pi}^0_\alpha \upharpoonright [Sft(T)]$. ⊣

Proof.

Suppose $A \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$. Show $Shift(A) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$.

Since $A \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$, $[T]\backslash A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. By sublemma 2.3.14,

$$([T] \backslash A)^+ \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)].$$

By observation 2.3.11,

$$Shift\,(A) = ([T] \backslash A)^+ \,.$$

Thus $Shift(A) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$.

A similar proof gives a proof of (2): Simply interchange $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$. □

In lemma 2.3.9, we computed the complexity of $Shift(A)$ for Borel sets A on $[T]$. Now, we will compute the complexity of $Shift(A)$ for projective sets A on $[T]$.

Lemma 2.3.15. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *Let* $n \in \omega\backslash\{0\}$.

1. *If* $A \in \mathbf{\Pi}^1_n \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Sigma}^1_n \upharpoonright [Sft(T)]$.

2. *If* $A \in \mathbf{\Sigma}^1_n \upharpoonright [T]$, *then* $Shift(A) \in \mathbf{\Pi}^1_n \upharpoonright [Sft(T)]$. ⊣

We will prove lemma 2.3.15 using sublemma 2.3.18. We will first prove sublemma 2.3.18 below by induction on n. We will use the following notation.

We generalize definition 2.3.10 as follows.

Definition 2.3.16. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Pick an arbitrary $k \in \omega$. Recall*

$$(\omega^\omega)^k = \underbrace{\omega^\omega \times \cdots \times \omega^\omega}_{k\ many}.$$

Given $S_k \subseteq [T] \times (\omega^\omega)^k$, define

$$S_k^+ = \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,| \langle e\,(h)\,, g_1, ..., g_k \rangle \in S_k \right\}. \qquad \dashv$$

In particular, similar to observation 2.3.11, if $k = 0$ and $S_0 = [T] \setminus A$, then

$$([T] \setminus A)^+ = S_0^+ = \{h \in [Sft\,(T)] \,| e\,(h) \in S_0\} = \{h \in [Sft\,(T)] \,| e\,(h) \in [T] \setminus A\} = Shift\,(A)\,.$$

The following is a similar result to sublemma 2.3.13. We will use the following sublemma to prove sublemma 2.3.18 below.

Sublemma 2.3.17. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. For any $k \in \omega$ and $S_k \subseteq [T] \times (\omega^\omega)^k$,*

$$\left([Sft\,(T)] \times (\omega^\omega)^k\right) \setminus S_k^+ = \left(\left([T] \times (\omega^\omega)^k\right) \setminus S_k\right)^+. \qquad \dashv$$

Proof.

Fix $k \in \omega$ and $S_k \subseteq [T] \times (\omega^\omega)^k$. Then

$$\left([Sft\,(T)] \times (\omega^\omega)^k\right) \setminus S_k^+ = \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,| \langle e\,(h)\,, g_1, ..., g_k \rangle \notin S_k \right\}$$

$$= \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,\middle| \langle e\,(h)\,, g_1, ..., g_k \rangle \in \left([T] \times (\omega^\omega)^k\right) \setminus S_k \right\}$$

$$= \left(\left([T] \times (\omega^\omega)^k\right) \setminus S_k\right)^+. \qquad \square$$

We will use sublemma 2.3.17 to prove sublemma 2.3.18 (2).

Sublemma 2.3.18. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. For any $k, n \in \omega$ and for any $S_k \subseteq [T] \times (\omega^\omega)^k$:*

1. *If $S_k \in \mathbf{\Sigma}^1_n \restriction \left([T] \times (\omega^\omega)^k\right)$, then $S_k^+ \in \mathbf{\Sigma}^1_n \restriction \left([Sft\,(T)] \times (\omega^\omega)^k\right)$.*

2. If $S_k \in \mathbf{\Pi}^1_n \restriction \left([T] \times (\omega^\omega)^k\right)$*, then* $S^+_k \in \mathbf{\Pi}^1_n \restriction \left([Sft\,(T)] \times (\omega^\omega)^k\right)$*.* $\dashv$

Proof.

We prove both (1) and (2) simultaneously by induction on n. (2) follows from (1) and sublemma 2.3.17.

Base case : $n = 0$. (Recall by definition 1.3.15: $\mathbf{\Sigma}^1_0 \restriction [T] = \mathbf{\Sigma}^0_1 \restriction [T]$, $\mathbf{\Pi}^1_0 = \mathbf{\Pi}^0_1 \restriction [T]$.) Pick an arbitrary $k \in \omega$.

Show that if $S_k \in \mathbf{\Sigma}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$, then $S^+_k \in \mathbf{\Sigma}^0_1 \restriction \left([Sft\,(T)] \times (\omega^\omega)^k\right)$.

Suppose $S_k \in \mathbf{\Sigma}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$. Pick an arbitrary $\langle h, g_1, ..., g_k\rangle \in S^+_k$. Then

$$\langle e\,(h)\,, g_1, ..., g_k\rangle \in S_k.$$

Since $S_k \in \mathbf{\Sigma}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$, there exist finite $E, F_1, ..., F_k$ such that for any $\langle x, y_1, , , , .y_k\rangle \in [T] \times (\omega^\omega)^k$, if $x \supseteq e(h) \restriction E$ and for all $1 \leq l \leq k$, $y_l \supseteq g_l \restriction F_l$ then $\langle x, y_1, ..., y_k\rangle \in S_k$. Define $E^+ = \{i+1\,|i \in E\}$. Then E^+ is finite. Pick an arbitrary $\langle \hat{h}, \hat{g}_1, ..., \hat{g}_k\rangle \in ([Sft(T)] \times (\omega^\omega)^k)$ such that $\hat{h} \supseteq h \restriction E^+$, and for all $1 \leq l \leq k$, $\hat{g}_l \supseteq g_l \restriction F_l$. Then for any $j \in E$,

$$(e(\hat{h}))(j) = \hat{h}(j+1) = h(j+1) = (e(h))(j).$$

Thus $e(\hat{h}) \supseteq e(h) \restriction E$ so that $\langle e(\hat{h}), \hat{g}_1, ..., \hat{g}_k\rangle \in S_k$. Thus $\langle \hat{h}, \hat{g}_1, ..., \hat{g}_k\rangle \in S^+_k$. Hence $S^+_k \in \mathbf{\Sigma}^0_1 \restriction ([Sft(T)] \times (\omega^\omega)^k)$.

Show that if $S_k \in \mathbf{\Pi}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$, then $S^+_k \in \mathbf{\Pi}^0_1 \restriction \left([Sft\,(T)] \times (\omega^\omega)^k\right)$. $\cdots (*)$

Suppose $S_k \in \mathbf{\Pi}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$. Then $\left([T] \times (\omega^\omega)^k\right) \backslash S_k \in \mathbf{\Sigma}^0_1 \restriction \left([T] \times (\omega^\omega)^k\right)$. Since we have already shown (1) for $n = 0$, we have:

$$\left(\left([T] \times (\omega^\omega)^k\right) \backslash S_k\right)^+ \in \mathbf{\Sigma}^0_1 \restriction ([Sft\,(T)] \times (\omega^\omega)^k).$$

By sublemma 2.3.17,

$$\left([Sft\,(T)] \times (\omega^\omega)^k\right) \backslash S_k^+ = \left(\left([T] \times (\omega^\omega)^k\right) \backslash S_k\right)^+.$$

Thus $S_k^+ \in \mathbf{\Pi}_1^0 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^k\right)$.

Induction Step :

Assume that, as an induction hypothesis, for all $l \in \omega$, if $S_l \in \mathbf{\Sigma}_n^1 \upharpoonright \left([T] \times (\omega^\omega)^l\right)$, then $S_l^+ \in \mathbf{\Sigma}_n^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^l\right)$ and if $S_l \in \mathbf{\Pi}_n^1 \upharpoonright \left([T] \times (\omega^\omega)^l\right)$, then $S_l^+ \in \mathbf{\Pi}_n^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^l\right)$.

Pick an arbitrary $k \in \omega$. Suppose $S_k \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright \left([T] \times (\omega^\omega)^k\right)$. Show $S_k^+ \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^k\right)$.

Since $S_k \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright \left([T] \times (\omega^\omega)^k\right)$, there exists $S_{k+1} \in \mathbf{\Pi}_n^1 \upharpoonright \left([T] \times (\omega^\omega)^{k+1}\right)$ such that for any $\langle x, y_1, ..., y_k \rangle \in \left([T] \times (\omega^\omega)^k\right)$, $\langle x, y_1, ..., y_k \rangle \in S_k$ if and only if there exists $y_{k+1} \in \omega^\omega$ such that $\langle x, y_1, ..., y_k, y_{k+1} \rangle \in S_{k+1}$.

$$\begin{aligned} S_k^+ &= \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,|\langle e\,(h)\,, g_1, ..., g_k \rangle \in S_k \right\} \\ &= \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,|\exists g_{k+1} \in \omega^\omega \left(\langle e\,(h)\,, g_1, ..., g_k, g_{k+1} \rangle \in S_{k+1}\right) \right\} \\ &= \left\{ \langle h, g_1, ..., g_k \rangle \in [Sft\,(T)] \times (\omega^\omega)^k \,\middle|\exists g_{k+1} \in \omega^\omega \left(\langle h, g_1, ..., g_k, g_{k+1} \rangle \in S_{k+1}^+\right) \right\}. \end{aligned}$$

Since $S_{k+1}^+ \in \mathbf{\Pi}_n^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^{k+1}\right)$ by induction hypothesis,

$$S_k^+ \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^k\right).$$

Show that if $S_k \in \mathbf{\Pi}_{n+1}^1 \upharpoonright \left([T] \times (\omega^\omega)^k\right)$, then $S_k^+ \in \mathbf{\Pi}_{n+1}^1 \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^k\right)$. We repeat the same proof of $(*)$ on page 69.

Suppose $S_k \in \mathbf{\Pi}_{n+1}^1 \upharpoonright \left([T] \times (\omega^\omega)^k\right)$. Then $\left([T] \times (\omega^\omega)^k\right) \backslash S_k \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright \left([T] \times (\omega^\omega)^k\right)$.

Since we have already shown (1) for the case $n+1$, we have:

$$\left(\left([T] \times (\omega^\omega)^k\right) \setminus S_k\right)^+ \in \mathbf{\Sigma}^1_{n+1} \upharpoonright ([Sft\,(T)] \times (\omega^\omega)^k).$$

By sublemma 2.3.17,

$$\left([Sft\,(T)] \times (\omega^\omega)^k\right) \setminus S_k^+ = \left(\left([T] \times (\omega^\omega)^k\right) \setminus S_k\right)^+.$$

Thus $S_k^+ \in \mathbf{\Pi}^1_{n+1} \upharpoonright \left([Sft\,(T)] \times (\omega^\omega)^k\right)$. □

Lemma 2.3.15 is obtained immediately from sublemma 2.3.18. Recall lemma 2.3.15.

Lemma 2.3.15. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Let $n \in \omega\setminus\{0\}$.*

1. *If $A \in \mathbf{\Pi}^1_n \upharpoonright [T]$, then $Shift(A) \in \mathbf{\Sigma}^1_n \upharpoonright [Sft(T)]$.*

2. *If $A \in \mathbf{\Sigma}^1_n \upharpoonright [T]$, then $Shift(A) \in \mathbf{\Pi}^1_n \upharpoonright [Sft(T)]$.* ⊣

We will prove lemma 2.3.15 using sublemma 2.3.18. The proof of lemma 2.3.15 is similar to the proof of lemma 2.3.9.

Proof.

Suppose $A \in \mathbf{\Pi}^1_n \upharpoonright [T]$. Show $Shift(A) \in \mathbf{\Sigma}^1_n \upharpoonright [Sft(T)]$.

Since $A \in \mathbf{\Pi}^1_n \upharpoonright [T]$, $[T]\setminus A \in \mathbf{\Sigma}^1_n \upharpoonright [T]$. By sublemma 2.3.18,

$$([T] \setminus A)^+ \in \mathbf{\Sigma}^1_n \upharpoonright [Sft(T)].$$

By observation 2.3.11,

$$Shift\,(A) = ([T] \setminus A)^+\,.$$

Thus $Shift(A) \in \mathbf{\Sigma}^1_n \upharpoonright [Sft(T)]$.

A similar proof gives a proof of (2): Simply interchange $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$. □

Next, we compute the complexity of $B^+ = X \times B$ for Borel set B on X^ω. Recall that X^ω is a special case of Type 1 trees $T^{\Psi,B}_{X,Y}$ with $B = \emptyset$. By observation 2.3.4, $T = Sft(T) = X^{<\omega}$.

Sublemma 2.3.19. *Suppose $\alpha \in \omega_1$.*

1. *If $B \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$.*
2. *If $B \in \mathbf{\Pi}^0_\alpha \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Pi}^0_\alpha \upharpoonright X^\omega$.*
3. *If $B \in \mathbf{\Delta}^0_\alpha \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Delta}^0_\alpha \upharpoonright X^\omega$.*
4. *If $B \in \mathbf{B} \upharpoonright X^\omega$, then $B^+ \in \mathbf{B} \upharpoonright X^\omega$.* ⊣

Proof.

Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$ with $B = \emptyset$. Then $T = X^{<\omega}$. We have (1) and (2) by sublemma 2.3.14. (3) and (4) easily follow from (1) and (2). □

Similarly, we obtain the following using sublemma 2.3.18.

Sublemma 2.3.20. *Suppose $n \in \omega$.*

1. *If $B \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$.*
2. *If $B \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$.*
3. *If $B \in \mathbf{\Delta}^1_n \upharpoonright X^\omega$, then $B^+ \in \mathbf{\Delta}^1_n \upharpoonright X^\omega$.*
4. *If $B \in (\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n) \upharpoonright X^\omega$, then $B^+ \in (\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n) \upharpoonright X^\omega$.*
5. *If $B \in \mathbf{P} \upharpoonright X^\omega$, then $B^+ \in \mathbf{P} \upharpoonright X^\omega$.* ⊣

Proof.

Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$ with $B = \emptyset$. Then $T = X^{<\omega}$. We have (1) and (2) by sublemma 2.3.18. Consequently, we have (3) and (5).

Show (4). Suppose $B \in (\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n) \restriction X^\omega$. Then $B = B_1 \cap B_2$ for some $B_1 \in \mathbf{\Sigma}^1_n \restriction X^\omega$ and $B_2 \in \mathbf{\Pi}^1_n \restriction X^\omega$. Thus $B_1^+ = X \times B_1 \in \mathbf{\Sigma}^1_n \restriction X^\omega$ and $B_2^+ = X \times B_2 \in \mathbf{\Pi}^1_n \restriction X^\omega$ by sublemma 2.3.18. Hence

$$B^+ = X \times B = X \times (B_1 \cap B_2) = (X \times B_1) \cap (X \times B_2) = B_1^+ \cap B_2^+ \in (\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n) \restriction X^\omega.$$

□

Finally, we compute the complexity of the function Ψ^+ when Ψ is $\mathbf{\Delta}^0_\gamma$-measurable and $\mathbf{\Delta}^1_n$-measurable.

Sublemma 2.3.21. *Suppose $n \in \omega$ and $\gamma \in \omega_1$.*

1. *If $\Psi \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$, then $\Psi^+ \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$.*

2. *If $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_n)$, then $\Psi^+ \in \Gamma(\omega, \mathbf{\Delta}^1_n)$.*

3. *If $\Psi \in \Gamma(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n)$, then $\Psi^+ \in \Gamma(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n)$.* [9] ⊣

Proof.

Show (1).

Suppose $\Psi \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$. Then Ψ is a function from X^ω into ω such that for every open set $O \subseteq \omega$, $\Psi^{-1}(O) \in \mathbf{\Delta}^0_\gamma \restriction X^\omega$. Show $\Psi^+ \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$.

[9] Recall definition 2.3.2 for Ψ^+.

Show $\Psi^{+^{-1}}(O) \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$.

Fix $O \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega$. Since $O = \bigcup_{i \in O} \{i\}$ and we are using the discrete topology on ω, each singleton $\{i\}$ is clopen.

First, compute the complexity of $\Psi^{+^{-1}}(i)$ for $i \in \omega$. [10]

Case 1 : $i = 0$.

By the definition of Ψ^+, $\Psi^{+^{-1}}(0) = \emptyset$.

Case 2 : $i \neq 0$.

$$\begin{aligned} \Psi^{+^{-1}}(i) &= X \times \{f \in X^\omega \,|\Psi(f) + 1 = i\} \\ &= X \times \Psi^{-1}(i-1). \end{aligned}$$

Since $\Psi \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$, $\Psi^{-1}(i-1) \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$. By sublemma 2.3.19, $\Psi^{+^{-1}}(i) \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$.

Thus, by cases 1 and 2,

$$\Psi^{+^{-1}}(O) = \Psi^{+^{-1}}\left(\bigcup_{i \in O} \{i\}\right) = \bigcup_{i \in O} \Psi^{+^{-1}}(i) \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega. \tag{2.3}$$

Show $\Psi^{+^{-1}}(O) \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$. Show $X^\omega \backslash \Psi^{+^{-1}}(O) \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$.

Since $\omega \backslash O = \bigcup_{i \in \omega \backslash O} \{i\}$, $\omega \backslash O \in \mathbf{\Sigma}^0_1 \upharpoonright \omega$. Thus

$$X^\omega \backslash \Psi^{+^{-1}}(O) = \Psi^{+^{-1}}(\omega \backslash O) \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega.$$

Hence $\Psi^{+^{-1}}(O) \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$.

Therefore, $\Psi^+ \in \Gamma(\omega, \mathbf{\Delta}^0_\gamma)$.

Show (2).

If $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_n)$, then $\Psi^{-1}(n-1) \in \mathbf{\Delta}^1_n \upharpoonright X^\omega$. By sublemma 2.3.20, $\Psi^{+^{-1}}(i) = X \times \Psi^{-1}(i-$

[10] Recall notation 1.5.4 Abuse of notation : we suppress $\{\}$ for $\Psi^{-1}(\{i\})$, i.e., we write $\Psi^{-1}(i)$ to mean $\Psi^{-1}(\{i\})$. $\Psi^{-1}(i)$ does not mean the inverse image of $\{0, 1, ..., i-1\}$ here.

1) $\in \mathbf{\Delta}^1_n \restriction X^\omega$. Since $\mathbf{\Sigma}^1_n \restriction X^\omega$ is closed under countable unions by lemma 2.3.22 below (page 75), $\Psi^+ \in \Gamma(\omega, \mathbf{\Delta}^1_n)$ (replace $\mathbf{\Sigma}^0_\gamma$ in equation (2.3) by $\mathbf{\Sigma}^1_n$).

Show (3).

A similar proof of sublemma 2.3.20 (4) gives (3): if $\Psi \in \Gamma(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n)$, then $\Psi^+ \in \Gamma(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n)$. □

We used the well-known closure property of projective sets in the proof of 2.3.21. Sierpinski proved this in 1928 (as cited in Moschovakis, 2009, p. 47). We will prove the closure under countable unions for $\mathbf{\Sigma}^1_n \restriction X^\omega$ and the closure under countable intersections for $\mathbf{\Pi}^1_n \restriction X^\omega$. Readers familiar with this proof may skip to theorem 2.3.1 on page 55.

Lemma 2.3.22. *Let $n \in \omega$.*

1. *$\mathbf{\Sigma}^1_n \restriction X^\omega$ is closed under countable unions.*
2. *$\mathbf{\Pi}^1_n \restriction X^\omega$ is closed under countable intersections.* ⊣

We will prove lemma 2.3.22 by using sublemma 2.3.25 below. The proof of lemma 2.3.22 is on page 79. We first define the following notation.

Definition 2.3.23. *Let $k \in \omega$. Suppose $\langle S^k_i | i \in \omega\rangle$ to be such that each $S^k_i \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$. Define*

$$\langle S^k_i | i \in \omega\rangle^- = \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h \restriction [1, \omega), g_1, ..., g_k\rangle \in S^k_{h(0)}\right\}. \quad ⊣$$

We will use the following sublemma to prove sublemma 2.3.25.

Sublemma 2.3.24. *Let $k \in \omega$. Suppose $\langle S^k_i | i \in \omega\rangle$ to be such that each $S^k_i \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$. Then*

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus \langle S^k_i | i \in \omega\rangle^- = \langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_i | i \in \omega\rangle^-. \quad ⊣$$

Proof.

Fix $k \in \omega$ and $\langle S_i^k | i \in \omega \rangle$ with each $S_i^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$.

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash \langle S_i^k | i \in \omega \rangle^-$$

$$= \left\{ \langle f, h, g_1, ..., g_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h \restriction [1, \omega), g_1, ..., g_k \rangle \notin S_{h(0)}^k \right\}$$

$$= \left\{ \langle f, h, g_1, ..., g_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h \restriction [1, \omega), g_1, ..., g_k \rangle \in \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S_{h(0)}^k \right\}$$

$$= \langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S_i^k | i \in \omega \rangle^-.$$ □

Sublemma 2.3.25. *Suppose $n, k \in \omega$, $n \geq 1$. Suppose $\langle S_i^k | i \in \omega \rangle$ to be such that each $S_i^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$.*

1. *If every $S_i^k \in \mathbf{\Sigma}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, then $\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Sigma}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.*

2. *If every $S_i^k \in \mathbf{\Pi}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, then $\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Pi}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.* ⊣

Proof.

We prove both (1) and (2) simultaneously by induction on n. (2) follows from (1) and sublemma 2.3.24.

Base Case : $n = 1$.

Pick an arbitrary $k \in \omega$ and fix $\langle S_i^k | i \in \omega \rangle$.

Suppose every $S_i^k \in \mathbf{\Sigma}_1^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Sigma}_1^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Pick an arbitrary $\langle f, h, g_1, ..., g_k \rangle \in \langle S_i^k | i \in \omega \rangle^-$. Then $\langle f, h \restriction [1, \omega), g_1, ..., g_k \rangle \in S_{h(0)}^k$. Let $h_1 = h \restriction [1, \omega)$. Since $S_{h(0)}^k \in \mathbf{\Sigma}_1^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, there exist finite $F \subseteq \omega$, $H_1 \subseteq \omega$ and $G_i \subseteq \omega$, $1 \leq i \leq k$ such that for all $\langle x, y, z_1, ..., z_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \restriction F$, $y \supseteq h_1 \restriction H_1$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \restriction G_i$, then $\langle x, y, z_1, ..., z_k \rangle \in S_{h(0)}^k$. Define

$$H = \{ n + 1 \,| n \in H_1 \} \cup \{0\}.$$

Suppose $y \in \omega^\omega$ and $y \supseteq h \upharpoonright H$. Then $y(0) = h(0)$. Also, for any $j \in H_i$,

$$(y \upharpoonright [1, \omega))(j) = y(j+1) = h(j+1) = (h \upharpoonright [1, \omega))(j) = h_1(j).$$

Thus $y \upharpoonright [1, \omega) \supseteq h_1 \upharpoonright H_1$. Thus for all $\langle x, y, z_1, ..., z_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \upharpoonright F$, $y \supseteq h \upharpoonright H$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \upharpoonright G_i$, then $\langle x, y \upharpoonright [1, \omega), z_1, ..., z_k \rangle \in S^k_{h(0)} = S^k_{y(0)}$. Hence, for all $\langle x, y, z_1, ..., z_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \upharpoonright F$, $y \supseteq h \upharpoonright H$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \upharpoonright G_i$, then $\langle x, y, z_1, ..., z_k \rangle \in \langle S^k_i | i \in \omega \rangle^-$. Thus $\langle S^k_i | i \in \omega \rangle^- \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Suppose every $S^k_i \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $\langle S^k_i | i \in \omega \rangle^- \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. $\cdots (*)$

Since every $S^k_i \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, each $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_i \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Since we have already shown (1) for $n = 1$, we have: $\langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_i | i \in \omega \rangle^- \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. By sublemma 2.3.24,

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus \langle S^k_i | i \in \omega \rangle^- = \langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_i | i \in \omega \rangle^-.$$

Thus $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus \langle S^k_i | i \in \omega \rangle^- \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Hence $\langle S^k_i | i \in \omega \rangle^- \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Induction Step : Assume that, as an induction hypothesis, for all $l \in \omega$, if every $S^l_i \in \mathbf{\Sigma}^1_n \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$, then $\langle S^l_i | i \in \omega \rangle^- \in \mathbf{\Sigma}^1_n \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$ and if $S^l_i \in \mathbf{\Pi}^1_n \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$, then $\langle S^l_i | i \in \omega \rangle^- \in \mathbf{\Pi}^1_n \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$.

Pick an arbitrary $k \in \omega$.

Suppose every $S^k_i \in \mathbf{\Sigma}^1_{n+1} \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $\langle S^k_i | i \in \omega \rangle^- \in \mathbf{\Sigma}^1_{n+1} \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Since each $S^k_i \in \mathbf{\Sigma}^1_{n+1} \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, there exists $S^{k+1}_i \in \mathbf{\Pi}^1_n \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$

such that for any $\langle x, y, z_1, ..., z_k \rangle \in (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, $\langle x, y, z_1, ..., z_k \rangle \in S_i^k$ if and only if there exists $z_{k+1} \in \omega^\omega$ such that $\langle x, y, z_1, ..., z_k, z_{k+1} \rangle \in S_i^{k+1}$.

$$\begin{aligned}
&\langle S_i^k | i \in \omega \rangle^- \\
&= \left\{ \langle f, h, g_1, ..., g_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h \restriction [1, \omega), g_1, ..., g_k \rangle \in S_{h(0)}^k \right\} \\
&= \left\{ \langle f, h, g_1, ..., g_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \exists g_{k+1} \in \omega^\omega \, \langle f, h \restriction [1, \omega), g_1, ..., g_k, g_{k+1} \rangle \in S_{h(0)}^{k+1} \right\} \\
&= \left\{ \langle f, h, g_1, ..., g_k \rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \exists g_{k+1} \in \omega^\omega \, \langle f, h, g_1, ..., g_k, g_{k+1} \rangle \in \langle S_i^{k+1} | i \in \omega \rangle^- \right\}.
\end{aligned}$$

Since each $S_i^{k+1} \in \mathbf{\Pi}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$, by induction hypothesis, $\langle S_i^{k+1} | i \in \omega \rangle^- \in \mathbf{\Pi}_n^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$. Thus $\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Sigma}_{n+1}^1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Suppose every $S_i^k \in \mathbf{\Pi}_{n+1}^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show

$$\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Pi}_{n+1}^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

We repeat the same proof of $(*)$ on page 77.

Since every $S_i^k \in \mathbf{\Pi}_{n+1}^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, each

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S_i^k \in \mathbf{\Sigma}_{n+1}^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

Since we have already shown (1) for the case $n+1$, we have:

$$\langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S_i^k | i \in \omega \rangle^- \in \mathbf{\Sigma}_{n+1}^0 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

By sublemma 2.3.24,

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus \langle S_i^k | i \in \omega \rangle^- = \langle \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S_i^k | i \in \omega \rangle^-.$$

Thus $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus \langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Sigma}^0_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Hence

$$\langle S_i^k | i \in \omega \rangle^- \in \mathbf{\Pi}^0_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k). \qquad \square$$

Using sublemma 2.3.25, we prove lemma 2.3.22. Recall lemma 2.3.22. (2) is obtained from (1).

Lemma 2.3.22. *Let $n \in \omega$.*

1. $\mathbf{\Sigma}^1_n \restriction X^\omega$ *is closed under countable unions.*

2. $\mathbf{\Pi}^1_n \restriction X^\omega$ *is closed under countable intersections.* ⊣

Proof.

Show (1). When $n = 0$, $\mathbf{\Sigma}^1_0 \restriction X^\omega = \mathbf{\Sigma}^0_1 \restriction X^\omega$. Since $\mathbf{\Sigma}^0_1 \restriction X^\omega$ is closed under countable unions, assume that $n > 0$. Show $\mathbf{\Sigma}^1_n \restriction X^\omega$ is closed under countable unions. Let $\langle A_i | i \in \omega \rangle$ be such that each $A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Since each $A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$, there exists $C_i \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega$ such that

$$f \in A_i \Leftrightarrow \exists g \in \omega^\omega \, \langle f, g \rangle \in C_i.$$

Show $\bigcup_{i \in \omega} A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$.

$$\begin{aligned} f \in \bigcup_{i \in \omega} A_i &\Leftrightarrow \exists i \in \omega \, (f \in A_i) \\ &\Leftrightarrow \exists i \in \omega \exists g \in \omega^\omega \, (\langle f, g \rangle \in C_i) \\ &\Leftrightarrow \exists \hat{h} \in \omega^\omega (\langle f, \hat{h} \restriction [1, \omega) \rangle \in C_{\hat{h}(0)}) \; (\hat{h}\,(0) = i \text{ and } \hat{h} \restriction [1, \omega) = g) \\ &\Leftrightarrow \exists \hat{h} \in \omega^\omega (\langle f, \hat{h} \rangle \in \langle C_i | i \in \omega \rangle^-). \end{aligned}$$

Since each $C_i \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega$, by sublemma 2.3.25,

$$\langle C_i | i \in \omega \rangle^- \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega.$$

Thus $\bigcup_{i\in\omega} A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$.

Show (2). Suppose $\langle A_i | i \in \omega \rangle$ be such that each $A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$. Show $\bigcap_{i\in\omega} A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$.

Since each $A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$, $X^\omega \backslash A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Since we have already shown (1), we have:

$$\bigcup_{i\in\omega} (X^\omega \backslash A_i) \in \mathbf{\Sigma}^1_n \restriction X^\omega.$$

Since $X^\omega \backslash \left(\bigcap_{i\in\omega} A_i\right) = \bigcup_{i\in\omega} (X^\omega \backslash A_i)$, $X^\omega \backslash \left(\bigcap_{i\in\omega} A_i\right) \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Thus $\bigcap_{i\in\omega} A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$. □

Finally, by using above lemmas and sublemmas, we will prove theorem 2.3.1 on page 55. Recall theorem 2.3.1.

Theorem 2.3.1. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Then for any X and Θ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_1\right) \tag{2.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_1\right) \tag{2.2}$$

for $\mathcal{T}_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ [11] *where:*

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* ⊣

Proof of Theorem 2.3.1.

Fix $\mathcal{T}_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ in the theorem with fixed complexities for $\mathcal{B}$ and $\mathcal{C}$.

Pick an arbitrary $T = T^{\Psi,B}_{X,Y} \in \mathcal{T}_1$. Show the equivalence (2.1).

[11] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

($\Rightarrow$) Suppose $A \in \mathbf{\Pi}^0_\alpha \upharpoonright [T]$. Since $B \in \mathcal{B} \upharpoonright X^\omega$, by sublemma 2.3.19 and sublemma 2.3.20, $B^+ \in \mathcal{B} \upharpoonright X^\omega$. Also, by sublemma 2.3.21, $\Psi^+ \in \Gamma(\omega, \mathcal{C})$. Therefore, $Sft(T) = T^{\Psi^+, B^+}_{X,Y} \in \mathcal{T}_1$. By lemma 2.3.9, $Shift(A) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [Sft(T)]$. Thus

$$G(Shift(A), Sft(T)) \in \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right).$$

Hence $G(Shift(A), Sft(T))$ is determined. By theorem 2.3.8, $G(A, T)$ is determined.

($\Leftarrow$) By switching $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$ in the above proof, we can obtain this direction.

Show the equivalence (2.2).

($\Rightarrow$) Suppose $A \in \mathbf{\Pi}^1_n \upharpoonright [T]$. Since $B \in \mathcal{B} \upharpoonright X^\omega$, by sublemma 2.3.19 and sublemma 2.3.20, $B^+ \in \mathcal{B} \upharpoonright X^\omega$. Also, by sublemma 2.3.21, $\Psi^+ \in \Gamma(\omega, \mathcal{C})$. Therefore, $Sft(T) = T^{\Psi^+, B^+}_{X,Y} \in \mathcal{T}_1$. By lemma 2.3.15, $Shift(A) \in \mathbf{\Sigma}^1_1 \upharpoonright [Sft(T)]$. Thus

$$G(Shift(A), Sft(T)) \in \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right).$$

Hence $G(Shift(A), Sft(T))$ is determined. By theorem 2.3.8, $G(A, T)$ is determined.

($\Rightarrow$) Switch $\mathbf{\Pi}^1_n \upharpoonright [T]$ and $\mathbf{\Sigma}^1_n \upharpoonright [T]$ in the ($\Rightarrow$) direction of the equivalence (2.2). By lemma 2.3.15, $Shift(A) \in \mathbf{\Pi}^1_n \upharpoonright [Sft(T)]$. $\square$

2.4 Using the determinacy of games on a $Tree_1$ collection to obtain the determinacy of games on $X^{<\omega}$

In this section, we will use the determinacy of games on a $Tree_1$ collection to obtain determinacy of games on $X^{<\omega}$.

In section 2.4.1, under ZF-P, we will focus on using $\mathbf{\Delta}^0_1$ determinacy on a $Tree_1$ collection to obtain finite Borel determinacy on $X^{<\omega}$. We will obtain the determinacy of finite Borel games on $X^{<\omega}$ from the determinacy of clopen games on a particular $Tree_1$ collection.

In section 2.4.2, we will focus on using $\mathbf{\Sigma}^0_1$ determinacy on a $Tree_1$ collection to obtain the determinacy of games on $X^{<\omega}$. In section 2.4.2.1, we will define a special open set *Long* on a Type 1 tree. *Long* includes all plays of the tree which have length greater than ω and excludes those of length ω. In section 2.4.2.2, we will define a special open set Max on a Type 1 tree. Max is defined only on Type 1 trees with paths having maximum length. We will obtain the determinacy results using Max in sections 2.4.2.3 and 2.4.2.4.

In section 2.4.3, we will obtain $\alpha+1$-$\mathbf{\Pi}^1_1$ determinacy on X^ω for even $\alpha \in \omega_1$ using α-$\mathbf{\Pi}^1_1$ determinacy on $Tree_1$ collection. We will again obtain the determinacy results using Max in this section.

2.4.1 (ZF-P) Using $\mathbf{\Delta}_1^0$ determinacy on a $Tree_1$ collection to obtain finite Borel determinacy on $X^{<\omega}$

In this section, we will focus on obtaining in ZF-P the determinacy of finite Borel games on $X^{<\omega}$ from the determinacy of games on a $Tree_1$ collection.[12]

First, consider the special case of Type 1 trees $T = T_{X,Y}^{\Psi,B}$ with $B = \emptyset$. Since $[T_{X,Y}^{\Psi,\emptyset}] = X^\omega$,

$$\mathcal{G}\left(\Xi; Tree_1\left(X, \Theta, \Gamma, \emptyset\right)\right) = \left\{G\left(A; X^{<\omega}\right) | A \in \Xi \restriction X^\omega\right\}$$

for any X, Θ and Γ. Thus, we have the following observation 2.4.1.

Observation 2.4.1. *For any X, Y, any function Ψ from X^ω into ω and any complexity Ξ (in which $\Xi \restriction [T] \subseteq \wp([T])$ is defined),*

$$Det\ \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Psi, \emptyset\right)\right) \Rightarrow Det\left(\Xi \restriction X^\omega\right). \quad \dashv$$

$\Xi \restriction X^\omega$ could be any subset of X^ω in observation 2.4.1.

As an example to observation 2.4.1, consider $\mathcal{T}_1 = Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$ for any X and Y. Recall that $\Gamma(\omega, \mathbf{\Delta}_1^0)$ is the set of continuous functions. Since $\emptyset \in \mathbf{\Delta}_1^0 \restriction X^\omega$, by observation 2.4.1, we have $\mathcal{G}\left(\mathbf{\Sigma}_1^0; \mathcal{T}_1\right)$ contains all open games on $X^{<\omega}$.

Corollary 2.4.2. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Fix nonempty sets X and Y.*
Let $\mathcal{T}_1 = Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$. Then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_\alpha^0; \mathcal{T}_1\right) \Rightarrow Det\left(\mathbf{\Sigma}_\alpha^0 \restriction X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_n^1; \mathcal{T}_1\right) \Rightarrow Det\left(\mathbf{\Sigma}_n^1 \restriction X^\omega\right). \quad \dashv$$

[12] The proof of $Det(\mathbf{B} \restriction X^\omega)$ in ZFC uses the power set axiom.

Proof.

Since $\emptyset \in \mathbf{\Delta}^0_1 \upharpoonright X^\omega$, we have the results by observation 2.4.1 when Ξ is $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Sigma}^1_n$. $\square$

Observation 2.4.3. *Assume that Γ contains all constant functions from X^ω into ω. Then for any X, Y and complexity Ξ (in which $\Xi \upharpoonright X^\omega \times Y^n \subseteq \wp(X^\omega \times Y^n)$ is defined for all $n \in \omega$),*

$$Det\ \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup\nolimits_{n\in\omega} \Xi \upharpoonright \left(X^\omega \times Y^n\right)\right). \qquad \dashv$$

Proof.

Fix X, Y and Ξ. Assume $Det\,\mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$. Pick an arbitrary $A \in \bigcup_{n\in\omega} \Xi \upharpoonright (X^\omega \times Y^n)$. Then $\exists n \in \omega$ such that $A \in \Xi \upharpoonright (X^\omega \times Y^n)$.

Case 1 : $n = 0$.

See observation 2.4.1.

Case 2 : $n \neq 0$.

Let $B = X^\omega$ and Ψ is the constant function at $n-1$. Then $\Psi \in \Gamma$. Note that $[T^{\Psi, X^\omega}_{X,Y}] = X^\omega \times Y^n$. Thus $G(A; X^{<\omega} \times Y^{\leq n}) = G(A; T^{\Psi, X^\omega}_{X,Y}) \in \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$. Hence $G(A; X^\omega \times Y^n)$ is determined. $\square$

Theorem 2.4.4. *(ZF-P)*

Suppose Γ contains all constant functions from X^ω into ω and Y is denumerable. Then $Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$ implies $Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \upharpoonright X^\omega\right)$, finite Borel determinacy on $X^{<\omega}$. $\dashv$

The proof of theorem 2.4.4 consists of the following two parts:

1. For any $n \in \omega$, $Det\left(\mathbf{\Sigma}^0_n \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \upharpoonright X^\omega \times Y^n\right)$.

2. Use observation 2.4.3 with $\Xi = \mathbf{\Delta}^0_1$.

In pages 85-104, we will prove (1) as theorem 2.4.5. Then we will prove theorem 2.4.4 on page 104. (1) is "well-known". Readers familiar with the proof of (1) may skip to page 104.

Under ZF-P (i.e., ZF - power set), we will see some general results about the finite Borel games on trees with fixed length. Recall the following well-known results.

Theorem 2.4.5. *ZF-P(folklore)*

Suppose $n \in \omega$ and Y is denumerable. Then for any $n \in \omega$,

$$Det\left(\mathbf{\Sigma}^0_n \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^n\right). \qquad \dashv$$

Theorem 2.4.5 follows from repeated application of lemma 2.4.6 below. After proving lemma 2.4.6, we prove theorem 2.4.5 on page 102.

Lemma 2.4.6. *(Main lemma)*

Suppose $m, l \in \omega$ and Y is denumerable.

If l is even, then

$$Det\left(\mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right), \tag{2.4}$$

$$Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right). \tag{2.5}$$

If l is odd, then

$$Det\left(\mathbf{\Pi}^0_m \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right), \tag{2.6}$$

$$Det\left(\mathbf{\Pi}^0_1 \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right). \tag{2.7}$$

$\dashv$

We will separate proof of lemma 2.4.6 by directions. First, we will prove ($\Rightarrow$) direction of lemma 2.4.6 on page 93. Then we will prove ($\Leftarrow$) direction of lemma 2.4.6 on page 101. For each direction, we will prove four sublemmas. None of these sublemmas use that Y is denumerable. The proof of the equivalences (2.4) through (2.7) from sublemmas use that Y is denumerable.

Towards the prof of ($\Rightarrow$) direction, we define A_a^{short}. We will use sublemma 2.4.8 to show the equivalences (2.4) and (2.5). We will use sublemma 2.4.9 to show the equivalences (2.6) and (2.7). We will find the complexity for A_a^{short} in sublemma 2.4.11 given the complexity of A.

Definition 2.4.7. *Suppose $l \in \omega$. Let $A \subseteq X^\omega \times Y^{l+1}$. For each $a \in Y$, define*

$$A_a^{short} = \left\{ f \in X^\omega \times Y^l \,|\, f^\frown\langle a\rangle \in A \right\}. \qquad \dashv$$

Sublemma 2.4.8. *Assume $l \in \omega$ is even.*

If $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined, then $G(A; X^\omega \times Y^{l+1})$ is determined. $\dashv$

Proof.

Suppose $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined. Then I^* or II^* has a winning strategy s^* for $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$.

Case 1 : s^* is a I^*'s winning strategy for $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$.

Define I's strategy s for $G(A; X^\omega \times Y^{l+1})$ by

$$s(p) = \begin{cases} s^*(p) & \text{if } lh(p) < \omega + l, \\ \mu a \in Y \left(p \in A_a^{short}\right)^{13} & \text{if } lh(p) = \omega + l \text{ and } p \in \bigcup_{a \in Y} A_a^{short}, \\ \emptyset & \text{otherwise.}^{14} \end{cases}$$

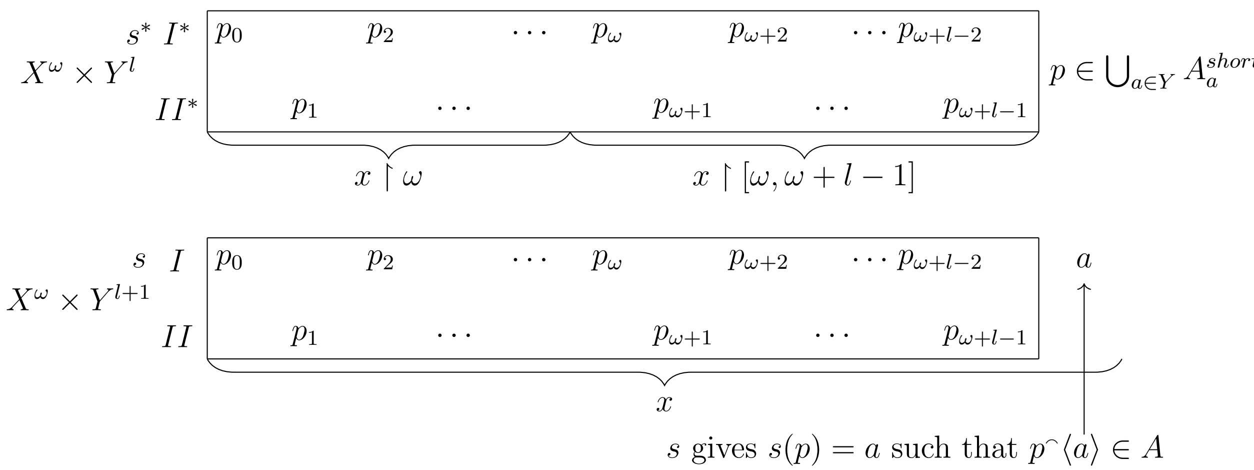

Figure 2.4.1: Illustration of $x \in X^\omega \times Y^{l+1}$ (l is even) according to I's strategy s (corresponding to the ($\Rightarrow$) direction of the equivalence (2.4) on page 85).

Show s is a winning strategy for I for $G(A; X^\omega \times Y^{l+1})$. Pick an arbitrary $x \in X^\omega \times Y^{l+1}$ according to s. Then $x \restriction (\omega+l)$ is according to I^*'s winning strategy s^* for $G(\bigcup_{a \in Y} A_a^{short}; X^\omega \times Y^l)$ so that $x \restriction (\omega + l) \in \bigcup_{a \in Y} A_a^{short}$. Since x is according to s, $x \restriction (\omega + l) \in A_{x(\omega+l)}^{short}$. Thus $x \in A$.

Case 2 : s^* is a II^*'s winning strategy for $G(\bigcup_{a \in Y} A_a^{short}; X^\omega \times Y^l)$.

Define II's strategy s for $G(A; X^\omega \times Y^{l+1})$ by $s = s^*$.

[13] μ represents "the least". If Y is well-orderable, fix a well-ordering of Y. Otherwise, pick any $a \in Y$ such that $p \in A_a^{short}$.

[14] This otherwise case does not occur for plays according to s.

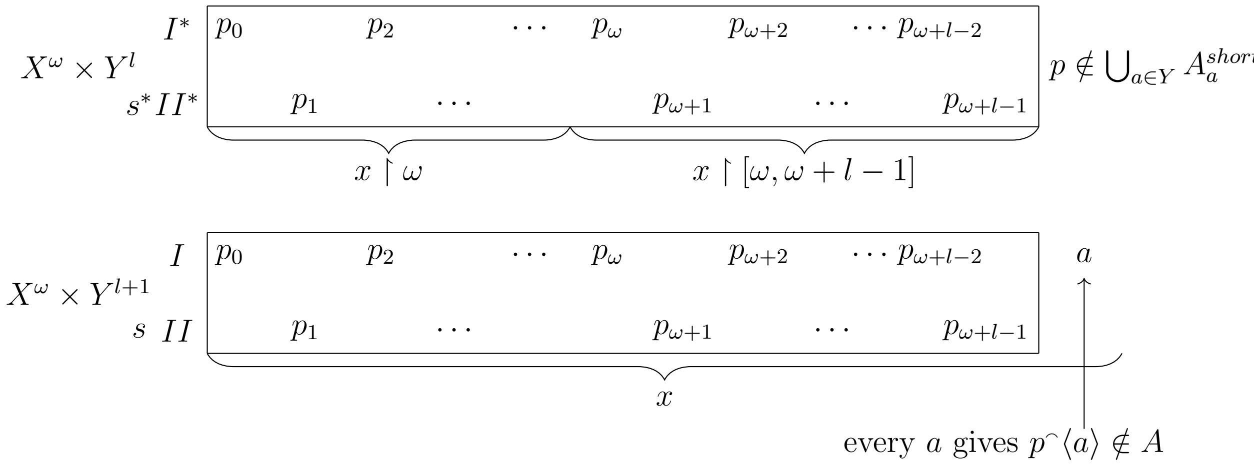

Figure 2.4.2: Illustration of $x \in X^\omega \times Y^{l+1}$ (l is even) according to II's strategy s (corresponding to the ($\Rightarrow$) direction of the equivalence (2.4) on page 85).

Show s is a winning strategy for II for $G(A; X^\omega \times Y^{l+1})$. Pick an arbitrary x according to s. Then $x \restriction (\omega + l)$ is according to s^*. Since s^* is a II^*'s winning strategy for $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$, $x \restriction (\omega+l) \notin \bigcup_{a\in Y} A_a^{short}$. Thus for any $a \in Y$, $(x \restriction (\omega+l))^\frown\langle a\rangle \notin A$, i.e., $x \notin A$. $\square$

Sublemma 2.4.9. *Assume l is odd.*

If $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined, then $G(A; X^\omega \times Y^{l+1})$ is determined. $\dashv$

Proof.

Assume $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined. Then I^* or II^* has a winning strategy s^* for $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$.

Case 1 : s^* is a I^*'s winning strategy for $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$.

Define I's strategy s for $G(A; X^\omega \times Y^{l+1})$ by $s = s^*$.

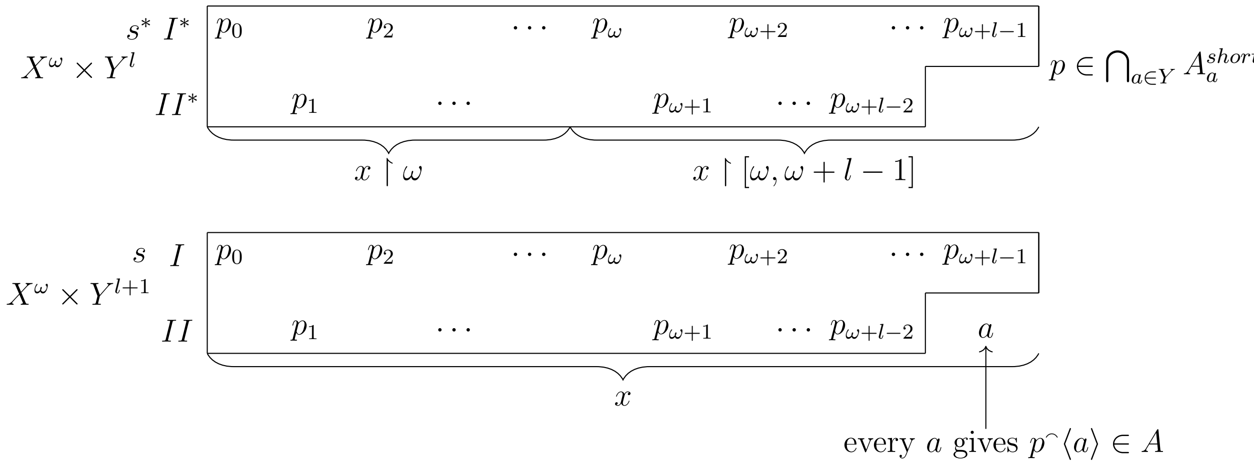

Figure 2.4.3: Illustration of $x \in X^\omega \times Y^{l+1}$ (l is odd) according to I's strategy s (corresponding to the ($\Rightarrow$) direction of the equivalence (2.6) on page 85).

Show s is a winning strategy for I for $G(A; X^\omega \times Y^{l+1})$. Pick an arbitrary $x \in X^\omega \times Y^{l+1}$ according to s. Then $x \restriction (\omega + l)$ is according to s^*. Since s^* is a I^*'s winning strategy for $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$, $x \restriction (\omega+l) \in \bigcap_{a\in Y} A_a^{short}$. Thus for any $a \in Y$, $x \restriction (\omega + l) \in A_a^{short}$. Thus $x \restriction (\omega + l) \in A_{x(\omega+l)}^{short}$. Hence $x \in A$.

Case 2 : s^* is a II^*'s winning strategy for $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$.

Define II's strategy s for $G(A; X^\omega \times Y^{l+1})$ by

$$s(p) = \begin{cases} s^*(p) & \text{if } lh(p) < \omega + l, \\ \mu a \in Y \left(p \notin A_a^{short}\right)^{15} & \text{if } lh(p) = \omega + l \text{ and } p \notin \bigcap_{a\in Y} A_a^{short}, \\ \emptyset & \text{otherwise.}^{16} \end{cases}$$

[15] μ represents "the least". If Y is well-orderable, fix a well-ordering of Y. Otherwise, pick any $a \in Y$ such that $p \notin A_a^{short}$.

[16] This otherwise case does not occur for plays according to s.

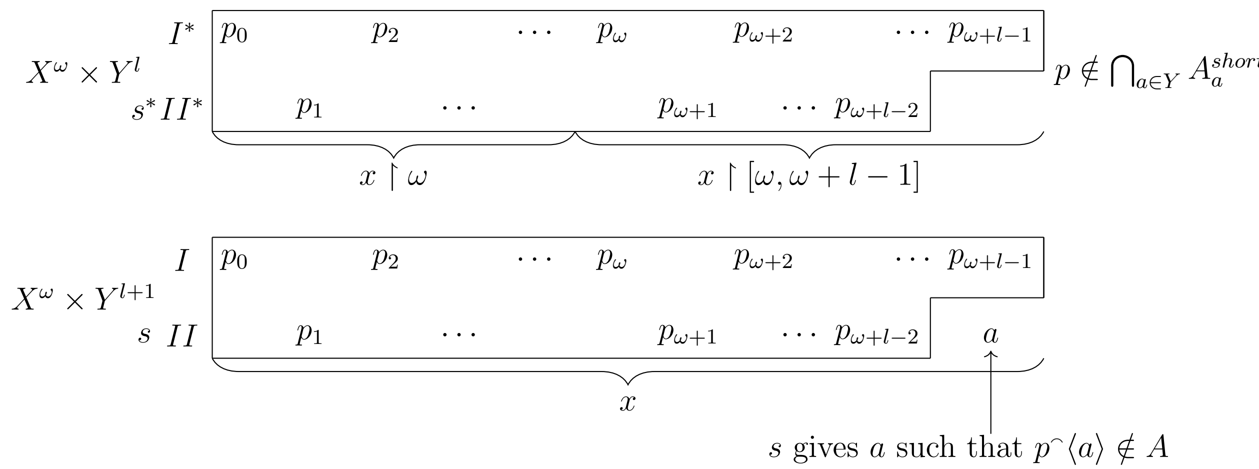

Figure 2.4.4: Illustration of $x \in X^\omega \times Y^{l+1}$ (l is odd) according to II's strategy s (corresponding to the ($\Rightarrow$) direction of the equivalence (2.6) on page 85).

Show s is a winning strategy for II for $G(A; X^\omega \times Y^{l+1})$. Pick an arbitrary x according to s. Then $x \upharpoonright (\omega + l)$ is according to s^*. Since s^* is a II^*'s winning strategy for $G(\bigcap_{a \in Y} A_a^{short}; X^\omega \times Y^l)$, $x \upharpoonright (\omega + l) \notin \bigcap_{a \in Y} A_a^{short}$. Since x is according to s, $x \upharpoonright (\omega + l) \notin A_{x(\omega+l)}^{short}$. Hence $x \notin A$. Therefore, $G(A; X^\omega \times Y^{l+1})$ is determined. $\square$

Recall that we are proving lemmas to obtain ($\Rightarrow$) direction of the equivalences (2.4) through (2.7) of the main lemma (lemma 2.4.6). Note that we have sublemmas 2.4.8 and 2.4.9, what remains is to compute the complexity of $(A)_a^{short}$ for $A \in \mathbf{\Sigma}_k^0 \upharpoonright X^\omega \times Y^{l+1}$ and $A \in \mathbf{\Pi}_k^0 \upharpoonright X^\omega \times Y^{l+1}$.

In sublemma 2.4.11, we will find the complexity of $(A)_a^{short}$ for for $A \in \mathbf{\Sigma}_k^0 \upharpoonright X^\omega \times Y^{l+1}$ and $A \in \mathbf{\Pi}_k^0 \upharpoonright X^\omega \times Y^{l+1}$, $a \in Y$. For X, Y and $a \in Y$, we first note that the function $A \mapsto (A)_a^{short}$ from subsets of $X^\omega \times Y^{l+1}$ to subsets of $X^\omega \times Y^l$ preserves complementation and unions:

Sublemma 2.4.10. *Let I be an index set. Suppose $l \in \omega$. For any $A, A_i \subseteq X^\omega \times Y^{l+1}$, $i \in I$,*

1. $\left((X^\omega \times Y^{l+1})\backslash A\right)_a^{short} = \left(X^\omega \times Y^l\right)\backslash A_a^{short}$.

2. $\left(\bigcup_{i\in I} A_i\right)_a^{short} = \bigcup_{i\in I} (A_i)_a^{short}$. ⊣

Proof.

Show (1).

$$\begin{aligned}\left(\left(X^\omega \times Y^{l+1}\right)\backslash A\right)_a^{short} &= \left\{f \in X^\omega \times Y^l \,\middle|\, f^\frown\langle a\rangle \in \left(X^\omega \times Y^{l+1}\right)\backslash A\right\} \\ &= \left(X^\omega \times Y^l\right)\backslash\left\{f \in X^\omega \times Y^l \,|\, f^\frown\langle a\rangle \in A\right\} \\ &= \left(X^\omega \times Y^l\right)\backslash A_a^{short}.\end{aligned}$$

Show (2).

$$\begin{aligned}\left(\bigcup_{i\in I} A_i\right)_a^{short} &= \left\{f \in X^\omega \times Y^l \,\middle|\, f\langle a\rangle \in \bigcup_{i\in I} A_i\right\} \\ &= \bigcup_{i\in I}\left\{f \in X^\omega \times Y^l \,|\, f\langle a\rangle \in A_i\right\} \\ &= \bigcup_{i\in I} (A_i)_a^{short}.\end{aligned}$$
□

By using sublemma 2.4.10, we will find the complexity of A_a^{short} for $A \in \mathbf{\Sigma}_k^0 \upharpoonright X^\omega \times Y^{l+1}$, $A \in \mathbf{\Pi}_k^0 \upharpoonright X^\omega \times Y^{l+1}$ and $A \in \mathbf{\Delta}_k^0 \upharpoonright X^\omega \times Y^{l+1}$ for any $a \in Y$.

Sublemma 2.4.11. *Suppose $l \in \omega$. Let $a \in Y$. Assume $A \subseteq X^\omega \times Y^{l+1}$. Then for any $k \in \omega$,*

1. *if $A \in \mathbf{\Sigma}_k^0 \upharpoonright X^\omega \times Y^{l+1}$, then each $A_a^{short} \in \mathbf{\Sigma}_k^0 \upharpoonright X^\omega \times Y^l$ and*

2. *if $A \in \mathbf{\Pi}_k^0 \upharpoonright X^\omega \times Y^{l+1}$, then each $A_a^{short} \in \mathbf{\Pi}_k^0 \upharpoonright X^\omega \times Y^l$.*

Consequently, if $A \in \mathbf{\Delta}_k^0 \upharpoonright X^\omega \times Y^{l+1}$, then each $A_a^{short} \in \mathbf{\Delta}_k^0 \upharpoonright X^\omega \times Y^l$. ⊣

Proof.

We prove both (1) and (2) simultaneously by induction on k. (2) follows from (1) and sublemma 2.4.10 (1).

Base Case : $k = 1$.

Show (1). Suppose $A \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^{l+1}$. Show $A^{short}_a \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l$. Pick an arbitrary $f \in A^{short}_a$. Then $f^\frown\langle a\rangle \in A$. Since A is open, there is a finite $E \subseteq \omega + l$ such that for any $g \in X^\omega \times Y^{l+1}$, if $g \supseteq f^\frown\langle a\rangle \restriction E$, then $g \in A$. Define $\hat{E} = E \backslash \{\omega + l\}$. Pick an arbitrary $\hat{f} \in X^\omega \times Y^l$ such that $\hat{f} \supseteq f \restriction \hat{E}$. Then $\hat{f}^\frown\langle a\rangle \supseteq f^\frown\langle a\rangle \restriction E$ so that $\hat{f}^\frown\langle a\rangle \in A$. Hence $\hat{f} \in A^{short}_a$. Thus $A^{short}_a \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l$.

Show (2). Suppose $A \in \mathbf{\Pi}^0_1 \restriction X^\omega \times Y^{l+1}$. Show $A^{short}_a \in \mathbf{\Pi}^0_1 \restriction X^\omega \times Y^l$. Since $A \in \mathbf{\Pi}^0_1 \restriction X^\omega \times Y^{l+1}$, $(X^\omega \times Y^{l+1})\backslash A \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^{l+1}$. By above (1), $\left((X^\omega \times Y^{l+1})\backslash A\right)^{short}_a \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l$. By sublemma 2.4.10 (1), $\left((X^\omega \times Y^{l+1})\backslash A\right)^{short}_a = \left(X^\omega \times Y^l\right) \backslash A^{short}_a$. Thus $A^{short}_a \in \mathbf{\Pi}^0_1 \restriction X^\omega \times Y^l$.

Induction Step : Assume as an induction hypothesis, if $A \in \mathbf{\Sigma}^0_k \restriction X^\omega \times Y^{l+1}$, then $A^{short}_a \in \mathbf{\Sigma}^0_k \restriction X^\omega \times Y^l$ and if $A \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^{l+1}$, then $A^{short}_a \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^l$. Show if $A \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^{l+1}$, then $A^{short}_a \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^l$ and if $A \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^{l+1}$, then $A^{short}_a \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^l$.

Show (1). Suppose $A \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^{l+1}$. Show $A^{short}_a \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^l$. Since $A \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^{l+1}$, there exists $\langle A_i \,|i \in \omega\rangle$ such that $A = \bigcup_{i\in\omega} A_i$ and each $A_i \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^{l+1}$. Since $A_i \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^{l+1}$, by induction hypothesis, $(A_i)^{short}_a \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^l$.

By sublemma 2.4.10 (2), $\left(\bigcup_{i\in\omega} A_i\right)_a^{short} = \bigcup_{i\in\omega} (A_i)_a^{short}$. Thus, we have

$$A_a^{short} = \left(\bigcup_{i\in\omega} A_i\right)_a^{short} = \bigcup_{i\in\omega} (A_i)_a^{short} \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^l.$$

Show (2). We repeat the same proof the base case for (2). Suppose $A \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^{l+1}$. Show $A_a^{short} \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^l$. Since $A \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^{l+1}$, $(X^\omega \times Y^{l+1})\backslash A \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^{l+1}$. By above (1), $\left((X^\omega \times Y^{l+1})\backslash A\right)_a^{short} \in \mathbf{\Sigma}^0_{k+1} \restriction X^\omega \times Y^l$. By sublemma 2.4.10 (1), $\left((X^\omega \times Y^{l+1})\backslash A\right)_a^{short} = \left(X^\omega \times Y^l\right)\backslash A_a^{short}$. Thus $A_a^{short} \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^l$.

Consequently, if $A \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^{l+1}$, then each $A_a^{short} \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^l$. □

Now, we prove the (⇒) direction of the main lemma (lemma 2.4.6) from the four sublemmas 2.4.8, 2.4.9 2.4.10 and 2.4.11.

Proof of (⇒) *direction of the main lemma (lemma 2.4.6 on page 85).*

Suppose Y is denumerable. Fix $m, l \in \omega$.

Suppose and l is even.

Show the (⇒) direction of the equivalence (2.4) :

$$Det\left(\mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l\right) \Rightarrow Det\left(\mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right).$$

Assume $Det\left(\mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l\right)$. Pick an arbitrary $A \in \mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}$. Then by sublemma 2.4.11, for all $a \in Y$, $A_a^{short} \in \mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^l$. Since Y is denumerable, $\bigcup_{a\in Y} A_a^{short} \in \mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l$. Then $G(\bigcup_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined. By sublemma 2.4.8, $G(A; X^\omega \times Y^{l+1})$ is determined.

A similar proof works for the (⇒) direction of the equivalence (2.5) :

$$Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l\right) \Rightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right).$$

Simply replace $\mathbf{\Sigma}^0_m$ by $\mathbf{\Sigma}^0_1$ and replace $\mathbf{\Pi}^0_{m-1}$ by $\mathbf{\Delta}^0_1$.

Suppose l is odd.

Show the ($\Rightarrow$) direction of the equivalence (2.6) :

$$Det\left(\mathbf{\Pi}^0_m \restriction X^\omega \times Y^l\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right).$$

Assume $Det\left(\mathbf{\Pi}^0_m \restriction X^\omega \times Y^l\right)$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}$. Then by sublemma 2.4.11, for all $a \in Y$, $A_a^{short} \in \mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^l$. Since Y is denumerable, $\bigcap_{a\in Y} A_a^{short} \in \mathbf{\Pi}^0_m \restriction X^\omega \times Y^l$. Then $G(\bigcap_{a\in Y} A_a^{short}; X^\omega \times Y^l)$ is determined. By sublemma 2.4.9, $G(A; X^\omega \times Y^{l+1})$ is determined.

A similar proof works for the ($\Rightarrow$) direction of the equivalence (2.7) :

$$Det\left(\mathbf{\Pi}^0_1 \restriction X^\omega \times Y^l\right) \Rightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right).$$

Simply replace $\mathbf{\Pi}^0_m$ by $\mathbf{\Pi}^0_1$ and replace $\mathbf{\Sigma}^0_{m-1}$ by $\mathbf{\Delta}^0_1$. □

Now that we have completed the proof of the ($\Rightarrow$) direction of the main lemma (lemma 2.4.6 on page 85), we will prove ($\Leftarrow$) direction of the main lemma (lemma 2.4.6) on page 101 after proving some sublemmas. Towards the proof of ($\Leftarrow$) direction, we will define $\langle A_a \,|a \in Y\rangle^{long}$. We will use sublemma 2.4.13 to show the remaining direction of the equivalences (2.4) and (2.5). We will use sublemma 2.4.14 to show the remaining direction of the equivalences (2.6) and (2.7). In sublemma 2.4.16, we will find complexity of $\langle A_a \,|a \in Y\rangle^{long}$ for the relevant A_a's.

Definition 2.4.12. *Suppose $l \in \omega$. Let $A_a \subseteq X^\omega \times Y^l$ for all $a \in Y$. Define*

$$\langle A_a \,|a \in Y \rangle^{long} = \left\{ h \in X^\omega \times Y^{l+1} \,\middle|\, h \restriction (\omega + l) \in A_{h(\omega+l)} \right\}. \qquad \dashv$$

Sublemma 2.4.13. *Suppose $l \in \omega$ is even and $A = \bigcup_{a \in Y} A_a \subseteq X^\omega \times Y^l$.*

If $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$ is determined, then $G(A; X^\omega \times Y^l)$ is determined. $\dashv$

Proof.

Assume $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$ is determined. Then I^* or II^* has a winning strategy s^* for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$.

Case 1 : s^* is a I^*'s winning strategy for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$.

Define I's strategy for $G(A; X^\omega \times Y^l)$ by $s = s^* \restriction T$ where $[T] = X^\omega \times Y^l$. [17]

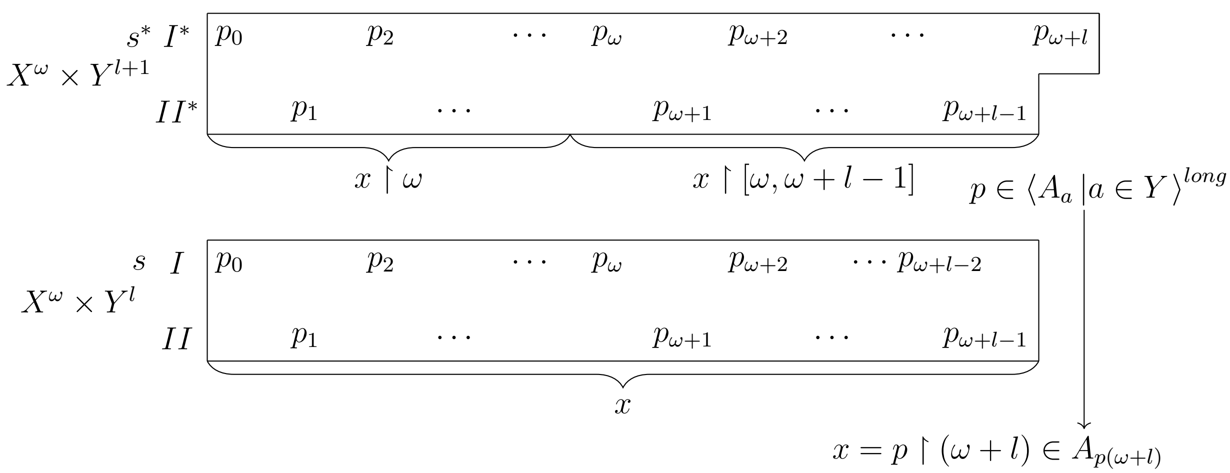

Figure 2.4.5: Illustration of $x \in X^\omega \times Y^l$ (l is even) according to I's strategy s (corresponding to the ($\Leftarrow$) direction of the equivalence (2.4) on page 85).

Show s is a winning strategy for I for $G(A; X^\omega \times Y^l)$. Pick an arbitrary $x \in X^\omega \times Y^l$ according to s. Then $x^\frown s^*(x)$ is according to s^* so that $s^*(x) \in Y$.

Since s^* is a I^*'s winning strategy for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$,

$x^\frown s^*(x) \in \langle A_a \,|a \in Y \rangle^{long}$. Hence $x \in A_{s^*(x)} \subseteq A$.

[17] $s^* \restriction T$ abbreviates $s^* \restriction (T \cap dom(s^*))$.

Case 2 : s^* is a II^*'s winning strategy for $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$.

Define II's strategy for $G(A; X^\omega \times Y^l)$ by $s = s^*$.

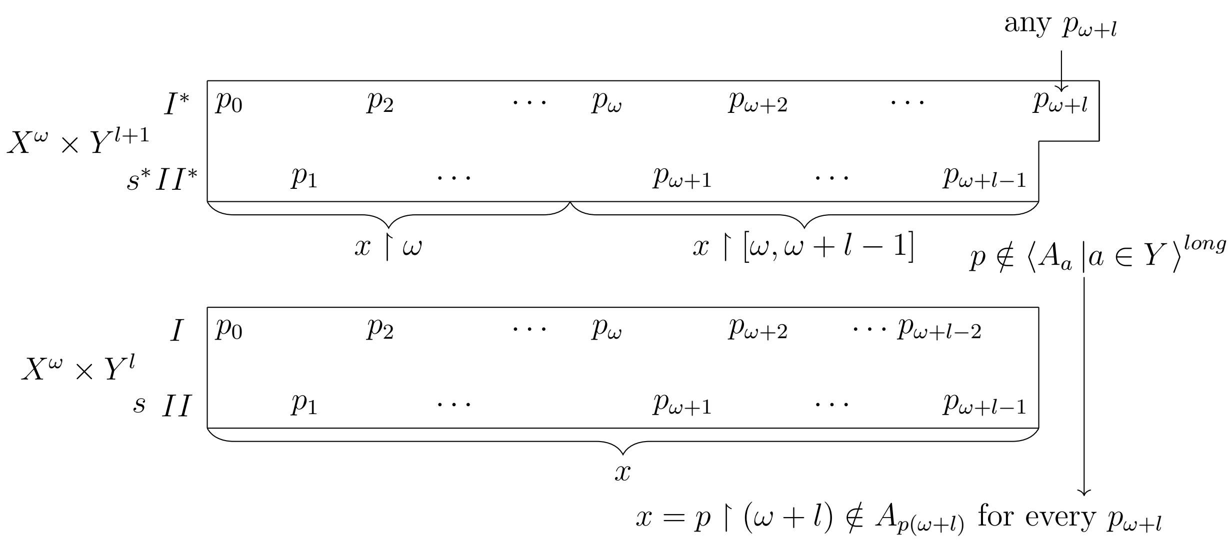

Figure 2.4.6: Illustration of $x \in X^\omega \times Y^l$ (l is even) according to II's strategy s (corresponding to the ($\Leftarrow$) direction of the equivalence (2.4) on page 85).

Show s is a winning strategy for II for $G(A; X^\omega \times Y^l)$. Pick an arbitrary $x \in X^\omega \times Y^l$ according to s. Then for any $a \in Y$, $x^\frown\langle a \rangle$ is according to s^*. Since s^* is a II^*'s winning strategy for $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$, for all $a \in Y$, $x^\frown\langle a \rangle \notin \langle A_a \,|a \in Y\,\rangle^{long}$. Hence for all $a \in Y$, $x \notin A_a$. Thus $x \notin \bigcup_{a \in Y} A_a = A$. Therefore, $G(A; X^\omega \times Y^l)$ is determined. □

Sublemma 2.4.14. *Suppose l is odd and $A = \bigcap_{a \in Y} A_a \subseteq X^\omega \times Y^l$.*

If $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$ is determined, then $G(A; X^\omega \times Y^l)$ is determined. ⊣

Proof.

Assume $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$ is determined. Then I^* or II^* has a winning strategy s^* for $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$.

Case 1 : s^* is a I^*'s winning strategy for $G(\langle A_a \,|a \in Y\,\rangle^{long}; X^\omega \times Y^{l+1})$.

Define I's strategy for $G(A; X^\omega \times Y^l)$ by $s = s^*$.

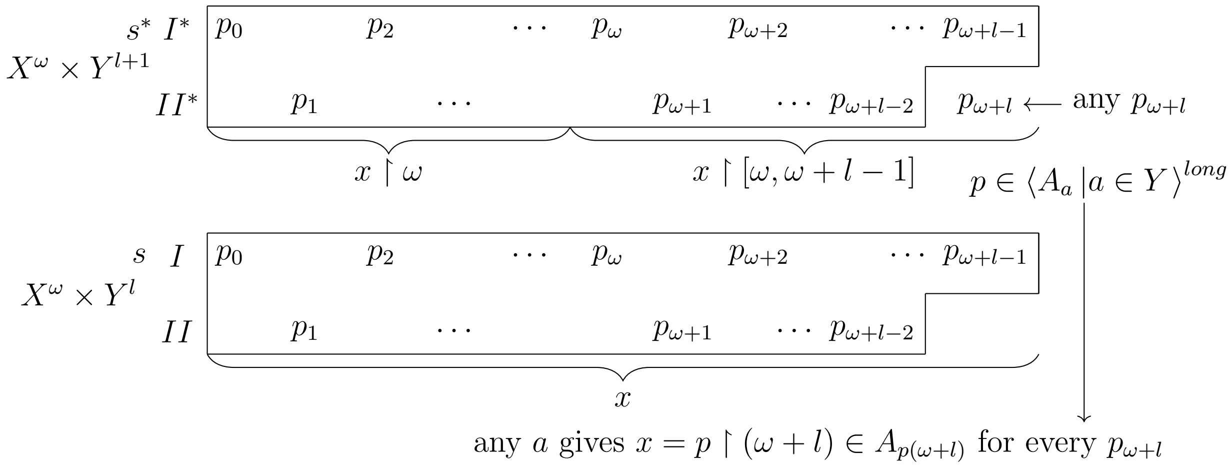

Figure 2.4.7: Illustration of $x \in X^\omega \times Y^l$ (l is odd) according to I's strategy s (corresponding to the ($\Leftarrow$) direction of the equivalence (2.6) on page 85).

Show s is a winning strategy for I for $G(A; X^\omega \times Y^l)$. Pick an arbitrary $x \in X^\omega \times Y^l$ according to s. Then for any $a \in Y$, $x^\frown\langle a\rangle$ is according to s^*. Since s^* is a I^*'s winning strategy for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$, for all $a \in Y$, $x^\frown\langle a\rangle \in \langle A_a \,|a \in Y \rangle^{long}$. Hence $x \in \bigcap_{a \in Y} A_a = A$.

Case 2 : s^* is a II^*'s winning strategy for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$.

Define II's strategy for $G(A; X^\omega \times \omega^l)$ by $s = s^* \restriction T$ where $[T] = X^\omega \times Y^l$. [18]

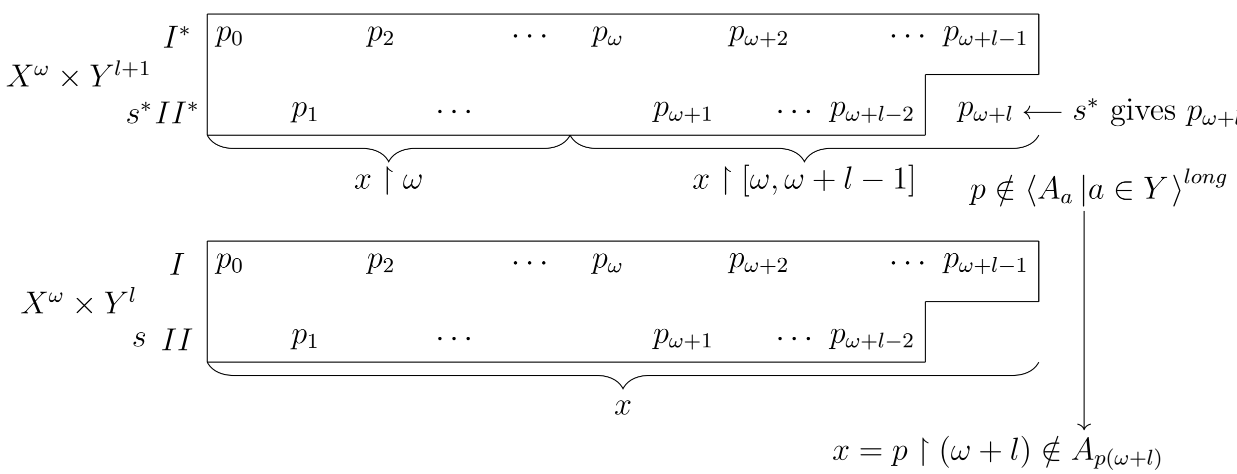

Figure 2.4.8: Illustration of $x \in X^\omega \times Y^l$ (l is odd) according to II's strategy s (corresponding to the ($\Leftarrow$) direction of the equivalence (2.6) on page 85).

[18] $s^* \restriction T$ abbreviates $s^* \restriction (T \cap dom(s^*))$.

Show s is a winning strategy for II for $G(A; X^\omega \times Y^l)$. Pick an arbitrary $x \in X^\omega \times Y^l$ according to s. Then $x^\frown s^*(x)$ is according to s^*. Since s^* is a II^*'s winning strategy for $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$, $x^\frown s^*(x) \notin \langle A_a \,|a \in Y \rangle^{long}$. Hence $x \notin A_{s^*(x)}$. Thus $x \notin \bigcap_{a \in Y} A_a = A$. Therefore, $G(A; X^\omega \times Y^l)$ is determined. □

Recall that we are proving lemmas to obtain the ($\Leftarrow$) direction of the equivalences (2.4) through (2.7) of the main lemma (lemma 2.4.6) on 85. Now that we have sublemmas 2.4.13 and 2.4.14, we next compute the complexity of $\langle A_a \,|a \in Y \rangle^{long}$ for relevant A_a's.

In sublemma 2.4.16, we will find the complexity of $\langle A_a \,|a \in Y \rangle^{long}$. For X, Y, we first note that the function $\langle A_a \,|a \in Y \rangle \mapsto \langle A_a \,|a \in Y \rangle^{long}$ from a sequence of subsets of $X^\omega \times Y^l$ to a subset of $X^\omega \times Y^{l+1}$ preserves complementation and unions:

Sublemma 2.4.15. *Let I be an index set. Suppose $l \in \omega$. Let $A_a, A_a^i \subseteq X^\omega \times Y^l$ for all $a \in Y$ and $i \in \omega$. Then*

1. $\left\langle \left(X^\omega \times Y^{l+1}\right) \setminus A_a \middle| \ a \in Y \right\rangle^{long} = \left(X^\omega \times Y^{l+1}\right) \setminus \langle A_a \,|a \in Y \rangle^{long}$.
2. $\left\langle \bigcup_{i \in I} A_a^i \,|a \in Y \right\rangle^{long} = \bigcup_{i \in I} \left(\langle A_a^i \,|a \in Y \rangle^{long} \right)$. ⊣

Proof.

Show (1).

$$\begin{aligned}\left\langle \left(X^\omega \times Y^{l+1}\right) \setminus A_a \,|a \in Y \right\rangle^{long} &= \left\{ h \in X^\omega \times Y^{l+1} \,\middle|\, h \restriction (\omega + l) \in \left(X^\omega \times Y^{l+1}\right) \setminus A_{h(\omega+l)} \right\} \\ &= \left(X^\omega \times Y^{l+1}\right) \setminus \left\{ h \in X^\omega \times Y^{l+1} \,\middle|\, h \restriction (\omega + l) \in A_{h(\omega+l)} \right\} \\ &= \left(X^\omega \times Y^{l+1}\right) \setminus \langle A_a \,|a \in Y \rangle^{long}.\end{aligned}$$

Show (2).

$$\left\langle \bigcup_{i\in I} A^i_a \,|a \in Y \right\rangle^{long} = \left\{ h \in X^\omega \times Y^{l+1} \,\middle|\, h \restriction (\omega + l) \in \bigcup_{i\in I} A^i_{h(\omega+l)} \right\}$$
$$= \bigcup_{i\in I} \left\{ h \in X^\omega \times Y^{l+1} \,\middle|\, h \restriction (\omega + l) \in A^i_{h(\omega+l)} \right\}$$
$$= \bigcup_{i\in I} \left(\langle A^i_a \,|a \in Y \rangle^{long} \right). \qquad \square$$

By using sublemma 2.4.15, we will find the complexity of $\langle A_a \,|a \in Y\rangle^{long}$ for the following cases: for all $a \in Y$, $A_a \in \mathbf{\Sigma}^0_k \restriction X^\omega \times Y^{l+1}$; for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^{l+1}$; and for all $a \in Y$, $A_a \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^{l+1}$.

Sublemma 2.4.16. *Suppose $l \in \omega$. Let $A_a \subseteq X^\omega \times Y^l$ for all $a \in Y$. Then for any $k \in \omega$:*

1. *If for all $a \in Y$, $A_a \in \mathbf{\Sigma}^0_k \restriction X^\omega \times Y^l$, then $\langle A_a \,|a \in Y\rangle^{long} \in \mathbf{\Sigma}^0_k \restriction X^\omega \times Y^{l+1}$.*

2. *If for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^l$, then $\langle A_a \,|a \in Y\rangle^{long} \in \mathbf{\Pi}^0_k \restriction X^\omega \times Y^{l+1}$.*

Consequently, if for all $a \in Y$, $A_a \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^l$, then $\langle A_a \,|a \in Y\rangle^{long} \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^{l+1}$.

$\dashv$

Proof.

We prove both (1) and (2) simultaneously by induction on k. By sublemma 2.4.15 (1), (2) follows from (1).

Base Case : $k = 1$.

Show (1). Suppose for all $a \in Y$, $A_a \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l$. Show $\langle A_a \,|a \in Y\rangle^{long} \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^{l+1}$. Pick an arbitrary $h \in \langle A_a \,|a \in Y\rangle^{long}$. Since $h(\omega + l) \in Y$, $h \restriction (\omega + l) \in A_{h(\omega+l)}$. Since $A_{h(\omega+l)}$ is open, there is a finite $E \subseteq \omega + l - 1$ such that for any $g \in X^\omega \times Y^l$, if $g \supseteq h \restriction E$, then $g \in A_{h(\omega+l)}$. Define $\hat{E} = E \cup \{\omega + l\}$. Pick an arbitrary $\hat{h} \in X^\omega \times Y^{l+1}$

such that $\hat{h} \supseteq h \upharpoonright \hat{E}$. Then $\hat{h} \upharpoonright (\omega + l) \in X^\omega \times Y^l$ and $\hat{h} \upharpoonright (\omega + l) \supseteq h \upharpoonright E$. Thus $\hat{h} \upharpoonright (\omega + l) \in A_{h(\omega+l)}$. Since $\hat{h} \supseteq h \upharpoonright \hat{E}$, $\hat{h}(\omega + l) = h(\omega + l)$. Hence $\hat{h} \upharpoonright (\omega + l) \in A_{\hat{h}(\omega+l)}$. Therefore, $\hat{h} \in \langle A_a \,|a \in Y \rangle^{long}$. Thus $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega \times Y^{l+1}$.

Show (2). Suppose for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_1 \upharpoonright X^\omega \times Y^l$. Show $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_1 \upharpoonright X^\omega \times Y^{l+1}$. Since for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_1 \upharpoonright X^\omega \times Y^l$, $(X^\omega \times Y^l)\backslash A_a \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega \times Y^l$ for all $a \in Y$. By above (1), $\left\langle \left(X^\omega \times Y^{l+1}\right) \backslash A_a \,|a \in Y \right\rangle^{long} \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega \times Y^{l+1}$. By sublemma 2.4.15, $\left\langle \left(X^\omega \times Y^{l+1}\right) \backslash A_a \,|a \in Y \right\rangle^{long} = \left(X^\omega \times Y^{l+1}\right) \backslash \langle A_a \,|a \in Y \rangle^{long}$. Thus $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_1 \upharpoonright X^\omega \times Y^{l+1}$.

Induction Step : As an induction hypothesis, assume for all $a \in Y$, if $A_a \in \mathbf{\Sigma}^0_k \upharpoonright X^\omega \times Y^l$, then $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Sigma}^0_k \upharpoonright X^\omega \times Y^{l+1}$ and for all $a \in Y$, if $A_a \in \mathbf{\Pi}^0_k \upharpoonright X^\omega \times Y^l$, then $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_k \upharpoonright X^\omega \times Y^{l+1}$.

Show(1). Suppose for all $a \in Y$, $A_a \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^l$. Show $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^{l+1}$. Since for all $a \in Y$, $A_a \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^l$, for each $a \in Y$, there exists $\langle A^i_a \,|i \in \omega \rangle$ such that $A_a = \bigcup_{i\in\omega} A^i_a$ and each $A^i_a \in \mathbf{\Pi}^0_k \upharpoonright X^\omega \times Y^l$. Since each $A^i_a \in \mathbf{\Pi}^0_k \upharpoonright X^\omega \times Y^l$, by induction hypothesis, $\langle A^i_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_k \upharpoonright X^\omega \times Y^{l+1}$. By sublemma 2.4.15, $\left\langle \bigcup_{i\in\omega} A^i_a \,|a \in Y \right\rangle^{long} = \bigcup_{i\in\omega} \langle A^i_a \,|a \in Y \rangle^{long}$. Thus

$$\langle A_a \,|a \in Y \rangle^{long} = \left\langle \bigcup_{i\in\omega} A^i_a \,|a \in Y \right\rangle^{long} = \bigcup_{i\in\omega} \left\langle A^i_a \,|a \in Y \right\rangle^{long} \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^{l+1}.$$

Show(2). We repeat the same proof of the base case for (2). Suppose for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_{k+1} \upharpoonright X^\omega \times Y^l$. Show $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_{k+1} \upharpoonright X^\omega \times Y^{l+1}$. Since for all $a \in Y$, $A_a \in \mathbf{\Pi}^0_{k+1} \upharpoonright X^\omega \times Y^l$, $(X^\omega \times Y^l)\backslash A_a \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^l$ for all $a \in Y$. By above (1), $\left\langle \left(X^\omega \times Y^{l+1}\right) \backslash A_a \,|a \in Y \right\rangle^{long} \in \mathbf{\Sigma}^0_{k+1} \upharpoonright X^\omega \times Y^{l+1}$. By sublemma 2.4.15,

$\left\langle \left(X^\omega \times Y^{l+1}\right) \setminus A_a \,|a \in Y \right\rangle^{long} = \left(X^\omega \times Y^{l+1}\right) \setminus \langle A_a \,|a \in Y \rangle^{long}$. Thus $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_{k+1} \restriction X^\omega \times Y^{l+1}$.

Consequently, if for all $a \in Y$, $A_a \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^l$, then $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Delta}^0_k \restriction X^\omega \times Y^{l+1}$. $\square$

Now, we prove ($\Leftarrow$) direction of the main lemma (lemma 2.4.6) on page 85 from sublemmas 2.4.13, 2.4.14 2.4.15 and 2.4.16.

Proof of the ($\Leftarrow$) direction of the main lemma (lemma 2.4.6) on page 85.

Suppose Y is denumerable. Fix $m, l \in \omega$.

Assume l is even. Show the ($\Leftarrow$) direction of the equivalence (2.4) :

$$Det\left(\mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right).$$

Recall we already have ($\Rightarrow$) direction on page 93.

($\Leftarrow$) Assume $Det\left(\mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right)$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_m \restriction X^\omega \times Y^l$. Since Y is denumerable and A could be written as a denumerable union of $\mathbf{\Pi}^0_{m-1}$ sets, there exists $\langle A_a \,|a \in Y \rangle$ such that $A = \bigcup_{a \in Y} A_a$ and each $A_a \in \mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^l$. Then by sublemma 2.4.16, $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Pi}^0_{m-1} \restriction X^\omega \times Y^{l+1}$. Thus $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$ is determined. By sublemma 2.4.13, $G(A; X^\omega \times Y^l)$ is determined.

A similar proof works for the ($\Leftarrow$) direction of the equivalence (2.5):

$$Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^l\right).$$

Simply replace $\mathbf{\Sigma}^0_m$ by $\mathbf{\Sigma}^0_1$ and replace $\mathbf{\Pi}^0_{m-1}$ by $\mathbf{\Delta}^0_1$.

Suppose l is odd. Show the ($\Leftarrow$) direction of the equivalence (2.6) :

$$Det\left(\mathbf{\Pi}^0_m \restriction X^\omega \times Y^l\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right).$$

Recall we already have ($\Rightarrow$) direction on page 94.

($\Leftarrow$) Assume $Det\left(\mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}\right)$. Pick an arbitrary $A \in \mathbf{\Pi}^0_m \restriction X^\omega \times Y^l$. Since Y is denumerable and A could be written as a denumerable intersection of $\mathbf{\Sigma}^0_{m-1}$ sets, there exists $\langle A_a \,|a \in Y \rangle$ such that $A = \bigcap_{a\in Y} A_a$ and each $A_a \in \mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^l$. Then by sublemma 2.4.16, $\langle A_a \,|a \in Y \rangle^{long} \in \mathbf{\Sigma}^0_{m-1} \restriction X^\omega \times Y^{l+1}$. Thus $G(\langle A_a \,|a \in Y \rangle^{long}; X^\omega \times Y^{l+1})$ is determined. By sublemma 2.4.14, $G(A; X^\omega \times Y^l)$ is determined.

A similar proof works for the ($\Leftarrow$) direction of the equivalence (2.7):

$$Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^{l+1}\right) \Rightarrow Det\left(\mathbf{\Pi}^0_1 \restriction X^\omega \times Y^l\right).$$

Simply replace $\mathbf{\Pi}^0_m$ by $\mathbf{\Pi}^0_1$ and replace $\mathbf{\Sigma}^0_{m-1}$ by $\mathbf{\Delta}^0_1$. □

By proofs in 93 and 101, we have lemma 2.4.6.

Now we will show theorem 2.4.5 by repeated application of the main lemma (lemma 2.4.6 on page 85). Recall theorem 2.4.5.

Theorem 2.4.5. *ZF-P(folklore)*

Suppose $n \in \omega$ and Y is denumerable. Then for any $n \in \omega$,

$$Det\left(\mathbf{\Sigma}^0_n \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^n\right). \qquad \dashv$$

Proof of theorem 2.4.5.

Pick an arbitrary $n \in \omega$.

By lemma 2.4.6 the equivalence (2.4) on page 85 with $l = 0$, we have

$$Det\left(\mathbf{\Sigma}_n^0 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Pi}_{n-1}^0 \upharpoonright X^\omega \times Y\right).$$

By lemma 2.4.6 the equivalence (2.6) on page 85 with $l = 1$, we have

$$Det\left(\mathbf{\Pi}_{n-1}^0 \upharpoonright X^\omega \times Y\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_{n-2}^0 \upharpoonright X^\omega \times Y^2\right).$$

Continue applying lemma 2.4.6 the equivalences (2.4) and (2.6) alternately.

Case 1 : n is even.

By lemma 2.4.6 the equivalence (2.4) with $l = n - 2$, we have

$$Det\left(\mathbf{\Sigma}_2^0 \upharpoonright X^\omega \times Y^{n-2}\right) \Leftrightarrow Det\left(\mathbf{\Pi}_1^0 \upharpoonright X^\omega \times Y^{n-1}\right).$$

By lemma 2.4.6 the equivalence (2.7) with $l = n - 1$, we have

$$Det\left(\mathbf{\Pi}_1^0 \upharpoonright X^\omega \times Y^{n-1}\right) \Leftrightarrow Det\left(\mathbf{\Delta}_1^0 \upharpoonright X^\omega \times Y^n\right).$$

Consequently, we have

$$Det\left(\mathbf{\Sigma}_n^0 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Pi}_{n-1}^0 \upharpoonright X^\omega \times Y\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_{n-2}^0 \upharpoonright X^\omega \times Y^2\right) \Leftrightarrow \cdots$$
$$\Leftrightarrow Det\left(\mathbf{\Sigma}_2^0 \upharpoonright X^\omega \times Y^{n-2}\right) \Leftrightarrow Det\left(\mathbf{\Pi}_1^0 \upharpoonright X^\omega \times Y^{n-1}\right) \Leftrightarrow Det\left(\mathbf{\Delta}_1^0 \upharpoonright X^\omega \times Y^n\right).$$

Case 2 : n is odd.

By lemma 2.4.6 the equivalence (2.7) with $l = n - 2$, we have

$$Det\left(\mathbf{\Pi}_2^0 \upharpoonright X^\omega \times Y^{n-2}\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright X^\omega \times Y^{n-1}\right).$$

By lemma 2.4.6 the equivalence (2.5) with $l = n - 1$, we have

$$Det\left(\mathbf{\Sigma}_1^0 \upharpoonright X^\omega \times Y^{n-1}\right) \Leftrightarrow Det\left(\mathbf{\Delta}_1^0 \upharpoonright X^\omega \times Y^n\right).$$

Consequently, we have

$$Det\left(\mathbf{\Sigma}^0_n \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Pi}^0_{n-1} \restriction X^\omega \times Y\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_{n-2} \restriction X^\omega \times Y^2\right) \Leftrightarrow$$

$$\cdots \Leftrightarrow Det\left(\mathbf{\Pi}^0_2 \restriction X^\omega \times Y^{n-2}\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \times Y^{n-1}\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^n\right). \qquad \square$$

Finally, by using the general results for the finite Borel sets for fixed length, we will prove the main theorem. Recall theorem 2.4.4.

Theorem 2.4.4. *(ZF-P)*

Suppose Γ *contains all constant functions from* X^ω *into* ω *and* Y *is denumerable. Then* $Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$ *implies* $Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right)$*, finite Borel determinacy on* $X^{<\omega}$*.* ⊣

Proof of theorem 2.4.4.

Assume Y is denumerable and $Det\left(\mathbf{\Delta}^0_1; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$. Pick an arbitrary $A \in \bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega$. Then there exists $n \in \omega$ such that $A \in \mathbf{\Sigma}^0_n \restriction X^\omega$. By theorem 2.4.5,

$$Det\left(\mathbf{\Sigma}^0_n \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times Y^n\right).$$

By observation 2.4.3, we have $Det\left(\bigcup_{n\in\omega} \mathbf{\Delta}^0_1 \restriction (X^\omega \times Y^n)\right)$. Thus $G(A; X^{<\omega})$ is determined. Hence $Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right)$. $\square$

Corollary 2.4.17.

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right).$$ [19] ⊣

Proof.

Since each constant functions is continuous, $\Gamma(\omega, \mathbf{\Delta}^0_1)$ contains all the constant functions

[19] Recall notation 1.5.10 for CTB and notation 1.5.8 for $\Gamma(\omega, \mathbf{\Delta}^0_1)$.

from X^ω into ω. Also $\emptyset, X^\omega \in \mathbf{\Delta}^0_1 \upharpoonright X^\omega$. Thus, we have the result by theorem 2.4.4. $\square$

2.4.2 Using $\mathbf{\Sigma}_1^0$ determinacy on a $Tree_1$ collection to obtain the determinacy of games on $X^{<\omega}$

In this section, we will obtain the determinacy of games on $X^{<\omega}$ from the determinacy of open games on a $Tree_2$ collection.

In section 2.4.2.1, we will define a open set *Long* on a Type 1 tree. *Long* includes all plays of the tree which have length greater than ω and excludes those of length ω.

In section 2.4.2.2, we will define an open set Max on a Type 1 tree. Max is defined only on Type 1 trees with paths having maximum length. We will obtain the determinacy results using Max in sections 2.4.2.3 and 2.4.2.4.

2.4.2.1 Definition of the open set $Long\,(B)$ on the body of a Type 1 tree and using the determinacy equivalence of open games $Long(A)$ on a $Tree_1$ collection to obtain the determinacy of games A on X^{ω}

In this section, we will define the open set *Long* on a Type 2 tree and obtain the determinacy of games A on $X^{<\omega}$ from the determinacy of open games $Long(A)$ on a $Tree_2$ collection.

For this section, it will be convenient to consider for any Type 1 tree T, the length function $lh_{[T]}$.

Definition 2.4.18. *(Definition of the length function $lh_{[T]}$)*

$$\begin{aligned} lh_{[T]}: \quad & [T] \to \omega + \omega \\ & h \mapsto lh\,(h)\,. \end{aligned}$$

⊣

In this section, we will define the open set *Long* on a Type 1 tree. By definition of

$T = T^{\Psi,B}_{X,Y}$, we can split the body of the tree into two pieces, the "short" piece, $lh^{-1}_{[T]}(\omega)$, which consists of paths of length ω, and the "long" piece which consists of paths of length greater than ω. *Long* is the subset of the body of a Type 1 tree that consists of all plays of length greater than ω. By using this open set, *Long*, we will obtain the determinacy of games on $X^{<\omega}$.

Definition 2.4.19. *Suppose $B \subseteq X^\omega$, Ψ is a function from X^ω into ω and Y is arbitrary. Define*

$$Long\,(B) = \{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) > \omega\}. \qquad \dashv$$

Then $Long\,(B) = \{h \in [T^{\Psi,B}_{X,Y}] \mid h \restriction \omega \in B\}$. This set is the set of plays of length longer than ω. It is easy to see that $Long(B)$ is open in $[T^{\Psi,B}_{X,Y}]$ for any $B \subseteq X^\omega$ by taking the finite set to be $\{\omega\}$: for any $h \in Long(B)$, every $g \in [T^{\Psi,B}_{X,Y}]$ with $g \supseteq h \restriction \{\omega\}$ has $\omega \in dom(g)$ and thus it is in $Long(B)$. $Long(B)$ is open in $[T^{\Psi,B}_{X,Y}]$ even if B is a collection of nondetermined sets. Also, the complement of $Long(B)$;

$$[T^{\Psi,B}_{X,Y}] \backslash Long\,(B) = \{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) = \omega\} = \{h \in [T^{\Psi,B}_{X,Y}] \mid h \restriction \omega \in X^\omega \backslash B\}$$

is closed in $[T^{\Psi,B}_{X,Y}]$ for any $B \subseteq X^\omega$. Note that the complement of $Long\,(B)$ is open in $[T^{\Psi,B}_{X,Y}]$ if and only if B is closed in X^ω. Hence $Long\,(B)$ is clopen if and only if B is closed. In general, for $n \in \omega$, $\{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) > \omega + n\}$ is open.

Theorem 2.4.20. *For any X, Y, $\Psi : X^\omega \to \omega$, for any $A \subseteq X^\omega$,*

$G\,(A; X^\omega)$ is determined if and only if $G(Long\,(A)\,; T^{\Psi,A}_{X,Y})$ is determined. $\dashv$

Proof.

Fix X, Y, $\Psi : X^\omega \to \omega$ and $A \subseteq X^\omega$ arbitrary. Recall

$$h \in \left[T_{X,Y}^{\Psi,A}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin A, \\ h \in X^\omega \times Y^{\Psi(h\restriction\omega)+1} & \text{if } h \restriction \omega \in A. \end{cases}$$

($\Leftarrow$) Assume $G(A; X^\omega)$ is determined. Then I or II has a winning strategy s for $G(A; X^\omega)$. Define s^* to be such that $s^* \restriction X^{<\omega} = s$ and play anything after that to finish the play. Show s^* is a winning strategy for $G(Long(A); T_{X,Y}^{\Psi,A})$. Pick an arbitrary $f \in [T_{X,Y}^{\Psi,A}]$ according to s^*.

Case I : s is a winning strategy for I.

Since $f \restriction \omega$ is according to s, $f \restriction \omega \in A$ so that $lh(f) > \omega$. Thus $f \in Long(A)$.

Case II : s is a winning strategy for II.

Since $f \restriction \omega$ is according to s, $f \restriction \omega \notin A$ so that $lh(f) = \omega$. Thus $f \notin Long(A)$. Hence $G(Long(A); T_{X,Y}^{\Psi,A})$ is determined.

($\Rightarrow$) Assume $G(Long(A);\ T_{X,Y}^{\Psi,A})$ is determined. Then I^* or II^* has a winning strategy s^* for $G(Long(A); T_{X,Y}^{\Psi,A})$. Define $s = s^* \restriction X^{<\omega}$. [20] Show s is a winning strategy. for $G(A; X^\omega)$. Pick an arbitrary $f \in X^\omega$ according to s. Play according to s^* after f, call it g, until $f^\frown g \in [T_{X,Y}^{\Psi,A}]$.

Case I : s^* is a winning strategy for I^*.

Then $f^\frown g \in Long(A)$ so that $g \neq \emptyset$, i.e., $f \in A$. Hence s is a winning strategy for I for $G(A; X^\omega)$.

Case II : s^* is a winning strategy for II^*.

Then $f^\frown g \notin Long(A)$ so that $f^\frown g \in X^\omega$, i.e., $g = \emptyset$, hence $f \notin A$. Hence s is a winning

[20] $s^* \restriction X^{<\omega}$ abbreviates $s^* \restriction (X^{<\omega} \cap dom(s^*))$.

strategy for II for $G(A; X^\omega)$. Thus $G(A; X^\omega)$ is determined. $\square$

The following three corollaries follow from theorem 2.4.20.

Corollary 2.4.21. *For any* X, Y, $\Psi : X^\omega \to \omega$ *and* Λ,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \Lambda \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\Lambda \upharpoonright X^\omega\right). \qquad \dashv$$

Proof.

Pick an arbitrary $A \in \Lambda \upharpoonright X^\omega$. Fix X, Y and $\Psi : X^\omega \to \omega$. Then, $Long(A) \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,A}_{X,Y}]$. Thus $G(Long(A); T^{\Psi,A}_{X,Y})$ is determined. By theorem 2.4.20, $G(A; X^{<\omega})$ is determined. $\square$

Corollary 2.4.22. *(Corollary to Corollary 2.4.21)*

For any $\alpha \in \omega_1$, $n \in \omega$, X, Y *and* $\Psi : X^\omega \to \omega$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \mathbf{\Sigma}^1_n \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \upharpoonright X^\omega\right). \qquad \dashv$$

Proof.

By corollary 2.4.21 with $\Lambda = \mathbf{\Sigma}^0_\alpha$ and $\Lambda = \mathbf{\Sigma}^1_n$. $\square$

Corollary 2.4.23. *(Corollary to Corollary 2.4.21)*

Suppose X *is a nonempty set,* Θ *is an arbitrary collection of sets,* Γ *is any collection of functions from* X^ω *into* ω *and* Λ *is a collection of nondetermined sets on* X^ω*. Then,*

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma, \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Proof.

Assume Λ is a collection of nondetermined sets. Then $\neg Det(\Lambda \upharpoonright X^\omega)$. By corollary 2.4.21,

$\neg Det\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, Y, \Psi, \Lambda \restriction X^\omega\right)\right)$ for any Y and $\Psi : X^\omega \to \omega$, i.e.,

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma, \Lambda \restriction X^\omega\right)\right).$$

In fact, $G(Long(A); T_{X,Y}^{\Psi,A})$ is not determined for any nondetermined set $A \subseteq X^\omega$ and any X, Y and function Ψ from X^ω into ω. $\square$

2.4.2.2 Definition of the open set $Max\,(\Psi, B)$ on a Type 1 tree

In this section, we will consider Type 1 trees $T^{\Psi,B}_{X,Y}$ such that $\Psi \restriction B$ is bounded below ω. We will define the open set $Max(\Psi, B)$ on a Type 1 tree. This open set is defined only on Type 1 trees with height bounded below $\omega+\omega$. Max consists of all plays of the maximum length. In sections 2.4.2.3 and 2.4.2.4, we will obtain some determinacy results using Max.

Definition 2.4.24. *(Definition of Max)*

Suppose $\Psi \restriction B$ is bounded below ω. Let $n^{\Psi,B}_{\max}$ be the maximum tail length determined from Ψ and B. ($n^{\Psi,B}_{\max} = \max(Im(\Psi \restriction B)) + 1$.) If Ψ and B are clear from the context, we suppress Ψ and B, i.e.,$n_{\max} = n^{\Psi,B}_{\max}$.

Define

$$Max\,(\Psi, B) = \{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) = \omega + n_{\max}\} = lh^{-1}_{[T^{\Psi,B}_{X,Y}]}(\omega + n_{\max}). \qquad \dashv$$

It is easy to see that $Max\,(\Psi, B)$ is open in $[T^{\Psi,B}_{X,Y}]$. In general, if n is not the maximal length, $\{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) = \omega + n\}$ may not be open.

2.4.2.3 Using the determinacy of open games $Max(\chi_A, X^\omega)$ on a $Tree_1$ collection to obtain the determinacy of games A on $X^{<\omega}$

In this section, we will obtain the determinacy of games A on $X^{<\omega}$ from the determinacy of open games $Max(\chi_A, X^\omega)$ on a $Tree_1$ collection. Let $A \subseteq X^\omega$. We will consider the case for the Type 1 tree $T^{\Psi,B}_{X,Y}$ in which Ψ is the characteristic function χ_A of A and $B = X^\omega$. By using χ_A, we can split the body of the tree $T = T^{\chi_A, X^\omega}_{X,Y}$ into two pieces, $lh^{-1}_{[T]}(\omega+1)$ and

$lh^{-1}_{[T]}(\omega+2)$. Recall

$$\chi_A: \quad X^\omega \to \{0,1\}$$
$$f \mapsto \begin{cases} 0 & \text{if } f \notin A, \\ 1 & \text{if } f \in A. \end{cases}$$

Note that

$$h \in \left[T^{\chi_A, X^\omega}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin X^\omega, \\ h \in X^\omega \times Y^{\chi_A(h\restriction\omega)+1} & \text{if } h \restriction \omega \in X^\omega. \end{cases}$$
$$\leftrightarrow \begin{cases} h \in X^\omega \times Y & \text{if } h \restriction \omega \notin A, \\ h \in X^\omega \times Y^2 & \text{if } h \restriction \omega \in A. \end{cases}$$

Thus $\forall h \in [T^{\chi_A, X^\omega}_{X,Y}]$,

$$lh\,(h) = \begin{cases} \omega+1 & \text{if } h \restriction \omega \in X^\omega \backslash A, \\ \omega+2 & \text{if } h \restriction \omega \in A. \end{cases}$$

Hence, for tree $T^{\chi_A, X^\omega}_{X,Y}$, $n_{\max} = 2$ and

$$Max\,(\chi_A, X^\omega) = \{h \in [T^{\chi_A, X^\omega}_{X,Y}] \mid lh\,(h) = \omega+2\} = \{h \in [T^{\chi_A, X^\omega}_{X,Y}] \mid h \restriction \omega \in A\}. \tag{2.8}$$

Theorem 2.4.25. *For any $A \subseteq X^\omega$,*

$G\,(A; X^\omega)$ is determined if and only if $G(Max(\chi_A, X^\omega);\, T^{\chi_A, X^\omega}_{X,Y})$ is determined. ⊣

Proof.

Pick an arbitrary $A \subseteq X^\omega$.

($\Rightarrow$) Assume $G(A; X^{<\omega})$ is determined. Thus I or II has a winning strategy s for $G(A; X^{<\omega})$. Define s^* to be such that $s^* \restriction X^{<\omega} = s$ and play anything after that to finish the play (note that there is no play of length ω in $T^{\Psi, X^\omega}_{X,Y}$). Show s^* is a winning strategy for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$. Pick an arbitrary $h \in [T_{X,Y}^{\chi_A, X^\omega}]$ according to s^*. Then $h \restriction \omega$ is according to s.

Case 1 : s is a winning strategy for I for $G(A; X^{<\omega})$.

Then $h \restriction \omega \in A$. Thus the length of h is $\omega + 2$ so that $h \in Max\,(\chi_A, X^\omega)$. Hence s^* is a winning strategy for I^* for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$.

Case 2 : s is a winning strategy for II for $G(A; X^{<\omega})$.

Then $h \restriction \omega \notin A$. Thus the length of h is $\omega + 1$ so that $h \notin Max\,(\chi_A, X^\omega)$. Hence s^* is a winning strategy for II^* for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$.

($\Leftarrow$) Assume $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$ is determined. Thus I^* or II^* has a winning strategy s^* for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$. Define $s = s^* \restriction X^{<\omega}$. [21] Show s is a winning strategy for $G(A; X^{<\omega})$. Pick an arbitrary $f \in X^\omega$ according to s. Then f is according to s^*. Thus $f \in T_{X,Y}^{\chi_A, X^\omega}$ (note that there is no play of length ω in $T_{X,Y}^{\Psi, X^\omega}$). Then play $g \in Y^{<\omega}$ according to s^* to get $f^\frown g \in [T_{X,Y}^{\chi_A, X^\omega}]$.

Case 1 : s^* is a winning strategy for I^* for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$.

Then $f^\frown g \in Max\,(\chi_A, X^\omega)$ so by equation (2.8) on page 112, $f \in A$. Thus s is a winning strategy for I for $G(A; X^{<\omega})$.

Case 2 : s^* is a winning strategy for II^* for $G(Max\,(\chi_A, X^\omega)\,; T_{X,Y}^{\chi_A, X^\omega})$.

Then $f^\frown g \notin Max(\chi_A, X^\omega)$ so by equation (2.8) on page 112, $f \notin A$. Thus s is a winning strategy for II for $G(A; X^{<\omega})$. Therefore, $G(A; X^{<\omega})$ is determined. □

Observation 2.4.26. *Suppose Ξ is a complexity (in which $\Xi \restriction X^\omega \subseteq \wp(X^\omega)$ is defined).*

[21] $s^* \restriction X^{<\omega}$ abbreviates $s^* \restriction (X^{<\omega} \cap dom(s^*))$.

For any $A \in \Xi \restriction X^\omega$, *the characteristic function* χ_A *on* A *is in* $\Gamma(\omega, 2\text{-}\Xi) = \Gamma(\omega, \Xi \wedge co\text{-}\Xi)$.

$\dashv$

Proof.

Pick an arbitrary $O \subseteq \omega$. Then $O = \bigcup_{n \in O} \{n\}$. Recall

$$\chi_A : \; X^\omega \to \{0,1\}$$
$$f \mapsto \begin{cases} 0 & \text{if } f \notin A, \\ 1 & \text{if } f \in A. \end{cases}$$

Case 1 : $0, 1 \in O$.

Then $\chi_A^{-1}(O) = X^\omega \in \mathbf{\Delta}_1^0 \restriction X^\omega$.

Case 2 : If $0, 1 \notin O$.

Then $\chi_A^{-1}(O) = \emptyset \in \mathbf{\Delta}_1^0 \restriction X^\omega$.

Case 3 : $0 \notin O$ and $1 \in O$.

Then $\chi_A^{-1}(O) = A \in \Xi \restriction X^\omega$. Since $A = A \cap X^\omega$, $\chi_A^{-1}(O) \in \Xi \restriction X^\omega \wedge co\text{-}\Xi \restriction X^\omega$.

Case 4 : $0 \in O$ and $1 \notin O$.

Then $\chi_A^{-1}(O) = X^\omega \backslash A \in co\text{-}\Xi \restriction X^\omega$. Since $X^\omega \backslash A = X^\omega \cap (X^\omega \backslash A)$, $\chi_A^{-1}(O) \in \Xi \restriction X^\omega \wedge co\text{-}\Xi \restriction X^\omega$.

Thus $\chi_A^{-1}(O) \in \Xi \restriction X^\omega \wedge co\text{-}\Xi \restriction X^\omega$ for any open $O \subseteq X$. Since $\Xi \restriction X^\omega \wedge co\text{-}\Xi \restriction X^\omega = 2\text{-}\Xi \restriction X^\omega$, $\chi_A \in \Gamma(\omega, 2\text{-}\Xi)$. $\square$

The following corollaries are direct results from theorem 2.4.25. Determinacy of any game A of length ω can be obtained from the determinacy of open games $Max(\chi_A, X^\omega)$ on a particular $Tree_1$ collection.

Corollary 2.4.27. *For any X, Y and complexity Ξ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi \restriction X^\omega\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \restriction X^\omega\right).$$

Thus,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \restriction X^\omega). \qquad \dashv$$

Proof.

Fix X, Y. Pick an arbitrary $A \in \Xi \restriction X^\omega$. Then

$$G(Max\left(\chi_A, X^\omega\right); T^{\chi_A, X^\omega}_{X,Y}) \in \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi \restriction X^\omega\}, X^\omega\right)\right).$$

Thus $G(Max(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,Y})$ is determined. By theorem 2.4.25, $G(A; X^\omega)$ is determined.

Hence

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi \restriction X^\omega\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \restriction X^\omega\right).$$

Since $\chi_A \in \Gamma(\omega, 2\text{-}\Xi)$ by observation 2.4.26,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \restriction X^\omega). \qquad \square$$

Corollary 2.4.28. *(Corollary to Corollary 2.4.27)*

For any $\alpha \in \omega_1$ and any X, Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Sigma}^0_\alpha \restriction X^\omega).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Delta}^0_\alpha \restriction X^\omega). \qquad \dashv$$

Proof.

By corollary 2.4.27 with Ξ is $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$. $\square$

2.4.2.4 Using the determinacy of open games $Max(\chi_A, B)$ on a $Tree_1$ collection to obtain the determinacy of games $A \cap B$ on $X^{<\omega}$

In this section, we will obtain the determinacy of games $A \cap B$ on $X^{<\omega}$ from the determinacy of open games $Max(\chi_A, B)$ on a $Tree_1$ collection. Let $A \subseteq X^\omega$. In section 2.4.2.3, as a special case of Type 1 tree, we considered Type 1 trees $T_{X,Y}^{\Psi,B}$ such that $B = X^\omega$ and Ψ to be the characteristic function χ_A of A. In this section, as a generalization of trees in section 2.4.2.3, we will consider Type 1 trees $T_{X,Y}^{\Psi,B}$ such that B is an arbitrary subset of X^ω and Ψ to be the characteristic function χ_A of A.

Suppose $A, B \subseteq X^\omega$. Note that

$$h \in \left[T_{X,Y}^{\chi_A,B}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times Y^{\chi_A(h \restriction \omega)+1} & \text{if } h \restriction \omega \in B. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times Y & \text{if } h \restriction \omega \in B \backslash A, \\ h \in X^\omega \times Y^2 & \text{if } h \restriction \omega \in A \cap B. \end{cases}$$

Thus $\forall h \in \left[T_{X,Y}^{\chi_A,B}\right]$,

$$lh\,(h) = \begin{cases} \omega & \text{if } h \restriction \omega \in X^\omega \backslash B, \\ \omega + 1 & \text{if } h \restriction \omega \in B \backslash A, \\ \omega + 2 & \text{if } h \restriction \omega \in A \cap B. \end{cases}$$

Hence for the tree $T_{X,Y}^{\chi_A,B}$, $n_{\max}^{\chi_A,B} = 2$ and

$$Max\,(\chi_A, B) = \{h \in [T_{X,Y}^{\chi_A,B}] \mid lh\,(h) = \omega + 2\} = \{h \in [T_{X,Y}^{\chi_A,B}] \mid h \restriction \omega \in A \cap B\}. \quad (2.9)$$

Thus, in this case, we consider the game $A \cap B$ on X^ω.

Theorem 2.4.29. *For any $A, B \subseteq X^\omega$,*

$G(A \cap B; X^\omega)$ is determined if and only if $G(Max(\chi_A, B); T_{X,Y}^{\chi_A;B})$ is determined. $\dashv$

Proof.

Pick arbitrary $A, B \subseteq X^\omega$.

($\Rightarrow$) Assume $G(A \cap B; X^{<\omega})$ is determined. Then I or II has a winning strategy s for $G(A \cap B; X^{<\omega})$. Define s^* to be such that $s^* \restriction X^{<\omega} = s$ and play anything after that (if needed) to finish the play. Show s^* is a winning strategy for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$. Pick an arbitrary $h \in [T_{X,Y}^{\chi_A;B}]$ according to s^*. Then $h \restriction \omega$ is according to s.

Case 1 : s is a winning strategy for I for $G(A \cap B; X^{<\omega})$.

Then $h \restriction \omega \in A \cap B$. Thus the length of h is $\omega + 2$ so that $h \in Max\,(\chi_A, B)$. Hence s^* is a winning strategy for I^* for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$.

Case 2 : s is a winning strategy for II for $G(A \cap B; X^{<\omega})$.

Then $h \restriction \omega \notin A \cap B$. Thus the length of h is ω or $\omega + 1$ so that $h \notin Max\,(\chi_A, B)$. Hence s^* is a winning strategy for II^* for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$.

($\Leftarrow$) Assume $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$ is determined. Then I^* or II^* has a winning strategy s^* for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$. Define $s = s^* \restriction X^{<\omega}$.[22] Show s is a winning strategy for $G(A \cap B; X^{<\omega})$. Pick an arbitrary $f \in X^\omega$ according to s. Then f is according to s^*. If $f \in [T_{X,Y}^{\chi_A;B}]$, then let $g = \emptyset$. If $f \in T_{X,Y}^{\chi_A;B}$ play $g \in Y^{<\omega}$ according to s to get $f^\frown g \in [T_{X,Y}^{\chi_A;B}]$.

Case 1 : s^* is a winning strategy for I^* for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$.

Then $f^\frown g \in Max\,(\chi_A, B)$ so by equation (2.9) on page 116, $f \in A \cap B$. Hence s is a winning strategy for I for $G(A \cap B; X^{<\omega})$.

Case 2 : s^* is a winning strategy for II^* for $G(Max\,(\chi_A, B)\,; T_{X,Y}^{\chi_A;B})$.

[22] $s^* \restriction X^{<\omega}$ abbreviates $s^* \restriction (X^{<\omega} \cap dom(s^*))$.

Then $f^\frown g \notin Max\,(\chi_A, B)$ so by equation (2.9) on page 116, $f \notin A \cap B$. Hence s is a winning strategy for II for $G(A \cap B; X^{<\omega})$. Therefore, $G(A \cap B; X^{<\omega})$ is determined. □

The following corollaries are direct results from theorem 2.4.29.

Corollary 2.4.30. *Suppose Ξ_1, Ξ_2 are complexities. Then for any X, Y,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi_1 \restriction X^\omega\}, \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.10}$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi_2 \restriction X^\omega\}, \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.11}$$

⊣

Proof.

Fix X, Y. Pick an arbitrary $A \in (\Xi_1 \wedge \Xi_2) \restriction X^\omega$. Then there exists $B \in \Xi_1 \restriction X^\omega$ and $C \in \Xi_2 \restriction X^\omega$ such that $A = B \cap C$.

Show the implication (2.10).

Since $\chi_B \in \{\chi_{\hat{A}} \,|\, \hat{A} \in \Xi_1 \restriction X^\omega\}$, we consider the tree $T_{X,Y}^{\chi_B;C}$. In this tree, $n_{\max}^{\chi_B;C} = 2$. Then

$$Max\,(\chi_B, C) = \{h \in [T_{X,Y}^{\chi_B;C}] \,|\, lh\,(h) = \omega + 2\} = \{h \in [T_{X,Y}^{\chi_B;C}] \,|\, h \restriction \omega \in B \cap C\}.$$

Since

$$G(Max(\chi_B, C); T_{X,Y}^{\chi_B;C}) \in \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, Y, \{\chi_A\,|A \in \Xi_1 \restriction X^\omega\}, \Xi_2 \restriction X^\omega\right)\right),$$

$G(Max(\chi_B, C);\ T_{X,Y}^{\chi_B;C})$ is determined. By theorem 2.4.29, $G(B \cap C; X^{<\omega})$ is determined. Hence $G(A; X^{<\omega})$ is determined.

Show the implication (2.11).

Since $\chi_C \in \{\chi_{\hat{A}} \mid \hat{A} \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\}$, we consider the tree $T^{\chi_C,B}_{X,Y}$. In this tree, $n^{\chi_C,B}_{\max} = 2$. Then

$$Max\,(\chi_C, B) = \{h \in [T^{\chi_C,B}_{X,Y}] \mid lh\,(h) = \omega + 2\} = \{h \in [T^{\chi_C,B}_{X,Y}] \mid h \upharpoonright \omega \in B \cap C\}.$$

Since

$$G(Max(\chi_C, B); T^{\chi_C,B}_{X,Y}) \in \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \{\chi_A \,|A \in \Xi_2 \upharpoonright X^\omega\}\,, \Xi_1 \upharpoonright X^\omega)\right).$$

$G(Max(\chi_C, B); T^{\chi_C,B}_{X,Y})$ is determined. By theorem 2.4.29, $G(C \cap B; X^{<\omega})$ is determined. Hence $G(A; X^{<\omega})$ is determined. □

Corollary 2.4.31. *(Corollary to Corollary 2.4.30)*

Suppose Ξ_1, Ξ_2 are complexities. Then for any X, Y,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, \Xi_1 \wedge co\text{-}\Xi_1), \Xi_2 \upharpoonright X^\omega)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \upharpoonright X^\omega). \qquad (2.12)$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, \Xi_2 \wedge co\text{-}\Xi_2), \Xi_1 \upharpoonright X^\omega)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \upharpoonright X^\omega). \qquad (2.13)$$

⊣

Proof.

Since $\{\chi_A \,|A \in \Xi_1 \upharpoonright X^\omega\} \subseteq \Gamma\,(\omega, \Xi_1 \wedge co\text{-}\Xi_1)$ by observation 2.4.26, we obtain the implication (2.12) from corollary 2.4.30 the implication (2.10). Since $\{\chi_A \,|A \in \Xi_2 \upharpoonright X^\omega\} \subseteq \Gamma\,(\omega, \Xi_2 \wedge co\text{-}\Xi_2)$ by observation 2.4.26, we obtain the implication (2.13) from corollary 2.4.30 the implication (2.11). □

We list some obvious special case of corollary 2.4.30. We obtain corollary 2.4.32 from replacing $\Xi_1 = \mathbf{\Sigma}^0_\alpha$ and $\Xi_2 = \mathbf{\Pi}^0_\beta$ in corollary 2.4.31. We obtain corollary 2.4.33 from replacing $\Xi_1 = \mathbf{\Sigma}^1_\alpha$ and $\Xi_2 = \mathbf{\Pi}^1_\beta$ in corollary 2.4.31.

Corollary 2.4.32. *(Corollary to Corollary 2.4.31)*

Suppose $\alpha, \beta \in \omega_1$. Then for any Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha\right), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega). \qquad \dashv$$

We get similar results for projective sets.

Corollary 2.4.33. *(Corollary to Corollary 2.4.31)*

Suppose $n, m \in \omega$. Then for any Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n\right), \mathbf{\Pi}^1_m \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m\right), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega). \qquad \dashv$$

In particular, if $\alpha = \beta = 1$, we can get a 2-$\mathbf{\Pi}^1_1$ set.

Corollary 2.4.34. *(Corollary to Corollary 2.4.33) For any Y,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Proof.

Note that $\mathbf{\Sigma}^1_1 \wedge \mathbf{\Pi}^1_1 = 2\text{-}\mathbf{\Pi}^1_1$. $\square$

Question 1. *By corollary 2.4.34 on page 120, corollary 2.4.21 on page 109 and corollary 2.4.1 on page 83, all of the following imply* $Det(2\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega)$*:*

$$
\begin{array}{ll}
(i) & Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \upharpoonright X^\omega\right)\right) \\
(ii) & Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \upharpoonright X^\omega\right)\right) \\
(iii) & Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), 2\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega\right)\right) \\
(iv) & Det\ \mathcal{G}\left(2\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right)
\end{array}
$$

What is the relation between $(i), (ii), (iii)$ *and* (iv)*?* ⊣

Recall definition 1.3.23 on page 23. For every $n \in \omega$, if $A \in (n+1)\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega$, then $A = A_0 \backslash A_1 = A_0 \cap X^\omega \backslash A_1$ where $A_0 \in \mathbf{\Pi}^1_1 \upharpoonright X^\omega$ and $A_1 \in n\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega$ (hence $X^\omega \backslash A_1 \in$ *co*-n-$\mathbf{\Pi}^1_1 \upharpoonright X^\omega$). We obtain corollary 2.4.35 the implication (2.14) from replacing $\Xi_1 = \mathbf{\Sigma}^1_1$ and $\Xi_2 =$ *co*-n-$\mathbf{\Pi}^1_1$ in corollary 2.4.31 the implication (2.12). We obtain corollary 2.4.35 the implication (2.14) from replacing $\Xi_1 =$ *co*-n-$\mathbf{\Pi}^1_1$ and $\Xi_2 = \mathbf{\Sigma}^1_1$ in corollary 2.4.31 the implication (2.13).

Corollary 2.4.35. *(Corollary to Corollary 2.4.31) For any* Y *and* $n \in \omega$*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), (co\text{-}n\text{-}\mathbf{\Pi}^1_1) \upharpoonright X^\omega\right)\right) \Rightarrow Det(n+1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega). \tag{2.14}$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(n+1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega\right). \tag{2.15}$$

⊣

Proof.

Since$\{\chi_A\,|A \in \mathbf{\Pi}^1_1 \restriction X^\omega\} \subseteq \Gamma\,(\omega, 2\text{-}\mathbf{\Pi}^1_1)$ and $\{\chi_A\,|A \in co\text{-}n\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\} \subseteq$ $\Gamma\,(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1)$ by observation 2.4.26, we have the results by corollary 2.4.31. □

Question 2. *By corollary 2.4.35 on page 121, corollary 2.4.21 on page 109 and corollary 2.4.1 on page 83, all of the following imply* $Det(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega)$*:*

(i) $Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), (co\text{-}n\text{-}\mathbf{\Pi}^1_1) \restriction X^\omega))$

(ii) $Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega))$

(iii) $Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega))$

(iv) $Det\ \mathcal{G}\,(n+1\text{-}\mathbf{\Pi}^1_1; Tree_1\,(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1))$

What is the relationship between $(i), (ii), (iii)$ *and* (iv)*?* ⊣

2.4.3 Using α-$\mathbf{\Pi}^1_1$ determinacy on $Tree_1$ collection to obtain α+1-$\mathbf{\Pi}^1_1$ determinacy on X^ω for even $\alpha \in \omega_1$

In section 2.4.2.4, we used Max on certain $Tree_1$ collections to obtain the determinacy of games on $X^{<\omega}$. In theorem 2.4.29, we obtained the determinacy equivalence of games $G(A \cap B; X^{<\omega})$ and $G(Max(\chi_A, B); T^{\chi_A, B}_{X,Y})$ for any $A, B \subseteq X^\omega$.

In this section, we will obtain $\alpha + 1$-$\mathbf{\Pi}^1_1$ determinacy on X^ω for even $\alpha \in \omega_1$ from α-$\mathbf{\Pi}^1_1$ determinacy on $Tree_1$ collection. Fix $\alpha \in \omega_1$ and $\langle A_\beta | \beta \leq \alpha \rangle$ where each $A_\beta \subseteq X^\omega$. By observation 2.4.38 below, $dk\left(\langle A_\beta \left| \beta \leq \alpha \right.\rangle\right) = dk\left(\langle A_\beta \left| \beta \in \alpha \right.\rangle\right) \cup (\bigcap_{\beta \leq \alpha} A_\beta)$. We set $A = A_\alpha$ and $B = \bigcap_{\beta \in \alpha} A_\beta$ so that $A \cap B = \bigcap_{\beta \leq \alpha} A_\beta$. Thus, we will consider a Type 1 tree $T^{\chi_{A_\alpha}, B}_{X,Y}$ with $B = \bigcap_{\beta \in \alpha} A_\beta$.

$$h \in \left[T^{\chi_A, \bigcap_{\beta \in \alpha} A_\beta}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin \bigcap_{\beta \in \alpha} A_\beta, \\ h \in X^\omega \times Y^{\chi_A(h \upharpoonright \omega)+1} & \text{if } h \upharpoonright \omega \in \bigcap_{\beta \in \alpha} A_\beta. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin \bigcap_{\beta \in \alpha} A_\beta, \\ h \in X^\omega \times Y & \text{if } h \upharpoonright \omega \in (\bigcap_{\beta \in \alpha} A_\beta) \backslash A_\alpha, \\ h \in X^\omega \times Y^2 & \text{if } h \upharpoonright \omega \in \bigcap_{\beta \leq \alpha} A_\beta. \end{cases}$$

Thus $\forall h \in \left[T^{\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta}_{X,Y}\right]$,

$$lh\,(h) = \begin{cases} \omega & \text{if } h \upharpoonright \omega \in X^\omega \backslash \bigcap_{\beta \in \alpha} A_\beta, \\ \omega + 1 & \text{if } h \upharpoonright \omega \in (\bigcap_{\beta \in \alpha} A_\beta) \backslash A_\alpha, \\ \omega + 2 & \text{if } h \upharpoonright \omega \in \bigcap_{\beta \leq \alpha} A_\beta. \end{cases}$$

For tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$, $n_{\max}=2$.

$$\begin{aligned} Max\left(\chi_{A_\alpha},\bigcap\nolimits_{\beta\in\alpha}A_\beta\right) &= \left\{h\in\left[T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}\right]\middle| lh\left(h\right)=\omega+2\right\} \\ &= \left\{h\in\left[T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}\right]\middle| h\restriction\omega\in\bigcap\nolimits_{\beta\leq\alpha}A_\beta\right\}. \end{aligned} \tag{2.16}$$

We will obtain the determinacy equivalences of a certain game for such Type 1 tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$ and a $dk(\langle A_\beta|\beta\leq\alpha\rangle)$ games on $X^{<\omega}$. In definition 2.4.36, we will define $dk_{<\alpha}\left(\langle A_\beta\,|\beta\leq\alpha\rangle\right)\subseteq[T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}]$. In observation 2.4.37, we will show that $dk_{<\alpha}\left(\langle A_\beta\,|\beta\leq\alpha\rangle\right)=dk\left(\langle A_\beta\,|\beta\in\alpha\rangle\right)$. Then in theorem 2.4.40, we will show that the determinacy equivalence of a $dk\left(\langle A_\beta\,|\beta\leq\alpha\rangle\right)$ game on $X^{<\omega}$ and a $dk_{<\alpha}\left(\langle A_\beta\,|\beta\leq\alpha\rangle\right)\cup T_{Max}(\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta)$ game on the tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$.

In particular, for even $\alpha\in\omega$ and sequences $\langle A_\beta\,|\beta\leq\alpha\rangle$ with each $A_\beta\in\mathbf{\Pi}_1^1\restriction X^\omega$, we will obtain $\alpha+1$-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from α-$\mathbf{\Pi}_1^1$ games on a particular $Tree_1$ collection in corollary 2.4.42. As a special case, when α is a limit ordinal and $A_\alpha\in\mathbf{\Sigma}_\lambda^0$ for some $\lambda\in\omega_1$, we will obtain a similar result for α-$\mathbf{\Pi}_1^1+\mathbf{\Sigma}_\lambda^0$ games on $X^{<\omega}$ from α-$\mathbf{\Pi}_1^1$ games on a particular $Tree_1$ collection in corollary 2.4.44.

First, recall definition 1.3.22 on page 23.

Definition 1.3.22. *(Definition of the difference kernel)(Hausdorff, 1944* [23]*)*
Denote the difference kernel of $\vec{A}=\langle A_\beta\,|\beta\in\alpha\rangle$ *by* $dk(\vec{A})$ *and define*

$$dk(\vec{A})=\{x\in[T]\,|\mu\beta\,(x\notin A_\beta\vee\beta=\alpha)\ \textit{ is odd}\}\,. \qquad \dashv$$

Given $\langle A_\beta\,|\beta\leq\alpha\rangle$ where each $A_\beta\subseteq X^\omega$, we define $dk_{<\alpha}\left(\langle A_\beta\,|\beta\leq\alpha\rangle\right)$ on the tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$.

[23] as cited in Welch (1996, p. 1).

Definition 2.4.36. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Fix a Type 1 tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha} A_\beta}$. Define*

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = \left\{ h \in \left[T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha} A_\beta}\right] \middle| \, h \restriction \omega \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \restriction \omega \notin A_\beta) \text{ is odd}\right\}. \dashv$$

Notice that if α is even, we have:

$$dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) = \left\{ f \in X^\omega \middle| f \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd}\right\}.$$

Thus

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \restriction X^\omega = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right).$$

In fact, we have the following.

Observation 2.4.37. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Then*

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) \subseteq X^\omega. \qquad \dashv$$

Proof.

Notice that since α is even,

$$dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) = \left\{ f \in X^\omega \middle| f \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd}\right\}.$$

Also, if $h \in dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right)$, then $h \in [T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha} A_\beta}]$ and $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ so that $h = h \restriction \omega$. Thus

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = \left\{h \in \left[T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}\right] \middle| h \restriction \omega \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \restriction \omega \notin A_\beta) \text{ is odd}\right\}$$

$$= \left\{h \in \left[T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}\right] \middle| h = h \restriction \omega \wedge h \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \notin A_\beta) \text{ is odd}\right\}$$

$$= \left\{f \in X^\omega \middle| f \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd}\right\}$$

$$= dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right). \qquad \square$$

Since $\alpha \in \omega_1$ is even, $dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right)$ could be express as a union of $dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right)$ and $\bigcap_{\beta\leq\alpha} A_\beta$. Thus we have the following.

Observation 2.4.38. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Then*

$$dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) \cup \left(\bigcap\nolimits_{\beta\leq\alpha} A_\beta\right). \qquad \dashv$$

Proof.

Since $\alpha \in \omega_1$ is even,

$$dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = \left\{f \in X^\omega \middle| \left(f \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd}\right) \vee f \in \bigcap\nolimits_{\beta\leq\alpha} A_\beta\right\}$$

$$= \left\{f \in X^\omega \middle| \left(f \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd}\right)\right\} \cup \left(\bigcap\nolimits_{\beta\leq\alpha} A_\beta\right)$$

$$= dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) \cup \left(\bigcap\nolimits_{\beta\leq\alpha} A_\beta\right). \qquad \square$$

Proposition 2.4.39. *Suppose $T = T_{X,Y}^{\Psi,B}$ is a Type 1 tree. Assume the following:*

1. *$C, D \subseteq X^\omega$ and $E, F \subseteq [T]$.*

2. *s is a strategy for $X^{<\omega}$.*

3. *s^* is a strategy for T such that $s^* \restriction X^{<\omega} = s$.*

4. *for any $h \in [T]$ according to s^*, $h \restriction \omega \in C$ if and only if $h \in E$.*

5. *for any* $h \in [T]$ *according to* s^*, $h \upharpoonright \omega \in D$ *if and only if* $h \in F$.

Then s *is a winning strategy for* I *for* $G(C \cup D; X^{<\omega})$ *if and only if* s^* *is a winning strategy for* I^* *for* $G(E \cup F; T)$. *Also* s *is a winning strategy for* II *for* $G(C \cup D; X^{<\omega})$ *if and only if* s^* *is a winning strategy for* II^* *for* $G(E \cup F; T)$. ⊣

Proof.

Show that s is a winning strategy for I for $G(C \cup D; X^{<\omega})$ if and only if s^* is a winning strategy for I for $G(E \cup F; T)$.

(⇒) Assume s is a winning strategy for I for $G(C \cup D; X^{<\omega})$. Show s^* is a winning strategy for I^* for $G(E \cup F; T)$. Pick an arbitrary $h \in [T]$ according to s^*. Then $h \upharpoonright \omega$ is according to s. Thus $h \upharpoonright \omega \in C \cup D$. If $h \upharpoonright \omega \in C$, then $h \in E$. If $h \upharpoonright \omega \in D$, then $h \in F$. Thus $h \in E \cup F$. Hence s^* is a winning strategy for I^* for $G(E \cup F; T)$.

(⇐) Assume s^* is a winning strategy for I^* for $G(E \cup F; T)$. Show s is a winning strategy for I for $G(C \cup D; X^{<\omega})$. Pick an arbitrary $f \in X^{\omega}$ according to s. Then f is according to s^*. Play according to s^* after f, call it g, so that $f^\frown g \in [T]$ (if $f \in [T]$, then $g = \emptyset$). Then $f^\frown g \in E \cup F$. Since $f^\frown g$ is according to s^*, if $f^\frown g \in E$, then $f \in C$ and if $f^\frown g \in F$, then $f \in D$. Thus $f \in C \cup D$. Hence s is a winning strategy for I for $G(C \cup D; X^{<\omega})$.

Show that s is a winning strategy for II for $G(C \cup D; X^{<\omega})$ if and only if s^* is a winning strategy for II^* for $G(E \cup F; T)$.

(⇒) Assume s is a winning strategy for II for $G(C \cup D; X^{<\omega})$. Show s^* is a winning strategy for II^* for $G(E \cup F; T)$. Pick an arbitrary $h \in [T]$ according to s^*. Then $h \upharpoonright \omega$ is according to s. Thus $h \upharpoonright \omega \notin C \cup D$. Hence $h \upharpoonright \omega \notin C$ and $h \upharpoonright \omega \notin D$. Therefore, $h \notin E$

and $h \notin F$. Thus $h \notin E \cup F$. Hence s^* is a winning strategy for II^* for $G(E \cup F; T)$.

($\Leftarrow$) Assume s^* is a winning strategy for II^* for $G(E \cup F; T)$. Show s is a winning strategy for II for $G(C \cup D; X^{<\omega})$. Pick an arbitrary $f \in X^\omega$ according to s. Then f is according to s^*. Play according to s^* after f, call it g, so that $f^\frown g \in [T]$ (if $f \in [T]$, then $g = \emptyset$). Then $f^\frown g \notin E \cup F$. Thus $f^\frown g \notin E$ and $f^\frown g \notin F$. Since $f^\frown g$ is according to s^*, $f \notin C$ and $f \notin D$. Thus $f \notin C \cup D$. Hence s is a winning strategy for II for $G(C \cup D; X^{<\omega})$. $\square$

By proposition 2.4.39, we obtain the following.

Theorem 2.4.40. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha \rangle$ where each $A_\beta \subseteq X^\omega$. Let $T = T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta}$. Then $G\left(dk\left(\langle A_\beta \,|\beta \leq \alpha \rangle\right); X^{<\omega}\right)$ is determined if and only if*

$$G\left(dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha \rangle\right) \cup Max\left(\chi_{A_\alpha}, \bigcap\nolimits_{\beta \in \alpha} A_\beta\right); T\right)$$

is determined. ⊣

Proof.

Use proposition 2.4.39 with:

- $C = dk\left(\langle A_\beta \,|\beta \in \alpha \rangle\right)$.
- $D = \bigcap_{\beta \leq \alpha} A_\beta$.
- $E = dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha \rangle\right)$
- $F = Max\left(\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta\right)$

Then this satisfies property (1) of proposition 2.4.39. By observation 2.4.38,

$$dk\left(\langle A_\beta \,|\beta \leq \alpha \rangle\right) = dk\left(\langle A_\beta \,|\beta \in \alpha \rangle\right) \cup \left(\bigcap\nolimits_{\beta \leq \alpha} A_\beta\right) = C \cup D.$$

By observation 2.4.37 and equation 2.16 shows that properties (4) and (5) of proposition 2.4.39 are satisfied.

Show $G\left(dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right); X^{<\omega}\right)$ is determined if and only if

$$G\left(dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \cup \; T_{Max}\left(\chi_{A_\alpha}, \bigcap\nolimits_{\beta\in\alpha} A_\beta\right); T\right)$$

is determined.

($\Rightarrow$) Suppose $G\left(dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right); X^{<\omega}\right) = G\left(C \cup D; X^{<\omega}\right)$ is determined. Then I or II has a winning strategy s for $G\left(C \cup D; X^{<\omega}\right)$. Define s^* to be such that $s^* \upharpoonright X^{<\omega} = s$ and play anything after that. Then this satisfies properties (2) and (3) of proposition 2.4.39. By proposition 2.4.39, If s is a winning strategy for I for $G(C \cup D; X^{<\omega})$, then s^* is a winning strategy for I^* for $G(E \cup F; T)$. If s is a winning strategy for II for $G(C \cup D; X^{<\omega})$, then s^* is a winning strategy for II^* for $G(E \cup F; T)$. Thus I^* or II^* has a winning strategy for $G(E \cup F; T)$. Therefore, $G\left(E \cup F; T\right) = G(dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \cup Max(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta); T)$ is determined.

($\Leftarrow$) Suppose $G(dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \cup Max(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta); T) = G\left(E \cup F; T\right)$ is determined. Then I^* or II^* has a winning strategy s^* for $G\left(E \cup F; T\right)$. Define s to be such that $s = s^* \upharpoonright X^{<\omega}$. Then this satisfies properties (2) and (3) of proposition 2.4.39. By proposition 2.4.39, If s^* is a winning strategy for I^* for $G(E \cup F; T)$, then s is a winning strategy for I for $G(C \cup D; X^{<\omega})$. If s^* is a winning strategy for II^* for $G(E \cup F; T)$, then s is a winning strategy for II for $G(C \cup D; X^{<\omega})$. Thus I or II has a winning strategy for $G(C \cup D; X^{<\omega})$. Therefore, $G\left(C \cup D; X^{<\omega}\right) = G\left(dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right); X^{<\omega}\right)$ is determined. □

Now, let's consider the complexity of each A_β. Recall definition 1.3.23 on page 23.

Definition 1.3.23. *Suppose Λ is a class of subsets of $[T]$ and Λ is closed under countable intersections. Suppose $\alpha \in \omega_1$. Define*

$$\alpha\text{-}\Lambda \restriction [T] = \left\{ A \subseteq [T] \middle| \exists \vec{A} = \langle A_\beta \,|\beta \in \alpha\rangle \left(\textit{each } A_\beta \in \Lambda \textit{ and } A = dk(\vec{A})\right)\right\}. \qquad \dashv$$

We will consider theorem 2.4.40 with $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$. Then $dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \in \alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$ where $\alpha \in \omega_1$ is even.

Lemma 2.4.41. *Suppose $\alpha \in \omega_1$ is even. Fix $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$. Then*

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \in \alpha\text{-}\mathbf{\Pi}^1_1 \restriction \left[T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}\right]. \qquad \dashv$$

Proof.

Suppose $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$. Since each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$, there exists $O_\beta \in \mathbf{\Sigma}^0_1 \restriction X^\omega \times \omega^\omega$ such that $x \in A_\beta$ if and only if for all $y \in \omega^\omega$ $\langle x, y\rangle \in O_\beta$. For all $\beta \in \alpha$, define

$$\hat{O}_\beta = \{\langle h, y\rangle \in [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}] \times \omega^\omega \mid \langle h \restriction \omega, y\rangle \in O_\beta\}.$$

Then each $\hat{O}_\beta \in \mathbf{\Sigma}^0_1 \restriction [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}] \times \omega^\omega$. Define

$$\hat{A}_\beta = \{\, h \in [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}] \,\Big|\, \forall y \in \omega^\omega \, \langle h, y\rangle \in \hat{O}_\beta\}.$$

Then $\hat{A}_\beta \in \mathbf{\Pi}^1_1 \restriction [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}]$. Also, for all $x \in [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}]$,

$$\begin{aligned} x \in \hat{A}_\beta &\Leftrightarrow \forall y \in \omega^\omega \, \langle x, y\rangle \in \hat{O}_\beta \\ &\Leftrightarrow \forall y \in \omega^\omega \, \langle x \restriction \omega, y\rangle \in O_\beta \\ &\Leftrightarrow x \restriction \omega \in A_\beta. \end{aligned}$$

Thus

$$
\begin{aligned}
dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) &= \left\{ h \in [T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}] \,\middle|\, \exists\beta \in \alpha\,(h \upharpoonright \omega \notin A_\beta) \wedge \mu\beta\,(h \upharpoonright \omega \notin A_\beta) \text{ is odd} \right\} \\
&= \left\{ h \in [T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}] \,\middle|\, \exists\beta \in \alpha(h \notin \hat{A}_\beta) \wedge \mu\beta(h \notin \hat{A}_\beta) \text{ is odd} \right\} \\
&\in \alpha\text{-}\mathbf{\Pi}^1_1 \upharpoonright [T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}].
\end{aligned}
$$

□

Using theorem 2.4.40 and lemma 2.4.41, we have the following.

Corollary 2.4.42. *Assume $\alpha \in \omega_1$ is even. Then for any Y,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \upharpoonright X^\omega\right)\right) \Rightarrow Det(\alpha + 1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega). \qquad \dashv$$

Proof.

Fix Y. Suppose $\alpha \in \omega_1$ is even $A \in \alpha + 1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega$. Then there exists a sequence $\vec{A} = \langle A_\beta \,|\beta \leq \alpha\rangle$ witness that $A = dk(\vec{A}) \in \alpha + 1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega$. Then $\bigcap_{\beta\leq\alpha} A_\beta \in \mathbf{\Pi}^1_1 \upharpoonright X^\omega$. By observation 2.4.26, $\chi_{A_\alpha} \in \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1)$. Let $T = T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}$. By lemma 2.4.41, $dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) \in \alpha\text{-}\mathbf{\Pi}^1_1 \upharpoonright [T]$. Since $Max(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta) \in \mathbf{\Sigma}^0_1 \upharpoonright [T]$,

$$dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) \cup Max\left(\chi_{A_\alpha}, \bigcap\nolimits_{\beta\in\alpha} A_\beta\right) \in \alpha\text{-}\mathbf{\Pi}^1_1 \upharpoonright [T].$$

Thus $G(dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) \cup Max(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta); T)$ is determined. By theorem 2.4.40, $G(dk(\vec{A}); X^\omega)$ is determined, i.e., $G(A; X^\omega)$ is determined. □

Question 3. *By corollary 2.4.1 on page 83, corollary 2.4.21 on page 109 and corollary 2.4.27 on page 114, all of the following imply $Det(\alpha + 1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega)$:*

(*i*) $Det\ \mathcal{G}(\alpha + 1\text{-}\mathbf{\Pi}^1_1; Tree_1(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1))$

(*ii*) $Det\ \mathcal{G}(\mathbf{\Sigma}^0_1; Tree_1(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \alpha + 1\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega))$

(*iii*) $Det\ \mathcal{G}(\mathbf{\Sigma}^0_1; Tree_1(X, Y, \Gamma(\omega, \alpha + 1\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}\alpha + 1\text{-}\mathbf{\Pi}^1_1), X^\omega))$

Moreover, if $\alpha \in \omega_1$ is even, then by corollary 2.4.42,

$$(iv) \quad Det\mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$$

implies $Det(\alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega)$.

What is the relationship between $(i), (ii), (iii)$ and (iv)? ⊣

One might notice that when we reduce the complexity of A, the payoff set for the player I, we raise the complexity of the function Ψ and/or the complexity of B.

Now, suppose $\alpha \in \omega_1$ is a limit ordinal.[24] As a special case of $\alpha + 1\text{-}\mathbf{\Pi}^1_1$, we will define $\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda$ sets over a tree T.

Definition 2.4.43. *(Definition of $\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \restriction [T]$)*

Suppose $\alpha \in \omega_1$ is a limit ordinal. Let $\lambda \in \omega_1$. Suppose T is a tree. Define $A \in (\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda) \restriction [T]$ if and only if there is a sequence $\vec{A} = \langle A_\beta \,|\beta \leq \alpha\rangle$ witness that $A = dk(\vec{A}) \in \alpha{+}1\text{-}\mathbf{\Pi}^1_1 \restriction [T]$ and $A_\alpha \in \mathbf{\Sigma}^0_\lambda \restriction [T]$, i.e.,

$$\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction [T] = \left\{ A \subseteq [T] \,\middle|\, \exists \vec{A} = \langle A_\beta \,|\beta \leq \alpha\rangle \begin{pmatrix} \forall \beta \in \alpha \left(A_\beta \in \mathbf{\Pi}^1_1 \restriction [T]\right), \\ A_\alpha \in \mathbf{\Sigma}^0_\lambda \restriction [T] \text{ and } A = dk(\vec{A}) \end{pmatrix} \right\}.$$

⊣

We have a similar result for $\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda$ sets to corollary 2.4.42.

Corollary 2.4.44. *Assume $\alpha \in \omega_1$ is a limit ordinal and $\lambda \in \omega$. Then for any Y,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction X^\omega). \quad ⊣$$

[24] Recall that limit ordinals are even.

Proof.

A similar proof of corollary 2.4.42 with $\chi_{A_\alpha} \in \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda)$ by observation 2.4.26. $\square$

Question 4. *Suppose $\alpha \in \omega_1$ is a limit ordinal. By corollary 2.4.1 on page 83, corollary 2.4.44, corollary 2.4.21 on page 109 and corollary 2.4.27 on page 114, all of the following imply $Det((\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda) \restriction X^\omega)$:*

(i) $Det\mathcal{G}\,(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda; Tree_1\,(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1))$

(ii) $Det\ \mathcal{G}\,(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\,(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega))$

(iii) $Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \restriction X^\omega))$

(iv) $Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, \alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda), X^\omega))$

What is the relationship between (i), (ii), (iii) and (iv)? $\dashv$

Through out this section, we set that α is even so that $\alpha + 1$ is odd. One might ask for the case that α is odd, i.e., the case for $\alpha + 1$ even.

In this section, we generalized the idea of using $A \cap B$ from section 2.4.2.4 setting $A = A_\alpha$ and $B = \bigcap_{\beta \in \alpha} A_\beta$ so that $A \cap B = \bigcap_{\beta \leq \alpha} A_\beta$. This is because when α is even, $dk\,(\langle A_\beta \,|\beta \leq \alpha\rangle) = dk\,(\langle A_\beta \,|\beta \in \alpha + 1\rangle)$ and thus we have

$$dk\,(\langle A_\beta \,|\beta \leq \alpha\rangle) = dk\,(\langle A_\beta \,|\beta \in \alpha\rangle) \cup \left(\bigcap\nolimits_{\beta \leq \alpha} A_\beta\right).$$

However, when α is odd, then $\alpha + 1$ is even so that

$$\begin{aligned} dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) &= dk\left(\langle A_\beta \,|\beta \in \alpha+1\rangle\right) \\ &= \{x \in [T] \,|\mu\beta\left(x \notin A_\beta \vee \beta = \alpha+1\right) \text{ is odd}\} \\ &= \{x \in [T] \,|\exists\beta \in \alpha+1 (x \notin A_\beta) \wedge \mu\beta\left(x \notin A_\beta\right) \text{ is odd}\} \\ &= \left\{x \in [T] \,\middle|\, x \notin \bigcap\nolimits_{\beta\leq\alpha} A_\beta \wedge \mu\beta\left(x \notin A_\beta\right) \text{ is odd}\right\} \end{aligned}$$

Thus

$$\bigcap\nolimits_{\beta\leq\alpha} A_\beta \nsubseteq dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right).$$

Hence we do not obtain the same determinacy result for α is odd using the method we described.

2.5 Getting the determinacy of the games on a $Tree_1$ collection from the determinacy of the games on $X^{<\omega}$ (Reversed direction of section 2.4)

In section 2.4, we obtained the determinacy of games on $X^{<\omega}$ from the determinacy of games on a certain $Tree_1$ collection. In this section, we will focus on the other direction, in some cases, results from section 2.4 leading to the determinacy equivalences. This section is the main section in this chapter.

In section 2.5.1 through section 2.5.4, we will obtain the determinacy of games on a certain $Tree_1$ collection such that each tree $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection has a countable Y, from the determinacy of games on $X^{<\omega}$.

In section 2.5.1, we will give definitions and notations. We will set up all the notations in this section, e.g., given $A \subseteq [T^{\Psi,B}_{X,Y}]$, we will define the following notations:

- A^n for all $n \in \omega$.
- $A_\emptyset$.
- A^n_g for all $n \in \omega$ and $g \in Y^{n+1}$.
- A^n_p for all $n \in \omega$ and $p \in Y^{<n+1}$.

We will use these notations in the later sections.

In section 2.5.2, we will obtain level by level results for the determinacy of open games on a certain $Tree_1$ collection from the determinacy of games on $X^{<\omega}$. The main theorems in this section are theorem 2.5.18 on page 156 and theorem 2.5.20 on page 160.

In section 2.5.3, we will obtain level by level results for the determinacy of Borel games on a certain $Tree_1$ collection from the determinacy of certain games on $X^{<\omega}$. The main theorems in this section are theorem 2.5.29 on page 167 and theorem 2.5.30 on page 168.

In section 2.5.4, we will obtain level by level results for the determinacy of projective games on a certain $Tree_1$ collection from the determinacy of certain games on $X^{<\omega}$. The main theorem in this section is theorem 2.5.38 on page 183.

In section 2.5.5, we will discuss the reason we focused on Y to be countable in earlier sections 2.5.1 - 2.5.4 by using well-known results about uncountable $Y = \mathcal{N}$.

2.5.1 Getting the determinacy of games on a $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$

Notation 2.5.1. *(Definition of a $Tree_1$ collection with/over countable Y)*

Let $\mathcal{T}_1$ be a $Tree_1$ collection. Suppose for every Type 1 tree $T^{\Psi,B}_{X,Y} \in \mathcal{T}_1$, Y is countable. Then we say $\mathcal{T}_1$ is a "$Tree_1$ collection with/over countable Y". [25] $\dashv$

In sections 2.5.2 through 2.5.4, we will obtain level by level results for the the determinacy of games on a particular $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In section 2.5.2, we will obtain the determinacy of open games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In section 2.5.3, we will obtain the determinacy of Borel games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In section 2.5.4, we will obtain the determinacy of projective games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In this section, we will give definitions and prove some lemmas which we will use throughout sections 2.5.2 through 2.5.4.

For each Type 1 tree $T^{\Psi,B}_{X,Y}$ and $A \subseteq [T^{\Psi,B}_{X,Y}]$, we will find $A^* \subseteq X^\omega$ which will satisfy the following:

$$f \in A^* \text{ if and only if}$$
$$\text{there is a winning strategy at } f \text{ in the Type 1 tree } T^{\Psi,B}_{X,Y} \text{ for } G(A; T^{\Psi,B}_{X,Y}).$$

[25] Note that possibly different Y's for different T's. A $Tree_1$ collection with/over countable Y does not mean that Y is fixed.

We will now describe our A^*. Recall from page 48,

$$[T_{X,Y}^{\Psi,B}] = \bigcup_{n\in\omega} ((B \cap \Psi^{-1}(n)) \times Y^{n+1}) \dot{\cup} (X^\omega \backslash B).$$

We will split A into pairwise disjoint pieces $A_\emptyset$ and A^n for $n \in \omega$. $A_\emptyset$ will be a subset of $X^\omega \backslash B$ and A^n will be a subset of B for each $n \in \omega$. Then we will define A_g^n for each $n \in \omega$ and $g \in Y^{n+1}$ such that if a play f in A^* is in A_g^n, then $f^\frown g$ will be in A. Then, by backwards induction, we will define $A_\emptyset^n$ from $\{A_g^n | g \in Y^{n+1}\}$ using $n+1$ many unions and intersections (countable unions and countable intersections when Y is countable). Whenever a play f of A^* is in $A_\emptyset^n$, there is a canonical strategy at f to get into A. Let $A^* = \bigcup_{n\in\omega} A_\emptyset^n \cup A_\emptyset$. We will show that:

- If $f \in A^*$, then I has a winning strategy at f to get into A.
- If $f \notin A^*$, then II has a winning strategy at f to avoid A.

Definition 2.5.2. *Suppose $A \subseteq [T_{X,Y}^{\Psi,B}]$. For each $n \in \omega$, define*

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times Y^{n+1}),$$

$$A_\emptyset = A \cap (X^\omega \backslash B).$$

Then $A = \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset$. ⊣

In definition 2.5.3, for each $g \in Y^{n+1}$, we define $A_g^n \subseteq X^\omega$ as a collection of $f \in X^\omega$ such that $f^\frown g \in A^n$. In definition 2.5.4, by backwards induction, we will define for each $i < n+1 = lh(g)$, $A_{g\restriction i}^n$ from $\{A_{(g\restriction i)^\frown\langle m\rangle}^n | m \in Y\}$.

Definition 2.5.3. *Suppose $A \subseteq [T^{\Psi,B}_{X,Y}]$. For every $n \in \omega$ and $g \in Y^{n+1}$, define*

$$A^n_g = \{f \in X^\omega \,|\, f^\frown g \in A^n\}. \qquad \dashv$$

Since $A^n \subseteq (B \cap \Psi^{-1}(n)) \times Y^{n+1}$, $A^n_g \subseteq B \cap \Psi^{-1}(n)$ for every $g \in Y^{n+1}$.

Definition 2.5.4. *Suppose $A \subseteq [T^{\Psi,B}_{X,Y}]$. For all $n \in \omega$ and for all $p \in Y^{<n+1}$, define*

$$A^n_p \overset{df}{=} \begin{cases} \bigcup_{m \in Y} A^n_{p^\frown\langle m\rangle} & \text{if } lh\,(p) \text{ is even,} \\ \bigcap_{m \in Y} A^n_{p^\frown\langle m\rangle} & \text{if } lh\,(p) \text{ is odd.} \end{cases} \qquad \dashv$$

Note that for all $n \in \omega$ and for all $p \in Y^{\leq n+1}$, we have A^n_p. Definition 2.5.3 applies if $lh(p) = n+1$.

Observation 2.5.5. *Suppose $A \subseteq [T^{\Psi,B}_{X,Y}]$. For all $n \in \omega$ and for all $p \in Y^{\leq n+1}$, $A^n_p \subseteq B \cap \Psi^{-1}(n)$.* $\dashv$

Proof.

Pick an arbitrary $n \in \omega$. Suppose $p \in Y^{\leq n+1}$. The proof is by backwards induction on the length of p.

Base case : $lh(p) = n+1$.

Then we have $A^n_p = \{f \in X^\omega \,|\, f^\frown p \in A^n\} \subseteq B \cap \Psi^{-1}(n)$ since $A^n \subseteq (B \cap \Psi^{-1}(n)) \times Y^{n+1}$.

Induction step : As an induction hypothesis, assume that for all $p \in Y^{\leq n+1}$, if $lh(p) = l+1 \leq n+1$, then $A^n_p \subseteq B \cap \Psi^{-1}(n)$. Suppose $lh(p) = l$. Show $A^n_p \subseteq B \cap \Psi^{-1}(n)$. Pick an arbitrary $f \in A^n_p$.

Case 1 : l is even. Then $A^n_p = \bigcup_{m \in Y} A^n_{p^\frown\langle m\rangle}$. Then $f \in A^n_{p^\frown\langle m\rangle}$ for some $m \in Y$. Since $lh(p^\frown\langle m\rangle) = l+1$, by induction hypothesis, we have $A^n_{p^\frown\langle m\rangle} \subseteq B \cap \Psi^{-1}(n)$. Thus $f \in B \cap \Psi^{-1}(n)$. Since $f \in A^n_p$ is arbitrary, $A^n_p \subseteq B \cap \Psi^{-1}(n)$.

Case 2 : l is odd. Then $A_p^n = \bigcap_{m \in Y} A^n_{p^\frown\langle m\rangle}$. Then $f \in A^n_{p^\frown\langle m\rangle}$ for every $m \in Y$. Since $lh(p^\frown\langle m\rangle) = l+1$ for every $m \in Y$, by induction hypothesis, we have $A^n_{p^\frown\langle m\rangle} \subseteq B \cap \Psi^{-1}(n)$ for every $m \in Y$. Thus $f \in B \cap \Psi^{-1}(n)$. Since $f \in A_p^n$ is arbitrary, $A_p^n \subseteq B \cap \Psi^{-1}(n)$. □

For each strategy s^* on $X^{<\omega}$, we define the canonical strategy s on a Type 1 tree T. First, we define the canonical strategy for player I.

Definition 2.5.6. *(Definition of the canonical tail strategy s for player I)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Let $\mathcal{S}_I(X^{<\omega})$ be the set of strategies for I on $X^{<\omega}$ and let $\mathcal{S}_I(T)$ be the set of strategies for I on T. Define

$$\varphi_I : \mathcal{S}_I\left(X^{<\omega}\right) \to \mathcal{S}_I\left(T\right).$$

For each $s^* \in \mathcal{S}_I(X^{<\omega})$, Define $s = \varphi_I(s^*)$ as follows: For $p \in T \backslash [T]$ such that either p is finite and $p \in dom(s^*)$, or p is infinite and $lh(p)$ is even,

$$s(p) = \begin{cases} s^*(p) & \text{if } p \text{ finite}, \\ \mu m \in Y \left(p\restriction\omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega, lh(p))^\frown\langle m\rangle}\right)^{26} & \text{if } p\restriction\omega \in B \text{ and } p\restriction\omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))} = \bigcup_{m\in Y} A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}, \\ \mu m(m \in Y) & \text{otherwise},^{27} \end{cases}$$

when Y is well-orderable. Then s is a strategy for I for T.

(We define for the case that Y is well-orderable. See footnote (26) for the case that Y is not well-orderable.) ⊣

Lemma 2.5.7. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$ and $A \subseteq [T]$. Suppose I^* has a winning strategy s^* for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$. Then the canonical tail strategy $s = \varphi_I(s^*)$ is a winning*

[26] μ represents "the least". If Y is well-orderable, fix a well-ordering of Y. Otherwise, pick any $m \in Y$ such that $p \restriction \omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}$.

[27] This otherwise case does not occur for plays of interest. If Y is not well-orderable, pick any $m \in Y$.

strategy for I for $G(A;T)$. ⊣

Proof.

Pick an arbitrary $h \in [T] = [T^{\Psi,B}_{X,Y}]$ according to s. Show $h \in A$. Since $h \upharpoonright \omega$ is according to s, $h \upharpoonright \omega$ is according to s^*. Since s^* is a I^*'s winning strategy for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$, $h \upharpoonright \omega \in \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

Case 1 : $h \upharpoonright \omega \in B$.

Then $h \upharpoonright \omega \notin A_\emptyset$. By observation 2.5.5, $h \upharpoonright \omega \notin A^l_\emptyset$ for any $l \neq \Psi(h \upharpoonright \omega)$. Thus $h \upharpoonright \omega \in A^{\Psi(h\upharpoonright\omega)}_\emptyset$. Since h is according to the canonical tail strategy $s = \varphi_I(s^*)$ for I, $h \upharpoonright \omega \in A^{\Psi(h\upharpoonright\omega)}_{\langle h(\omega)\rangle}$. Since $lh(h \upharpoonright (\omega+1)) = \omega + 1$, by definition, $A^{\Psi(h\upharpoonright\omega)}_{\langle h(\omega)\rangle} = \bigcap_{m\in Y} A^n_{\langle h(\omega)\rangle^\frown\langle m\rangle}$. Thus for any II's move $m \in Y$, $h \upharpoonright \omega \in A^{\Psi(h\upharpoonright\omega)}_{\langle h(\omega),m\rangle}$. In particular, $h \upharpoonright \omega \in A^{\Psi(h\upharpoonright\omega)}_{\langle h(\omega),h(\omega+1)\rangle}$. Repeat this argument. Eventually, we get $h \upharpoonright \omega \in A^{\Psi(h\upharpoonright\omega)}_{h\upharpoonright[\omega,lh(h))}$. Since $lh(h \upharpoonright [\omega, lh(h))) = \Psi(h \upharpoonright \omega) + 1$, $h = (h \upharpoonright \omega)^\frown h \upharpoonright [\omega, lh(h)) \in A^{\Psi(h\upharpoonright\omega)} \subseteq A$.

Case 2 : $h \upharpoonright \omega \notin B$.

By observation 2.5.5, $h \upharpoonright \omega \notin A^n_\emptyset$ for any $n \in \omega$. Thus $h = h \upharpoonright \omega \in A_\emptyset \subseteq A$.

In either case, $h \in A$. Hence the canonical tail strategy $s = \varphi_I(s^*)$ is a winning strategy for I for $G(A;T)$. □

Now, we define the canonical strategy for player II.

Definition 2.5.8. *(Definition of the canonical tail strategy s for player II)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Let $\mathcal{S}_{II}(X^{<\omega})$ be the set of strategies for II on $X^{<\omega}$ and let $\mathcal{S}_{II}(T)$ be the set of strategies for II on T. Define

$$\varphi_{II} : \mathcal{S}_{II}\left(X^{<\omega}\right) \to \mathcal{S}_{II}\left(T\right).$$

For each $s^* \in \mathcal{S}_{II}(X^{<\omega})$, define $s = \varphi_{II}(s^*)$ as follows: For $p \in T \backslash [T]$ such that either p is finite and $p \in dom(s^*)$, or p is infinite and $lh(p)$ is odd,

$$s(p)=\begin{cases} s^*(p) & \text{if } p \text{ finite,} \\ \mu m{\in}Y\left(p{\restriction}\omega{\notin}A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}\right)^{28} & \text{if } p{\restriction}\omega{\in}B \text{ and } p{\restriction}\omega{\notin}A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))}{=}\bigcap_{m\in Y}A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}, \\ \mu m(m{\in}Y) & \text{otherwise,}^{29} \end{cases}$$

when Y is well-orderable. Then s is a strategy for II for T.

(We define for the case that Y is well-orderable. See footnote (28) for the case that Y is not well-orderable.) ⊣

Lemma 2.5.9. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$ and $A \subseteq [T]$. Suppose s^* is a II^*'s winning strategy for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$. Then the canonical tail strategy $s = \varphi_{II}(s^*)$ is a winning strategy for II for $G(A;T)$.* ⊣

Proof.

Pick an arbitrary $h \in [T] = [T^{\Psi,B}_{X,Y}]$ according to s. Show $h \notin A$. Then $h \restriction \omega$ is according to s^*. Since s^* is II^*'s winning strategy for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$, $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

Case 1 : $h \restriction \omega \in B$.

Since $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_\emptyset$. By definition, $A^{\Psi(h\restriction\omega)}_\emptyset = \bigcup_{m\in Y} A^{\Psi(h\restriction\omega)}_{\langle m\rangle}$. Thus for any I's move $m \in Y$, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle m\rangle}$. In particular, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle h(\omega)\rangle}$. By definition, $A^{\Psi(h\restriction\omega)}_{\langle h(\omega)\rangle} = \bigcap_{m\in Y} A^n_{\langle h(\omega)\rangle^\frown\langle m\rangle}$. Since h is according to the canonical tail strategy $s = \varphi_{II}(s^*)$ for II, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle h(\omega),h(\omega+1)\rangle}$. Repeat this argument. Eventually, we get $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{h\restriction[\omega,lh(h))}$. Since $lh(h \restriction [\omega, lh(h))) = \Psi(h \restriction \omega) + 1$, $h = (h \restriction \omega)^\frown h \restriction [\omega, lh(h)) \notin A^{\Psi(h\restriction\omega)}$. By observation 2.5.5, $h \restriction \omega \notin A^l_\emptyset$ for any $l \neq \Psi(h \restriction \omega)$. Hence $h \notin \bigcup_{n\in\omega} A^n \dot\cup A_\emptyset = A$.

[28] μ represents "the least". If Y is well-orderable, fix a well-ordering of Y. Otherwise, pick any $m \in Y$ such that $p \restriction \omega \notin A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}$.

[29] This otherwise case does not occur for plays of interest. If Y is not well-orderable, pick any $m \in Y$.

Case 2 : $h \restriction \omega \notin B$.

Since $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$, $h = h \restriction \omega \notin A_\emptyset$. By observation 2.5.5, $h \restriction \omega \notin A^n_\emptyset$ for any $n \in \omega$. Hence $h \notin \bigcup_{n\in\omega} A^n \dot\cup A_\emptyset = A$.

In either case, $h \notin A$. Hence the canonical tail strategy $s = \varphi_{II}(s^*)$ is a winning strategy for II for $G(A;T)$. $\square$

Let $\varphi = \varphi_I \dot\cup \varphi_{II}$. Then φ takes strategies on $X^{<\omega}$ to strategies on $T^{\Psi,B}_{X,Y}$. By lemmas 2.5.7 and 2.5.9, we have the following.

Theorem 2.5.10. *If $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined, then $G(A; T^{\Psi,B}_{X,Y})$ is determined.*

$\dashv$

Now, we will find the complexity of $A^n_\emptyset$ for each $n \in \omega$ assuming some fixed complexity for each A^n_g.

Lemma 2.5.11. *Suppose $n, m \in \omega$, $m > 1$ and $\alpha \in \omega_1$.*

1. *If for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ and Y is finite, then $A^n_\emptyset \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$.*

2. *If for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ and Y is denumerable, $A^n_\emptyset \in \mathbf{\Sigma}^0_{\alpha+\omega} \restriction X^\omega$.*

3. *If for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Sigma}^1_m \restriction X^\omega$ and Y is countable, then $A^n_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$.*

4. *If for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Pi}^1_m \restriction X^\omega$ and Y is countable, then $A^n_\emptyset \in \mathbf{\Pi}^1_m \restriction X^\omega$.*

5. *If for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Delta}^1_m \restriction X^\omega$ and Y is countable, then $A^n_\emptyset \in \mathbf{\Delta}^1_m \restriction X^\omega$.*

6. *If Λ is an algebra, for all $g \in Y^{n+1}$, $A^n_g \in \Lambda \restriction X^\omega$ and Y is finite, then $A^n_\emptyset \in \Lambda \restriction X^\omega$.*

7. *If Λ is a σ-algebra, for all $g \in Y^{n+1}$, $A^n_g \in \Lambda \restriction X^\omega$ and Y is countable, then $A^n_\emptyset \in \Lambda \restriction X^\omega$.* $\dashv$

Proof.

Fix $n \in \omega$. Then

$$A^n_\emptyset = \begin{cases} \bigcup_{a_0 \in Y} \bigcap_{a_1 \in Y} \cdots \bigcup_{a_n \in Y} A^n_{\langle a_0,\ldots,a_n\rangle} & \text{if } n \text{ is even,} \\ \bigcup_{a_0 \in Y} \bigcap_{a_1 \in Y} \cdots \bigcap_{a_n \in Y} A^n_{\langle a_0,\ldots,a_n\rangle} & \text{if } n \text{ is odd.} \end{cases}$$

Show (1). Suppose for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ and Y is finite. Prove $A^n_p \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ by backwards induction on the length of $p \in Y^{\leq n+1}$.

Base Case : $lh(p) = n + 1$.

Then $p \in Y^{n+1}$. Thus $A^n_p \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$.

Induction Step : Let $k \leq n$. Assume, as an induction hypothesis, for any $p \in Y^{\leq n+1}$ such that $lh(p) = k + 1$, $A^n_p \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ for some $l \in \omega$. Pick an arbitrary $p \in Y^{\leq n+1}$ such that $lh(p) = k$.

Case 1 : $lh(p)$ is even.

Then $\bigcup_{m \in Y} A^n_{p^\frown\langle m\rangle} \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ since $\mathbf{\Sigma}^0_\alpha \restriction X^\omega$ is closed under finite unions.

Case 2 : $lh(p)$ is odd.

Then $\bigcap_{m \in Y} A^n_{p^\frown\langle m\rangle} \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ since $\mathbf{\Sigma}^0_\alpha \restriction X^\omega$ is closed under finite intersections.

In particular, when $k = 0$, $A^n_\emptyset \in \mathbf{\Sigma}^0_{\alpha+\omega} \restriction X^\omega$.

Show (2). Suppose for all $g \in Y^{n+1}$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$. Suppose Y is denumerable. Prove $A^n_p \in \mathbf{\Sigma}^0_{\alpha+l} \restriction X^\omega$ for some $l \in \omega$ by backwards induction on the length of $p \in Y^{\leq n+1}$.

Base Case : $lh(p) = n + 1$.

$A^n_p \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$.

Induction Step : Let $k \leq n$. Assume for any $p \in Y^{\leq n+1}$ such that $lh(p) = k + 1$,

$A_p^n \in \mathbf{\Sigma}^0_{\alpha+l} \restriction X^\omega$ for some $l \in \omega$. Pick an arbitrary $p \in Y^{\leq n+1}$ such that $lh(p) = k$.

$$A_p^n = \begin{cases} \bigcup_{m \in Y} A^n_{p^\frown\langle m\rangle} \in \mathbf{\Sigma}^0_{\alpha+l} \restriction X^\omega & \text{if } k \text{ is even,} \\ \bigcap_{m \in Y} A^n_{p^\frown\langle m\rangle} \in \mathbf{\Pi}^0_{\alpha+l+1} \restriction X^\omega \subseteq \mathbf{\Sigma}^0_{\alpha+l+2} \restriction X^\omega & \text{if } k \text{ is odd.} \end{cases}$$

Thus $A_p^n \in \mathbf{\Sigma}^0_{\alpha+l+2} \restriction X^\omega$.

In particular, when $k = 0$, $A^n_\emptyset \in \mathbf{\Sigma}^0_{\alpha+\omega} \restriction X^\omega$.

Show (3). Suppose each $A^n_g \in \mathbf{\Sigma}^1_m \restriction X^\omega$ and Y is countable. Since $\mathbf{\Sigma}^1_m$ is closed under countable unions by lemma 2.3.22 and $\mathbf{\Sigma}^1_m$ is closed under countable intersections by lemma 2.5.13 below, by the similar argument as (1) (replace $\mathbf{\Sigma}^0_\alpha$ to $\mathbf{\Sigma}^1_m$ and finite to countable), we have $A^n_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$. Similarly for (4), the case for $\mathbf{\Pi}^1_m$ and (5), the case for $\mathbf{\Delta}^1_m$.

Show (6). Suppose Λ is an algebra, each $A^n_g \in \Lambda \restriction X^\omega$ and Y is finite. Since Λ is closed under finite unions and finite intersections, by the similar argument as (1) (replace $\mathbf{\Sigma}^0_\alpha$ to Λ), we have $A^n_\emptyset \in \Lambda \restriction X^\omega$.

Show (7). Suppose Λ is a σ-algebra, each $A^n_g \in \Lambda \restriction X^\omega$ and Y is countable. Since Λ is closed under countable unions and countable intersections, by the similar argument as (1) (replace $\mathbf{\Sigma}^0_\alpha$ to Λ and finite to countable), we have $A^n_\emptyset \in \Lambda \restriction X^\omega$. $\square$

Next, we will find the complexity of $A_\emptyset$ and A^n_g for each $g \in Y^{n+1}$.

Lemma 2.5.12. *Let $n \in \omega$ be arbitrary. Suppose Λ_0 and Λ_1 are complexities. Suppose $\Psi^{-1}(n) \in \Lambda_0 \restriction X^\omega$, $B \in \Lambda_1 \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. Then for every $g \in Y^{n+1}$,*

$$A^n_g \in \left(\mathbf{\Sigma}^0_1 \wedge \Lambda_0 \wedge \Lambda_1\right) \restriction X^\omega \text{ and } A_\emptyset \in (\mathbf{\Sigma}^0_1 \wedge co\text{-}\Lambda_1) \restriction X^\omega. \qquad \dashv$$

Proof.

Pick arbitrary $n \in \omega$ and $g \in Y^{n+1}$. First, we consider A^n_g. Since $g \in Y^{n+1}$, $g \neq \emptyset$. Since

$A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$,

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times Y^{n+1}) \in \mathbf{\Sigma}^0_1 \restriction (B \cap \Psi^{-1}(n)) \times Y^{n+1}.$$

Thus there exist $\langle O_i \,|i \in \omega\rangle$ such that $A^n = \bigcup_{i\in\omega} O_i$ where each O_i is a basic open neighborhood of $(B \cap \Psi^{-1}(n)) \times Y^{n+1}$, i.e., there exists $p_i \in X^{<\omega}$ and $q_i \in Y^{n+1}$ such that

$$O_i = \{h \in (B \cap \Psi^{-1}(n)) \times Y^{n+1} \mid h \restriction \omega \supseteq p_i \wedge h \restriction [\omega, \omega + n + 1) = q_i\}.$$

Define $G = \{i \in \omega \,|q_i = g\}$. Then

$$\begin{aligned} A^n_g &\overset{df}{=} \{f \in X^\omega \,|f^\frown g \in A^n\} \\ &= \left\{ f \in X^\omega \,\middle|\, f^\frown g \in \bigcup_{i\in\omega} O_i \right\} \\ &= \bigcup_{i\in\omega} \{f \in X^\omega \,|f^\frown g \in O_i\} \\ &= \bigcup_{i\in G} \{f \in X^\omega \,|f^\frown g \in O_i\} \\ &\overset{(*)}{=} \underbrace{\bigcup_{i\in G} \{f \in X^\omega \,|f \supseteq p_i\}}_{\mathbf{\Sigma}^0_1\restriction X^\omega} \cap \underbrace{\Psi^{-1}(n)}_{\Lambda_0\restriction X^\omega} \cap \underbrace{B}_{\Lambda_1\restriction X^\omega} \\ &\in \left(\mathbf{\Sigma}^0_1 \wedge \Lambda_0 \wedge \Lambda_1\right) \restriction X^\omega. \end{aligned}$$

[Proof of $(*)$]

($\subseteq$) Pick an arbitrary $\hat{f} \in \bigcup_{i\in G} \{f \in X^\omega \,|f^\frown g \in O_i\}$. Then $\exists i \in G$ such that $\hat{f}^\frown g \in O_i$. Thus $\hat{f} \supseteq p_i$ and $g = q_i$. Since $f^\frown g \in [T^{\Psi,B}_{X,Y}]$ and $g \in Y^{n+1}$, $\hat{f} \in \Psi^{-1}(n) \cap B$. Hence $\hat{f} \in \bigcup_{i\in G} \{f \in X^\omega \,|f \supseteq p_i\} \cap \Psi^{-1}(n) \cap B$.

($\supseteq$) Pick an arbitrary $\hat{f} \in \bigcup_{i\in G}\{f \in X^\omega \mid f \supseteq p_i\} \cap \Psi^{-1}(n) \cap B$. Then $\exists i \in G$ such that $\hat{f} \supseteq p_i$ and $\hat{f} \in \Psi^{-1}(n) \cap B$. By the tail exchange property of the tree $T^{\Psi,B}_{X,Y}$, $\forall \hat{g} \in Y^{n+1}$, $\hat{f}^\frown \hat{g} \in [T^{\Psi,B}_{X,Y}]$. In particular, $\hat{f}^\frown g \in [T^{\Psi,B}_{X,Y}]$. Since $i \in G$, $q_i = g$. Thus $\hat{f}^\frown g \in O_i$. Hence

$\hat{f} \in \bigcup_{i \in G} \{f \in X^\omega \,|\, f^\frown g \in O_i\}$. $\square(*)$

Now, we consider $A_\emptyset$. Define $J = \{i \in \omega \,|\, q_i = \emptyset\}$. Then for all $f \in X^\omega$,

$$f \in A_\emptyset \Leftrightarrow f \in A \cap \underbrace{(X^\omega \backslash B)}_{co\text{-}\Lambda_1 \restriction X^\omega} \Leftrightarrow f \in \underbrace{\exists j \in J\,(f \supseteq p_j)}_{\mathbf{\Sigma}^0_1 \restriction X^\omega} \wedge \underbrace{X^\omega \backslash B}_{co\text{-}\Lambda_1 \restriction X^\omega}.$$

Thus $A_\emptyset \in (\mathbf{\Sigma}^0_1 \wedge co\text{-}\Lambda_1) \restriction X^\omega$. $\square$

By lemmas 2.5.12 and 2.5.11, we obtain the complexity of $A_\emptyset$ and A^n_g for all $n \in \omega$ and $g \in Y^{n+1}$ from the complexity of B and Ψ. In the next section, we will obtain the determinacy of open games on $Tree_1$ collections from the determinacy of games on $X^{<\omega}$ by using theorem 2.5.10 lemma 2.5.11 and lemma 2.5.12.

We used lemma 2.5.13, well-known closure property of projective sets, in the proof of lemma 2.5.11. Sierpinski showed this property in 1928 (as cited in Moschovakis, 2009, p. 47). The following is a proof for this property. Readers familiar with this proof may skip the rest of this section.

Lemma 2.5.13. *Let $n \in \omega \backslash \{0\}$.*

1. *$\mathbf{\Sigma}^1_n \restriction X^\omega$ is closed under countable intersections.*
2. *$\mathbf{\Pi}^1_n \restriction X^\omega$ is closed under countable unions.* $\dashv$

First, we prove sublemma 2.5.14. Given $S^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$, we will define $S^k_{i\pm}$ in definition 2.5.15 by using sublemma 2.5.14. Then we will prove sublemma 2.5.17 by using sublemma 2.5.16. We will use sublema 2.5.17 to prove lemma 2.5.13. The proof of lemma 2.5.13 is on page 154.

Sublemma 2.5.14.

1. *$\mathcal{N}^\omega$ is homeomorphic to $\mathcal{N}$.*

2. *For any $k \in \omega$, $\mathcal{N}^k$ is homeomorphic to $\mathcal{N}$.* ⊣

Proof.

Show (1). Fix a bijection $\pi : \omega \times \omega \xrightarrow[onto]{1-1} \omega$. Define

$$\varphi : \qquad \mathcal{N} \to \mathcal{N}^\omega$$

$$f \mapsto \langle f_n \,| n \in \omega \rangle$$

where each $f_n(i) = f(\pi(\langle n, i\rangle))$.

Show φ is a homeomorphism.

1. Show φ is one to one. Suppose $f, g \in \mathcal{N}$ such that $f \neq g$. Show $\varphi(f) \neq \varphi(g)$.

   Since $f \neq g$, there exists $k \in \omega$ such that $f(k) \neq g(k)$. Since π is a bijection, there exists $n, i \in \omega$ such that $\pi^{-1}(k) = \langle n, i\rangle$. Then

   $$f_n(i) = f(\pi(\langle n, i\rangle)) = f(k) \neq g(k) = g(\pi(\langle n, i\rangle)) = g_n(i).$$

   Thus $f_n \neq g_n$. Hence

   $$\varphi(f) = \langle f_n \,| n \in \omega\rangle \neq \langle g_n \,| n \in \omega\rangle = \varphi(g).$$

2. Show φ is onto. Pick an arbitrary $\langle f_n \,| n \in \omega\rangle \in \mathcal{N}^\omega$. Since π is a bijection, for each $k \in \omega$, there exists $n_k, i_k \in \omega$ such that $\pi^{-1}(k) = \langle n_k, i_k\rangle$. For each $k \in \omega$, define

   $$f(k) = f_{n_k}(i_k).$$

   Then $f \in \mathcal{N}$. Since each $f_{n_k}(i_k) = f(k) = f(\pi(\langle n_k, i_k\rangle))$, $\varphi(f) = \langle f_n \,| n \in \omega\rangle$.

3. Show φ is continuous. Pick an arbitrary $O \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}^\omega$. Show $\varphi^{-1}(O) \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}$. Since $O \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}^\omega$, $O = \bigcup_{l \in \omega} O_l$ where each

$$O_l = \prod_{j \in \omega} U_j^l$$

and

- each $U_j^l \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}$.
- for each $l \in \omega$, $E^l = \left\{ j \in \omega \,\middle|\, U_j^l \neq \mathcal{N} \right\}$ is finite.

Pick an arbitrary $f \in \varphi^{-1}(O)$. Then $\varphi(f) = \langle f_n \,|n \in \omega \rangle \in O$. There exists $j \in \omega$ such that $\varphi(f) = \langle f_n \,|n \in \omega \rangle \in O_l$. For each $j \in E^l$, $f_j \in U_j^l$. Since $U_j^l \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}$, there exists finite $F_j^l \subseteq \omega$ such that for any $g_j \in \mathcal{N}$ with $g_j \supseteq f_j \upharpoonright F_j^l$, $g_j \in U_j^l$. Define

$$G^l = \left\{ k \,\middle|\, \exists j \in E^l \left(\pi^{-1}(k) \in E^l \times F_j^l \right) \right\}.$$

Since E^l is finite, each F_j^l is finite, and π is a bijection, $G^l \subseteq \omega$ is finite. Pick an arbitrary $g \in \mathcal{N}$ such that $g \supseteq f \upharpoonright G^l$. Show $g \in \varphi^{-1}(O)$. Show $\varphi(g) = \langle g_n \,|n \in \omega \rangle \in O_l$. Pick arbitrary $j \in E^l$ and $i \in F_j^l$. Then $\pi(\langle j, i \rangle) \in G^l$. Thus

$$g_j(i) = g(\pi(\langle j, i \rangle)) = f(\pi(\langle j, i \rangle)) = f_j(i).$$

Since $i \in F_j^l$ is arbitrary, $g_j \supseteq f_j \upharpoonright F_j^l$. Thus $g_j \in U_j^l$. Since $j \in E^l$ is arbitrary, for each $j \in E^l$, $g_j \in U_j^l$. Thus $\varphi(g) = \langle g_n \,|n \in \omega \rangle \in O_l \subseteq O$. Therefore, $g \in \varphi^{-1}(O)$.

4. Show φ^{-1} is continuous. Pick an arbitrary $O \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}$. Show $\varphi(O) \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}^\omega$. Pick an arbitrary $\langle f_n \,|n \in \omega \rangle \in \varphi(O)$. Then there exists $f \in O$ such that $\varphi(f) = \langle f_n \,|n \in \omega \rangle$. Since $O \in \mathbf{\Sigma}_1^0 \upharpoonright \mathcal{N}$, there exists finite $G \subseteq \omega$ such that for any $g \in \mathcal{N}$, if $g \supseteq f \upharpoonright G$, $g \in O$. Let $F = \{ j \,| \exists i \in \omega \, (\pi(\langle i, i \rangle) \in G) \}$. Since G is finite, F is finite.

For any $j \in F$, define

$$U_j = \{h \in \mathcal{N} \,|\forall i \in \omega \text{ if } \pi(\langle j, i\rangle) \in G \text{ then } h(i) = f_j(i)\}.$$

Since G is finite, each $U_j \in \mathbf{\Sigma}^0_1 \restriction \mathcal{N}$. Define

$$E = \prod_{j \in \omega} H_j$$

where

$$H_j = \begin{cases} U_j & \text{if } j \in F, \\ \mathcal{N} & \text{otherwise.} \end{cases}$$

Then $E \in \mathbf{\Sigma}^0_1 \restriction \mathcal{N}^\omega$. Show $E \subseteq \varphi(O)$. Pick an arbitrary $\langle g_n \,| n \in \omega\rangle \in E$. Then for any $j \in F$, $g_j \in U_j$. Let $\varphi(g) = \langle g_n \,| n \in \omega\rangle$. Show $g \in O$. Since for every $j \in F$, $g_j \in U_j$, for any $j \in F$ and $i \in \omega$, if $\pi(\langle j, i\rangle) \in G$, then $g_j(i) = f_j(i)$. Thus for any $i, j \in \omega$, if $\pi(\langle i, i\rangle) \in G$, then $g_j(i) = f_j(i)$. Hence for any $k \in G$, $g(k) = f(k)$. Therefore, $g \in O$ so that $E \subseteq \varphi(O)$. Hence $\varphi(O) \in \mathbf{\Sigma}^0_1 \restriction \mathcal{N}^\omega$.

Consequently, by (1)-(4), φ is a homeomorphism.

Show (2).

Pick an arbitrary $k \in \omega$. Fix a bijection $\pi : k \times \omega \xrightarrow[onto]{1-1} \omega$. The rest of proof is the same as the proof of (1). □

Definition 2.5.15. *Let $k, i \in \omega$. Suppose φ is the homeomorphism defined in 2.5.14 and for each $h \in \omega^\omega$, $\varphi(h) = \langle h_n \,| n \in \omega\rangle$. Suppose $S^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$. Define*

$$S^k_{i\pm} = \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h_i, g_1, ..., g_k\rangle \in S^k\right\}. \qquad \dashv$$

Sublemma 2.5.16. *Let $k, i \in \omega$. Suppose φ is the homeomorphism defined in 2.5.14 and for each $h \in \omega^\omega$, $\varphi(h) = \langle h_n \,|n \in \omega \rangle$. Suppose $S^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$. Then*

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k_{i\pm} = \left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k\right)_{i\pm}. \qquad \dashv$$

Proof.

Fix $k \in \omega$ and $S^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$.

$$\begin{aligned}
&\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k_{i\pm} \\
&= \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h_i, g_1, ..., g_k\rangle \in S^k\right\} \\
&= \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h_i, g_1, ..., g_k\rangle \in \left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k\right\} \\
&= \left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k\right)_{i\pm}
\end{aligned} \qquad \square$$

Sublemma 2.5.17. *Suppose $n, k, i \in \omega$. Suppose φ is the homeomorphism defined in 2.5.14 and for each $h \in \omega^\omega$, $\varphi(h) = \langle h_n \,|n \in \omega \rangle$. Suppose $S^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$.*

1. *If $S^k \in \mathbf{\Sigma}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, then $S^k_{i\pm} \in \mathbf{\Sigma}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.*
2. *If $S^k \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, then $S^k_{i\pm} \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.* $\dashv$

Proof.

We prove both (1) and (2) simultaneously by induction on n. (2) follows from (1) and sublemma 2.5.16.

Base Case : $n = 0$.

Pick an arbitrary $k \in \omega$.

Suppose $S^k \in \mathbf{\Sigma}^0_1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $S^k_{i\pm} \in \mathbf{\Sigma}^0_1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Pick an arbitrary $\langle f, h, g_1, ..., g_k\rangle \in S^k_{\pm}$. Then $\langle f, h_i, g_1, ..., g_k\rangle \in S^k$. Since $S^k \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, there exist finite $F \subseteq \omega$, $H_i \subseteq \omega$ and $G_i \subseteq \omega$, $1 \leq i \leq k$ such that for all $\langle x, y, z_1, ..., z_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \upharpoonright F$, $y \supseteq h_i \upharpoonright H_i$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \upharpoonright G_i$, then $\langle x, y, z_1, ..., z_k\rangle \in S^k$. Define

$$O = \prod_{j\in\omega} U_j$$

where

$$U_j = \begin{cases} \{y \in \omega^\omega \,|y \supseteq h_i \upharpoonright H_i\} & j = i, \\ \mathcal{N} & \text{otherwise.} \end{cases}$$

Then $O \in \mathbf{\Sigma}^0_1 \upharpoonright \mathcal{N}^\omega$. Since $\varphi : \mathcal{N} \to \mathcal{N}^\omega$ is continuous, $\varphi^{-1}(O) \in \mathbf{\Sigma}^0_1 \upharpoonright \mathcal{N}$. Let $H = \varphi^{-1}(O)$. Suppose $y \in \omega^\omega$ and $y \supseteq h \upharpoonright H$. Then $\varphi(y) = \langle y_n \,|n \in \omega\rangle \in O$ so that $y_i \supseteq h_i \upharpoonright H_i$. Thus for all $\langle x, y, z_1, ..., z_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \upharpoonright F$, $y \supseteq h \upharpoonright H$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \upharpoonright G_i$, then $\langle x, y_i, z_1, ..., z_k\rangle \in S^k$. Hence, for all $\langle x, y, z_1, ..., z_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k$, if $x \supseteq f \upharpoonright F$, $y \supseteq h \upharpoonright H$ and for all $1 \leq i \leq k$, $z_i \supseteq g_i \upharpoonright G_i$, then $\langle x, y, z_1, ..., z_k\rangle \in S^k_{i\pm}$. Thus $S^k_{i\pm} \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Suppose $S^k \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $S^k_{i\pm} \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. $\cdots (*)$

Since $S^k \in \mathbf{\Pi}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Since we have already shown (1) for $n = 0$, we have:

$$\left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k\right)_{i\pm} \in \mathbf{\Sigma}^0_1 \upharpoonright (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

By sublemma 2.5.16,

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_{i\pm} = \left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k\right)_{i\pm}$$

Thus $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k_{i\pm} \in \mathbf{\Sigma}^0_1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Hence

$$S^k_{i\pm} \in \mathbf{\Pi}^0_1 \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

Induction Step : Assume that, as an induction hypothesis, for all $l \in \omega$, if $S^l \in \mathbf{\Sigma}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$, then $S^l_{i\pm} \in \mathbf{\Sigma}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$ and if $S^l \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$, then $S^k_{i\pm} \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^l)$.

Pick an arbitrary $k \in \omega$.

Suppose $S^k \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $S^k_{i\pm} \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Since $S^k \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, there exists $S^{k+1} \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$ such that for any $\langle x, y, z_1, ..., z_k\rangle \in (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, $\langle x, y, z_1, ..., z_k\rangle \in S^k$ if and only if there exists $z_{k+1} \in \omega^\omega$ such that $\langle x, y, z_1, ..., z_k, z_{k+1}\rangle \in S^{k+1}$.

$$\begin{aligned} S^k_{i\pm} &= \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h_i, g_1, ..., g_k\rangle \in S^k\right\} \\ &= \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \exists g_{k+1} \in \omega^\omega \, \langle f, h_i, g_1, ..., g_k, g_{k+1}\rangle \in S^{k+1}\right\} \\ &= \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \exists g_{k+1} \in \omega^\omega \, \langle f, h, g_1, ..., g_k, g_{k+1}\rangle \in S^{k+1}_{i\pm}\right\}. \end{aligned}$$

Since each $S^{k+1} \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$, by induction hypothesis, $S^{k+1}_{i\pm} \in \mathbf{\Pi}^1_n \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^{k+1})$. Thus $S^k_{i\pm} \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$.

Suppose $S^k \in \mathbf{\Pi}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Show $S^k_{i\pm} \in \mathbf{\Pi}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. We repeat the same proof of $(*)$ on page 152.

Since $S^k \in \mathbf{\Pi}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$, $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Since we have already shown (1) for the case $n+1$, we have:

$$\left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \setminus S^k\right)_{i\pm} \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k).$$

By sublemma 2.5.16,

$$\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k_{i\pm} = \left(\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k\right)_{i\pm}$$

Thus $\left(X^\omega \times \omega^\omega \times (\omega^\omega)^k\right) \backslash S^k_{i\pm} \in \mathbf{\Sigma}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k)$. Hence

$$S^k_{i\pm} \in \mathbf{\Pi}^1_{n+1} \restriction (X^\omega \times \omega^\omega \times (\omega^\omega)^k). \qquad \square$$

Using sublemma 2.5.16, we prove lemma 2.5.13. Recall lemma 2.5.13. (2) is obtained from (1).

Lemma 2.5.13. *Let* $n \in \omega\backslash\{0\}$.

1. $\mathbf{\Sigma}^1_n \restriction X^\omega$ *is closed under countable intersections.*

2. $\mathbf{\Pi}^1_n \restriction X^\omega$ *is closed under countable unions.* $\dashv$

Proof.

Suppose φ is the homeomorphism defined in 2.5.14 and for each $h \in \omega^\omega$, $\varphi(h) = \langle h_n \,| n \in \omega\rangle$. Show (1). Assume that $n > 0$. Show $\mathbf{\Sigma}^1_n \restriction X^\omega$ is closed under countable intersections. Let $\langle A_i | i \in \omega\rangle$ be such that each $A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Since each $A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$, there exists $C_i \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega$ such that

$$f \in A_i \Leftrightarrow \exists g \in \omega^\omega \, \langle f, g\rangle \in C_i.$$

Show $\bigcap_{i\in\omega} A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$.

$$\begin{aligned} f \in \bigcap_{i\in\omega} A_i &\Leftrightarrow \forall i \in \omega \, (f \in A_i) \\ &\Leftrightarrow \forall i \in \omega \exists g \in \omega^\omega \, (\langle f, g\rangle \in C_i) \\ &\Leftrightarrow \exists h \in \omega^\omega \forall i \in \omega \, (\langle f, h_i\rangle \in C_i) \\ &\Leftrightarrow \exists h \in \omega^\omega \langle f, h\rangle \in \bigcap_{i\in\omega} (C_i)_{i\pm} \end{aligned}$$

Since each $C_i \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega$, by sublemma 2.5.16,

$$(C_i)_{i\pm} \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega$$

so that

$$\bigcap_{i\in\omega} (C_i)_{i\pm} \in \mathbf{\Pi}^0_{n-1} \restriction X^\omega \times \omega^\omega.$$

Thus $\bigcap_{i\in\omega} A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$.

Show (2). Suppose $\langle A_i | i \in \omega\rangle$ be such that each $A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$. Show $\bigcup_{i\in\omega} A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$.

Since each $A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$, $X^\omega \backslash A_i \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Since we have already shown (1), we have:

$$\bigcap\nolimits_{i\in\omega} (X^\omega \backslash A_i) \in \mathbf{\Sigma}^1_n \restriction X^\omega.$$

Since $X^\omega \backslash \left(\bigcup_{i\in\omega} A_i\right) = \bigcap_{i\in\omega} (X^\omega \backslash A_i)$, $X^\omega \backslash \left(\bigcup_{i\in\omega} A_i\right) \in \mathbf{\Sigma}^1_n \restriction X^\omega$. Thus $\bigcup_{i\in\omega} A_i \in \mathbf{\Pi}^1_n \restriction X^\omega$. □

2.5.2 Obtaining the determinacy of open games on a $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$

In section 2.5.1, we defined notations and proved some lemmas. In this section, we will obtain open determinacy on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$ by using theorem 2.5.10, lemma 2.5.11 and lemma 2.5.12. The main theorems of this section are theorem 2.5.18, theorem 2.5.20 and theorem 2.5.26. In theorem 2.5.18, we will obtain the determinacy of open games on a $Tree_1$ collection such that each $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection having finite Y, a Borel function Ψ and a Borel set B from the determinacy of Borel games on $X^{<\omega}$. In theorem 2.5.20, we will obtain the determinacy of open games on a $Tree_1$ collection such that each $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection having countable Y, a Borel function Ψ and a Borel set B from the determinacy of Borel games on $X^{<\omega}$. In theorem 2.5.26, we will obtain the determinacy of open games on a $Tree_1$ collection such that each $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection having countable Y, a projective function Ψ and a projective set B from the determinacy of projective games on $X^{<\omega}$.

The proofs of theorem 2.5.18, theorem 2.5.20, corollary 2.5.23, corollary 2.5.24, corollary 3.5.20 and theorem 2.5.26 are similar. First, we consider $Tree_1$ collections over FIN. Then we obtain results for $Tree_1$ collections over CTB on page 160.

Theorem 2.5.18. *Suppose $\beta, \gamma \in \omega_1$.*

If $\beta, \gamma > 1$, then

$$Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right).^{30} \tag{2.17}$$

[30] Recall notation 1.5.10 for FIN.

If $\beta < \gamma$,

$$Det\left(\mathbf{\Delta}^0_\gamma \upharpoonright X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). & (2.18) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right). & (2.19) \end{cases}$$

If $\beta \geq \gamma$,

$$Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \upharpoonright X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). & (2.20) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right). & (2.21) \end{cases}$$

Also,

$$Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right). \tag{2.22}$$

$\dashv$

The implications (2.18) through (2.21) state that we set

$$\mathcal{T}_1 = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \left(\mathbf{\Sigma}^0_\beta \cup \mathbf{\Pi}^0_\beta\right) \upharpoonright X^\omega\right),$$

then $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_1\right)$ follows from

$$\begin{cases} Det\left(\mathbf{\Delta}^0_\gamma \upharpoonright X^\omega\right) & \text{when } \beta < \gamma, \\ Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \upharpoonright X^\omega\right) & \text{when } \beta \geq \gamma. \end{cases}$$

Proof.

Show the implication (2.17). Fix $\beta, \gamma \in \omega_1$ greater than 1. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Delta}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,Y}]$. Assume $Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \upharpoonright X^\omega)$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Delta}^0_\beta \upharpoonright X^\omega$. Since Y is finite, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, we have

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right).$$

The proofs for the implications (2.18) through (2.22) are similar. Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$ in the appropriate $Tree_1$ collection. We only need to check the complexity of $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

For the implication (2.18), pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Since Y is finite, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$.

For the implication (2.19), pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Since Y is finite, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$.

For the implication (2.20), pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Since Y is finite, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in (\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta) \upharpoonright X^\omega$.

For the implication (2.21), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}_\gamma^0\right)$ and $B \in \mathbf{\Pi}_\beta^0 \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \restriction [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A_g^n \in \mathbf{\Pi}_\beta^0 \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}_\beta^0 \restriction X^\omega$. Since Y is finite, by lemma 2.5.11, each $A_\emptyset^n \in \mathbf{\Pi}_\beta^0 \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A_\emptyset^n \cup A_\emptyset \in (\mathbf{\Sigma}_\beta^0 \vee \mathbf{\Pi}_\beta^0) \restriction X^\omega$.

For the implication (2.22), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}_1^0\right)$ and $B \in \mathbf{\Delta}_1^0 \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \restriction [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A_g^n \in \mathbf{\Sigma}_1^0 \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}_1^0 \restriction X^\omega$. Since Y is finite, by lemma 2.5.11, each $A_\emptyset^n \in \mathbf{\Sigma}_1^0 \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A_\emptyset^n \cup A_\emptyset \in \mathbf{\Sigma}_1^0 \restriction X^\omega$. □

Combining corollary 2.4.32 on page 120 and theorem 2.5.18, we have the following.

Corollary 2.5.19. *Suppose $\beta, \gamma \in \omega_1$. Then for any $\beta \geq \gamma$,*

$$\left.\begin{array}{l} ① \ Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, FIN, \Gamma\left(\omega, \mathbf{\Sigma}_\beta^0 \wedge \mathbf{\Pi}_\beta^0\right), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \\ ② \ Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, FIN, \Gamma\left(\omega, \mathbf{\Sigma}_\beta^0 \wedge \mathbf{\Pi}_\beta^0\right), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)\right) \end{array}\right\}$$

$$\Rightarrow ③ \ Det((\mathbf{\Sigma}_\beta^0 \wedge \mathbf{\Pi}_\beta^0) \restriction X^\omega)$$

$$\Leftrightarrow ④ \ Det\left(\left(\mathbf{\Sigma}_\beta^0 \vee \mathbf{\Pi}_\beta^0\right) \restriction X^\omega\right)$$

$$\Rightarrow \left\{\begin{array}{l} ⑤ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)\right). \\ ⑥ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right). \end{array}\right.$$

That is : ① implies ③, ② implies ③, ③ if and only if ④ and ④ implies both ⑤ and ⑥. ⊣

So far, we focused on getting the determinacy on $Tree_1$ collections over FIN. Now, we consider $Tree_1$ collections over CTB.

Theorem 2.5.20. *Suppose $\beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right).^{31} & (2.23) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right). & (2.24) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). & (2.25) \end{cases}$$

⊣

Proof.

Show the implication (2.23). Fix $\beta, \gamma \in \omega_1$ greater than 1. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. Assume $Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega)$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Pi}^0_\beta \restriction X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$. (If Ψ is bounded, then there exists $m \in \omega$ such that $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+m} \restriction X^\omega$.) Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, we have $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right)$.

The proofs for the implications (2.24) and (2.25) are similar. Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$ in the appropriate $Tree_1$ collection. We only need to check the complexity of $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

For the implication (2.24), pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right).$$

[31] Recall notation 1.5.10 for CTB.

Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Pi}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Pi}^0_{\max\{\beta,\gamma\}} \restriction X^\omega \subseteq \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+1} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$.

For the implication (2.25), pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.12, for every $n \in \omega$ and every $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega \subseteq \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+1} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Delta}^0_\beta \restriction X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega$. □

Combining corollary 2.4.17 on page 104 and and theorem 2.5.20, we have the following.

Corollary 2.5.21. *For any finite n and m,*

$$\begin{aligned}
&Det\left(\mathbf{\Sigma}^0_\omega \restriction X^\omega\right)\\
&\Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \restriction X^\omega\right)\right)\\
&\Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right)\\
&\Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right).
\end{aligned}$$ ⊣

Question 5. *Are any of the collections in corollary 2.5.21 determinacy equivalent?* ⊣

Combining observation 2.4.1 on page 83 and theorem 2.5.20, we have the following.

Corollary 2.5.22.

$$
\begin{aligned}
&Det\mathcal{G}\left(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega}; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \emptyset\right)\right)\\
&\Rightarrow Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega)\\
&\Rightarrow Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right).
\end{aligned}
\qquad \dashv
$$

Corollary 2.5.23. *Suppose* Λ *is an algebra. Then*

$$
Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv
$$

Proof.

Pick an arbitrary Type 1 tree $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)$. Then Y is finite, $\Psi \in \Gamma(\omega, \Lambda)$ and $B \in \Lambda \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. Assume $Det(\Lambda \restriction X^\omega)$. By lemma 2.5.12, for all $n \in \omega$ and for all $g \in Y^{n+1}$, each $A^n_g \in \Lambda \restriction X^\omega$ and since Λ is closed under complement, $A_\emptyset \in \Lambda \restriction X^\omega$. Since Y is finite and Λ is closed under finite unions and finite intersections, by lemma 2.5.11, each $A^n_\emptyset \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. $\square$

Corollary 2.5.24. *Suppose* Λ *is* σ*-algebra. Then*

$$
Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv
$$

Proof.

Pick an arbitrary Type 1 tree $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)$. Then Y is countable, $\Psi \in \Gamma(\omega, \Lambda)$ and $B \in \Lambda \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. Assume $Det(\Lambda \restriction X^\omega)$. By lemma 2.5.12, for all $n \in \omega$ and for all $g \in Y^{n+1}$, each $A^n_g \in \Lambda \restriction X^\omega$ and since Λ is closed under complement, $A_\emptyset \in \Lambda \restriction X^\omega$. Since Y is countable and Λ is closed under countable unions and countable intersections, by lemma 2.5.11, each $A^n_\emptyset \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. □

Corollary 2.5.25.

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right) \end{cases}$$

⊣

Proof.

Corollary 2.5.25 follows immediately from corollaries 2.5.23 and 2.5.24 since **B** is σ-algebra.

□

So far, we focused on getting the determinacy on $Tree_1$ collections such that each Type 2 tree $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection satisfying $Y \in CTB$, Ψ is a Borel function and B is a Borel set. Now, we we focus on getting the determinacy on a $Tree_1$ collection such that each Type 2 tree $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection satisfying $Y \in CTB$, Ψ is a projective function and B is a projective set.

Theorem 2.5.26. *Suppose $m, n \in \omega\backslash\{0\}$.*

$$Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right)\right).^{32} \tag{2.26}$$

If $n < m$,

$$Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (2.27) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (2.28) \end{cases}$$

[32] Recall notation 1.5.10 for CTB.

If $n \geq m$,

$$Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (2.29) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (2.30) \end{cases}$$

⊣

The implications (2.27) through (2.30) state that we set

$$\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \left(\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n\right) \restriction X^\omega\right),$$

then $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_1\right)$ follows from

$$\begin{cases} Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) & \text{when } n < m, \\ Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) & \text{when } n \geq m. \end{cases}$$

Proof.

Show the implication (2.26). Fix $n, m \in \omega_1$ greater than 1. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^1_m\right)$ and $B \in \mathbf{\Delta}^1_n \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,Y}]$. Assume $Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega)$. By lemma 2.5.12, for each $i \in \omega$ and $g \in Y^{i+1}$, $A^i_g \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Delta}^1_n \restriction X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^i_\emptyset \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$. Hence $G(\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, we have

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right)\right).$$

The proofs for the implications (2.27) through (2.30) are similar. Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$ in the appropriate $Tree_1$ collection. We only need to check the complexity of $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset$.

For the implication (2.27), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_m^1), \mathbf{\Sigma}_n^1 \upharpoonright X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma(\omega, \mathbf{\Delta}_m^1)$ and $B \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for each $i \in \omega$ and $g \in Y^{i+1}$, $A_g^i \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A_\emptyset^i \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A_\emptyset^i \cup A_\emptyset \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$.

For the implication (2.28), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_m^1), \mathbf{\Pi}_n^1 \upharpoonright X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma(\omega, \mathbf{\Delta}_m^1)$ and $B \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for each $i \in \omega$ and $g \in Y^{i+1}$, $A_g^i \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A_\emptyset^i \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A_\emptyset^i \cup A_\emptyset \in \mathbf{\Delta}_m^1 \upharpoonright X^\omega$.

For the implication (2.29), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_m^1), \mathbf{\Sigma}_n^1 \upharpoonright X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma(\omega, \mathbf{\Delta}_m^1)$ and $B \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for each $i \in \omega$ and $g \in Y^{i+1}$, $A_g^i \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A_\emptyset^i \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A_\emptyset^i \cup A_\emptyset \in (\mathbf{\Sigma}_n^1 \vee \mathbf{\Pi}_n^1) \upharpoonright X^\omega$.

For the implication (2.30), pick an arbitrary Type 1 tree

$$T_{X,Y}^{\Psi,B} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_m^1), \mathbf{\Pi}_n^1 \upharpoonright X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma(\omega, \mathbf{\Delta}_m^1)$ and $B \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,Y}^{\Psi,B}]$. By lemma 2.5.12, for each $i \in \omega$ and $g \in Y^{i+1}$, $A_g^i \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}_n^1 \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A_\emptyset^i \in \mathbf{\Pi}_n^1 \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A_\emptyset^i \cup A_\emptyset \in (\mathbf{\Sigma}_n^1 \vee \mathbf{\Pi}_n^1) \upharpoonright X^\omega$. $\square$

Corollary 2.5.27. *(Corollary to Theorem 2.5.26)*

$Det\,(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_1), (\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1) \restriction X^\omega)).$ ⊣

By combining corollary 2.4.34 on page 120 and corollary 3.5.20, we have the following.

Corollary 2.5.28. *For any nonempty X and Y,*

$$\left.\begin{array}{l} ① \ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega)) \\ ② \ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega)) \end{array}\right\}$$

$$\Rightarrow ③ \ Det\left(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right)$$

$$\Rightarrow ④ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_1), \left(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1\right) \restriction X^\omega\right)\right).$$

That is : ① implies ③, ② implies ③, and ③ implies ④. ⊣

Question 6. *With respect to corollary 2.5.28, does ④ imply ① or ②?* ⊣

2.5.3 Obtaining the determinacy of Borel games on a $Tree_1$ collection with countable Y from the determinacy of Borel games on $X^{<\omega}$

In section 2.5.2, we focused on obtaining the determinacy of open games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In this section, as a general case of open games on a $Tree_1$ collection, we will consider games which are more higher complexity. The main theorems in this section are theorems 2.5.29 and 2.5.30. We will obtain level by level results for the determinacy of Borel games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$.

Theorem 2.5.29. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{2.31}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.32}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.33}$$

⊣

The implications (2.32) and (2.33) states that when we set

$$\mathcal{T}_1 = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right)$ follows from

$$\begin{cases} Det\left(\mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega\right) & \text{when } \beta < \gamma, \\ Det\left(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega\right) & \text{when } \beta \geq \gamma. \end{cases}$$

We will prove this theorem on page 178.

Theorem 2.5.30. *Suppose* $\alpha, \beta, \gamma \in \omega_1$. *Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.34}$$

Moreover, if $\beta < \gamma$, *then*

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.35}$$

If $\beta \geq \gamma$, *then*

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.36}$$

⊣

The implications (2.35) and (2.36) states that when we set

$$\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right),$$

$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right)$ follows from

$$\begin{cases} Det(\mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \upharpoonright X^\omega) & \text{when } \beta < \gamma, \\ Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \upharpoonright X^\omega) & \text{when } \beta \geq \gamma. \end{cases}$$

We will prove this theorem on page 180.

The idea of the proofs are similar as in section 2.5.2. We will use the same definition of A^n, A^n_g and $A_\emptyset$ from section 2.5.1. We will find the complexity of each A^n_g and $A_\emptyset$ in lemma

2.5.33. Then, by using lemma 2.5.11 and theorem 2.5.10, we will obtain the determinacy results in theorems 2.5.29 and 2.5.30. To obtain the complexity of each A^n_g and $A_\emptyset$, we will define a function Fix form X^ω into $[T^{\Psi,B}_{X,Y}]$ and find the complexity of Fix in lemma 2.5.32 This Fix will be the key to find the complexity of A^n_g and $A_\emptyset$. For each $g \in Y^{n+1}$, we will collect all of $f \in X^\omega$ such that $f^\frown g \in [T^{\Psi,B}_{X,Y}]$ by using Fix. Fix will be the identity map for any $f \in X^\omega \backslash B$ and if $f \in B$, then it will fix the tail.

Definition 2.5.31. *(Definition of "Fix")*

For all $m \in \omega$, fix $a_m \in Y^{m+1}$. Define

$$Fix\,\langle a_m : m \in \omega\rangle : \quad X^\omega \to [T^{\Psi,B}_{X,Y}]$$
$$f \mapsto \begin{cases} f & \text{if } f \notin B, \\ f^\frown a_{\Psi(f)} & \text{otherwise.} \end{cases}$$

If $\langle a_m : m \in \omega\rangle$ is clear from the context, we will denote Fix to mean $Fix\,\langle a_m : m \in \omega\rangle$. $\dashv$

We will compute the complexity of Fix.

Lemma 2.5.32. *(Finding the complexity of Fix)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Suppose Y is countable. For all $m \in \omega$, fix $a_m \in Y^{m+1}$. Suppose $\alpha, \beta, \gamma_n \in \omega_1$, $n \in \omega$.

1. *Suppose:*

   - $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$,
   - *for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,*

   then $Fix \in \Gamma([T], \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega} \gamma_n\}})$.[33]

[33]Recall notation 1.5.8 for $\Gamma([T], \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}})$.

2. *Suppose:*

- *for all* $n \in \omega$, $\beta \geq \gamma_n$,
- $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$,
- *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,

then $Fix \in \Gamma([T], \mathbf{\Sigma}^0_{\beta+1})$.

3. *Suppose:*

- *there exists* $n \in \omega$ *such that* $\gamma_n > \beta$,
- $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$,
- *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,

then $Fix \in \Gamma([T], \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n})$.

4. *Suppose* Λ *is* σ*-algebra and:*

- $B \in \Lambda \restriction X^\omega$,
- $\Psi \in \Gamma(\omega, \Lambda)$,

then $Fix \in \Gamma([T], \Lambda)$. ⊣

Proof.

Pick an arbitrary $O \in \mathbf{\Sigma}^0_1 \restriction [T]$. Then there exists $\langle O_i \,|i \in \omega\rangle$ such that $O = \bigcup_{i\in\omega} O_i$ where each O_i is a basic open neighborhood of $[T]$, i.e., there exists $p_i \in X^{<\omega}$ and $q_i \in Y^{<\omega}$ such that

$$O_i = \{h \in [T] \,|h \restriction \omega \supseteq p_i \wedge h \restriction [\omega, lh\,(h)) \supseteq q_i\}\,.$$

Since each tail has finite length and Y is countable, there are countably many tails. Thus each O_i can be written as $\bigcup_{j\in\omega}\hat{O}_{i,j}$ where each

$$\hat{O}_{i,j}=\{h\in[T]\mid h\restriction\omega\supseteq\hat{p}_{i,j}=p_i\wedge h\restriction[\omega,lh(h))=\hat{q}_{i,j}\}$$

for some $\hat{q}_{i,j}\in Y^{<\omega}$. Then $O=\bigcup_{i\in\omega}O_i=\bigcup_{i\in\omega}\bigcup_{j\in\omega}\hat{O}_{i,j}=\bigcup_{k\in\omega}\hat{O}_k$ where $\hat{O}_k$'s enumerate $\hat{O}_{i,j}$'s, etc. $\hat{O}_k=\{h\in[T]\mid h\restriction\omega\supseteq\hat{p}_k\wedge h\restriction[\omega,lh(h))=\hat{q}_k\}$. Since

$$Fix^{-1}(O)=Fix^{-1}(\bigcup_{k\in\omega}\hat{O}_k)=\bigcup_{k\in\omega}Fix^{-1}(\hat{O}_k),$$

we find the complexity of each $Fix^{-1}(\hat{O}_k)$.

Case 1 : $\hat{q}_k=\emptyset$.

$$Fix^{-1}(\hat{O}_k)=\underbrace{\{h\in X^\omega\,|h\restriction\omega\supseteq\hat{p}_k\}}_{\mathbf{\Sigma}^0_1\restriction X^\omega}\cap(X^\omega\backslash B).$$

If $B\in\mathbf{\Delta}^0_\beta\restriction X^\omega$, $Fix^{-1}(\hat{O}_k)\in\left(\mathbf{\Sigma}^0_1\wedge\mathbf{\Delta}^0_\beta\right)\restriction X^\omega$.

If $B\in\mathbf{\Sigma}^0_\beta\restriction X^\omega$, $Fix^{-1}(\hat{O}_k)\in(\mathbf{\Sigma}^0_1\wedge\mathbf{\Pi}^0_\beta)\restriction X^\omega$.

If Λ is σ-algebra and $B\in\Lambda\restriction X^\omega$, then $Fix^{-1}(\hat{O}_k)\in\Lambda\restriction X^\omega$.

Case 2 : $\hat{q}_k=a_{l_k}$ for some $l_k\in\omega$.

$$Fix^{-1}(\hat{O}_k)=\underbrace{\{h\in X^\omega\,|h\restriction\omega\supseteq\hat{p}_k\}}_{\mathbf{\Sigma}^0_1\restriction X^\omega}\cap\underbrace{\Psi^{-1}\,(l_k)}_{\mathbf{\Delta}^0_{\gamma_{l_k}}\restriction X^\omega}\cap B.$$

If $B\in\mathbf{\Delta}^0_\beta\restriction X^\omega$ and $\Psi^{-1}(l_k)\in\mathbf{\Delta}^0_{\gamma_{l_k}}\restriction X^\omega$, then $Fix^{-1}(\hat{O}_k)\in\left(\mathbf{\Sigma}^0_1\wedge\mathbf{\Delta}^0_{\max\{\beta,\gamma_{l_k}\}}\right)\restriction X^\omega$.

If $B\in\mathbf{\Sigma}^0_\beta\restriction X^\omega$ and $\Psi^{-1}(l_k)\in\mathbf{\Delta}^0_{\gamma_{l_k}}\restriction X^\omega$, then $Fix^{-1}(\hat{O}_k)\in\mathbf{\Sigma}^0_{\max\{\beta,\gamma_{l_k}\}}\restriction X^\omega$.

If Λ is σ-algebra, $B\in\Lambda\restriction X^\omega$ and $\Psi\in\Gamma(\omega,\Lambda)$, $Fix^{-1}(\hat{O}_k)\in\Lambda\restriction X^\omega$.

Case 3 : $\hat{q}_k \neq \emptyset$ and $\hat{q}_k \neq a_l$ for any l.

$$Fix^{-1}(\hat{O}_k) = \emptyset.$$

Show (1). Suppose $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$. Then

$$Fix^{-1}(O) = \bigcup_{k\in\omega} \underbrace{Fix^{-1}(\hat{O}_k)}_{\left(\mathbf{\Sigma}^0_1 \wedge \mathbf{\Delta}^0_{\max\left\{\beta,\gamma_{l_k}\right\}}\right)\restriction X^\omega} \in \mathbf{\Sigma}^0_{\max\{\beta,\sup\limits_{k\in\omega}\gamma_{l_k}\}} \restriction X^\omega \subseteq \mathbf{\Sigma}^0_{\max\{\beta,\sup\limits_{n\in\omega}\gamma_n\}} \restriction X^\omega.$$

Since $O \in \mathbf{\Sigma}^0_1 \restriction [T]$ is arbitrary, Fix is $\mathbf{\Sigma}^0_{\max\{\beta,\sup\limits_{n\in\omega}\gamma_n\}}$-measurable.

Show (2). Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$. Then $Fix^{-1}(O) \in \mathbf{\Sigma}^0_{\beta+1} \restriction X^\omega$. Since $O \in \mathbf{\Sigma}^0_1 \restriction [T]$ is arbitrary, Fix is $\mathbf{\Sigma}^0_{\beta+1}$-measurable.

Show(3). Suppose there exists $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$. Then $Fix^{-1}(O) \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n} \restriction X^\omega$. Since $O \in \mathbf{\Sigma}^0_1 \restriction [T]$ is arbitrary, Fix is $\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable.

Show (4). Suppose Λ is σ-algebra, $B \in \Lambda \restriction X^\omega$ and $\Psi \in \Gamma(\omega, \Lambda)$. Then $Fix^{-1}(\hat{O}_k) \in \Lambda$ for any $k \in \omega$ and thus $Fix^{-1}(O) \in \Lambda$. Since $O \in \mathbf{\Sigma}^0_1 \restriction [T]$ is arbitrary, Fix is Λ-measurable. □

Using the complexity of Fix computed in lemma 2.5.32, we find the complexity of A^n_g and $A_\emptyset$. In the proof of lemma 2.5.33, we use sublemma 2.5.34 on page 176.

Lemma 2.5.33. *(Finding the complexity of A^n_g and $A_\emptyset$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Assume Y is countable. Suppose $\alpha, \beta \in \omega_1$, $\alpha > 1$, and $m \in \omega$. Assume that for all $n \in \omega$, $\gamma_n \in \omega_1$.

1. *Suppose:*

   - $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$,

- *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,

- $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$,

then $A_\emptyset, A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha}$ *for any* $n \in \omega$ *and for any* $g \in Y^{n+1}$.

2. *Suppose:*

   - *for all* $n \in \omega$, $\beta \geq \gamma_n$,

   - $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$,

   - *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,

   - $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$,

   then $A_\emptyset$, $A^n_g \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in Y^{n+1}$.

3. *Suppose:*

   - *there is* $n \in \omega$ *such that* $\gamma_n > \beta$,

   - $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$,

   - *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,

   - $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$,

   then $A_\emptyset$, $A^n_g \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in Y^{n+1}$.

4. *Suppose* Λ *is* σ*-algebra, closed under* Λ*-substitution and:*

   - $B \in \Lambda \restriction X^\omega$,

   - $\Psi \in \Gamma(\omega, \Lambda)$,

- $A \in \Lambda \restriction [T]$,

then $A_\emptyset$, $A^n_g \in \Lambda \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in Y^{n+1}$. $\dashv$

Proof.

Fix $n \in \omega$ and $g \in Y^{n+1}$. First, we will find the complexity of A^n_g. We will use Fix with $a_n = g$. Show

$$A^n_g \overset{df}{=} \{f \in X^\omega \,|\, f^\frown g \in A^n\} = Fix^{-1}(A) \cap \Psi^{-1}(n) \cap B.$$

Recall $A^n = A \cap ((B \cap \Psi^{-1}(n)) \times Y^{n+1})$. [34]

($\subseteq$) Suppose $f \in A^n_g$. Since $g \in Y^{n+1}$ and $f^\frown g \in A^n$, $f \in \Psi^{-1}(n) \cap B$ and $f^\frown g \in A$. Since $f \in B$ and $\Psi(f) = n$,

$$Fix(f) = f^\frown a_{\Psi(f)} = f^\frown a_n = f^\frown g.$$

Thus $Fix(f) \in A$ so that $f \in Fix^{-1}(A)$.

($\supseteq$) Suppose $f \in Fix^{-1}(A) \cap \Psi^{-1}(n) \cap B$. Since $f \in Fix^{-1}(A)$ and $f \in B$,

$$Fix(f) = f^\frown a_{\Psi(f)} = f^\frown a_n = f^\frown g \in A.$$

Since $g \in Y^{n+1}$, $f^\frown g \in A \cap ((B \cap \Psi^{-1}(n)) \times Y^{n+1}) = A^n$. Hence $f \in A^n_g$.

First, we consider the complexity of A^n_g.

$$A^n_g \overset{df}{=} \{f \in X^\omega \,|\, f^\frown g \in A^n\} = Fix^{-1}(A) \cap \Psi^{-1}(n) \cap B.$$

Show (1) for A^n_g. Suppose $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$, for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 2.5.32, Fix is $\mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}}$-measurable. Note that since

[34] Recall definitions 2.5.2 through 2.5.4.

ω_1 is regular, $\sup_{n\in\omega}\gamma_n \in \omega_1$. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34 below,[35]

$$Fix^{-1}(A) \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha} \restriction X^\omega.$$

Thus $A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha} \restriction X^\omega$.

Show (2) for A^n_g. Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$, for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 2.5.32, Fix is $\mathbf{\Sigma}^0_{\beta+1}$-measurable. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34 below, $Fix^{-1}(A) \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$. Thus $A^n_g \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$.

Show (3) for A^n_g. Suppose there is $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 2.5.32, Fix is $\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34 below, $Fix^{-1}(A) \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$. Thus $A^n_g \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$.

Show (4) for A^n_g. Suppose Λ is σ-algebra, closed under Λ-substitution. Suppose $\Psi \in \Gamma(\omega, \Lambda), B \in \Lambda \restriction X^\omega$ and $A \in \Lambda \restriction [T]$. Since Λ is σ-algebra, by lemma 2.5.32, Fix is Λ-measurable. Since Λ is closed under Λ-substitution, $Fix^{-1}(A) \in \Lambda$. Since $\Psi \in \Gamma(\omega, \Lambda)$ and $B \in \Lambda \restriction X^\omega$, each $A^n_g \in \Lambda \restriction X^\omega$.

Now, we consider the complexity of $A_\emptyset$. Recall $long(B) = \{h \in [T] \,|\, lh(h) > \omega\}$. Then $long(B) \in \mathbf{\Sigma}^0_1 \restriction [T]$.

$$\underbrace{([T]\backslash long(B))}_{\mathbf{\Pi}^0_1 \restriction [T]} \cap \underbrace{A}_{\mathbf{\Sigma}^0_\alpha \restriction [T]} \in \mathbf{\Sigma}^0_\alpha \restriction [T] \text{ for } \alpha > 1.$$

$$A_\emptyset = \{f \in X^\omega \backslash B \,|\, f \in A\} = Fix^{-1}\left(([T]\backslash long(B)) \cap A\right).$$

Show (1) for $A_\emptyset$. Suppose $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 2.5.32, Fix

[35] See sublemma 2.5.34 on page 176.

is $\mathbf{\Delta}^0_{\max\{\beta,\sup_{n\in\omega}\gamma_n\}}$-measurable. By sublemma 2.5.34 below, $A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\sup_{n\in\omega}\gamma_n\}+\alpha} \upharpoonright X^\omega$.

Show (2) for $A_\emptyset$. Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Then by lemma 2.5.32, Fix is $\mathbf{\Sigma}^0_{\beta+1}$-measurable. By sublemma 2.5.34 below, $A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega$.

Show (3) for $A_\emptyset$. Suppose there is $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Then by lemma 2.5.32, Fix is $\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable. By sublemma 2.5.34 below, $A_\emptyset \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \upharpoonright X^\omega$.

Show (4) for $A_\emptyset$. Suppose Λ is σ-algebra and closed under Λ-substitution, $\Psi \in \Gamma(\omega,\Lambda)$, $B \in \Lambda \upharpoonright X^\omega$ and $A \in \Lambda \upharpoonright [T]$. Then by lemma 2.5.32, Fix is Λ measurable and $([T]\backslash long(B)) \cap A \in \Lambda \upharpoonright [T]$. Since Λ is closed under Λ-substitution and $([T]\backslash long(B)) \cap A \in \Lambda \upharpoonright [T]$, $A_\emptyset \in \Lambda \upharpoonright X^\omega$. □

We used to the following sublemma 2.5.34 for the prove of lemma 2.5.33 to find the complexity of A^n_g and $A_\emptyset$. We prove the following well-known property about the measurable functions. This is listed in Moschovakis (2009, p. 43, Exercise 1G.7.).

Sublemma 2.5.34. *Suppose $\alpha, \gamma \in \omega_1\backslash\{0\}$. Suppose $f : X_1 \to X_2$ is $\mathbf{\Sigma}^0_\gamma$-measurable.*

1. *If $P \in \mathbf{\Sigma}^0_\alpha \upharpoonright X_2$, then $f^{-1}(P) \in \mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X_1$.*

2. *If $P \in \mathbf{\Pi}^0_\alpha \upharpoonright X_2$, then $f^{-1}(P) \in \mathbf{\Pi}^0_{\gamma+\alpha} \upharpoonright X_1$.*

Consequently, $\mathbf{B}$ *is closed under Borel-substitution.* ⊣

Proof.

Fix $\gamma \in \omega_1\backslash\{0\}$. We prove this by induction both (1) and (2) simultaneously on α .

Base Case : $\alpha = 1$. Pick arbitrary $\mathbf{\Sigma}^0_\gamma$-measurable $f : X_1 \to X_2$

Suppose $P \in \mathbf{\Sigma}^0_1 \upharpoonright X_2$. Then by definition of $\mathbf{\Sigma}^0_\gamma$-measurable, $f^{-1}(P) \in \mathbf{\Sigma}^0_\gamma \upharpoonright X_1$.

Suppose $P \in \mathbf{\Pi}^0_1 \restriction X_2$. Show $f^{-1}(P) \in \mathbf{\Pi}^0_\gamma \restriction X_1$. Since $P \in \mathbf{\Pi}^0_1 \restriction X_2$, $X_2 \backslash P \in \mathbf{\Sigma}^0_1 \restriction X_2$. Since we have already shown (1) for $\alpha = 1$, we have: $f^{-1}(X_2 \backslash P) \in \mathbf{\Sigma}^0_\gamma \restriction X_1$. Since $f^{-1}(X_2 \backslash P) = X_1 \backslash f^{-1}(P)$, $f^{-1}(P) \in \mathbf{\Pi}^0_\gamma \restriction X_1$. $\cdots(*)$

Induction Step : As an induction hypothesis, assume for all $\mathbf{\Sigma}^0_\gamma$-measurable $f : X_1 \to X_2$, $\forall \beta \in \alpha$,

if $P \in \mathbf{\Sigma}^0_\beta \restriction X_2$ then $f^{-1}(P) \in \mathbf{\Sigma}^0_{\gamma+\beta} \restriction X_1$ and if $P \in \mathbf{\Pi}^0_\beta \restriction X_2$ then $f^{-1}(P) \in \mathbf{\Pi}^0_{\gamma+\beta} \restriction X_1$.

Suppose $P \in \mathbf{\Sigma}^0_\alpha \restriction X_2$. Show $f^{-1}(P) \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X_1$. Since $P \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$, there exists $\langle P^i \,|i \in \omega \rangle$ such that each $P^i \in \mathbf{\Pi}^0_{\beta_i} \restriction X_2$, $\beta_i \in \alpha$ and $P = \bigcup_{i \in \omega} P^i$.

$$f^{-1}(P) = f^{-1}(\bigcup_{i \in \omega} P^i) = \bigcup_{i \in \omega} \underbrace{f^{-1}(P^i)}_{\mathbf{\Pi}^0_{\gamma+\beta_i}} \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X_1.$$

Suppose $P \in \mathbf{\Pi}^0_\alpha \restriction X_2$. Show $f^{-1}(P) \in \mathbf{\Pi}^0_{\gamma+\alpha} \restriction X_1$. We repeat the same proof of $(*)$ on page 177.

Since $P \in \mathbf{\Pi}^0_\alpha \restriction X_2$, $X_2 \backslash P \in \mathbf{\Sigma}^0_\alpha \restriction X_2$. Since we have already shown (1) for the case α, we have: $f^{-1}(X_2 \backslash P) \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X_1$. Since $f^{-1}(X_2 \backslash P) = X_1 \backslash f^{-1}(P)$,
$f^{-1}(P) \in \mathbf{\Pi}^0_{\gamma+\alpha} \restriction X_1$.

Suppose $f : X_1 \to X_2$ is Borel-measurable. Then if $P \in \mathbf{B} \restriction X_2$, then $f^{-1}(P) \in \mathbf{B} \restriction X_1$. Consequently, $\mathbf{B}$ is closed under Borel-substitution. □

We computed the complexity of each A^n_g and $A_\emptyset$ in lemma 2.5.33. Using lemma 2.5.11 and theorem 2.5.10, we obtain the determinacy results in theorems 2.5.29 and 2.5.30. First, we consider $Tree_1$ collections over FIN. Recall theorem 2.5.29.

Theorem 2.5.29. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.31}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.32}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.33}$$

$\dashv$

Proof of theorem 2.5.29.

Fix $\alpha, \beta, \gamma \in \omega_1$ such that $\alpha > 1$.

Show the implication (2.31). Assume $Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha})$. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Delta}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \upharpoonright X^\omega$. Since Y is finite, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right).$$

Similarly, for the implication (2.32), suppose $\beta \geq \gamma$. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega$. By lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega$ and thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right)$.

For the implication (2.33), suppose $\gamma \geq \beta$. Pick an arbitrary Type 1 tree

$$T^{\Psi,B}_{X,Y} \in Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right).$$

Then Y is finite, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega$. Then by lemma 2.5.33, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega$ and thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right)$. □

Now, we consider $Tree_1$ collections over CTB. Recall theorem 2.5.30.

Theorem 2.5.30. *Suppose $\alpha, \beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.34}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.35}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{2.36}$$

$\dashv$

Proof of Theorem 2.5.30.

Fix $\alpha, \beta, \gamma \in \omega_1$.

Show the implication (2.34). Assume $Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega})$. Pick an arbitrary Type 1 tree $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)$. Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha}$ and $A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+1}$ (1 is added for the case $\alpha = 1$). Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega} \restriction X^\omega$. Thus

$$\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega} \restriction X^\omega.$$

Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right)$.

Similarly, show the implication (2.35). Suppose $\gamma > \beta$. Assume $Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \restriction X^\omega)$. Pick an arbitrary Type 1 tree $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)$. Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha+1} \restriction X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \restriction X^\omega$ and thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right)$.

Show the implication (2.36). Suppose $\beta \geq \gamma$. Assume $Det(\mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \restriction X^\omega)$. Pick an arbitrary Type 1 tree $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)$. Then Y is countable, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$ and $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{\Psi,B}_{X,Y}]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha+1} \restriction X^\omega$.

Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \upharpoonright X^\omega$ and thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore, $\mathit{Det}\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right)$. □

Combining corollary 2.5.21 on page 161 and theorem 2.5.30, we have the following.

Corollary 2.5.35. *For any finite n, m and k,*

$$\begin{aligned}
&\mathit{Det}\left(\mathbf{\Sigma}^0_\omega \upharpoonright X^\omega\right)\\
&\Rightarrow \mathit{Det}\ \mathcal{G}\left(\mathbf{\Sigma}^0_k; \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \upharpoonright X^\omega\right)\right)\\
&\Rightarrow \mathit{Det}\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \upharpoonright X^\omega\right)\right)\\
&\Rightarrow \mathit{Det}\ \mathcal{G}\left(\mathbf{\Delta}^0_1; \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right)\\
&\Rightarrow \mathit{Det}\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \upharpoonright X^\omega\right).
\end{aligned}$$

⊣

Question 7. *Are any of the collections in corollary 2.5.35 determinacy equivalent?* ⊣

Corollary 2.5.36. *Suppose Λ is a σ-algebra and Λ is closed under Λ-substitution. Then*

$$\mathit{Det}(\Lambda \upharpoonright X^\omega) \Rightarrow \mathit{Det}\ \mathcal{G}\left(\Lambda; \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

⊣

Proof.

Assume $\mathit{Det}(\Lambda \upharpoonright X^\omega)$. Pick an arbitrary Type 1 tree

$$T = T^{\Psi,B}_{X,Y} \in \mathit{Tree}_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right).$$

Then Y is countable, $\Psi \in \Gamma(\omega, \Lambda)$ and $B \in \Lambda \upharpoonright X^\omega$. Pick an arbitrary $A \in \Lambda \upharpoonright [T]$. By lemma 2.5.33, for all $n \in \omega$ and $g \in Y^{n+1}$, each A^n_g and $A_\emptyset$ are in $\Lambda \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A_\emptyset^n \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A_\emptyset^n \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A_\emptyset^n \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A;T)$ is determined. Therefore,

$$Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, CTB, \Gamma(\omega,\Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \square$$

Corollary 2.5.37. *(Corollary to Corollary 2.5.36)*

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_1\left(X, CTB, \Gamma(\omega,\mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

Since $\mathbf{B}$ is σ-algebra and closed under Borel-substitution, by corollary 2.5.36, we have the result. $\square$

2.5.4 Obtaining the determinacy of projective games on a $Tree_1$ collection with countable Y from the determinacy of projective games on $X^{<\omega}$

In section 2.5.2, we focused on obtaining the determinacy of open games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. In section 2.5.3, we focused on obtaining the determinacy of Borel games on a certain $Tree_1$ collection with countable Y from the determinacy of Borel games on $X^{<\omega}$. In this section, we will generalize Borel games on a $Tree_1$ collection to projective games on a particular $Tree_1$ collection. We will obtain the determinacy of projective games on a certain $Tree_1$ collection with countable Y from the determinacy of projective games on $X^{<\omega}$. The main theorem in this section is theorem 2.5.38.

Theorem 2.5.38. *Suppose* $m \in \omega$. *Suppose* $\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^{\omega}\right)$. *Then*

$$Det(\mathbf{\Sigma}^1_m \restriction X^{\omega}) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Pi}^1_m \restriction X^{\omega}) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Delta}^1_m \restriction X^{\omega}) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_1\right).$$

Note that $\mathbf{\Delta}^1_1 \restriction X^{\omega}$ *is Borel if* X *is countable and if* X *is uncountable, it is the quasi-Borel.*

$\dashv$

The idea of the proof is similar as in sections 2.5.2 and 2.5.3. We will use the same definition of A^n, A^n_g and $A_{\emptyset}$ from section 2.5.1. We will find the complexity of each A^n_g and $A_{\emptyset}$ in lemma 2.5.39 using sublemma 2.5.40 on page 185. Then, by using lemma 2.5.11 and theorems 2.5.10, we will obtain the determinacy results in theorems 2.5.29 and 2.5.30. The proof of the theorem is on page 185.

Lemma 2.5.39. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,Y}$. *Assume* Y *is countable. Suppose* $m \in \omega$, $\alpha, \beta, \gamma \in \omega_1$, $\Psi \in \Gamma(\omega, \mathbf{B})$ *and* $B \in \mathbf{B} \restriction X^\omega$.

1. *If* $A \in \mathbf{\Sigma}^1_m \restriction [T]$, *then for all* $n \in \omega$ *and for all* $g \in Y^{n+1}$, *each* $A^n_g, A_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$.

2. *If* $A \in \mathbf{\Pi}^1_m \restriction [T]$, *then for all* $n \in \omega$ *and for all* $g \in Y^{n+1}$, *each* $A^n_g, A_\emptyset \in \mathbf{\Pi}^1_m \restriction X^\omega$.

3. *If* $A \in \mathbf{\Delta}^1_m \restriction [T]$, *then for all* $n \in \omega$ *and for all* $g \in Y^{n+1}$, *each* $A^n_g, A_\emptyset \in \mathbf{\Delta}^1_m \restriction X^\omega$. ⊣

Proof.

The proof is similar to the proof of lemma 2.5.33. We will show the case for (1): $A \in \mathbf{\Sigma}^1_m \restriction [T]$. The proofs are similar for cases (2): $A \in \mathbf{\Pi}^1_m \restriction [T]$ and (3): $A \in \mathbf{\Delta}^1_m \restriction [T]$.

Suppose $A \in \mathbf{\Sigma}^1_m \restriction [T]$. By lemma 2.5.32, Fix is Borel-measurable under $\Psi \in \Gamma(\omega, \mathbf{B})$ and $B \in \mathbf{B} \restriction X^\omega$. By sublemma 2.5.40 below,[36] $Fix^{-1}(A) \in \mathbf{\Sigma}^1_m \restriction X^\omega$.

First, we consider A^n_g for $g \in Y^{n+1}$.

$$A^n_g \stackrel{df}{=} \{f \in X^\omega \,|\, f^\frown g \in A^n\} = \underbrace{Fix^{-1}(A)}_{\mathbf{\Sigma}^1_m \restriction X^\omega} \cap \underbrace{\Psi^{-1}(n)}_{\mathbf{B} \restriction X^\omega} \cap \underbrace{B}_{\mathbf{B} \restriction X^\omega} \in \mathbf{\Sigma}^1_m \restriction X^\omega.$$

Now, we consider $A_\emptyset$. Recall $long(B) = \{h \in [T] \,|\, lh(h) > \omega\}$. Then $long(B) \in \mathbf{\Sigma}^0_1 \restriction [T]$.

$$\underbrace{([T] \backslash long(B))}_{\mathbf{\Pi}^0_1 \restriction [T]} \cap \underbrace{A}_{\mathbf{\Sigma}^1_m \restriction [T]} \in \mathbf{\Sigma}^1_m \restriction [T].$$

$$A_\emptyset = \{f \in X^\omega \backslash B \,|\, f \in A\} = Fix^{-1}\left(([T] \backslash long(B)) \cap A\right) \in \mathbf{\Sigma}^1_m \restriction X^\omega$$

by sublemma 2.5.40 below.

Similarly for the cases (2) and (3). □

[36]See sublemma 2.5.40 on page 185.

We computed the complexity of each A^n_g for $n \in \omega$ and $g \in Y^{n+1}$, and $A_\emptyset$ in lemma 2.5.39. Using lemma 2.5.11 and theorem 2.5.10, we will obtain the determinacy results in theorem 2.5.38. Recall theorem 2.5.38

Theorem 2.5.38. *Suppose $m \in \omega$. Suppose $\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)$. Then*

$$Det(\mathbf{\Sigma}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Pi}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Delta}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_1\right).$$

Note that $\mathbf{\Delta}^1_1 \upharpoonright X^\omega$ is Borel if X is countable and if X is uncountable, it is the quasi-Borel.

⊣

Proof of Theorem 2.5.38.

We will show the case for $\mathbf{\Sigma}^1_m$. The proofs are similar for case $\mathbf{\Pi}^1_m$ and $\mathbf{\Delta}^1_m$.

Show $Det(\mathbf{\Sigma}^1_m \upharpoonright X^\omega)$ implies $Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_1\right)$. Assume $Det(\mathbf{\Sigma}^1_m \upharpoonright X^\omega)$. Pick an arbitrary Type 1 tree $T = T^{\Psi,B}_{X,Y} \in \mathcal{T}_1$. Then $Y \in CTB$, $\Psi \in \Gamma(\omega, \mathbf{B})$ and $B \in \mathbf{B} \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^1_m \upharpoonright [T]$. By lemma 2.5.39, for all $n \in \omega$ and $g \in Y^{n+1}$, each $A^n_g \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$. Since Y is countable, by lemma 2.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 2.5.10, $G(A;T)$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_1\right)$. □

Now, we will show that for each $n \in \omega$, $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ are closed under Borel-substitution. Readers familiar with the proof of sublemma 2.5.40 may skip the rest of the section.

Sublemma 2.5.40. *Suppose $n \in \omega \backslash \{0\}$.*

Then $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ are closed under Borel-substitutions. ⊣

We will prove sublemma 2.5.40 on page 190. First, for each $k \in \omega$ and a function f from X_1 to X_2, we define a function φ_k^f from $X_1 \times (\omega^\omega)^k \times \omega^\omega$ into $X_2 \times (\omega^\omega)^k \times \omega^\omega$. In sublemma 2.5.43, we will show that if f is Borel-measurable, then φ_k^f is also Borel-measurable for every $k \in \omega$ by using sublemma 2.5.42. Using sublemmas 2.5.43 and 2.5.44, we will prove sublemma 2.5.40.

Definition 2.5.41. *Suppose $f : X_1 \to X_2$. Define*

$$\begin{aligned} \varphi_k^f : \quad & X_1 \times (\omega^\omega)^k \to X_2 \times (\omega^\omega)^k \\ & \langle x, y_1, ..., y_k \rangle \mapsto \langle f(x), y_1, ..., y_k \rangle \end{aligned} \qquad \dashv$$

Note that if $k = 0$, then $\varphi_0^f = f$ for any $f : X_1 \to X_2$.

Sublemma 2.5.42. *Suppose $f : X_1 \to X_2$. Assume that $E \subseteq X_2$ and for $1 \leq j \leq k$, $F^j \subseteq \omega^\omega$. Then*

$$(\varphi_k^f)^{-1}\left(E \times F^1 \times \cdots \times F^k\right) = f^{-1}(E) \times F^1 \times \cdots \times F^k. \qquad \dashv$$

Proof.

Suppose $E \subseteq X_2$ and for $1 \leq j \leq k$, $F^j \subseteq \omega^\omega$.

$$\begin{aligned} & \langle x, y_1, ..., y_k \rangle \in (\varphi_k^f)^{-1}\left(E \times F^1 \times \cdots \times F^k\right) \\ & \Leftrightarrow \langle f(x), y_1, ..., y_k \rangle = \varphi_k^f(\langle x, y_1, ..., y_k \rangle) \in E \times F^1 \times \cdots \times F^k \\ & \Leftrightarrow f(x) \in E \wedge y_1 \in F^1 \wedge \cdots \wedge y_k \in F^k \\ & \Leftrightarrow x \in f^{-1}(E) \wedge y_1 \in F^1 \wedge \cdots \wedge y_k \in F^k \\ & \Leftrightarrow \langle x, y_1, ..., y_k \rangle \in f^{-1}(E) \times F^1 \times \cdots \times F^k. \end{aligned} \qquad \square$$

We prove the following sublemma using sublemma 2.5.42.

Sublemma 2.5.43. *For any $f : X_1 \to X_2$ and $k \in \omega$, if f is Borel-measurable, then φ_k^f is Borel-measurable.* $\dashv$

Proof.

Suppose Borel-measurable $f : X_1 \to X_2$ and $k \in \omega$. Suppose $S_k \in \mathbf{B} \restriction \left(X_2 \times (\omega^\omega)^k\right)$. Show $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$. We prove this by induction on the complexity of S_k.

Base Case : $S_k \in \mathbf{\Sigma}_1^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$.

Show $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$. Since $S_k \in \mathbf{\Sigma}_1^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, there exist $E_i \in \mathbf{\Sigma}_1^0 \restriction X_2$ and $F_i^j \in \mathbf{\Sigma}_1^0 \restriction \omega^\omega$, $1 \leq j \leq k$ and $i \in \omega$ such that

$$S_k = \bigcup\nolimits_{i\in\omega} \left(E_i \times F_i^1 \times \cdots \times F_i^k\right).$$

$$\begin{aligned}
(\varphi_k^f)^{-1}(S_k) &= (\varphi_k^f)^{-1}\left(\bigcup\nolimits_{i\in\omega} \left(E_i \times F_i^1 \times \cdots \times F_i^k\right)\right) \\
&= \bigcup_{i\in\omega} (\varphi_k^f)^{-1}\left(E_i \times F_i^1 \times \cdots \times F_i^k\right) \\
&= \bigcup_{i\in\omega} \underbrace{(\underbrace{f^{-1}(E_i)}_{\mathbf{B}\restriction X_1} \times F_i^1 \times \cdots \times F_i^k)}_{\mathbf{B}\restriction(X_1\times(\omega^\omega)^k)} \text{ by sublemma 2.5.42} \\
&\in \mathbf{B} \restriction (X_1 \times (\omega^\omega)^k).
\end{aligned}$$

Induction Step : Assume, as an induction hypothesis, for any $\beta \in \alpha$, if $S_k \in \mathbf{\Sigma}_\beta^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, then $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$ and if $S_k \in \mathbf{\Pi}_\beta^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, then $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$.

Suppose $S_k \in \mathbf{\Sigma}_\alpha^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$. Show $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$.

Since $S_k \in \mathbf{\Sigma}_\alpha^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, there exists $\langle A_i \,|i \in \omega\rangle$ such that each $A_i \in \mathbf{\Pi}_{\beta_i}^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, $\beta_i \in \alpha$, and $S_k = \bigcup_{i\in\omega} A_i$. By induction hypothesis, for each $i \in \omega$,

$(\varphi_k^f)^{-1}(A_i) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$. Since

$$(\varphi_k^f)^{-1}(S_k) = (\varphi_k^f)^{-1}\left(\bigcup_{i\in\omega} A_i\right) = \bigcup_{i\in\omega} (\varphi_k^f)^{-1}(A_i)$$

and $\mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$ is closed under countable union, $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$.

Suppose $S_k \in \mathbf{\Pi}_\alpha^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$. Show $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$. Since $S_k \in \mathbf{\Pi}_\alpha^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$, $(X_1 \times (\omega^\omega)^k)\backslash S_k \in \mathbf{\Sigma}_\alpha^0 \restriction \left(X_2 \times (\omega^\omega)^k\right)$. Since we have already shown (1) for the case α, we have:

$$(\varphi_k^f)^{-1}\left((X_1 \times (\omega^\omega)^k)\backslash S_k\right) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right).$$

Since $(X_1 \times (\omega^\omega)^k)\backslash(\varphi_k^f)^{-1}(S_k) = (\varphi_k^f)^{-1}\left((X_2 \times (\omega^\omega)^k)\backslash S_k\right)$ and $\mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$ is closed under complement, $(\varphi_k^f)^{-1}(S_k) \in \mathbf{B} \restriction \left(X_1 \times (\omega^\omega)^k\right)$. □

Now, we show that there is a homeomorphism between $X_1 \times (\omega^\omega)^{k+1}$ and $X_1 \times (\omega^\omega)^k$.

Sublemma 2.5.44. *Let $k \in \omega$ be arbitrary. By sublemma 2.5.14, there exists a homeomorphism π from $(\omega^\omega)^k \times \omega^\omega$ to $(\omega^\omega)^k$. Suppose π is a homeomorphism from $(\omega^\omega)^k \times \omega^\omega$ to $(\omega^\omega)^k$. Define*

$$\rho: \quad X_1 \times (\omega^\omega)^k \times \omega^\omega \to X_1 \times (\omega^\omega)^k$$

$$\langle x, y, z\rangle \to \langle x, \pi(y, z)\rangle$$

Then ρ is a homeomorphism from $X_1 \times (\omega^\omega)^{k+1}$ to $X_1 \times (\omega^\omega)^k$. ⊣

Proof.

Since π is a bijection, ρ is a bijection.

Show ρ is continuous. Pick an arbitrary $O \in \mathbf{\Sigma}_1^0 \restriction \left(X_1 \times (\omega^\omega)^k\right)$. Then there exist

$E_i \in \mathbf{\Sigma}^0_1 \upharpoonright X_1$ and $F^j_i \in \mathbf{\Sigma}^0_1 \upharpoonright \omega^\omega$, $i \le j \le k$ for $i \in \omega$ such that

$$O = \bigcup\nolimits_{i\in\omega} \left(E_i \times F^1_i \times \cdots \times F^k_i\right)$$

$$\begin{aligned}
\rho^{-1}(O) &= \rho^{-1}\left(\bigcup\nolimits_{i\in\omega} \left(E_i \times F^1_i \times \cdots \times F^k_i\right).\right) \\
&= \bigcup\nolimits_{i\in\omega} \rho^{-1}\left(E_i \times F^1_i \times \cdots \times F^k_i\right) \\
&= \bigcup\nolimits_{i\in\omega} \underbrace{\left(\underbrace{E_i}_{\mathbf{\Sigma}^0_1 \upharpoonright X_1} \times \underbrace{\pi^{-1}\left(F^1_i \times \cdots \times F^k_i\right)}_{\mathbf{\Sigma}^0_1 \upharpoonright (\omega^\omega)^k \times \omega^\omega}\right)}_{\mathbf{\Sigma}^0_1 \upharpoonright \left(X_1 \times (\omega^\omega)^k \times \omega^\omega\right)} \\
&\in \mathbf{\Sigma}^0_1 \upharpoonright \left(X_1 \times (\omega^\omega)^k \times \omega^\omega\right).
\end{aligned}$$

Thus ρ is continuous.

Show ρ is an open map. Pick an arbitrary $E \in \mathbf{\Sigma}^0_1 \upharpoonright \left(X_1 \times (\omega^\omega)^k \times \omega^\omega\right)$. Then there exist $O_i \in \mathbf{\Sigma}^0_1 \upharpoonright X_1$, $P^j_i \in \mathbf{\Sigma}^0_1 \upharpoonright \omega^\omega$, $1 \le j \le k$ and $Q_i \in \mathbf{\Sigma}^0_1 \upharpoonright \omega^\omega$ for $i \in \omega$ such that

$$E = \bigcup\nolimits_{i\in\omega} \left(O_i \times P^1_i \times \cdots \times P^k_i \times Q_i\right).$$

Since ρ is a bijection,

$$\begin{aligned}
\rho(E) &= \rho\left(\bigcup\nolimits_{i\in\omega} \left(O_i \times P^1_i \times \cdots \times P^k_i \times Q_i\right)\right) \\
&= \bigcup\nolimits_{i\in\omega} \rho\left(O_i \times P^1_i \times \cdots \times P^k_i \times Q_i\right) \\
&= \bigcup\nolimits_{i\in\omega} \left(\underbrace{O_i}_{\mathbf{\Sigma}^0_1 \upharpoonright X_1} \times \underbrace{\pi\left(P^1_i \times \cdots \times P^k_i \times Q_i\right)}_{\mathbf{\Sigma}^0_1 \upharpoonright (\omega^\omega)^k}\right) \\
&\in \mathbf{\Sigma}^0_1 \upharpoonright \left(X_1 \times (\omega^\omega)^k\right)
\end{aligned}$$

Thus ρ is an open map.

Therefore, ρ is a homeomorphism. $\square$

Finally, we prove sublemma 2.5.40 using sublemmas 2.5.43 and 2.5.44. Recall sublemma 2.5.40.

Sublemma 2.5.40. *Suppose* $n \in \omega \backslash \{0\}$.

Then $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ *are closed under Borel-substitutions.* $\dashv$

Proof of Sublemma 2.5.40.

Suppose $f : X_1 \to X_2$ is Borel-measurable. Let $k \in \omega$. Show that

1. for all $P \in \mathbf{\Sigma}^1_n \restriction (X_2 \times (\omega^\omega)^k)$, $(\varphi^f_k)^{-1}(P) \in \mathbf{\Sigma}^1_n \restriction (X_1 \times (\omega^\omega)^k)$,

2. for all $P \in \mathbf{\Pi}^1_n \restriction (X_2 \times (\omega^\omega)^k)$, $(\varphi^f_k)^{-1}(P) \in \mathbf{\Pi}^1_n \restriction (X_1 \times (\omega^\omega)^k)$,

3. for all $P \in \mathbf{\Delta}^1_n \restriction (X_2 \times (\omega^\omega)^k)$, $(\varphi^f_k)^{-1}(P) \in \mathbf{\Delta}^1_n \restriction (X_1 \times (\omega^\omega)^k)$,

by induction on n.

Show both (1) and (2) simultaneously. (2) follows from (1).

Base Case : $n = 1$:

Pick an arbitrary $k \in \omega$. Show that for all $P \in \mathbf{\Sigma}^1_1 \restriction (X_2 \times (\omega^\omega)^k)$, $(\varphi^f_k)^{-1}(P) \in \mathbf{\Sigma}^1_1 \restriction (X_1 \times (\omega^\omega)^k)$.

Pick an arbitrary $P \in \mathbf{\Sigma}^1_1 \restriction (X_2 \times (\omega^\omega)^k)$. Then there is a closed $C \subseteq X_2 \times (\omega^\omega)^{k+1}$ such that for all $x \in X_2 \times (\omega^\omega)^k$,

$$x \in P \leftrightarrow \exists y \in \omega^\omega \, \langle x, y \rangle \in C.$$

Then

$$\begin{aligned}\langle x, y_1, ..., y_k\rangle \in (\varphi_k^f)^{-1}(P) &\Leftrightarrow (\varphi_k^f)(\langle x, y_1, ..., y_k\rangle) \in P \\ &\Leftrightarrow (\langle f(x), y_1, ..., y_k\rangle) \in P \\ &\Leftrightarrow \exists z \in \omega^\omega \langle f(x), y_1, ..., y_k, z\rangle \in C \\ &\Leftrightarrow \exists z \in \omega^\omega \langle x, y_1, ..., y_k, z\rangle \in (\varphi_{k+1}^f)^{-1}(C).\end{aligned}$$

By sublemma 2.5.43, since $C \in \mathbf{\Pi}_1^0 \restriction (X_2 \times (\omega^\omega)^{k+1})$, $(\varphi_{k+1}^f)^{-1}(C) \in \mathbf{B} \restriction (X_1 \times (\omega^\omega)^{k+1})$. Since $\mathbf{B} \restriction (X_1 \times (\omega^\omega)^{k+1}) \subseteq \mathbf{\Sigma}_1^1 \restriction (X_1 \times (\omega^\omega)^{k+1})$, there is a closed $D \subseteq X_1 \times (\omega^\omega)^{k+2}$ such that for all $a \in X_1 \times (\omega^\omega)^{k+1}$,

$$a \in (\varphi_{k+1}^f)^{-1}(C) \leftrightarrow \exists b \in \omega^\omega \langle a, b\rangle \in D.$$

By sublemma 2.5.44,

$$\begin{aligned}\rho: \quad & X_1 \times (\omega^\omega)^{k+1} \times \omega^\omega \to X_1 \times (\omega^\omega)^{k+1} \\ & \langle x, y, z\rangle \to \langle x, \pi(y, z)\rangle\end{aligned}$$

is a homeomorphism from $X_1 \times (\omega^\omega)^{k+2}$ to $X_1 \times (\omega^\omega)^{k+1}$. Since $D \in \mathbf{\Pi}_1^0 \restriction (X_1 \times (\omega^\omega)^{k+2})$, $\rho(D) \in \mathbf{\Pi}_1^0 \restriction (X_1 \times (\omega^\omega)^{k+1})$.

$$
\begin{aligned}
\langle x, y_1, ..., y_k \rangle \in (\varphi_k^f)^{-1}(P) &\Leftrightarrow (\varphi_k^f)(\langle x, y_1, ..., y_k \rangle) \in P \\
&\Leftrightarrow (\langle f(x), y_1, ..., y_k \rangle) \in P \\
&\Leftrightarrow \exists z \in \omega^\omega \langle f(x), y_1, ..., y_k, z \rangle \in C \\
&\Leftrightarrow \exists z \in \omega^\omega \langle x, y_1, ..., y_k, z \rangle \in (\varphi_{k+1}^f)^{-1}(C) \\
&\Leftrightarrow \exists z \in \omega^\omega \exists w \in \omega^\omega \langle x, y_1, ..., y_k, z, w \rangle \in D \\
&\Leftrightarrow \exists h \in \omega^\omega \langle x, y_1, ..., y_k, h \rangle \in \rho(D).
\end{aligned}
$$

Therefore, $(\varphi_k^f)^{-1}(P) \in \mathbf{\Sigma}_1^1 \restriction (X_1 \times (\omega^\omega)^k)$.

Show if $P \in \mathbf{\Pi}_1^1 \restriction (X_2 \times (\omega^\omega)^k)$, then $(\varphi_k^f)^{-1}(P) \in \mathbf{\Pi}_1^1 \restriction (X_1 \times (\omega^\omega)^k)$. Suppose $P \in \mathbf{\Pi}_1^1 \restriction (X_2 \times (\omega^\omega)^k)$. Then $\left(X_2 \times (\omega^\omega)^k\right) \backslash P \in \mathbf{\Sigma}_1^1 \restriction (X_2 \times (\omega^\omega)^k)$. Since we have already shown (1) for $n = 1$, we have: $(\varphi_k^f)^{-1}\left(\left(X_2 \times (\omega^\omega)^k\right) \backslash P\right) \in \mathbf{\Sigma}_1^1 \restriction (X_1 \times (\omega^\omega)^k)$. Since

$$
\left(X_1 \times (\omega^\omega)^k\right) \backslash (\varphi_k^f)^{-1}(P) = (\varphi_k^f)^{-1}\left(\left(X_2 \times (\omega^\omega)^k\right) \backslash P\right),
$$

$(\varphi_k^f)^{-1}(P) \in \mathbf{\Pi}_1^1 \restriction (X_1 \times (\omega^\omega)^k)$.

Induction Step : As an induction hypothesis, suppose that for all $l \in \omega$, if $P \in \mathbf{\Sigma}_n^1 \restriction (X_2 \times (\omega^\omega)^l)$, then $(\varphi_l^f)^{-1}(P) \in \mathbf{\Sigma}_n^1 \restriction (X_1 \times (\omega^\omega)^l)$ and if $P \in \mathbf{\Pi}_n^1 \restriction (X_2 \times (\omega^\omega)^l)$, then $(\varphi_l^f)^{-1}(P) \in \mathbf{\Pi}_n^1 \restriction (X_1 \times (\omega^\omega)^l)$.

Pick an arbitrary $k \in \omega$. Show if $P \in \mathbf{\Sigma}_{n+1}^1 \restriction (X_2 \times (\omega^\omega)^k)$, then $(\varphi_k^f)^{-1}(P) \in \mathbf{\Sigma}_{n+1}^1 \restriction (X_1 \times (\omega^\omega)^k)$. Suppose $P \in \mathbf{\Sigma}_{n+1}^1 \restriction (X_2 \times (\omega^\omega)^k)$. Then there exists $C \in \mathbf{\Pi}_n^1 \restriction \left(X_2 \times (\omega^\omega)^{k+1}\right)$ such that for all $x \in X_2 \times (\omega^\omega)^k$,

$$
x \in P \leftrightarrow \exists y \in \omega^\omega \langle x, y \rangle \in C.
$$

By induction hypothesis, $(\varphi_{k+1}^{f})^{-1}(C) \in \mathbf{\Sigma}_{n}^{1} \upharpoonright (X_1 \times (\omega^\omega)^{k+1})$.

$$\begin{aligned}\langle x, y_1, ..., y_k \rangle \in (\varphi_k^f)^{-1}(P) &\Leftrightarrow (\varphi_k^f)(\langle x, y_1, ..., y_k \rangle) \in P \\ &\Leftrightarrow (\langle f(x), y_1, ..., y_k \rangle) \in P \\ &\Leftrightarrow \exists z \in \omega^\omega \langle f(x), y_1, ..., y_k, z \rangle \in C \\ &\Leftrightarrow \exists z \in \omega^\omega \langle x, y_1, ..., y_k, z \rangle \in (\varphi_{k+1}^f)^{-1}(C).\end{aligned}$$

Therefore, $(\varphi_k^f)^{-1}(P) \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright (X_1 \times (\omega^\omega)^k)$.

Show if $P \in \mathbf{\Pi}_{n+1}^1 \upharpoonright (X_2 \times (\omega^\omega)^k)$, then $(\varphi_k^f)^{-1}(P) \in \mathbf{\Pi}_{n+1}^1 \upharpoonright (X_1 \times (\omega^\omega)^k)$. Suppose $P \in \mathbf{\Pi}_{n+1}^1 \upharpoonright (X_2 \times (\omega^\omega)^k)$. Then $\left(X_2 \times (\omega^\omega)^k\right) \backslash P \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright (X_2 \times (\omega^\omega)^k)$. Since we have already shown (1) for the case $n+1$, we have: $f_k^{-1}\left(\left(X_2 \times (\omega^\omega)^k\right) \backslash P\right) \in \mathbf{\Sigma}_{n+1}^1 \upharpoonright (X_1 \times (\omega^\omega)^k)$. Since

$$\left(X_1 \times (\omega^\omega)^k\right) \backslash (\varphi_k^f)^{-1}(P) = f_k^{-1}\left(\left(X_2 \times (\omega^\omega)^k\right) \backslash P\right),$$

$(\varphi_k^f)^{-1}(P) \in \mathbf{\Pi}_{n+1}^1 \upharpoonright (X_1 \times (\omega^\omega)^k)$.

Therefore, we have (1) and (2). Consequently, we have (3).

In particular, if $k = 0$, then $\varphi_0^f = f$.

1. for all $P \in \mathbf{\Sigma}_n^1 \upharpoonright X_2$, $f^{-1}(P) \in \mathbf{\Sigma}_n^1 \upharpoonright X_1$,

2. for all $P \in \mathbf{\Pi}_n^1 \upharpoonright X_2$, $f^{-1}(P) \in \mathbf{\Pi}_n^1 \upharpoonright X_1$,

3. for all $P \in \mathbf{\Delta}_n^1 \upharpoonright X_2$, $f^{-1}(P) \in \mathbf{\Delta}_n^1 \upharpoonright X_1$.

Therefore, $\mathbf{\Sigma}_n^1, \mathbf{\Pi}_n^1, \mathbf{\Delta}_n^1$ are closed under Borel-substitution. □

2.5.5 Well-known results about uncountable $Y = \mathcal{N}$

In sections 2.5.1 through 2.5.4, we obtained the determinacy of games on a certain $Tree_1$ collection with countable Y from the determinacy of games on $X^{<\omega}$. The way we obtained the determinacy results in these sections are using the fact that each $T^{\Psi,B}_{X,Y}$ in the $Tree_1$ collection having a countable Y. Without this restriction, we need to have the determinacy of games on X^ω with higher complexity, even just an open game on a Type 1 tree. Since $\mathbf{B} \restriction X^\omega$, $\mathbf{\Sigma}^1_n \restriction X^\omega$ and $\mathbf{\Pi}^1_n \restriction X^\omega$ are closed under countable unions and countable intersections, we were able to conclude the results.

Consider the special case of uncountable Y where $Y = \mathcal{N}(= \omega^\omega)$. Suppose C_0 to be the constant function from X^ω at 0. Then $T^{C_0,X^\omega}_{X,\mathcal{N}} = X^\omega \times \mathcal{N}$. In this particular tree, it is well-known that

$$Det\left(\mathbf{\Pi}^1_1 \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction (X^\omega \times \mathcal{N})\right).$$ [37]

Hence

$$Det\left(\mathbf{\Pi}^1_1 \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction [T^{C_0,X^\omega}_{X,\mathcal{N}}]\right).$$

Since $Tree_1\left(X, \mathcal{N}, C_0, X^\omega\right) = \left\{T^{C_0,X^\omega}_{X,\mathcal{N}}\right\}$,

$$Det\left(\mathbf{\Pi}^1_1 \restriction X^\omega\right) \Leftrightarrow \mathcal{G}\ \left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, C_0, X^\omega\right)\right).$$

In general,

$$Det\left(\mathbf{\Sigma}^1_n \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction (X^\omega \times \mathcal{N}^n)\right).$$

[37] see outline of the proof for Fraker, 2001, pp.59-62, Corollary 5.3.

Hence for $n \in \omega$ and constant function C_n from X^ω at n,

$$Det\left(\mathbf{\Sigma}^1_n \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \upharpoonright [T^{C_{n-1},X^\omega}_{X,\mathcal{N}}]\right).$$

Since $Tree_1\left(X, \mathcal{N}, C_{n-1}, X^\omega\right) = \left\{T^{C_{n-1},X^\omega}_{X,\mathcal{N}}\right\}$,

$$Det\left(\mathbf{\Sigma}^1_n \upharpoonright X^\omega\right) \Leftrightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, C_{n-1}, X^\omega\right)\right).$$

Therefore, if Γ is the set of the constant functions from X^ω into ω,

$$Det\left(\mathbf{P} \upharpoonright X^\omega\right) \Leftrightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, \Gamma, X^\omega\right)\right).$$

Since

$$Tree_1\left(X, \mathcal{N}, \Gamma, X^\omega\right) \subseteq Tree_1\left(X, \mathcal{N}, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right),$$

we have the following.

Observation 2.5.45.

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right) \Rightarrow Det\left(\mathbf{P} \upharpoonright X^\omega\right). \qquad \dashv$$

Question 8. *Which class over X^ω is equivalent to the determinacy of*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right)? \qquad \dashv$$

2.6 Determinacy equivalences between games on $X^{<\omega}$ and games on $Tree_1$ collections

In sections 2.3 through 2.5, we observed the determinacy strength on games on $Tree_1$ collections. In section 2.3, by shifting, we compared the determinacy of $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Sigma}^1_n$) games on a particular $Tree_1$ collection and $\mathbf{\Pi}^0_\alpha$ (respectively, $\mathbf{\Pi}^1_n$) games on the same $Tree_1$ collection, for $\alpha \in \omega_1$ and $n \in \omega$. In section 2.4, we used the determinacy of a fixed complexity of games on a certain $Tree_1$ collection to obtain the determinacy of a certain complexity of games on $X^{<\omega}$. In section 2.5, we obtained the determinacy of Borel and projective games on particular $Tree_1$ collections from the determinacy of a fixed complexity of games on $X^{<\omega}$.

In this section, we will combine results from section 2.3, section 2.4 and section 2.5. we will conclude this chapter with the resulting determinacy equivalences from the earlier determinacy results between games on $X^{<\omega}$ and games on a $Tree_1$ collection.

In section 2.6.1, we will obtain the determinacy equivalences between Borel games on $X^{<\omega}$ and games on particular $Tree_1$ collections.

In section 2.6.2, we will obtain the determinacy equivalences between projective games on $X^{<\omega}$ and games on particular $Tree_1$ collections.

2.6.1 Determinacy equivalence between Borel games on $X^{<\omega}$ and games on $Tree_1$ collections

In this section, we will obtain the determinacy equivalences between Borel games on $X^{<\omega}$ and games on particular $Tree_1$ collections.

Theorem 2.6.1. *For any nonempty $\Theta \subseteq FIN$, the determinacy of following (2.37) through (2.42) are all equivalent to $Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.37}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.38}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.39}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.40}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.41}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.42}$$

$\dashv$

Theorem 2.6.1 says that if we let

$$\mathcal{T}^1_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right),$$

$$\mathcal{T}^2_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right),$$

$$\mathcal{T}^3_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1;\mathcal{T}^i_1\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1;\mathcal{T}^j_1\right)$$

for any $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof.

Suppose $\emptyset \neq \Theta \subseteq FIN$. We obtain $Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right)$ if and only if the determinacy of (2.37) by theorem 2.5.18 and corollary 2.4.2.

($\Rightarrow$) We obtain this from theorem 2.5.18.

($\Leftarrow$) We obtain this from corollary 2.4.2.

By observation 2.2.10, (2.37), (2.38) and (2.39) are the same set. Similarly, (2.40), (2.41) and (2.42) are the same set. The determinacy of (2.37) and the determinacy of (2.40) are equivalent by theorem 2.3.1. Consequently, the determinacy of (2.37) through (2.42) are all equivalent to $Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right)$. □

Theorem 2.6.2. *Suppose $\beta, \gamma \in \omega_1$ and $\beta \geq \gamma$. Then for any nonempty $\Theta \subseteq FIN$, the determinacy of following (2.43) through (2.48) are all equivalent to Det $\left(\mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{2.43}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{2.44}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{2.45}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{2.46}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.47}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.48}$$

⊣

Theorem 2.6.2 says that if we let

$$\mathcal{T}^1_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}^2_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}^3_1 = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}^i_1\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_\beta \restriction X^\omega\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; \mathcal{T}^j_1\right)$$

for any $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof.

Suppose $\emptyset \neq \Theta \subseteq FIN$. We obtain $Det\left(\mathbf{\Delta}^0_\beta \restriction X^\omega\right)$ if and only if the determinacy of (2.43) by theorem 2.5.18 and corollary 2.4.21.

(⇒) This is obtained from theorem 2.5.18.

(⇐) This is obtained from corollary 2.4.21

By observation 2.2.10, (2.43), (2.44) and (2.45) are the same set. Similarly, (2.46), (2.47) and (2.48) are the same set. The determinacy of (2.43) and the determinacy of (2.46) are equivalent by theorem 2.3.1. Consequently, the determinacy of (2.43) through (2.48) are all equivalent to $Det\left(\mathbf{\Delta}^0_\beta \restriction X^\omega\right)$. □

Theorem 2.6.3. *Suppose $\beta, \gamma \in \omega_1$ and $1 \leq \beta < \gamma$. Then for any nonempty $\Theta \subseteq FIN$, the determinacy of following (2.49) through (2.66) are all equivalent to Det $\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.49}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.50}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.51}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.52}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.53}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.54}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.55}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.56}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.57}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.58}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.59}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{2.60}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.61}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.62}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.63}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.64}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.65}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.66}$$

$\dashv$

Theorem 2.6.3 says that if we let

$$\mathcal{T}_1^{\mathbf{\Delta\Sigma}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Sigma\Sigma}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Pi\Sigma}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Delta\Pi}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Sigma\Pi}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Pi\Pi}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Delta\Delta}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Sigma\Delta}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_1^{\mathbf{\Pi\Delta}} = Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_1^{ij}\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; \mathcal{T}_1^{kl}\right)$$

for any $i, j, k, l \in \{\mathbf{\Delta}, \mathbf{\Sigma}, \mathbf{\Pi}\}$.

Proof.

Suppose $\emptyset \neq \Theta \subseteq FIN$. We obtain $Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$ if and only if the determinacy of (2.49) from theorem 2.5.18 and corollary 2.4.27.

($\Rightarrow$) We obtain this from theorem 2.5.18.

($\Leftarrow$) We obtain this from corollary 2.4.27.

Similarly, we obtain $Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$ if and only if the determinacy of (2.55); and $Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$ if and only if the determinacy of (2.61).

By observation 2.2.10, (2.49), (2.50) and (2.51) are the same sets. Similarly, (2.52), (2.53) and (2.54) are the same sets. The determinacy of (2.49) and the determinacy of (2.52) are equivalent by theorem 2.3.1. Consequently, the determinacy of (2.49) through (2.54) are all equivalent to $Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$.

By observation 2.2.10, (2.55), (2.56) and (2.57) are the same sets. Similarly, (2.58), (2.59) and (2.60) are the same sets. The determinacy of (2.55) and the determinacy of (2.58) are equivalent by theorem 2.3.1. Consequently, the determinacy of (2.55) through (2.60) are all equivalent to $Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$.

By observation 2.2.10, (2.61), (2.62) and (2.63) are the same sets. Similarly, (2.64), (2.65) and (2.66) are the same sets. The determinacy of (2.61) and the determinacy of (2.64) are equivalent by theorem 2.3.1. Consequently, the determinacy of (2.61) through (2.66) are all equivalent to $Det\left(\boldsymbol{\Delta}^0_\gamma \restriction X^\omega\right)$. □

Corollary 2.6.4. *Suppose Λ is an algebra. Then for any nonempty $\Theta \subseteq FIN$,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\boldsymbol{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \quad \dashv$$

Proof.

($\Rightarrow$) We obtain this from corollary 2.5.23.

($\Leftarrow$) We obtain this from corollary 2.4.21. □

Corollary 2.6.5. *Suppose Λ is a σ-algebra. Then for any nonempty $\Theta \subseteq CTB$,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\boldsymbol{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \quad \dashv$$

Proof.

($\Rightarrow$) We obtain this from corollary 2.5.24.

($\Leftarrow$) We obtain this from corollary 2.4.21. □

Corollary 2.6.6. *For any $\alpha, \beta, \gamma \in \omega_1$,*

$$Det(\mathbf{B} \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right)$$

where:

- $\emptyset \neq \Theta \subseteq CTB$,

- $\begin{cases} \mathcal{A} = \mathbf{B}, \ \ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\} \quad for\ \alpha > 1, \ \ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1\} \qquad\quad for\ \alpha = 1, \end{cases}$
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,
- *at least one of* $\mathcal{A}, \mathcal{B}$ *or* $\mathcal{C}$ *is* $\mathbf{B}$. ⊣

Proof.

(⇒) Corollary 2.5.37 gives

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right).$$

Under the condition for Θ, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$,

$$\mathcal{G}\left(\mathcal{A}; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \subseteq \mathcal{G}\left(\mathbf{B}; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right).$$

Thus we have (⇒) direction.

(⇐) Recall that at least one of $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ is $\mathbf{B}$.

Case 1 : $\mathcal{A} = \mathbf{B}$. By observation 2.4.1, we have

$$Det\ \mathcal{G}\left(\mathcal{A}; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Case 2 : $\mathcal{B} = \mathbf{B}$.

Subcase 1 : $\mathbf{\Sigma}^0_1 \subseteq \mathcal{A}$. By corollary 2.4.21, we have the results.

Subcase 2 : $\mathcal{A} = \mathbf{\Pi}^0_1$. By theorem 2.3.1,

$$Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega)\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega)\right).$$

By subcase 1 for $\mathcal{A} = \mathbf{\Sigma}^0_1$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \upharpoonright X^\omega)$$

Hence

$$Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \upharpoonright X^\omega).$$

Case 3 : $\mathcal{C} = \mathbf{B}$. By corollary 2.4.27, we have

$$Det\ \mathcal{G}\left(\mathcal{A}; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \upharpoonright X^\omega). \qquad \square$$

2.6.2 Determinacy equivalence between projective games on $X^{<\omega}$ and games on $Tree_1$ collections

In this section, we will obtain the determinacy equivalences between projective games on $X^{<\omega}$ and games on particular $Tree_1$ collections.

Theorem 2.6.7. *Suppose $n \in \omega\backslash\{0\}$, $\beta, \gamma \in \omega_1$.*

For any nonempty $\Theta \subseteq CTB$, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (2.67) and (2.68) are equivalent to $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.67}$$

$$\mathcal{G}\left(\mathbf{\Pi}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.68}$$

For any nonempty $\Theta \subseteq CTB$, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (2.69)is equivalent to $Det(\mathbf{\Delta}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Delta}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.69}$$

$\dashv$

Proof.

We obtain

1. $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$ if and only if the determinacy of (2.67),

2. $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$ if and only if the determinacy of (2.68),

3. $Det(\mathbf{\Delta}^1_n \restriction X^\omega)$ if and only if the determinacy of (2.69).

(⇒) Let $\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)$. By theorem 2.5.38,

$$Det(\mathbf{\Sigma}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Delta}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_n; \mathcal{T}_1\right).$$

By theorem 2.3.1,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_1\right).$$

Thus we have

$$Det(\mathbf{\Sigma}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_1\right).$$

By observation 2.2.4, for any $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

$$Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right) \subseteq \mathcal{T}_1.$$

Thus, we have (⇒) of (1) through (3).

(⇐) By corollary 2.4.2, for any $Y \in \Theta$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \restriction X^\omega\right).$$

By observation 2.2.4, for any $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

$$Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right) \subseteq Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right).$$

Thus, we have (⇐) of (1) through (3). □

Corollary 2.6.8. *Suppose Λ is σ-algebra and closed under Λ-substitution. Then for any nonempty $\Theta \subseteq CTB$,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right) \qquad \dashv$$

Proof.

($\Rightarrow$) By corollary 2.5.36,

$$Det(\Lambda \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, CTB, \Gamma(\omega,\Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

Since $Tree_1\left(X, \Theta, \Gamma(\omega,\Lambda), \Lambda \upharpoonright X^\omega\right) \subseteq Tree_1\left(X, CTB, \Gamma(\omega,\Lambda), \Lambda \upharpoonright X^\omega\right)$ for any $\Theta \subseteq CTB$,

$$Det(\Lambda \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, \Theta, \Gamma(\omega,\Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

($\Leftarrow$) Since $\emptyset \in \Lambda \upharpoonright X^\omega$, by observation 2.4.1, we have the result. $\square$

2.7 Generalization of a Type 1 tree

Instead fixing the first part as X^ω, we can take X^α for any limit ordinal α. The following is the generalization of a Type 1 tree, named an α-Type 1 tree.

Definition 2.7.1. *(Definition of an α-Type 1 tree)*

Suppose α is a limit ordinal. Suppose X and Y are nonempty sets. Let B be a subset of X^α and let Ψ be a function from X^α into ω. For any $h \in X^\alpha \times Y^{<\omega}$, define $[{}_\alpha T^{\Psi,B}_{X,Y}]$ by :

$$h \in \left[{}_\alpha T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\alpha & \text{if } h \restriction \alpha \notin B, \\ h \in X^\alpha \times Y^{\Psi(h\restriction\alpha)+1} & \text{if } h \restriction \alpha \in B. \end{cases}$$

⊣

Thus a Type 1 tree is an ω-Type 1 tree. Notice that we can obtain the similar results for α-Type 1 trees. Simply replace X^ω by X^α (respectively, $X^{<\omega}$ by $X^{<\alpha}$).

Chapter 3

Type 2 Tree : $T^{\Psi,B}_{X,T_{sq}}$

In chapter 2, we introduced a "Type 1" tree and obtained the determinacy results between games on particular $Tree_1$ collections and games on $X^{<\omega}$. In this chapter, we will introduce a "Type 2" tree and consider games on Type 2 trees. Type 2 tree will generalize Type 1 tree: all Type 1 trees are Type 2 trees.

In section 3.1, we will define a Type 2 tree. This section will be a Type 2 tree version of section 2.1. Recall that every Type 1 tree $T = T^{\Psi,B}_{X,Y}$ satisfies the following properties:

1. Every $h \in [T]$ has $lh(h) \geq \omega$.

2. For any $h \in [T]$, every move of $h \upharpoonright \omega$ is from X.

3. For every $h \in [T]$, if $h \upharpoonright \omega \in B$, then $lh(h) > \omega$ and if $h \upharpoonright \omega \notin B$, then $lh(h) = \omega$.

4. For any $h \in [T]$, if $lh(h) > \omega$, then $h \upharpoonright [\omega, lh(h)) \in Y^{\Psi(h \upharpoonright \omega)+1}$ so that $lh(h) = \Psi(h \upharpoonright \omega) + 1 < \omega + \omega$.

Given an ω-sequence of nonempty trees $T_{sq} = \langle T_n | n \in \omega \rangle$ and X, Ψ, B as before, a Type

2 tree $T = T^{\Psi,B}_{X,T_{sq}}$ satisfies the following conditions: Properties 1-3 above and 4* below.

4*. For any $h \in [T]$, if $lh(h) > \omega$, then $h \restriction [\omega, lh(h)) \in [T_{\Psi(h\restriction\omega)}]$.

Thus the play of the tails is different in Type 1 tree and Type 2 tree. If every path in T_n has the same fixed length, then every path of a Type 2 tree constructed by these trees will have the fixed length. Typically, the paths of a Type 2 tree will have variable lengths. In observation 3.1.3, we will show that every Type 1 tree is a Type 2 tree.

There are Type 2 trees which are not Type 1 trees. These are differences between properties 4 and 4*.

1. We do allow the trees T_n's on $T_{sq} = \langle T_n | n \in \omega \rangle$ to have any height greater than 0 (including greater than ω). Recall that the height of every Type 1 tree is $\leq \omega + \omega$, so that there are "long" Type 2 tree which are not Type 1 trees.

2. There are Type 2 trees in which each T_n from $T_{sq} = \langle T_n | n \in \omega \rangle$ has finite height that are not Type 1 trees. Moves of positions in each T_n doesn't necessarily come from a same set. For example, $[T_n]$ could be $Y \times Z$ for some $Y \neq Z$. Moreover, for distinct $p, q \in T_n$ with $lh(p) = lh(q)$, the set of possible moves at p and the set of possible moves at q could be different.

3. There are Type 2 trees in which each T_n from $T_{sq} = \langle T_n | n \in \omega \rangle$ has finite height and every move of each T_n's are from the same set that are not Type 1 trees. The length of every play h with $lh(h) > \omega$ in a Type 1 tree is computed from $\Psi(h \restriction \omega)$. For a Type 2 tree, $h \restriction \omega$ decides that the tail $h \restriction [\omega, lh(h)) \in [T_{\Psi(h\restriction\omega)}]$. If $T_{\Psi(h\restriction\omega)}$ has paths

of different lengths i and j, then this Type 2 tree contains paths h_1 of length $\omega + i$ and h_2 of length $\omega + j$ such that $h_1 \restriction \omega = h_2 \restriction \omega$.

We will also provide a separate characterization of Type 2 trees called (X,countable tail trees)-$[\omega, \infty)$ trees.

In section 3.2, we will define a $Tree_2$ collection and games on a $Tree_2$ collection. This section will be a Type 2 tree version of section 2.2.

In sections 3.3 through 3.6, we will observe the determinacy strength on games on $Tree_2$ collections. To be consistent, we will observe the determinacy comparison in the same order as in chapter 2. We can obtain similar results the same way as we proved in chapter 2. In section 3.3, by shifting, we will compare the determinacy of $\mathbf{\Sigma}^0_\alpha$ games and $\mathbf{\Pi}^0_\alpha$ games on a particular $Tree_2$ collection and the determinacy of $\mathbf{\Sigma}^1_n$ games and $\mathbf{\Pi}^1_n$ games on the same $Tree_2$ collection, for $\alpha \in \omega_1$ and $n \in \omega$. This section will be a Type 2 tree version of section 2.3. In sections 3.4 through 3.6, we will compare the determinacy strength of games on a $Tree_2$ collection and games on $X^{<\omega}$:

- In section 3.4, we will use the determinacy of games on a particular $Tree_2$ collection to obtain the determinacy of certain games on $X^{<\omega}$. This section will be a Type 2 tree version of section 2.4.

- In section 3.5, we will obtain the determinacy of Borel and projective games on certain $Tree_2$ collections from the determinacy of games on $X^{<\omega}$. Some of these results are converses of results from section 3.4. This section will be a Type 2 tree version of section 2.5.

- In section 3.6, we will conclude this chapter with the resulting determinacy equivalences from the earlier determinacy results between games on $X^{<\omega}$ and on $Tree_2$ collections. This section will be a Type 2 tree version of section 2.6.

Lastly, in section 3.7, we will generalize a Type 2 tree to an α-Type 1 tree for a limit ordinal α. This section will be a Type 2 tree version of section 2.7.

3.1 Definition of a Type 2 tree

In this section, we will give a definition of a Type 2 tree. Throughout this chapter, we will assume the following notation 3.1.1.

Notation 3.1.1.

- T_{sq} *will always denote some* ω*-sequence of nonempty trees.*

As in chapter 2, we will have the following notational conventions throughout chapter 3:

- X *will always denote a nonempty set.*
- B *will always denote a subset of* X^ω*.*
- Ψ *will always denote a function from* X^ω *into* ω*.* $\dashv$

A Type 2 tree is a tree with the following form.

Definition 3.1.2. *(Definition of a Type 2 tree)*

Suppose X *is a nonempty set,* Ψ *is a function from* X^ω *into* ω*,* B *is a subsets of* X^ω *and* $T_{sq} = \langle T_n \,|n \in \omega \rangle$ *where each* T_n *is a tree. Define* $[T^{\Psi,B}_{X,T_{sq}}]$ *by :*

$$h \in \left[T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin B, \\ h \in X^\omega \times \left[T_{\Psi(h \upharpoonright \omega)}\right] & \text{if } h \upharpoonright \omega \in B. \end{cases}$$

A tree T *is a Type 2 tree if and only if* $T = T^{\Psi,B}_{X,T_{sq}}$ *for some nonempty set* X*, a function* Ψ *from* X^ω *into* ω*, a subset* B *of* X^ω*and some* $T_{sq} = \langle T_n \,|n \in \omega \rangle$*, where each* T_n *is a nonempty tree.* $\dashv$

As in definition 3.1.2, fix X, $\Psi : X^\omega \to \omega$, $B \subseteq X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega \rangle$. $[T^{\Psi,B}_{X,T_{sq}}]$ is the disjoint union of plays of the short play (length ω) and the long play (length greater than ω). Thus we have $[T^{\Psi,B}_{X,T_{sq}}] = \bigcup_{n\in\omega}[(B\cap\Psi^{-1}(n))\times[T_n]]\dot{\cup}(X^\omega\backslash B)$. [1] In particular, $[T^{\Psi,\emptyset}_{X,Y_{sq}}] = X^\omega$.

Then for all $n\in\omega$, for all $f\in B\cap\Psi^{-1}(n)$ and for all $g\in[T_n]$,

$$f^\frown g\in[T^{\Psi,B}_{X,T_{sq}}].$$

Each element of T_n corresponds to a "tail" of a play. For each $f\in B\cap\Psi^{-1}(n)$, we can attach any tail in T_n. This is the reason we use/abuse the cross product notation.

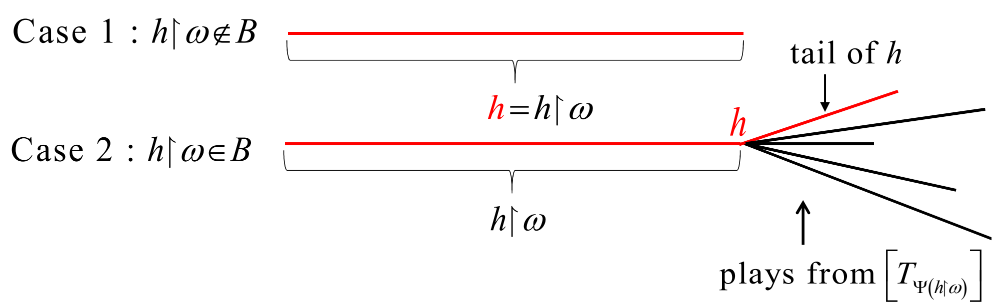

Figure 3.1.1: Illustration of paths $h\in[T]$ for a Type 2 tree $T=T^{\Psi,B}_{X,T_{sq}}$ for $B\neq\emptyset$.

Now, we observe that Type 1 trees are a special case of Type 2 trees. Suppose Y is a nonempty set, Ψ is a function from X^ω into ω and B is a subset of X^ω. Define $T_n = Y^{\leq n+1}$ for any $n\in\omega$. Then $[T_n]=Y^{n+1}$. Let $T_{sq}=\langle T_n\,|n\in\omega\rangle$. Then

[1] Recall notation 1.5.3 : abuse of product notation.

$$h \in \left[T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin B, \\ h \in X^\omega \times Y^{\Psi(h\upharpoonright\omega)+1} & \text{if } h \upharpoonright \omega \in B. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin B, \\ h \in X^\omega \times \left[T_{\Psi(h\upharpoonright\omega)}\right] & \text{if } h \upharpoonright \omega \in B. \end{cases}$$

$$\leftrightarrow h \in \left[T^{\Psi,B}_{X,T_{sq}}\right]$$

Thus $[T^{\Psi,B}_{X,Y}] = [T^{\Psi,B}_{X,T_{sq}}]$, i.e., $T^{\Psi,B}_{X,Y} = T^{\Psi,B}_{X,T_{sq}}$ for $T_{sq} = \left\langle Y^{\leq n+1} \,|n \in \omega \right\rangle$.

Observation 3.1.3. *For each Type 1 tree $T^{\Psi,B}_{X,Y}$, $[T^{\Psi,B}_{X,Y}] = [T^{\Psi,B}_{X,T_{sq}}]$ where $T_{sq}{=}\left\langle Y^{\leq n+1}|n{\in}\omega\right\rangle$. Thus any Type 1 tree is a Type 2 tree.* $\dashv$

Note that for each Type 1 tree, for any two functions f_1 and f_2 in X^ω such that $\Psi(f_1) \neq \Psi(f_2)$, it is impossible to share the same tail since the length of play is depending on Ψ. However, unlike Type 1 trees, for Type 2 trees, it is possible for two functions f_1 and f_2 in X^ω to be followed (in play) by the same tail, even if $\Psi(f_1) \neq \Psi(f_2)$. For example, suppose $[T_n] \cap [T_m] \neq \emptyset$ for some $n \neq m$. Then for all $f_1 \in B \cap \Psi^{-1}(n)$ and for all $f_2 \in B \cap \Psi^{-1}(m)$, if $g \in [T_n] \cap [T_m]$ then $f_1^\frown g, f_2^\frown g \in [T^{\Psi,B}_{X,T_{sq}}]$. Thus, for $h \in [T^{\Psi,B}_{X,T_{sq}}]$, $h \upharpoonright [\omega, lh(h)) \in [T_n]$ does not necessarily imply $\Psi(h \upharpoonright \omega) = n$.

Next, we provide an alternate description of Type 2 trees. In definition 3.1.4 below, we will give a definition of the "tail tree" T^f of f.

Definition 3.1.4. *(Definition of the tail tree T^f of f)*

Suppose T is a tree. For each $f \in X^\omega$, define $[T^f]$ to be the set of tails for f, i.e., for any

$f \in X^\omega$ and for any $g \in [T^f]$, $f^\frown g \in T$. Then $f \in X^\omega \cap [T]$ if and only if $[T^f] = \emptyset$. Notice that each T^f is a tree. Define T^f to be the tail tree of f. ⊣

We will define "the countable tail trees property" on a tree T.

Definition 3.1.5. *(Definition of the countable tail trees property)*

Suppose T is a tree. Define that T has "the countable tail trees property" if and only if $\left\{ [T^f] \,|f \in X^\omega \right\}$ is countable. ⊣

Definition 3.1.6. *(Definition of $(X$,countable tail trees)-$[\omega, \infty)$)*

Suppose X is a nonempty set. Define that a tree T is "$(X$,countable tail trees)-$[\omega, \infty)$" if $[T]$ satisfies the following three properties:

1. *for all $y \in [T]$, $y \upharpoonright \omega \in X^\omega$.*
2. *for all $y \in [T]$, $lh(y) \geq \omega$.*
3. *T satisfies the countable tail trees property.* ⊣

Observation 3.1.7. *For any X, $\Psi : X^\omega \to \omega$, for any nonempty $B \subseteq X^\omega$ and any ω-sequence of nonempty trees T_{sq}, $T^{\Psi,B}_{X,T_{sq}}$ satisfies $(X$,countable tail trees)-$[\omega, \infty)$. Conversely, for any $(X$,countable tail trees)-$[\omega, \infty)$ tree T, there exists $\Psi : X^\omega \to \omega$ and a unique $B \subseteq X^\omega$ such that $T = T^{\Psi,B}_{X,T_{sq}}$.* ⊣

Proof.

(⇒) Show $T^{\Psi,B}_{X,T_{sq}}$ is $(X$,countable tail trees)-$[\omega, \infty)$. Clearly, (1) and (2) holds. Show (3). By definition of the Type 2 tree, for each $f \in B \cap \Psi^{-1}(n)$, $[T^f] = [T_n]$. Since Ψ maps form X^ω into ω, there are at most ω many distinct $[T^f]$s.

($\Leftarrow$) Suppose T is (X,countable tail trees)-$[\omega, \infty)$. Let $B = \{f \restriction \omega \in T \mid lh(f) > \omega\}$. Then $B \subseteq X^\omega$ by (1). Since there are countable many distinct $[T^f]$'s, let $\langle [T_\beta] | \beta \in \omega \rangle$ be an enumeration of nonempty $[T^f]$'s. Define

$$\begin{aligned} \Psi : \quad & X^\omega \to \omega \\ & f \mapsto n \text{ where } [T_n] = \left[T^f\right]. \end{aligned}$$

Show $[T] = [T^{\Psi,B}_{X,T_{sq}}]$.

($\subseteq$) Pick an arbitrary $h \in [T]$.

By (1), $h \restriction \omega \in X^\omega$. Case 1 : $lh(h) = \omega$.

Then $h \restriction \omega \notin B$ so that $h = h \restriction \omega \in [T^{\Psi,B}_{X,T_{sq}}]$.

Case 2 : $lh(h) > \omega$.

Then $h \restriction \omega \in B$. Since $h \restriction [\omega, lh(\omega)) \in [T^{h \restriction \omega}] = [T_{\Psi(h \restriction \omega)}]$, $h \in [T^{\Psi,B}_{X,T_{sq}}]$.

($\supseteq$) Pick an arbitrary $h \in [T^{\Psi,B}_{X,T_{sq}}]$.

Case 1 : $h \restriction \omega \notin B$.

Then $lh(h) = \omega$ so that $h = h \restriction \omega$. Hence $[T^{h \restriction \omega}] = \emptyset$. Thus $h \restriction \omega \in T$.

Case 2 : $h \restriction \omega \in B$.

Then $lh(h) > \omega$. By definition of $[T^{\Psi,B}_{X,T_{sq}}]$, $h \restriction [\omega, lh(h)) \in [T_{\Psi(h \restriction \omega)}] = [T^{h \restriction \omega}]$. Thus $h \in [T]$.

Show uniqueness. Suppose $\hat{B} \subseteq X^\omega$ such that $T = T^{\Psi,B}_{X,T_{sq}} = T^{\hat{\Psi},\hat{B}}_{X,\hat{T}_{sq}}$ for some $\hat{\Psi}$ from X^ω into ω and $\hat{T}_{sq}$. Show $\hat{B} = B$. Suppose, for a contradiction, $\hat{B} \neq B$. Then there exists $h \in [T]$ such that $h \restriction \omega \in \hat{B} \backslash B$ or $h \restriction \omega \in B \backslash \hat{B}$. Suppose $h \restriction \omega \in \hat{B} \backslash B$. Since $h \in [T] = [T^{\Psi,B}_{X,T_{sq}}]$ and $h \restriction \omega \notin B$, $lh(h) = \omega$. Since $h \in [T] = [T^{\hat{\Psi},\hat{B}}_{X,\hat{T}_{sq}}]$ and $h \restriction \omega \in \hat{B}$, $lh(h) > \omega$, a contradiction. Similar for the case $h \restriction \omega \in B \backslash \hat{B}$. Thus $\hat{B} = B$. □

3.2 Definition of a $Tree_2$ collection and a collection of games on a $Tree_2$ collection with complexity Ξ

In this section, we will define a $Tree_2$ collection of Type 2 trees. We will use Υ to be a nonempty collection of nonempty trees[2]. We will be considering a collection of trees $T^{\Psi,B}_{X,T_{sq}}$ for which T_{sq} varies over Υ^ω, Ψ varies over Γ and B varies over Λ, while X is fixed. Thus, each $Tree_2$ collection will be defined from X, Υ, Γ and Λ. We will denote a $Tree_2$ collection by $Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$ constructed from some X, Υ, Γ and Λ. Throughout the rest of this chapter, we will assume notation 3.2.1. Then, in definition 3.2.10, we will define a collection of games on a $Tree_2$ collection with complexity Ξ. We will use $\Xi \upharpoonright [T]$ for each tree T in $Tree_2$ collection. We will also make some observations of $Tree_2$ collections and collections of games on $Tree_2$ collections.

Notation 3.2.1. *We will assume the following notation throughout chapter 3.*

- Υ *will always denote a nonempty collection of nonempty trees.*

As in chapter 2:

- Γ*, respectively,* Γ_i *will always denote nonempty collection of functions from* X^ω *into* ω*.*

- Λ*, respectively,* Λ_i *will always denote nonempty collection of subsets of* X^ω*.* $\dashv$

We next define a collection of Type 2 trees constructed by X, Υ, Γ and Λ.

[2] Υ (a Greek letter uppercase Upsilon) corresponds to the letter Y. The position for Y in a Type 1 tree $T^{\Psi,B}_{X,Y}$ is replaced by $T_{sq} \in \Upsilon^\omega$ in a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$. I choose Υ just because its shape resembles the trees.

Definition 3.2.2. *(Definition of a $Tree_2$ collection)*

Fix a nonempty set X. Let $\Upsilon \neq \emptyset$ be any collection of nonempty trees. Suppose $\Lambda \neq \emptyset$ is any collection of subsets of X^ω, $\Gamma \neq \emptyset$ is a collection of functions from X^ω to ω. Define

$$Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda) = \left\{ T^{\Psi,B}_{X,T_{sq}} \middle| T_{sq} \in \Upsilon^\omega, \Psi \in \Gamma, B \in \Lambda \right\}.$$

A collection is a $Tree_2$ collection if and only if it is $Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$ for some nonempty set X, a nonempty collection Υ of nonempty trees, a nonempty collection Γ of functions from X^ω into ω and a nonempty collection Λ of subsets of X^ω.

We sometimes let $\mathcal{T}_2$ be a $Tree_2$ collection when we wish to suppress $X, \Upsilon^\omega, \Gamma$ and Λ, i.e., $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. $\dashv$

Notation 3.2.3. *If Υ is a singleton $\{T\}$, we will write $Tree_2(X, \{T\}^\omega, \Gamma, \Lambda)$. To avoid confusion, we do not supress the brackets $\{\}$ in $\{T\}^\omega$, we do not write $Tree_2(X, T^\omega, \Gamma, \Lambda)$. If we fix an ω-sequence of trees T_{sq}, we will write $Tree_2(X, T_{sq}, \Gamma, \Lambda)$. When dealing with the singletons for any of the last two components of $Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$, we will suppress $\{\}$, i.e., if Γ is a singleton $\{f\}$, $Tree_2(X, \Upsilon^\omega, f, \Lambda)$ abbreviates $Tree_2(X, \Upsilon^\omega, \{f\}, \Lambda)$ and if Λ is a singleton $\{B\}$, $Tree_2(X, \Upsilon^\omega, \Gamma, B)$ abbreviates $Tree_2(X, \Upsilon^\omega, \Gamma, \{B\})$.* $\dashv$

Now, we will observe a relation between $Tree_1$ and $Tree_2$. By observation 3.1.3, each Type 1 tree is a Type 2 tree. Thus, we have the following inclusion.

Observation 3.2.4. *Suppose Y is a nonempty set and $\left\{ Y^{\leq n+1} \middle| n \in \omega \right\} \subseteq \Upsilon$. Then*

$$Tree_1(X, Y, \Gamma, \Lambda) \subseteq Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda).$$

In general, if $\left\{Y^{\leq n+1} \,|Y \in \Theta, n \in \omega\right\} \subseteq \Upsilon$, *then*

$$Tree_1\left(X, \Theta, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, \Upsilon^{\omega}, \Gamma, \Lambda\right). \quad \dashv$$

Proof.

Suppose $\left\{Y^{\leq n+1} \,|n \in \omega\right\} \subseteq \Upsilon$. Pick an arbitrary $T^{\Psi,B}_{X,Y} \in Tree_1\left(X, Y, \Gamma, \Lambda\right)$. Then $\Psi \in \Gamma$ and $B \in \Lambda$. Define $T^n = Y^{\leq n+1}$ for all $n \in \omega$. Let $T_{sq} = \langle T_n \,|n \in \omega\rangle$. Then $T_{sq} \in \Upsilon^{\omega}$. By observation 3.1.3, $[T^{\Psi,B}_{X,Y}] = [T^{\Psi,B}_{X,T_{sq}}]$. Thus $T^{\Psi,B}_{X,Y} = T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, \Upsilon^{\omega}, \Gamma, \Lambda\right)$.

Suppose $\left\{Y^{\leq n+1} \,|Y \in \Theta, n \in \omega\right\} \subseteq \Upsilon$. Then for any $Y \in \Theta$, $T_{sq} = \left\langle Y^{\leq n+1} \,|n \in \omega\right\rangle \in \Upsilon^{\omega}$. Thus we have

$$Tree_1\left(X, \Theta, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, \Upsilon^{\omega}, \Gamma, \Lambda\right). \quad \square$$

We will consider the Type 2 tree in which T_{sq} is an ω-sequence of well-founded trees. Recall the following notation.

Notation 1.5.11. *Let WF be the set of nonempty well-founded trees. Let $CWF \subseteq WF$ be the set of nonempty well founded trees such that each move is from some countable set. Similarly, let $FWF \subseteq CWF$ be the set of nonempty well-founded trees such that each move is from some finite set.* $\dashv$

Observation 3.2.5. *Suppose Ψ is a function from X^{ω} into ω and $B \subseteq X^{\omega}$ is nonempty. Then $T_{sq} \in WF^{\omega}$ implies $T^{\Psi,B}_{X,T_{sq}}$ is (X,countable tail trees)-$[\omega, \omega + \omega)$.*

Observation 3.2.6.

$$Tree_1\left(X, FIN, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, FWF^{\omega}, \Gamma, \Lambda\right).$$

$$Tree_1\left(X, CTB, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, CWF^{\omega}, \Gamma, \Lambda\right). \quad \dashv$$

Proof.

For any $Y \in FIN$, each $Y^{\leq n+1} \in FWF$ so that $T_{sq} = \left\langle Y^{\leq n+1} \middle| n \in \omega \right\rangle \in FWF^{\omega}$. Similarly, for any $Y \in CTB$, each $Y^{\leq n+1} \in CWF$ so that $T_{sq} = \left\langle Y^{\leq n+1} \middle| n \in \omega \right\rangle \in CWF^{\omega}$. By observation 3.2.4, we have the results. $\square$

We have the following inclusions similar to $Tree_1$. The following is a $Tree_2$ version of observation 2.2.4 on page 51.

Observation 3.2.7. *Fix a nonempty set X. Suppose $\Upsilon, \Upsilon_1, \Upsilon_2$ are collections of nonempty trees; $\Gamma, \Gamma_1, \Gamma_2$ are collections of functions from X^{ω} into ω; and $\Lambda, \Lambda_1, \Lambda_2$ are collections of subsets of X^{ω}. If $\Upsilon_1 \subseteq \Upsilon_2$, then*

$$Tree_2 \left(X, \Upsilon_1^{\omega}, \Gamma, \Lambda\right) \subseteq Tree_2 \left(X, \Upsilon_2^{\omega}, \Gamma, \Lambda\right).$$

Similarly, if $\Gamma_1 \subseteq \Gamma_2$, then

$$Tree_2 \left(X, \Upsilon^{\omega}, \Gamma_1, \Lambda\right) \subseteq Tree_2 \left(X, \Upsilon^{\omega}, \Gamma_2, \Lambda\right),$$

and if $\Lambda_1 \subseteq \Lambda_2$, then

$$Tree_2 \left(X, \Upsilon^{\omega}, \Gamma, \Lambda_1\right) \subseteq Tree_2 \left(X, \Upsilon^{\omega}, \Gamma, \Lambda_2\right).$$

Thus $Tree_2$ is an increasing operation on last three components. $\dashv$

The following is a $Tree_2$ version of observation 2.2.5 on page 52.

Observation 3.2.8. *Let Υ be a collection of nonempty trees and $\Lambda \subseteq X^{\omega}$. Suppose we have Ξ such that $\Xi \restriction X^{\omega}$ is defined (e.g., $\mathbf{\Sigma}^0_{\alpha}, \mathbf{\Pi}^0_{\alpha}, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $Tree_2 \left(X, \Upsilon^{\omega}, \Gamma(\omega, \Xi), \Lambda\right)$

- $Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, co\text{-}\Xi), \Lambda)$
- $Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda)$ ⊣

We will define a collection of games played on a $Tree_2$ collection similar to a $Tree_1$ collection. The following is a $Tree_2$ version of definition 2.2.6 on page 52.

Definition 3.2.9. *(Definition of "games on a $Tree_2$ collection")*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2\,(X, \Upsilon^\omega, \Gamma, \Lambda)$ for some X, Υ, Γ and Λ. Define "games on the $Tree_2$ collection $\mathcal{T}_2$" by

$$\bigcup_{T \in \mathcal{T}_2} \{G\,(A; T)\,|A \subseteq [T]\}.$$ ⊣

If Ξ is a complexity (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$), we define Ξ games on a $Tree_2$ collection as follows. The following is a $Tree_2$ version of definition 2.2.7 on page 52.

Definition 3.2.10. *(Definition of Ξ games on a $Tree_2$ collection)*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2\,(X, \Upsilon^\omega, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in Tree_2$, $\Xi \restriction [T] \subseteq \wp\,([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ games on a $Tree_2$ collection $\mathcal{T}_2$ by

$$\mathcal{G}\,(\Xi; \mathcal{T}_2) = \bigcup_{T \in \mathcal{T}_2} \{G\,(A; T)\,|A \in \Xi \restriction [T]\}.$$

⊣

We will use $\mathcal{G}$ for a collection of games.

Though often Ξ will be a standard classes (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$), note that $G(\Xi; T)$ is defined as long as we have defined $\Xi \restriction [T] \subseteq \wp\,([T])$. The following is a $Tree_2$ version of observation 2.2.8 on page 53.

Definition 3.2.11. *(Definition of Ξ determinacy on a $Tree_2$ collection)*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in$

$Tree_2$, $\Xi \upharpoonright [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ determinacy on the $Tree_2$ collection $\mathcal{T}_2$ by

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_2),$$

i.e., for any X, $T_{sq} \in \Upsilon^\omega, \Psi \in \Gamma$, $B \in \Lambda$ and $A \in \Xi \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$, every game $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. ⊣

Next, similar to Type 1 trees, we will make three observations about games on Type 2 trees. The following is a $Tree_2$ version of observation 2.2.9 on page 53.

Observation 3.2.12. *Suppose X is a nonempty set, Υ is a collection of trees, Γ is a collection of functions from X^ω into ω, Λ is a collection of subsets of X^ω and Ξ_1, Ξ_2 are complexities. Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. If for any $T \in \mathcal{T}_2$, $\Xi_1 \upharpoonright [T] \subseteq \Xi_2 \upharpoonright [T]$, then*

$$\mathcal{G}\left(\Xi_1; \mathcal{T}_2\right) \subseteq \mathcal{G}\left(\Xi_2; \mathcal{T}_2\right).$$

Thus $\mathcal{G}$ is an increasing operation on the first component. ⊣

The following is a $Tree_2$ version of observation 2.2.10 on page 54.

Observation 3.2.13. *Let Υ be a collection of trees and $\Lambda \subseteq X^\omega$. Suppose we have Ξ such that $\Xi \upharpoonright X^\omega \subseteq \wp(X^\omega)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Suppose we have Ξ_1 such that for each $T \in Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi), \Lambda\right)$, $\Xi_1 \upharpoonright [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi), \Lambda\right)\right)$
- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, co\text{-}\Xi), \Lambda\right)\right)$
- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda\right)\right)$ ⊣

3.3 Equivalence between $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ determinacy on Type 2 trees and equivalence between $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ determinacy on Type 2 trees

As in section 2.3, we will obtain the determinacy equivalence of particular games on $Tree_2$ collections. All of the results in this section is the Type 2 version of results in section 2.3.

The main theorem of this section is theorem 3.3.8, which is a $Tree_2$ version of theorem 2.3.1 on page 55. In this theorem, we will show that the determinacy of $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ games on certain $Tree_2$ collections of Type 2 trees are equivalent. We will also obtain the determinacy equivalence of $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ games on certain $Tree_2$ collections. For each Type 2 tree T, we will define a corresponding "Shift tree" $Sft_2(T)$ in definition 3.3.2 (We will use the subscript 2 to represent Type 2 trees). $Sft_2(T)$ is similar to $Sft(T)$ for a Type 1 tree T, and it is slightly modified for a Type 2 tree. In lemma 3.3.9 and lemma 3.3.10, we will find the complexity of $Shift_2(A)$ for A being Borel (respectively, a projective set) on a $Tree_2$ collection.

For each Type 2 tree, there is a natural Shift tree which is also a Type 2 tree. In order to define a Shift tree for each Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$, we define Ψ^+_2 and T^+_{sq} from Ψ and T_{sq} which satisfying a "shift" relation. Ψ^+_2 is a function on X^ω into ω and for any $f \in X^\omega$ and for any $a \in X$, $\Psi^+_2(\langle a\rangle^\frown f) = \Psi(f)$. Thus Ψ^+_2 is just ignoring the first move. Recall that Ψ^+ in section 2.3.2 had an extra 1; i.e., for any $a \in X$, $\Psi^+(\langle a\rangle^\frown f) = \Psi(f) + 1$. Instead, we will define T^+_n for each tree T_n in T_{sq} which has an extra move.

Definition 3.3.1. *(Definition of Ψ^+_2 and T^+_{sq})*

Fix a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$. Then $B \subseteq X^\omega$, $\Psi : X^\omega \to \omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle$ is an ω-sequence of nonempty trees. Define

1. $\Psi^+_2 : X^\omega \to \omega$ *such that* $\Psi^+_2(f) = \Psi(f \restriction [1, \omega))$.

2. *For each* $n \in \omega$, $T^+_n = Y_n \times T_n$ *for some nonempty set* Y_n *and* $T^+_{sq} = \langle T^+_n \,|n \in \omega\rangle$. ⊣

Recall we have defined B^+ for $B \subseteq X^\omega$ as $B^+ = X \times B \subseteq X^\omega$ by definition 2.3.2 on page 56. [3] By using B^+, Ψ^+ and T^+_{sq}, we will define a Shift tree as follows.

Definition 3.3.2. *(Definition of a Shift tree $Sft_2(T)$)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define a Shift tree $Sft_2(T)$ by

$$Sft_2(T) = T^{\Psi^+_2,B^+}_{X,T^+_{sq}}. \qquad \dashv$$

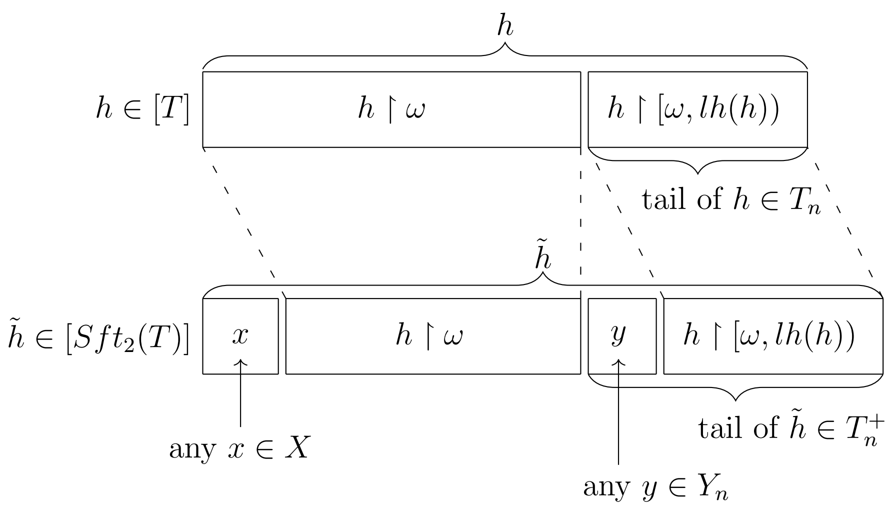

Figure 3.3.1: Illustration of $\tilde{h} \in [Sft_2(T)]$ with $lh(\tilde{h}) > \omega$.

For any Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$ with $B = \emptyset$, $T = X^{<\omega} = Sft_2(T)$.

Unlike observation 2.3.4, $T = Sft_2(T)$ doesn't imply $B = \emptyset$. Set $X = Y = \omega$. Let $T_n = \omega^\omega$ for all $n \in \omega$. Then for $B = \omega^\omega$ and any function Ψ from ω^ω into ω, $T = T^{\Psi,B}_{\omega,\omega} = \omega^{\omega+\omega}$.

[3] Recall abuse of notation 1.5.3 on page 42.

Since $B^+ = \omega \times \omega^\omega = \omega^\omega$, $Sft_2(T) = T^{\Psi_2^+, B^+}_{\omega,\omega} = \omega^{\omega+\omega}$. Hence we have $T = Sft_2(T)$.

Notice that for each Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$ and for each $h \in [Sft_2(T)]$, there is a unique $f \in [T]$ such that $h(i+1) = f(i)$ for every $i \in lh(f)$ (e.g., $h(1) = f(0), h(2) = f(1), ..., h(\omega+1) = f(\omega), h(\omega+2) = f(\omega+1), ...$ for $h \restriction \omega \in B$).

Proposition 3.3.3. *Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Then for every $h \in [Sft_2(T)]$,*

$$\begin{cases} h \restriction [1,\omega) \in [T] & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) \in [T] & \text{if } h \restriction \omega \in B^+. \end{cases}$$

⊣

Proof.

Pick an arbitrary $h \in [Sft_2(T)]$. Notice that $h \restriction \omega \in B^+$ if and only if $h \restriction [1,\omega) \in B$. Also, $\Psi_2^+(h \restriction \omega) = \Psi_2^+(h(0)^\frown h \restriction [1,\omega)) = \Psi(h \restriction [1,\omega))$. Thus

$$\begin{aligned} h \in [Sft_2(T)] &\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B^+, \\ h \in X^\omega \times T^+_{\Psi^+(h\restriction\omega)} & \text{if } h \restriction \omega \in B^+. \end{cases} \\ &\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction [1,\omega) \notin B, \\ h \in X^\omega \times T^+_{\Psi^+(h\restriction\omega)} & \text{if } h \restriction [1,\omega) \in B. \end{cases} \\ &\leftrightarrow \begin{cases} h \restriction [1,\omega) \in X^\omega & \text{if } h \restriction [1,\omega) \notin B, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) \in X^\omega \times T_{\Psi(h\restriction[1,\omega))} & \text{if } h \restriction [1,\omega) \in B. \end{cases} \\ &\leftrightarrow \begin{cases} h \restriction [1,\omega) \in [T] & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) \in [T] & \text{if } h \restriction \omega \in B^+. \end{cases} \end{aligned}$$

□

Proposition 3.3.3 give us a natural erasing function e_2 from $[Sft_2(T)]$ into $[T]$.

Definition 3.3.4. *(Definition of the erasing function $e_2 : [Sft_2(T)] \to [T]$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define the erasing function e from $[Sft_2(T)]$ into $[T]$ by

$$e_2 : \ [Sft_2(T)] \to [T]$$
$$h \mapsto \begin{cases} h \restriction [1,\omega) & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) & \text{if } h \restriction \omega \in B^+. \end{cases} \dashv$$

Now, we define a function "Shift" which maps subsets A of $[T]$ for a Type 2 tree T to a particular subset of $[Sft_2(T)]$.

Definition 3.3.5. *(Definition of $Shift_2$)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define

$$Shift_2 : \ \wp([T]) \to \wp([Sft_2(T)])$$
$$A \mapsto \{h \in [Sft_2(T)] \mid e_2(h) \in [T] \setminus A\}. \dashv$$

Theorem 3.3.6. *For any Type 2 tree T, the determinacy of $G(Shift_2(A); Sft_2(T))$ implies the determinacy of $G(A,T)$.* $\dashv$

Proof.

Pick an arbitrary Type 12 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Assume $G(Shift_2(A); Sft_2(T))$ is determined. Then I^* or II^* has a winning strategy s^* for $G(Shift_2(A); Sft_2(T))$. Show that $G(A;T)$ is determined.

Case I : s^* is a I^*'s winning strategy for $G(Shift_2(A); Sft_2(T))$. Define a strategy s for II for $G(A;T)$ as follows: Suppose $a_0 = s^*(\emptyset)$.

For $p \in T$ such that p is finite and $\langle a_0\rangle^\frown p \in dom(s^*)$ or

p is infinite and $\langle a_0\rangle^\frown p \restriction \omega^\frown \langle a_\omega\rangle^\frown p \restriction [\omega, lh(p)) \in dom(s^*)$ where $a_\omega = s^*\big(\langle a_0\rangle^\frown p \restriction \omega\big)$,

$p \in dom(s)$ and

$$s(p) = \begin{cases} s^*\left(\langle a_0\rangle^\frown p\right) & \text{if } p \text{ is finite,} \\ s^*\left(\langle a_0\rangle^\frown p \restriction \omega^\frown \langle a_\omega\rangle^\frown p \restriction [\omega, lh(p))\right) & \text{if } p \text{ is infinite.} \end{cases}$$

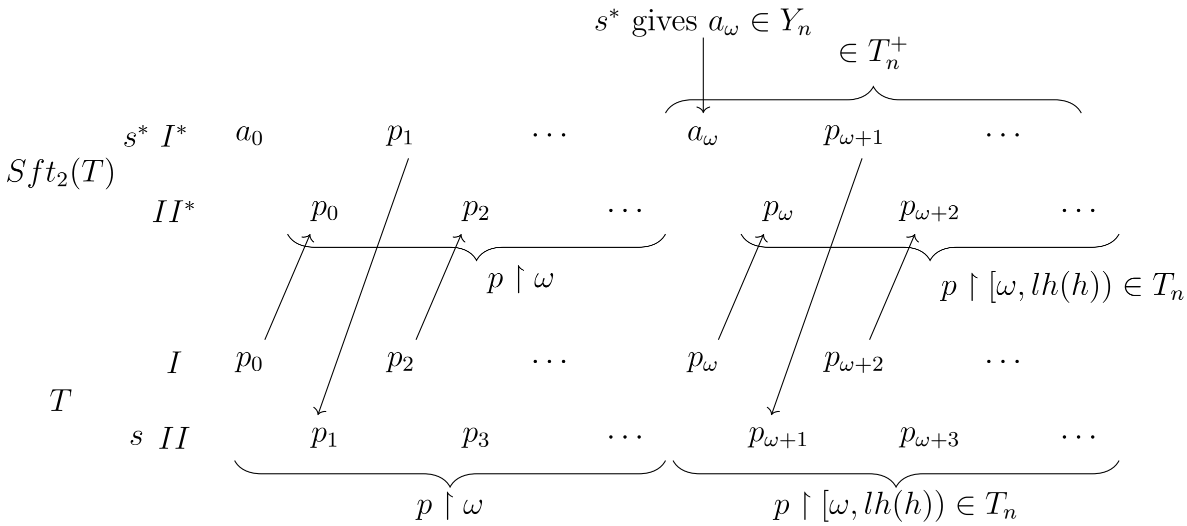

Figure 3.3.2: Illustration of $p \in T$, $lh(p) > \omega$ according to II's strategy s.

Show s is a winning strategy for II for $G(A;T)$. Pick an arbitrary $x \in [T]$ according to s.

Subcase 1 : $x \restriction \omega \notin B$.

Then $x = x \restriction \omega$ and $s^*(\emptyset)^\frown x \notin B^+$. Thus $s^*(\emptyset)^\frown x \in [Sft_2(T)]$ and it is according to s^*. Hence $s^*(\emptyset)^\frown x \in Shift_2(A)$ and thus $x = e_2(s^*(\emptyset)^\frown x) \notin A$. [4]

Subcase 2 : $x \restriction \omega \in B$.

Then $s^*(\emptyset)^\frown x \restriction \omega \in B^+$ and $\Psi_2^+(s^*(\emptyset)^\frown x \restriction \omega) = \Psi(x \restriction \omega)$. [5] Let

$$n = \Psi_2^+(s^*(\emptyset)^\frown x \restriction \omega) = \Psi(x \restriction \omega).$$

[4]Recall definition 3.3.4 for the erasing function e_2.
[5]Recall definition 3.3.1 for Ψ_2^+.

Then $x \upharpoonright [\omega, lh(x)+1) \in T^n$. Let

$$h = s^*(\emptyset)^\frown(x \upharpoonright \omega)^\frown s^*\big(s^*(\emptyset)^\frown x \upharpoonright \omega\big)^\frown x \upharpoonright [\omega, lh(x)) \,.$$

Since x is according to s, h is according to s^*. Since s^* is a strategy for I^* for $G(Shift_2(A); Sft_2(T))$, $s^*\left(s^*(\emptyset)^\frown x \upharpoonright \omega\right) \in Y_n$. Thus

$$s^*\big(s^*(\emptyset)^\frown x \upharpoonright \omega\big)^\frown x \upharpoonright [\omega, lh(x)+1) \in Y_n \times T_n = T_n^+.$$

Hence $h \in [Sft_2(T)]$. Since s^* is a winning strategy for I^* for $G(Shift_2(A); Sft_2(T))$, $h \in Shift_2(A)$. Hence $x = e(h) \notin A$. Therefore, s is a winning strategy for II for $G(A;T)$.

Case II : s^* is a II^*'s winning strategy for $G(Shift_2(A); Sft_2(T))$.

Define a strategy s for I for $G(A;T)$ as follows: Suppose $a_0 \in X$ and $a_\omega^n \in Y_n$, $n \in \omega$ are arbitrary.

For $p \in T$ such that p is finite and $\langle a_0\rangle^\frown p \in dom(s^*)$ or

p is infinite and $\langle a_0\rangle^\frown p \upharpoonright \omega^\frown \langle a_\omega^n\rangle^\frown p \upharpoonright [\omega, lh\,(p)) \in dom(s^*)$, $p \in dom(s)$ and

$$s\,(p) = \begin{cases} s^*\left(\langle a_0\rangle^\frown p\right) & \text{if } p \text{ is finite,} \\ s^*\left(\langle a_0\rangle^\frown p \upharpoonright \omega^\frown \langle a_\omega^n\rangle^\frown p \upharpoonright [\omega, lh\,(p))\right) & \text{if } p \text{ is infinite.} \end{cases}$$

Note that for $\langle a_0\rangle^\frown p \upharpoonright \omega^\frown \langle a_\omega^n\rangle^\frown p \upharpoonright [\omega, lh\,(p)) \in Sft_2(T)$, $a_\omega^n \in Y_n$ and $p \upharpoonright [\omega, lh\,(p)) \in T_n$ where $n = \Psi_2^+(\langle a_0\rangle^\frown p \upharpoonright \omega) = \Psi(p \upharpoonright \omega)$.

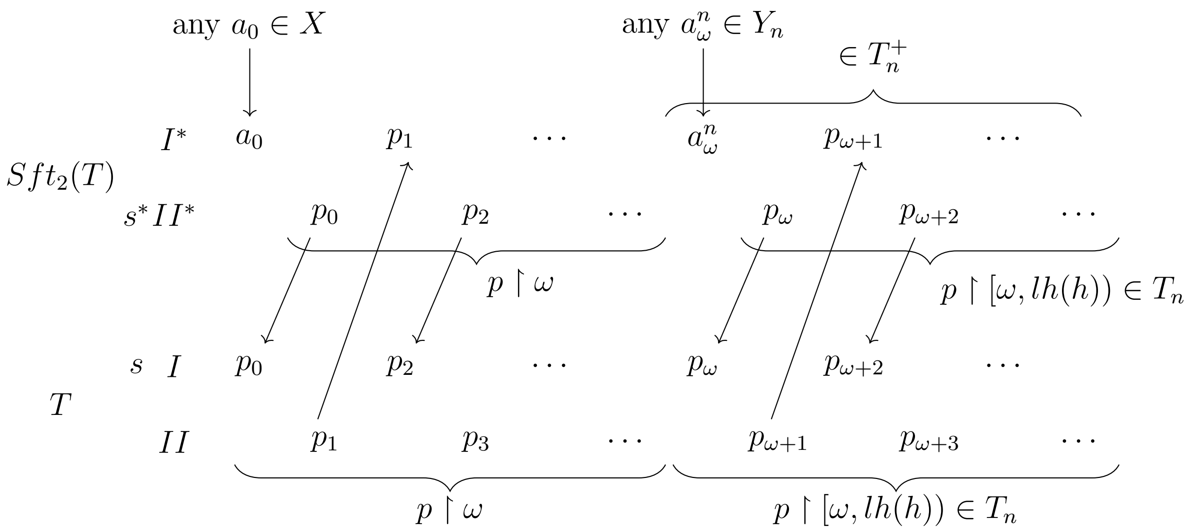

Figure 3.3.3: Illustration of $p \in T$, $lh(p) > \omega$ according to I's strategy s.

Show s is a winning strategy for I for $G(A;T)$. Pick an arbitrary $x \in [T]$ according to s. Let $a_0 \in X$ and $a_\omega^n \in Y_n$, $n \in \omega$ be arbitrary.

Subcase 1 : $x \restriction \omega \notin B$.

Then $x = x \restriction \omega$ and $\langle a_0 \rangle^\frown x \notin B^+$. Thus $\langle a_0 \rangle^\frown x \in [Sft_2(T)]$ and it is according to s^*. Hence $\langle a_0 \rangle^\frown x \notin Shift_2(A)$ and thus $x = e_2(\langle a_0 \rangle^\frown x) \in A$.

Subcase 2 : $x \restriction \omega \in B$.

Then $\langle a_0 \rangle^\frown x \restriction \omega \in B^+$ and $\Psi^+(\langle a_0 \rangle^\frown x \restriction \omega) = \Psi(x \restriction \omega)$. Let $n = \Psi^+(\langle a_0 \rangle^\frown x \restriction \omega) = \Psi(x \restriction \omega)$. Let

$$h = \langle a_0 \rangle^\frown (x \restriction \omega)^\frown \langle a_\omega^n \rangle^\frown x \restriction [\omega, lh(x))\,.$$

Then $\langle a_\omega^n \rangle^\frown x \restriction [\omega, lh(x)) \in T_n^+$. Since x is according to s, h is according to s^*. Since s^* is a winning strategy for II^* for $G(Shift_2(A); Sft_2(T))$, $h \notin Shift(A)$. Hence $x = e(h) \in A$. Therefore, s is a winning strategy for I for $G(A;T)$.

By cases I and II, $G(A;T)$ is determined. $\square$

Definition 3.3.7. *(Definition of a "shifting tree")*

For a tree T, define a shifting tree to be $Y \times T$ for some nonempty Y. Suppose Υ is a collection of nonempty trees. Define Υ to be closed under shifting trees if for each $T \in \Upsilon$, there is a shifting tree $Y \times T \in \Upsilon$ for some nonempty Y. $\dashv$

Using theorem 3.3.6, we have the following theorem 3.3.8.

Theorem 3.3.8. *Suppose $n \in \omega$ and $\alpha \in \omega_1$. Suppose Υ is closed under shifting trees. Then for any X,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_2\right) \tag{3.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_2\right) \tag{3.2}$$

for $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)$ [6] *where:*

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* $\dashv$

Similar to theorem 2.3.1, for the equivalences in theorem 3.3.8, we won't be obtaining the determinacy of a game $G(A;T)$ from the same tree T (except for the case when $B = \emptyset$). We will instead use two trees T and $Sft_2(T)$ in the same $Tree_2$ collection.

We have similar results as lemmas 2.3.9 and 2.3.15: lemma 3.3.9 and lemma 3.3.10. The similar proof as lemma 2.3.9 gives us lemma 3.3.9 and the similar proof as lemma 2.3.15 gives us lemma 3.3.10. In lemma 3.3.9, we obtain the complexity of $Shift_2(A)$ for each

[6] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

$A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$ and $A \in \mathbf{\Pi}^0_\alpha \restriction [T]$. In lemma 3.3.10, we obtain the complexity of $Shift_2(A)$ for each $A \in \mathbf{\Sigma}^1_1 \restriction [T]$ and $A \in \mathbf{\Pi}^1_1 \restriction [T]$.

Lemma 3.3.9. *Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Then, for any $\alpha \in \omega_1$:*

1. *If $A \in \mathbf{\Pi}^0_\alpha \restriction [T]$, then $Shift_2(A) \in \mathbf{\Sigma}^0_\alpha \restriction [Sft_2(T)]$.*

2. *If $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, then $Shift_2(A) \in \mathbf{\Pi}^0_\alpha \restriction [Sft_2(T)]$.* ⊣

Proof.

The proof is similar to lemma 2.3.9. □

Lemma 3.3.10. *Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Then for any $n \in \omega\backslash\{0\}$:*

1. *If $A \in \mathbf{\Pi}^1_n \restriction [T]$, then $Shift_2(A) \in \mathbf{\Sigma}^1_n \restriction [Sft_2(T)]$.*

2. *If $A \in \mathbf{\Sigma}^1_n \restriction [T]$, then $Shift_2(A) \in \mathbf{\Pi}^1_n \restriction [Sft_2(T)]$.* ⊣

Proof.

The proof is similar to lemma 2.3.15. □

Now, we prove theorem 3.3.8 on page 232. Recall theorem 3.3.8.

Theorem 3.3.8. *Suppose $n \in \omega$ and $\alpha \in \omega_1$. Suppose Υ is closed under shifting trees. Then for any X,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_2\right) \tag{3.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_2\right) \tag{3.2}$$

for $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ [7] *where:*

[7] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\gamma \in \omega_1$ *and* $m \in \omega$.

- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\beta \in \omega_1$ *and* $m \in \omega$. ⊣

Proof of Theorem 3.3.8.

The proof is similar to theorem 2.3.1. Readers familiar with the proof of theorem 2.3.1 may skip the proofs.

Pick an arbitrary nonempty set X. Suppose Υ is closed under shifting trees. Fix $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$ in the theorem with fixed complexities for $\mathcal{B}$ and $\mathcal{C}$. Pick an arbitrary $T = T^{\Psi,B}_{X,T_{sq}} \in \mathcal{T}_2$. Let $T_{sq} = \langle T_n | n \in \omega\rangle$. Show the equivalence (3.1).

($\Rightarrow$) Pick arbitrary $A \in \mathbf{\Pi}^0_\alpha \restriction [T]$. Since Υ is closed under shifting trees, for each T_n, there exists a nonempty set Y_n such that $T^+_n = Y_n \times T_n \in \Upsilon$. Then $T^+_{sq} = \langle Y_n \times T_n \,| n \in \omega\rangle \in \Upsilon^\omega$. Since $B \in \mathcal{B} \restriction X^\omega$, by sublemma 2.3.19 and sublemma 2.3.20, $B^+ = X \times B \in \mathcal{B} \restriction X^\omega$. Also, by sublemma 2.3.21, $\Psi^+_2 \in \Gamma(\omega, \mathcal{C})$. Therefore, $Sft(T) = T^{\Psi^+,B^+}_{X,T_{sq}} \in \mathcal{T}_2$. Since $A \in \mathbf{\Pi}^0_\alpha \restriction [T]$, by lemma 3.3.9, $Shift_2(A) \in \mathbf{\Sigma}^0_\alpha \restriction [Sft_2(T)]$. Thus

$$G(Shift_2(A), Sft_2(T)) \in \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right).$$

Hence $G(Shift_2(A), Sft_2(T))$ is determined. By theorem 3.3.6, $G(A,T)$ is determined.

($\Leftarrow$) By switching $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$ in the above proof, we can obtain this direction.

Show the equivalence (3.2).

($\Rightarrow$) Suppose $A \in \mathbf{\Pi}^1_n \restriction [T]$. Since $B \in \mathcal{B} \restriction X^\omega$, by sublemma 2.3.19 and sublemma 2.3.20, $B^+ \in \mathcal{B} \restriction X^\omega$. Also, by sublemma 2.3.21, $\Psi^+ \in \Gamma(\omega, \mathcal{C})$. Therefore, $Sft(T) = T^{\Psi^+,B^+}_{X,T_{sq}} \in \mathcal{T}_2$. By lemma 3.3.9, $Shift(A) \in \mathbf{\Sigma}^1_n \restriction [Sft(T)]$. Thus

$$G(Shift_2(A), Sft_2(T)) \in \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right).$$

Hence $G(Shift_2(A), Sft_2(T))$ is determined. By theorem 2.3.8, $G(A,T)$ is determined.

($\Rightarrow$) Switch $\mathbf{\Pi}^1_n \restriction [T]$ and $\mathbf{\Sigma}^1_n \restriction [T]$ in the ($\Rightarrow$) direction of the equivalence (3.1). By lemma 2.3.15, $Shift(A) \in \mathbf{\Pi}^1_n \restriction [Sft(T)]$.

($\Leftarrow$) By switching $\mathbf{\Pi}^1_n$ and $\mathbf{\Sigma}^1_n$ in the above proof, we can obtain this direction. $\square$

The following is a corollary to theorem 2.3.1.

Corollary 3.3.11. *Suppose $n \in \omega$ and $\alpha \in \omega_1$. Let:*

- $\mathcal{T}^1_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$.
- $\mathcal{T}^2_2 = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$. [8]

Then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}^1_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}^1_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}^2_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}^2_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}^1_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}^1_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}^2_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}^2_2\right)$$

for any $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\gamma \in \omega_1$ and $m \in \omega$;

and any $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\beta \in \omega_1$ and $m \in \omega$. $\dashv$

Proof.

Since FWF and CWF are closed under shifting trees, we have the results by theorem 3.3.8. $\square$

[8] Recall notation 1.5.11 for FWF and CWF.

3.4 Using the determinacy of games on a $Tree_2$ collection to obtain the determinacy of games on $X^{<\omega}$

In this section, we will use the determinacy of games on a $Tree_2$ collection to obtain the determinacy of games on $X^{<\omega}$. We will obtain similar results to those in section 2.4 on page 82 for $Tree_2$ collections except sections 3.4.1.2 and 3.4.4. These sections will be distinct as the type of trees we will use in these sections will be Type 2 trees which are not Type 1 trees.

In section 3.4.1, under ZF-P, we will focus on using $\mathbf{\Delta}_1^0$ determinacy on a $Tree_2$ collection to obtain Borel determinacy on $X^{<\omega}$. The results in section 3.4.1.1 are Type 2 tree version of the results in section 2.4.1. We will obtain the determinacy of finite Borel games on $X^{<\omega}$ from the determinacy of clopen games on a particular $Tree_2$ collection. In section 3.4.1.2, we will use the determinacy of clopen games on a certain $Tree_2$ collection to obtain the determinacy of Borel games on $X^{<\omega}$. We will use the tree defined in Yost's thesis (Yost, n.d.).

In section 3.4.2, we will focus on using $\mathbf{\Sigma}_1^0$ determinacy on a $Tree_2$ collection to obtain the determinacy of games on $X^{<\omega}$. The results in this section are Type 2 tree version of the results in section 2.4.2. We will obtain similar results to section 2.4.2.1 through 2.4.2.4 in sections 3.4.2.1 through 3.4.2.4. In section 3.4.2.1, we will obtain the similar results as sections 2.4.2.1. We will define $Long_2$, which is the Type 2 tree version of $Long$. In sections 3.4.2.2, 3.4.2.3 and 3.4.2.4, we will obtain the similar results as sections 2.4.2.2, 2.4.2.3 and 2.4.2.4, respectively. We will define T_{Max} on Type 2 trees which corresponds to Max for

Type 1 trees. T_{Max} will be defined on Type 2 trees with Ψ bounded over B.

In section 3.4.3, we will obtain $\alpha+1$-$\mathbf{\Pi}_1^1$ determinacy on X^ω for even $\alpha \in \omega_1$ from the determinacy of α-$\mathbf{\Pi}_1^1$ games on $Tree_2$ collection. The results in this section are Type 2 tree version of the results in section 2.4.3. We will again use T_{Max}.

In section 3.4.4, we will obtain the determinacy of α-$\mathbf{\Pi}_1^1$, ($\alpha \in \omega_1$) games on X^ω from the determinacy of open games on a $Tree_2$ collection. We will define $Tail$ which is a generalization of T_{Max}. $Tail$ will be defined on any Type 2 trees. Using $Tail$, we will obtain the determinacy of α-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from the determinacy of open games on a certain $Tree_2$ collection.

3.4.1 (ZF-P) Using $\mathbf{\Delta}^0_1$ determinacy on a $Tree_2$ collection to obtain Borel determinacy on $X^{<\omega}$

In this section, we will focus on obtaining the determinacy of Borel games on $X^{<\omega}$ from the determinacy of a certain $Tree_2$ collection.[9] First, in section 3.4.1.1, similar to section 2.4.1, we will obtain the determinacy of finite Borel games on $X^{<\omega}$ is obtained from the determinacy of open games on a particular $Tree_2$ collection. In section 3.4.1.2, we will obtain the determinacy of Borel games on $X^{<\omega}$ from the determinacy of clopen games on a certain $Tree_2$ collection by using the tree defined in Yost's thesis (Yost, n.d.).

3.4.1.1 (ZF-P) Using $\mathbf{\Delta}^0_1$ determinacy on a $Tree_2$ collection to obtain finite Borel determinacy on $X^{<\omega}$

We will obtain the determinacy of finite Borel games on $X^{<\omega}$ from the determinacy of open games on a $Tree_2$ collection $Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right)$ where Γ contains all the constant functions from X^ω into ω and Υ contains $Y^{\leq n+1}$ for all $n \in \omega$ for some countable Y. The results of section are similar to section 2.4.1.

Observation 3.4.1 below is the special case of a Type 2 tree with $B = \emptyset$. Since $[T^{\Psi,\emptyset}_{X,T_{sq}}] = X^\omega$, $\mathcal{G}\left(\Xi; Tree_2\left(X, \Upsilon^\omega, \Gamma, \emptyset\right)\right) = \{G\left(A; X^{<\omega}\right) | A \in \Xi \restriction X^\omega\}$ for any Υ and Γ. Thus, for example, $\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right)$ contains all open games on $X^{<\omega}$ for any Υ since $\emptyset \in \mathbf{\Delta}^0_1 \restriction X^\omega$ and $\Gamma(\omega, \mathbf{\Delta}^0_1)$ is the set of continuous functions from X^ω into ω. The following is a $Tree_2$ collection version of observation 2.4.1 on page 83.

Observation 3.4.1. *For any X, any ω-sequence of nonempty trees T_{sq}, any function f from*

[9] The proof of $Det(\mathbf{B} \restriction X^\omega)$ in ZFC uses the power set axiom.

X^ω into ω, and any complexity Ξ (in which for any $T \in Tree_2(X, T_{sq}, f, \emptyset)$, $\Xi \upharpoonright [T] \subseteq \wp([T])$ is defined),

$$Det\ \mathcal{G}(\Xi; Tree_2(X, T_{sq}, f, \emptyset)) \Rightarrow Det(\Xi \upharpoonright X^\omega). \qquad \dashv$$

$\Xi \upharpoonright X^\omega$ in observation 3.4.1 could be any subset of X^ω.

Corollary 3.4.2. *Fix nonempty X and nonempty Υ.*

Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega)$. For any complexity Ξ,

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_1) \Rightarrow Det(\Xi \upharpoonright X^\omega). \qquad \dashv$$

Proof.

Note $\emptyset \in \mathbf{\Delta}^0_1 \upharpoonright X^\omega$. Since observation 3.4.1 is true for any function f from X^ω into ω, by taking f to be the continuous function, we have $f \in \Gamma(\omega, \mathbf{\Delta}^0_1)$. Since observation 3.4.1 is true for any ω-sequence of nonempty trees T_{sq}, for any nonempty Υ, we have

$$Tree_2(X, T_{sq}, f, \emptyset) \subseteq Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right) = \mathcal{T}_2.$$

Thus, by observation 3.4.1, we have the result. □

By replacing Ξ by $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Sigma}^1_n$ in corollary 3.4.2, we obtain corollary 3.4.3. The following is a $Tree_2$ collection version of corollary 2.4.2 on page 83.

Corollary 3.4.3. *(Corollary to Corollary 3.4.2)*

Suppose $\alpha \in \omega_1$ and $n \in \omega$. Fix nonempty X and nonempty Υ.

Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega)$. Then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \upharpoonright X^\omega\right). \qquad \dashv$$

The following is a $Tree_2$ collection version of observation 2.4.3 on page 84.

Observation 3.4.4. *Suppose* Γ *contains all constant functions from* X^ω *into* ω. *Suppose* Y *is a nonempty set and* Υ *contains* $Y^{\leq n+1}$ *for all* $n \in \omega$. *Then for any complexity* Ξ *(in which* $\Xi \upharpoonright X^\omega \times Y^n \subseteq \wp(X^\omega \times Y^n)$ *is defined for all* $n \in \omega$*),*

$$Det\ \mathcal{G}\left(\Xi; Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \Xi \upharpoonright \left(X^\omega \times Y^n\right)\right). \qquad \dashv$$

Before we prove observation 3.4.4, recall observation 3.2.4.

Observation 3.2.4. *Suppose* Y *is a nonempty set and* $\left\{Y^{\leq n+1} \middle| n \in \omega\right\} \subseteq \Upsilon$. *Then*

$$Tree_1\left(X, Y, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right).$$

In general, if $\left\{Y^{\leq n+1} \middle| Y \in \Theta, n \in \omega\right\} \subseteq \Upsilon$, *then*

$$Tree_1\left(X, \Theta, \Gamma, \Lambda\right) \subseteq Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right). \qquad \dashv$$

Proof of observation 3.4.4.

Since Υ contains $Y^{\leq n+1}$ for all $n \in \omega$, by observation 3.2.4,

$$Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right) \subseteq Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right).$$

Thus

$$\mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right) \subseteq \mathcal{G}\left(\Xi; Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right)\right).$$

Since Γ contains all constant functions from X^ω into ω, by observation 2.4.3,

$$Det\ \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \Xi \upharpoonright \left(X^\omega \times Y^n\right)\right)$$

Thus

$$Det\ \mathcal{G}\left(\Xi; Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \Xi \upharpoonright \left(X^\omega \times Y^n\right)\right). \qquad \square$$

The following theorem 3.4.5 is a $Tree_2$ version of theorem 2.4.4 on page 84. Recall theorem 2.4.4.

Theorem 2.4.4. *(ZF-P)*

Suppose Γ *contains all constant functions from* X^ω *into* ω *and* Y *is denumerable. Then* $Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$ *implies* $Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right)$*, finite Borel determinacy on* $X^{<\omega}$*.* ⊣

By theorem 2.4.4 and observation 3.2.4, we have the following theorem.

Theorem 3.4.5. *(ZF-P)*

Suppose Γ *contains all constant functions from* X^ω *into* ω *and* Υ *contains* $Y^{\leq n+1}$ *for all* $\in \omega$ *for some countable* Y*. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right). \qquad \dashv$$

By using theorem 3.4.5, we can obtain corollary 3.4.6. The following is a $Tree_2$ collection version of corollary 2.4.17 on page 104.

Corollary 3.4.6.

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right). \qquad \dashv$$

Proof.

Since each constant functions is continuous, $\Gamma(\omega, \mathbf{\Delta}^0_1)$ contains all the constant functions from X^ω into ω. Also $\emptyset, X^\omega \in \mathbf{\Delta}^0_1 \restriction X^\omega$ and CWF contains $Y^{\leq n+1}$ for all $n \in \omega$ for all countable Y. Thus, we have the result by theorem 2.4.4 and and observation 3.2.4. □

3.4.1.2 (ZF-P) Using $\mathbf{\Delta}^0_1$ determinacy on a $Tree_2$ collection to obtain Borel determinacy on $X^{<\omega}$

In section 3.4.1.1, we obtained finite Borel determinacy on $X^{<\omega}$ from the determinacy of clopen games on a particular $Tree_2$ collection. In this section, we will obtain Borel determinacy on $X^{<\omega}$ from the determinacy of clopen games on a particular $Tree_2$ collection.

Corollary 3.4.7. *Suppose Γ is a nonempty collection of functions from X^ω into ω. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Gamma, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega). \qquad \dashv$$

This result follows from a result in Yost's thesis (Yost, n.d.) In Yost (n.d.), for each $\alpha \in \omega_1$, Yost defines a tree $T^\alpha_{g.t}$ (which I shall call a Yost tree). We give a definition of $T^\alpha_{g.t.}$ in appendix C.6 on page 365.

For each limit ordinal $\alpha \in \omega_1$, the Yost tree $T^\alpha_{g.t.}$ is constructed by the following manner:

1. Fix a decomposition of α.

2. Each play in $T^\alpha_{g.t.}$ is a finite sequence from ω.

3. The length of a play is determined by certain moves for player I. These moves to calculate the length of each play depends on the decomposition of α.

Yost proves the following theorem in Yost (n.d.) (it might be open instead of clopen).

Theorem 3.4.8. *(Yost, n.d.) For each limit ordinal $\alpha \in \omega_1$,*

$$Det\left(\mathbf{\Delta}^0_1 \restriction (X^\omega \times [T^\alpha_{g.t.}])\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right).$$

Thus

$$Det\left(\bigcup_{\alpha\in\omega_1} \mathbf{\Delta}^0_1 \restriction (X^\omega \times [T^\alpha_{g.t.}])\right) \Leftrightarrow Det\left(\mathbf{B} \restriction X^\omega\right). \qquad \dashv$$

In the next lemma, we show that $X^\omega \times \left[T^\alpha_{g.t.}\right]$ is in fact a Type 2 tree.

Lemma 3.4.9. *For each limit ordinal $\alpha \in \omega_1$, $X^\omega \times \left[T^\alpha_{g.t.}\right]$ is a Type 2 tree.* $\dashv$

Proof.

Pick an arbitrary function Ψ from X^ω into ω. Let $T_{sq} = \langle T_n \,| n \in \omega \rangle$ be such that for every $n \in \omega$, $T_n = T^\alpha_{g.t.}$. Show

$$X^\omega \times \left[T^\alpha_{g.t.}\right] = [T^{\Psi, X^\omega}_{X, T_{sq}}].$$

$$\begin{aligned} f \in X^\omega \times \left[T^\alpha_{g.t.}\right] &\Leftrightarrow f \in X^\omega \times \left[T_{\Psi(f)}\right] \\ &\Leftrightarrow f \in [T^{\Psi, X^\omega}_{X, T_{sq}}]. \end{aligned}$$

so that $X^\omega \times \left[T^\alpha_{g.t.}\right] = [T^{\Psi, X^\omega}_{X, T_{sq}}]$. Therefore, $X^\omega \times \left[T^\alpha_{g.t.}\right]$ is a Type 2 tree. $\square$

There are multiple ways to prove lemma 3.4.9. We could use Ψ to be a constant function at 0 and $T_{sq} = \langle T_n \,| n \in \omega \rangle$ to be such that $T_0 = T^\alpha_{g.t.}$ and T_n could be any tree for $n > 0$ instead. We can use on any function Ψ from X^ω into ω if we set T_{sq} to be in the proof of lemma 3.4.9.

Using theorem 3.4.8 and lemma 3.4.9, we obtain corollary 3.4.10.

Corollary 3.4.10. *Suppose Ψ is any function from X^ω into ω and $\left\{T^\alpha_{g.t.} \,| \alpha \in \omega_1\right\} \subseteq \Upsilon$.*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega). \quad \dashv$$

Proof.

Assume $Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right)$. Pick arbitrary $A \in \mathbf{B} \restriction X^\omega$. Then $A \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ for some limit ordinal $\alpha \in \omega_1$. By theorem 3.4.8,

$$Det\left(\mathbf{\Delta}^0_1 \restriction (X^\omega \times [T^\alpha_{g.t.}])\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right).$$

By lemma 3.4.9, $X^\omega \times \left[T^\alpha_{g.t.}\right] = [T^{\Psi,X^\omega}_{X,T_{sq}}]$ where $T_{sq} = \langle T_n \,| n \in \omega \rangle$ and for every $n \in \omega$, $T_n = T^\alpha_{g.t.}$. Thus $T_{sq} \in \left\{T^\alpha_{g.t.} \,| \alpha \in \omega_1 \right\}^\omega = \Upsilon^\omega$. Since

$$G\left(\mathbf{\Delta}^0_1, T^{\Psi,X^\omega}_{X,T_{sq}}\right) \in \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right),$$

$\mathbf{\Delta}^0_1 \restriction [T^{\Psi,X^\omega}_{X,T_{sq}}]$ is determined. Since

$$Det\left(\mathbf{\Delta}^0_1 \restriction [T^{\Psi,X^\omega}_{X,T_{sq}}]\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right),$$

$\mathbf{\Sigma}^0_\alpha \restriction X^\omega$ is determined. Since $\alpha \in \omega_1$ is arbitrary, we have

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

□

By corollary 3.4.10, we obtain corollary 2.4.32. Recall corollary 3.4.7.

Corollary 3.4.7. *Suppose Γ is a nonempty collection of functions from X^ω into ω. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Gamma, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

⊣

Proof of Corollary 3.4.7.

Fix a collection Γ of functions from X^ω into ω. Notice that for any $\alpha \in \omega_1$, $T^\alpha_{g.t.} \in CWF$. Thus $\left\{T^\alpha_{g.t.} \,| \alpha \in \omega_1\right\} \subseteq CWF$. By corollary 3.4.10, for any $\Psi \in \Gamma$,

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Psi, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Since for any $\Psi \in \Gamma$,

$$Tree_2\left(X, CWF^\omega, \Psi, X^\omega\right) \subseteq Tree_2\left(X, CWF^\omega, \Gamma, \mathbf{\Delta}^0_1 \restriction X^\omega\right),$$

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Gamma, \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

□

3.4.2 Using $\mathbf{\Sigma}_1^0$ determinacy on a $Tree_2$ collection to obtain the determinacy of games on $X^{<\omega}$

In this section, we will obtain the determinacy of games on $X^{<\omega}$ from the determinacy of open games on a $Tree_2$ collection. We will obtain similar results to sections 2.4.2: We will obtain similar results to sections 2.4.2.1 through 2.4.2.4 in sections 3.4.2.1 through 3.4.2.4, respectively.

In section 3.4.2.1, we will define $Long_2$, which is the Type 2 tree version of $Long$.

In section 3.4.2.2, We will define T_{Max} on Type 2 trees which corresponds to Max for Type 1 trees. In sections 3.4.2.3 and 3.4.2.4, we will consider T_{Max} on two special cases of Type 2 trees.

3.4.2.1 Definition of the open set $Long_2(B)$ on a Type 2 tree and using the determinacy of open games $Long_2(A)$ on a $Tree_2$ collection to obtain the determinacy of games A on X^ω

In this section, we will define the open set $Long_2$ on the body of a Type 2 tree and obtain the determinacy of games A on $X^{<\omega}$ from the determinacy of open games $Long_2(A)$ on a $Tree_2$ collection. We will obtain the similar results as section 2.4.2.1. $Long_2$ will be the Type 2 tree version of $Long$. The only difference will be that $Long$ is defined on the body of Type 1 trees and $Long_2$ is defined the body of on Type 2 trees. Recall by observation 3.1.3, every Type 1 tree is a Type 2 tree. $Long_2$ will be a generalization of $Long$.

The following definition is a Type 2 tree version of definition 2.4.19 on page 107.

Definition 3.4.11. *Suppose $B \subseteq X^\omega$, Ψ is a function from X^ω into ω and T_{sq} is an ω-*

sequence of nonempty trees. Define

$$Long_2\,(B) = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid lh(h) > \omega\}. \qquad \dashv$$

Then $Long_2\,(B) = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \restriction \omega \in B\}$. It is easy to see that $Long_2(B)$ is open. We also have a Type 2 tree version of theorem 2.4.20 on 107.

Theorem 3.4.12. *For any ω-sequence T_{sq} of nonempty trees, $\Psi : X^\omega \to \omega$, for any $A \subseteq X^\omega$, $G\,(A; X^\omega)$ is determined if and only if $G(Long_2\,(A)\,; T^{\Psi,A}_{X,T_{sq}})$ is determined .* $\dashv$

The proof of this theorem is similar to the proof of theorem 2.4.20. Simply replace a Type 1 tree $T^{\Psi,A}_{X,T_{sq}}$ by a Type 2 tree $T^{\Psi,A}_{X,T_{sq}}$.

The following is a $Tree_2$ collection version of corollary 2.4.21 on page 109.

Corollary 3.4.13. *For any nonempty collection Υ of nonempty trees, $\Psi : X^\omega \to \omega$ and Λ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\,(X, \Upsilon^\omega, \Psi, \Lambda \restriction X^\omega)\right) \Rightarrow Det\,(\Lambda \restriction X^\omega)\,. \qquad \dashv$$

Proof.

Pick an arbitrary $A \in \Lambda \restriction X^\omega$. Then, for any $T_{sq} \in \Upsilon^\omega$, $Long_2(A) \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,A}_{X,T_{sq}}]$. Thus $G(Long_2(A); T^{\Psi,A}_{X,T_{sq}})$ is determined. By theorem 3.4.12, $G(A; X^{<\omega})$ is determined. □

The following is a $Tree_2$ collection version of corollary 2.4.22 on page 109.

Corollary 3.4.14. *For any $\alpha \in \omega_1$, Υ and $\Psi : X^\omega \to \omega$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right). \qquad \dashv$$

Proof.

By corollary 3.4.13 with $\Lambda = \mathbf{\Sigma}^0_\alpha$. □

The following is a $Tree_2$ collection version of corollary 2.4.23 on page 109.

Corollary 3.4.15. *Suppose* Υ *is an arbitrary nonempty collection of nonempty trees,* Γ *is any collection of functions from* X^ω *into* ω *and* Λ *is a collection of nondetermined sets. Then,*

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Proof.

Assume Λ is a collection of nondetermined sets. Then $\neg Det(\Lambda \upharpoonright X^\omega)$. By corollary 3.4.13, $\neg Det\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, \Lambda \upharpoonright X^\omega\right)\right)$ for any Υ and $\Psi : X^\omega \to \omega$, i.e.,

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda \upharpoonright X^\omega\right)\right).$$

In fact, $G(Long_2(A); T^{\Psi,A}_{X,T_{sq}})$ is not determined for any nondetermined set A and any T_{sq} and function Ψ from X^ω into ω. $\square$

Corollaries 3.4.13, 3.4.14 and 3.4.15 state that the respective relation holds for every Υ. Since by observation 3.1.3, every Type 1 tree $T^{\Psi,B}_{X,Y}$ is a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ where $T_{sq} = \langle Y^{\leq n+1} | n \in \omega\rangle$, by letting $\Upsilon \supseteq \{Y^{\leq n+1} | n \in \omega\}$, we see that these corollaries are generalization of corollaries 2.4.21, 2.4.22 and 2.4.23, respectively.

3.4.2.2 Definition of the open set $T_{Max}(\Psi, B)$ on a Type 2 tree

In this section, we will consider Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ such that $\Psi \upharpoonright B$ is bounded below ω. We will define $T_{Max}(\Psi, B)$ on appropriate Type 2 trees which corresponds to $Max(\Psi, B)$ defined on appropriate Type 1 trees in section 2.4.2.2. In sections 3.4.2.3 and 3.4.2.4, we will obtain some determinacy results using Max. First, recall definition 2.4.24 on page 111.

Definition 2.4.24. *(Definition of Max)*

Suppose $\Psi \upharpoonright B$ is bounded below ω. Let $n^{\Psi,B}_{\max}$ be the maximum tail length determined from Ψ and B. ($n^{\Psi,B}_{\max} = \max(Im(\Psi \upharpoonright B)) + 1$.) If Ψ and B are clear from the context, we suppress Ψ and B, i.e.,$n_{\max} = n^{\Psi,B}_{\max}$.

Define

$$Max(\Psi, B) = \{h \in [T^{\Psi,B}_{X,Y}] \mid lh(h) = \omega + n_{\max}\} = lh^{-1}_{[T^{\Psi,B}_{X,Y}]}(\omega + n_{\max}). \qquad \dashv$$

Let's consider the set Max for a Type 2 tree. Suppose we define a Type 2 tree version of Max the same way as we defined Max on a Type 1 tree. Even when $\Psi \upharpoonright B$ will be bounded below ω, the tail lengths in $[T^{\Psi,B}_{X,T_{sq}}]$ may not be bounded. Notice that unlike Type 1 trees, the maximum length is not determined by Ψ and B. It is determined by the definition of each tree T_n in $T_{sq} = \langle T_n | n \in \omega\rangle$. Therefore, we modify this definition as follows. Same as Max, T_{Max} will be defined on Type 2 trees with Ψ bounded over B.

Definition 3.4.16. *Suppose $\Psi \upharpoonright B$ is bounded below ω and let $\Psi_{\max} \in \omega$ be the maximum value of Ψ over B. Define*

$$T_{Max}(\Psi, B) = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \upharpoonright [\omega, lh(h)) \in [T_{\Psi_{\max}}]\}. \qquad \dashv$$

Recall that $h \in T_{Max}(\Psi, B)$ doesn't necessary imply $\Psi(h \upharpoonright \omega) = \Psi_{\max}$. See comment below observation 3.1.3 on page 216.

Now, we observe the relationship between $Max(\Psi, B)$ and $T_{Max}(\Psi, B)$. Recall by observation 3.1.3 on page 216, every Type 1 tree is a Type 2 tree.

Observation 3.1.3. *For each Type 1 tree* $T_{X,Y}^{\Psi,B}$, $[T_{X,Y}^{\Psi,B}] = [T_{X,T_{sq}}^{\Psi,B}]$ *where* $T_{sq}{=}\left\langle Y^{\leq n+1} | n{\in}\omega \right\rangle$. *Thus any Type 1 tree is a Type 2 tree.* ⊣

Fix a Type 1 tree $T = T_{X,Y}^{\Psi,B}$. Set $T_n = Y^{\leq n+1}$ for each $n \in \omega$ and $T_{sq} = \langle T_n | n \in \omega \rangle$. Then $T = T_{X,Y}^{\Psi,B} = T_{X,T_{sq}}^{\Psi,B}$. Suppose $\Psi \upharpoonright B$ is bounded below ω. Then $Max(\Psi, B)$ and $T_{Max}(\Psi, B)$ are both defined on T. Note that $n_{\max} = \Psi_{\max} + 1$. Thus

$$\begin{aligned} Max\left(\Psi, B\right) &= \left\{ h \in \left[T_{X,Y}^{\Psi,B}\right] | lh\left(h\right) = \omega + n_{\max} \right\} \\ &= \left\{ h \in \left[T_{X,Y}^{\Psi,B}\right] | h \upharpoonright [\omega, lh(h)) \in Y^{n_{\max}} \right\} \\ &= \left\{ h \in \left[T_{X,T_{sq}}^{\Psi,B}\right] | h \upharpoonright [\omega, lh(h)) \in Y^{\Psi_{\max}+1} \right\} \\ &= \left\{ h \in \left[T_{X,T_{sq}}^{\Psi,B}\right] | h \upharpoonright [\omega, lh(h)) \in [T_{\Psi_{\max}}] \right\} \\ &= T_{Max}\left(\Psi, B\right). \end{aligned}$$

Hence $T_{Max}(\Psi, B)$ is a Type 2 version of $Max(\Psi, B)$.

Recall that $Max(\Psi, B)$ is open on Type 1 trees $T_{X,Y}^{\Psi,B}$ for Ψ bounded over B (see page 111). For an arbitrary ω-sequence of trees T_{sq}, T_{Max} may not be open. We will restrict the trees T_n as follows.

For each finite set of nonempty trees Υ and an enumeration $\langle T_n \, | n \leq N \rangle$ of Υ, we will define the two properties, "the N maximal tree property", which will be defined in definition

3.4.17 and "the N disjoint tree property", which will be defined in definition 3.4.19.

In order to satisfy the N maximal tree property, we require that T_N is a well-founded tree. It is possible for T_n, $n \neq N$, to be ill-founded. We also require that there is no path in T_n for $n < N$ such that its proper initial segment is in T_n. Thus it is possible for $[T_n] \cap [T_N] \neq \emptyset$ but there is no $g \in [T_N] \cap T_n \backslash [T_n]$ for every $n < N$.

Definition 3.4.17. *(Definition of the N maximal tree property)*

Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N+1$ satisfying that: there is an enumeration $\langle T_n \,|n \leq N \rangle$ of Υ such that

1. *T_N is well-founded.*

2. *For each $n < N$ and for every $g_n \in [T_n]$, g_n does not properly extend g for every $g \in [T_N]$.*

Then we say Υ satisfies the N maximal tree property. We also say $\langle T_n \,|n \leq N \rangle$ has the N maximal tree property. We say $T_{sq} = \langle T_n \,|n \in \omega \rangle$ has the N maximal tree property if $\langle T_n \,|n \leq N \rangle$ satisfies the N maximal tree property. $\dashv$

In proposition 3.4.21 below, we will show that for Ψ from X^ω into ω such that $\Psi \restriction B$ is bounded below ω and T_{sq} satisfying $\Psi_{\max}$ maximal tree property, $T_{Max}(\Psi, B)$ is open in the Type 2 tree $[T^{\Psi,B}_{X,T_{sq}}]$. Recall that for $\langle T_n \,|n \leq N \rangle$ to satisfy N maximal tree property, we required T_N being a well-founded tree. One may wish to get rid of this restriction. We will define "N disjoint tree property" below in definition 3.4.19. For $\langle T_n \,|n \leq N \rangle$ to satisfy this property, none of the trees are required to be well-founded. Instead, we will restrict the set of the possible first moves of T_N to be disjoint from the set of possible first moves of T_n for

$n < N$. First, we will define a notation corresponds to the set of each possible move at a position p in T_n. A similar definition was given in definition 1.2.3 on page 8 for a fixed tree T.

Notation 3.4.18. *Suppose for each $n \in \omega$, T_n is a tree. For each $n \in \omega$ and for any $p \in T_n$, define*

$$M_p^n = \{m \,|p^\frown\langle m\rangle \in T_n\}\,. \qquad \dashv$$

In particular, if $p = \emptyset$, then $M_\emptyset^n$ is the possible initial moves of T_n. Now, we define "N disjoint tree property" for a set of trees Υ and an enumeration $\langle T_n \,|n \leq N\rangle$ of Υ.

Definition 3.4.19. *(Definition of the N disjoint tree property)*
Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N+1$ satisfying that: there is an enumeration $\langle T_n \,|n \leq N\rangle$ of Υ such that

$$\text{for every } n < N,\ M_\emptyset^n \cap M_\emptyset^N = \emptyset.$$

Then we say Υ satisfies the N disjoint tree property. We also say $\langle T_n \,|n \leq N\rangle$ has the N disjoint tree property. We say $T_{sq} = \langle T_n \,|n \in \omega\rangle$ has the N disjoint tree property if $\langle T_n \,|n \leq N\rangle$ satisfies the N disjoint tree property. If $T_{sq} = \langle T_n \,|n \in \omega\rangle$ satisfies that each $M_\emptyset^n$ is pairwise disjoint, then we say T_{sq} has the disjoint tree property. $\dashv$

Note that the set $\langle T_n \,|n \leq N\rangle$ to satisfy N disjoint tree property, for $n, m < N$ with $n \neq m$, $M_\emptyset^n$ and $M_\emptyset^m$ need not to be disjoint. Also, if $T_{sq} = \langle T_n \,|n \in \omega\rangle$ satisfies the disjoint tree property, then each $M_\emptyset^n$ is pairwise disjoint so that each T_n is pairwise disjoint. Thus, we have the following.

Observation 3.4.20. *Fix a Type 2 tree* $T = T^{\Psi,B}_{X,T_{sq}}$. *Suppose* $T_{sq} = \langle T_n | n \in \omega \rangle$ *satisfies the disjoint tree property. Then each* $(X^\omega \times [T_n]) \cap [T]$ *is pairwise disjoint.* $\dashv$

Proposition 3.4.21. *Fix a Type 2 tree* $T^{\Psi,B}_{X,T_{sq}}$. *If* $\Psi \upharpoonright B$ *is bounded below* ω *and* $T_{sq} = \langle T_n | n \in \omega \rangle$ *satisfies the* $\Psi_{\max}$ *maximal tree property or a* $\Psi_{\max}$ *disjoint tree property, then* $T_{Max}(\Psi, B)$ *is open in* $[T^{\Psi,B}_{X,T_{sq}}]$. $\dashv$

Proof.

Pick an arbitrary $h \in T_{Max}(\Psi, B)$. Then $h \upharpoonright [\omega, lh(h)) \in [T_{\Psi_{\max}}]$.

Case 1 : T_{sq} satisfies the $\Psi_{\max}$ maximal tree property.

Then for every $n < \Psi_{\max}$ and for every $g \in [T_n]$, $g \not\supseteq h \upharpoonright [\omega, lh(h))$. Let $E = [\omega, lh(h))$. By property (1) of definition 3.4.17, $h \upharpoonright [\omega, lh(h))$ is finite. Thus E is finite. Pick an arbitrary $\hat{h}$ such that $\hat{h} \supseteq h \upharpoonright E$. Then $\hat{h} \upharpoonright [\omega, lh(h)) = h \upharpoonright [\omega, lh(h))$. Thus $lh(\hat{h}) \geq lh(h)$. Show $lh(\hat{h}) = lh(h)$. Suppose, for a contradiction, $lh(\hat{h}) > lh(h)$. Then by property (2) in definition 3.4.17, $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \notin [T_n]$ for every $n \leq \Psi_{\max}$. By the definition of $T^{\Psi,B}_{X,T_{sq}}$ and the fact that $\Psi \upharpoonright B$ is bounded below ω, $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_n]$ for $n \leq \Psi_{\max}$, a contradiction. Thus $lh(\hat{h}) = lh(h)$ and thus $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_{\Psi_{\max}}]$. Therefore, $\hat{h} \in T_{Max}(\Psi, B)$. (Note that $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_{\Psi_{\max}}]$ does not guarantee that $\hat{h} \in (B \cap \Psi^{-1}(\Psi_{\max})) \times [T_{\Psi_{\max}}]$. It is possible that $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_{\Psi_{\max}}] \cap [T_n]$ for some $n < \Psi_{\max}$.)

Case 2 : T_{sq} satisfies the $\Psi_{\max}$ disjoint tree property.

Let $F = \{\omega\}$. Then F is finite. Pick an arbitrary $\hat{h} \in [T^{\Psi,B}_{X,T_{sq}}]$ such that $\hat{h} \supseteq h \upharpoonright F$. Then $\hat{h}(\omega) = h(\omega)$. Since $h(\omega) \in M^{\Psi_{\max}}_{\emptyset}$, $\hat{h}(\omega) \in M^{\Psi_{\max}}_{\emptyset}$. Thus $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_{\Psi_{\max}}]$. Hence $\hat{h} \in T_{Max}(\Psi, B)$. (In this case, we have $\hat{h} \in (B \cap \Psi^{-1}(\Psi_{\max})) \times [T_{\Psi_{\max}}]$. Note that for each $n \in \Psi_{\max}$, $M^n_{\emptyset} \cap M^{\Psi_{\max}}_{\emptyset} = \emptyset$. Thus $\hat{h}(\omega) \notin M^n_{\emptyset}$ for any $n \in \Psi_{\max}$. Hence $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \notin [T_n]$

for any $n \in \Psi_{\max}$.)

Therefore, $T_{Max}(\Psi, B)$ is open in $[T^{\Psi,B}_{X,T_{sq}}]$. $\square$

By proposition 3.4.21, the maximal tail tree property is enough to make $T_{Max}(\Psi, B)$ open in $[T^{\Psi,B}_{X,T_{sq}}]$. However, for a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ with T_{sq} satisfying the maximal tail tree property, $h \in T_{Max}(\Psi, B)$ doesn't necessary imply $\Psi(h \upharpoonright \omega) = \Psi_{\max}$ (it is possible that $[T_{\Psi_{\max}}] \cap [T_n] \neq \emptyset$ for some $n < \Psi_{\max}$). We will slightly restrict the maximal tail tree property to satisfy the condition $[T_{\Psi_{\max}}] \cap [T_n] = \emptyset$ for every $n < \Psi_{\max}$.

Definition 3.4.22. *(Definition of the modified N maximal tree property)*

Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N + 1$ satisfying that: there is an enumeration $\langle T_n \,|n \leq N \rangle$ of Υ such that

1. *T_N is well-founded.*

2. *for each $n < N$ and for every $g_n \in [T_n]$, g_n does not extend g for every $g \in [T_N]$ (This gives us $[T_N] \cap [T_n] = \emptyset$ for every $n < N$.)*

Then we say Υ satisfies the modified N maximal tree property. We also say $\langle T_n \,|n \leq N \rangle$ satisfies the modified N maximal tree property. We say $T_{sq} = \langle T_n \,|n \in \omega \rangle$ has the modified N maximal tree property if $\langle T_n \,|n \leq N \rangle$ satisfies the N maximal tree property. $\dashv$

By the property (2) of definition 3.4.22, if $\langle T_n \,|n \leq N \rangle$ satisfies the modified N maximal property, then we have:

$$\text{if } g \in [T_N], \text{ then } g \notin T_n \cup [T_n] \text{ for every } n < N.$$

Thus, we have the following.

Observation 3.4.23. *Fix a Type 2 tree* $T = T^{\Psi,B}_{X,T_{sq}}$*. Suppose* $\Psi \upharpoonright B$ *is bounded below* ω*. Suppose* $T_{sq} = \langle T_n | n \in \omega \rangle$ *satisfies a the modified* $\Psi_{\max}$ *maximal tree property. Then* $(X^\omega \times [T_{\Psi_{\max}}]) \cap [T]$ *is disjoint from each* $(X^\omega \times [T_n]) \cap [T]$*.* ⊣

A similar proof of proposition 3.4.21 gives us that if $\Psi \upharpoonright B$ is bounded below ω and T_{sq} satisfy the modified $\Psi_{\max}$ maximal tree property, $T_{Max}(\Psi, B)$ is open in $[T^{\Psi,B}_{X,T_{sq}}]$.

In sections 3.4.2.3 and 3.4.2.4, we will be looking at Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ such that $T_{sq} = \langle T_n | n \in \omega \rangle$ satisfying the modified 1 maximal tree property or the 1 disjoint tree property. To prepare for these sections, we will observe the following special case.

Observation 3.4.24. *Suppose* $T_{sq} = \langle T_n | n \in \omega \rangle$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then*

$$[T_0] \cap [T_1] = \emptyset. \qquad ⊣$$

3.4.2.3 Using the determinacy of open games $T_{Max}(\chi_A, X^\omega)$ on a $Tree_2$ collection to obtain the determinacy of games A on $X^{<\omega}$

In this section, we will obtain the determinacy of games A on $X^{<\omega}$ from the determinacy of open games $T_{Max}(\chi_A, X^\omega)$ on a $Tree_2$ collection. We will obtain the similar results as section 2.4.2.3. Let $A \subseteq X^\omega$. As a special case of Type 2 tree, we will consider Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ such that $B = X^\omega$ and Ψ to be the characteristic function χ_A of A. Recall

$$\chi_A : \; X^\omega \to \{0,1\}$$
$$f \mapsto \begin{cases} 0 & \text{if } f \notin A, \\ 1 & \text{if } f \in A. \end{cases}$$

Then

$$h \in \left[T^{\chi_A, X^\omega}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin X^\omega, \\ h \in X^\omega \times [T_{\chi_A(h\restriction\omega)}] & \text{if } h \restriction \omega \in X^\omega. \end{cases}$$
$$\leftrightarrow \begin{cases} h \in X^\omega \times [T_0] & \text{if } h \restriction \omega \notin A, \\ h \in X^\omega \times [T_1] & \text{if } h \restriction \omega \in A. \end{cases}$$

Note that it is possible that $[T_0] \cap [T_1] \neq \emptyset$ (T_0 could be equal to T_1). $h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}]$ such that $h \restriction [\omega, lh(h)) \in [T_0] \cap [T_1]$ does not mean $h \restriction \omega \notin A$ and $h \restriction \omega \in A$. Recall that for Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ with $h \in [T^{\Psi,B}_{X,T_{sq}}]$, $h \restriction [\omega, lh(h)) \in [T_n]$ does not imply $\Psi(h \restriction \omega) = n$ (see comment under observation 3.1.3 on page 216).

Observation 3.4.25. *Let $A \subseteq X^\omega$ and χ_A be the characteristic function of A. Suppose T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tail tree property. Then*

$$T_{Max}\left(\chi_A, X^\omega\right) = \left\{ h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|h \restriction \omega \in A \right\}. \qquad \dashv$$

Proof.

Suppose $T_{sq} = \langle T_n \,|n \leq N \rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tail tree property. Since $(\chi_A)_{\max} = 1$, [10] by definition 3.4.16,

$$T_{Max}\left(\chi_A, X^\omega\right) = \{h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|h \restriction [\omega, lh(h)) \in [T_1]\}.$$

Show

$$\left\{ h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|h \restriction [\omega, lh(h)) \in [T_1] \right\} = \left\{ h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|h \restriction \omega \in A \right\}.$$

($\subseteq$) Pick an arbitrary $h \in \{h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|h \restriction [\omega, lh(h)) \in [T_1]\}$. Then $h \restriction [\omega, lh(h)) \in [T_1]$. Show $h \restriction \omega \in A$. Suppose, for a contradiction, $h \restriction \omega \notin A$. Then $\chi_A(h \restriction \omega) = 0$. Thus

[10] Recall definition 3.4.16 for $\Psi_{\max}$.

$h \upharpoonright [\omega, lh(h)) \in [T_0]$. Hence $h \upharpoonright [\omega, lh(h)) \in [T_0] \cap [T_1]$. Since T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tail tree property, by observation 3.4.24, $[T_0] \cap [T_1] = \emptyset$. This is a contradiction. Thus $h \in \{h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|\, h \upharpoonright \omega \in A\}$.

($\supseteq$) Pick an arbitrary $h \in \{h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|\, h \upharpoonright \omega \in A\}$. Then $\chi_A(h \upharpoonright \omega) = 1$. Thus $h \upharpoonright [\omega, lh(h)) \in [T_1]$ by definition of $[T^{\chi_A, X^\omega}_{X,T_{sq}}]$. Hence $h \in \{h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}] \,|\, h \upharpoonright [\omega, lh(h)) \in [T_1]\}$.

□

The following is a Type 2 tree version of theorem 2.4.25 on page 112.

Theorem 3.4.26. *Suppose $T^{\chi_A, X^\omega}_{X,T_{sq}}$ is a Type 2 tree such that T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property. Then for any $A \subseteq X^\omega$, $G(A; X^\omega)$ is determined if and only if $G(T_{Max}(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,T_{sq}})$ is determined.* ⊣

Proof.

Pick an arbitrary $A \subseteq X^\omega$.

($\Rightarrow$) Assume $G(A; X^{<\omega})$ is determined. Thus I or II has a winning strategy s for $G(A; X^{<\omega})$. Define s^* to be such that $s^* \upharpoonright X^{<\omega} = s$ and play anything after that to finish the play. Show s^* is a winning strategy for $G(T_{Max}(\chi_A, X^\omega);\, T^{\chi_A, X^\omega}_{X,T_{sq}})$. Pick an arbitrary $h \in [T^{\chi_A, X^\omega}_{X,T_{sq}}]$ according to s^*.

Case 1 : s is a winning strategy for I for $G(A; X^{<\omega})$.

Then $h \upharpoonright \omega \in A$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $h \in T_{Max}(\chi_A, X^\omega)$. Hence s^* is a winning strategy for I^* for $G(Max(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,T_{sq}})$.

Case 2 : s is a winning strategy for II for $G(A; X^{<\omega})$.

Then $h \upharpoonright \omega \notin A$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $h \notin T_{Max}(\chi_A, X^\omega)$. Hence s^* is a winning strategy for II^* for $G(Max(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,T_{sq}})$.

($\Leftarrow$) Assume $G(T_{Max}(\chi_A, X^\omega);\ T^{\chi_A, X^\omega}_{X,T_{sq}})$ is determined. Thus I^* or II^* has a winning strategy s^* for $G(T_{Max}(\chi_A, X^\omega);\ T^{\chi_A, X^\omega}_{X,T_{sq}})$. Define $s = s^* \upharpoonright X^{<\omega}$. Show s is a winning strategy for $G(A; X^{<\omega})$. Pick an arbitrary $f \in X^\omega$ according to s. Then f is according to s^*. Note that $f \in T^{\chi_A, X^\omega}_{X,T_{sq}}$ since there is no path of length ω in $T^{\chi_A, X^\omega}_{X,T_{sq}}$. Play g according to s^* after f to get $f^\frown g \in [T^{\chi_A, X^\omega}_{X,T_{sq}}]$.

Case 1 : s^* is a winning strategy for I^* for $G(T_{Max}(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,T_{sq}})$.
Then $f^\frown g \in T_{Max}(\chi_A, X^\omega)$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $f \in A$. Hence s is a winning strategy for I for $G(A; X^{<\omega})$.

Case 2 : s^* is a winning strategy for II^* for $G(T_{Max}(\chi_A, X^\omega); T^{\chi_A, X^\omega}_{X,T_{sq}})$.
Then $f^\frown g \notin T_{Max}(\chi_A, X^\omega)$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $f \notin A$. Hence s is a winning strategy for II for $G(A; X^{<\omega})$. Therefore, $G(A; X^{<\omega})$ is determined. □

By theorem 3.4.26, the determinacy of any game A of length ω can be obtained from the determinacy of open games $T_{Max}(\chi_A, X^\omega)$ of a Type 2 tree.[11] The following corollaries 3.4.27 and 3.4.28 are direct results from theorem 3.4.26.

The following is a $Tree_2$ collection version of corollary 2.4.27 on page 114.

Corollary 3.4.27. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1*

[11] In particular, if A is a nondetermined game of length ω, then our result gives a corresponding nondetermined game on a Type 2 tree.

disjoint tree property. Then for any complexity Ξ *and for any* $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \left| A \in \Xi \upharpoonright X^\omega\right.\right\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \upharpoonright X^\omega\right).$$

Thus,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \upharpoonright X^\omega). \qquad \dashv$$

Proof.

Pick an arbitrary $A \in \Xi \upharpoonright X^\omega$. Since $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property, $\langle T_0, T_1\rangle$ or $\langle T_1, T_0\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Without loss of generality, assume $\langle T_0, T_1\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Define $T_{sq} = \langle \hat{T}_n \left| n \in \omega\right.\rangle$ to be such that $\hat{T}_0 = T_0$, $\hat{T}_1 = T_1$ and for any $n > 1$, $\hat{T}_n \in \Upsilon$ to be arbitrary. Then T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Since $(\chi_A)_{\max} = 1$, by proposition 3.4.21, $T_{Max}\left(\chi_A, X^\omega\right)$ is open in $[T_{X,T_{sq}}^{\chi_A, X^\omega}]$. Thus

$$G(T_{Max}\left(\chi_A, X^\omega\right); T_{X,T_{sq}}^{\chi_A, X^\omega}) \in \mathcal{G}(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \left| A \in \Xi \upharpoonright X^\omega\right.\right\}, X^\omega\right)).$$

Hence $G(T_{Max}\left(\chi_A, X^\omega\right); T_{X,T_{sq}}^{\chi_A, X^\omega})$ is determined. By theorem 3.4.26, $G(A; X^\omega)$ is determined. Therefore, if $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property, then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \left| A \in \Xi \upharpoonright X^\omega\right.\right\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \upharpoonright X^\omega\right).$$

Since $\chi_A \in \Gamma(\omega, 2\text{-}\Xi)$ by observation 2.4.26,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \upharpoonright X^\omega). \qquad \square$$

The following is a $Tree_2$ collection version of corollary 2.4.28 on page 115.

Corollary 3.4.28. *(Corollary to Corollary 3.4.27)*

Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Delta}^0_\alpha \upharpoonright X^\omega). \qquad \dashv$$

Proof.

By corollary 3.4.14 with Ξ is $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$. □

Corollaries 3.4.27 and 3.4.28 state that the respective relation holds for every $\Upsilon \supseteq \{T_0, T_1\}$ where $\{T_0, T_1\}$ satisfying the modified 1 maximal tree property or the 1 disjoint tree property. Notice that $\langle Y^{\leq 1}, Y^{\leq 2}\rangle$ satisfies the modified 1 maximal tree property. Let $T_{sq} = \langle T_n | n \in \omega\rangle$ be such that $T_0 = Y^{\leq 1}$, $T_1 = Y^{\leq 2}$ and for any $n > 1$, $T_n \in \Upsilon$ to be arbitrary. Then for $A \subseteq X^\omega$, a Type 1 tree $T^{\chi_A, X^\omega}_{X,Y}$ is a Type 2 tree $T^{\chi_A, X^\omega}_{X,T_{sq}}$. Therefore, by letting Υ to be arbitrary such that $\Upsilon \supseteq \{Y^{\leq 1}, Y^{\leq 2}\}$, we see that these corollaries are generalization of corollaries 2.4.27, and 2.4.28, respectively.

3.4.2.4 Using the determinacy of open games $T_{Max}(\chi_A, B)$ on a $Tree_2$ collection to obtain the determinacy of games $A \cap B$ on $X^{<\omega}$

In this section, we will obtain the determinacy of games $A \cap B$ on $X^{<\omega}$ from the determinacy of open games $T_{Max}(\chi_A, B)$ on a $Tree_2$ collection. We will obtain similar results as section 2.4.2.4. Let $A \subseteq X^\omega$. In section 3.4.2.3, as a special case of Type 2 tree, we considered Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ such that $B = X^\omega$ and Ψ to be the characteristic function χ_A of A. In this section, as a generalization of trees in section 3.4.2.3, we will consider Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ such that B is an arbitrary subset of X^ω and Ψ to be the characteristic function χ_A of A.

Suppose $A, B \subseteq X^\omega$. Then

$$h \in \left[T^{\chi_A,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times T_{\chi_A(h\restriction\omega)} & \text{if } h \restriction \omega \in B. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times [T_0] & \text{if } h \restriction \omega \in B \backslash A, \\ h \in X^\omega \times [T_1] & \text{if } h \restriction \omega \in A \cap B. \end{cases}$$

Note that it is possible that $[T_0] \cap [T_1] \neq \emptyset$ (T_0 could be equal to T_1). $h \in [T^{\chi_A,B}_{X,T_{sq}}]$ such that $h \restriction [\omega, lh(h)) \in [T_0] \cap [T_1]$ does not mean $h \restriction \omega \in B\backslash A$ and $h \restriction \omega \in A \cap B$. Recall that for Type 2 trees $T^{\Psi,B}_{X,T_{sq}}$ with $h \in [T^{\Psi,B}_{X,T_{sq}}]$, $h \restriction [\omega, lh(h)) \in [T_n]$ does not imply $\Psi(h \restriction \omega) = n$ (see comment under observation 3.1.3 on page 216).

A similar proof of observation 3.4.25 gives us the following.

Observation 3.4.29. *Let $A, B \subseteq X^\omega$ and χ_A be the characteristic function of A. Suppose T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tail tree property. Then*

$$T_{Max}(\chi_A, B) = \left\{ h \in [T^{\chi_A,B}_{X,T_{sq}}] \mid h \restriction \omega \in A \cap B \right\}. \qquad \dashv$$

The following theorem is a Type 2 tree version of theorem 2.4.29 on page 116.

Theorem 3.4.30. *Suppose $T^{\chi_A,B}_{X,T_{sq}}$ is a Type 2 tree such that T_{sq} satisfying the modified 1 maximal tail tree property or the 2 disjoint tail tree property. Then for all $A, B \subseteq X^\omega$, $G(A \cap B; X^\omega)$ is determined if and only if $G(T_{Max}(\chi_A, B); T^{\chi_A,B}_{X,T_{sq}})$ is determined.* $\dashv$

Proof.

Pick arbitrary $A, B \subseteq X^{\omega}$.

($\Rightarrow$) Assume $G(A \cap B; X^{<\omega})$ is determined. Then I or II has a winning strategy s for $G(A \cap B; X^{<\omega})$. Define s^* to be such that $s^* \restriction X^{<\omega} = s$ and play anything after that (if necessary) to finish the play. Show s^* is a winning strategy for $G(T_{Max}(\chi_A, B)\,;\, T^{\chi_A,B}_{X,T_{sq}})$. Pick an arbitrary $h \in [T^{\chi_A,B}_{X,T_{sq}}]$ according to s^*.

Case 1 : s is a winning strategy for I for $G(A \cap B; X^{<\omega})$.

Then $h \restriction \omega \in A \cap B$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.29, $h \in T_{Max}(\chi_A, B)$. Hence s^* is a winning strategy for I^* for $G(Max(\chi_A, B)\,; T^{\chi_A,B}_{X,T_{sq}})$.

Case 2 : s is a winning strategy for II for $G(A \cap B; X^{<\omega})$.

Then $h \restriction \omega \notin A \cap B$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.29, $h \notin T_{Max}(\chi_A, B)$. Hence s^* is a winning strategy for II^* for $G(Max(\chi_A, B)\,; T^{\chi_A,B}_{X,T_{sq}})$.

($\Leftarrow$) Assume $G(T_{Max}(\chi_A, B)\,;\, T^{\chi_A,B}_{X,T_{sq}})$ is determined. Then I^* or II^* has a winning strategy s^* for $G(T_{Max}(\chi_A, B)\,;\, T^{\chi_A,B}_{X,T_{sq}})$. Define $s = s^* \restriction X^{<\omega}$. Show s is a winning strategy for $G(A \cap B; X^{<\omega})$. Pick an arbitrary $f \in X^{\omega}$ according to s. Then f is according to s^*. If $f \in [T^{\chi_A,B}_{X,T_{sq}}]$ then let $g = \emptyset$. If $f \in T^{\chi_A,B}_{X,T_{sq}}$, play g according to s^* to get g such that $f^\frown g \in [T^{\chi_A,B}_{X,T_{sq}}]$.

Case 1 : s^* is a winning strategy for I^* for $G(T_{Max}(\chi_A, B)\,;\, T^{\chi_A,B}_{X,T_{sq}})$.

Then $f^\frown g \in T_{Max}(\chi_A, B)$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $f \in A \cap B$. Hence s is a winning strategy for I for $G(A \cap B; X^{<\omega})$.

Case 2 : s^* is a winning strategy for II^* for $G(T_{Max}(\chi_A, B)\,;\, T^{\chi_A,B}_{X,T_{sq}})$.

Then $f^\frown g \notin T_{Max}(\chi_A, B)$. Since T_{sq} satisfying the modified 1 maximal tail tree property or the 1 disjoint tail tree property, by observation 3.4.25, $f \notin A$. Hence s is a winning strategy for II for $G(A \cap B; X^{<\omega})$. Therefore, $G(A \cap B; X^{<\omega})$ is determined. □

The following is a $Tree_2$ collection version of 2.4.30 on page 118.

Corollary 3.4.31. *Suppose $\alpha, \beta \in \omega_1$ and Ξ_1, Ξ_2 are complexities. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \{\chi_A \,|A \in \Xi_1 \restriction X^\omega\}, \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \quad (3.3)$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \{\chi_A \,|A \in \Xi_2 \restriction X^\omega\}, \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \quad (3.4)$$

⊣

Proof.

Pick an arbitrary $A \in (\Xi_1 \wedge \Xi_2) \restriction X^\omega$. Then there exists $B \in \Xi_1 \restriction X^\omega$ and $C \in \Xi_2 \restriction X^\omega$ such that $A = B \cap C$.

Show the implication (3.3).

Since $\chi_B \in \{\chi_{\hat{A}} \mid \hat{A} \in \Xi_1 \restriction X^\omega\}$, we consider the tree $T^{\chi_B, C}_{X, T_{sq}}$. Since $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property, $\langle T_0, T_1\rangle$ or $\langle T_1, T_0\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Without loss of generality, assume $\langle T_0, T_1\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Define $T_{sq} = \langle \hat{T}_n \,|n \in \omega\rangle$ to be such that $\hat{T}_0 = T_0$, $\hat{T}_1 = T_1$ and for any $n > 1$, $\hat{T}_n \in \Upsilon$ to be arbitrary. Then T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tree property. By observation 3.4.29,

$$Max\,(\chi_B, C) = \{h \in [T^{\chi_B,C}_{X,T_{sq}}] \mid h \upharpoonright \omega \in B \cap C\}.$$

Since

$$G(Max(\chi_B, C); T^{\chi_B,C}_{X,T_{sq}}) \in \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \,|A \in \Xi_1 \upharpoonright X^\omega\right\}, \Xi_2 \upharpoonright X^\omega\right)\right),$$

$G(Max(\chi_B, C); T^{\chi_B,C}_{X,T_{sq}})$ is determined. By theorem 3.4.30, $G(B \cap C; X^{<\omega})$ is determined. Hence $G(A; X^{<\omega})$ is determined.

Show the implication (3.4).

Since $\chi_C \in \{\chi_{\hat{A}} \mid \hat{A} \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\}$, we consider the tree $T^{\chi_C,B}_{X,T_{sq}}$ where T_{sq} is defined above. Then T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tree property. By observation 3.4.29,

$$Max\,(\chi_C, B) = \{h \in [T^{\chi_C,B}_{X,T_{sq}}] \mid h \upharpoonright \omega \in B \cap C\}.$$

Since

$$G(Max(\chi_C, B); T^{\chi_C,B}_{X,T_{sq}}) \in \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \,|A \in \Xi_2 \upharpoonright X^\omega\right\}, \Xi_1 \upharpoonright X^\omega\right)\right).$$

$G(Max(\chi_C, B); T^{\chi_C,B}_{X,T_{sq}})$ is determined. By theorem 3.4.30, $G(C \cap B; X^{<\omega})$ is determined. Hence $G(A; X^{<\omega})$ is determined. □

The following is a $Tree_2$ collection version of corollary 2.4.31 on page 119.

Corollary 3.4.32. *(Corollary to Corollary 3.4.31)*

Suppose Ξ_1, Ξ_2 are complexities. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree prop-

erty or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi_1 \wedge co\text{-}\Xi_1), \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{3.5}$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi_2 \wedge co\text{-}\Xi_2), \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{3.6}$$

⊣

Proof.

Since $\{\chi_A \,|A \in \Xi_1 \restriction X^\omega\} \subseteq \Gamma\left(\omega, \Xi_1 \wedge co\text{-}\Xi_1\right)$ by observation 2.4.26, we obtain the implication (3.5) from corollary 3.4.31 the implication (3.3). Since $\{\chi_A \,|A \in \Xi_2 \restriction X^\omega\} \subseteq \Gamma\left(\omega, \Xi_2 \wedge co\text{-}\Xi_2\right)$ by observation 2.4.26, we obtain the implication (3.6) from corollary 3.4.31 the implication (3.4). □

We list some obvious special case of corollary 3.4.32. We obtain corollary 3.4.33 from replacing $\Xi_1 = \mathbf{\Sigma}^0_\alpha$ and $\Xi_2 = \mathbf{\Pi}^0_\beta$ in corollary 3.4.32. This is a $Tree_2$ collection version of corollary 2.4.32 on page 120. We also obtain corollary 3.4.34 from replacing $\Xi_1 = \mathbf{\Sigma}^1_\alpha$ and $\Xi_2 = \mathbf{\Pi}^1_\beta$ in corollary 3.4.32. This is a $Tree_2$ collection version of corollary 2.4.33 on page 120.

Corollary 3.4.33. *(Corollary to Corollary 3.4.32)*

Suppose $\alpha, \beta \in \omega_1$. *Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 2 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha\right), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega). \qquad \dashv$$

Proof.

Since $\{\chi_A \,|A \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega\} \subseteq \Gamma\left(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha\right)$ and $\left\{\chi_A \,\middle|A \in \mathbf{\Pi}^0_\beta \restriction X^\omega\right\} \subseteq \Gamma(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta)$ by observation 2.4.26, we have the results by corollary 3.4.32. □

Corollary 3.4.34. *(Corollary to Corollary 3.4.32)*

Suppose $n, m \in \omega$. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 2 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n\right), \mathbf{\Pi}^1_m \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega).$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m\right), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega). \qquad \dashv$$

Proof.

Since $\{\chi_A \,|A \in \mathbf{\Sigma}^1_n \restriction X^\omega\} \subseteq \Gamma\left(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n\right)$ and $\{\chi_A \,|A \in \mathbf{\Pi}^1_m \restriction X^\omega\} \subseteq \Gamma(\omega, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m)$ by observation 2.4.26, we have the results by corollary 3.4.32. □

In particular, intersection of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets are 2-$\mathbf{\Pi}^1_1$. Thus, the following is a special case of corollary 3.4.34 and this is a $Tree_2$ collection version of corollary 2.4.34 on page 120.

Corollary 3.4.35. *(Corollary to Corollary 3.4.34)*

Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Proof.

Note that $\mathbf{\Sigma}^1_1 \wedge \mathbf{\Pi}^1_1 = 2\text{-}\mathbf{\Pi}^1_1$. □

The following is a $Tree_2$ collection version of question 1 on page 121.

Question 9. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property and $\Upsilon \supseteq \{T_0, T_1\}$. By corollary 3.4.35 on page 265, corollary 3.4.13 on page 246 and corollary 3.4.1 on page 238, all of the following imply $Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega)$:*

$$(i) \quad Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega\right)\right)$$

$$(ii) \quad Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$$

$$(iii) \quad Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), 2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$$

$$(iv) \quad Det\ \mathcal{G}\left(2\text{-}\mathbf{\Pi}^1_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right)$$

What is the relationship between $(i), (ii), (iii)$ and (iv)? $\dashv$

Let's consider $(n+1)\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$ sets for finite n. Recall definition 1.3.23 on page 23. For every $n \in \omega$, if $A \in (n+1)\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$, then $A = A_0 \backslash A_1 = A_0 \cap X^\omega \backslash A_1$ where $A_0 \in \mathbf{\Pi}^1_1 \restriction X^\omega$ and $A_1 \in n\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$ (hence $X^\omega \backslash A_1 \in co\text{-}n\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$). Thus we can express A as an intersection of a $\mathbf{\Pi}^1_1 \restriction X^\omega$ set and a $co\text{-}n\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$ set. We obtain corollary 3.4.36 the implication (3.7) by replacing $\Xi_1 = \mathbf{\Sigma}^1_1$ and $\Xi_2 = co\text{-}n\text{-}\mathbf{\Pi}^1_1$ in corollary 3.4.32 the implication (3.5). We obtain corollary 3.4.36 the implication (3.8) by replacing $\Xi_1 = co\text{-}n\text{-}\mathbf{\Pi}^1_1$ and $\Xi_2 = \mathbf{\Sigma}^1_1$ in corollary 3.4.32 the implication (3.6). This is also a $Tree_2$ collection version of corollary 2.4.35 on page 121.

Corollary 3.4.36. *(Corollary to Corollary 3.4.32)*

Suppose $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$ *and for any* $n \in \omega$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), (co\text{-}n\text{-}\mathbf{\Pi}^1_1) \restriction X^\omega\right)\right) \Rightarrow Det(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \quad (3.7)$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right). \quad (3.8)$$

Proof.

Since$\{\chi_A \,|A \in \mathbf{\Pi}^1_1 \restriction X^\omega\} \subseteq \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1)$ and $\{\chi_A \,|A \in co\text{-}n\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\} \subseteq$ $\Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1)$ by observation 2.4.26, we have the results by corollary 3.4.36. □

3.4.3 Using α-$\mathbf{\Pi}^1_1$ determinacy on $Tree_2$ collection to obtain $\alpha+1$-$\mathbf{\Pi}^1_1$ determinacy on $X^{<\omega}$ for even $\alpha \in \omega_1$

In this section, we will obtain similar results for Type 2 trees as section 2.4.3. The only difference is the trees are Type 2 trees instead of Type 1 trees.

In section 3.4.2.4, we used T_{Max} on certain $Tree_2$ collections to obtain the determinacy of games on $X^{<\omega}$. In theorem 3.4.30, we obtained the determinacy equivalence of games $G(A \cap B; X^{<\omega})$ and $G(T_{Max}(\chi_A, B); T^{\chi_A, B}_{X,T_{sq}})$ for any $A, B \subseteq X^\omega$ and T_{sq} satisfying a certain condition.

In this section, we will obtain $\alpha+1$-$\mathbf{\Pi}^1_1$ determinacy on X^ω for even $\alpha \in \omega_1$ from α-$\mathbf{\Pi}^1_1$ determinacy on $Tree_2$ collection. Fix $\alpha \in \omega_1$ and $\langle A_\beta | \beta \leq \alpha \rangle$ where each $A_\beta \subseteq X^\omega$. Recall that by observation 2.4.38, $dk(\langle A_\beta | \beta \leq \alpha \rangle) = dk(\langle A_\beta | \beta \in \alpha \rangle) \cup (\bigcap_{\beta \leq \alpha} A_\beta)$. We set $A = A_\alpha$ and $B = \bigcap_{\beta \in \alpha} A_\beta$ so that $A \cap B = \bigcap_{\beta \leq \alpha} A_\beta$. Thus, we will consider a Type 2 tree $T^{\chi_{A_\alpha}, B}_{X,T_{sq}}$ with $B = \bigcap_{\beta \in \alpha} A_\beta$ and T_{sq} satisfying a certain condition.

$$h \in \left[T^{\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin \bigcap_{\beta \in \alpha} A_\beta, \\ h \in X^\omega \times [T_{\chi_A(h \upharpoonright \omega)}] & \text{if } h \upharpoonright \omega \in \bigcap_{\beta \in \alpha} A_\beta. \end{cases}$$

$$\leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \upharpoonright \omega \notin \bigcap_{\beta \in \alpha} A_\beta, \\ h \in X^\omega \times [T_0] & \text{if } h \upharpoonright \omega \in (\bigcap_{\beta \in \alpha} A_\beta) \backslash A_\alpha, \\ h \in X^\omega \times [T_1] & \text{if } h \upharpoonright \omega \in \bigcap_{\beta \leq \alpha} A_\beta. \end{cases}$$

We will obtain the determinacy equivalence of a certain game for such Type 2 tree $T^{\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta}_{X,T_{sq}}$ and a $dk(\langle A_\beta | \beta \leq \alpha \rangle)$ game on $X^{<\omega}$. In definition 3.4.37, we will define $dk^2_{<\alpha}(\langle A_\beta | \beta \leq \alpha \rangle) \subseteq [T^{\chi_{A_\alpha}, \bigcap_{\beta \in \alpha} A_\beta}_{X,T_{sq}}]$ (we will use the "superscript 2" to represent that this set is defined on Type 2 trees). This is a Type 2 tree version of $dk^2_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) \subseteq [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}]$ which we defined in definition 2.4.36 on page 125. In theorem 3.4.41, we will show that the determinacy equivalence of a $dk(\langle A_\beta \,|\beta \leq \alpha\rangle)$ game on $X^{<\omega}$ and a $dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) \cup\ T_{Max}(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta)$ game on the tree $T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}$.

In particular, for even $\alpha \in \omega$ and sequences $\langle A_\beta \,|\beta \leq \alpha\rangle$ with each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$, we will obtain $\alpha + 1$-$\mathbf{\Pi}^1_1$ games on $X^{<\omega}$ from α-$\mathbf{\Pi}^1_1$ games on a particular $Tree_2$ collection in corollary 3.4.43. As a special case, when α is a limit ordinal and $A_\alpha \in \mathbf{\Sigma}^0_\lambda$ for some $\lambda \in \omega_1$, we will obtain a similar result for α-$\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda$ games on $X^{<\omega}$ from α-$\mathbf{\Pi}^1_1$ games on a particular $Tree_2$ collection in corollary 3.4.44.

First, recall definition 1.3.22 on page 23.

Definition 1.3.22. *(Definition of the difference kernel)(Hausdorff, 1944 [12])*
Denote the difference kernel of $\vec{A} = \langle A_\beta \,|\beta \in \alpha\rangle$ by $dk(\vec{A})$ and define

$$dk(\vec{A}) = \{x \in [T] \,|\mu\beta\,(x \notin A_\beta \vee \beta = \alpha) \text{ is odd}\}. \qquad \dashv$$

Given $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$, we define $dk_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle)$ on the tree $T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}$.

The following is a Type 2 tree version of definition 2.4.36 on page 125.

Definition 3.4.37. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Define*

$$dk^2_{<\alpha}(\langle A_\beta \,|\beta \leq \alpha\rangle) = \left\{h \in \left[T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}\right] \middle|\, h \restriction \omega \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \restriction \omega \notin A_\beta) \text{ is odd}\right\}.$$

[12] as cited in Welch (1996, p. 1).

$\dashv$

Notice that if α is even,

$$dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) = \left\{ f \in X^\omega \,\middle|\, f \notin \bigcap_{\beta \in \alpha} A_\beta \wedge \mu\beta(f \notin A_\beta) \text{ is odd} \right\}.$$

Thus

$$dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \restriction X^\omega = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right).$$

In fact, we have the following. This is a Type 2 tree version of observation 2.4.37.

Observation 3.4.38. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Then*

$$dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) \subseteq X^\omega. \qquad \dashv$$

Proof.

A similar proof to observation 2.4.37 gives observation 3.4.38. Simply replace a Type 1 tree $T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,Y}$ by a Type 2 tree $T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}$. □

The following is a special case of observation 3.4.29 for $B = \bigcap_{\beta\in\alpha} A_\beta$.

Corollary 3.4.39. *(Corollary to Observation 3.4.29)*

Suppose $\alpha \in \omega_1$ is even, $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$ and χ_{A_α} to be the characteristic function of A_α. Suppose T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tail tree property. Then

$$T_{Max}\left(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta\right) = \left\{ h \in [T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}] \,\middle|\, h \restriction \omega \in \bigcap_{\beta\leq\alpha} A_\beta \right\}. \qquad \dashv$$

Since $\alpha \in \omega_1$ is even, $dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right)$ could be express as a union of $dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right)$ and $(\bigcap_{\beta\leq\alpha} A_\beta)$. Recall 2.4.38.

Observation 2.4.38. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Then*

$$dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right) \cup \left(\bigcap\nolimits_{\beta \leq \alpha} A_\beta\right). \qquad \dashv$$

The following is a Type 2 tree version of proposition 2.4.39 on page 126.

Proposition 3.4.40. *Suppose $T = T^{\Psi,B}_{X,T_{sq}}$ is a Type 2 tree. Assume the following:*

1. *$C, D \subseteq X^\omega$ and $E, F \subseteq [T]$.*

2. *s is a strategy for $X^{<\omega}$.*

3. *s^* is a strategy for T such that $s^* \restriction X^{<\omega} = s$.*

4. *for any $h \in [T]$ according to s^*, $h \restriction \omega \in C$ if and only if $h \in E$.*

5. *for any $h \in [T]$ according to s^*, $h \restriction \omega \in D$ if and only if $h \in F$.*

Then s is a winning strategy for I for $G(C \cup D; X^{<\omega})$ if and only if s^* is a winning strategy for I^* for $G(E \cup F; T)$. Also s is a winning strategy for II for $G(C \cup D; X^{<\omega})$ if and only if s^* is a winning strategy for II^* for $G(E \cup F; T)$. $\dashv$

Proof.

A similar proof of proposition 2.4.39 gives proposition 3.4.40. Simply replace a Type 1 tree $T^{\Psi,B}_{X,Y}$ by a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$. □

By proposition 3.4.40, we obtain the following. This is a Type 2 tree version of theorem 2.4.40 on page 128.

Theorem 3.4.41. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \subseteq X^\omega$. Let $T = T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}$. Then $G\left(dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right); X^{<\omega}\right)$ is determined if and only if*

$$G\left(dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \cup\ T_{Max}\left(\chi_{A_\alpha}, \bigcap\nolimits_{\beta\in\alpha} A_\beta\right); T\right)$$

is determined. ⊣

Proof.

Use proposition 3.4.40 with:

- $C = dk\left(\langle A_\beta \,|\beta \in \alpha\rangle\right)$.
- $D = \bigcap_{\beta\leq\alpha} A_\beta$.
- $E = dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right)$
- $F = T_{Max}\left(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta\right)$

The rest of the proof is similar to the proof of theorem 2.4.40. □

Now, let's consider the complexity of each A_β. Recall definition 1.3.23 on page 23.

Definition 1.3.23. *Suppose Λ is a class of subsets of $[T]$ and Λ is closed under countable intersections. Suppose $\alpha \in \omega_1$. Define*

$$\alpha\text{-}\Lambda \restriction [T] = \left\{A \subseteq [T] \,\middle|\, \exists \vec{A} = \langle A_\beta \,|\beta \in \alpha\rangle \left(\text{each } A_\beta \in \Lambda \text{ and } A = dk(\vec{A})\right)\right\}. \quad ⊣$$

We will consider theorem 3.4.41 with $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$. Then $dk\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \in \alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$ where $\alpha \in \omega_1$ is even.

Lemma 3.4.42. *Suppose $\alpha \in \omega_1$ is even. Fix $\langle A_\beta \,|\beta \leq \alpha\rangle$ where each $A_\beta \in \mathbf{\Pi}^1_1 \restriction X^\omega$. Then*

$$dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha\rangle\right) \in \alpha\text{-}\mathbf{\Pi}^1_1 \restriction \left[T^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}_{X,T_{sq}}\right]. \quad ⊣$$

Proof.

A similar to lemma 2.4.41 gives lemma 3.4.42. Simply replace a Type 1 tree $T_{X,Y}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$ by a Type 2 tree $T_{X,T_{sq}}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$. $\square$

Using theorem 3.4.41 and lemma 3.4.42, we have the following.

Corollary 3.4.43. *Assume $\alpha \in \omega_1$ is even. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}_1^1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}_1^1), \mathbf{\Pi}_1^1 \restriction X^\omega\right)\right) \Rightarrow Det(\alpha+1\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega). \qquad \dashv$$

Proof.

Suppose $\alpha \in \omega_1$ is even and $A \in \alpha+1\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega$. Then there exists a sequence $\vec{A} = \langle A_\beta \,|\, \beta \leq \alpha\rangle$ witness that $A = dk(\vec{A}) \in \alpha+1\text{-}\mathbf{\Pi}_1^1 \restriction X^\omega$. Since $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property, $\langle T_0, T_1\rangle$ or $\langle T_1, T_0\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Without loss of generality, assume $\langle T_0, T_1\rangle$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Define $T_{sq} = \langle \hat{T}_n \,|\, n \in \omega\rangle$ to be such that $\hat{T}_0 = T_0$, $\hat{T}_1 = T_1$ and for any $n > 1$, $\hat{T}_n \in \Upsilon$ to be arbitrary. Then T_{sq} satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Let $T = T_{X,T_{sq}}^{\chi_{A_\alpha},\bigcap_{\beta\in\alpha}A_\beta}$. We have $\bigcap_{\beta\leq\alpha} A_\beta \in \mathbf{\Pi}_1^1 \restriction X^\omega$. By observation 2.4.26, $\chi_{A_\alpha} \in \Gamma(\omega, 2\text{-}\mathbf{\Pi}_1^1)$. By lemma 3.4.42, $dk_{<\alpha}(\langle A_\beta \,|\, \beta \leq \alpha\rangle) \in \alpha\text{-}\mathbf{\Pi}_1^1 \restriction [T]$. By proposition 3.4.21, $T_{Max}(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta) \in \mathbf{\Sigma}_1^0 \restriction [T]$. Hence

$$dk_{<\alpha}(\langle A_\beta \,|\, \beta \leq \alpha\rangle) \cup T_{Max}\left(\chi_{A_\alpha}, \bigcap\nolimits_{\beta\in\alpha} A_\beta\right) \in \alpha\text{-}\mathbf{\Pi}_1^1 \restriction [T].$$

Thus $G(dk_{<\alpha}(\langle A_\beta \,|\, \beta \leq \alpha\rangle) \cup T_{Max}(\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta); T)$ is determined. By theorem 3.4.41, $G(dk(\vec{A}); X^{<\omega})$ is determined, i.e., $G(A; X^{<\omega})$ is determined. $\square$

Question 10. *Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Assume that* $\Upsilon \supseteq \{T_0, T_1\}$. *By corollary 3.4.2 on page 239, corollary 3.4.13 on page 246 and corollary 3.4.27 on page 257, all of the following imply* $Det(\alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega)$*:*

$$(i) \quad Det\ \mathcal{G}\left(\alpha + 1\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right)$$

$$(ii) \quad Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$$

$$(iii) \quad Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Upsilon^\omega, \Gamma(\omega, \alpha + 1\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}\alpha + 1\text{-}\mathbf{\Pi}^1_1), X^\omega\right)\right)$$

Moreover, if $\alpha \in \omega_1$ *is even, then by corollary 3.4.43,*

$$(iv) \quad Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$$

implies $Det(\alpha + 1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega)$.

What is the relationship between $(i), (ii), (iii)$ *and* (iv)*?* $\dashv$

Suppose $\alpha \in \omega_1$ is a limit ordinal. As a special case of $\alpha + 1\text{-}\mathbf{\Pi}^1_1$, we will define $\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda$ sets over a tree T.

Recall definition 2.4.43 on page 132.

Definition 2.4.43. *(Definition of* $\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \restriction [T]$*)*

Suppose $\alpha \in \omega_1$ *is a limit ordinal. Let* $\lambda \in \omega_1$. *Suppose* T *is a tree. Define* $A \in (\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda) \restriction [T]$ *if and only if there is a sequence* $\vec{A} = \langle A_\beta \,|\, \beta \leq \alpha \rangle$ *witness that* $A = dk(\vec{A}) \in \alpha{+}1\text{-}\mathbf{\Pi}^1_1 \restriction [T]$ *and* $A_\alpha \in \mathbf{\Sigma}^0_\lambda \restriction [T]$, *i.e.,*

$$\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction [T] = \left\{ A \subseteq [T] \,\middle|\, \exists \vec{A} = \langle A_\beta \,|\, \beta \leq \alpha \rangle \begin{pmatrix} \forall \beta \in \alpha \left(A_\beta \in \mathbf{\Pi}^1_1 \restriction [T]\right), \\ A_\alpha \in \mathbf{\Sigma}^0_\lambda \restriction [T] \text{ and } A = dk(\vec{A}) \end{pmatrix} \right\}.$$

$\dashv$

We have a similar result for α-$\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda$ sets to corollary 3.4.43.

Corollary 3.4.44. *Assume $\alpha \in \omega_1$ is a limit ordinal and $\lambda \in \omega$. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction X^\omega). \quad \dashv$$

Proof.

Similar proof for corollary 3.4.43 with $\chi_{A_\alpha} \in \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda)$ by observation 2.4.26. □

Question 11. *Assume $\alpha \in \omega_1$ is a limit ordinal and $\lambda \in \omega$. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Assume $\Upsilon \supseteq \{T_0, T_1\}$. By corollary 3.4.2 on page 239, corollary 3.4.44, corollary 3.4.13 on page 246 and corollary 3.4.27 on page 257, all of the following imply $Det((\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda) \restriction X^\omega)$:*

(i) $Det\mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right)$

(ii) $Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right)$

(iii) $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \restriction X^\omega\right)\right)$

(iv) $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda), X^\omega\right)\right)$

What is the relationship between (i), (ii), (iii) and (iv)? $\dashv$

As we discussed on page 133, through out this section, we set that α is even so that $\alpha+1$ is odd. In this section, we obtained the determinacy of α+1-$\mathbf{\Pi}^1_1$ games on $X^{<\omega}$ from the determinacy of α-$\mathbf{\Pi}^1_1$ games on a particular $Tree_2$ collection for even $\alpha \in \omega_1$.

In the next section 3.4.4, we will obtain the determinacy of α-$\mathbf{\Pi}^1_1$ games on $X^{<\omega}$ from the determinacy of open games on some $Tree_2$ collections for $\alpha \in \omega_1$ for any countable α.

3.4.4 Using the determinacy of open games on a $Tree_2$ collection to obtain the determinacy of α-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$

In this section, we will focus on obtaining the determinacy of α-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from the determinacy of open games on some $Tree_2$ collections for $\alpha \in \omega_1$ for any countable α. In sections 3.4.2.2, 3.4.2.3, 3.4.2.4 and 3.4.3, we obtained the determinacy results using T_{Max}. In corollary 3.4.36 of section 3.4.2.4 , we obtained the determinacy of $n+1$-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from the determinacy of open games on a particular $Tree_2$ collection for all $n \in \omega$. In section 3.4.3, we obtained the determinacy of α+1-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from the determinacy of α-$\mathbf{\Pi}_1^1$ games on a particular $Tree_2$ collection for even $\alpha \in \omega_1$. There are two things to notice here:

- We only obtained the determinacy results for even α (so that α+1 is odd).
- We obtained the determinacy of α+1-$\mathbf{\Pi}_1^1$ games using the determinacy of α-$\mathbf{\Pi}_1^1$ games a particular $Tree_2$ collection, not open games on a $Tree_2$ collection.

In this section, we will obtain the determinacy of α-$\mathbf{\Pi}_1^1$ games on $X^{<\omega}$ from the determinacy of open games on a certain $Tree_2$ collection for any $\alpha \in \omega_1$. We will define $Tail$. Recall that $T_{Max}(\Psi, B)$ was defined on a Type 2 tree $T_{X,T_{sq}}^{\Psi,B}$ with Ψ having the maximum value $\Psi_{\max}$. $Tail(\Psi, n, B)$ will be the generalization of $T_{Max}(\Psi, B)$ for any function Ψ from X^ω into ω. For any $h \in Tail(\Psi, n, B)$, the tail of h will be in $[T_n]$ (the converse may be false).

Definition 3.4.45. *For any $n \in \omega$, define*

$$Tail\left(\Psi, n, B\right) = \left(B \cap \Psi^{-1}\left(n\right)\right) \times [T_n]. \qquad \dashv$$

Recall that for a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$, $[T^{\Psi,B}_{X,T_{sq}}] = \bigcup_{n\in\omega} ((B \cap \Psi^{-1}(n)) \times [T_n]) \cup (X^\omega \backslash B)$. [13] Thus using $Tail(\Psi, n, B)$, we can express the body of a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ as following.

$$\begin{aligned}[T^{\Psi,B}_{X,T_{sq}}] &= \bigcup\nolimits_{n\in\omega} \left(\left(B \cap \Psi^{-1}(n)\right) \times [T_n]\right) \cup (X^\omega \backslash B) \\ &= \bigcup\nolimits_{n\in\omega} Tail(\Psi, n, B) \cup (X^\omega \backslash B).\end{aligned}$$

Note that if Ψ has a maximum value $\Psi_{\max}$,

$$\begin{aligned}T_{Max}(\Psi, B) &= \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \upharpoonright [\omega, lh(h)) \in [T_{\Psi_{\max}}]\} \\ &= \left(B \cap \Psi^{-1}(\Psi_{\max})\right) \times [T_{\Psi_{\max}}] \\ &= Tail(\Psi, \Psi_{\max}, B).\end{aligned}$$

A similar comment to the comment on page 216, $Tail(\Psi, n, B)$ is not the collection of $h \in [T^{\Psi,B}_{X,T_{sq}}]$ such that $h \upharpoonright [\omega, lh(h)) \in [T_n]$. Recall that if $[T_n] \cap [T_m] \neq \emptyset$ for $n \neq m$, there exists $h \in [T^{\Psi,B}_{X,T_{sq}}]$ such that $h \upharpoonright [\omega, lh(h)) \in [T_n]$ but $h \notin Tail(\Psi, n, B)$ (for the case $h \upharpoonright \omega \in B$ with $\Psi(h \upharpoonright \omega) = m$).

Proposition 3.4.46. *Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Suppose T satisfies at least one of the following conditions:*

1. *$B \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega$ and Ψ is a continuous function from X^ω into ω.*
2. *$T_{sq} = \langle T_n \mid n \in \omega \rangle$ satisfies the disjoint tree property.*[14]

Then for any $n \in \omega$, $Tail(\Psi, n, B) \in \mathbf{\Sigma}^0_1 \upharpoonright [T]$. ⊣

Proof.

Pick an arbitrary $n \in \omega$. Pick an arbitrary $h \in Tail(\Psi, n, B)$. Then $h \upharpoonright \omega \in B \cap \Psi^{-1}(n)$.

[13] Recall page 215.

[14] Recall definition 3.4.19 on page 251.

Case 1 : Suppose T satisfies condition (1).

Since Ψ is a continuous function, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_1 \upharpoonright X^\omega$. Since $B \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega$, $B \cap \Psi^{-1}(n) \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega$. Since $B \cap \Psi^{-1}(n) \in \mathbf{\Sigma}^0_1 \upharpoonright X^\omega$, there exists a finite $E \subseteq \omega$ such that for any $f \in X^\omega$ if $f \supseteq h \upharpoonright E$, then $f \in B \cap \Psi^{-1}(n)$. Thus for any $\hat{h} \in [T]$, if $\hat{h} \supseteq h \upharpoonright E$, then $\hat{h} \in (B \cap \Psi^{-1}(n)) \times [T_n] = Tail(\Psi, n, B)$.

Case 2 : Suppose T satisfies condition (2).

Since $T_{sq} = \langle T_n \,|n \in \omega\rangle$ satisfies the disjoint tree property, each $M^n_\emptyset$ is pairwise disjoint. Let $F = \{\omega\}$. Then F is finite. Pick an arbitrary $\hat{h} \in [T]$ such that $\hat{h} \supseteq h \upharpoonright F$. Then $\hat{h}(\omega) = h(\omega)$. Since $h(\omega) \in M^n_\emptyset$, $\hat{h}(\omega) \in M^n_\emptyset$. Thus $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \in [T_n]$. Note that for each $m \in \omega$ such that $m \neq n$, we have $M^n_\emptyset \cap M^m_\emptyset = \emptyset$. Thus $\hat{h}(\omega) \notin M^m_\emptyset$ for any $m \in \omega$ with $m \neq n$. Thus $\hat{h} \upharpoonright [\omega, lh(\hat{h})) \notin [T_m]$ for any $m \in \omega$ with $m \neq n$. Hence $h \upharpoonright \omega \notin \Psi^{-1}(m)$ for any $m \in \omega$ with $m \neq n$. Thus $\hat{h} \in (B \cap \Psi^{-1}(n)) \times [T_n] = Tail(\Psi, n, B)$.

Therefore, for T satisfying conditions (1) or (2), we have $Tail(\Psi, n, B) \in \mathbf{\Sigma}^0_1 \upharpoonright [T]$. $\square$

Next, we define functions $least_\alpha$ and Ψ_α for each countable α.

Definition 3.4.47. *(Definition of $least_\alpha$ and Ψ_α)*

Suppose $\vec{A} = \langle A_\beta \,|\beta \in \alpha\rangle$ is a sequence of sets. Define

$$least_\alpha : \quad X^\omega \to \alpha + 1$$

$$f \mapsto \begin{cases} \mu\beta\,(f \notin A_\beta) & \text{if } f \notin \bigcap_{\beta\in\alpha} A_\beta, \\ \alpha & \text{otherwise.} \end{cases}$$

Then define

$$\Psi_\alpha : \quad X^\omega \to \omega$$

$$f \mapsto n \text{ where } least_\alpha(f) = \gamma + n,\ \gamma = 0 \text{ or } \gamma \text{ is a limit ordinal.} \qquad \dashv$$

In particular, suppose $\langle A_\beta \,|\beta \in \alpha\rangle$ is a sequence of $\mathbf{\Pi}^1_1 \restriction X^\omega$ sets. We calculate the complexity of Ψ_α.

Lemma 3.4.48. *Let $\alpha \in \omega_1$ be even. Suppose $\langle A_\beta \,|\beta \in \alpha\rangle$ is a sequence of $\mathbf{\Pi}^1_1 \restriction X^\omega$ sets. Then $\Psi_\alpha \in \Gamma\left(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)\right)$.* [15] ⊣

Proof.

Pick an arbitrary $O \in \mathbf{\Sigma}^0_1 \restriction \omega$. Then $O = \bigcup_{n\in O}\{n\}$. First, let's find the complexity of each $\Psi^{-1}_\alpha(n)$. Fix $n \in \omega$.

Case 1 : $\alpha = \hat{\gamma} + n$ for some limit ordinal $\hat{\gamma}$ or $\alpha = n$.

$$\Psi^{-1}_\alpha(n) = \left(\bigcup_{\gamma=0 \text{ or } \gamma\in\alpha \text{ limit}} (\underbrace{(\bigcap\nolimits_{m\in\gamma+n} A_m)}_{\mathbf{\Pi}^1_1\restriction X^\omega} \setminus \underbrace{A_{\gamma+n}}_{\mathbf{\Pi}^1_1\restriction X^\omega})\right) \cup \underbrace{\bigcap\nolimits_{\beta\in\alpha} A_\beta}_{\mathbf{\Pi}^1_1\restriction X^\omega} \in \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right) \restriction X^\omega.$$

Case 2 : $\alpha \neq \hat{\gamma} + n$ for any limit ordinal $\hat{\gamma}$ and $\alpha \neq n$.

$$\Psi^{-1}_\alpha(n) = \left(\bigcup_{\gamma=0 \text{ or } \gamma\in\alpha \text{ limit}} (\underbrace{(\bigcap\nolimits_{m\in\gamma+n} A_m)}_{\mathbf{\Pi}^1_1\restriction X^\omega} \setminus \underbrace{A_{\gamma+n}}_{\mathbf{\Pi}^1_1\restriction X^\omega})\right) \in \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right) \restriction X^\omega.$$

Thus

$$\Psi^{-1}_\alpha(O) = \Psi^{-1}_\alpha\left(\bigcup\nolimits_{n\in O}\{n\}\right) = \bigcup\nolimits_{n\in O}\Psi^{-1}_\alpha(n) \in \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right) \restriction X^\omega.$$ □

By taking union of $Tail(\Psi_\alpha, n, X^\omega)$ for all odd n, we have the following determinacy equivalence.

Theorem 3.4.49. *Suppose $\alpha \in \omega_1$. Fix $T_{sq} = \langle T_n \,|n \in \omega\rangle$. Let $T = T^{\Psi_\alpha, X^\omega}_{X, T_{sq}}$. Let $\vec{A} = \langle A_\beta \,|\beta \in \alpha\rangle$. $G(\bigcup_{odd\ n} Tail\left(\Psi_\alpha, n, X^\omega\right); T)$ is determined if and only if $G(dk(\vec{A}); X^\omega)$ is determined.* ⊣

[15] Recall definition 1.3.25 on page 25 for $\mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)$.

Proof.

($\Rightarrow$) Suppose $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$ is determined. Then I^* or II^* has a winning strategy s^* for $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$. Define $s = s^* \restriction X^{<\omega}$. Pick an arbitrary f according to s. Then $f \in X^\omega$ according to s^*. Since there is no path of length ω in T, $f \in T$. Play according to s^* after f, call it g, so that $f^\frown g \in [T]$.

Case 1 : s^* is a winning strategy for I^* for $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$.

Show s^* is a winning strategy for I for $G(dk(\vec{A}); X^\omega)$. Since $f^\frown g$ is according to s^*, $f^\frown g \in \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$. Since $Tail\,(\Psi_\alpha, n, X^\omega) = (X^\omega \cap \Psi_\alpha^{-1}(n)) \times [T_n]$ by definition, there exists a unique odd $m \in \omega$ such that $f^\frown g \in Tail\,(\Psi_\alpha, m, X^\omega) = \Psi_\alpha^{-1}(m) \times [T_m]$. Hence $f \in \Psi_\alpha^{-1}(m)$. Thus $least_\alpha(f) = \gamma + m$ for some limit ordinal $\gamma \in \alpha$ or $\gamma = 0$.

Subcase 1 : α is even.

Then $least_\alpha(f) = \gamma + m \neq \alpha$ since m is odd. Thus $f \notin \bigcap_{\beta\in\alpha} A_\beta$. Hence $\gamma + m$ is the least β such that $f \notin A_\beta$. Since m is odd, $f \in dk(\vec{A})$.

Subcase 2 : α is odd.

Subsubcase a : $least_\alpha(f) = \gamma + m \neq \alpha$.

Then $f \notin \bigcap_{\beta\in\alpha} A_\beta$. Hence $\gamma+m$ is the least β such that $f \notin A_\beta$. Since m is odd, $f \in dk(\vec{A})$.

Subsubcase b : $least_\alpha(f) = \gamma + m = \alpha$.

Then $f \in \bigcap_{\beta\in\alpha} A_\beta$, or $f \notin \bigcap_{\beta\in\alpha} A_\beta$ and $\gamma + m$ is the least β such that $f \notin A_\beta$. Since m and α are odd, $f \in dk(\vec{A})$.

Case 2 : s^* is a winning strategy for II^* for $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$.

Show s^* is a winning strategy for II for $G(dk(\vec{A}); X^\omega)$. Since $f^\frown g$ is according to s^*, $f^\frown g \notin \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$. Thus for all odd $n \in \omega$, $f^\frown g \notin Tail\,(\Psi_\alpha, n, X^\omega) = \Psi_\alpha^{-1}(n) \times [T_n]$. Hence $f \notin \Psi_\alpha^{-1}(n)$ for all odd n. Since Ψ_α is a function, there exists a unique even m such that $f \in \Psi_\alpha^{-1}(m)$. Thus $least_\alpha(f) = \gamma + m$ for some limit ordinal $\gamma \in \alpha$ or $\gamma = 0$.

Subcase 1 : α is even.

Subsubcase a : $least_\alpha(f) = \gamma + m \neq \alpha$.

Then $f \notin \bigcap_{\beta\in\alpha} A_\beta$. Hence $\gamma+m$ is the least β such that $f \notin A_\beta$. Since m is even, $f \notin dk(\vec{A})$.

Subsubcase b : $least_\alpha(f) = \gamma + m = \alpha$.

Then $f \in \bigcap_{\beta\in\alpha} A_\beta$, or $f \notin \bigcap_{\beta\in\alpha} A_\beta$ and $\gamma + m$ is the least β such that $f \notin A_\beta$. Since m and α are even, $f \notin dk(\vec{A})$.

Subcase 2 : α is odd.

Then $least_\alpha(f) = \gamma + m \neq \alpha$ since m is even. Thus $f \notin \bigcap_{\beta\in\alpha} A_\beta$. Hence $\gamma + m$ is the least β such that $f \notin A_\beta$. Since m is even, $f \in dk(\vec{A})$.

Hence $G(dk(\vec{A}); X^\omega)$ is determined.

($\Leftarrow$) Suppose $G(dk(\vec{A}); X^\omega)$ is determined. Then I or II has a winning strategy s for $G(dk(\vec{A}); X^\omega)$. Define a strategy s^* in T to be such that $s^* \upharpoonright X^\omega = s$ and play anything after that. Show s^* is a winning strategy for $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$. Pick an arbitrary $h \in [T]$ according to s^*. Then $h \upharpoonright \omega$ is according to s.

Case 1 : s is a winning strategy for I for $G(dk(\vec{A}); X^\omega)$.

Then $h \upharpoonright \omega \in dk(\vec{A})$.

Subcase 1 : α is even.

Then $h \upharpoonright \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \upharpoonright \omega \notin A_\beta$ is odd. Thus $least_\alpha(h \upharpoonright \omega)$ is odd so that $\Psi_\alpha(h \upharpoonright \omega) = m$ for some odd $m \in \omega$.

$$h \in \left(X^\omega \cap \Psi_\alpha^{-1}(m)\right) \times [T_m] = Tail\,(\Psi_\alpha, m, X^\omega)\,.$$

Since m is odd, $h \in \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$.

Subcase 2 : α is odd.

Then $f \in \bigcap_{\beta\in\alpha} A_\beta$, or $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \restriction \omega \notin A_\beta$ is odd. If $f \in \bigcap_{\beta\in\alpha} A_\beta$, then $least_\alpha(h \restriction \omega) = \alpha$. If $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \restriction \omega \notin A_\beta$ is odd, then by definition, $least_\alpha(h \restriction \omega) = \beta$. In either case, $m = \Psi_\alpha(h \restriction \omega)$ is odd. Thus

$$h \in \left(X^\omega \cap \Psi_\alpha^{-1}(m)\right) \times [T_m] = Tail\,(\Psi_\alpha, m, X^\omega)\,.$$

Since m is odd, $h \in \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$.

Case 2 : s is a winning strategy for II for $G(dk(\vec{A}); X^\omega)$.

Then $h \restriction \omega \notin dk(\vec{A})$.

Subcase 1 : α is even.

Then $f \in \bigcap_{\beta\in\alpha} A_\beta$, or $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \restriction \omega \notin A_\beta$ is even. If $f \in \bigcap_{\beta\in\alpha} A_\beta$, then $least_\alpha(h \restriction \omega) = \alpha$. If $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \restriction \omega \notin A_\beta$ is even, then by definition, $least_\alpha(h \restriction \omega) = \beta$. In either case, $least_\alpha(h \restriction \omega)$ is even. Thus $\Psi_\alpha(h \restriction \omega) = m$ for some even $m \in \omega$. Thus

$$h \in \left(X^\omega \cap \Psi_\alpha^{-1}(m)\right) \times [T_m] = Tail\,(\Psi_\alpha, m, X^\omega)\,.$$

Since m is even, $h \notin \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$.

Subcase 2 : α is odd.

Then $h \restriction \omega \notin \bigcap_{\beta\in\alpha} A_\beta$ and the least β such that $h \restriction \omega \notin A_\beta$ is even. Thus $least_\alpha(h \restriction \omega)$ is even so that $\Psi_\alpha(h \restriction \omega) = m$ for some even $m \in \omega$. Thus

$$h \in \left(X^\omega \cap \Psi_\alpha^{-1}(m)\right) \times [T_m] = Tail\,(\Psi_\alpha, m, X^\omega)\,.$$

Since m is even, $h \notin \bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega)$. □

We obtain the following corollary from theorem 3.4.49.

Corollary 3.4.50. *Suppose $\alpha \in \omega_1$. Suppose T_{sq} satisfies the disjoint tree property. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, T_{sq}, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right)\right) \Rightarrow Det(\alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Proof.

Pick an arbitrary $A \in \alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$. Then there exists $\vec{A} = \langle A_\beta\,|\beta \in \alpha\rangle$ witness that $A = dk(\vec{A}) \in \alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega$. By lemma 3.4.48, $\Psi_\alpha \in \Gamma\,(\omega, \mathbf{\Sigma}^0_1\,(\mathbf{\Pi}^1_1))$. Let $T = T^{\Psi_\alpha, X^\omega}_{X, T_{sq}}$. Then $T \in Tree_2(X, T_{sq}, \Gamma(\omega, \mathbf{\Sigma}^0_1\,(\mathbf{\Pi}^1_1)), X^\omega)$. Since T_{sq} satisfies the disjoint tree property, by proposition 3.4.46, $\bigcup_{odd\ n\in\omega} Tail\,(\Psi_\alpha, n, X^\omega) \in \mathbf{\Sigma}^0_1 \restriction [T]$. Thus

$$G\left(\bigcup_{odd\ n} Tail(\Psi_\alpha, n, X^\omega); T\right) \in \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, T_{sq}, \Gamma\left(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)\right), X^\omega\right)\right).$$

Hence $G(\bigcup_{odd\ n} Tail\,(\Psi_\alpha, n, X^\omega); T)$ is determined. By theorem 3.4.49, $G(dk(\vec{A}); X^\omega)$ is determined. □

In particular, FWF^ω contains T_{sq} which satisfies the disjoint tree property.

Corollary 3.4.51. *Suppose $\alpha \in \omega_1$.*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right)\right) \Rightarrow Det(\alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Proof.

$\{\emptyset, \langle n\rangle\} \in FWF$. Take $T_n = \{\emptyset, \langle n\rangle\}$ for all $n \in \omega$. Then $T_{sq} = \langle T_n | n \in \omega\rangle$ satisfies the disjoint tree property. Since $T_{sq} \in WFW^\omega$, by observation 3.2.7, we have

$$Tree_2\left(X, T_{sq}, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right) \subseteq Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right).$$

Recall notation 3.2.3. We use the fact that $\{T_{sq}\} \subseteq FWF^{\omega}$. Thus we have the result by corollary 3.4.50. $\square$

Recall that in section 3.4.2.3, we obtained corollary 3.4.27:

Corollary 3.4.27. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any complexity Ξ and for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^{\omega}, \{\chi_A \,|A \in \Xi \restriction X^{\omega}\}, X^{\omega}\right)\right) \Rightarrow Det\left(\Xi \restriction X^{\omega}\right).$$

Thus,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^{\omega}, \Gamma(\omega, 2\text{-}\Xi), X^{\omega}\right)\right) \Rightarrow Det(\Xi \restriction X^{\omega}). \quad \dashv$$

In particular, if we take Ξ to be α-$\mathbf{\Pi}_1^1$ for $\alpha \in \omega$, we have

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, \Upsilon^{\omega}, \Gamma(\omega, \alpha\text{-}\mathbf{\Pi}_1^1 \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}_1^1), X^{\omega}\right)\right) \Rightarrow Det(\alpha\text{-}\mathbf{\Pi}_1^1 \restriction X^{\omega}).$$

Question 12. *Fix T_{sq} satisfying the disjoint tree property. Then both of*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, T_{sq}, \Gamma(\omega, \mathbf{\Sigma}_1^0\left(\mathbf{\Pi}_1^1\right)), X^{\omega}\right)\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, T_{sq}, \Gamma(\omega, \alpha\text{-}\mathbf{\Pi}_1^1 \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}_1^1), X^{\omega}\right)\right)$$

implies $Det(\alpha\text{-}\mathbf{\Pi}_1^1 \restriction X^{\omega})$. What is the relationship between $\mathbf{\Sigma}_1^0\left(\mathbf{\Pi}_1^1\right) \restriction X^{\omega}$ and $(\alpha\text{-}\mathbf{\Pi}_1^1 \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}_1^1) \restriction X^{\omega}$? More precisely, for some α,

$$\left(\alpha\text{-}\mathbf{\Pi}_1^1 \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}_1^1\right) \restriction X^{\omega} \subsetneq \mathbf{\Sigma}_1^0\left(\mathbf{\Pi}_1^1\right) \restriction X^{\omega} \ \ or$$

$$\left(\alpha\text{-}\mathbf{\Pi}_1^1 \wedge co\text{-}\alpha\text{-}\mathbf{\Pi}_1^1\right) \restriction X^{\omega} = \mathbf{\Sigma}_1^0\left(\mathbf{\Pi}_1^1\right) \restriction X^{\omega}\ ? \quad \dashv$$

3.5 Getting the determinacy of games on a $Tree_2$ collection from the determinacy of games on $X^{<\omega}$ (Reversed direction of section 3.4)

In section 3.4, we obtained the determinacy of games on $X^{<\omega}$ from the determinacy of games on a certain $Tree_2$ collection. In this section, we will focus on the other direction, in some cases, results from section 3.4, leading to the determinacy equivalences. This section will be the Type 2 tree version of section 2.5.

Throughout this section, we will use the same notation A^n and A^n_g we defined on definition 2.5.2 even if it is defined over a Type 2 tree. (Recall that in section 3.4, we defined notation with "superscript 2" to represent that the set is defined on a Type 2 tree. We will avoid using "superscript 2" to simplify the notation.)

In section 3.5.1 through section 3.5.3, we will obtain level by level results for the determinacy of games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$ (recall notation 1.5.11 of FWF and CWF below). Recall notation 1.5.11 on page 44.

Notation 1.5.11. *Let WF be the set of nonempty well-founded trees. Let $CWF \subseteq WF$ be the set of nonempty well founded trees such that each move is from some countable set. Similarly, let $FWF \subseteq CWF$ be the set of nonempty well-founded trees such that each move is from some finite set.* ⊣

In section 3.5.1, we will give definitions and notations for Type 2 trees which are similar to the one in section 2.5.1 for Type 1 trees. We will set up all the notations in this section;

e.g., suppose $T_{sq} = \langle T_n \, | n \in \omega \rangle \in WF^{\omega}$. Given $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$, we will define the following notations:

- A^n for all $n \in \omega$.
- $A_{\emptyset}$.
- A^n_g for all $n \in \omega$ and $g \in [T_n]$.
- A^n_p for all $n \in \omega$ and $p \in T_n$.

We will use these notations in the later sections.

In section 3.5.2, we will consider open games on a certain $Tree_2$ collection and in section 3.5.3, we will consider Borel games on a certain $Tree_2$ collection. In section 3.5.4, we will consider projective games on a certain $Tree_2$ collection. The proofs for these sections are similar to the one in section 2.5.2, 2.5.3 and 2.5.4 respectively. The key to this direction is that we will take any $T_{sq} = \langle T_n | n \in \omega \rangle$ with each T_n being well-founded and each move is from a finite or a countable set.

The way we obtained the determinacy results in sections 3.5.1 through 3.5.4 are using the fact that each $T^{\Psi,B}_{X,T_{sq}}$ in the $Tree_2$ collection having $T_{sq} = \langle T_n | n \in \omega \rangle \in CWF^{\omega}$, i.e.,

1. each T_n is well-founded,
2. for every position $p \in T_n \backslash [T_n]$, the set of moves at p is countable.

In section 3.5.5, we will observe examples of open games on particular $Tree_2$ collections. In section 3.5.5.1, we will observe particular examples of the case for a $Tree_2$ collection does not satisfy the condition (1). In section 3.5.5.2, we will observe particular examples of the case for a $Tree_2$ collection does not satisfy the condition (2). This section corresponds to section 2.5.4 on page 183 for Type 1 trees.

3.5.1 Getting the determinacy of games on a $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$

Notation 3.5.1. *(Definition of $Tree_2$ collection with/over FWF and CWF)*

Let $\mathcal{T}_2$ be a $Tree_2$ collection. Suppose for every Type 2 tree $T^{\Psi,B}_{X,T_{sq}} \in \mathcal{T}_2$, $T_{sq} \in FWF^\omega$. Then we say $\mathcal{T}_2$ is a "$Tree_2$ collection with/over FWF". Similarly, if for every Type 2 tree $T^{\Psi,B}_{X,T_{sq}} \in \mathcal{T}_2$, $T_{sq} \in CWF^\omega$, then we say $\mathcal{T}_2$ is a "$Tree_2$ collection with/over CWF". ⊣

In sections 3.5.2 through 3.5.4, we will obtain level by level results for the determinacy of games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$. In section 3.5.2, we will obtain the determinacy of open games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$. In section 3.5.3, we will obtain the determinacy of Borel games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$. In section 3.5.4, we will obtain the determinacy of projective games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$. In this section, we will give definitions and some lemmas which we will use throughout sections 3.5.2 through 3.5.4.

In this section, we will discuss results similar to section 2.5.1 by replacing all $Y^{\leq n}$ to $[T_n]$. We will modify the notation in section 2.5.1 for Type 2 trees. The difference between Type 1 trees and Type 2 trees will appear in lemma 2.5.11 and lemma 3.5.11 for the countable tail trees.

For each Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ and $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$, we will find $A^* \subseteq X^\omega$ which will satisfy the following:

$$f \in A^* \text{ if and only if}$$

$$\text{there is a winning strategy at } f \text{ in the Type 2 tree } T^{\Psi,B}_{X,T_{sq}} \text{ for } G(A; T^{\Psi,B}_{X,T_{sq}}).$$

We will describe our A^*. Recall from page 215,

$$[T^{\Psi,B}_{X,T_{sq}}] = \bigcup_{n\in\omega} ((B \cap \Psi^{-1}(n)) \times [T_n])\dot{\cup}(X^\omega \backslash B).$$

We will split A into pairwise disjoint pieces $A_\emptyset$ and A^n for $n \in \omega$. $A_\emptyset$ will be a subset of $X^\omega \backslash B$ and A^n will be a subset of B for each $n \in \omega$. Then we will define A^n_g for each $n \in \omega$ and $g \in [T_n]$ such that if a play f is in A^n_g, then $f^\frown g$ will be in A. Then, by backwards induction, we will define $A^n_\emptyset$ from $\{A^n_g | g \in [T_n]\}$ using unions and intersections. (For the cases that we are interested, it will be the countable unions and intersections.) Whenever a play f of A^* is in $A^n_\emptyset$, there is a canonical strategy at f to get into A. Let $A^* = \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$. We will show that:

- if $f \in A^*$, then I has a winning strategy at f to get into A.
- if $f \notin A^*$, then II has a winning strategy at f to avoid A.

The following is a Type 2 tree version of definition 2.5.2 on page 138.

Definition 3.5.2. *Suppose $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$. For each $n \in \omega$, define*

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times [T_n]),$$

$$A_\emptyset = A \cap (X^\omega \backslash B).$$

Then $A = \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset$. ⊣

In definition 3.5.3, we will define $A^n_g \subseteq X^\omega$ as a collection of $f \in X^\omega$ such that $f^\frown g \in A^n$. The following is a Type 2 tree version of definition 2.5.3 on page 138.

Definition 3.5.3. *Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$ *and assume* $A_\emptyset, A^n$ *for all* $n \in \omega$ *defined in definition 3.5.2. For every* $n \in \omega$ *and* $g \in [T_n]$*, define*

$$A^n_g = \{ f \in X^\omega \,|\, f^\frown g \in A^n \}. \qquad \dashv$$

Since $A^n \subseteq (B \cap \Psi^{-1}(n)) \times [T_n]$, $A^n_g \subseteq B \cap \Psi^{-1}(n)$ for every $g \in [T_n]$.

Recall notation 3.4.18.

Notation 3.4.18. *Suppose for each* $n \in \omega$*,* T_n *is a tree. For each* $n \in \omega$ *and for any* $p \in T_n$*, define*

$$M^n_p = \{ m \,|\, p^\frown \langle m \rangle \in T_n \}. \qquad \dashv$$

Suppose $T_{sq} = \langle T_n \,|\, n \in \omega \rangle \in WF^\omega$. Then each T_n is a nonempty well-founded tree. In definition 3.5.4, by backwards induction, we will define for each $i \in lh(g)$, $A^n_{g \upharpoonright i}$ from $\{A^n_{(g \upharpoonright i)^\frown \langle m \rangle} | m \in M^n_{g \upharpoonright i}\}$. The following is a Type 2 tree version of definition 2.5.4 on page 139.

Definition 3.5.4. *Let* $T_{sq} = \langle T_n \,|\, n \in \omega \rangle \in WF^\omega$*. Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$*. Define*

$$A^n_p \overset{df}{=} \begin{cases} \bigcup_{m \in M^n_p} A^n_{p^\frown \langle m \rangle} & \text{if } lh\,(p) \text{ is even,} \\ \bigcap_{m \in M^n_p} A^n_{p^\frown \langle m \rangle} & \text{if } lh\,(p) \text{ is odd.} \end{cases}$$

Since T_n *is well-founded, each* A^n_p *is well-defined.* $\dashv$

Note that for all $p \in T_n$, we have A^n_p. Definition 3.5.3 applies if $p \in [T_n]$.

The following is a Type 2 tree version of observation 2.5.5 on page 139.

Observation 3.5.5. *Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$ *and* $T_{sq} = \langle T_n \,|\, n \in \omega \rangle \in WF^\omega$*. Then for all* $n \in \omega$ *and for all* $p \in T_n$*,*

$$A^n_p \subseteq B \cap \Psi^{-1}(n). \qquad \dashv$$

Proof.

Suppose $T_{sq} = \langle T_n \,|n \in \omega\rangle \in WF^\omega$. Then for all $n \in \omega$, $T_n \cap [T_n] \neq \emptyset$. Fix $n \in \omega$. Let $p \in T_n$. We prove this by backwards induction on the length of p.

Base case : $p \in [T_n]$.

Then we have $A^n_p = \{f \in X^\omega \,|f^\frown p \in A^n\} \subseteq B \cap \Psi^{-1}(n)$ since $A^n \subseteq (B \cap \Psi^{-1}(n)) \times [T_n]$.

Induction step : As an induction hypothesis, assume that for all $p \in T_n$, if $lh(p) = l+1$, then $A^n_p \subseteq B \cap \Psi^{-1}(n)$. Suppose $lh(p) = l$. Show $A^n_p \subseteq B \cap \Psi^{-1}(n)$. Pick an arbitrary $f \in A^n_p$.

Case 1 : l is even. Then $A^n_p = \bigcup_{m \in M^n_p} A^n_{p^\frown\langle m\rangle}$. Then $f \in A^n_{p^\frown\langle m\rangle}$ for some $m \in M^n_p$. Since $lh(p^\frown\langle m\rangle) = l+1$, by induction hypothesis, we have $A^n_{p^\frown\langle m\rangle} \subseteq B \cap \Psi^{-1}(n)$. Thus $f \in B \cap \Psi^{-1}(n)$. Since $f \in A^n_p$ is arbitrary, $A^n_p \subseteq B \cap \Psi^{-1}(n)$.

Case 2 : l is odd. Then $A^n_p = \bigcap_{m \in M^n_p} A^n_{p^\frown\langle m\rangle}$. Then $f \in A^n_{p^\frown\langle m\rangle}$ for every $m \in M^n_p$. Since $lh(p^\frown\langle m\rangle) = l+1$ for every $m \in M^n_p$, by induction hypothesis, we have $A^n_{p^\frown\langle m\rangle} \subseteq B \cap \Psi^{-1}(n)$ for every $m \in M^n_p$. Thus $f \in B \cap \Psi^{-1}(n)$. Since $f \in A^n_p$ is arbitrary, $A^n_p \subseteq B \cap \Psi^{-1}(n)$. □

For each strategy s^* on $X^{<\omega}$ we define the canonical strategy s on a Type 2 tree T. First, we define the canonical strategy for player I. This is a Type 2 tree version of definition 2.5.6 on page 140.

Definition 3.5.6. *(Definition of the canonical tail strategy s for player I)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. *Let* $\mathcal{S}_I(X^{<\omega})$ *be the set of strategies for* I *on* $X^{<\omega}$ *and let* $\mathcal{S}^2_I(T)$ *be the set of strategies for* I *on* T *(we use the "superscript 2" to represent a Type 2 tree). Define*

$$\varphi^2_I : \mathcal{S}_I\left(X^{<\omega}\right) \to \mathcal{S}^2_I\left(T\right).$$

For each $s^* \in \mathcal{S}_I(X^{<\omega})$*, Define* $s = \varphi^2_I(s^*)$ *as follows: For* $p \in T\backslash[T]$ *such that either* p *is* *finite and $p \in dom(s^*)$, or p is infinite and $lh(p)$ is even,*

$$s(p) = \begin{cases} s^*(p) & \text{if } p \text{ finite,} \\ \mu m \in M_p^{\Psi(p\restriction\omega)} \left(p\restriction\omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}\right)^{16} & \text{if } p\restriction\omega \in B \text{ and } \\ & p\restriction\omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))} = \bigcup_{m\in M_p^{\Psi(p\restriction\omega)}} A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}, \\ \mu m(m \in M_p^{\Psi(p\restriction\omega)}) & \text{otherwise,}^{17} \end{cases}$$

when $M_p^{\Psi(p\restriction\omega)}$ is well-orderable. Then s is a strategy for I for T.

(We define for the case that $M_p^{\Psi(p\restriction\omega)}$ is well-orderable. See footnote (16) for the case that $M_p^{\Psi(p\restriction\omega)}$ is not well-orderable.) ⊣

The following is a Type 2 tree version of lemma 2.5.7 on page 140.

Lemma 3.5.7. *Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$ and $A \subseteq [T]$. Suppose I^* has a winning strategy s^* for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$. Then the canonical tail strategy $s = \varphi^2_I(s^*)$ is a winning strategy for I for $G(A;T)$.* ⊣

Proof.

Pick an arbitrary $h \in [T] = [T^{\Psi,B}_{X,T_{sq}}]$ according to s. Show $h \in A$. Since $h \restriction \omega$ is according to s, $h \restriction \omega$ is according to s^*. Since s^* is a I^*'s winning strategy for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$, $h \restriction \omega \in \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

Case 1 : $h \restriction \omega \in B$.

[16] μ represents "the least". If $M_p^{\Psi(p\restriction\omega)}$ is well-orderable, fix a well-ordering of $M_p^{\Psi(p\restriction\omega)}$. Otherwise, pick any $m \in M_p^{\Psi(p\restriction\omega)}$ such that $p \restriction \omega \in A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}$.

[17] This otherwise case does not occur for plays of interest. If $M_p^{\Psi(p\restriction\omega)}$ is not well-orderable, pick any $m \in M_p^{\Psi(p\restriction\omega)}$.

Then $h \restriction \omega \notin A_\emptyset$. By observation 3.5.5, $h \restriction \omega \notin A_\emptyset^l$ for any $l \neq \Psi(h \restriction \omega)$. Thus $h \restriction \omega \in A_\emptyset^{\Psi(h\restriction\omega)}$. Since h is according to the canonical tail strategy $s = \varphi_I^2(s^*)$ for I, $h \restriction \omega \in A_{\langle h(\omega)\rangle}^{\Psi(h\restriction\omega)}$. Since $lh(h \restriction (\omega+1)) = \omega + 1$, by definition, $A_{\langle h(\omega)\rangle}^{\Psi(h\restriction\omega)} = \bigcap_{m \in M_{\langle h(\omega)\rangle}^{\Psi(h\restriction\omega)}} A^n_{\langle h(\omega)\rangle ^\frown \langle m\rangle}$. Thus for any II's move $m \in M_{\langle h(\omega)\rangle}^{\Psi(h\restriction\omega)}$, $h \restriction \omega \in A_{\langle h(\omega),m\rangle}^{\Psi(h\restriction\omega)}$. In particular, $h \restriction \omega \in A_{\langle h(\omega),h(\omega+1)\rangle}^{\Psi(h\restriction\omega)}$. Repeat this argument. Eventually, we get $h \restriction \omega \in A_{h\restriction[\omega,lh(h))}^{\Psi(h\restriction\omega)}$. Thus $h = (h \restriction \omega)^\frown h \restriction [\omega, lh(h)) \in A^{\Psi(h\restriction\omega)} \subseteq A$.

Case 2 : $h \restriction \omega \notin B$.

By observation 3.5.5, $h \restriction \omega \notin A_\emptyset^n$ for any $n \in \omega$. Thus $h = h \restriction \omega \in A_\emptyset \subseteq A$.

In either case, $h \in A$. Hence the canonical tail strategy $s = \varphi_I(s^*)$ is a winning strategy for I for $G(A;T)$. □

Now, we define the canonical strategy for player II. The following is a Type 2 tree version of definition 2.5.8 on page 141.

Definition 3.5.8. *(Definition of the canonical tail strategy s for player II)*

Fix a Type 2 tree $T = T_{X,T_{sq}}^{\Psi,B}$. Let $\mathcal{S}_{II}(X^{<\omega})$ be the set of strategies for II on $X^{<\omega}$ and let $\mathcal{S}_{II}^2(T)$ be the set of strategies for II on T. Define

$$\varphi_{II}^2 : \mathcal{S}_{II}\left(X^{<\omega}\right) \to \mathcal{S}_{II}^2\left(T\right).$$

For each $s^* \in \mathcal{S}_{II}(X^{<\omega})$, define $s = \varphi_{II}^2(s^*)$ as follows: For $p \in T\backslash[T]$ such that either p is *finite and* $p \in dom(s^*)$, *or* p *is infinite and* $lh(p)$ *is odd,*

$$s(p)=\begin{cases} s^*(p) & \text{if } p \text{ finite,} \\ \mu m \in M_p^{\Psi(p\restriction\omega)}\left(p\restriction\omega \notin A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}\right)^{18} & \text{if } p\restriction\omega\in B \text{ and } \\ & p\restriction\omega\notin A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))} = \bigcap_{m\in M_p^{\Psi(p\restriction\omega)}} A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}, \\ \mu m(m\in M_p^{\Psi(p\restriction\omega)}) & \text{otherwise,}^{19} \end{cases}$$

when $M_p^{\Psi(p\restriction\omega)}$ *is well-orderable. Then* s *is a strategy for* II *for* T.

(We define for the case that $M_p^{\Psi(p\restriction\omega)}$ *is well-orderable. See footnote (18) for the case that* $M_p^{\Psi(p\restriction\omega)}$ *is not well-orderable.)* ⊣

The following is a Type 2 tree version of lemma 2.5.9.

Lemma 3.5.9. *Fix a Type 1 tree* $T = T^{\Psi,B}_{X,T_{sq}}$ *and* $A \subseteq [T]$. *Suppose* s^* *is a* II^**'s winning strategy for* $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$. *Then the canonical tail strategy* $s = \varphi^2_{II}(s^*)$ *is a winning strategy for* II *for* $G(A;T)$. ⊣

Proof.

Pick an arbitrary $h \in [T] = [T^{\Psi,B}_{X,T_{sq}}]$ according to s. Show $h \notin A$. Then $h \restriction \omega$ is according to s^*. Since s^* is a II^*'s winning strategy for $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^{<\omega})$, $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

Case 1 : $h \restriction \omega \in B$.

Since $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_\emptyset$. By definition, $A^{\Psi(h\restriction\omega)}_\emptyset = \bigcup_{m\in M_\emptyset^{\Psi(p\restriction\omega)}} A^{\Psi(h\restriction\omega)}_{\langle m\rangle}$. Thus for any I's move $m \in M_\emptyset^{\Psi(p\restriction\omega)}$, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle m\rangle}$. In particular, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle h(\omega)\rangle}$.

[18] μ represents "the least". If $M_p^{\Psi(p\restriction\omega)}$ is well-orderable, fix a well-ordering of Y. Otherwise, pick any $m \in M_p^{\Psi(p\restriction\omega)}$ such that $p \restriction \omega \notin A^{\Psi(p\restriction\omega)}_{p\restriction[\omega,lh(p))^\frown\langle m\rangle}$.

[19] This otherwise case does not occur for plays of interest. If $M_p^{\Psi(p\restriction\omega)}$ is not well-orderable, pick any $m \in M_p^{\Psi(p\restriction\omega)}$.

By definition, $A^{\Psi(h\restriction\omega)}_{\langle h(\omega)\rangle} = \bigcap_{m\in M^{\Psi(p\restriction\omega)}_{\langle h(\omega)\rangle}} A^n_{\langle h(\omega)\rangle^\frown\langle m\rangle}$. Since h is according to the canonical tail strategy $s = \varphi^2_{II}(s^*)$ for II, $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{\langle h(\omega),h(\omega+1)\rangle}$. Repeat this argument. Eventually, we get $h \restriction \omega \notin A^{\Psi(h\restriction\omega)}_{h\restriction[\omega,lh(h))}$. Thus $h = (h \restriction \omega)^\frown h \restriction [\omega, lh(h)) \notin A^{\Psi(h\restriction\omega)}$. By observation 3.5.5, $h \restriction \omega \notin A^l_\emptyset$ for any $l \neq \Psi(h \restriction \omega)$. Hence $h \notin \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset = A$.

Case 2 : $h \restriction \omega \notin B$.

Since $h \restriction \omega \notin \bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$, $h = h \restriction \omega \notin A_\emptyset$. By observation 3.5.5, $h \restriction \omega \notin A^n_\emptyset$ for any $n \in \omega$. Hence $h \notin \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset = A$.

In either case, $h \notin A$. Hence the canonical tail strategy $s = \varphi^2_{II}(s^*)$ is a winning strategy for II for $G(A;T)$. □

Let $\varphi^2 = \varphi^2_I \dot{\cup} \varphi^2_{II}$. Then φ^2 takes strategies on $X^{<\omega}$ to strategies on $T^{\Psi,B}_{X,T_{sq}}$. By lemmas 3.5.7 and 3.5.9, we have the following. The following is a Type 2 tree version of theorem 2.5.10 on page 143.

Theorem 3.5.10. *If $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined, then $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined.*

⊣

The following is a Type 2 tree version of lemma 2.5.11 on page 143.

Lemma 3.5.11. *Suppose $n, m \in \omega$, $m > 1$ and $\alpha \in \omega_1$.*

1. *If for all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ and $T_n \in FWF$, then $A^n_\emptyset \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$.*

2. *If for all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ and $T_n \in CWF$, then $A^n_\emptyset \in \mathbf{B} \restriction X^\omega$.*

3. *If for all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^1_m \restriction X^\omega$ and $T_n \in CWF$, then $A^n_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$.*

4. *If for all $g \in [T_n]$, $A^n_g \in \mathbf{\Pi}^1_m \restriction X^\omega$ and $T_n \in CWF$, then $A^n_\emptyset \in \mathbf{\Pi}^1_m \restriction X^\omega$.*

5. If for all $g \in [T_n]$, $A^n_g \in \mathbf{\Delta}^1_m \restriction X^\omega$ and $T_n \in CWF$, then $A^n_\emptyset \in \mathbf{\Delta}^1_m \restriction X^\omega$.

6. If Λ is an algebra, for all $g \in [T_n]$, $A^n_g \in \Lambda \restriction X^\omega$ and $T_n \in FWF$, then $A^n_\emptyset \in \Lambda \restriction X^\omega$.

7. If Λ is a σ-algebra, for all $g \in [T_n]$, $A^n_g \in \Lambda \restriction X^\omega$ and $T_n \in CWF$, then $A^n_\emptyset \in \Lambda \restriction X^\omega$.

$\dashv$

Proof.

Fix $n \in \omega$. Notice that $FWF \subseteq WF$ and $CWF \subseteq WF$ so that for all (1)-(7), T_n is well-founded. Thus for each $n \in \omega$, there is a rank function of T_n defined in definition 1.2.6 on page 9.

Definition 1.2.6. *(Definition of the rank of a well-founded tree)*

Suppose T is a well-founded tree. Then $[T] \subseteq T$. Define the rank of T recursively.

$$rank_T : T \to \omega$$
$$p \mapsto \begin{cases} 0 & \text{if } p \in [T]\,, \\ \sup\{rank_T(p^\frown\langle k\rangle) + 1 \mid p^\frown\langle k\rangle \in T\} & \text{if } p \in T\setminus[T]\,. \end{cases}$$

$\dashv$

Recall definition 3.5.4.

Definition 3.5.4. *Let $T_{sq} = \langle T_n \,|n \in \omega\rangle \in WF^\omega$. Suppose $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$. Define*

$$A^n_p \overset{df}{=} \begin{cases} \bigcup_{m \in M^n_p} A^n_{p^\frown\langle m\rangle} & \text{if } lh(p) \text{ is even,} \\ \bigcap_{m \in M^n_p} A^n_{p^\frown\langle m\rangle} & \text{if } lh(p) \text{ is odd.} \end{cases}$$

Since T_n is well-founded, each A^n_p is well-defined. $\dashv$

Fix $n \in \omega$. Show (1). Assume all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$.

Show that for any $p \in T_n$, $A^n_p \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ by backwards induction on the rank of T_n.

Base Case : $rank\,(p) = 0$.

Then $p \in [T_n]$. Thus $A^n_p \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$.

Induction Step : Assume, as an induction hypothesis, $\forall q \in T_n$, if $rank\,(q) < rank\,(p)$ then $A^n_q \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$.

Show that $A^n_p \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$. Note that $\forall k \in M^n_p$, $rank\,(p^\frown\langle k\rangle) < rank\,(p)$. Thus, by induction hypothesis, $\forall k \in M^n_p$, $A^n_{p^\frown\langle k\rangle} \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$. Since $T_n \in FWF$, for each $p \in T_n$, M^n_p is finite.

Case 1 : $lh\,(p)$ is even.

Then $A^n_p = \bigcup_{k \in M^n_p} A^n_{p^\frown\langle k\rangle} \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ since $\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ is closed under finite unions.

Case 2 : $lh\,(p)$ is odd.

Then $A^n_p = \bigcap_{k \in M^n_p} A^n_{p^\frown\langle k\rangle} \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ since $\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ is closed under finite intersections.

In particular, when $k = 0$, $A^n_\emptyset \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$.

Show (2). Assume all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$. Since $T_n \in CWF$, for each $p \in T_n$, M^n_p is countable. By the similar argument as above (replace $\mathbf{\Sigma}^0_\alpha$ to $\mathbf{B}$ and finite to countable), we have $A^n_\emptyset \in \mathbf{B} \upharpoonright X^\omega$.

Show (3). Assume all $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$. Since $T_n \in CWF$, for each $p \in T_n$, M^n_p is countable. Since $\mathbf{\Sigma}^1_m \upharpoonright X^\omega$ is closed under countable unions and countable intersections, by the similar argument as above (replace $\mathbf{\Sigma}^0_\alpha$ to $\mathbf{\Sigma}^1_m$ and finite to countable), we have $A^n_\emptyset \in \mathbf{\Sigma}^1_m \upharpoonright X^\omega$. Similarly for (4), the case for $\mathbf{\Pi}^1_m$ and (5), the case for $\mathbf{\Delta}^1_m$.

Show (6). Suppose Λ is an algebra and each $A^n_g \in \Lambda \upharpoonright X^\omega$. Since $T_n \in FWF$, for each $p \in T_n$, M^n_p is finite. Since Λ is closed under countable unions and countable intersections, by the similar argument as above (replace $\mathbf{\Sigma}^0_\alpha$ to Λ), we have $A^n_\emptyset \in \Lambda \upharpoonright X^\omega$.

Show (7). Suppose Λ is a σ-algebra and each $A_g^n \in \Lambda \upharpoonright X^\omega$. Since $T_n \in CWF$, for each $p \in T_n$, M_p^n is countable. Since Λ is closed under countable unions and countable intersections, by the similar argument as above (replace $\mathbf{\Sigma}_\alpha^0$ to Λ and finite to countable), we have $A_\emptyset^n \in \Lambda \upharpoonright X^\omega$. □

Next, we will find the complexity of $A_\emptyset$ and A_g^n for each $g \in T_n$. This is a Type 2 tree version of lemma 2.5.12. The proof is similar to the proof of lemma 2.5.12.

Lemma 3.5.12. *Suppose Λ_0 and Λ_1 are complexities. Let $\Psi \in \Gamma(\omega, \Lambda_0), B \in \Lambda_1 \upharpoonright X^\omega$, $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$ and $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,Tsq}^{\Psi,B}]$. Then for every $n \in \omega$ and $g \in [T_n]$,*

$$A_g^n \in (\mathbf{\Sigma}_1^0 \wedge \Lambda_0 \wedge \Lambda_1) \upharpoonright X^\omega \text{ and } A_\emptyset \in (\mathbf{\Sigma}_1^0 \wedge co\text{-}\Lambda_1) \upharpoonright X^\omega. \quad \dashv$$

Proof.

Pick arbitrary $n \in \omega$ and $g \in [T_n]$. Then $g \neq \emptyset$. Since $A \in \mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}}^{\Psi,B}]$,

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times [T_n]) \in \mathbf{\Sigma}_1^0 \upharpoonright (B \cap \Psi^{-1}(n)) \times [T_n].$$

Thus there exists $\langle O_i \,|i \in \omega\rangle$ such that $A^n = \bigcup_{i\in\omega} O_i$ where each O_i is a basic open neighborhood of $(B \cap \Psi^{-1}(n)) \times [T_n]$, i.e., there exists $p_i \in X^{<\omega}$ and $q_i \in [T_n]$ such that

$$O_i = \{h \in (B \cap \Psi^{-1}(n)) \times [T_n] \,|\, h \upharpoonright \omega \supseteq p_i \wedge h \upharpoonright [\omega, lh(h)) \supseteq q_i\}.$$

Since each $T_n \in CWF$, each tail has finite length and for each $p \in T_n$, E_p^n is countable. Thus there are countably many tails. Hence each O_i can be written as $\bigcup_{j\in\omega} \hat{O}_{i,j}$ where each $\hat{O}_{i,j} = \{h \in (B \cap \Psi^{-1}(n)) \times [T_n] \,|\, h \upharpoonright \omega \supseteq \hat{p}_{i,j} = p_i \wedge h \upharpoonright [\omega, lh(h)) = \hat{q}_{i,j}\}$ for some $\hat{q}_{i,j} \in [T_n]$. Then $O = \bigcup_{i\in\omega} O_i = \bigcup_{i\in\omega}\bigcup_{j\in\omega} \hat{O}_{i,j} = \bigcup_{k\in\omega} \hat{O}_k$ where $\hat{O}_k$'s enumerate $\hat{O}_{i,j}$'s, etc. $\hat{O}_k = \{h \in (B \cap \Psi^{-1}(n)) \times [T_n] \,|\, h \upharpoonright \omega \supseteq \hat{p}_k \wedge h \upharpoonright [\omega, lh(h)) = \hat{q}_k\}$. Note that $\hat{O}_k$ may not be open.

Define $G = \{k \in \omega \,|\hat{q}_k = g\}$.

Then

$$\begin{aligned} A_g^n &\stackrel{df}{=} \{f \in X^\omega \,| f^\frown g \in A^n\} \\ &= \left\{ f \in X^\omega \,\middle|\, f^\frown g \in \bigcup_{k\in\omega} \hat{O}_k \right\} \\ &= \bigcup_{k\in\omega} \left\{ f \in X^\omega \,\middle|\, f^\frown g \in \hat{O}_k \right\} \\ &= \bigcup_{k\in G} \left\{ f \in X^\omega \,\middle|\, f^\frown g \in \hat{O}_k \right\} \\ &= \underbrace{\bigcup_{k\in G} \{f \in X^\omega \,| f \supseteq \hat{p}_k\}}_{\mathbf{\Sigma}_1^0 \restriction X^\omega} \cap \underbrace{\Psi^{-1}(n)}_{\Lambda_0 \restriction X^\omega} \cap \underbrace{B}_{\Lambda_1 \restriction X^\omega} \\ &\in \left(\mathbf{\Sigma}_1^0 \wedge \Lambda_0 \wedge \Lambda_1\right) \restriction X^\omega \end{aligned}$$

Now, we consider $A_\emptyset$. Define $J = \{k \in \omega \,|\hat{q}_k = \emptyset\}$. Then for all $f \in X^\omega$,

$$f \in A_\emptyset \Leftrightarrow f \in \underbrace{(X^\omega \backslash B)}_{co\text{-}\Lambda_1 \restriction X^\omega} \cap A \Leftrightarrow f \in \underbrace{X^\omega \backslash B}_{co\text{-}\Lambda_1 \restriction X^\omega} \wedge \underbrace{\exists k \in J\,(f \supseteq \hat{p}_k)}_{\mathbf{\Sigma}_1^0 \restriction X^\omega}.$$

Thus $A_\emptyset \in (\mathbf{\Sigma}_1^0 \wedge co\text{-}\Lambda_1) \restriction X^\omega$. □

By lemmas 3.5.12 and 3.5.11, we obtain the complexity of $A_\emptyset$ and A_g^n for all $n \in \omega$ and $g \in [T_n]$ from the complexity of B and Ψ. In the next section, we will obtain the determinacy of open games on $Tree_2$ collections from the determinacy of games on $X^{<\omega}$ by using theorem 3.5.10 lemma 3.5.11 and lemma 3.5.12.

3.5.2 Obtaining the open determinacy on $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$

In section 3.5.1, we defined notations and prove some lemmas. In this section, we will obtain open determinacy on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$ by using theorem 3.5.10, lemma 3.5.11 and lemma 3.5.12. The proof of this section is similar to the one in section 2.5.2. The difference between Type 1 trees and Type 2 trees will appear in proofs of lemma 2.5.12 and lemma 3.5.12 under the discussion about the basic open neighborhood.

The main theorems of this section are theorem 3.5.13 and theorem 3.5.15. The following is a $Tree_2$ version of theorem 2.5.18 on page 156.

Theorem 3.5.13. *Suppose $\beta, \gamma \in \omega_1$. If $\beta, \gamma > 1$, then*

$$Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{3.9}$$

If $\beta < \gamma$,

$$Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) & (3.10) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) & (3.11) \end{cases}$$

If $\beta \geq \gamma$,

$$Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) & (3.12) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) & (3.13) \end{cases}$$

Also,

$$Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega\right) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right). \tag{3.14}$$

⊣

The implications (3.10) through (3.13) state that we set

$$\mathcal{T}_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \left(\mathbf{\Sigma}^0_\beta \cup \mathbf{\Pi}^0_\beta\right) \restriction X^\omega\right),$$

then $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_2\right)$ follows from

$$\begin{cases} Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right) & \text{when } \beta < \gamma, \\ Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \restriction X^\omega\right) & \text{when } \beta \geq \gamma. \end{cases}$$

Proof.

Show the implication (3.9). Fix $\beta, \gamma \in \omega_1$ greater than 1. Pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,| n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. Assume $Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega)$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Delta}^0_\beta \restriction X^\omega$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_\emptyset \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right).$$

The proofs for the implications (3.10) through (3.14) are similar. Fix a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ in the appropriate $Tree_2$ collection. We only need to check the complexity of $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset$.

For the implication (3.10), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,| n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Delta}^0_\gamma \restriction X^\omega$

and $A_\emptyset \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$.

For the implication (3.11), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^0_\gamma \upharpoonright X^\omega$.

For the implication (3.12), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_\emptyset \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in (\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta) \upharpoonright X^\omega$.

For the implication (3.13), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_\emptyset \in \mathbf{\Pi}^0_\beta \upharpoonright X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in (\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta) \upharpoonright X^\omega$.

For the implication (3.14), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^{\omega}\right)$$

. Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_1\right)$, $B \in \mathbf{\Delta}^0_1 \restriction X^{\omega}$ and $T_{sq} = \langle T_n \,| n \in \omega\rangle \in FWF^{\omega}$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$,for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Sigma}^0_1 \restriction X^{\omega}$ and $A_{\emptyset} \in \mathbf{\Sigma}^0_1 \restriction X^{\omega}$. Since for any $p \in T_n$ each M^n_p is finite, each $A^n_{\emptyset} \in \mathbf{\Sigma}^0_1 \restriction X^{\omega}$. Thus $\bigcup_{n\in\omega} A^n_{\emptyset} \cup A_{\emptyset} \in \mathbf{\Sigma}^0_1 \restriction X^{\omega}$. □

Combining corollary 3.4.33 on page 264 and theorem 3.5.13, we have the following. The following is a $Tree_2$ version of corollary 2.5.19 on page 159.

Corollary 3.5.14. *Suppose $\beta, \gamma \in \omega_1$. Then for any $\beta \geq \gamma$,*

$$\left.\begin{aligned} &① \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^{\omega}, \Gamma\left(\omega, \mathbf{\Sigma}^0_{\beta} \wedge \mathbf{\Pi}^0_{\beta}\right), \mathbf{\Pi}^0_{\beta} \restriction X^{\omega}\right)\right) \\ &② \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^{\omega}, \Gamma\left(\omega, \mathbf{\Sigma}^0_{\beta} \wedge \mathbf{\Pi}^0_{\beta}\right), \mathbf{\Sigma}^0_{\beta} \restriction X^{\omega}\right)\right) \end{aligned}\right\}$$

$$\Rightarrow ③ \ Det((\mathbf{\Sigma}^0_{\beta} \wedge \mathbf{\Pi}^0_{\beta}) \restriction X^{\omega})$$

$$\Leftrightarrow ④ \ Det\left(\left(\mathbf{\Sigma}^0_{\beta} \vee \mathbf{\Pi}^0_{\beta}\right) \restriction X^{\omega}\right)$$

$$\Rightarrow \left\{\begin{aligned} &⑤ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Delta}^0_{\gamma}), \mathbf{\Sigma}^0_{\beta} \restriction X^{\omega}\right)\right). \\ &⑥ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Delta}^0_{\gamma}), \mathbf{\Pi}^0_{\beta} \restriction X^{\omega}\right)\right). \end{aligned}\right.$$

That is : ① implies ③, ② implies ③, ③ if and only if ④ and ④ implies both ⑤ and ⑥. ⊣

So far, we focused on getting the determinacy on $Tree_2$ collections such that each Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ in the $Tree_2$ collection satisfying $T_{sq} \in FWF^{\omega}$. Now, we consider $Tree_2$ collections over CWF. The following is a $Tree_2$ version of theorem 2.5.20 on page 160.

Theorem 3.5.15. *Suppose $\beta, \lambda \in \omega_1$. Then*

$$Det\,(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}(\mathbf{\Sigma}^0_1; Tree_2(X, CWF^\omega, \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), \mathbf{\Sigma}^0_\beta \restriction X^\omega)). \qquad \dashv$$

Proof.

Fix $\beta, \gamma \in \omega_1$. Assume $Det(\mathbf{B} \restriction X^\omega)$. Pick arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2(X, CWF^\omega, \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), \mathbf{\Sigma}^0_\beta \restriction X^\omega).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,| n \in \omega\rangle \in CWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By Lemma 3.5.12, for all $n \in \omega$ and $g \in [T_n]$, $A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^0_1 \wedge \mathbf{\Pi}^0_\beta \restriction X^\omega$. Since $T_{sq} \in CWF^\omega$, each $A^n_\emptyset \in \mathbf{B} \restriction X^\omega$ by lemma 3.5.11. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{B} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,Y})$ is determined. Therefore,

$$Det\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \qquad \square$$

Since β and γ are arbitrary, we have the following. The following is a $Tree_2$ version of corollary 2.5.25 on page 163.

Corollary 3.5.16. *(Corollary to Theorem 3.5.15)*

$$Det\,(\mathbf{B} \restriction X^\omega) \Rightarrow Det\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right). \qquad \dashv$$

The following is a $Tree_2$ version of corollary 2.5.23 on page 162.

Corollary 3.5.17. *Suppose Λ an algebra. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

Pick arbitrary Type 2 tree $T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)$. Then $\Psi \in \Gamma(\omega, \Lambda)$,

$B \in \Lambda \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and for all $g \in [T_n]$, each $A^n_g \in \Lambda \restriction X^\omega$ and since Λ is closed under complement, $A_\emptyset \in \Lambda \restriction X^\omega$. Since Λ is closed under finite unions and intersections, each $A^n_\emptyset \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. □

The following is a $Tree_2$ version of corollary 2.5.24 on page 162.

Corollary 3.5.18. *Suppose* Λ *is* σ*-algebra. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, CWF^\omega, \Gamma(\omega,\Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

Pick arbitrary $T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega,\Lambda), \Lambda \restriction X^\omega\right)$. Then $\Psi \in \Gamma(\omega,\Lambda)$, $B \in \Lambda \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}_1^0 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.12, for all $n \in \omega$ and for all $g \in [T_n]$, each $A^n_g \in \Lambda \restriction X^\omega$ and $A_\emptyset \in \Lambda \restriction X^\omega$. Since Λ is a σ-algebra, each $A^n_\emptyset \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. □

So far, we focused on getting the determinacy on $Tree_2$ collections such that each Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ in the $Tree_2$ collection satisfying $T_{sq} \in CWF^\omega$, Ψ is a Borel function and B is a Borel set. Now, we we focus on getting the determinacy on a $Tree_2$ collection such that each Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ in the $Tree_2$ collection satisfying $T_{sq} \in CWF^\omega$, Ψ is a projective function and B is a projective set. The following is a $Tree_2$ version of theorem 2.5.26 on page 163.

Theorem 3.5.19. *Suppose $m, n \in \omega\backslash\{0\}$.*

$$Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right)\right). \tag{3.15}$$

If $n < m$,

$$Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (3.16) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (3.17) \end{cases}$$

If $n \geq m$,

$$Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (3.18) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (3.19) \end{cases}$$

$\dashv$

The implications (3.16) through (3.19) state that we set

$$\mathcal{T}_2 = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \left(\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n\right) \restriction X^\omega\right),$$

then $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_2\right)$ follows from

$$\begin{cases} Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) & \text{when } n < m, \\ Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) & \text{when } n \geq m. \end{cases}$$

Proof.

Show the implication (3.15). Fix $n, m \in \omega_1$ greater than 1. Pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right).$$

Then $T_{sq} \in CWF^\omega$, $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^1_m\right)$ and $B \in \mathbf{\Delta}^1_n \restriction X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]$. Assume $Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega)$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.12, for each $i \in \omega$ and $g \in [T_i]$, $A^i_g \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Delta}^1_n \restriction X^\omega$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.11, each $A^i_\emptyset \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega$. Hence $G(\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore, we have

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \upharpoonright X^\omega\right)\right).$$

The proofs for the implications (3.16) through (3.19) are similar. Fix a Type 1 tree $T^{\Psi,B}_{X,T_{sq}}$ in the appropriate $Tree_2$ collection. We only need to check the complexity of $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset$.

For the implication (3.16), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \upharpoonright X^\omega\right).$$

Then $T_{sq} \in CWF^\omega$, $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_m)$ and $B \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.12, for each $i \in \omega$ and $g \in [T_i]$, $A^i_g \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.11, each $A^i_\emptyset \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$.

For the implication (3.17), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \upharpoonright X^\omega\right).$$

Then $T_{sq} \in CWF^\omega$, $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_m)$ and $B \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.12, for each $i \in \omega$ and $g \in [T_i]$, $A^i_g \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.11, each $A^i_\emptyset \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in \mathbf{\Delta}^1_m \upharpoonright X^\omega$.

For the implication (3.18), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \upharpoonright X^\omega\right).$$

Then $T_{sq} \in CWF^\omega$, $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_m)$ and $B \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.12, for each $i \in \omega$ and $g \in [T_i]$, $A^i_g \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$

and $A_\emptyset \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.11, each $A^i_\emptyset \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in (\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n) \upharpoonright X^\omega$.

For the implication (3.19), pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \upharpoonright X^\omega\right).$$

Then $T_{sq} \in CWF^\omega$, $\Psi \in \Gamma(\omega, \mathbf{\Delta}^1_m)$ and $B \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.12, for each $i \in \omega$ and $g \in [T_i]$, $A^i_g \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^1_n \upharpoonright X^\omega$. Since $T_{sq} \in CWF^\omega$, by lemma 3.5.11, each $A^i_\emptyset \in \mathbf{\Pi}^1_n \upharpoonright X^\omega$. Thus $\bigcup_{i\in\omega} A^i_\emptyset \cup A_\emptyset \in (\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n) \upharpoonright X^\omega$. □

The following is a $Tree_2$ version of corollary 3.5.20 on page 308.

Corollary 3.5.20. *(Corollary to Theorem 3.5.19)*

$Det\,(2\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_1), (\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1) \upharpoonright X^\omega))$. ⊣

By combining corollary 3.4.35 on page 265 and corollary 3.5.20, we have the following. The following is a $Tree_2$ version of corollary 2.5.28 on page 166.

Corollary 3.5.21. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$\left.\begin{array}{l} ①\ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \upharpoonright X^\omega)) \\ ②\ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \upharpoonright X^\omega)) \end{array}\right\}$$

$$\Rightarrow ③\ Det\left(2\text{-}\mathbf{\Pi}^1_1 \upharpoonright X^\omega\right)$$

$$\Rightarrow ④\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_1), \left(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1\right) \upharpoonright X^\omega\right)\right).$$

That is : ① implies ③, ② implies ③, and ③ implies ④. ⊣

Question 13. *With respect to corollary 3.5.21, does ④ imply ① or ②?* ⊣

3.5.3 Obtaining the determinacy of Borel games on a $Tree_2$ collection with FWF and CWF from the determinacy of Borel games on $X^{<\omega}$

In section 3.5.2, we focused on obtaining the determinacy of open games on a certain $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$. In this section, as a general case of open games on a $Tree_2$ collection, we will consider games which are more higher complexity. The idea is similar as in section 2.5.3. The difference between Type 1 trees and Type 2 trees will appear in lemma 2.5.32 and lemma 3.5.25 under the discussion about the basic open neighborhood.

The main theorems in this section are theorems 3.5.22 and 3.5.23. We will obtain level by level results for the determinacy of Borel games on a $Tree_2$ collection with FWF and CWF from the determinacy of games on $X^{<\omega}$.

The following is a $Tree_2$ version of theorem 2.5.29 on page 167.

Theorem 3.5.22. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \quad (3.20)$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \quad (3.21)$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \quad (3.22)$$

⊣

The implications (3.21) and (3.22) states that when we set

$$\mathcal{T}_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right),$$

$Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2)$ follows from

$$\begin{cases} Det\left(\mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega\right) & \text{when } \beta < \gamma, \\ Det\left(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega\right) & \text{when } \beta \geq \gamma. \end{cases}$$

We will prove this theorem on page 319.

The following is a $Tree_2$ version of theorem 2.5.30 on page 168.

Theorem 3.5.23. *Suppose $\alpha, \beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{B} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right). \qquad \dashv$$

We will prove this theorem on page 320.

The idea of the proofs are similar as in section 3.5.2. We will use the same definition of A_n, A^n_g and $A_\emptyset$ from section 3.5.1. We will find the complexity of each A^n_g and $A_\emptyset$ in lemma 3.5.26. Then, by using lemma 3.5.11 and theorem 3.5.10, we will obtain the determinacy results in theorems 3.5.22 and 3.5.23. To obtain the complexity of each A^n_g and $A_\emptyset$, we will define a function Fix from X^ω into $[T^{\Psi,B}_{X,T_{sq}}]$ and find the complexity of Fix in lemma 3.5.25. This Fix will be the key to find the complexity of A^n_g and $A_\emptyset$. For each $g \in [T_n]$, we will collect all of $f \in X^\omega$ such that $f^\frown g \in [T^{\Psi,B}_{X,T_{sq}}]$ by using Fix. Fix will be the identity map for any $f \in X^\omega \backslash B$ and if $f \in B$, then it will fix the tail.

The following is a Type 2 tree version of definition 2.5.31 on page 169.

Definition 3.5.24. *(Definition of "Fix_2")*

For all $m \in \omega$*, fix* $a_m \in [T_m]$*. Define*

$$Fix_2 \langle a_m : m \in \omega \rangle : \quad X^\omega \to [T^{\Psi,B}_{X,T_{sq}}]$$
$$f \mapsto \begin{cases} f & \text{if } f \notin B, \\ f^\frown a_{\Psi(f)} & \text{otherwise.} \end{cases}$$

If $\langle a_m : m \in \omega \rangle$ *is clear from the context, we will denote* Fix_2 *to mean* $Fix_2 \langle a_m : m \in \omega \rangle$*.* $\dashv$

We will compute the complexity of Fix_2. The following is a Type 2 tree version of lemma 2.5.32 on page 169.

Lemma 3.5.25. *(Finding the complexity of* Fix_2*)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$ *such that* $T_{sq} = \langle T_n \,|n \in \omega \rangle \in CWF^\omega$*. For all* $m \in \omega$*, fix* $a_m \in [T_m]$*. Suppose* $\alpha, \beta \in \omega_1$*, for all* $n \in \omega$*,* $\gamma_n \in \omega_1$

1. *Suppose:*

   - $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$*,*
   - *for all* $n \in \omega$*,* $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$*,*

   then $Fix_2 \in \Gamma([T], \mathbf{\Delta}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}})$*.* [20]

2. *Suppose:*

   - *for all* $n \in \omega$*,* $\beta \geq \gamma_n$*,*
   - $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$*,*
   - *for all* $n \in \omega$*,* $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n}$*,*

   then $Fix_2 \in \Gamma([T^{\Psi,B}_{X,T_{sq}}], \mathbf{\Sigma}^0_{\beta+1})$*.*

[20] Recall notation 1.5.8 for $\Gamma([T], \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}})$.

3. Suppose:

- *there exists* $n \in \omega$ *such that* $\gamma_n > \beta$,
- $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$,
- *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n}$,

then $Fix_2 \in \Gamma([T^{\Psi,B}_{X,T_{sq}}], \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n})$.

4. Suppose Λ *is* σ*-algebra and:*

- $B \in \Lambda \upharpoonright X^\omega$,
- $\Psi \in \Gamma(\omega, \Lambda)$,

then $Fix_2 \in \Gamma([T^{\Psi,B}_{X,T_{sq}}], \Lambda)$. $\dashv$

Proof.

Pick an arbitrary $O \in \mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,B}_{X,T_{sq}}]$. Then there exists $\langle O_i \,|i \in \omega\rangle$ such that $O = \bigcup_{i\in\omega} O_i$ where each O_i is a basic open neighborhood of $[T^{\Psi,B}_{X,T_{sq}}]$, i.e., there exists $p_i \in X^{<\omega}$ and $q_i \in [T_n]$ for some $n \in \omega$ or $q_i = \emptyset$ such that

$$O_i = \left\{h \in [T^{\Psi,B}_{X,T_{sq}}] \,|h \upharpoonright \omega \supseteq p_i \wedge h \upharpoonright [\omega, lh\,(h)) \supseteq q_i\right\}.$$

Since $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$, for every $n \in \omega$, each $p \in [T_n]$ has finite length and for each $p \in T_n$, M^n_p is countable, there are countably many tails. Thus each O_i can be written as $\bigcup_{j\in\omega} \hat{O}_{i,j}$ where each

$$\hat{O}_{i,j} = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \upharpoonright \omega \supseteq \hat{p}_{i,j} = p_i \wedge h \upharpoonright [\omega, lh(h)) = \hat{q}_{i,j}\}$$

for some $\hat{q}_{i,j} \in Y^{<\omega}$. Then $O = \bigcup_{i\in\omega} O_i = \bigcup_{i\in\omega}\bigcup_{j\in\omega} \hat{O}_{i,j} = \bigcup_{k\in\omega} \hat{O}_k$ where $\hat{O}_k$'s enumerate $\hat{O}_{i,j}$'s, etc. $\hat{O}_k = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \upharpoonright \omega \supseteq \hat{p}_k \wedge h \upharpoonright [\omega, lh(h)) = \hat{q}_k\}$.

Case 1 : $\hat{q}_k = \emptyset$.

$$Fix_2^{-1}(\hat{O}_k) = \underbrace{\{h \in X^\omega \,|h \upharpoonright \omega \supseteq \hat{p}_k\}}_{\mathbf{\Sigma}_1^0 \upharpoonright X^\omega} \cap (X^\omega \backslash B).$$

If $B \in \mathbf{\Delta}_\beta^0 \upharpoonright X^\omega$, $Fix_2^{-1}(\hat{O}_k) \in \left(\mathbf{\Sigma}_1^0 \wedge \mathbf{\Delta}_\beta^0\right) \upharpoonright X^\omega$.

If $B \in \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega$, $Fix_2^{-1}(\hat{O}_k) \in (\mathbf{\Sigma}_1^0 \wedge \mathbf{\Pi}_\beta^0) \upharpoonright X^\omega$.

Case 2 : $\hat{q}_k = a_{l_k}$ for some $l_k \in \omega$.

$$Fix_2^{-1}(\hat{O}_k) = \underbrace{\{h \in X^\omega \,|h \upharpoonright \omega \supseteq \hat{p}_k\}}_{\mathbf{\Sigma}_1^0 \upharpoonright X^\omega} \cap \underbrace{\Psi^{-1}(l_k)}_{\mathbf{\Delta}_{\gamma_{l_k}}^0 \upharpoonright X^\omega} \cap \underbrace{B}_{\mathbf{\Delta}_\beta^0 \upharpoonright X^\omega \text{ or } \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega}.$$

If $B \in \mathbf{\Delta}_\beta^0 \upharpoonright X^\omega$ and $\Psi^{-1}(l_k) \in \mathbf{\Delta}_{\gamma_{l_k}}^0$, $Fix_2^{-1}(\hat{O}_k) \in \left(\mathbf{\Sigma}_1^0 \wedge \mathbf{\Delta}_{\max\left\{\beta, \gamma_{l_k}\right\}}^0\right) \upharpoonright X^\omega$.

If $B \in \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega$ and $\Psi^{-1}(l_k) \in \mathbf{\Delta}_{\gamma_{l_k}}^0$, $Fix_2^{-1}(\hat{O}_k) \in \mathbf{\Sigma}_{\max\left\{\beta, \gamma_{l_k}\right\}}^0 \upharpoonright X^\omega$.

If Λ is σ-algebra, $B \in \Lambda \upharpoonright X^\omega$ and $\Psi \in \Gamma(\omega, \Lambda)$, then $Fix^{-1}(\hat{O}_k) \in \Lambda \upharpoonright X^\omega$.

Case 3 : $\hat{q}_k \neq \emptyset$ and $\hat{q}_k \neq a_l$ for any l.

$$Fix_2^{-1}(\hat{O}_k) = \emptyset.$$

Show (1). Suppose $B \in \mathbf{\Delta}_\beta^0 \upharpoonright X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}_{\gamma_n}^0$. Then

$$Fix_2^{-1}(O) = \bigcup_{k \in \omega} \underbrace{Fix_2^{-1}(\hat{O}_k)}_{\left(\mathbf{\Sigma}_1^0 \wedge \mathbf{\Delta}_{\max\left\{\beta, \gamma_{l_k}\right\}}^0\right) \upharpoonright X^\omega} \in \mathbf{\Sigma}_{\max\{\beta, \sup_{k \in \omega} \gamma_{l_k}\}}^0 \upharpoonright X^\omega \subseteq \mathbf{\Sigma}_{\max\{\beta, \sup_{n \in \omega} \gamma_n\}}^0 \upharpoonright X^\omega.$$

Since $O \in \mathbf{\Sigma}_1^0 \upharpoonright [T]$ is arbitrary, Fix_2 is $\mathbf{\Sigma}_{\max\{\beta, \sup_{n \in \omega} \gamma_n\}}^0$-measurable.

Show (2). Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}_{\gamma_n}^0$. Then $Fix_2^{-1}(O) \in \mathbf{\Sigma}_{\beta+1}^0 \upharpoonright X^\omega$. Since $O \in \mathbf{\Sigma}_1^0 \upharpoonright [T]$ is arbitrary, Fix_2 is $\mathbf{\Sigma}_{\beta+1}^0$-measurable.

Show(3). Suppose there exists $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega$ and for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}_{\gamma_n}^0$. Then $Fix_2^{-1}(O) \in \mathbf{\Sigma}_{\sup_{n \in \omega} \gamma_n}^0 \upharpoonright X^\omega$. Since $O \in \mathbf{\Sigma}_1^0 \upharpoonright [T]$ is arbitrary, Fix_2 is

$\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable.

Show (4). Suppose Λ is σ-algebra, $B \in \Lambda \restriction X^\omega$ and $\Psi \in \Gamma(\omega, \Lambda)$. Then $Fix_2^{-1}(\hat{O}_k) \in \Lambda$ for any $k \in \omega$ and thus $Fix_2^{-1}(O) \in \Lambda$. Since $O \in \mathbf{\Sigma}^0_1 \restriction [T]$ is arbitrary, Fix_2 is Λ-measurable. $\square$

Using the complexity of Fix_2 computed in lemma 3.5.25, we find the complexity of A^n_g and $A_\emptyset$. In the proof of 3.5.26, we use sublemma 2.5.34 on page 176. The following is a Type 2 tree version of lemma 2.5.33 on page 172.

Lemma 3.5.26. *(Finding the complexity of A^n_g and $A_\emptyset$)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,Y}$ such that $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$. Suppose $\alpha, \beta \in \omega_1$, $\alpha > 1$, for all $n \in \omega$, $\gamma_n \in \omega_1$.

1. *Suppose:*
   - $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$,
   - *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,
   - $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$,

   then $A_\emptyset, A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta,\sup_{n\in\omega}\gamma_n\}+\alpha}$ *for any* $n \in \omega$ *and for any* $g \in [T_n]$.

2. *Suppose:*
   - *for all* $n \in \omega$, $\beta \geq \gamma_n$,
   - $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$,
   - *for all* $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$,
   - $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$,

then $A_\emptyset$*,* $A_g^n \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in [T_n]$*.*

3. *Suppose:*

   - *there exists* $n \in \omega$ *such that* $\gamma_n > \beta$*,*
   - $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$*,*
   - *for all* $n \in \omega$*,* $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$*,*
   - $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$*,*

   then $A_\emptyset$*,* $A_g^n \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in [T_n]$*.*

4. *Suppose* Λ *is* σ*-algebra, closed under* Λ*-substitution and:*

   - $B \in \Lambda \restriction X^\omega$*,*
   - $\Psi \in \Gamma(\omega, \Lambda)$*,*
   - $A \in \Lambda \restriction [T]$*,*

   then $A_\emptyset$*,* $A_g^n \in \Lambda \restriction X^\omega$ *for any* $n \in \omega$ *and for any* $g \in [T_n]$*.* ⊣

Proof.

Fix $n \in \omega$ and $g \in [T_n]$. First, we will find the complexity of A_g^n. We will use Fix_2 with $a_n = g$. Show

$$A_g^n \stackrel{df}{=} \{f \in X^\omega \,|f^\frown g \in A^n\} = Fix_2^{-1}(A) \cap \Psi^{-1}(n) \cap B.$$

Recall $A^n = A \cap ((B \cap \Psi^{-1}(n)) \times [T_n])$.

($\subseteq$) Suppose $f \in A_g^n$. Since $g \in [T_n]$ and $f^\frown g \in A^n$, $f \in \Psi^{-1}(n) \cap B$ and $f^\frown g \in A$. Since $f \in B$ and $\Psi(f) = n$, $Fix_2(f) = f^\frown a_{\Psi(f)} = f^\frown a_n = f^\frown g$. Thus $Fix_2(f) \in A$ so that $f \in Fix_2^{-1}(A)$.

($\supseteq$) Suppose $f \in Fix_2^{-1}(A) \cap \Psi^{-1}(n) \cap B$. Since $f \in Fix_2^{-1}(A)$ and $f \in B$, $Fix_2(f) = f^\frown a_{\Psi(f)} = f^\frown a_n = f^\frown g \in A$. Since $g \in [T_n]$, $f^\frown g \in A \cap ((B \cap \Psi^{-1}(n)) \times [T_n] = A^n$. Hence $f \in A_g^n$.

First, we will consider the complexity of A_g^n.

$$A_g^n \overset{df}{=} \{f \in X^\omega \,|\, f^\frown g \in A^n\} = Fix_2^{-1}(A) \cap \underbrace{\Psi^{-1}(n)}_{\mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega} \cap \underbrace{B}_{\mathbf{\Delta}^0_\beta \restriction X^\omega \text{ or } \mathbf{\Sigma}^0_\beta \restriction X^\omega}.$$

Show (1) for A_g^n. Suppose $B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$, for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}}$-measurable. Note that since ω_1 is regular, $\sup_{n\in\omega}\gamma_n \in \omega_1$. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34,

$$Fix_2^{-1}(A) \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha} \restriction X^\omega.$$

Thus $A_g^n \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha} \restriction X^\omega$.

Show (2) for A_g^n. Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$, for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Sigma}^0_{\beta+1}$-measurable. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34, $Fix_2^{-1}(A) \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$. Thus $A_g^n \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$.

Show (3) for A_g^n. Suppose there is $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ for all $n \in \omega$, $\Psi^{-1}(n) \in \mathbf{\Delta}^0_{\gamma_n} \restriction X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable. Since $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, by sublemma 2.5.34, $Fix_2^{-1}(A) \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$. Thus $A_g^n \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \restriction X^\omega$.

Show (4) for A_g^n. Suppose Λ is σ-algebra, closed under Λ-substitution. Suppose $\Psi \in \Gamma(\omega, \Lambda), B \in \Lambda \restriction X^\omega$ and $A \in \Lambda \restriction [T]$. Since Λ is σ-algebra, by lemma 3.5.25, Fix_2 is Λ-measurable. Since Λ is closed under Λ-substitution, $Fix_2^{-1}(A) \in \Lambda$. Since $\Psi \in \Gamma(\omega, \Lambda)$ and $B \in \Lambda \restriction X^\omega$, each $A_g^n \in \Lambda \restriction X^\omega$.

Now, we consider the complexity of $A_\emptyset$. Recall $long_2(B) = \{h \in [T] \mid lh(h) > \omega\}$. Then $long_2(B) \in \mathbf{\Sigma}^0_1 \upharpoonright [T]$.

$$\underbrace{([T]\backslash long_2(B))}_{\mathbf{\Pi}^0_1 \upharpoonright [T]} \cap \underbrace{A}_{\mathbf{\Sigma}^0_\alpha \upharpoonright [T]} \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T] \text{ for } \alpha > 1.$$

$$A_\emptyset = \{f \in X^\omega \backslash B \,|\, f \in A\} = Fix_2^{-1}\left(([T]\backslash long_2(B)) \cap A\right).$$

Show (1) for $A_\emptyset$. Suppose $B \in \mathbf{\Delta}^0_\beta \upharpoonright X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Delta}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}}$-measurable. By sublemma 2.5.34, $A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta, \sup_{n\in\omega}\gamma_n\}+\alpha} \upharpoonright X^\omega$.

Show (2) for $A_\emptyset$. Suppose for all $n \in \omega$, $\beta \geq \gamma_n$, $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Sigma}^0_{\beta+1}$-measurable. By sublemma 2.5.34, $A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega$.

Show (3) for $A_\emptyset$. Suppose there is $n \in \omega$ such that $\gamma_n > \beta$, $B \in \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. Then by lemma 3.5.25, Fix_2 is $\mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n}$-measurable. By sublemma 2.5.34, $A_\emptyset \in \mathbf{\Sigma}^0_{\sup_{n\in\omega}\gamma_n+\alpha} \upharpoonright X^\omega$.

Show (4) for $A_\emptyset$. Suppose Λ is σ-algebra and closed under Λ-substitution, $\Psi \in \Gamma(\omega, \Lambda)$, $B \in \Lambda \upharpoonright X^\omega$ and $A \in \Lambda \upharpoonright [T]$. Then by lemma 3.5.25, Fix_2 is Λ measurable and $([T]\backslash long(B)) \cap A \in \Lambda \upharpoonright [T]$. Since Λ is closed under Λ-substitution and $([T]\backslash long(B)) \cap A \in \Lambda \upharpoonright [T]$, $A_\emptyset \in \Lambda \upharpoonright X^\omega$. □

We computed the complexity of each A^n_g and $A_\emptyset$ in lemma 3.5.26. Using lemma 3.5.11 and theorem 3.5.10, we obtain the determinacy results in theorems 3.5.22 and 3.5.23. First, we consider $Tree_2$ collections over FWF. Recall theorem 3.5.22.

Theorem 3.5.22. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right). \quad (3.20)$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{3.21}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{3.22}$$

⊣

Proof of Theorem 3.5.22.

Fix $\alpha, \beta, \gamma \in \omega_1$ such that $\alpha > 1$.

Show the implication (3.20). Assume $Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega)$. Pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.26, since $FWF \subseteq CWF$, for each $n \in \omega$ and $g \in [T_n]$, $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega$. Since $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$, for each $n \in \omega$ and for each $p \in T_n$, M^n_p is finite. By lemma 3.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega$. Thus

$$\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega.$$

Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right)$.

Similarly, for the implication (3.21), suppose $\beta \geq \gamma$. Pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. By lemma 3.5.26, for each $n \in \omega$ and $g \in [T_n]$, $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$. Since $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$, by lemma 3.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right)$.

For the implication (3.22), suppose $\gamma \geq \beta$. Pick an arbitrary Type 2 tree

$$T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right).$$

Then $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right)$, $B \in \mathbf{\Sigma}^0_\beta \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$. By lemma 3.5.26, for each $n \in \omega$ and $g \in [T_n]$, $A^n_g, A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega$. Since $T_{sq} = \langle T_n \,|n \in \omega\rangle \in FWF^\omega$, by lemma 3.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right)$. $\square$

Now, we consider $Tree_2$ collections over CWF. Recall theorem 3.5.23.

Theorem 3.5.23. *Suppose $\alpha, \beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \quad \dashv$$

Proof of Theorem 3.5.23.

Fix $\alpha, \beta, \gamma \in \omega_1$. Assume $Det(\mathbf{B} \restriction X^\omega)$. Pick arbitrary $\Psi \in \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), B \in \mathbf{\Delta}^0_\beta \restriction X^\omega$, $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$ and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{\Psi,B}_{X,T_{sq}}]$. By lemma 3.5.26, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha}$ and $A_\emptyset \in \mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+1}$ (plus 1 when $\alpha = 1$). Since for each $n \in \omega$ and for each $p \in T_n$, M^n_p is countable, by lemma 3.5.11, each $A^n_\emptyset \in \mathbf{B} \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{B} \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. Therefore, $Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right)$. □

The following is a $Tree_2$ version of corollary 2.5.36 on page 181.

Corollary 3.5.27. *Suppose* Λ *is a* σ*-algebra and* Λ *is closed under* Λ*-substitution. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\mathcal{G}\left(\Lambda; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

Assume $Det(\Lambda \restriction X^\omega)$. Pick an arbitrary Type 2 tree

$$T = T^{\Psi,B}_{X,T_{sq}} \in Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right).$$

Then $\Psi \in \Gamma(\omega, \Lambda)$, $B \in \Lambda \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$. Pick an arbitrary $A \in \Lambda \restriction [T]$. By lemma 3.5.26, for all $n \in \omega$ and $g \in [T_n]$, each A^n_g and $A_\emptyset$ are in $\Lambda \restriction X^\omega$. Since for each $n \in \omega$ and for each $p \in T_n$, M^n_p is countable, by lemma 3.5.11, each $A^n_\emptyset \in \Lambda \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \Lambda \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A;T)$ is determined. Therefore, $Det\mathcal{G}\left(\Lambda; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right)$. □

The following is a $Tree_2$ version of corollary 2.5.37 on page 182.

Corollary 3.5.28. *(Corollary to Corollary 3.5.27)*

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

Since $\mathbf{B}$ is σ-algebra and closed under Borel-substitution, by corollary 3.5.27, we have the result. □

3.5.4 Obtaining the determinacy of projective games on a $Tree_2$ collection with CWF from the determinacy of projective games on $X^{<\omega}$

In section 3.5.2, we focused on obtaining the determinacy of open games on a certain $Tree_2$ collection with CWF from the determinacy of games on $X^{<\omega}$. In section 3.5.3, we focused on obtaining the determinacy of Borel games on a certain $Tree_2$ collection with CWF from the determinacy of Borel games on $X^{<\omega}$. In this section, we will generalize Borel games on a $Tree_2$ collection to projective games on a particular $Tree_2$ collection. We will obtain the determinacy of projective games on a certain $Tree_2$ collection with CWF from the determinacy of projective games on $X^{<\omega}$. The idea is similar as in section 2.5.4. The main theorem in this section is theorem 3.5.29.

The following is a $Tree_2$ version of theorem 2.5.38 on page 183.

Theorem 3.5.29. *Suppose $m \in \omega$. Let $\mathcal{T}_2 = Tree_2\,(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega)$. Then*

$$Det(\mathbf{\Sigma}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Pi}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Delta}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_2\right).$$

Note that $\mathbf{\Delta}^1_1 \restriction X^\omega$ is Borel if X is countable and if X is uncountable, it is the quasi-Borel.

$\dashv$

The idea of the proof is similar as in sections 3.5.2 and 3.5.3. We will use the same definition of A^n, A^n_g and $A_\emptyset$ from section 3.5.1. We will find the complexity of each A^n_g and $A_\emptyset$ in lemma 3.5.30 using sublemma 2.5.40 on page 185. Then, by using lemma 3.5.11 and theorem 3.5.10, we will obtain the determinacy results in theorem 3.5.29. The proof of the theorem is on page 324.

The following is a Type 2 tree version of lemma 2.5.39 on page 184.

Lemma 3.5.30. *Fix a Type 2 tree* $T = T^{\Psi,B}_{X,T_{sq}}$ *such that* $T_{sq} = \langle T_n | n \in \omega \rangle \in CWF^{\omega}$*. Suppose* $m \in \omega$*,* $\alpha, \beta, \gamma \in \omega_1$*,* $\Psi \in \Gamma(\omega, \mathbf{B})$*,* $B \in \mathbf{B} \restriction X^{\omega}$*.*

1. *If* $A \in \mathbf{\Sigma}^1_m \restriction [T]$*, then for any* $n \in \omega$ *and for any* $g \in [T_n]$*, each* $A^n_g, A_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^{\omega}$*.*

2. *If* $A \in \mathbf{\Pi}^1_m \restriction [T]$*, then for any* $n \in \omega$ *and for any* $g \in [T_n]$*, each* $A^n_g, A_\emptyset \in \mathbf{\Pi}^1_m \restriction X^{\omega}$*.*

3. *If* $A \in \mathbf{\Delta}^1_m \restriction [T]$*, then for any* $n \in \omega$ *and for any* $g \in [T_n]$*, each* $A^n_g, A_\emptyset \in \mathbf{\Delta}^1_m \restriction X^{\omega}$*.*

$\dashv$

Proof.

The proof is similar to the proof of lemma 3.5.26. We will show the case for (1): $A \in \mathbf{\Sigma}^1_m \restriction [T]$. The proofs are similar for cases (2): $A \in \mathbf{\Pi}^1_m \restriction [T]$ and (3): $A \in \mathbf{\Delta}^1_m \restriction [T]$.

Suppose $A \in \mathbf{\Sigma}^1_m \restriction [T]$. By lemma 3.5.25, Fix_2 is Borel-measurable under $\Psi \in \Gamma(\omega, \mathbf{B})$ and $B \in \mathbf{B} \restriction X^{\omega}$. By sublemma 2.5.40, $Fix_2^{-1}(A) \in \mathbf{\Sigma}^1_m \restriction X^{\omega}$. Then

$$A^n_g \stackrel{df}{=} \{f \in X^{\omega} \,| f^\frown g \in A^n\} = \underbrace{Fix_2^{-1}(A)}_{\mathbf{\Sigma}^1_m \restriction X^{\omega}} \cap \underbrace{\Psi^{-1}(n)}_{\mathbf{B} \restriction X^{\omega}} \cap \underbrace{B}_{\mathbf{B} \restriction X^{\omega}} \in \mathbf{\Sigma}^1_m \restriction X^{\omega}.$$

Now, consider $A_\emptyset$. Recall $Long_2(B) = \{h \in [T] \,|\, lh(h) > \omega\}$. Then $Long_2(B) \in \mathbf{\Sigma}^0_1 \restriction [T]$.

$$\underbrace{([T] \backslash Long_2(B))}_{\mathbf{\Pi}^0_1 \restriction [T]} \cap \underbrace{A}_{\mathbf{\Sigma}^1_m \restriction [T]} \in \mathbf{\Sigma}^1_m \restriction [T].$$

Thus

$$A_\emptyset = \{f \in X^{\omega} \backslash B \,| f \in A\} = Fix_2^{-1}\left(([T] \backslash Long_2(B)) \cap A\right) \in \mathbf{\Sigma}^1_m \restriction X^{\omega}$$

by sublemma 2.5.40.

Similarly, for the cases (2) and (3). □

We computed the complexity of each A^n_g for all $n \in \omega$ and $g \in [T_n]$, and $A_\emptyset$ in lemma 3.5.30. Using lemma 3.5.11 and theorem 3.5.10, we obtain the determinacy results in theorem 3.5.29. Recall theorem 3.5.29.

Theorem 3.5.29. *Suppose $m \in \omega$. Let $\mathcal{T}_2 = Tree_2\,(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega)$. Then*

$$Det(\mathbf{\Sigma}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Pi}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Delta}^1_m \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_2\right).$$

Note that $\mathbf{\Delta}^1_1 \restriction X^\omega$ is Borel if X is countable and if X is uncountable, it is the quasi-Borel.

⊣

Proof of Theorem 3.5.29.

We will show the case for $\mathbf{\Sigma}^1_m$. The proofs are similar for cases $\mathbf{\Pi}^1_m$ and $\mathbf{\Delta}^1_m$.

Show $Det(\mathbf{\Sigma}^1_m \restriction X^\omega)$ implies $Det\ \mathcal{G}\,(\mathbf{\Sigma}^1_m; \mathcal{T}_2)$. Assume $Det(\mathbf{\Sigma}^1_m \restriction X^\omega)$. Pick an arbitrary $T = T^{\Psi,B}_{X,T_{sq}} \in \mathcal{T}_2$. Then $\Psi \in \Gamma\,(\omega, \mathbf{B})$, $B \in \mathbf{B} \restriction X^\omega$ and $T_{sq} = \langle T_n \,|n \in \omega\rangle \in CWF^\omega$. Pick an arbitrary $A \in \mathbf{\Sigma}^1_m \restriction [T]$. By lemma 3.5.30, for all $n \in \omega$ and $g \in [T_n]$, each $A^n_g \in \mathbf{\Sigma}^1_m \restriction X^\omega$ and $A_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$. Since for each $n \in \omega$, for each $p \in T_n$, M^n_p is countable, by lemma 3.5.11, each $A^n_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$. Thus $\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset \in \mathbf{\Sigma}^1_m \restriction X^\omega$. Hence $G(\bigcup_{n\in\omega} A^n_\emptyset \cup A_\emptyset; X^\omega)$ is determined. By theorem 3.5.10, $G(A;T)$ is determined. Therefore, $Det\ \mathcal{G}\,(\mathbf{\Sigma}^1_m; \mathcal{T}_2)$. □

3.5.5 Comment about a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ for T_{sq} with not well-founded trees or moves over an uncountable set

In sections 3.5.1 through 3.5.4, we obtained the determinacy of games on a certain $Tree_2$ collection with CWF from the determinacy of games on $X^{<\omega}$. The way we obtained the determinacy results in these sections are using the fact that each $T^{\Psi,B}_{X,T_{sq}}$ in the $Tree_2$ collection having $T_{sq} = \langle T_n | n \in \omega \rangle \in CWF^{\omega}$, i.e.,

1. each T_n is well-founded,
2. for every position $p \in T_n \backslash [T_n]$, the set of moves at p is countable.

Without these restrictions, we need to have the determinacy of games on X^{ω} with higher complexity, even just an open game on a Type 2 tree.

In section 3.5.5.1, we will observe particular examples of the case for a $Tree_2$ collection does not satisfy the condition (1). In section 3.5.5.2, we will observe particular examples of the case for a $Tree_2$ collection does not satisfy the condition (2).

3.5.5.1 Without the well-foundedness, each move is from a countable set

Let's consider the case $T_{sq} = \langle T_n | n \in \omega \rangle$ with some T_n being ill-founded. Suppose $T^{\Psi,B}_{X,T_{sq}}$ has $T_{sq} = \langle T_n | n \in \omega \rangle$ such that

- T_{sq} contains trees that are not well-founded,
- for every $n \in \omega$, T_n has height $\leq \omega$,
- for every position $p \in T_n \backslash [T_n]$, the set of moves at p is countable.

If T_n contains plays with finite length, add arbitrary moves after that to create a modified tree T_n^{mod} with all plays having length ω. If all of the plays in T_n have length ω, then define $T_n^{mod} = T_n$. Define $T_{sq}^{mod} = \langle T_n^{mod} | n \in \omega \rangle$. Then each open set in $[T_{X,T_{sq}}^{\Psi,B}]$ has a corresponding open set in $[T_{X,T_{sq}^{mod}}^{\Psi,B}]$ (the converse is false). Thus

$$Det\left(\mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}^{mod}}^{\Psi,B}]\right) \Rightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}}^{\Psi,B}]\right).$$

As a special case, suppose $T_{X,T_{sq}}^{\Psi,X^\omega}$ where $T_{sq} = \langle T_n | n \in \omega \rangle$, each $T_n \subseteq \omega^{<\omega}$ and there is $n \in \omega$ such that T_n contains a play with length ω. Take $T_n^{mod} = \omega^{<\omega}$ for every $n \in \omega$. Then $[T_{X,T_{sq}^{mod}}^{\Psi,X^\omega}] = X^\omega \times \omega^\omega$. It is well-known that

$$Det\left(\mathbf{\Sigma}_1^1 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright (X^\omega \times \mathcal{N})\right).$$ [21]

Hence

$$Det\left(\mathbf{\Sigma}_1^1 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}^{mod}}^{\Psi,X^\omega}]\right).$$

Thus

$$Det\left(\mathbf{\Sigma}_1^1 \upharpoonright X^\omega\right) \Rightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}}^{\Psi,X^\omega}]\right).$$

In general, suppose $T_{X,T_{sq}}^{\Psi,X^\omega}$ where $T_{sq} = \langle T_n | n \in \omega \rangle$, each $T_n \subseteq \omega^{<\omega \cdot k}$ for some $k \in \omega$. Take $T_n^{mod} = \omega^{<\omega \cdot k}$. Then $[T_{X,T_{sq}^{mod}}^{\Psi,X^\omega}] = X^\omega \times \omega^{\omega \cdot k} = X^\omega \times \mathcal{N}^k$. Since

$$Det\left(\mathbf{\Sigma}_k^1 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright (X^\omega \times \mathcal{N}^k)\right),$$

$$Det\left(\mathbf{\Sigma}_k^1 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \upharpoonright [T_{X,T_{sq}^{mod}}^{\Psi,X^\omega}]\right).$$

[21] see outline of the proof for Fraker, 2001, pp.59-62, Corollary 5.3.

Thus

$$Det\left(\mathbf{\Sigma}^1_k \upharpoonright X^\omega\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_1 \upharpoonright [T^{\Psi,X^\omega}_{X,T_{sq}}]\right).$$

Therefore, we have the following. Suppose $\Upsilon = \{T \,\big|\, T \text{ is a tree and } T \subseteq \omega^{<\omega\cdot k}\}$. Then for any nonempty collection Γ of functions from X^ω into ω,

$$Det\left(\mathbf{\Sigma}^1_k \upharpoonright X^\omega\right) \Rightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_2\left(X, \Upsilon^\omega, \Gamma, X^\omega\right)\right).$$

In particular, if $\Upsilon = \{T \,\big|\, T \text{ is a tree and } T \subseteq \omega^{<\omega^2}\}$, then we have the following.

Observation 3.5.31. *Suppose $\Upsilon = \{T \,\big|\, T \text{ is a tree and } T \subseteq \omega^{<\omega^2}\}$. Then for any nonempty collection Γ of functions from X^ω into ω,*

$$Det\left(\mathbf{P} \upharpoonright X^\omega\right) \Rightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_2\left(X, \Upsilon^\omega, \Gamma, X^\omega\right)\right). \qquad \dashv$$

3.5.5.2 Moves over an uncountable set, each T_n is well-founded

As we mentioned in section 2.5.5, if we have $p \in T_n$ such that the set of moves at p is uncountable, we will need a higher determinacy on games of X^ω, just to get an open determinacy of simple Type 2 trees.

Suppose $T^{\Psi,B}_{X,T_{sq}}$ has $T_{sq} = \langle T_n | n \in \omega\rangle$ such that

- every T_n is well-founded,
- there exists a position $p \in T_n \backslash [T_n]$ such that the set of moves at p is uncountable.

Consider the special such case; every $T_n = \mathcal{N}^{\leq 1}$, the tree of height 1 and each move is $f \in \mathcal{N}$. Then each $[T_n] = \mathcal{N}$. Thus for any Ψ from X^ω into ω, $[T^{\Psi,X^\omega}_{X,T_{sq}}] = X^\omega \times \mathcal{N}$. Since we have

$$Det\left(\mathbf{\Sigma}^1_1 \upharpoonright X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \upharpoonright \left(X^\omega \times \mathcal{N}\right)\right),$$

$$Det\left(\mathbf{\Sigma}^1_1 \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction \left[T^{\Psi,X^\omega}_{X,T_{sq}}\right]\right).$$

Suppose T_{sq} have every $T_n = \mathcal{N}^{\leq k}$ for some $k \in \omega$. Then each $[T_n] = \mathcal{N}^k$. Thus for any Ψ from X^ω into ω, $[T^{\Psi,X^\omega}_{X,T_{sq}}] = X^\omega \times \mathcal{N}^k$. Since we have

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction (X^\omega \times \mathcal{N}^k)\right),$$

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction \left[T^{\Psi,X^\omega}_{X,T_{sq}}\right]\right).$$

Fix $k \in \omega$. Let's take T_{sq} to be each $T_n = \mathcal{N}^{\leq i}$ for some $i \leq k$. Then each $[T_n] = \mathcal{N}^i$. Create $T^{mod}_n = \mathcal{N}^k$ by extending each play if $i < k$. Define $T^{mod}_{sq} = \langle T^{mod}_n | n \in \omega\rangle$. Then for any Ψ from X^ω into ω, $[T^{\Psi,X^\omega}_{X,T^{mod}_{sq}}] = X^\omega \times \mathcal{N}^k$. Since we have

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction (X^\omega \times \mathcal{N}^k)\right),$$

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Leftrightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction \left[T^{\Psi,X^\omega}_{X,T^{mod}_{sq}}\right]\right).$$

Since

$$Det\left(\mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T^{mod}_{sq}}]\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction [T^{\Psi,B}_{X,T_{sq}}]\right),$$

we have

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_1 \restriction \left[T^{\Psi,X^\omega}_{X,T_{sq}}\right]\right).$$

Let $\Upsilon = \left\{\mathcal{N}^{\leq i} | i \leq k\right\}$. Then for any nonempty collection Γ of functions from X^ω into ω,

$$Det\left(\mathbf{\Sigma}^1_k \restriction X^\omega\right) \Rightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_2\left(X, \Upsilon^\omega, \Gamma, X^\omega\right)\right).$$

In particular, if $\Upsilon = \left\{\mathcal{N}^{\leq i} | i \in \omega\right\}$, we have the following.

Observation 3.5.32. *Suppose $\Upsilon = \left\{\mathcal{N}^{\leq i} | i \in \omega\right\}$. Then for any nonempty collection Γ of* *functions from* X^ω *into* ω*, we have*

$$Det\left(\mathbf{P}\restriction X^\omega\right)\Rightarrow\mathcal{G}\left(\mathbf{\Sigma}_1^0, Tree_2\left(X,\Upsilon^\omega,\Gamma,X^\omega\right)\right). \qquad \dashv$$

3.6 Determinacy equivalence between games on $X^{<\omega}$ and games on $Tree_2$ collections

In sections 3.3 through 3.5, we observed the determinacy strength on games on $Tree_1$ collections. In section 3.3, by shifting, we compared the determinacy of $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Sigma}^1_n$) games on a particular $Tree_2$ collection and $\mathbf{\Pi}^0_\alpha$ (respectively, $\mathbf{\Pi}^1_n$) games on the same $Tree_2$ collection, for $\alpha \in \omega_1$ and $n \in \omega$. In section 3.4, we used the determinacy of a fixed complexity of games on a certain $Tree_2$ collection to obtain the determinacy of a certain complexity of games on $X^{<\omega}$. In section 3.5, we obtained the determinacy of Borel and projective games on particular $Tree_2$ collections from the determinacy of a fixed complexity of games on $X^{<\omega}$.

In this section, we will combine results from section 3.3, section 3.4 and section 3.5. Although results from section 2.4 and section 3.4 are slightly different, since results from section 3.5 are similar to the results in section 2.5, we will have the similar results as section 2.6. The only difference is corollary 3.6.6. It is slightly different from corollary 2.6.6.

In section 3.6.1, we will obtain the determinacy equivalences between Borel games on $X^{<\omega}$ and games on particular $Tree_2$ collections.

In section 3.6.2, we will obtain the determinacy equivalences between projective games on $X^{<\omega}$ and games on particular $Tree_2$ collections.

3.6.1 Determinacy equivalence between Borel games on $X^{<\omega}$ and games on $Tree_2$ collections

In this section, we will obtain the determinacy equivalences between Borel games on $X^{<\omega}$ and games on particular $Tree_1$ collections. We will obtain the similar results as section 2.6. Corollary 3.6.6 is slightly different from corollary 2.6.6.

The following is a $Tree_2$ version of theorem 2.6.1 on page 197.

Theorem 3.6.1. *The determinacy of following (3.23) through (3.28) are all equivalent to* $Det\,(\mathbf{\Sigma}^0_1 \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.23}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.24}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.25}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.26}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.27}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.28}$$

$\dashv$

Theorem 3.6.3 says that if we let

$$\mathcal{T}_1^1 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right),$$

$$\mathcal{T}_1^2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right),$$

$$\mathcal{T}_1^3 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; \mathcal{T}_2^i\right) \Leftrightarrow Det\left(\mathbf{\Sigma}_1^0 \restriction X^\omega\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}_1^0; \mathcal{T}_2^j\right)$$

for any $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof.

We obtain $Det\,(\mathbf{\Sigma}_1^0 \restriction X^\omega)$ if and only if the determinacy of (3.23) by theorem 3.5.13 and corollary 3.4.14.

($\Rightarrow$) We obtain this from theorem 3.5.13.

($\Leftarrow$) We obtain this from corollary 3.4.3.

By observation 3.2.13, (3.23), (3.24) and (3.25) are the same set. Similarly, (3.26), (3.27) and (3.28) are the same set. The determinacy of (3.23) and the determinacy of (3.26) are equivalent by theorem 3.3.8. Consequently, the determinacy of (3.23) through (3.28) are all equivalent to $Det\,(\mathbf{\Sigma}_1^0 \restriction X^\omega)$. $\square$

The following is a $Tree_2$ version of theorem 2.6.2 on page 198.

Theorem 3.6.2. *Suppose $\beta, \gamma \in \omega_1$ and $\beta \geq \gamma$. Then the following (3.29) through (3.34) are all equivalent to $Det\left(\mathbf{\Delta}_\beta^0 \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Delta}_\beta^0 \restriction X^\omega\right)\right) \tag{3.29}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.30}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.31}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.32}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.33}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.34}$$

$\dashv$

Theorem 3.6.2 says that if we let

$$\mathcal{T}^1_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right),$$

$$\mathcal{T}^2_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right),$$

$$\mathcal{T}^3_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}^i_2\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; \mathcal{T}^j_2\right)$$

for any $i = 1, 2, 3$ and $j = 1, 2, 3$.

Proof.

We obtain $Det\left(\mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)$ if and only if the determinacy of (3.29) by theorem 3.5.13 and corollary 3.4.13.

($\Rightarrow$) This is obtained from theorem 3.5.13.

($\Leftarrow$) This is obtained from corollary 3.4.13.

By observation 3.2.13, (3.29), (3.30) and (3.31) are the same sets. Similarly, (3.32), (3.33) and (3.34) are the same sets. The determinacy of (3.29) and the determinacy of (3.32) are equivalent by theorem 3.3.8. □

The following is a $Tree_2$ version of theorem 2.6.3 on page 199.

Theorem 3.6.3. *Suppose $\beta, \gamma \in \omega_1$ and $1 \leq \beta < \gamma$. Then the determinacy of following (3.35) through (3.52) are all equivalent to Det $\left(\mathbf{\Delta}^0_\gamma \upharpoonright X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.35}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.36}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.37}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.38}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.39}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.40}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.41}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.42}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.43}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{3.44}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{3.45}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \tag{3.46}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.47}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.48}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.49}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.50}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.51}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.52}$$

$\dashv$

Theorem 3.6.3 says that if we let

$$\mathcal{T}_2^{\mathbf{\Delta\Sigma}} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_2^{\mathbf{\Sigma\Sigma}} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_2^{\mathbf{\Pi\Sigma}} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right),$$

$$\mathcal{T}_2^{\mathbf{\Delta\Pi}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Delta}^0_{\gamma}), \mathbf{\Pi}^0_{\beta} \upharpoonright X^{\omega}\right),$$

$$\mathcal{T}_2^{\mathbf{\Sigma\Pi}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Sigma}^0_{\gamma}), \mathbf{\Pi}^0_{\beta} \upharpoonright X^{\omega}\right),$$

$$\mathcal{T}_2^{\mathbf{\Pi\Pi}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Pi}^0_{\gamma}), \mathbf{\Pi}^0_{\beta} \upharpoonright X^{\omega}\right),$$

$$\mathcal{T}_2^{\mathbf{\Delta\Delta}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Delta}^0_{\gamma}), \mathbf{\Delta}^0_{\beta} \upharpoonright X^{\omega}\right),$$

$$\mathcal{T}_2^{\mathbf{\Sigma\Delta}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Sigma}^0_{\gamma}), \mathbf{\Delta}^0_{\beta} \upharpoonright X^{\omega}\right),$$

$$\mathcal{T}_2^{\mathbf{\Pi\Delta}} = Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathbf{\Pi}^0_{\gamma}), \mathbf{\Delta}^0_{\beta} \upharpoonright X^{\omega}\right),$$

then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; \mathcal{T}_2^{ij}\right) \Leftrightarrow Det\left(\mathbf{\Delta}^0_{\gamma} \upharpoonright X^{\omega}\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_1; \mathcal{T}_2^{kl}\right)$$

for any $i, j, k, l \in \{\mathbf{\Delta}, \mathbf{\Sigma}, \mathbf{\Pi}\}$.

Proof.

We obtain $Det\left(\mathbf{\Delta}^0_{\gamma} \upharpoonright X^{\omega}\right)$ if and only if the determinacy of (3.35) from theorem 3.5.13 and corollary 3.4.14.

($\Rightarrow$) We obtain this from theorem 3.5.13.

($\Leftarrow$) We obtain this from corollary 3.4.14.

Similarly, we obtain $Det\left(\mathbf{\Delta}^0_{\gamma} \upharpoonright X^{\omega}\right)$ if and only if the determinacy of 3.41;

and $Det\left(\mathbf{\Delta}^0_{\gamma} \upharpoonright X^{\omega}\right)$ if and only if the determinacy of 3.47.

By observation 3.2.13, (3.35), (3.36) and (3.37) are the same set. Similarly, (3.38), (3.39) and (3.40) are the same set. The determinacy of (3.35) and the determinacy of (3.38) are equivalent by theorem 3.3.8. Consequently, the determinacy of (3.35) through (3.40) are all equivalent to $Det\left(\mathbf{\Delta}^0_{\gamma} \upharpoonright X^{\omega}\right)$.

By observation 3.2.13, (3.41), (3.42) and (3.43) are the same set. Similarly, (3.44), (3.45) and (3.46) are the same set. The determinacy of (3.41) and the determinacy of (3.44) are equivalent by theorem 3.3.8. Consequently, the determinacy of (3.41) through (3.46) are all equivalent to $Det\left(\mathbf{\Delta}^0_\gamma \upharpoonright X^\omega\right)$.

By observation 3.2.13, (3.47), (3.48) and (3.49) are the same set. Similarly, (3.50), (3.51) and (3.52) are the same set. The determinacy of (3.47) and the determinacy of (3.50) are equivalent by theorem 3.3.8. Consequently, the determinacy of (3.47) through (3.52) are all equivalent to $Det\left(\mathbf{\Delta}^0_\gamma \upharpoonright X^\omega\right)$. □

The following is a $Tree_2$ version of corollary 2.6.4 on page 203.

Corollary 3.6.4. *Suppose Λ is an algebra. Then for any nonempty $\Upsilon \subseteq FWF$,*

$$Det(\Lambda \upharpoonright X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Proof.

(⇒) We obtain this from corollary 3.5.17.

(⇐) We obtain this from corollary 3.4.13. □

The following is a $Tree_2$ version of corollary 2.6.5 on page 203.

Corollary 3.6.5. *Suppose Λ is σ-algebra. Then for any nonempty $\Upsilon \subseteq CWF$,*

$$Det(\Lambda \upharpoonright X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Proof.

(⇒) We obtain this from corollary 3.5.18.

(⇐) We obtain this from corollary 3.4.13. □

The following corollary is slightly different from corollary 2.6.6. Since we have theorem 3.4.7, we can obtain the determinacy equivalence between Borel games over $X^{<\omega}$ and $\mathbf{\Delta}^0_1$ games on certain $Tree_2$ collections. The following is a $Tree_2$ version of corollary 2.6.6 on page 203.

Corollary 3.6.6. *For any $\alpha, \beta, \gamma \in \omega_1$,*

$$Det(\mathbf{B} \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right)$$

where:

- $\Upsilon \subseteq CWF$ *is closed under shifting trees*[22] *and*

  1. *if* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \subseteq \Upsilon$, *then* $\mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha, \mathbf{B}\}$,[23]
  2. *if* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \nsubseteq \Upsilon$, *then*

$$\begin{cases} \mathcal{A} = \mathbf{B}, \ \textit{or} \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\} & \textit{for } \alpha > 1, \ \textit{or} \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1\} & \textit{for } \alpha = 1, \end{cases}$$

- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,
- *at least one of $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ is* $\mathbf{B}$ *if* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \nsubseteq \Upsilon$. ⊣

Proof.

Fix $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ as above. Let $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)$.

[22] Recall definition 3.3.7 on page 231 for the closure property under shifting trees.
[23] Recall Yost tree $T^\alpha_{g.t.}$ for section 3.4.1.2 on page 242.

(⇒) Corollary 3.6.5 gives

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right).$$

Under the condition for $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$,

$$\mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right) \subseteq \mathcal{G}\left(\mathbf{B}; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right).$$

Thus we have

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right).$$

(⇐) Case 1 : $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \subseteq \Upsilon$.

By corollary 3.4.10, for any $\Psi \in \Gamma(\omega, \mathcal{C})$,

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Since

$$\mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, X^\omega\right)\right) \subseteq \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right),$$

$$Det\ \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Case 2 : $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \nsubseteq \Upsilon$.

Then at least one of $\mathcal{A}, \mathcal{B}$ or $\mathcal{C}$ is $\mathbf{B}$.

Subcase 1 : $\mathcal{A} = \mathbf{B}$. We obtain

$$Det\ \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

from observation 3.4.1.

Subcase 2 : $\mathcal{B} = \mathbf{B}$.

If $\mathcal{A}_1 \neq \mathbf{\Pi}_1^0$, by the choice of $\mathcal{A}$, we have $\mathbf{\Sigma}_1^0 \restriction \mathcal{T}_2 \subseteq \mathcal{A}_1 \restriction \mathcal{T}_2$, we obtain

$$Det\ \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

from corollary 3.4.13.

If $\mathcal{A}_1 = \mathbf{\Pi}_1^0$, since Υ is closed under shifting trees, by theorem 3.3.8,

$$Det\ \mathcal{G}\left(\mathbf{\Pi}_1^0; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}_1^0; \mathcal{T}_2\right).$$

By corollary 3.4.13,

$$Det\mathcal{G}\left(\mathbf{\Sigma}_1^0; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Thus we have

$$Det\ \mathcal{G}\left(\mathbf{\Pi}_1^0; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

Subcase 3 : $\mathcal{C} = \mathbf{B}$. We obtain

$$Det\ \mathcal{G}\left(\mathcal{A}; \mathcal{T}_2\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega).$$

from observation 3.4.1 from corollary 3.4.14. □

Since FWF is closed under shifting trees and $\left\{T^\alpha_{g.t.}\, |\alpha \in \omega_1\right\} \nsubseteq FWF$, we obtain the following from corollary 3.6.6.

Corollary 3.6.7. *(Corollary to Corollary 3.6.6)*

For any $\alpha, \beta, \gamma \in \omega_1$,

$$Det(\mathbf{B} \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right)$$

where:

- $\begin{cases} \mathcal{A} = \mathbf{B}, \ \ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\} \quad for\ \alpha > 1, \ \ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1\} \qquad\quad for\ \alpha = 1, \end{cases}$
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,
- *at least one of* $\mathcal{A}, \mathcal{B}$ *or* $\mathcal{C}$ *is* $\mathbf{B}$. ⊣

Since CWF is closed under shifting trees and $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \subseteq CWF$, we obtain the following from corollary 3.6.6.

Corollary 3.6.8. *(Corollary to Corollary 3.6.6)*

For any $\alpha, \beta, \gamma \in \omega_1$,

$$Det(\mathbf{B} \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right)$$

where:

- $\mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha, \mathbf{B}\}$,
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$. ⊣

3.6.2 Determinacy equivalence between projective games on $X^{<\omega}$ and games on $Tree_2$ collections

In this section, we will obtain the determinacy equivalences between projective games on $X^{<\omega}$ and games on particular $Tree_2$ collections.

The following is a $Tree_2$ version of theorem 2.6.7 on page 206.

Theorem 3.6.9. *Suppose $n \in \omega \backslash \{0\}$, $\beta, \gamma \in \omega_1$.*

For any nonempty $\Upsilon \subseteq CWF$ such that Υ is closed under shifting trees, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (3.53) and (3.54) are equivalent to $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{3.53}$$

$$\mathcal{G}\left(\mathbf{\Pi}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{3.54}$$

For any nonempty $\Upsilon \subseteq CWF$ such that Υ is closed under shifting trees, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (3.55) is equivalent to $Det(\mathbf{\Delta}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Delta}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{3.55}$$

$\dashv$

Proof.

We obtain

1. $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$ if and only if the determinacy of (3.53),

2. $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$ if and only if the determinacy of (3.54),

3. $Det(\mathbf{\Delta}^1_n \restriction X^\omega)$ if and only if the determinacy of (3.55).

($\Rightarrow$) Let $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)$. By theorem 3.5.29,

$$Det(\mathbf{\Sigma}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Delta}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_n; \mathcal{T}_2\right).$$

By theorem 3.3.8, since Υ is closed under shifting trees,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_2\right).$$

Thus we have

$$Det(\mathbf{\Sigma}^1_n \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_2\right).$$

By observation 3.2.7, for any $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

$$Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right) \subseteq \mathcal{T}_2.$$

Thus, we have ($\Rightarrow$) of (1) through (3).

($\Leftarrow$) By corollary 3.4.2,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \restriction X^\omega\right).$$

By observation 3.2.7, for any $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

$$Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right) \subseteq Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right).$$

Thus, we have ($\Leftarrow$) of (1) through (3).

□

The following is a $Tree_2$ version of corollary 2.6.8 on page 207.

Corollary 3.6.10. *Suppose* Λ *is* σ*-algebra and closed under* Λ*-substitution. Then for any nonempty* $\Upsilon \subseteq CWF$*,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\mathcal{G}\left(\Lambda; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Proof.

($\Rightarrow$) By corollary 3.5.27,

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\Lambda; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right).$$

Since $\mathcal{G}\left(\Lambda; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right) \subseteq \mathcal{G}\left(\Lambda; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right)$, for any $\Upsilon \subseteq CWF$,

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\Lambda; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right).$$

($\Leftarrow$) Since $\emptyset \in \Lambda \restriction X^\omega$, by observation 3.4.1, we have the result. $\square$

3.7 Generalization of a Type 2 tree

As in section 2.7.1, we can generalize Type 2 trees into α-Type 2 trees. Instead fixing the first part as X^ω, we can take X^ω for any limit ordinal α. The following is the generalization of a Type 2 tree, named an α-Type 2 tree.

Definition 3.7.1. *(Definition of an α-Type 1 tree)*

Suppose α is a limit ordinal. Suppose X is a nonempty set, Ψ is a function from X^α into ω, B is a subsets of X^α and $T_{sq} = \langle T_n \,|n \in \omega\rangle$ where each T_n is a tree. Define $[{}_\alpha T^{\Psi,B}_{X,T_{sq}}]$ by :

$$h \in \left[{}_\alpha T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\alpha & \textit{if } h \restriction \alpha \notin B, \\ h \in X^\alpha \times \left[T_{\Psi(h\restriction\alpha)}\right] & \textit{if } h \restriction \alpha \in B. \end{cases}$$

⊣

Thus a Type 2 tree is an ω-Type 2 tree. Notice that we can obtain the similar results for α-Type 2 trees.

Chapter 4

Definitions of Type 3, Type 4, Type 5 trees and future questions

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In this dissertation, we only consider Type 1 and Type 2 trees. In chapter 2, we defined a Type 1 tree and $Tree_1$ collections, collections of Type 1 trees. We also observed the determinacy strength of games on $Tree_1$ collections by comparing the determinacy of games on $X^{<\omega}$. In chapter 3, we defined a Type 2 tree and $Tree_2$ collections, collections of Type 2 trees. We also observed the determinacy strength of games on $Tree_2$ collections by comparing the determinacy of games on $X^{<\omega}$.

In this section, we will define a Type 3, Type 4 and Type 5 trees. In section 4.1, as a generalization of a Type 2 tree, we will define a Type 3 tree. Type 3 trees are defined only when the tree satisfies a certain property In section 4.2, we modify a Type 3 tree and define a Type 4 tree. Unlike Type 3 trees, Type 4 trees are always defied. In section 4.3, we define a Type 5 tree. For Type 3, Type 4 and Type 5 trees, we don't have determinacy results. We shall leave this to readers as future questions.

4.1 Definition of a Type 3 tree

In this section, as a generalization of a Type 2 tree, we will define a Type 3 tree. In a Type 2 tree, we used Ψ to decide the tail tree. For a Type 3 tree, we will fix a continuous function Φ from X^ω into ω^ω. Φ maps the first omega moves f into a sequence g in ω^ω. Then each component of g will create a tree T_g^π by using a bijection π from ω into $\omega^{<\omega}$. π takes $g(i)$ into a finite sequence $h_i \in \omega^{<\omega}$. Thus $T_g^\pi = \{h_i | i \in \omega\}$. If T_g^π is a well-founded tree, we define a Type 3 tree. Otherwise, Type 3 is undefined. Note that we includes empty trees here. In theorem 4.2.1, we will observe that some of Type 2 trees are Type 3 trees.

Definition 4.1.1. *Fix a continuous function Φ from X^ω into ω^ω. Fix a bijection π from ω into $\omega^{<\omega}$. For each $f \in X^\omega$, define*

$$T^\pi_{\Phi(f)} = \{\pi(\Phi(f)(n)) | n \in \omega\}.$$

$\dashv$

Then $T^\pi_{\Phi(f)} \subseteq \omega^{<\omega}$.

Definition 4.1.2. *(Definition of a Type 3 tree)*

Fix a continuous function Φ from X^ω into ω^ω. Fix a bijection π from ω into $\omega^{<\omega}$. If for all $f \in X^\omega$, every $T^\pi_{\Phi(f)}$ is a well-founded tree, then we define a Type 3 tree $T^{\Phi,\pi}$ by

$$\left[T_3^{\Phi,\pi}\right] = \bigcup_{f \in X^\omega} \left(\{f\} \times \left[T^\pi_{\Phi(f)}\right]\right).$$

$\dashv$

Theorem 4.1.3. *Let $\Upsilon = \left\{\omega^{\leq n} | n \in \omega\right\}$. Then every Type 2 tree in*

$$Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Pi}_1^0 \restriction X^\omega)$$

is a Type 3 tree.

$\dashv$

Proof.

Pick an arbitrary Type 2 tree $T^{\Psi,B}_{X,T_{sq}} \in Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Pi}^0_1 \restriction X^\omega)$. Pick an arbitrary bijection π from ω into $\omega^{<\omega}$. Since π is a bijection, $\pi^{-1}(\emptyset) \in \omega$. Define $s_\emptyset \in \omega^\omega$ to be the sequence with every component $\pi^{-1}(\emptyset)$. Then

$$T^\pi_{s_\emptyset} = \{\pi(s_\emptyset(m)) \,|m \in \omega\} = \left\{\pi\left(\pi^{-1}(\emptyset)\right)\right\} = \{\emptyset\}.$$

We will define a continuous function Φ from X^ω into ω^ω. Since $T_{sq} = \langle T_n | n \in \omega\rangle \in \Upsilon^\omega$, each T_n is well-founded. We decompose each T_n to find the set of natural numbers N_n such that $T_n = \{\pi(i) | i \in N_n\}$. Since π is a bijection, each N_n is uniquely defined from T_n. Since $|T_n|$ is countable and π is a bijection, N_n is countable. Thus we can order N_n as an ω-sequence. Fix such a sequence $s^n \in \omega^\omega$ for each T_n. Then we have

$$T^\pi_{s^n} = \{\pi(s^n(m)) \,|m \in \omega\} = \{\pi(i) \,|i \in N_n\} = T_n.$$

Define

$$\begin{aligned} \Phi : \quad & X^\omega \to \omega^\omega \\ & f \mapsto \begin{cases} s^{\Psi(f)} & \text{if } f \in B, \\ s_\emptyset & \text{if } f \notin B. \end{cases} \end{aligned}$$

Recall that Ψ is a function from X^ω into ω fixed by $T^{\Psi,B}_{X,T_{sq}}$. Thus there are ω many $s^{\Psi(f)}$'s.

Show Φ is a continuous function. Pick an arbitrary $O \in \mathbf{\Sigma}^0_1 \restriction \omega^\omega$. Then there exists $\langle p_i | i \in \omega\rangle$ such that each $p_i \in \omega^{<\omega}$ and $O = \bigcup_{i\in\omega} O_i$ where each $O_i = \{h \in \omega^\omega \,|h \supseteq p_i\}$. Let $M_{p_i} = \{m \in \omega \,|s^m \supseteq p_i\}$ for each $i \in \omega$. Then

$$\Phi^{-1}(O_i) = \Phi^{-1}\{h \in \omega^\omega \mid h \supseteq p_i\} = \begin{cases} \bigcup_{m \in M_{p_i}} \underbrace{\Psi^{-1}(m)}_{\boldsymbol{\Delta}^0_1 \restriction X^\omega} & \text{if } p_i \nsubseteq s_\emptyset, \\ \bigcup_{m \in M_{p_i}} \underbrace{\Psi^{-1}(m)}_{\boldsymbol{\Delta}^0_1 \restriction X^\omega} \cup \underbrace{(X^\omega \backslash B)}_{\boldsymbol{\Sigma}^0_1 \restriction X^\omega} & \text{if } p_i \subseteq s_\emptyset. \end{cases}$$

Hence $\Phi^{-1}(O_i) \in \boldsymbol{\Sigma}^0_1 \restriction X^\omega$ and thus $\Phi^{-1}(O) \in \boldsymbol{\Sigma}^0_1 \restriction X^\omega$. Therefore Φ is a continuous function.

Under this Φ, we have

$$\begin{aligned} \left[T_3^{\Phi,\pi}\right] &= \bigcup_{f \in X^\omega} \left(\{f\} \times \left[T^\pi_{\Phi(f)}\right]\right) \\ &= \bigcup_{n \in \omega} \left(\Psi^{-1}(n) \times [T^\pi_{s^n}]\right) \cup \Big((X^\omega \backslash B) \times [T^\pi_{s_\emptyset}]\Big) \\ &= \bigcup_{n \in \omega} \left(\Psi^{-1}(n) \times [T_n]\right) \cup (X^\omega \backslash B) \\ &= \left[T^{\Psi,B}_{X,T^{sq}}\right]. \end{aligned}$$

□

Notice that if we raise the complexity of B's, we need to raise the complexity of Φ's.

Question 14. *What is the relation between Type 2 trees and Type 3 trees?* ⊣

4.2 Definition of a Type 4 tree

In section 4.1, we define a Type 3 tree. In this section, we will modify the definition of a Type 3 tree to define a Type 4 tree. Unlike Type 3 trees, Type 4 trees are always defined.

First, recall definition 4.1.1.

Definition 4.1.1. *Fix a continuous function Φ from X^ω into ω^ω. Fix a bijection π from ω into $\omega^{<\omega}$. For each $f \in X^\omega$, define*

$$T^{\pi}_{\Phi(f)} = \{ \pi(\Phi(f)(n)) | \, n \in \omega \} .$$

$\dashv$

Definition 4.2.1. *(Definition of a Type 4 tree)*

Suppose Φ is a continuous function from X^ω into ω^ω and π is a bijection from ω to $\omega^{<\omega}$. Define

$$h \in \left[T_4^{\Phi,\pi}\right] \overset{df}{\Leftrightarrow} \begin{cases} h \in X^\omega \times \left[T^{\pi}_{\Phi(h\restriction\omega)}\right] & \text{if } T^{\pi}_{\Phi(h\restriction\omega)} \text{ is well-founded,} \\ h \in X^\omega & \text{otherwise.} \end{cases}$$

$\dashv$

It is easy to see that every Type 3 tree is a Type 4 tree.

4.3 Definition of a Type 5 tree

In section 4.1, we defined a Type 3 tree. In section 4.2, we defined a Type 4 tree by modifying the definition of a Type 3 tree. In this section, we will define a Type 5 tree. This tree is different from Type 3 and Type 4 trees. Type 5 trees are generalization of Type 2 trees for countable X.

Definition 4.3.1. *(Definition of a Type 5 tree)*

Assume X is countable. Suppose $\langle T_\alpha \,|\alpha \in 2^\omega \rangle$ to be such that each T_α is well-founded (T_α could be the empty tree). Suppose $\langle f_\alpha \,|\alpha \in 2^\omega \rangle$ is an enumeration of X^ω. Define a Type 5 tree $T^{\langle f_\alpha |\alpha \in 2^\omega \rangle}_{\langle T_\alpha |\alpha \in 2^\omega \rangle}$ by

$$\left[T^{\langle f_\alpha |\alpha \in 2^\omega \rangle}_{\langle T_\alpha |\alpha \in 2^\omega \rangle}\right] = \bigcup_{\alpha \in 2^\omega} (\{f_\alpha\} \times [T_\alpha]). \qquad \dashv$$

Observation 4.3.2. *Every Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ with countable X is a Type 5 tree.* $\dashv$

Proof.

Pick an arbitrary Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ with countable X. Suppose $\langle f_\alpha \,|\alpha \in 2^\omega \rangle$ is an enumeration of X^ω. For each $\alpha \in 2^\omega$, define

$$T_\alpha = \begin{cases} \{\emptyset\} & \text{if } f_\alpha \notin B, \\ T_{\Psi(f_\alpha)} & \text{if } f_\alpha \in B. \end{cases}$$

Then

$$h \in \left[T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times \left[T_{\Psi(h\restriction\omega)}\right] & \text{if } h \restriction \omega \in B. \end{cases}$$

$$\leftrightarrow \exists \alpha \in 2^\omega \text{ such that } h \restriction \omega = f_\alpha \text{ and } \begin{cases} h \in \{f_\alpha\} \times \{\emptyset\} & \text{if } h \restriction \omega \notin B, \\ h \in \{f_\alpha\} \times \left[T_{\Psi(f_\alpha)}\right] & \text{if } h \restriction \omega \in B. \end{cases}$$

$$\leftrightarrow h \in \bigcup_{\alpha \in 2^\omega} \left(\{f_\alpha\} \times [T_\alpha]\right)$$

$$\leftrightarrow h \in \left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}\right].$$

Thus $T^{\Psi,B}_{X,T_{sq}} = T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$. □

Observation 4.3.3. *There is a Type 5 tree with countable X which is not a Type 2 tree.* ⊣

Proof.

Fix a countable X. Let $WF(\omega)$ be the set of all well-founded trees on ω. Show $|WF(\omega)| = 2^\omega$.

Show $|WF(\omega)| \leq 2^\omega$.

$$|WF(\omega)| \leq |\wp(\omega^{<\omega})| = |\wp(\omega)| = 2^\omega.$$

Show $|WF(\omega)| \geq 2^\omega$.

$\forall \alpha \in 2^\omega$, define $[T_\alpha] = \{\langle n, \alpha(n)\rangle \,| n \in \omega\}$. Then each $T_\alpha \in WF(\omega)$ and for all $\alpha, \beta \in 2^\omega$ such that $\alpha \neq \beta$, $T_\alpha \neq T_\beta$ so that there are 2^ω many distinct T_αs.

Thus we have

$$|WF(\omega)| = 2^\omega.$$

Hence there are 2^ω many distinct well-founded trees on ω. Let $\langle T_\alpha \,| \alpha \in 2^\omega\rangle$ be an enumeration of $WF(\omega)$ excluding the empty tree. Consider a Type 5 tree $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$. Show that $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$ is not a Type 2 tree.

Suppose, for a contradiction, there exists a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$ such that $T^{\Psi,B}_{X,T_{sq}} = T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$. Let $T_{sq} = \langle T^n | n \in \omega \rangle$. Fix $\alpha \in 2^\omega$ such that for all $n \in \omega$, $T_\alpha \neq T^n$.

Case 1 : $f_\alpha \notin B$.

Then $f_\alpha \in [T^{\Psi,B}_{X,T^{sq}}] \setminus \left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle} \right]$ since every T_α is not the empty tree. Thus $T^{\Psi,B}_{X,T_{sq}} \neq T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$.

Case 2 : $f_\alpha \in B$.

By the definition of $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$, for all $g \in \omega^{<\omega}$,

$$f_\alpha{}^\frown g \in \left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle} \right] \Leftrightarrow g \in [T_\alpha].$$

Since for all $n \in \omega$, $T_\alpha \neq T^n$,

$$\left([T_\alpha] \setminus \left[T^{\Psi(f_\alpha)} \right] \right) \cup \left(\left[T^{\Psi(f_\alpha)} \right] \setminus [T_\alpha] \right) \neq \emptyset.$$

Assume $\hat{g} \in \left([T_\alpha] \setminus \left[T^{\Psi(f_\alpha)} \right] \right)$. Then $f_\alpha{}^\frown g \in \left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle} \right] \setminus \left[T^{\Psi,B}_{X,T^{sq}} \right]$. This is a contradiction. Hence $\hat{g} \in \left(\left[T^{\Psi(f_\alpha)} \right] \setminus [T_\alpha] \right)$. Then $f_\alpha{}^\frown g \in \left[T^{\Psi,B}_{X,T^{sq}} \right] \setminus \left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle} \right]$ This is a contradiction. Therefore, $T^{\Psi,B}_{X,T_{sq}} \neq T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$.

Hence $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$ is not a Type 2 tree. $\square$

The following is a question throughout this chapter.

Question 15. *What are the determinacy strength of games on Type 3, Type 4 and type 5 trees?* $\dashv$

Appendix A

Big picture

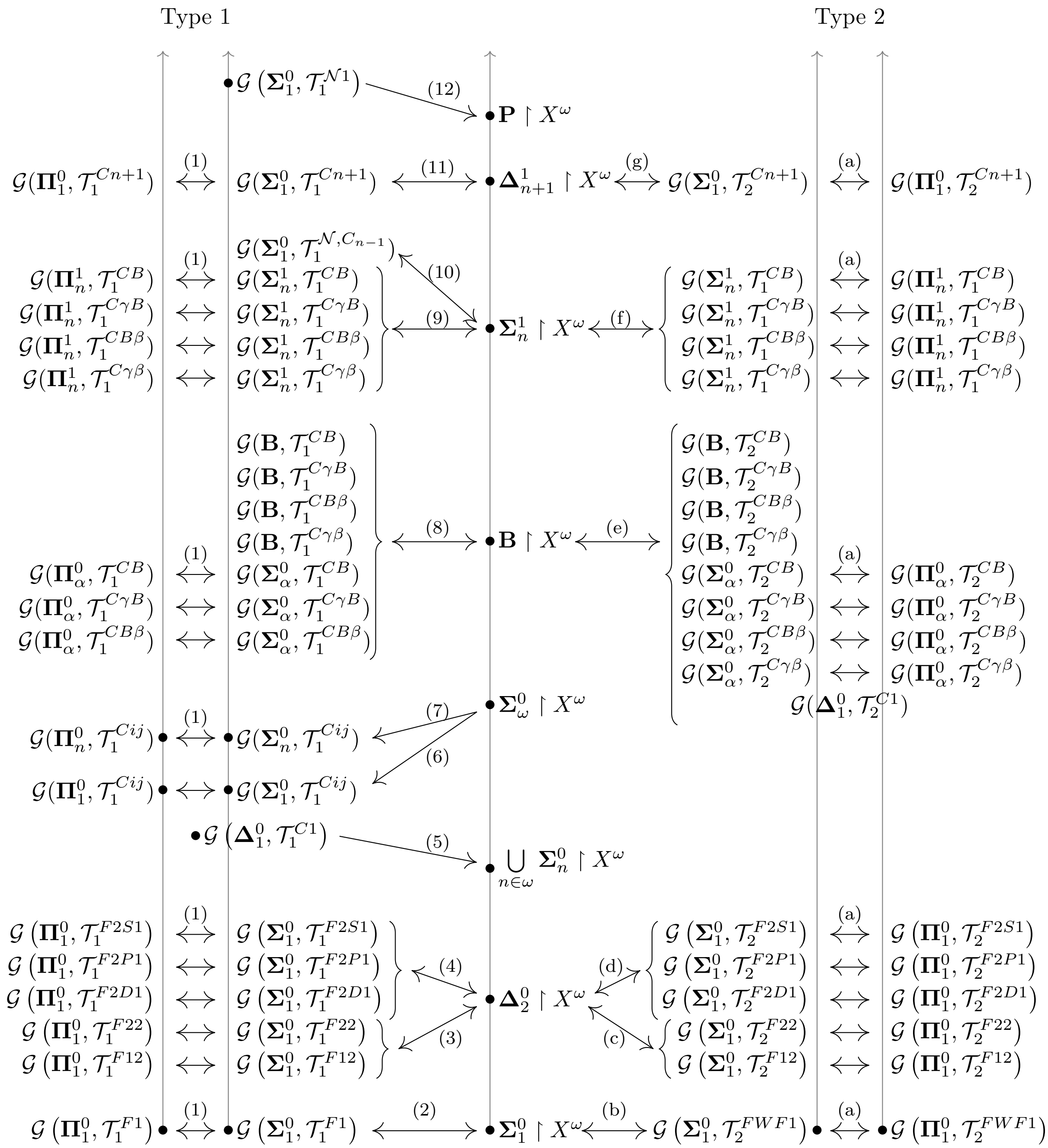
In this section, we draw a picture of some of the determinacy results proved in this dissertation. The symbols $\rightarrow$, $\leftarrow$ and $\leftrightarrow$ are used to compare the determinacy strength of each collection. For example,

$$A \leftrightarrow \begin{cases} B \\ C \\ D \end{cases}$$

illustrates that $det(A) \leftrightarrow det(B)$, $det(A) \leftrightarrow det(C)$ and $det(A) \leftrightarrow det(D)$. Similarly,

$$\left.\begin{matrix} B \\ C \\ D \end{matrix}\right\} \leftrightarrow A$$

illustrates that $det(A) \leftrightarrow det(B)$, $det(A) \leftrightarrow det(C)$ and $det(A) \leftrightarrow det(D)$. The symbols shown in Figure A.0.1 is listed on pages 357 and 358.



$n, i, j \in \omega$, $\alpha, \beta, \gamma \in \omega_1$, Symbols are listed on the next page.

(1) theorem 2.3.1, (2) theorem 2.6.1, (3) theorem 2.6.2, (4) theorem 2.6.3, (5) corollary 2.4.17, (6) theorem 2.5.20, (7) theorem 2.5.30, (8) corollary 2.6.6, (9) theorem 2.6.7, (10) comment above observation 2.5.45, (11) corollary 2.6.5, (12) observation 2.5.45, (a) corollary 3.3.11, (b) theorem 3.6.3, (c) theorem 3.6.2, (d) theorem 3.6.3, (e) corollary 3.6.8, (f) theorem 3.6.9, (g) corollary 3.6.5.

Figure A.0.1: Illustration of the determinacy equivalences between well-known results and some of the results in this dissertation.

Symbols shown in the Figure A.0.1 under Type 1:

$$\mathcal{T}_1^{F1} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{F22} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Delta}_2^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{F12} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_2^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{F2D1} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{F2S1} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Sigma}_1^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{F2P1} = Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Pi}_1^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{C1} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{Cij} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_i^0), \mathbf{\Sigma}_j^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{CB\beta} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{C\gamma B} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{B} \restriction X^\omega\right)$$

$$\mathcal{T}_1^{C\gamma\beta} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}_\alpha^0), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)$$

$$\mathcal{T}_1^{CB} = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)$$

$$\mathcal{T}_1^{\mathcal{N}, C_{n-1}} = Tree_1\left(X, \mathcal{N}, C_{n-1}, X^\omega\right)$$

$$\mathcal{T}_1^{\mathcal{N}1} = Tree_1\left(X, \mathcal{N}, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \restriction X^\omega\right)$$

Symbols shown in the Figure A.0.1 under Type 2:

$$\mathcal{T}_2^{F1} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{F22} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Delta}_2^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{F12} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_2^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{F2D1} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Delta}_1^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{F2S1} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Sigma}_1^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{F2P1} = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}_2^0), \mathbf{\Pi}_1^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{C1} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}_1^0), \mathbf{\Delta}_1^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{Cij} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}_i^0), \mathbf{\Sigma}_j^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{CB\beta} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{C\gamma B} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{B} \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{C\gamma\beta} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}_\alpha^0), \mathbf{\Sigma}_\beta^0 \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{CB} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)$$

$$\mathcal{T}_2^{Cn+1} = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}_{n+1}^1), \mathbf{\Delta}_{n+1}^1 \upharpoonright X^\omega\right)$$

Appendix B

List of Symbols

The following symbols are specific to this dissertation.

B.1 Letters with special meanings

FIN : the collection of nonempty finite sets 44

CTB : the collection of nonempty countable sets 44

WF : the set of nonempty well-founded trees 44

FWF : the set of nonempty well-founded trees with each move is from some finite set · · 44

CWF : the set of nonempty well founded trees with each move is from some countable set 44

X : a nonempty set 47, 214

Y : a nonempty set 47

B.2 Symbols related to Type 1 and Type 2 trees

B.2.1 Symbols related to Type 1 trees

B.2.2 Symbols related to Type 2 trees

B.3 Other notations

Appendix C

Definition of Trees

In this dissertation, we only consider Type 1 and Type 2 trees. For Type 3, Type 4 and Type 5 trees, we don't have results. We will list those trees. See more details in chapter 4.

C.1 Type 1 tree : $T^{\Psi,B}_{X,Y}$

Definition 2.1.2. *(Definition of a Type 1 tree)*

Suppose X and Y are nonempty sets. Let B be a subset of X^ω and let Ψ be a function from X^ω into ω. For any $h \in X^\omega \times Y^{<\omega}$, define $[T^{\Psi,B}_{X,Y}]$ by :

$$h \in \left[T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times Y^{\Psi(h\restriction\omega)+1} & \text{if } h \restriction \omega \in B. \end{cases}$$

A tree T is a Type 1 tree if and only if $T = T^{\Psi,B}_{X,Y}$ for some nonempty sets X and Y, a function Ψ from X^ω into ω and a subset B of X^ω. (Possibly $X = Y$ and also B could be the empty set.) $\dashv$

Definition 2.7.1. *(Definition of an α-Type 1 tree)*

Suppose α is a limit ordinal. Suppose X and Y are nonempty sets. Let B be a subset of X^α and let Ψ be a function from X^α into ω. For any $h \in X^\alpha \times Y^{<\omega}$, define $[{}_\alpha T^{\Psi,B}_{X,Y}]$ by :

$$h \in \left[{}_\alpha T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\alpha & \text{if } h \restriction \alpha \notin B, \\ h \in X^\alpha \times Y^{\Psi(h\restriction\alpha)+1} & \text{if } h \restriction \alpha \in B. \end{cases}$$

⊣

C.2 Type 2 tree : $T^{\Psi,B}_{X,T_{sq}}$

Definition 3.1.2. *(Definition of a Type 2 tree)*

Suppose X is a nonempty set, Ψ is a function from X^ω into ω, B is a subsets of X^ω and $T_{sq} = \langle T_n \,|n \in \omega\rangle$ where each T_n is a tree. Define $[T^{\Psi,B}_{X,T_{sq}}]$ by :

$$h \in \left[T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times \left[T_{\Psi(h\restriction\omega)}\right] & \text{if } h \restriction \omega \in B. \end{cases}$$

A tree T is a Type 2 tree if and only if $T = T^{\Psi,B}_{X,T_{sq}}$ for some nonempty set X, a function Ψ from X^ω into ω, a subset B of $X^\omega$and some $T_{sq} = \langle T_n \,|n \in \omega\rangle$, where each T_n is a nonempty tree. ⊣

Definition 3.7.1. *(Definition of an α-Type 1 tree)*

Suppose α is a limit ordinal. Suppose X is a nonempty set, Ψ is a function from X^α into ω, B is a subsets of X^α and $T_{sq} = \langle T_n \,|n \in \omega\rangle$ where each T_n is a tree. Define $[{}_\alpha T^{\Psi,B}_{X,T_{sq}}]$ by :

$$h \in \left[{}_\alpha T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\alpha & \text{if } h \restriction \alpha \notin B, \\ h \in X^\alpha \times \left[T_{\Psi(h\restriction\alpha)}\right] & \text{if } h \restriction \alpha \in B. \end{cases}$$

$\dashv$

C.3 Type 3 tree : $T_3^{\Phi,\pi}$

Definition 4.1.2. *(Definition of a Type 3 tree)*

Fix a continuous function Φ from X^ω into ω^ω. Fix a bijection π from ω into $\omega^{<\omega}$. If for all $f \in X^\omega$, every $T^\pi_{\Phi(f)}$ is a well-founded tree, then we define a Type 3 tree $T^{\Phi,\pi}$ by

$$\left[T_3^{\Phi,\pi}\right] = \bigcup_{f \in X^\omega} \left(\{f\} \times \left[T^\pi_{\Phi(f)}\right]\right).$$

$\dashv$

C.4 Type 4 tree : $T_4^{\Phi,\pi}$

Definition 4.2.1. *(Definition of a Type 4 tree)*

Suppose Φ is a continuous function from X^ω into ω^ω and π is a bijection from ω to $\omega^{<\omega}$. Define

$$h \in \left[T_4^{\Phi,\pi}\right] \overset{df}{\Leftrightarrow} \begin{cases} h \in X^\omega \times \left[T^\pi_{\Phi(h\restriction\omega)}\right] & \textit{if } T^\pi_{\Phi(h\restriction\omega)} \textit{ is well-founded,} \\ h \in X^\omega & \textit{otherwise.} \end{cases}$$

$\dashv$

C.5 Type 5 tree : $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$

Definition 4.3.1. *(Definition of a Type 5 tree)*

Assume X is countable. Suppose $\langle T_\alpha \,|\alpha \in 2^\omega\rangle$ to be such that each T_α is well-founded (T_α could be the empty tree). Suppose $\langle f_\alpha \,|\alpha \in 2^\omega\rangle$ is an enumeration of X^ω. Define a Type 5 *tree* $T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}$ *by*

$$\left[T^{\langle f_\alpha | \alpha \in 2^\omega \rangle}_{\langle T_\alpha | \alpha \in 2^\omega \rangle}\right] = \bigcup_{\alpha \in 2^\omega} (\{f_\alpha\} \times [T_\alpha]). \qquad \dashv$$

C.6 Yost tree $T^{\alpha}_{g.t.}$

Definition C.6.1. *(Definition of a Yost tree $T^{\alpha}_{g.t}$)(See more details in Yost, n.d.)*

For each limit ordinal $\alpha \in \omega_1$, the Yost tree $T^{\alpha}_{g.t.}$ is constructed by the following manner:

1. *Fix a limit ordinal $\alpha \in \omega_1$ and fix an α's decomposition as below.*
2. *Each play in $T^{\alpha}_{g.t.}$ is finite and moves from ω (a finite sequence from ω).*
3. *The length of a play is determined by a certain player I's move depending on the decomposition.*

Fix a decomposition of α as follows and construct an ordinal α decomposition tree H_α.

1. *Fix $\langle \alpha_{n_1} + 2n_1 + 1 \,| n_1 \in \omega \rangle$ such that*
   - $\sup_{n_1 \in \omega} (\alpha_{n_1} + 2n_1 + 1) = \alpha$,
   - *each α_{n_1} is a limit ordinal.*
2. *Repeat this for each α_{n_1} i.e.,*

   fix $\langle \alpha_{n_1,n_2} + 2n_2 + 1 \,| n_2 \in \omega \rangle$ such that
   - $\sup_{n_2 \in \omega} (\alpha_{n_1,n_2} + 2n_2 + 1) = \alpha_{n_1}$,
   - *each α_{n_1,n_2} is a limit ordinal.*
3. *Repeat this for each $\alpha_{n_1,n_2}, \ldots$ until we get down to $\alpha_{n_1,n_2,\ldots,n_{l-1}} = \omega$ for some $l - 1 \in \omega$.*

Then define $\alpha_{n_1,n_2,...,n_l} = 0$ *for each* $n_l \in \omega$ *We have* $\sup_{n_l \in \omega} \left(\alpha_{\alpha_{n_1,n_2,...,n_l}} + 2n_l + 1\right) = \omega$. *Define the ordinal* α *decomposition tree* H_α *to be such that* $[H_\alpha]$ *consists of such sequences* $\langle n_1, n_2, ..., n_l \rangle$. *We define the Yost tree* $g \in [T^{\alpha}_{g.t.}]$ *if and only if*

1. *each move of* g *is from* ω,

2. *there is a sequence* $\langle n_1, n_2, ..., n_l \rangle \in [H_\alpha]$ *such that*

   - $g(0) = n_1$,
   - $g\left(\sum_{i=1}^{j} (2n_j + 2)\right) = n_{j+1}$,
   - $lh(g) = \sum_{i=1}^{l} (2n_j + 2)$. ⊣

Appendix D

Well-known Determinacy Results

D.1 Determinacy results from ZFC

Theorem 1.4.2. *(Gale and Stewart, 1953)*

Suppose T is a tree. If T is well-founded, then for any $A \subseteq [T]$, $G(A;T)$ is determined. ⊣

Theorem 1.4.3. *(AC)(Gale and Stewart, 1953)(as cited in Moschovakis, 2009, p. 222, 6A.6)*

There exists $A \subseteq \omega^\omega$ such that $G(A;\omega^{<\omega})$ is not determined. ⊣

Theorem 1.4.6. *(Gale and Stewart, 1953)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{\Sigma}^0_1 \restriction [T])$ and $Det(\mathbf{\Pi}^0_1 \restriction [T])$. ⊣

Theorem 1.4.7. *(Wolfe, 1955)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{\Sigma}^0_2 \restriction [T])$. ⊣

Theorem 1.4.8. *(Martin, 1975; Martin, 1990)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then $Det(\mathbf{B} \restriction [T])$. ⊣

Theorem 1.4.9. *(Martin, 1990)*

Suppose $T = X^{<\omega}$ for some nonempty X. Then Det$(\mathbf{qB} \restriction [T])$. ⊣

D.2 Results related to the existence of measurable cardinals

Theorem 1.4.12. *(Martin, 1970)*

If there is a measurable cardinal, then Det$(\mathbf{\Pi}^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.13. *(Martin, 1970)(as cited in Martin, 2017 draft, p.187, Theorem 4.1.6)*

Let T be a game tree. Assume there is a measurable cardinal larger than $|T|$. Then Det$(\mathbf{\Pi}^1_1 \restriction [T])$. ⊣

Theorem 1.4.14. *(Martin, 1990, p. 287, Theorem 3)*

If there is a measurable cardinal, then Det$(\omega^2\text{-}\mathbf{\Pi}^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.15. *(Martin, 1990, p. 292, Theorem 4)*

If there is a measurable cardinal, then Det $\Delta((\omega^2 + 1)\text{-}\mathbf{\Pi}^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.16. *(Martin, 2017 draft, p.241, Chapter 5 Theorem 5.2.32)*

Let α be a countable ordinal and $T = X^{<\omega}$. If the class of measurable cardinals greater than $|T|$ has order type $\geq \alpha$, then Det $\Delta((\omega^2 \cdot \alpha + 1)\text{-}\mathbf{\Pi}^1_1 \restriction [T])$. ⊣

Theorem 1.4.17. *(Simms 1979* [1]*)*

Let $T = X^{<\omega}$. If there is a measurable limit of measurable cardinals that is larger than $|T|$, then Det$(\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1) \restriction [T])$. ⊣

[1] as cited in Martin (2017 draft, p. 281, Chapter 5 Theorem 5.4.5).

D.3 Projective determinacy

Theorem 1.4.22. *(Martin and Steel, 1985)*

For $n \in \omega$, if there exist n Woodin cardinals with a measurable cardinal above them, then $Det(\mathbf{\Pi}^1_{n+1} \restriction \omega^\omega)$. ⊣

Theorem 1.4.23. *(Martin and Steel, 1985)*

Suppose there are infinitely many Woodin cardinals. Then $Det(\mathbf{P} \restriction \omega^\omega)$. ⊣

D.4 Lightface results related to the existence of $0^\#$

Theorem 1.4.27. *(Martin, 1970; Martin, 2017 draft, p.2.9, Theorem 4.4.3)*

If $0^\#$ exists, then $Det(\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.28. *(Friedman, 1971 [2])*

If $0^\#$ exists, then $Det(3\text{-}\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.29. *(Martin, early 1970's [3])*

If $0^\#$ exists, then $Det(\bigcup_{\beta \in \omega^2} \beta\text{-}\Pi^1_1 \restriction \omega^\omega)$. ⊣

Theorem 1.4.30. *(Martin, 1975)*

$Det(3\text{-}\Pi^1_1 \restriction \omega^\omega)$ implies $0^\#$ exists. ⊣

Theorem 1.4.31. *(Harrington, 1978 [4])*

$Det(\Pi^1_1 \restriction \omega^\omega)$ implies $0^\#$ exists. ⊣

[2] as cited in DuBose (1990, p. 504).
[3] as cited in DuBose (1990, p. 512).
[4] as cited in DuBose (1990, p. 512); Martin (2017 draft, p. 209).

Theorem 1.4.32. *(Martin and Harrington)*

$Det(\Pi^1_1 \restriction \omega^\omega)$ *if and only if* $0^\#$ *exists if and only if* $Det(\bigcup_{\beta \in \omega^2} \beta\text{-}\Pi^1_1 \restriction \omega^\omega)$. $\dashv$

Appendix E

Definitions and Notations

The following are lists of definitions and notations specific to this dissertation.

E.1 Chapter 1

Notation 1.3.1. *(Definition of a complexity)*

In this dissertation, whenever we mention a "complexity" in chapters 2 and 3, we mean the complexities defined in this section, i.e., Borel, projective and difference hierarchy, unless specified. More precisely, the definition of a complexity in this dissertation is the following: Suppose we have Ξ *such that for each tree* T, $\Xi \restriction [T] \subseteq \wp([T])$ *is defined (e.g.,* $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$*). Then we say* Ξ *is a complexity.* $\dashv$

Notation 1.5.3. *(Abuse of product notation)*

Suppose T, T_1, T_2 *are trees and satisfies the following properties.*

1. *every path of* $[T_1]$ *has length* α *for a fixed* α,
2. *for any* $\langle f, g\rangle \in [T_1] \times [T_2]$, $f^\frown g \in [T]$ *and*

3. for any $h \in [T]$, $\langle h \restriction \alpha, h \restriction [\alpha, lh(h))\rangle \in [T_1] \times [T_2]$.

Then to simplify notation, we abuse the cross product notation and express

$$[T] = [T_1] \times [T_2].$$

Caution :

We will use the actual cross product in some places. Readers should identify them from the context. $\dashv$

Notation 1.5.4. *(Abuse of inverse image notation)*

Suppose f *is a function from* A *to* B*. If* $b \in B$ *is a singleton, we suppress* $\{\}$ *for* $f^{-1}(\{b\})$*, i.e., we write* $f^{-1}(b)$ *to mean* $f^{-1}(\{b\})$. $\dashv$

E.2 Chapter 2

Definition 2.1.2. *(Definition of a Type 1 tree)*

Suppose X and Y are nonempty sets. Let B be a subset of X^ω and let Ψ be a function from X^ω into ω. For any $h \in X^\omega \times Y^{<\omega}$, define $[T^{\Psi,B}_{X,Y}]$ by :

$$h \in \left[T^{\Psi,B}_{X,Y}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times Y^{\Psi(h\restriction\omega)+1} & \text{if } h \restriction \omega \in B. \end{cases}$$

A tree T is a Type 1 tree if and only if $T = T^{\Psi,B}_{X,Y}$ for some nonempty sets X and Y, a function Ψ from X^ω into ω and a subset B of X^ω. (Possibly $X = Y$ and also B could be the empty set.) ⊣

Definition 2.1.3. *(Definition of the (X,Y)-TEP-$[\omega, \omega+\omega)$ property)*

Suppose X and Y are nonempty sets. Let T be a tree. T satisfies (X,Y)-TEP-$[\omega, \omega+\omega)$ property if for all $y \in [T]$, y satisfies the following four properties:

1. *$y \restriction \omega \in X^\omega$.*

2. *$lh(y) \in [\omega, \omega+\omega)$.*

3. *If $lh(y) > \omega$, then each move of the tail of h is from Y.*

4. *If $lh(y) > \omega$, then there exists a unique $n \in \omega \setminus \{0\}$ such that*

$$\forall g \in Y^n \;\; (y \restriction \omega)^\frown g \in [T] \quad \textit{(tail exchange property)}$$ ⊣

Definition 2.2.2. *(Definition of a $Tree_1$ collection)*

Fix a nonempty set X. Let $\Theta \neq \emptyset$ be any collection of nonempty sets. Suppose $\Lambda \neq \emptyset$ is any *collection of subsets of* X^ω *and* $\Gamma \neq \emptyset$ *is a collection of functions from* X^ω *to* ω. *Define*

$$Tree_1\left(X,\Theta,\Gamma,\Lambda\right) = \left\{T^{\Psi,B}_{X,Y}\,|Y \in \Theta, \Psi \in \Gamma, B \in \Lambda\right\}.$$

A collection is a $Tree_1$ *collection if and only if it is* $Tree_1(X,\Theta,\Gamma,\Lambda)$ *for some nonempty set* X*, a nonempty collection* Θ *of* Y*'s, a nonempty collection* Γ *of functions from* X^ω *into* ω *and a nonempty collection* Λ *of subsets of* X^ω.

We sometimes let $\mathcal{T}_1$ *be a* $Tree_1$ *collection when we wish to suppress* X, Θ, Γ *and* Λ*, i.e.,* $\mathcal{T}_1 = Tree_1\left(X,\Theta,\Gamma,\Lambda\right)$. ⊣

Notation 2.2.3. *When dealing with singletons for any of the last three components of* $Tree_1\left(X,\Theta,\Gamma,\Lambda\right)$*, we will suppress* $\{\}$*, i.e., if* Θ *is a singleton* $\{Y\}$*,* $Tree_1\left(X,Y,\Gamma,\Lambda\right)$ *abbreviates* $Tree_1\left(X,\{Y\},\Gamma,\Lambda\right)$. *Similarly, if* Γ *is a singleton* $\{f\}$*,* $Tree_1\left(X,\Theta,f,\Lambda\right)$ *abbreviates* $Tree_1\left(X,\Theta,\{f\},\Lambda\right)$*and if* Λ *is a singleton* $\{B\}$*,* $Tree_1\left(X,\Theta,\Gamma,B\right)$ *abbreviates* $Tree_1(X, \Theta,\Gamma,\{B\})$. ⊣

Definition 2.2.6. *(Definition of "games on a* $Tree_1$ *collection")*

Let $Tree_1$ *collection* $\mathcal{T}_1 = Tree_1(X,\Theta,\Gamma,\Lambda)$ *for some* X,Θ,Γ *and* Λ. *Define "games on the* $Tree_1$ *collection* $\mathcal{T}_1$*" by*

$$\bigcup_{T\in\mathcal{T}_1}\left\{G\left(A;T\right)|A \subseteq [T]\right\}$$ ⊣

Definition 2.2.7. *(Definition of* Ξ *games on a* $Tree_1$ *collection)*

Let $Tree_1$ *collection* $\mathcal{T}_1 = Tree_1(X,\Theta,\Gamma,\Lambda)$. *Suppose we have* Ξ *such that for each* $T \in Tree_1$*,* $\Xi \restriction [T] \subseteq \wp\left([T]\right)$ *is defined (e.g.,* $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$*). Define* Ξ *games on a* $Tree_1$ *collection* $\mathcal{T}_1$ *by*

$$\mathcal{G}\left(\Xi;\mathcal{T}_1\right) = \bigcup_{T\in\mathcal{T}_1}\left\{G\left(A;T\right)|A \in \Xi \restriction [T]\right\}.$$

As a notation, we will use $\mathcal{G}$ for a collection of games on a $Tree_1$ collection. $\dashv$

Definition 2.2.8. *(Definition of Ξ determinacy on a $Tree_1$ collection)*

Let $Tree_1$ collection $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in \mathcal{T}_1$, $\Xi \restriction [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ determinacy on the $Tree_1$ collection $\mathcal{T}_1$ by

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_1),$$

i.e., for any $T \in \mathcal{T}_1$ and $A \in \Xi \restriction [T]$, every game $G(A;T)$ is determined. $\dashv$

Definition 2.3.2. *(Definition of B^+ and Ψ^+)*

Fix a Type 1 tree $T^{\Psi,B}_{X,Y}$. Then $B \subseteq X^\omega$ and $\Psi : X^\omega \to \omega$. Define

1. *$B^+ = X \times B \subseteq X^\omega$,* [1]

2. *$\Psi^+ : X^\omega \to \omega$ such that $\Psi^+(f) = \Psi(f \restriction [1, \omega)) + 1$.* $\dashv$

Definition 2.3.3. *(Definition of a Shift tree $Sft(T)$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Define a Shift tree $Sft(T)$ by

$$Sft(T) = T^{\Psi^+,B^+}_{X,Y}. \quad \dashv$$

Proposition 2.3.5. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Then for every $h \in [Sft(T)]$,*

$$\begin{cases} h \restriction [1,\omega) \in [T] & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) \in [T] & \text{if } h \restriction \omega \in B^+. \end{cases} \quad \dashv$$

[1]Recall abuse of notation 1.5.3 on page 42.

Definition 2.3.6. *(Definition of the erasing function $e : [Sft(T)] \to [T]$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Define the erasing function e from $[Sft(T)]$ into $[T]$ by

$$e: \ [Sft(T)] \to [T]$$
$$h \mapsto \begin{cases} h \restriction [1,\omega) & \text{if } h \restriction \omega \notin B^{+}, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) & \text{if } h \restriction \omega \in B^{+}. \end{cases} \quad \dashv$$

Definition 2.3.7. *(Definition of Shift)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Define

$$Shift: \ \wp([T]) \to \wp([Sft(T)])$$
$$A \mapsto \{h \in [Sft(T)] \mid e(h) \in [T] \setminus A\}. \quad \dashv$$

Definition 2.3.10. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Given $S \subseteq [T]$, define*

$$S^{+} = \{h \in [Sft(T)] \, | e(h) \in S\}. \quad \dashv$$

Definition 2.3.16. *Fix a Type 1 tree $T = T^{\Psi,B}_{X,Y}$. Pick an arbitrary $k \in \omega$. Recall*

$$(\omega^\omega)^k = \underbrace{\omega^\omega \times \cdots \times \omega^\omega}_{k \text{ many}}.$$

Given $S_k \subseteq [T] \times (\omega^\omega)^k$, define

$$S_k^{+} = \left\{\langle h, g_1, ..., g_k\rangle \in [Sft(T)] \times (\omega^\omega)^k \, | \langle e(h), g_1, ..., g_k\rangle \in S_k \right\}. \quad \dashv$$

Definition 2.3.23. *Let $k \in \omega$. Suppose $\langle S_i^k | i \in \omega\rangle$ to be such that each $S_i^k \subseteq X^\omega \times \omega^\omega \times (\omega^\omega)^k$.*

Define

$$\langle S_i^k | i \in \omega\rangle^{-} = \left\{\langle f, h, g_1, ..., g_k\rangle \in X^\omega \times \omega^\omega \times (\omega^\omega)^k \,\middle|\, \langle f, h \restriction [1,\omega), g_1, ..., g_k\rangle \in S^k_{h(0)} \right\}. \quad \dashv$$

Definition 2.4.7. *Suppose* $l \in \omega$*. Let* $A \subseteq X^\omega \times Y^{l+1}$*. For each* $a \in Y$*, define*

$$A_a^{short} = \left\{ f \in X^\omega \times Y^l \,|\, f^\frown \langle a \rangle \in A \right\}. \qquad \dashv$$

Definition 2.4.12. *Suppose* $l \in \omega$*. Let* $A_a \subseteq X^\omega \times Y^l$ *for all* $a \in Y$*. Define*

$$\langle A_a \,|a \in Y \rangle^{long} = \left\{ h \in X^\omega \times Y^{l+1} \left| h \restriction (\omega + l) \in A_{h(\omega+l)} \right. \right\}. \qquad \dashv$$

Definition 2.4.18. *(Definition of the length function* $lh_{[T]}$*)*

$$\begin{aligned} lh_{[T]} :\ & [T] \to \omega + \omega \\ & h \mapsto lh\,(h)\,. \end{aligned} \qquad \dashv$$

Definition 2.4.19. *Suppose* $B \subseteq X^\omega$*,* Ψ *is a function from* X^ω *into* ω *and* Y *is arbitrary. Define*

$$Long\,(B) = \{h \in [T_{X,Y}^{\Psi,B}] \,|\, lh(h) > \omega\}. \qquad \dashv$$

Definition 2.4.24. *(Definition of* Max*)*

Suppose $\Psi \restriction B$ *is bounded below* ω*. Let* $n_{\max}^{\Psi,B}$ *be the maximum tail length determined from* Ψ *and* B*. (*$n_{\max}^{\Psi,B} = \max(Im(\Psi \restriction B)) + 1$*.) If* Ψ *and* B *are clear from the context, we suppress* Ψ *and* B*, i.e.,*$n_{\max} = n_{\max}^{\Psi,B}$*.*

Define

$$Max\,(\Psi, B) = \{h \in [T_{X,Y}^{\Psi,B}] \,|\, lh(h) = \omega + n_{\max}\} = lh^{-1}_{[T_{X,Y}^{\Psi,B}]}(\omega + n_{\max}). \qquad \dashv$$

Definition 2.4.36. *Suppose* $\alpha \in \omega_1$ *is even and* $\langle A_\beta \,|\beta \leq \alpha \rangle$ *where each* $A_\beta \subseteq X^\omega$*. Fix a Type 1 tree* $T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta}$*. Define*

$$dk_{<\alpha}\left(\langle A_\beta \,|\beta \leq \alpha \rangle\right) = \left\{ h \in \left[T_{X,Y}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta} \right] \middle| \, h \restriction \omega \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \restriction \omega \notin A_\beta) \text{ is odd} \right\}. \dashv$$

Definition 2.5.2. *Suppose* $A \subseteq [T^{\Psi,B}_{X,Y}]$*. For each* $n \in \omega$*, define*

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times Y^{n+1}),$$

$$A_\emptyset = A \cap (X^\omega \backslash B).$$

Then $A = \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset$. ⊣

Definition 2.5.3. *Suppose* $A \subseteq [T^{\Psi,B}_{X,Y}]$*. For every* $n \in \omega$ *and* $g \in Y^{n+1}$*, define*

$$A^n_g = \{ f \in X^\omega \,|\, f^\frown g \in A^n \} . \quad ⊣$$

Observation 2.5.5. *Suppose* $A \subseteq [T^{\Psi,B}_{X,Y}]$*. For all* $n \in \omega$ *and for all* $p \in Y^{\leq n+1}$*,* $A^n_p \subseteq B \cap \Psi^{-1}(n)$. ⊣

Definition 2.5.31. *(Definition of "Fix")*

For all $m \in \omega$*, fix* $a_m \in Y^{m+1}$*. Define*

$$Fix\,\langle a_m : m \in \omega\rangle : \quad X^\omega \to [T^{\Psi,B}_{X,Y}]$$
$$f \mapsto \begin{cases} f & \text{if } f \notin B, \\ f^\frown a_{\Psi(f)} & \text{otherwise.} \end{cases}$$

If $\langle a_m : m \in \omega\rangle$ *is clear from the context, we will denote* Fix *to mean* $Fix\,\langle a_m : m \in \omega\rangle$. ⊣

Sublemma 2.5.42. *Suppose* $f : X_1 \to X_2$*. Assume that* $E \subseteq X_2$ *and for* $1 \leq j \leq k$*,* $F^j \subseteq \omega^\omega$*. Then*

$$(\varphi^f_k)^{-1}\left(E \times F^1 \times \cdots \times F^k\right) = f^{-1}(E) \times F^1 \times \cdots \times F^k. \quad ⊣$$

E.3 Chapter 3

Definition 3.1.2. *(Definition of a Type 2 tree)*

Suppose X is a nonempty set, Ψ is a function from X^ω into ω, B is a subsets of X^ω and $T_{sq} = \langle T_n \, | n \in \omega \rangle$ where each T_n is a tree. Define $[T^{\Psi,B}_{X,T_{sq}}]$ by :

$$h \in \left[T^{\Psi,B}_{X,T_{sq}}\right] \leftrightarrow \begin{cases} h \in X^\omega & \text{if } h \restriction \omega \notin B, \\ h \in X^\omega \times \left[T_{\Psi(h\restriction\omega)}\right] & \text{if } h \restriction \omega \in B. \end{cases}$$

A tree T is a Type 2 tree if and only if $T = T^{\Psi,B}_{X,T_{sq}}$ for some nonempty set X, a function Ψ from X^ω into ω, a subset B of $X^\omega$and some $T_{sq} = \langle T_n \, | n \in \omega \rangle$, where each T_n is a nonempty tree. $\dashv$

Definition 3.1.4. *(Definition of the tail tree T^f of f)*

Suppose T is a tree. For each $f \in X^\omega$, define $[T^f]$ to be the set of tails for f, i.e., for any $f \in X^\omega$ and for any $g \in [T^f]$, $f^\frown g \in T$. Then $f \in X^\omega \cap [T]$ if and only if $[T^f] = \emptyset$. Notice that each T^f is a tree. Define T^f to be the tail tree of f. $\dashv$

Definition 3.1.5. *(Definition of the countable tail trees property)*

Suppose T is a tree. Define that T has "the countable tail trees property" if and only if $\left\{ \left[T^f\right] | f \in X^\omega \right\}$ is countable. $\dashv$

Definition 3.1.6. *(Definition of $(X$,countable tail trees)-$[\omega, \infty))$*

Suppose X is a nonempty set. Define that a tree T is "$(X$,countable tail trees)-$[\omega, \infty)$" if $[T]$ satisfies the following three properties:

1. *for all $y \in [T]$, $y \restriction \omega \in X^\omega$.*

2. *for all $y \in [T]$, $lh(y) \geq \omega$.*

3. T satisfies the countable tail trees property. ⊣

Definition 3.2.2. *(Definition of a $Tree_2$ collection)*

Fix a nonempty set X. Let $\Upsilon \neq \emptyset$ be any collection of nonempty trees. Suppose $\Lambda \neq \emptyset$ is any collection of subsets of X^ω, $\Gamma \neq \emptyset$ is a collection of functions from X^ω to ω. Define

$$Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right) = \left\{ T^{\Psi,B}_{X,T_{sq}} \left| T_{sq} \in \Upsilon^\omega, \Psi \in \Gamma, B \in \Lambda \right. \right\}.$$

A collection is a $Tree_2$ collection if and only if it is $Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$ for some nonempty set X, a nonempty collection Υ of nonempty trees, a nonempty collection Γ of functions from X^ω into ω and a nonempty collection Λ of subsets of X^ω.

We sometimes let $\mathcal{T}_2$ be a $Tree_2$ collection when we wish to suppress $X, \Upsilon^\omega, \Gamma$ and Λ, i.e., $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. ⊣

Notation 3.2.3. *If Υ is a singleton $\{T\}$, we will write $Tree_2\left(X, \{T\}^\omega, \Gamma, \Lambda\right)$. To avoid confusion, we do not supress the brackets $\{\}$ in $\{T\}^\omega$, we do not write $Tree_2\left(X, T^\omega, \Gamma, \Lambda\right)$. If we fix an ω-sequence of trees T_{sq}, we will write $Tree_2\left(X, T_{sq}, \Gamma, \Lambda\right)$. When dealing with the singletons for any of the last two components of $Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right)$, we will suppress $\{\}$, i.e., if Γ is a singleton $\{f\}$, $Tree_2\left(X, \Upsilon^\omega, f, \Lambda\right)$ abbreviates $Tree_2\left(X, \Upsilon^\omega, \{f\}, \Lambda\right)$ and if Λ is a singleton $\{B\}$, $Tree_2\left(X, \Upsilon^\omega, \Gamma, B\right)$ abbreviates $Tree_2\left(X, \Upsilon^\omega, \Gamma, \{B\}\right)$.* ⊣

Definition 3.2.9. *(Definition of "games on a $Tree_2$ collection")*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right)$ for some X, Υ, Γ and Λ. Define "games on the $Tree_2$ collection $\mathcal{T}_2$" by

$$\bigcup_{T \in \mathcal{T}_2} \left\{ G\left(A; T\right) \left| A \subseteq [T] \right. \right\}.$$ ⊣

Definition 3.2.10. *(Definition of Ξ games on a $Tree_2$ collection)*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda\right)$. Suppose we have Ξ such that for each $T \in Tree_2$, $\Xi \restriction [T] \subseteq \wp\left([T]\right)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ games on a $Tree_2$ collection $\mathcal{T}_2$ by

$$\mathcal{G}\left(\Xi; \mathcal{T}_2\right) = \bigcup_{T \in \mathcal{T}_2} \{G\left(A; T\right) | A \in \Xi \restriction [T]\}.$$

$\dashv$

We will use $\mathcal{G}$ for a collection of games.

Definition 3.2.11. *(Definition of Ξ determinacy on a $Tree_2$ collection)*

Let $Tree_2$ collection $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. Suppose we have Ξ such that for each $T \in Tree_2$, $\Xi \restriction [T] \subseteq \wp\left([T]\right)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Define Ξ determinacy on the $Tree_2$ collection $\mathcal{T}_2$ by

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_2),$$

i.e., for any X, $T_{sq} \in \Upsilon^\omega, \Psi \in \Gamma$, $B \in \Lambda$ and $A \in \Xi \restriction [T^{\Psi,B}_{X,T_{sq}}]$, every game $G(A; T^{\Psi,B}_{X,T_{sq}})$ is determined. $\dashv$

Definition 3.3.1. *(Definition of Ψ^+_2 and T^+_{sq})*

Fix a Type 2 tree $T^{\Psi,B}_{X,T_{sq}}$. Then $B \subseteq X^\omega$, $\Psi : X^\omega \to \omega$ and $T_{sq} = \langle T_n | n \in \omega\rangle$ is an ω-sequence of nonempty trees. Define

1. *$\Psi^+_2 : X^\omega \to \omega$ such that $\Psi^+_2(f) = \Psi(f \restriction [1, \omega))$.*

2. *For each $n \in \omega$, $T^+_n = Y_n \times T_n$ for some nonempty set Y_n and $T^+_{sq} = \langle T^+_n | n \in \omega\rangle$.* $\dashv$

Definition 3.3.2. *(Definition of a Shift tree $Sft_2(T)$)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define a Shift tree $Sft_2(T)$ by

$$Sft_2(T) = T^{\Psi^+_2, B^+}_{X, T^+_{sq}}.$$

$\dashv$

Definition 3.3.4. *(Definition of the erasing function $e_2 : [Sft_2(T)] \to [T]$)*

Fix a Type 1 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define the erasing function e from $[Sft_2(T)]$ into $[T]$ by

$$e_2 : \ [Sft_2(T)] \to [T]$$
$$h \mapsto \begin{cases} h \restriction [1,\omega) & \text{if } h \restriction \omega \notin B^+, \\ h \restriction [1,\omega)^\frown h \restriction [\omega+1, lh(h)) & \text{if } h \restriction \omega \in B^+. \end{cases} \qquad \dashv$$

Definition 3.3.5. *(Definition of $Shift_2$)*

Fix a Type 2 tree $T = T^{\Psi,B}_{X,T_{sq}}$. Define

$$Shift_2 : \ \wp([T]) \to \wp([Sft_2(T)])$$
$$A \mapsto \{h \in [Sft_2(T)] \mid e_2(h) \in [T] \setminus A\}. \qquad \dashv$$

Definition 3.3.7. *(Definition of a "shifting tree")*

For a tree T, define a shifting tree to be $Y \times T$ for some nonempty Y. Suppose Υ is a collection of nonempty trees. Define Υ to be closed under shifting trees if for each $T \in \Upsilon$, there is a shifting tree $Y \times T \in \Upsilon$ for some nonempty Y. $\dashv$

Definition 3.4.11. *Suppose $B \subseteq X^\omega$, Ψ is a function from X^ω into ω and T_{sq} is an ω-sequence of nonempty trees. Define*

$$Long_2(B) = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid lh(h) > \omega\}. \qquad \dashv$$

Definition 3.4.16. *Suppose $\Psi \restriction B$ is bounded below ω and let $\Psi_{\max} \in \omega$ be the maximum value of Ψ over B. Define*

$$T_{Max}(\Psi, B) = \{h \in [T^{\Psi,B}_{X,T_{sq}}] \mid h \restriction [\omega, lh(h)) \in [T_{\Psi_{\max}}]\}. \qquad \dashv$$

Definition 3.4.17. *(Definition of the N maximal tree property)*

Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N+1$ satisfying that: there is an enumeration $\langle T_n \,|n \leq N \rangle$ of Υ such that

1. *T_N is well-founded.*

2. *For each $n < N$ and for every $g_n \in [T_n]$, g_n does not properly extend g for every $g \in [T_N]$.*

Then we say Υ satisfies the N maximal tree property. We also say $\langle T_n \,|n \leq N \rangle$ has the N maximal tree property. We say $T_{sq} = \langle T_n \,|n \in \omega \rangle$ has the N maximal tree property if $\langle T_n \,|n \leq N \rangle$ satisfies the N maximal tree property. ⊣

Notation 3.4.18. *Suppose for each $n \in \omega$, T_n is a tree. For each $n \in \omega$ and for any $p \in T_n$, define*

$$M^n_p = \{m \,|p^\frown \langle m \rangle \in T_n \} \,. \qquad \dashv$$

Definition 3.4.19. *(Definition of the N disjoint tree property)*

Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N+1$ satisfying that: there is an enumeration $\langle T_n \,|n \leq N \rangle$ of Υ such that

$$\text{for every } n < N,\ M^n_\emptyset \cap M^N_\emptyset = \emptyset.$$

Then we say Υ satisfies the N disjoint tree property. We also say $\langle T_n \,|n \leq N \rangle$ has the N disjoint tree property. We say $T_{sq} = \langle T_n \,|n \in \omega \rangle$ has the N disjoint tree property if $\langle T_n \,|n \leq N \rangle$ satisfies the N disjoint tree property. If $T_{sq} = \langle T_n \,|n \in \omega \rangle$ satisfies that each $M^n_\emptyset$ is pairwise disjoint, then we say T_{sq} has the disjoint tree property. ⊣

Definition 3.4.22. *(Definition of the modified N maximal tree property)*

Fix $N \in \omega$. Suppose Υ is a set of nonempty trees of size $N+1$ satisfying that: there is an enumeration $\langle T_n \,|n \le N \rangle$ of Υ such that

1. *T_N is well-founded.*

2. *for each $n < N$ and for every $g_n \in [T_n]$, g_n does not extend g for every $g \in [T_N]$ (This gives us $[T_N] \cap [T_n] = \emptyset$ for every $n < N$.)*

Then we say Υ satisfies the modified N maximal tree property. We also say $\langle T_n \,|n \le N \rangle$ satisfies the modified N maximal tree property. We say $T_{sq} = \langle T_n \,|n \in \omega \rangle$ has the modified N maximal tree property if $\langle T_n \,|n \le N \rangle$ satisfies the N maximal tree property. ⊣

Definition 3.4.37. *Suppose $\alpha \in \omega_1$ is even and $\langle A_\beta \,|\beta \le \alpha \rangle$ where each $A_\beta \subseteq X^\omega$. Define*

$$dk^2_{<\alpha}\left(\langle A_\beta \,|\beta \le \alpha \rangle\right) = \left\{ h \in \left[T_{X,T_{sq}}^{\chi_{A_\alpha}, \bigcap_{\beta\in\alpha} A_\beta} \right] \middle| \, h \restriction \omega \notin \bigcap\nolimits_{\beta\in\alpha} A_\beta \wedge \mu\beta(h \restriction \omega \notin A_\beta) \text{ is odd} \right\}. \dashv$$

Definition 3.4.45. *For any $n \in \omega$, define*

$$Tail\,(\Psi, n, B) = \left(B \cap \Psi^{-1}(n)\right) \times [T_n]. \qquad \dashv$$

Definition 3.4.47. *(Definition of $least_\alpha$ and Ψ_α)*

Suppose $\vec{A} = \langle A_\beta \,|\beta \in \alpha \rangle$ is a sequence of sets. Define

$$\begin{aligned} least_\alpha : \quad & X^\omega \to \alpha + 1 \\ & f \mapsto \begin{cases} \mu\beta\,(f \notin A_\beta) & \text{if } f \notin \bigcap_{\beta\in\alpha} A_\beta, \\ \alpha & \text{otherwise.} \end{cases} \end{aligned}$$

Then define

$$\begin{aligned} \Psi_\alpha : \quad & X^\omega \to \omega \\ & f \mapsto n \text{ where } least_\alpha(f) = \gamma + n,\ \gamma = 0 \text{ or } \gamma \text{ is a limit ordinal.} \end{aligned} \qquad \dashv$$

Definition 3.5.2. *Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$. *For each* $n \in \omega$, *define*

$$A^n = A \cap ((B \cap \Psi^{-1}(n)) \times [T_n]),$$

$$A_\emptyset = A \cap (X^\omega \backslash B).$$

Then $A = \bigcup_{n\in\omega} A^n \dot{\cup} A_\emptyset$. ⊣

Definition 3.5.3. *Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$ *and assume* $A_\emptyset, A^n$ *for all* $n \in \omega$ *defined in definition 3.5.2. For every* $n \in \omega$ *and* $g \in [T_n]$, *define*

$$A^n_g = \{ f \in X^\omega \,|\, f^\frown g \in A^n \}. \qquad ⊣$$

Definition 3.5.4. *Let* $T_{sq} = \langle T_n \,|n \in \omega \rangle \in WF^\omega$. *Suppose* $A \subseteq [T^{\Psi,B}_{X,T_{sq}}]$. *Define*

$$A^n_p \overset{df}{=} \begin{cases} \bigcup\limits_{m \in M^n_p} A^n_{p^\frown \langle m \rangle} & \text{if } lh\,(p) \text{ is even,} \\ \bigcap\limits_{m \in M^n_p} A^n_{p^\frown \langle m \rangle} & \text{if } lh\,(p) \text{ is odd.} \end{cases}$$

Since T_n *is well-founded, each* A^n_p *is well-defined.* ⊣

Definition 3.5.24. *(Definition of "*Fix_2*")*

For all $m \in \omega$, *fix* $a_m \in [T_m]$. *Define*

$$Fix_2 \langle a_m : m \in \omega \rangle : \quad X^\omega \to [T^{\Psi,B}_{X,T_{sq}}]$$

$$f \mapsto \begin{cases} f & \text{if } f \notin B, \\ f^\frown a_{\Psi(f)} & \text{otherwise.} \end{cases}$$

If $\langle a_m : m \in \omega \rangle$ *is clear from the context, we will denote* Fix_2 *to mean* $Fix_2 \langle a_m : m \in \omega \rangle$. ⊣

Appendix F

Determinacy Results

In this section, we will list all the determinacy results in this dissertation involving games on $Tree_1$ collections and $Tree_2$ collections.

F.1 Chapter 2

(2.2) Definition of a $Tree_1$ collection and a collection of games on a $Tree_1$ collection with complexity Ξ

Observation 2.2.9. *Suppose X is a nonempty set, Θ is a collection of sets, Γ is a collection of functions from X^ω into ω, Λ is a collection of subsets of X^ω. Let $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma, \Lambda)$. Suppose we have Ξ_1, Ξ_2 such that for each $T \in \mathcal{T}_1$, $\Xi_1 \upharpoonright [T] \subseteq \wp([T])$ and $\Xi_2 \upharpoonright [T] \subseteq \wp([T])$ are defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). If for any $T \in \mathcal{T}_1$, $\Xi_1 \upharpoonright [T] \subseteq \Xi_2 \upharpoonright [T]$, then*

$$\mathcal{G}(\Xi_1; \mathcal{T}_1) \subseteq \mathcal{G}(\Xi_2; \mathcal{T}_1).$$

Thus $\mathcal{G}$ is an increasing operation on the first component. ⊣

Observation 2.2.10. *Let Θ be a collection of sets and $\Lambda \subseteq X^\omega$. Suppose we have Ξ such that $\Xi \upharpoonright X^\omega \subseteq \wp(X^\omega)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Suppose we have Ξ_1 such that for each $T \in Tree_1(X, \Theta, \Gamma(\omega, \Xi), \Lambda)$, $\Xi_1 \upharpoonright [T] \subseteq \wp([T])$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, \Xi), \Lambda))$
- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, co\text{-}\Xi), \Lambda))$
- $\mathcal{G}(\Xi_1; Tree_1(X, \Theta, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda))$ ⊣

(2.3) Equivalence between $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ determinacy on a $Tree_1$ collection and equivalence between $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ determinacy on a $Tree_1$ collection

Theorem 2.3.1. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Then for any X and Θ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_1\right) \tag{2.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_1\right) \tag{2.2}$$

for $\mathcal{T}_1 = Tree_1(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega)$ [1] where:

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* ⊣

[1] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

(2.4) Using the determinacy of games on a $Tree_1$ collection to obtain the determinacy of games on $X^{<\omega}$

Observation 2.4.1. *For any X, Y, any function Ψ from X^ω into ω and any complexity Ξ (in which $\Xi \restriction [T] \subseteq \wp([T])$ is defined),*

$$Det\ \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Psi, \emptyset\right)\right) \Rightarrow Det\left(\Xi \restriction X^\omega\right). \qquad \dashv$$

Corollary 2.4.2. *Suppose $\alpha \in \omega_1$ and $n \in \omega$. Fix nonempty sets X and Y. Let $\mathcal{T}_1 = Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)$. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_1\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_1\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \restriction X^\omega\right). \qquad \dashv$$

Observation 2.4.3. *Assume that Γ contains all constant functions from X^ω into ω. Then for any X, Y and complexity Ξ (in which $\Xi \restriction X^\omega \times Y^n \subseteq \wp\left(X^\omega \times Y^n\right)$ is defined for all $n \in \omega$),*

$$Det\ \mathcal{G}\left(\Xi; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \Xi \restriction \left(X^\omega \times Y^n\right)\right). \qquad \dashv$$

Theorem 2.4.4. *(ZF-P)*

Suppose Γ contains all constant functions from X^ω into ω and Y is denumerable. Then $Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, Y, \Gamma, \{\emptyset, X^\omega\}\right)\right)$ implies $Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right)$, finite Borel determinacy on $X^{<\omega}$. $\dashv$

Corollary 2.4.17.

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right).^{[2]} \qquad \dashv$$

[2] Recall notation 1.5.10 for CTB and notation 1.5.8 for $\Gamma(\omega, \mathbf{\Delta}^0_1)$.

Corollary 2.4.21. *For any X, Y, $\Psi : X^\omega \to \omega$ and Λ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \Lambda \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\Lambda \upharpoonright X^\omega\right). \qquad \dashv$$

Corollary 2.4.22. *(Corollary to Corollary 2.4.21)*

For any $\alpha \in \omega_1$, $n \in \omega$, X, Y and $\Psi : X^\omega \to \omega$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Psi, \mathbf{\Sigma}^1_n \upharpoonright X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \upharpoonright X^\omega\right). \qquad \dashv$$

Corollary 2.4.23. *(Corollary to Corollary 2.4.21)*

Suppose X is a nonempty set, Θ is an arbitrary collection of sets, Γ is any collection of functions from X^ω into ω and Λ is a collection of nondetermined sets on X^ω. Then,

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma, \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Corollary 2.4.27. *For any X, Y and complexity Ξ,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \left\{\chi_A \,|A \in \Xi \upharpoonright X^\omega\right\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \upharpoonright X^\omega\right).$$

Thus,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \upharpoonright X^\omega). \qquad \dashv$$

Corollary 2.4.28. *(Corollary to Corollary 2.4.27)*

For any $\alpha \in \omega_1$ and any X, Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Delta}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Delta}^0_\alpha \upharpoonright X^\omega). \qquad \dashv$$

Corollary 2.4.30. *Suppose Ξ_1, Ξ_2 are complexities. Then for any X, Y,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \left\{\chi_A \,|A \in \Xi_1 \restriction X^\omega\right\}, \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.10}$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \left\{\chi_A \,|A \in \Xi_2 \restriction X^\omega\right\}, \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.11}$$

⊣

Corollary 2.4.31. *(Corollary to Corollary 2.4.30)*

Suppose Ξ_1, Ξ_2 are complexities. Then for any X, Y,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \Xi_1 \wedge co\text{-}\Xi_1), \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.12}$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, \Xi_2 \wedge co\text{-}\Xi_2), \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \tag{2.13}$$

⊣

Corollary 2.4.32. *(Corollary to Corollary 2.4.31)*

Suppose $\alpha, \beta \in \omega_1$. Then for any Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha\right), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega).$$ ⊣

Corollary 2.4.33. *(Corollary to Corollary 2.4.31)*

Suppose $n, m \in \omega$. Then for any Y,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n\right), \mathbf{\Pi}^1_m \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma\left(\omega, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m\right), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega). \qquad \dashv$$

Corollary 2.4.34. *(Corollary to Corollary 2.4.33) For any Y,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Corollary 2.4.35. *(Corollary to Corollary 2.4.31) For any Y and $n \in \omega$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), (co\text{-}n\text{-}\mathbf{\Pi}^1_1) \restriction X^\omega\right)\right) \Rightarrow Det(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \tag{2.14}$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, Y, \Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right). \tag{2.15}$$

$\dashv$

Corollary 2.4.42. *Assume $\alpha \in \omega_1$ is even. Then for any Y,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\alpha+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Corollary 2.4.44. *Assume $\alpha \in \omega_1$ is a limit ordinal and $\lambda \in \omega$. Then for any Y,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_1\left(X, Y, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction X^\omega). \qquad \dashv$$

(2.5) Getting the determinacy of the games on a $Tree_1$ collection from the determinacy of the games on $X^{<\omega}$ (Reversed direction of section 2.4)

Theorem 2.5.18. *Suppose* $\beta, \gamma \in \omega_1$.

If $\beta, \gamma > 1$, *then*

$$Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{2.17}$$

[3]

If $\beta < \gamma$,

$$Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). & (2.18) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right). & (2.19) \end{cases}$$

If $\beta \geq \gamma$,

$$Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). & (2.20) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right). & (2.21) \end{cases}$$

Also,

$$Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega\right) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right). \tag{2.22}$$

$\dashv$

Theorem 2.5.20. *Suppose* $\beta, \gamma \in \omega_1$. *Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). & (2.23) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right). & (2.24) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). & (2.25) \end{cases}$$

[4] (on 2.23)

$\dashv$

[3] Recall notation 1.5.10 for FIN.
[4] Recall notation 1.5.10 for CTB.

Corollary 2.5.21. *For any finite n and m,*

$$
\begin{aligned}
& Det\left(\mathbf{\Sigma}^0_\omega \restriction X^\omega\right) \\
& \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \restriction X^\omega\right)\right) \\
& \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \\
& \Rightarrow Det\left(\bigcup\nolimits_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right). \qquad \dashv
\end{aligned}
$$

Corollary 2.5.22.

$$
\begin{aligned}
& Det\mathcal{G}\left(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega}; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \emptyset\right)\right) \\
& \Rightarrow Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\omega} \restriction X^\omega) \\
& \Rightarrow Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \qquad \dashv
\end{aligned}
$$

Corollary 2.5.23. *Suppose Λ is an algebra. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Corollary 2.5.24. *Suppose Λ is σ-algebra. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Corollary 2.5.25.

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right) \end{cases} \qquad \dashv$$

Theorem 2.5.26. *Suppose $m, n \in \omega\backslash\{0\}$.*

$$Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right)\right).[5] \qquad (2.26)$$

[5]Recall notation 1.5.10 for CTB.

If $n < m$,

$$Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (2.27) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (2.28) \end{cases}$$

If $n \geq m$,

$$Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (2.29) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (2.30) \end{cases}$$

$\dashv$

Corollary 2.5.27. *(Corollary to Theorem 2.5.26)*

$Det\,(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_1), (\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1) \restriction X^\omega)).$ $\dashv$

Corollary 2.5.28. *For any nonempty X and Y,*

$$\left.\begin{array}{l} ①\ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega)) \\ ②\ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_1\,(X, Y, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega)) \end{array}\right\}$$

$$\Rightarrow ③\ Det\left(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right)$$

$$\Rightarrow ④\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^1_1), \left(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1\right) \restriction X^\omega\right)\right).$$

That is : ① implies ③, ② implies ③, and ③ implies ④. $\dashv$

Theorem 2.5.29. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{2.31}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.32}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, FIN, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.33}$$

⊣

Theorem 2.5.30. *Suppose $\alpha, \beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha+\omega} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{2.34}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha+\omega} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.35}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha+\omega} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right). \tag{2.36}$$

⊣

Corollary 2.5.35. *For any finite n, m and k,*

$$\begin{aligned}
&Det\left(\mathbf{\Sigma}^0_\omega \restriction X^\omega\right) \\
&\Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_k; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \restriction X^\omega\right)\right) \\
&\Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_n), \mathbf{\Sigma}^0_m \restriction X^\omega\right)\right) \\
&\Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \\
&\Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right).
\end{aligned}$$

⊣

Corollary 2.5.36. *Suppose Λ is a σ-algebra and Λ is closed under Λ-substitution. Then*

$$Det(\Lambda \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, CTB, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right).$$

⊣

Corollary 2.5.37. *(Corollary to Corollary 2.5.36)*

$$Det(\mathbf{B} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \restriction X^\omega\right)\right).$$

⊣

Theorem 2.5.38. *Suppose* $m \in \omega$. *Suppose* $\mathcal{T}_1 = Tree_1\left(X, CTB, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)$. *Then*

$$Det(\mathbf{\Sigma}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Pi}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_1\right).$$

$$Det(\mathbf{\Delta}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_1\right).$$

Note that $\mathbf{\Delta}^1_1 \upharpoonright X^\omega$ *is Borel if* X *is countable and if* X *is uncountable, it is the quasi-Borel.*

⊣

Observation 2.5.45.

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_1\left(X, \mathcal{N}, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1\right)\right) \Rightarrow Det\left(\mathbf{P} \upharpoonright X^\omega\right). \quad ⊣$$

(2.6) Determinacy equivalences between games on $X^{<\omega}$ and games on $Tree_1$ collections

Theorem 2.6.1. *For any nonempty* $\Theta \subseteq FIN$, *the determinacy of following (2.37) through (2.42) are all equivalent to* $Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.37}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.38}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.39}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.40}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right) \tag{2.41}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{2.42}$$

$\dashv$

Theorem 2.6.2. *Suppose $\beta, \gamma \in \omega_1$ and $\beta \geq \gamma$. Then for any nonempty $\Theta \subseteq FIN$, the determinacy of following (2.43) through (2.48) are all equivalent to Det $\left(\mathbf{\Delta}^0_\beta \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.43}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.44}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.45}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.46}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.47}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.48}$$

$\dashv$

Theorem 2.6.3. *Suppose $\beta, \gamma \in \omega_1$ and $1 \leq \beta < \gamma$. Then for any nonempty $\Theta \subseteq FIN$, the determinacy of following (2.49) through (2.66) are all equivalent to Det $\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.49}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.50}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{2.51}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)\right) \tag{2.52}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}_\gamma^0), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)\right) \tag{2.53}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}_\gamma^0), \mathbf{\Sigma}_\beta^0 \restriction X^\omega\right)\right) \tag{2.54}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.55}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.56}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.57}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.58}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.59}$$

$$\mathcal{G}\left(\mathbf{\Pi}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}_\gamma^0), \mathbf{\Pi}_\beta^0 \restriction X^\omega\right)\right) \tag{2.60}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}_\gamma^0), \mathbf{\Delta}_\beta^0 \restriction X^\omega\right)\right) \tag{2.61}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}_\gamma^0), \mathbf{\Delta}_\beta^0 \restriction X^\omega\right)\right) \tag{2.62}$$

$$\mathcal{G}\left(\mathbf{\Sigma}_1^0; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}_\gamma^0), \mathbf{\Delta}_\beta^0 \restriction X^\omega\right)\right) \tag{2.63}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.64}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.65}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{2.66}$$

$\dashv$

Corollary 2.6.4. *Suppose Λ is an algebra. Then for any nonempty $\Theta \subseteq FIN$,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Corollary 2.6.5. *Suppose Λ is a σ-algebra. Then for any nonempty $\Theta \subseteq CTB$,*

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Corollary 2.6.6. *For any $\alpha, \beta, \gamma \in \omega_1$,*

$$Det(\mathbf{B} \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right)$$

where:

- $\emptyset \neq \Theta \subseteq CTB$,
- $\begin{cases} \mathcal{A} = \mathbf{B}, & or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\} & for\ \alpha > 1,\ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1\} & for\ \alpha = 1, \end{cases}$
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

- *at least one of* $\mathcal{A}, \mathcal{B}$ *or* $\mathcal{C}$ *is* $\mathbf{B}$. ⊣

Theorem 2.6.7. *Suppose* $n \in \omega\backslash\{0\}$, $\beta, \gamma \in \omega_1$.

For any nonempty $\Theta \subseteq CTB$, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ *and* $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, *the determinacy of following (2.67) and (2.68) are equivalent to* $Det(\mathbf{\Sigma}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.67}$$

$$\mathcal{G}\left(\mathbf{\Pi}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.68}$$

For any nonempty $\Theta \subseteq CTB$, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ *and* $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, *the determinacy of following (2.69)is equivalent to* $Det(\mathbf{\Delta}^1_n \restriction X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Delta}^1_n; Tree_1\left(X, \Theta, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^\omega\right)\right) \tag{2.69}$$

⊣

Corollary 2.6.8. *Suppose* Λ *is* σ*-algebra and closed under* Λ*-substitution. Then for any nonempty* $\Theta \subseteq CTB$,

$$Det(\Lambda \restriction X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\Lambda; Tree_1\left(X, \Theta, \Gamma(\omega, \Lambda), \Lambda \restriction X^\omega\right)\right)$$ ⊣

F.2 Chapter 3

(3.2) Definition of a $Tree_2$ collection and a collection of games on a $Tree_2$ collection with complexity Ξ

Observation 3.2.12. *Suppose X is a nonempty set, Υ is a collection of trees, Γ is a collection of functions from X^ω into ω, Λ is a collection of subsets of X^ω and Ξ_1, Ξ_2 are complexities. Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma, \Lambda)$. If for any $T \in \mathcal{T}_2$, $\Xi_1 \restriction [T] \subseteq \Xi_2 \restriction [T]$, then*

$$\mathcal{G}\left(\Xi_1; \mathcal{T}_2\right) \subseteq \mathcal{G}\left(\Xi_2; \mathcal{T}_2\right).$$

Thus $\mathcal{G}$ is an increasing operation on the first component. ⊣

Observation 3.2.13. *Let Υ be a collection of trees and $\Lambda \subseteq X^\omega$. Suppose we have Ξ such that $\Xi \restriction X^\omega \subseteq \wp\left(X^\omega\right)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Suppose we have Ξ_1 such that for each $T \in Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi), \Lambda\right)$, $\Xi_1 \restriction [T] \subseteq \wp\left([T]\right)$ is defined (e.g., $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$). Then the following are equal.*

- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi), \Lambda\right)\right)$
- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, co\text{-}\Xi), \Lambda\right)\right)$
- $\mathcal{G}\left(\Xi_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}(\Xi)), \Lambda\right)\right)$ ⊣

(3.3) Equivalence between $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ determinacy on Type 2 trees and equivalence between $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ determinacy on Type 2 trees

Theorem 3.3.8. *Suppose $n \in \omega$ and $\alpha \in \omega_1$. Suppose Υ is closed under shifting trees. Then for any X,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}_2\right) \tag{3.1}$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}_2\right) \tag{3.2}$$

for $\mathcal{T}_2 = Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)$ [6] *where:*

- $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\gamma \in \omega_1$ and $m \in \omega$.*
- $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, *$\beta \in \omega_1$ and $m \in \omega$.* ⊣

Corollary 3.3.11. *Suppose $n \in \omega$ and $\alpha \in \omega_1$. Let:*

- $\mathcal{T}^1_2 = Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)$.
- $\mathcal{T}^2_2 = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)$. [7]

Then

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}^1_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}^1_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}^2_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^0_\alpha; \mathcal{T}^2_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}^1_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}^1_2\right)$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}^2_2\right) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_n; \mathcal{T}^2_2\right)$$

[6] Recall notation 1.5.8 for $\Gamma(\omega, \mathcal{C})$.

[7] Recall notation 1.5.11 for FWF and CWF.

for any $\mathcal{C} \in \{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\gamma \in \omega_1$ *and* $m \in \omega$*;*

and any $\mathcal{B} \in \{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}, \mathbf{\Sigma}^1_m, \mathbf{\Pi}^1_m, \mathbf{\Delta}^1_m, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m, \mathbf{P}\}$, $\beta \in \omega_1$ *and* $m \in \omega$*.* ⊣

(3.4) Using the determinacy of games on a $Tree_2$ collection to obtain the determinacy of games on $X^{<\omega}$

Observation 3.4.1. *For any* X*, any* ω*-sequence of nonempty trees* T_{sq}*, any function* f *from* X^ω *into* ω*, and any complexity* Ξ *(in which for any* $T \in Tree_2(X, T_{sq}, f, \emptyset)$*,* $\Xi \restriction [T] \subseteq \wp([T])$ *is defined),*

$$Det\ \mathcal{G}(\Xi; Tree_2(X, T_{sq}, f, \emptyset)) \Rightarrow Det(\Xi \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.2. *Fix nonempty* X *and nonempty* Υ*.*

Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega)$*. For any complexity* Ξ*,*

$$Det\ \mathcal{G}(\Xi; \mathcal{T}_1) \Rightarrow Det(\Xi \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.3. *(Corollary to Corollary 3.4.2)*

Suppose $\alpha \in \omega_1$ *and* $n \in \omega$*. Fix nonempty* X *and nonempty* Υ*.*

Let $\mathcal{T}_2 = Tree_2(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega)$*. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; \mathcal{T}_2\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_n; \mathcal{T}_2\right) \Rightarrow Det\left(\mathbf{\Sigma}^1_n \restriction X^\omega\right). \qquad \dashv$$

Observation 3.4.4. *Suppose* Γ *contains all constant functions from* X^ω *into* ω*. Suppose* Y *is a nonempty set and* Υ *contains* $Y^{\leq n+1}$ *for all* $n \in \omega$*. Then for any complexity* Ξ *(in which*

$\Xi \restriction X^\omega \times Y^n \subseteq \wp(X^\omega \times Y^n)$ *is defined for all* $n \in \omega$*),*

$$Det\ \mathcal{G}(\Xi; Tree_2(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\})) \Rightarrow Det\left(\bigcup_{n\in\omega} \Xi \restriction (X^\omega \times Y^n)\right). \qquad \dashv$$

Theorem 3.4.5. *(ZF-P)*

Suppose Γ *contains all constant functions from* X^ω *into* ω *and* Υ *contains* $Y^{\leq n+1}$ *for all* $\in \omega$ *for some countable* Y*. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2(X, \Upsilon^\omega, \Gamma, \{\emptyset, X^\omega\})\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right). \qquad \dashv$$

Corollary 3.4.6.

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(\bigcup_{n\in\omega} \mathbf{\Sigma}^0_n \restriction X^\omega\right). \qquad \dashv$$

Corollary 3.4.7. *Suppose* Γ *is a nonempty collection of functions from* X^ω *into* ω*. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2(X, CWF^\omega, \Gamma, X^\omega)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.10. *Suppose* Ψ *is any function from* X^ω *into* ω *and* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \subseteq \Upsilon$*.*

$$Det\ \mathcal{G}\left(\mathbf{\Delta}^0_1; Tree_2(X, \Upsilon^\omega, \Psi, X^\omega)\right) \Rightarrow Det(\mathbf{B} \restriction X^\omega). \qquad \dashv$$

Theorem 3.4.12. *For any* ω*-sequence* T_{sq} *of nonempty trees,* $\Psi : X^\omega \to \omega$*, for any* $A \subseteq X^\omega$*,* $G(A; X^\omega)$ *is determined if and only if* $G(Long_2(A); T^{\Psi,A}_{X,T_{sq}})$ *is determined .* $\dashv$

Corollary 3.4.13. *For any nonempty collection* Υ *of nonempty trees,* $\Psi : X^\omega \to \omega$ *and* Λ*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2(X, \Upsilon^\omega, \Psi, \Lambda \restriction X^\omega)\right) \Rightarrow Det(\Lambda \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.14. *For any* $\alpha \in \omega_1$*,* Υ *and* $\Psi : X^\omega \to \omega$*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Psi, \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_\alpha \restriction X^\omega\right). \qquad \dashv$$

Corollary 3.4.15. *Suppose Υ is an arbitrary nonempty collection of nonempty trees, Γ is any collection of functions from X^ω into ω and Λ is a collection of nondetermined sets. Then,*

$$\neg Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma, \Lambda \restriction X^\omega\right)\right). \qquad \dashv$$

Corollary 3.4.27. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any complexity Ξ and for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \left| A \in \Xi \restriction X^\omega\right.\right\}, X^\omega\right)\right) \Rightarrow Det\left(\Xi \restriction X^\omega\right).$$

Thus,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\Xi), X^\omega\right)\right) \Rightarrow Det(\Xi \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.28. *(Corollary to Corollary 3.4.27)*

Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Sigma}^0_\alpha \restriction X^\omega).$$

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\alpha), X^\omega\right)\right) \Rightarrow Det(\mathbf{\Delta}^0_\alpha \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.31. *Suppose $\alpha, \beta \in \omega_1$ and Ξ_1, Ξ_2 are complexities. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \left\{\chi_A \left| A \in \Xi_1 \restriction X^\omega\right.\right\}, \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \qquad (3.3)$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \{\chi_A | A \in \Xi_2 \restriction X^\omega\}, \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \quad (3.4)$$

⊣

Corollary 3.4.32. *(Corollary to Corollary 3.4.31)*

Suppose Ξ_1, Ξ_2 *are complexities. Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi_1 \wedge co\text{-}\Xi_1), \Xi_2 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \quad (3.5)$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Xi_2 \wedge co\text{-}\Xi_2), \Xi_1 \restriction X^\omega\right)\right) \Rightarrow Det((\Xi_1 \wedge \Xi_2) \restriction X^\omega). \quad (3.6)$$

⊣

Corollary 3.4.33. *(Corollary to Corollary 3.4.32)*

Suppose $\alpha, \beta \in \omega_1$. *Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 2 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\alpha\right), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Sigma}^0_\alpha \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^0_\alpha \wedge \mathbf{\Pi}^0_\beta) \restriction X^\omega). \quad ⊣$$

Corollary 3.4.34. *(Corollary to Corollary 3.4.32)*

Suppose $n, m \in \omega$. *Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 2* *disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_n\right), \mathbf{\Pi}^1_m \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega).$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^1_m \wedge \mathbf{\Pi}^1_m\right), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right) \Rightarrow Det((\mathbf{\Sigma}^1_n \wedge \mathbf{\Pi}^1_m) \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.35. *(Corollary to Corollary 3.4.34)*

Suppose $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega).$$

Similarly,

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.36. *(Corollary to Corollary 3.4.32)*

Suppose $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$ *and for any* $n \in \omega$*,*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), (co\text{-}n\text{-}\mathbf{\Pi}^1_1) \restriction X^\omega\right)\right) \Rightarrow Det(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \tag{3.7}$$

Similarly,

$$Det\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, n\text{-}\mathbf{\Pi}^1_1 \wedge co\text{-}n\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det\left(n+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right). \tag{3.8}$$ $\dashv$

Corollary 3.4.43. *Assume* $\alpha \in \omega_1$ *is even. Suppose* $\{T_0, T_1\}$ *satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any* $\Upsilon \supseteq \{T_0, T_1\}$*,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\alpha+1\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \qquad \dashv$$

Corollary 3.4.44. *Assume $\alpha \in \omega_1$ is a limit ordinal and $\lambda \in \omega$. Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$Det\ \mathcal{G}\left(\alpha\text{-}\mathbf{\Pi}^1_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\lambda \wedge \mathbf{\Pi}^0_\lambda), \mathbf{\Pi}^1_1 \restriction X^\omega\right)\right) \Rightarrow Det(\left(\alpha\text{-}\mathbf{\Pi}^1_1 + \mathbf{\Sigma}^0_\lambda\right) \restriction X^\omega). \quad \dashv$$

Corollary 3.4.50. *Suppose $\alpha \in \omega_1$. Suppose T_{sq} satisfies the disjoint tree property. Then*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, T_{sq}, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right)\right) \Rightarrow Det(\alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \quad \dashv$$

Corollary 3.4.51. *Suppose $\alpha \in \omega_1$.*

$$Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1\left(\mathbf{\Pi}^1_1\right)), X^\omega\right)\right) \Rightarrow Det(\alpha\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega). \quad \dashv$$

(3.5) Getting the determinacy of games on a $Tree_2$ collection from the determinacy of games on $X^{<\omega}$ (Reversed direction of section 3.4)

Theorem 3.5.13. *Suppose $\beta, \gamma \in \omega_1$. If $\beta, \gamma > 1$, then*

$$Det(\mathbf{\Delta}^0_{\max\{\beta,\gamma\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{3.9}$$

If $\beta < \gamma$,

$$Det\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) & (3.10) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) & (3.11) \end{cases}$$

If $\beta \geq \gamma$,

$$Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) & (3.12) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \restriction X^\omega\right)\right) & (3.13) \end{cases}$$

Also,

$$Det\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega\right) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \upharpoonright X^\omega\right)\right). \tag{3.14}$$

⊣

Corollary 3.5.14. *Suppose $\beta, \gamma \in \omega_1$. Then for any $\beta \geq \gamma$,*

$$\left.\begin{array}{l} ① \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \\ ② \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma\left(\omega, \mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta\right), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \end{array}\right\}$$

$$\Rightarrow ③ \ Det((\mathbf{\Sigma}^0_\beta \wedge \mathbf{\Pi}^0_\beta) \upharpoonright X^\omega)$$

$$\Leftrightarrow ④ \ Det\left(\left(\mathbf{\Sigma}^0_\beta \vee \mathbf{\Pi}^0_\beta\right) \upharpoonright X^\omega\right)$$

$$\Rightarrow \left\{\begin{array}{l} ⑤ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \\ ⑥ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right). \end{array}\right.$$

That is : ① implies ③, ② implies ③, ③ if and only if ④ and ④ implies both ⑤ and ⑥. ⊣

Theorem 3.5.15. *Suppose $\beta, \lambda \in \omega_1$. Then*

$$Det\left(\mathbf{B} \upharpoonright X^\omega\right) \Rightarrow Det\ \mathcal{G}(\mathbf{\Sigma}^0_1; Tree_2(X, CWF^\omega, \Gamma\left(\omega, \mathbf{\Delta}^0_\gamma\right), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega)).$$

⊣

Corollary 3.5.16. *(Corollary to Theorem 3.5.15)*

$$Det\left(\mathbf{B} \upharpoonright X^\omega\right) \Rightarrow Det\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)\right).$$

⊣

Corollary 3.5.17. *Suppose Λ an algebra. Then*

$$Det(\Lambda \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

⊣

Corollary 3.5.18. *Suppose Λ is σ-algebra. Then*

$$Det(\Lambda \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

⊣

Theorem 3.5.19. *Suppose $m, n \in \omega \backslash \{0\}$.*

$$Det(\mathbf{\Delta}^1_{\max\{n,m\}} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Delta}^1_n \restriction X^\omega\right)\right). \tag{3.15}$$

If $n < m$,

$$Det\left(\mathbf{\Delta}^1_m \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (3.16) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (3.17) \end{cases}$$

If $n \geq m$,

$$Det\left(\left(\mathbf{\Sigma}^1_n \vee \mathbf{\Pi}^1_n\right) \restriction X^\omega\right) \Rightarrow \begin{cases} Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Sigma}^1_n \restriction X^\omega\right)\right). & (3.18) \\ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_m), \mathbf{\Pi}^1_n \restriction X^\omega\right)\right). & (3.19) \end{cases}$$

$\dashv$

Corollary 3.5.20. *(Corollary to Theorem 3.5.19)*

$Det\,(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_1), (\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1) \restriction X^\omega)).$ $\dashv$

Corollary 3.5.21. *Suppose $\{T_0, T_1\}$ satisfies the modified 1 maximal tree property or the 1 disjoint tree property. Then for any $\Upsilon \supseteq \{T_0, T_1\}$,*

$$\left.\begin{array}{l} ① \ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Sigma}^1_1 \restriction X^\omega)) \\ ② \ Det\ \mathcal{G}\,(\mathbf{\Sigma}^0_1; Tree_2\,(X, \Upsilon^\omega, \Gamma(\omega, 2\text{-}\mathbf{\Pi}^1_1), \mathbf{\Pi}^1_1 \restriction X^\omega)) \end{array}\right\}$$

$$\Rightarrow ③ \ Det\left(2\text{-}\mathbf{\Pi}^1_1 \restriction X^\omega\right)$$

$$\Rightarrow ④ \ Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_1\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^1_1), \left(\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1\right) \restriction X^\omega\right)\right).$$

That is : ① implies ③, ② implies ③, and ③ implies ④. $\dashv$

Theorem 3.5.22. *Suppose $\alpha, \beta, \gamma \in \omega_1$ and $\alpha > 1$. Then*

$$Det(\mathbf{\Sigma}^0_{\max\{\beta,\gamma\}+\alpha} \restriction X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right). \tag{3.20}$$

Moreover, if $\beta < \gamma$, then

$$Det(\mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{3.21}$$

If $\beta \geq \gamma$, then

$$Det(\mathbf{\Sigma}^0_{(\beta+1)+\alpha} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right). \tag{3.22}$$

$\dashv$

Theorem 3.5.23. *Suppose $\alpha, \beta, \gamma \in \omega_1$. Then*

$$Det(\mathbf{B} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_\alpha; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right).$$

$\dashv$

Corollary 3.5.27. *Suppose Λ is a σ-algebra and Λ is closed under Λ-substitution. Then*

$$Det(\Lambda \upharpoonright X^\omega) \Rightarrow Det\mathcal{G}\left(\Lambda; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right).$$

$\dashv$

Corollary 3.5.28. *(Corollary to Corollary 3.5.27)*

$$Det(\mathbf{B} \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{B}; Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)\right).$$

$\dashv$

Theorem 3.5.29. *Suppose $m \in \omega$. Let $\mathcal{T}_2 = Tree_2\left(X, CWF^\omega, \Gamma(\omega, \mathbf{B}), \mathbf{B} \upharpoonright X^\omega\right)$. Then*

$$Det(\mathbf{\Sigma}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Pi}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Pi}^1_m; \mathcal{T}_2\right).$$

$$Det(\mathbf{\Delta}^1_m \upharpoonright X^\omega) \Rightarrow Det\ \mathcal{G}\left(\mathbf{\Delta}^1_m; \mathcal{T}_2\right).$$

Note that $\mathbf{\Delta}^1_1 \upharpoonright X^\omega$ is Borel if X is countable and if X is uncountable, it is the quasi-Borel.

$\dashv$

Observation 3.5.31. *Suppose* $\Upsilon = \{T \,\big|\, T \text{ is a tree and } T \subseteq \omega^{<\omega^2}\}$. *Then for any nonempty collection* Γ *of functions from* X^ω *into* ω,

$$Det\left(\mathbf{P} \restriction X^\omega\right) \Rightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_2\left(X, \Upsilon^\omega, \Gamma, X^\omega\right)\right). \qquad \dashv$$

Observation 3.5.32. *Suppose* $\Upsilon = \left\{\mathcal{N}^{\leq i} \,|i \in \omega\right\}$. *Then for any nonempty collection* Γ *of functions from* X^ω *into* ω, *we have*

$$Det\left(\mathbf{P} \restriction X^\omega\right) \Rightarrow \mathcal{G}\left(\mathbf{\Sigma}^0_1, Tree_2\left(X, \Upsilon^\omega, \Gamma, X^\omega\right)\right). \qquad \dashv$$

(3.6) Determinacy equivalence between games on $X^{<\omega}$ and games on $Tree_2$ collections

Theorem 3.6.1. *The determinacy of following (3.23) through (3.28) are all equivalent to* $Det\left(\mathbf{\Sigma}^0_1 \restriction X^\omega\right)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.23}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.24}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.25}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.26}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.27}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_1), \mathbf{\Delta}^0_1 \restriction X^\omega\right)\right) \tag{3.28}$$

$\dashv$

Theorem 3.6.2. *Suppose $\beta, \gamma \in \omega_1$ and $\beta \geq \gamma$. Then the following (3.29) through (3.34) are all equivalent to Det $\left(\mathbf{\Delta}^0_\beta \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.29}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.30}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.31}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.32}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.33}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \restriction X^\omega\right)\right) \tag{3.34}$$

$\dashv$

Theorem 3.6.3. *Suppose $\beta, \gamma \in \omega_1$ and $1 \leq \beta < \gamma$. Then the determinacy of following (3.35) through (3.52) are all equivalent to Det $\left(\mathbf{\Delta}^0_\gamma \restriction X^\omega\right)$.*

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{3.35}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{3.36}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{3.37}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Sigma}^0_\beta \restriction X^\omega\right)\right) \tag{3.38}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.39}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Sigma}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.40}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.41}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.42}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.43}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.44}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.45}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Pi}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.46}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.47}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.48}$$

$$\mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.49}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Delta}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.50}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Sigma}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.51}$$

$$\mathcal{G}\left(\mathbf{\Pi}^0_1; Tree_2\left(X, FWF^\omega, \Gamma(\omega, \mathbf{\Pi}^0_\gamma), \mathbf{\Delta}^0_\beta \upharpoonright X^\omega\right)\right) \tag{3.52}$$

$\dashv$

Corollary 3.6.4. *Suppose Λ is an algebra. Then for any nonempty $\Upsilon \subseteq FWF$,*

$$Det(\Lambda \upharpoonright X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Corollary 3.6.5. *Suppose Λ is σ-algebra. Then for any nonempty $\Upsilon \subseteq CWF$,*

$$Det(\Lambda \upharpoonright X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathbf{\Sigma}^0_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

Corollary 3.6.6. *For any $\alpha, \beta, \gamma \in \omega_1$,*

$$Det(\mathbf{B} \upharpoonright X^\omega) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}_1; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right)$$

where:

- $\Upsilon \subseteq CWF$ *is closed under shifting trees*[8] *and*

  1. *if* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \subseteq \Upsilon$, *then* $\mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha, \mathbf{B}\}$,[9]
  2. *if* $\left\{T^\alpha_{g.t.} \,|\alpha \in \omega_1\right\} \nsubseteq \Upsilon$, *then*

$$\begin{cases} \mathcal{A} = \mathbf{B}, & or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha\} & for\ \alpha > 1,\ or \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1\} & for\ \alpha = 1, \end{cases}$$

- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$,

[8] Recall definition 3.3.7 on page 231 for the closure property under shifting trees.
[9] Recall Yost tree $T^\alpha_{g.t.}$ for section 3.4.1.2 on page 242.

- *at least one of* $\mathcal{A}, \mathcal{B}$ *or* $\mathcal{C}$ *is* $\mathbf{B}$ *if* $\left\{T^{\alpha}_{g.t.} \,|\alpha \in \omega_1\right\} \nsubseteq \Upsilon$. ⊣

Corollary 3.6.7. *(Corollary to Corollary 3.6.6)*

For any $\alpha, \beta, \gamma \in \omega_1$,

$$Det(\mathbf{B} \restriction X^{\omega}) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_2\left(X, FWF^{\omega}, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^{\omega}\right)\right)$$

where:

- $\begin{cases} \mathcal{A} = \mathbf{B}, \ \textit{or} \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_{\alpha}, \mathbf{\Pi}^0_{\alpha}, \mathbf{\Delta}^0_{\alpha}\} & \textit{for } \alpha > 1, \ \textit{or} \\ \mathcal{A} \in \{\mathbf{\Sigma}^0_{1}, \mathbf{\Pi}^0_{1}\} & \textit{for } \alpha = 1, \end{cases}$
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_{\beta}, \mathbf{\Pi}^0_{\beta}, \mathbf{\Delta}^0_{\beta}, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_{\gamma}, \mathbf{\Pi}^0_{\gamma}, \mathbf{\Delta}^0_{\gamma}, \mathbf{B}\right\}$,
- *at least one of* $\mathcal{A}, \mathcal{B}$ *or* $\mathcal{C}$ *is* $\mathbf{B}$. ⊣

Corollary 3.6.8. *(Corollary to Corollary 3.6.6)*

For any $\alpha, \beta, \gamma \in \omega_1$,

$$Det(\mathbf{B} \restriction X^{\omega}) \Leftrightarrow Det\ \mathcal{G}\left(\mathcal{A}; Tree_2\left(X, CWF^{\omega}, \Gamma(\omega, \mathcal{C}), \mathcal{B} \restriction X^{\omega}\right)\right)$$

where:

- $\mathcal{A} \in \{\mathbf{\Sigma}^0_{\alpha}, \mathbf{\Pi}^0_{\alpha}, \mathbf{\Delta}^0_{\alpha}, \mathbf{B}\}$,
- $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_{\beta}, \mathbf{\Pi}^0_{\beta}, \mathbf{\Delta}^0_{\beta}, \mathbf{B}\right\}$,
- $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_{\gamma}, \mathbf{\Pi}^0_{\gamma}, \mathbf{\Delta}^0_{\gamma}, \mathbf{B}\right\}$. ⊣

Theorem 3.6.9. *Suppose $n \in \omega \backslash \{0\}$, $\beta, \gamma \in \omega_1$.*

For any nonempty $\Upsilon \subseteq CWF$ such that Υ is closed under shifting trees, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (3.53) and (3.54) are equivalent to $Det(\mathbf{\Sigma}^1_n \upharpoonright X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Sigma}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \tag{3.53}$$

$$\mathcal{G}\left(\mathbf{\Pi}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \tag{3.54}$$

For any nonempty $\Upsilon \subseteq CWF$ such that Υ is closed under shifting trees, $\mathcal{B} \in \left\{\mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Delta}^0_\beta, \mathbf{B}\right\}$ and $\mathcal{C} \in \left\{\mathbf{\Sigma}^0_\gamma, \mathbf{\Pi}^0_\gamma, \mathbf{\Delta}^0_\gamma, \mathbf{B}\right\}$, the determinacy of following (3.55) is equivalent to $Det(\mathbf{\Delta}^1_n \upharpoonright X^\omega)$.

$$\mathcal{G}\left(\mathbf{\Delta}^1_n; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \mathcal{C}), \mathcal{B} \upharpoonright X^\omega\right)\right) \tag{3.55}$$

$\dashv$

Corollary 3.6.10. *Suppose Λ is σ-algebra and closed under Λ-substitution. Then for any nonempty $\Upsilon \subseteq CWF$,*

$$Det(\Lambda \upharpoonright X^\omega) \Leftrightarrow Det\mathcal{G}\left(\Lambda; Tree_2\left(X, \Upsilon^\omega, \Gamma(\omega, \Lambda), \Lambda \upharpoonright X^\omega\right)\right). \qquad \dashv$$

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Curriculum Vitae

Date: May 2017

Emi Ikeda

Department of Mathematical Sciences
University of Nevada, Las Vegas
4505 S. Maryland Pkwy.
Las Vegas, NV 89154-4020
Office: CDC-09 923
Phone: 702-895-5253
Email: ikedae2@unlv.nevada.edu

Education

University of Nevada, Las Vegas,
Department of Mathematical Sciences
Doctor of Philosophy, 2017
Major: Pure Mathematics
Focus of Study: Set Theory

Yamaguchi University, Japan
Master of Science, 2006
Major: Pure Mathematics
Focus of Study: Algebraic Topology

Yamaguchi University, Japan
Bachelor of Science, 2004
Major: Pure Mathematics
Focus of Study: Algebraic Topology