May 2017

Effectiveness of ST Math in College Remedial Mathematics Students of Learning Fraction Concepts

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EFFECTIVENESS OF ST MATH IN COLLEGE REMEDIAL MATHMATICS STUDENTS

LEARNING OF FRACTION CONCEPTS

BY

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A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Philosophy – Curriculum and Instruction

Department of Teaching and Learning
College of Education
The Graduate College

University of Nevada, Las Vegas
May 2017

Dissertation Approval

The Graduate College
The University of Nevada, Las Vegas

March 28, 2017

This dissertation prepared by

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entitled

Effectiveness of ST Math in College Remedial Mathematics Students Learning of Fraction Concepts

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy – Curriculum and Instruction
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Abstract

This study examined the extent to which the iPad app, Spatial Temporal Mathematics (ST Math), diminished college remedial mathematics students' natural number bias and deepened their fraction conceptual understanding. In this quasi-experimental study one class played the ST Math fraction games for 8 weeks, and they were compared to a control class who were taught without technology. The frameworks for this study included the framework theory of conceptual change, the reorganization theory, the microworld and the Lesh Translation Model. Pre and post-tests were used to examine the fraction conceptual understanding of the students in the ST Math class and the non-ST Math class before and after the intervention. Also, interviews were conducted to investigate how the ST Math students developed their fraction conceptual understand before and after the game play, compared to the non-ST Math students.

As a whole, the students in both classes exhibited the natural number bias in all three areas: magnitude, density, and computation at the pre-test stage. However at the post-test stage, they diminished the bias in magnitude but in the other two areas, the bias was persistent. Overall, the statistical result was not significant between the ST Math class and the non-ST Math class on the post-test. However, the ST Math class had more students who answered correctly regarding the fraction density concept on the post-test. Similarly, the post-interview results revealed that the ST Math students exhibited conceptual gain in fraction magnitude, the fraction addition concept, and the fraction multiplication concept. However, although they showed the conceptual gain in fraction multiplication, they had difficulty with the concept of fraction division. The students in both classes did not show any conceptual gain in fraction division.

Acknowledgments

I was supported by so many people to compete this dissertation and I cannot find words to describe my gratitude to each and every individual who impacted my journey to get to this point. I would like to show my deep appreciation to those who had a significant influence on this study and this journey.

First and foremost, I would like to thank my advisors Micah Stohlmann and Satish Bhatnagar for their support, patience and guidance throughout this entire study. Secondly, I would like to show my deep appreciation to my other doctoral committee members, Travis Olson and Howard Gordon, for providing me the precise advice in the dissertation and William Speer who offered me the classes so that I could conduct my dissertation study.

Also, thanks to my fellow graduate colleagues, Marissa Nichols, Amy Adkins and Daniel Corral, for encouraging, supporting and assisting me throughout the journey. Without each one of you, I would not be able to finish this study.

I would like to thank my parents for their unconditional support, love, patient and understanding. Similarly, thank my congregational pastors and families, Richard Hill and Richard Cooper, for their diligent prayers and spiritual support.

Most importantly, all my praises go to Elohim Yeshua who saved me through His grace and resides in me as the spirit. He orchestrated everything and everyone in this journey so that I could accomplish this work through His guidance. It is my desire that my Elohim is going to use me for His purpose through my completion of this degree to make a difference in my future students' life so that His name will be glorified.

Dedication

I dedicate this dissertation to my Elohim, without Him, I would not be able to accomplish this. I also dedicate to all the professors in my doctoral committee, the colleagues, my parents and the church family. Each one of them has a special place in my heart and without their support, I certainly could not get this done.

Philippians 4:13

I have strength to do all, through Messiah who empowers me.

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CHAPTER ONE

Introduction

Many students face great difficulty understanding the meaning of fractions (Anthony & Walshaw, 2007; Lamon, 2007; Verschaffel, Greer & Torbrynes, 2006; Young-Loveridge, Taylor, Hawera & Sharma, 2007). This is because it requires a deeper understanding of numbers than whole numbers. Understanding fractions is vital because it can be a strong indicator of students' mathematical achievement later years in their academic career (Baily, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012), since understanding the concept of fractions is a foundation for learning algebra, geometry and other higher mathematics (Fazio & Siegler, 2011). However, mastering fractions has been a major obstacle for students (Pantziara & Philippou, 2012).

Too many students who enter college education are not prepared for college level mathematics courses as the number of enrollees in remedial mathematics courses have been increasing (La Joy, 2013). Also, a majority of students in remedial mathematics classes do not go on to pass their required mathematics courses (Wiseley, 2011). In remedial mathematics courses, one of the main focuses is mastering fractions because students have failed to master this in their previous education (Gal, 2000). Therefore, finding a solution to assist college remedial students to master fraction concepts should be a priority of college mathematics departments so that students can be successful in college level mathematics courses to meet their graduation requirement.

Usually college mathematics classes are taught procedurally because remedial courses have to cover many topics and instructors are under pressure of covering the topics quickly and shallowly in a procedural way (Hinds, 2009). This means that they do not utilize interventions to

deepen students' conceptual understanding such as manipulatives, models and technology. This study investigates how technology usage in a college remedial mathematics course can be a solution to the high number of students in remedial mathematics classes that are not successful. The use of technology could assist students to better understand the fraction concepts that are foundations for other mathematical concepts.

Natural Number Bias

Why do so many students face great difficulty developing and understanding the concepts connected to fractions? One of the main reasons for the difficulty is that many properties, which are true for whole numbers, are not necessary true for fractions (Fazio & Siegler, 2011). Children develop initial concepts based on their experience with their environment. This prior knowledge of initial concepts formed in childhood can have a lasting effect into high school and beyond (DeWolf & Vosniadou, 2011). Vosniadou and Verschaffel state that children have formed an initial concept of number, which is built on counting and resembles the mathematical concept of natural number before they are exposed to rational numbers (2004). This leads even students who are in high school and above to have expectations about what counts as a number and how the number is supposed to behave (Vamvakoussi & Vosniadou, 2004). Similarly, misconceptions happen when students add new conflicting information of fractions to their initial concept of natural numbers (DeWolf & Vosniadou, 2011). For instance, multiplication of fractions does not always lead to a value larger than the multiplicands, division of fractions does not necessarily lead to an answer smaller than the dividend, and fractions do not have unique successors as natural numbers have (e.g., the successor of 3 is always 4) (Fazio & Siegler, 2011). It is very challenging even for high school students to overcome the belief that properties, which are true to whole numbers are not true to all numbers. They do not understand that there are infinitely

many numbers between two fractions (Vamvakoussi & Vosniadou, 2010). This is because the initial concept of natural numbers is persistent to learners and can be very difficult to overcome when they are in process of building understanding of new concepts (Inagaki & Hatano, 2008). This is called natural number bias. Learners tend to apply the natural number properties to rational numbers and this bias is one of the major obstacles in learning fractions (Ni & Zhou, 2005). This is one of the main reasons that the number of college students who are enrolled in remedial mathematics courses has been increasing (Ramussen et al., 2011). For instance, in California community colleges, just 9% of students enrolled in remedial math courses are going to complete a subsequent college level mathematics course, after the remedial classes (Wisely, 2011). This certainly affects their academic career and some of them will never complete a degree because of not being able to complete the remedial course.

In a study of the natural number bias in fraction learning, Vamvakoussi and Vosniadou (2010) constructed a theory called framework theory of conceptual change, which is that learners tend to construct their knowledge through daily experiences in logical categories. This was introduced to explain how students' misconceptions are constructed and how the misconception learners have will be changed through getting exposed to new mathematical concepts. Stigler and his colleagues indicate that community college students have difficulties with understanding fractions (Stigler et al., 2010), since many students in a remedial math courses have failed to master the fraction concepts in their previous schooling (Gal, 2000). According to the following researchers (Vamvakoussi & Vosniadou, 200; Inagaki & Hatano, 2008; Ramussen et al., 2011), one of the possible reasons for students having challenges with mastering the concept is the natural number bias is persistent throughout secondary education. Therefore, it makes sense to apply this theory to this study that investigates how learners reconstruct their initial fractional

concepts, which could be influenced by the natural numbers, through experiencing fraction problems that do not hold the natural number rules. This framework theory in conceptual change has been utilized in different studies that investigated how the natural number bias prevents secondary students from understanding the fractional concepts (Prediger, 2008; Vamvakoussi & Voaniadou, 2010).

Conceptual Understanding in Fraction Learning

Not only has the topic of fractions been taught procedurally, but also most of mathematics content has been. This is seen in the video studies of American eighth grade classrooms and other studies that students spent much of the instructional time on practicing routines and procedures introduced by the teacher (Hiebert et al., 2003; Jacobs et al., 2006; Stigler & Hiebert, 1997). Heibert and Stigler mentioned that many American teachers think that mathematics is a set of procedures and the goal is helping students to execute the procedures (1997). The problem of relying on the procedural way is that students use steps that lead to a solution without having conceptual understanding, which is the ability to see connections among representations for the steps (Byrnes, 1992). This way of learning leads learners to memorize a meaningless series of steps and often they forget some of the steps or mix up the steps, which leads to errors (Tirosh, 2000; Freiman & Volkov, 2004).

Teachers often emphasize algorithms for fraction operations at the cost of developing children's conceptual understanding of rational numbers (Lubinski & Fox, 1998; Moss & Case, 1999). Because of the lack of conceptual understanding, for instance, when students add or subtract fractions, they tend to add the numerators and denominators separately without having a common denominator as if fractions were natural numbers. If they understood the reason why they need to have a common denominator for the operations, they would not add or subtract

fractions without having a common denominator (Fazio & Siegler, 2011). This happens because students do not have a solid understanding of fraction magnitude, which could help them estimate a reasonable answer. Also, many students learn correct procedures but never seem to learn the principles that underlie them (Byrnes & Wasik, 1991). This is why learners tend to forget procedures easily and end up adding and subtracting without having a common denominator. Therefore, they often fail to transfer the procedural skills to a new problem (McNeil & Alibali, 2000) and procedural skills without conceptual grounding lead to inflexible procedural knowledge (Byrnes, 1992).

The concepts of mathematics are continuous and one is built on another. The fraction concepts are the foundation for the other mathematical concepts learned in high school and higher education from algebra to calculus (Siegler, Thompson, & Schneider, 2011; Siegler, et al., 2012). The existing studies regarding students' fraction conceptual understanding have targeted mainly elementary children. Hence, there have not been many studies conducted, which targeted secondary or college students' conceptual understanding of fractions.

Therefore fraction instruction needs to aim at learners' conceptual understanding by implementing effective ways of pedagogies to reduce the natural number bias and strengthen learners' fractional concepts. There are different elements to make fractional lessons effective for understanding. For instance, utilizing manipulatives, visual models and technology during the fraction lesson are very helpful for learners to develop understanding and reduce the natural number bias (Fazio & Siegler, 2011).

The Lesh Translation Model (LTM) (Lesh, Landau, & Hamilton, 1983) describes a definition of conceptual understanding that can be used with fractions and this study applies the model to investigate how the students' fraction conceptual understanding will be changed

through the usage of a iPad app. This model claims that understanding of concepts can be defined as an ability to make translations (connections) between and within representations such as manipulative, picture, written symbols, real-world situations and language. The deeper a student's understanding is, the more translations the students can perform. Utilizing this model for investing how students' fraction conceptual understanding becomes deeper throughout the usage of the iPad app is reasonable as the LTM has been used in a number of studies on conceptual understanding (Moore, Miller, Lesh, Stohlmann, & Kim, 2013; Stohlmann, Moore, & Cramer, 2013).

Technology Usage in Fraction Learning

Technology usage in fraction learning has gained more attention in recent years. We live in the technology era and new mobile technologies such as the iPad and iPhone can provide fun and engaging learning (Riconscenete, 2013). Similarly, by using a touchscreen mathematical learning app, children's learning efficiency can be improved. Technology apps can also facilitate the development of students' conceptual understanding because technology can offer a variety of tools and constraints and reduce cognitive load (Bertolo et al., 2014; Mavrikis, Hansen & Geraniou, 2016). Hence, there is no reason not to take advantage of technology for effective fraction learning. However, there are not many studies conducted regarding effectiveness of technology in learning fractions.

In a study of Steffe and Wiegel (1994), they analyzed children's mathematical learning through their cognitive play activity by using technology they created called computer microworld. With this technology children engage cognitive play activities and could transform the play activities to mathematical play activities. This computer microworld was inspired by Papert's concept of microworld that provides a medium where children can learn mathematics

and express what they have learned (1980). Papert's microworld is an independent world where children can "learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction" (p. 117). Since children realize things through playing and playing motivates them, this idea was applied to their study. The nature of playful computer microworld captured the children's interest and they were able to use their mathematical schemes in ways that are not possible with a workbook. On microworld, children were able to establish number sequence in the context of a playful orientation and establish the ownership of the mathematical activities. Steffe and Wiegel also suggest that the cognitive play activities in a mathematical context can establish a new concept by utilizing another concept. This emergence of a new concept can be explained by the reorganization hypothesis that is when a new scheme is established by using another scheme in a novel way, the new scheme can be regarded as a reorganization of the prior scheme (Steffe, 2001, p.268). The framework theory, the concept of microworld and the reorganization hypothesis in these studies mentioned above can be utilized for a study aiming at adults' fraction conceptual understanding change with technology usage. Therefore, this study uses framework theory, the concept of microworld and the reorganization hypothesis as the frameworks.

Although, usage of technology could provide a great learning opportunity to develop the concepts of fraction, there are not many studies that have investigated how technology usage, specifically iPad apps, can enhance students' fraction concepts. The result of this study will contribute to college and junior college mathematics departments, which are in charge of curriculum of remedial mathematics courses. The results of this study may point how to implement technology to provide more effective learning opportunities for fraction conceptual understanding, instead of the traditional lecture based learning, so that the students can be

successful in subsequent college level mathematics courses.

Benefit of ST Math

Computer based instruction (CBI) can have a positive impact on students learning (Tran et al., 2012). Shyu (1999) conducted an experimental study examining how CBI influences students' attitude toward mathematics instruction and their problem solving skills. The results revealed that video-based mathematics instruction was positively associated with students' mathematical achievement. Another quasi-experimental study (Malouf, et al., 1990) was conducted to investigate the effectiveness of a computer software program toward 52 middle school special education students and the result indicated that it had a positive influence on their mathematics state standardized exam performance.

It has been found that CBI implementing the spatial contiguity principal, which is to align words to corresponding graphics (Clark & Mayer, 2011), has more desirable effects on learners' achievement than the non-spatial CBI (Harter & Ku, 2008). Spatial Temporal Mathematics (ST Math) uses images to assist students to develop spatial temporal reasoning, which can lead to deepened mathematical conceptual understanding such as fractions, proportions, symmetry and other arithmetic operations (Tran et al., 2012). Spatial-Temporal reasoning means instinctive ability to visualize and manipulate images through ordered steps in space and time; which is essential to solve problems in mathematics, science and other subject areas (MIND Research Institute, 2016).

ST Math is built on the concept of assisting students to develop the capability to visualize underlying concepts of mathematics and to connect between the concepts and mathematical problems (Rutherford et al., 2010). Similarly the software aims at promoting the learning process, which is students could gain conceptual understanding along with procedural and

algebraic skills by understanding the meaning behind procedures (Shaw & Peterson, 2000). According to Geary (1995, 2011), the relation between spatial representations and an implicit ability to understand quantity of non- symbolic representations could be a successful path to enhance learners' numerical magnitudes and support to connect between symbolic and non-symbolic representations. Hence, visual representations can enhance the conceptual understanding of both the fraction magnitude and its manipulation (Siegler et al., 2010; Siegler, Thompson, & Schneider, 2011).

The visual learning approach not only supports differentiated instruction to reach students of all abilities and language proficiency, but also provides engaging learning opportunities to students who struggle with learning the subject with procedural methods and traditional materials. ST Math includes hundreds of computer games that slowly scaffold in language and promote mastery-based learning and understanding. The program can be combined with any textbook and classroom instruction. Students can learn at their own pace. The program also offers instructive feedback and data-driven reports (MIND Research Institute, 2014). Rutherford and researchers (2010) conducted a randomized experimental study to examine the correlation between the ST Math usage and 2nd through 5th grade students' mathematics achievement on the California Standards Test (CST). The result indicates that ST Math usage and their CST achievement were positively correlated.

Above mentioned, CBI with Spatial Temporal principle promotes more favorable results on students' mathematical achievement and ST Math is created to assist them to visualize mathematical concepts and connect the concepts to mathematical problems along with procedural and computational skills. Hence, the software could be an effective medium for remedial mathematics students to master the fractions concepts.

Statement of the Problem

Many previous studies (e.g. Vamvakoussi, & Vosniadou, 2010, 2012; Van Hoof et.al., 2015) have investigated how the natural number bias has been persistent and how it interferes with learners acquiring understanding of fraction concepts. However, these studies have been mainly focused on elementary school children and have been conducted outside of the U.S. In all areas of the world though the usage of manipulatives and models has been found to provide learners very effective learning opportunities (Burns, 2004; Cramer, Wyberg, & Leavitt, 2008; Fazio & Siegler, 2011; Naiser, Wright, & Capraro, 2004; O'Shea; 1993).

For the recent years in mathematics learning, mobile technology such as the iPad and iPhone, has been catching mathematics educators' attention because it provides instructional designers more options to create more effective learning opportunities that go beyond dry presentation, limited interaction and formal school setting (Riconscenete, 2013). However, there are not enough studies conducted that examined how mathematics apps usage can assist learners to deepen their mathematical understanding, particularly in the area of fractions. Although there are studies utilizing the mathematical learning software in learning the concept of unit fraction, the researchers also integrated the interactional aspect with the teacher and the other students in their studies (Steffe & Olive, 2002; Tzur, 1999, 2004). According to the researchers, the interactions are also a critical element in the children's mathematical schemes with the usage of technology.

The study of Riconscenete (2013) aimed at investigating the effectiveness of the iPad app called Motion Math, and had children use the app alone in their fraction learning. However, the study focused more on the students' motivational aspect than on their fraction conceptual understanding. Hence, a study that aims at investigating how a certain mathematical app can

deepen learners' fraction conceptual understanding is necessary.

It would be very useful to know whether a particular iPad mathematics app called ST Math that can be more effective because it utilizes the spatial contiguity principal (Harter & Ku, 2008), reduces the misconceptions originating from the natural number bias and strengthens fractional conceptual understanding of college undergraduate students in a remedial mathematics course. This study will fill the gap in the existing literature mentioned above. The research problem therefore is to investigate how the usage of a iPad app, ST Math, can strengthen college remedial mathematics students' fraction concepts while reducing the natural number bias they have.

Purpose of the Study

For years in the U.S., mathematics has been considered one of the most difficult subjects and test scores indicate that American students are performing just slightly above the international average on the Trends in International Mathematics and Science (TIMSS) assessment (Aljami, 2012). As the TIMSS result shows, the U.S. is losing their competitive advantage in the field of mathematics, which contributes significantly to a technological society (La Joy, 2013). In order for this country to maintain the leading position in the field of technology, it is necessary for more students to pursue their higher education degree in the fields of science, mathematics, and engineering (National Research Council, 2001). However, the number of students who are enrolled in remedial mathematics courses is rising (Ramussen et al., 2011). The main reason for this trend is that deficits and gaps in mathematical understanding are persistent throughout students' educational careers (National Research Council, 2001). The subject is continuous and one concept is built on another concept and once learners do not understand one concept, subsequent concepts increase difficulties of understanding. Among

many mathematical concepts to be learned before college education; one of the key concepts, which is a foundation for mathematical skills is fractions (Siegler, Scherider, & Thmopson, 2011; Siegler et al., 2012). Understanding the concept of fractions is a foundation for learning algebra, geometry and other higher mathematics (Fazio & Siegler, 2011) and understanding fractions can be a strong indicator of students' mathematical achievement later years in their academic career (Baily, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012). However, understanding fractions has been a major obstacle in mathematics education (Pantzianra & Philippou, 2012).

College students who need to take remedial mathematics courses have to complete the course(s) to take required mathematics courses such as college level algebra, for graduation. However, many students do not complete the remedial courses successfully (Wisely, 2011) and this holds them back from moving forward in their academic career (La Joy, 2013). The worst case is some students give up on pursing their degree because they do not succeed in the remedial courses. Many students are forced to drop out of community college since they are unable to be successful in their mathematics course (Stigler, Givvin, & Thmopson, 2010).

According to different researchers, students in remedial mathematics courses have difficulties with fractions (Stigler et al., 2010), as the topic of fractions continues to be a main focus of study in remedial courses since many students are not able to master the fraction concepts in their K-12 education (Gal, 2000). Therefore even if students move onto the subsequent required math courses after the remedial courses, they may not succeed in the courses because the concept of fractions is the foundation for these courses, such as college level algebra and pre-calculus. This is seen as, "Algebra teachers rank poor understanding of fractions as one of students' biggest weakness in preparation for the study of algebra" (Stigler et al., 2010, p.1).

Therefore it is clear that fraction understanding is a key element for college students to be successful in college level mathematics courses and the instruction should focus on developing the fractional conceptual understanding, instead of just procedural aspects. However, the instruction mainly copies traditional elementary, secondary and post-secondary mathematics classrooms that focus on “lecturing and listening” (National Research Council, 1989, p. 57) and transmitting procedural knowledge to students (Grubb, 2001; Hinds, 2009); instead of helping students to understand conceptually by utilizing manipulatives and visual models or technology (Bachman, 2013). Therefore, if the research can show how technology usage such as an iPad app in college remedial math courses can develop college students’ fraction concepts, the result will indicate that the technology usage in fraction learning is beneficial not only for children but also for adults. This research’s result could be one of the solutions for reducing the number of college remedial mathematics students who will retake the course and for preparing them to succeed in the subsequent mathematics courses.

There are many mathematical learning apps on the market in these days. However, there are not enough studies, which discern effective apps from ineffective apps and many apps focus on the entertainment aspects more than educational content (Riconscente, 2013). Although the study of Riconscente (2013) investigated the effectiveness of an app called Motion Math in fraction learning, the study focused more on the children’s motivational aspect by using the app. In this regard, this current study, which focuses on students’ mathematical learning by using the app ST Math, fills the gaps of the existing studies. Lastly, there has not been enough experimental research studies conducted to ascertain whether iPad apps enhance students’ mathematical learning (Riconscente, 2013). Therefore, by using an experimental design, the effectiveness of the ST Math can be better ascertained.

Significance of the Study

Although fractions are a foundation for college level mathematics courses, which are a requirement for graduation, the students in remedial mathematics courses have trouble mastering fraction concepts. If they do not master fractions, it is clear that they will not succeed in the required mathematics classes they have to take. This indicates that they have to retake the required courses and this will certainly hold them back from earning a college degree. The worst scenario is that some students might give up earning a degree because the mathematics courses become an obstacle. It is fair to state that understanding fractional concepts in remedial courses is very crucial and it could decide students' academic career. Hence, it is critical for a college mathematics department to identify how they can support these students to master the fraction concepts. Since this study utilizes technology, ST Math, that has not been investigated before in remedial mathematics classes, the study's results could provide a college mathematics department with a new solution. Similarly, the study's result could provide support for secondary school mathematics educators to implement ST Math so that their students could master fraction concepts better and these students will not go through remedial mathematics courses when they get in college.

Research Design

The study used a nonrandomized control group pre- and post- tests quasi-experimental design to compare two remedial math courses because the researcher used already existing classes and could not assign students randomly into these two classes. The ST Math class used ST Math 30 minutes every class meeting for 8 weeks and the non ST Math class completed worksheet problems during this time. The ST Math class used ST Math during the normal

instruction, which lasted 75 minutes and was held twice a week, after they received a lecture that covered the required mathematical topics in the course. Since there were eight sequences of fraction games to be played, the students were to finish each sequence of the game each week. To make sure they were on the schedule, those who were not able to finish the sequence for the week during the class meeting, they were to play the game outside of the classroom for another 1 hour. The non ST Math class received the normal instruction that covered the same topics but did not use the ST Math fraction games. Instead of playing the ST Math games, they worked on a worksheet that contained fraction problems and other problems related to what they have learned in the class for the day. The researcher taught both classes. This entire study lasted for 12 weeks including the pre and post-tests and the pre and post- interviews. Before this experiment started, a pre-test was administered to examine students' initial fraction understanding and at the end of the study, a post-test was administered to investigate statistically how much fraction conceptual understanding the ST Math class had been able to gain compared to the control class.

Because the study was a quasi-experimental design, there were potential threats to its internal validity and external validity. These threats are explained in further detail in chapter 3. The possible threats to internal validity for this design were experimenter effect, diffusion, differential selection, subject effects and interaction of selection and regression. Since external validity influences the generalizability of the study result, the possible threats to the external validity were subject effects and selection. To identify the effectiveness of ST Math statistically, by comparing the post-test of the non ST Math class to the ST Math class, an independent two-sample t-test was conducted.

However, a statistical result does not tell everything, because students can have a wrong reasoning to solve a problem, they still can come up with a right answer. Hence, this study

included interviews with 3 selected students from the ST Math class and with 3 selected students from the non ST Math class before the usage of the app and after so that the researcher could examine how the 3 ST Math students' fractional conceptual understanding had changed from the pre-test to the post-test by them explaining their rationale for their solutions verbally, compared to the 3 non ST Math students' reasoning on both pre and post-interviews.

Research Question and Hypotheses

1. *To what extent do college remedial mathematics students possess whole number misconceptions? In addition, to what extent can each class, ST Math and non-ST Math, express fractions and their operations, pictorially, realistically, symbolically and verbally?*
2. *To what extent does the usage of ST Math eliminate the natural number misconceptions and deepen college remedial mathematics students' conceptual understanding in learning fractions compared to a non ST Math class of remedial mathematics students taught traditionally without technology?*

The researcher hypothesized at the pre-test stage, that there would be no statistical significant difference in the fractional understanding for the students in both classes. However, after the intervention, the ST Math class students would show statistical significance in their understanding on the assessment test. Also, through the pre-interviews, the selected 3 students from the ST Math class and the selected 3 students from non ST Math class would not explain reasons for their answers verbally in an effective manner and show their fraction concept was biased by the natural number concepts. This would carry for the selected 3 non ST Math students in the post-interview. In the post-interview, the 3 ST Math students would connect the fraction concepts they had gained through the ST Math play with the problems on the assessment and

would be able to explain the reasons for their answers verbally more effectively. They would be able to express their understanding pictorially, realistically and symbolically and would demonstrate the misconceptions caused by the natural number bias were diminished.

Assumptions

This dissertation study was conducted under the several assumptions listed below.

1. All participants were to answer the pre and post- tests to their best abilities.
2. Those who participated in the interview were motivated participants.
3. Assumed that learners usually employed additive actions such as accommodation, to deepen their initial fraction concepts of fractions with new information of non-natural numbers.
4. Acquiring new information of fractions built on the initial fraction concepts could facilitate reconstruction of fraction knowledge.
5. Assumed that deeper fractional conceptual understanding would emerge when students use their previous fraction understanding influenced by the natural number bias, to solve novel fraction problems.
6. That conceptual change was a time consuming process.

Expected Findings

The learners' interaction with the medium of ST Math would provide them opportunity to discover and make sense of the fraction concepts. This would lead them to diminish the natural number bias and assist their conceptual change of fraction knowledge. The conceptual change would be reflected in the post-test score and the interviews, significantly. Those who were in the ST Math class were going to be able to explain the concepts behind the fundamental fraction algorithms and express a fraction pictorially, realistically and symbolically. The usage of ST

Math provides the effective fraction learning opportunity even to college undergraduate level students and usage of the app should be considered even in higher education institutions.

Organization of the study

Chapter 1 presented the introduction, statement of the problem, purpose of and significance of the study, research design, research questions, assumptions, and expected outcomes. Chapter 2 includes the review of related literature to the problem being investigated: mathematical understanding, natural number bias, and effective pedagogies for conceptual understanding of fractions and ST Math. The methodology and procedures used to collect data for the study are obtained in Chapter 3. The result of the analyses and findings from the study are presented in Chapter 4. Chapter 5 contains a summary of the study and findings, conclusions drawn from the findings, a discussion, and recommendations for future study.

CHAPTER TWO

Literature Review

Introduction

In this chapter, the literature relating to the natural number bias that is an obstacle for students to understand the fractional concepts is addressed as well as conceptual understanding and effective fractional instruction. Particularly, the important elements to make instruction more effective so that learners' misconception caused by the natural number bias is reduced and their fractional conceptual understanding will be strengthened are addressed.

Literature relevant to this study includes:

- Theoretical frameworks
- Mathematical understanding
- Natural number bias
- Elements for effective fraction pedagogy
- College remedial mathematics
- Spatial Temporal Mathematics

Theoretical Frameworks

If a learner has the persistent natural number bias, when a learner acquires the fractional concepts, the learner may construct the fractional concepts on the already existing natural number knowledge. To explain the conceptual change from the natural number to fractions, Steffe's (2002) reorganization hypothesis and the framework theory of conceptual change by Vamvakoussi and Vosniadou (2004; 2010) are the core ideas of the fractional concepts acquisition.

The concept of the microworld (Papert, 1980) is that learners can construct not only mathematical concepts but also express the concepts they have constructed creatively through learners' play in a mathematical context by using technology as a medium that can support mathematical concepts and operations, since learners construct their reality through playing.

Hence, the concepts of the microworld are indispensable for this study integrating the technology usage for fraction understanding.

Framework Theory of Conceptual Change

This theory was initially developed to explain learners' misconceptions in their science learning, but has been applied to mathematics. A main assumption of this theory is that children coordinate their explanations of daily experiences in the context of what they experience into relatively ordered categories or framework theories (Vamvakoussi & Vosniadou, 2010). In here, the term theory is used to imply a relatively ordered system, which is generative and helps children to predict, explain and deal with brand new problems.

In students' initial framework theory of numbers, the theory assumes that they have beliefs and expectations about what counts as a number and how it is to behave. At the initial framework, numbers imply natural numbers and they follow the natural number rules. When students encounter fraction concepts which are apparently not numbers in their initial framework theory of numbers, the question arises: how do they make sense of the new entity, fraction concepts?

The framework theory of conceptual change is a constructivist approach and in acquisition of the fraction concepts, it assumes that learners usually use additional learning mechanisms such as accommodation, assimilation and internalization, to deepen their initial framework theory of natural numbers with new information of fractions. Thus, the theory of conceptual change explains that the students' misconceptions as a result of wrong natural number reasoning (Moss, 2005; Ni &Zhou, 2005) attribute to students inferring their natural number concepts heavily to make sense fractions. That is the transition process from the natural number knowledge to the fraction concepts would generate synthetic conceptions of fractions,

which bridge the gap between the natural number knowledge and the fraction concepts.

Reorganization Hypothesis

Steffe and Olive (1990) state that when children construct the arithmetic of rational numbers, their fractional schemes can accommodate their numerical counting schemes. This is called reorganization hypothesis; when a new scheme is constructed by using an already existing knowledge in a different way, the new scheme will emerge, reorganizing the prior scheme. In the case of fraction learning, children's fractional schemes emerge as accommodations in their numerical counting schemes.

Steffe (2001) claims that when learners acquire fractional schemes, the schemes are going to supersede their prior counting schemes and the fractional schemes assist them to solve the problems better than the counting schemes solved and also to solve new problems the counting schemes were unable to solve. Hence, in this hypothesis children's fractional schemes will supersede their numerical counting schemes and the counting schemes are considered as constructive vessels in the construction of fractional schemes (Kieren, 1980).

Microworld

Papert's microworld is an independent world where children can "learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction" (p. 117). Children build their reality through playing and this idea can be applicable to mathematical learning as well. Playing in a context of mathematics could encourage their construction of a mathematical reality and serve as a source of motivation to do mathematics (Steffe & Wiegel, 1999). According to Hilgard and Bower (1966), motivation is a core element in any learning model and is dichotomized into two kinds of motivation, intrinsic and extrinsic motivation. The motivation by playing is extrinsic motivation and although this motivation cannot supersede the

internal motivation coming from understanding mathematics, the external reinforcement cannot be ignored in learning.

The usage of medium such as manipulatives, certainly offer children an opportunity to engage in cognitive play activities with the aim that they perform fundamental conceptual operations as enjoyment. To enhance mathematical learning, the medium needs to not only support constructing mathematical concepts and operations but also provide the opportunities to express the concepts and operations creatively to attain learning goals (Steffe & Wiegel, 1994). The concept of microworlds is a promising way to create an active medium where children can learn mathematics and express what they have learned creatively (Papert, 1980). A microworld provides a world where children learn to apply habits of exploration from their daily life to the domain of scientific construction. Steffe and Wiegel (1994) created a computer based medium where children engage in cognitive play and learn mathematical play, based on the concept of microworld.

Mathematical Understanding

Conceptual and Procedural Knowledge

The concept of mathematical understanding is complex although the term is often used (Stienstra, 2013). In the field of mathematics education, some could relate the understanding to successful performance on exams, quizzes and homework, while others could define it is about knowing why and not just how (Brown, Cooney, & Jones, 1990). The National Council of Teachers of Mathematics (NCTM) indicates the importance of understanding in mathematical learning by stating that students need to learn mathematics with understanding by building new knowledge from experience and previous knowledge (NCTM, 2000).

Contrary to this, focusing on skills and procedures in mathematical learning caused an

assumption that producing the correct solutions was the sign of possessing mathematical understanding (Stienstra, 2013). However, mathematical understanding was insufficiently defined under rote memorization or meaningless manipulation (Byers, 1980). Brownell (1935) claimed that students have to be able to explain their thought process while solving problems and mathematical understanding is developed by recognizing the relationships within mathematics. Brownell also suggested that arithmetic needs to be less challenging to learners' memory and more challenging to their intelligence.

Skemp discussed mathematical understanding from the perspective of instrumental and relational understanding (1976). For his definition, relational understanding means deeper understanding and mentions knowing what to do and why and this understanding more likely promotes transfer. On the contrary, instrumental understanding means knowing and using rules and procedures. Skemp suggested that mathematical understanding was dichotomous although he acknowledged that when the two types of understanding had some connections, partial understanding could happen (Byers, 1980). Byers and Herscovics also recognized different levels and magnitudes of understanding (1977). They added intuitive and formal understanding to the Skemp's model. Intuitive understanding is the ability to solve problems with instinct without any sort of instruction and formal understanding is the ability to think logically and make connections between mathematical symbols and ideas.

According to Byrnes (1992), conceptual knowledge is defined as "knowing that and more precisely as relational representations, which consists of two or more represented entities, which are mentally connected through a relation of some sort" (p. 236). Hibert and Lefevere (1986) described that conceptual understanding is described more clearly as knowledge that has rich relationships. Similarly, Kieren (1993) defined it as "the interweaving of the intuitive and formal

knowledge on a personal basis" (p. 49). These different words such as "interweaving", "connecting" and "rich in relationships" define conceptual knowledge not as the memorization of separate valuable facts but as the ability to see connections between knowledge (Hallett, Nunes & Bryant, 2010).

On the other hand, procedural knowledge is characterized as the "knowing how" to do something (Hallett, Nunnes, & Bryant, 2010). Byrnes (1992) defined the knowledge as 'goal directed action-sequences" (p. 236) and this includes things as memory strategies, mathematical algorithms, and grammatical rules. Procedural knowledge contains a series of certain defined actions to produce a desired result, which is usually the right answer to a problem in mathematical learning (Hallett, Nunes, & Bryant, 2010). This is what Hiebert and Lefevre (1986) described procedural knowledge as "algorithms or rules from completing mathematical tasks" (p. 6). The distinctive aspect of procedures is that they are executed and independent of meaning and this implies that an individual should not need to reflect on what concepts are implemented in the procedures (Hallett, Nunes, & Bryant, 2010). While conceptual understanding consists of many different kinds of relationships, procedural knowledge is linear and has minimal connections to link each step of the procedure and the knowledge can be obtained by rote but conceptual understanding cannot be (Biesenthal, 2006).

Iterative Process

Since mathematical proficiency has been considered as procedural fluency, mathematics educators focus on students attaining procedural skills first (Biesenthal, 2006). However, more than a few studies indicate (Hiebert & Lefevre, 1986; Moss & Case, 1999; Schneider & Stern, 2010) there is an iterative process between conceptual understanding and procedural knowledge.

Carpenter and his colleagues suggest that teaching should aim at students developing

conceptual understanding first and then link this to procedures and symbols (Carpenter & Lehrer, 1999; Stigler & Carpenter, 1992; Carpenter, 1986). Recent studies show that there is a positive correlation between high level of conceptual knowledge and execution of procedures. It also has been found that individuals with high conceptual knowledge are able to (a) detect errors (Hasselhorn & Korkel, 1986) (b) memorize things in a manner of making sense (Bjorklund & Buchanan, 1989; Schneider, 1986), and (c) use mathematical algorithms properly (Byrnes & Wasik, 1991). Similarly, according to Hiebert and Wearn (1996), in their study of 70 elementary school students grade 1 through 3, mathematics instruction which emphasizes more on both conceptual understanding and procedural knowledge, is more effective than instruction, which focuses only on procedural skills alone.

Another study that aimed at investigating the relations between conceptual and procedural knowledge of mathematical equivalence toward 4th and 5th graders, Rittle-Johnson and Alibali found that the two types of knowledge interact with each other and conceptual knowledge may have more influence on procedural knowledge than the other way around (1999).

Lastly, the study of Osana and Pitsolantis (2013) investigated two different instructions, one made an explicit link between conceptual understanding and procedural knowledge and the other one did not make any. The results indicated that the students who were in the instruction that made the external link gained significantly more conceptual understanding, compared to the other instruction. These studies suggest that the key for mathematical learning is conceptual understanding and once learners learn the concepts, they are able to connect the concepts with procedural knowledge to gain deeper mathematical understanding.

Mathematical Conversation

Interviews with the participants and student-student conversations are a critical part of this study to investigate their conceptual change in fraction learning. Conversations with learners are critical tools because they inform what the learners do and do not understand (Vanderhye & Demers, 2008). The conversations could always expose a variety of wrong reasoning, computational mistakes, and misunderstandings that teachers can correct to guide the learners' mathematical learning (Chapin, O'Connor & Anderson, 2003). While students explain their thinking and their mistakes are revealed, their understanding can be deepened by correcting errors and internalizing newly obtained concepts (Chapin & O'Connor, 2003).

The NCTM also suggests it is very valuable to gather information about students through questions and interviews (NCTM, 2000) so that teachers can learn about their students' mathematical understanding (Vanderhye & Demers, 2008). In one to one diagnostic interviews with a student, a teacher can examine a learner's mathematical understanding (Bräuning & Steinbring, 2011). According to Krainer (1988), the purpose of the interview is so a teacher can know his/her student's mathematical thinking more clearly and where they lack mathematical understanding.

When teachers question their students' mathematical understanding, the conversation could be a teacher-centered conversation and their students tend to respond in the way to be accepted by the teachers (Falle, 2004). According to Barnes' study (1999), the conversations among students are much freer and the conversations can disclose a lot about their understanding. A few studies suggest that when teachers promote the conversations among students, the questions should be non-routine ones because these questions can stimulate their thought and expand higher order thinking to get their mathematical knowledge deeper and more

conceptual (Biggs, 1991; Ellerton & Clements, 1996). Hence, the pre- and post- tests in this study are non-routine questions that focus on fraction concepts and the students are given the opportunity to discuss any fraction question that will happen during this study among them.

Lesh Translation Model

The Lesh Translation Model (Post, Behr, & Lesh, 1986) suggests that fundamental mathematical ideas can be expressed in five different modes: manipulatives, pictures, real life situations, spoken language, and written symbols (Figure 1). This emphasizes that mathematical understanding is shown in the capability of make connections among and within the different modes. Hence, the model indicates that developing deeper mathematical understanding requires experiences in different modes and experience making connections between and within these modes.

For instance, a translation within written symbols implies that students can conceptualize 3/5 is equal to 6/10, when the same size of pizzas are divided into 5ths and 10ths, the total amount of 3 slices of pizzas from the pizza sliced into 5 is equal to the total amount of 6 slices of pizzas from the pizza sliced into 10 by drawing 3/5 and 6/10 pictorially.

It is important to mention that a translation within and between the modes requires students to reinterpret a concept from one mode to another. Mathematical conversation is also a key element of displaying learners' understanding. When they understand mathematical concepts in the other modes, they should be able to express the understanding verbally.

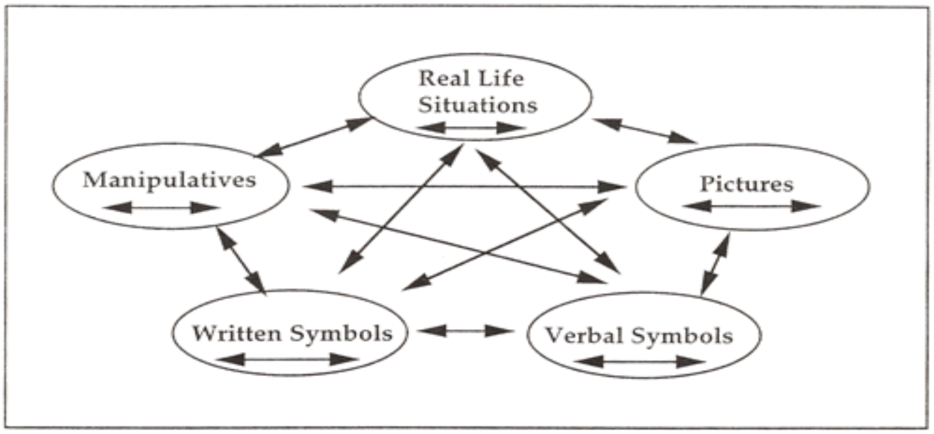


Figure 1. The Lesh Translation Model

Since this study aims at how ST Math, a manipulative in the form of technology, usage strengthens fractional conceptual understanding of college undergraduate students in a remedial mathematics course by investigating how they can transfer the fraction concepts they have gained by using the app to a verbal communication and to a post- test that contains real life situation problems, pictures and written symbols, applying this model to the study is essential.

The Natural Number Bias

The natural number bias is defined as the inappropriate application of natural number properties in rational number tasks (Van Hoof, Vandewalle, Vershaffel, & Van Dooren, 2014). As mentioned earlier, even before children actually receive a formal introduction of rational numbers, they have already constructed an intuitive idea of what a number is, based on their experiences with natural numbers (Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010). This is normal because children encounter natural numbers much more often than rational numbers in their daily life such as figure counting (Van Hoof et al., 2015). This intuitive idea of numbers as natural numbers is solidified and systematized by elementary school students' first years of mathematics learning (Greer, 2006). In the middle years of elementary education, when they are exposed to rational numbers, the rules and principals of natural numbers are no longer applied to rational numbers and many students fall into confusion, applying the natural number

reasoning to rational numbers that leads to an incorrect solution (Van Hoof et al., 2015). Once they built the intuitive idea of natural numbers, it is a major obstacle for students to overcome the belief, which is the natural number rules and principals are applicable to all numbers (Vamvakoussi & Vosniadou, 2010).

For instance, if learners are asked to choose the larger number out of 25/100 and 7/10, they would say 25/100 is greater because 25 is larger than 7 by reasoning in terms of natural numbers, which leads to the incorrect answer (Van Hoof et al., 2015). Similarly, when an individual learns addition of fractions, the individual has to be able to categorize a fraction different from a whole number. Without it, the person would likely apply incorrect whole number specific procedures to calculate fractions such as 1/5 + 3/7 = 4/12 (Byrnes, 1992).

Because of this, once students are exposed to the mathematical concept of rational numbers in the classroom, misconceptions and difficulties happen when they face situations with rational numbers where the natural number principles and rules are no longer applicable (Gelman, 2000; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2004). This causes learners to rely on the natural number knowledge for rational number tasks and this leads to incorrect answers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2004) and this natural number bias is persistent throughout the secondary level education (Van Hoof et al. 2014). Gelman and Williams (1998, p. 618) also state that, "children's knowledge of natural numbers as a core domain serves as a conceptual barrier to later learning about other numbers and their mathematical structures."

Based on the literature mentioned above, the natural number bias leads learners to misconceptions and becomes the main obstacle for them to understand the fractional concepts. If fractional instruction does not make a distinction between the conceptual differences between the

natural numbers and the rational numbers (Van Hoof et al., 2014) and focuses on procedural aspects of fractions, learners tend to apply the natural number properties and their fractional conceptual understanding remains fragile (Gelman & Williams, 1998; Van Hoof et al., 2014). There are three main aspects discovered where the properties of natural numbers are inappropriately applied to rational numbers: the dense structure, the numerical size, and the effect of operations (Moss, 2005; Ni, & Zhou, 2005; Vamvakoussi, 2015).

The Dense Structure

According to Van Hoof, Janssen, Verschaffel and Van Dooren, the dense structure concerns the structural differences between natural and rational numbers (2015). Natural numbers have the discreteness in their character and there is always a unique discrete successor number of any natural number (e.g., the successor of 2 is always 3). On the contrary, rational numbers are characterized by a dense structure. This means that there is no unique successor of a given rational numbers since there are always infinitely many numbers between two rational numbers.

Understanding this difference between natural and rational numbers is very challenging for learners including adults (Vamvakoussi, Christou, Mertens, & Van Dooren, 2011) because this concept requires understanding the various representations of rational numbers and the way they are connected to each other and the interrelations between the different subsets of the set of rational numbers (Vamvakoussi and Vosniadou, 2004). When students learn about rational numbers, they tend to use the properties of whole numbers to interpret rational numbers (Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2007). Vamvakoussi and Vosniadou discovered that students treat fractions and decimals as natural numbers, which are countable and discrete (2010). Gelman and Williams (1998) claim that that the main reasons

hindering fraction learning are, (1) children tend to assume each number has a unique successor, (2) children are exposed to sets, which can be counted by assigning numbers to objects in a 1 to 1 manner, and (3) the last number in a count can be used to represent the cardinality of the set. Therefore, they use the knowledge of whole numbers as a framework for interpreting the properties of fractions and decimals (Vamvakoussi & Vosniadou, 2012).

Quite a few studies indicate that different age groups of students state that there is a finite number of numbers between two rational numbers of points to the successor of a particular number (Giannaloulias, Souyoul, & Zachariades, 2007; Hartnett & Gleman, 1998; Hannula, Pehkonen, Maijala, & Soro, 2006; Malara, 2002; Merenluoto & Lehtinen, 2002; Tirosh, Fischbein, Graeber, & Wilson, 1999; Vamvakoussi & Vosniadou, 2004, 2007). These findings indicate that learners wrongly use the property of discreteness to rational numbers. Therefore, students' difficulties with density can be placed in the more general framework of the problems they face in the transition from natural to rational and real numbers (Vamvakousi et. al, 2011). This shift is characterized by the interference of natural number knowledge in rational number tasks by relying on natural number reasoning, which is no longer applicable and leads to systematic errors (Vamvakousi, et al., 2011).

Mathematic educators and cognitive psychologists acknowledge this phenomenon where prior knowledge holds learners back from further learning and this calls for substantial changes to students' conceptual organization of number (Desme, Gregoire, & Mussolin, 2010; Gelman, 2000; Hartnett & Gelman, 1998; Merenluoto & Lehtinen, 2002, Moss, 2005: Ni & Zhou, 2005; Smith, Solmon, & Carey, 2005; Stafylidou, & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004, 2007). To explain how prior knowledge of number holds back further learning, Vamvakoussi and Vosniadou introduced the framework theory approach to conceptual change

(2010). This theory assumes that children organize their everyday experiences in the context of the experiences in domain-specific conceptual structures named framework theories, based on the evidence from cognitive developmental research (Vamvakousi et al., 2011). Therefore, the framework theory approach to conceptual change focuses on coherence rather than fragmentation of children's thinking on their conceptual changes and was originally developed to account for the challenges students face in regard to the learning of certain concepts (Vosniadou, Vamvakoussi, & Skopeliti, 2008). The theory assumes that students have formed a coherent domain-specific, naïve theory of number, before they are exposed to rational number instruction (Vamvakousi et.al., 2011)

Vamvakoussi and Vosniadou (2004) applied the framework theory to provide an empirical evidence in conceptual change in understanding density of rational numbers. They assumed that understanding of density was a slow and thorough process and learners would acquire new information of rational numbers on their existing knowledge of natural numbers. Therefore, the researchers expected the learners to make errors in acquiring the new information and the learners to show conceptual change in rational numbers through students' efforts to understand the new information about rational numbers on their already existing natural number concepts.

The participants were 16 ninth graders in a middle class school in Athens, Greece and they were given a paper- and-pencil questionnaire regarding the natural and the rational numbers concepts during the interviews. When they were asked how many numbers existed between two successive fractions, 3/8 and 5/8, 14 of them answered as there is only one number, there are infinitely many numbers that are all equal to 4/8 or there is finite number of numbers. This result shows that students' idea of discreetness is built on their previous knowledge of natural numbers.

Although through the interview process, the students have developed some knowledge of fractions and the other rational numbers by getting exposed to the problems that did not hold the natural number rules, the conceptual change from natural numbers to rational numbers was a slow process and math educators need to keep this in mind. Similarly, the educators need to offer learning environments where students can express their thought process so that they can be more aware of their misconceptions.

Vamvakoussi & Vosniadou (2010) also conducted another study to investigate the secondary school students' understanding of density in an interval with different endpoints such as natural numbers, decimals, or fractions, based on the framework theory. To investigate that, the researchers administered a multiple choice questionnaires such as: When the interval is 1/8 – 1/7,

a) There is no other number
b) There is a finite number of decimals
c) There are infinitely many decimals
d) There is a finite number of fractions
e) There are infinitely many fractions
f) There are infinitely many numbers and they can take different forms, such as decimals, fractions, and non-terminations decimals.
g) I do not agree with any of above.

The participants were 181 seventh graders, 166 ninth graders, and 202 eleventh graders. They were selected from middle class Greek schools. The authors hypothesized that they would answer there is a finite number of numbers between the fractions and their answer choice would be affected by the symbolic representation of the intervals.

The results indicated that the idea of discreteness in fractions was strong in all grades and was a major obstacle for students to understand the concept of discreetness and they tended to believe that the intermediate number need to be the same kind as the interval endpoints. Within the framework theory of conceptual change, this result shows that learners will build the new

concept of rational numbers on their prior knowledge of natural numbers to make sense of rational numbers by getting exposed to the tasks during the interviews. The conceptual change will happen slowly, fragmentary and eventually become more coherent. Younger students were more controlled by the view of discreetness and answered that there were finite numbers between any type of intervals, natural, decimals and fractions. However, this idea became fragmented and some students answered finitely many numbers and others responded infinitely many numbers.

Then finally, it became more coherent but still fragmented and more students responded infinitely many numbers. Therefore, the process of conceptual change is a slow process. Ninth graders and eleventh graders performed significantly better than the seventh graders in all items. Since this study was conducted for a very short period, the result cannot provide insight about whether the understanding will be more sophisticated or not in the long run. The study examined only the ordering aspect of rational numbers. Thus a wider range of tasks is necessary to investigate to provide an overall picture of learners' development of the number concept.

Similarly, this study did not have students' own explanation through interviews and this certainly would have provided a better insight of their conceptual change. From the results of this study, the authors suggest that instruction needs to give opportunities for students to make sense of rational numbers on their own by making the differences between natural and rational numbers explicit.

Vamvakoussi and her colleagues (2011) replicated their earlier study of Vamvakoussi and Vosniadou (2010) by aiming at a different population so that the researchers could draw a general conclusion about the natural number bias throughout the different students' population. The participants were 9th graders, 84 Greek students and 128 Flemish students and the same questionnaire (Vamvakoussi & Vosniadou, 2010) was administered. Although the Flemish

students outscored the Greek students, the different type of endpoints affected both groups of the students. This result implies that there are conceptual difficulties involved in the shift from natural to rational numbers and this is possibly attributed to the mathematics instructions in both countries. It appeared that the 9th graders of both countries were not really aware of the differences between natural and rational numbers. Only 28.3% of the Greek and Flemish 9th graders answered correctly to the question, which is whether there are infinitely many numbers between two numbers and whether these numbers can have both a fractions and a decimal form. This study did not have a qualitative evidence because it did not conduct an interview with the students, although the qualitative aspect is necessary to provide more accurate information regarding students' reasoning behind each task.

These results aligned with the studies conducted prior to the studies mentioned above showed that the idea of discreteness is a major obstacle on students' understandings of density (Ginnakoulias, Souyoul, & Zachariades, 2007; Hartnett & Gelman, 1998; Malara, 2001; Merenluoto & Lehtinen, 2002; Neumann, 2001; Pehkonen et al., 2006; Tirosh et al., 1999; Vamvakoussi & Vosniadou, 2004, 2007). They argue that the idea of discreteness is one of the factors, which needs to be taken into account to explain students' difficulties with density of rational numbers since the density property of rational numbers is challenging for students to understand (Ginnakoulias, Souyoul, Zachariades, 2007; Tirosh et al., 1999; Neumann, 2001).

These results also imply that the type of the interval endpoints such as natural numbers, decimals and fractions, have a huge effect on student's judgment of the number as well as the type of intermediate numbers. Students have the tendency to believe that the intermediate numbers have to be the same type as the interval endpoints and they did not treat natural numbers, decimals and fractions in the same way, regarding the number of intermediate

numbers. This indicates that students have difficulty of grasping the nature of the relation between decimals and fractions (Mitchell, 2005; Neumann, 2001; O'Connor, 2001). This agrees with the statement that it is difficult for students to see the rational numbers set as a unified system of numbers (Kilpatrick, Swafford, & Findell, 2001). Wynn (1995) claimed "There are also limits to the kinds of numerical entities the accumulator mechanism represents. It does not represent numbers other than positive numbers...For example, children have great difficulty learning to think of fractions as numerical entities" (p.176).

The Numerical Size

The next aspect of fractional misunderstanding is related to the numerical size of rational numbers. Students' wrong reasoning of the size of rational numbers usually originates from the wrong idea that if the natural numbers in the symbolic representation are larger, the magnitude of the rational number is similarly larger (Gabriel et al., 2013). This is why they think that 4/7 is larger than 3/5. They tend to think of each component of a fraction as different natural numbers instead of thinking of a fraction as a number and this misconception leads them to assume that the fraction value increases as the value of the numerator or denominator gets larger (Clarke & Roche, 2009). Similarly, one common misconception happens in comparing decimal numbers, where learners have the wrong assumption which originates from the natural number rule that the longer the number is, the larger the number is and the shorter the number is, the smaller the number is.

Learners tend to apply the rule to decimals, generalizing the longer the decimal is, the lager the decimal is and the shorter the decimal is, the smaller the decimal is (Resnick et al., 1989). The study conducted by Smith, Solomon and Carey (2005) of 50 upper elementary school students indicates that common mistakes shown were that they incorrectly assumed that 0.65 was

larger than 0.8 since 65 is larger than 8 and similarly, they thought that 2.09 was larger than 2.9 since 209 is greater than 29.

De Wolf and Vosniadou (2011) examined how the initial concepts constructed in childhood have lasting effects into adulthood in the area of fraction magnitude. Twenty-eight undergraduate students from Carnegie Mellon University who enrolled in introductory psychology courses took part in this study and two conditions were used; the inconsistent condition and the consistent condition. In the consistent condition, the fraction magnitude followed the whole number reasoning, which is the larger magnitude fractions have larger whole number parts. For instance, 5/7 is larger than 2/5. In the inconsistent condition, the reasoning was not applicable. The larger fractions had smaller whole number parts. For instance, 2/3 is larger than 3/7. This study measured accuracy and reaction time for the fraction comparison in the two conditions.

The result showed that the undergraduate students were more accurate and faster for the consistent condition than the inconsistent condition and it could be said that even undergraduate students can have the whole number ordering reasoning and it could cause them to delay their response or have erroneous responses. Therefore, the natural number bias is persistent even in college students who are supposed to have more sophisticated understanding of fractions. This result assures that the natural number bias can be reduced with age but it may not completely disappear.

Van Hoof, Janssen, Verschaffel and Van Dooren (2015) aimed at characterizing the development of the natural number bias in the three aspects of natural number bias: density, size and operations between 4th and 12th graders. The authors expanded the participants of the study up to secondary school to address the issue that there have not been many studies regarding the

bias conducted for secondary education so far. They created their own comprehensive paper-pencil test for this study to investigate these ideas: the overall chances of occurring the natural number bias, how strong the bias is in the decimal versus fraction format, how strong the natural number bias is across the density, size and operation and how the bias develops with age in the three aspects. There were 1343 participants: 213 fourth graders, 230 sixth graders, 293 eighth graders, 302 tenth graders, and 305 twelfth graders from 21 schools, 9 primary schools and 12 secondary schools, in Flanders, Belgium.

The result of the study agreed with their previous research conducted (Vamvakoussi & Vosniadou, 2012), which is the bias was equally strong in tasks with decimal numbers and with fractions. Similarly, students tend to judge the size of fractions based on the numerical value of each component of the fractions. For instance, the value of fractions increases as its denominator, numerator or both increases (Mamede, Nunes, & Bryant, 2005). According to the result, an overall decrease of the strength of the natural number bias was discovered with grade. While the natural number bias was strong in 4th and 6th grades, it was significantly decreased in 8th grade and got lower in 10th grade. There was no significant natural number bias observed between 10th and 12th grades. However, according to the authors, this does not imply that the bias disappeared, since clear remarks of the bias were still found in 12th graders in density and operation items.

Among these three aspects, density, size and operation, the students had the strongest bias in the fraction density, then in the size and they had the weakest bias in the operations. This result aligned with the previous research (Vamvakoussi & Vosniadou, 2012). Lastly, the result of this study showed that the strength of natural number bias decreased gradually throughout the three aspects of a rational number and in the size aspect, the bias almost disappeared by grade

eight. This study was a cross-sectional design and longitudinal design would be interesting to analyze how students' rational number understanding will change over the long haul. As the study results show the natural number bias is persistent even through secondary school. The authors suggested that rational number instruction needs to focus on the conceptual differences between natural numbers and rational number, instead of focusing on their similarities.

Stafylidou & Vosniadou (2004) aimed at development of students' understanding of the fraction magnitude from the conceptual change theoretical framework that explains how learners develop science and mathematics (Vosniadou, 1994, 2001, 2003). The authors conducted this study with 40 fifth graders, 40 sixth graders, 40 seventh graders, 40 eighth graders and 40 secondary students from different middle class public schools. A questionnaire was given to the students to examine the development of their ideas of the numerical value of a fraction and there were two sets. In the first set, they were asked to write fractions from the smallest to the largest and explain their answers. In the second set, they ordered and compared different types of fractions. The result was aligned with the research hypothesis, which presuppositions of natural number would hinder the acquisition of fractions and lead them to misconceptions. The participants tended to treat a fraction as consisting of two independent numbers and they concluded the magnitude of a fraction based on the idea. They also had a very strong belief that a fraction is a part of a unit and this strong belief caused them to have the idea that a fraction cannot be greater than the unit. Therefore, they came to the misconception that a fraction always represents a quantity smaller than the unit.

Other studies coincide with the above studies results in fraction comparison tasks, since learners mistakenly assume that the magnitude of a fraction increases as its denominator and numerator or both increases (Mamede, Nunes, & Bryant, 2005; Meert, Gregoire, & Noel, 2010).

The study of Clarke and Roche shows that 77% of the 6th graders interviewed on fraction comparison tasks of choosing the larger fractions with the same denominators answered correctly but only 37% of them were able to answer correctly when the fractions had different denominators (2009). The literature above indicate that the natural number bias influences the concepts of fraction magnitudes heavily and although this bias subsides with age, it may not completely disappear. Even college undergraduate students who could have much more sophisticated thought process are susceptible to the bias.

The Effect of Operations

This section summarizes the effect of arithmetic operations influenced by the natural number bias. In operations of rational numbers, certain characteristics, which are applicable to natural numbers, are not applicable any more (Van Hoof et al., 2015). For instance, in natural number operation, addition and multiplication always result in larger values, while subtraction and division always result in smaller values. On the contrary to these, in the operations of rational numbers, these rules do not stand, but learners tend to wrongly assume that these are true (Hasemann, 1981; Vamvakoussi & Vosniadou, 2012). In the study of Van Hoof et al. (2014), the authors asked 8^{th}, 10^{th} and 12^{th} graders to judge the right algebraic expressions to address the effect of operations. One of the items was asking $x/4 < x$, when x is a natural number and/or a rational number. Since this statement is true for both the x as a natural number and as a positive rational number, even those students who had the natural number bias were able to answer correctly. However, another item asked that $3 < 3/x$ is true or not. To examine this statement, the students had to have understanding of natural numbers and rational numbers. Those who had the natural number bias and thought that a division always gives a smaller value were not able to answer this question. In their study, overall natural number bias was exhibited clearly.

In the case of fraction addition and subtraction, students tend to treat fractions numerators and denominators as separate natural numbers (Fazio & Seigler, 2011). This is attributed to not understanding the concept of fraction magnitude as mentioned above. Hence students often add and subtract the numerators and the denominators without having a common denominator and these students fail to treat fractions as one established number (Clarke & Roche, 2009). Another reason for this is that in elementary education, students specifically do arithmetic with natural numbers alone and the natural number rules are typically not stated explicitly during the instruction (Van Hoof et al., 2014).

Van Hoof et al (2014) examined the natural number bias in arithmetic operation. The first study investigated to what extent 8^{th} graders who just learned expressions involving literal symbols struggled with the natural number bias. In the second study, the researchers aimed at whether the natural number bias found in study 1 decreased and ultimately disappeared towards the end of the secondary education, by replicating study 1. In study 1, the researchers collected two kinds of data. A paper and pencil test was taken by a large group (N= 291) 8^{th} graders from six different secondary schools in Flanders, Belgium and afterward, the data of the interviews with a small group of students (N=10) who solved the similar tasks individually, were collected.

For selecting the participants, they made sure to choose students of the second year of general secondary education who were 8^{th} graders in Flanders and were more capable of interpreting algebraic expressions since the study conducted by Gabriel et al (2013) collected their data from 7^{th} graders in Flanders who were exposed to algebraic expression the first time. The result of the study 1 showed that the students performed with higher accuracy on congruent items which the natural number rules can be used to obtain answers such as $x < x+2$ and $x/4 < x$ than on incongruent items which the natural number rules cannot be applicable such as $3 < 3/x$.

This was the clear evidence of existence of natural number bias.

In study 2, the participants were 10th and 12th graders. Three- hundred and one tenth graders were chosen from eight different schools and three- hundred and five twelfth graders were chosen from ten different schools in Flanders, Belgium. The result showed that the bias clearly remained for the multiplication and division items through the upper secondary grades as well. From the results, the authors suggest that since the natural number bias is persistent even throughout the secondary education, secondary teachers need to be aware of the bias as well and continuously assist leaners to understand the differences of the two types of numbers by discerning rational numbers from natural numbers. Since the interview was conducted with the small sample size, the finding needs to be generalized with a larger sample size.

Similarly, Christou (2015) investigated that students have a tendency to anticipate answers of specific arithmetic operations based on the natural number bias. The participants were 189 fifth and sixth graders from public schools in Greece and 104 were boys and 85 were girls. A paper-pencil test was administered and the fraction problems on the test were such as 10 _ ½ < 10 or 10 _ ¾ > 10, for students to fill in multiplication or division. The other type of questions asked students to put fractions in order from smallest to largest. The result showed that when students selected the operations, they tended to apply the natural number concept, which was the result of multiplication always gets larger than the numbers involved and the result of division always gets smaller. There is a natural number bias and their ways of rationale of the arithmetic operations is related to their understanding of the number concept and this result is aligned with the study of Van Hoof et al. (2014).

The empirical study of Prediger (2008) investigated the persistent conception "multiplication makes bigger" by dealing with students' competencies, content knowledge and

fraction concepts and fraction operations under the framework theory of conceptual change (Vamvakoussi & Vosniadou, 2010) with the idea of Grundvorstellungen (GV) (Kleine, Jordan & Harvey, 2005). The GV is considered very critical for acquiring mathematical concepts and it has three aspects in the acquisition: (1) Construction of meaning of mathematical concepts based on familiar experiences and contexts, (2) Generalization of mental representations of concept that facilitates operative thinking and (3) Ability to apply a concept to reality by recognition of the respective structure in real life contexts or modeling a real life context with the help of mathematical structure (vom Hofe, Klein, Blum, & Pekrun, 2005). The researcher was specifically interested in students' individual models for fraction multiplication and their implications for students' intuitive knowledge (multiplication makes bigger) and its connections with the algorithm.

The participants were seventh graders with 44 boys and 37 girls who were from Germany. The students took a paper and pencil test and afterward interviews were conducted. The test questions aimed at examining their fraction multiplication procedure, and its concept such as "Which statement is true, what happens when you multiply two fractions? The answer choices will be 1. Always larger the two fractions, 2. Always smaller than two fractions and 3. The answer is sometimes bigger, sometimes smaller than the two fractions." 84% of the students showed their procedural proficiency in the calculation tasks. However, most students agreed with the statement of "always larger than two fractions" in the conceptual tasks. The question that aimed at an individual model asked them to come up with a situation implying $3/4 * 1/3 = 1/4$. Only 5 students were able to come up with an adequate model. This is because they had the intuitive knowledge of multiplication (multiplication makes bigger) and this implies that the natural number bias is the major obstacle to overcome. The result of the interviews also

confirmed that the idea of “multiplication makes bigger” was connected to an inappropriate model of multiplication and their algorithm. Many students answered that multiplication makes it bigger and multiplying two numbers mean repeating addition, for instance 3*4 meaning adding 3 four times, and they applied this algorithm to the fraction multiplication. Therefore, the majority of the students needed to construct new models in the transition from natural numbers to fractions. This misconception in multiplication is the major obstacle for learners to overcome.

One of the interesting findings is that learners’ fraction competency did not reflect on their conceptual understanding. From the view of conceptual change, treating the misconception is not enough in instruction, but it is critical to aim at learners’ conceptual change. Since this study had done for the short period and with a relatively small sample size, a future study needs to adjust the limitations.

As the results of these studies show, the natural number bias affects learners’ fraction operation and this bias is persistent throughout secondary education as well. One of the main reasons for this is the learners do not fully understand the concept of fraction magnitude and the conceptual change from natural numbers to fractions does not happen speedily.

Key Elements for Effective Pedagogies for Conceptual Understanding of Fractions

As mentioned above, all three fractional domains, density, size and operations, influenced by the natural number bias, are attributed to fragile conceptual understanding of fractions because the topic of fractions has been taught procedurally. Therefore, the heartbeat of the effective pedagogies need to be shifted away from the procedural way of teaching and need to aim at distinguishing the rational numbers from the natural numbers and deepening students’ fractional concepts. Although, reducing the misconceptions originated from the natural number bias could be a long process for learners (Vamvakousi & Vsoniadou, 2007), this can be attained

by differentiating instructional methods.

One of the most important elements of teaching for understanding is providing students opportunities to make sense of natural and rational numbers on their own (Steffe, 2003; Tzur, 2004), instead of teachers intending to transmit their knowledge of these two types of numbers to their students (Moss & Case, 1999). Steffe and Tzur claimed that although teachers can be a part of children's mathematics learning, they cannot simply make the teachers' knowledge their own. They need to interiorize the knowledge through their activities (1994). It is critical to create learning environments, which allow students to express and develop their opinions so that they can be sure of their beliefs (Vamvakousi & Vosniadou, 2004). "It is important to encourage students to represent their ideas in ways that make sense to them, even if their first representations are not conventional ones" (NCTM 2000, p.67). The following provides key elements that could make fractional instruction more effective, which can give opportunities for students to make sense of fractional concepts on their own and strengthen learners' fractional conceptual understanding so that their number sense will not be susceptible to the misconceptions that originate from the natural number bias.

Building on Informal Knowledge

Informal knowledge of fractions is knowledge students build on their natural number concepts throughout their life up to the point of a formal fraction instruction (Mack, 1993). If fraction knowledge is constructed on learners' informal understanding of fractions, the fraction content can be taught in the elementary grades (Powell & Hunting, 2003). Educators can begin to teach elementary students by making students represent fractions by language alone, starting with having students sharing items instead of using fraction symbols (Powell & Hunting, 2003). According to a few studies, to develop students' understanding of the true fractional meaning, it

should be presented verbally without using fraction symbols first (Mack, 1993, 1995; Saenz-Ludlow, 1994). Since it takes time to make the transition from the natural number meanings to fraction meanings, Mack, in her study, started teaching fractions by introducing real life problems first and she introduced the fraction symbols later on (1993).

According to Hiebert and Carpenter (1992), students tend to construct meaning for symbolic representations, which are foreign to them, by drawing on previous knowledge of other mathematical symbols and as a result they could overgeneralize and construct inappropriate meanings of the new representations. Hence, Mack noticed that it was critical to visit problems that were represented symbolically and in the context of real life situations back and forth, as it is stated that understanding new things will happen when we see connections with other things we know (Hiebert et al., 1997), and it was also important to eliminate symbolic representations when her students were not ready to work with them.

In the study of Mack (1990), 8 six graders were placed in the individualized instruction for fraction addition and subtraction in a one to one setting for 6 weeks and the instruction was aimed at building on the learners' prior fractional knowledge. All the participants in this study had fractional knowledge based on the natural number knowledge and fractional knowledge as a part of a whole in a real world situation. The aim of the study was to investigate the development of students' understanding about fractions during the instruction from two angles: (1) The way students built new fraction knowledge on their prior fraction knowledge and (2) Investigating how the prior knowledge of procedures influences students' ability to relate new fraction knowledge to the prior fraction knowledge. Her study found that her students were able to connect their informal fraction knowledge to the symbolic representation of fractions if the symbolic representations were very clear. Although students were able to relate the prior

knowledge to the symbolic representation of fractions and procedures, there was a danger of the prior knowledge of procedures that was characterized by misconceptions interfering. Hence teaching fractional concepts before fractional procedures is necessary so that learners can build new fraction concept on prior informal fraction knowledge meaningfully.

Mack (2001) also investigated the possibility for students to use their informal knowledge of partitioning that involves a part-whole perspective focusing on fractional quantities as counting units, as the foundation to understand fraction multiplication. This study was conducted based on the idea and hypothesis, which in mathematics and science, understanding of complex content domain could be developed by the aid of students' initial understanding (di Sessa, 1993; Hatano, 1996) and Steffe's (1988) reconceptualization hypothesis. 6 fifth graders received one to one instruction over a 3-month period. All the students had fundamental knowledge of partitioning, which is the smaller the parts become, the more parts a whole is divided into, but they had never received fraction multiplication instruction before this study. Each student received one to one instruction with the researcher. During the instruction, these students were encouraged to build on their informal knowledge of partitioning while they were solving problems. For instance, when a student solved a problem involving finding one fourth of four fifths of a cookie, the researcher asked him/her to solve a problem asking to find three fourths of two thirds.

The result shows that they were able to build on their informal knowledge of partitioning to re-conceptualize and partition units in the meaningful ways. For instance, initially all six students thought that a unit could not be repartitioned to make a composite unit. They viewed composite unit consists of two independent pieces rather than composed of fractional portions of a referent whole. However, as the instruction went on, all students started building on their

informal partitioning knowledge by seeing units to be partitioned and the results of the partitioning as fractional amounts regarding to a referent whole. They started conceptualizing the results of partitioning as unit fractions and also thinking about meaning of partitioning a unit into a fractional amount. This results show that the informal knowledge of partitioning could be a foundation to develop the concept of fraction multiplication. However, this process was not simple and happened gradually. Students' informal knowledge is a foundation to connect more complex ideas while they develop understanding. Therefore, building on previous fractional knowledge is an effective way to deepen complex fractional concepts.

Empson (1999), conducted two studies to investigate how first graders developed fractional conceptual understanding by using informal knowledge of equal- sharing and partitioning by using the framework of the reconceptualization theory and sociocognitive perspective because the author thinks that social interaction is an analytic medium to explain children's cognitive change. This was a 5-week long case study and the researcher observed fraction lessons that were planned by her, the teacher and her colleagues, and she also conducted the interviews with children before and after the instruction. The lessons were built based on the students' understanding of equal-sharing and partitioning and focused on problem solving and classroom discussions.

The author particularly focused on Ms. Kolan's class that had 19 children and her class focused on sharing, comparing and justifying their thinking regarding the problems and their solutions. Before the instruction, during the interviews, 14 children solved equal sharing problems, which included halving or repeated halving, but only 4 were able to so solve equal sharing problems besides repeated halving. After they went through the 15 days of instruction, 16 students were able to solve the two most challenging equal sharing problems that were not

able to be solved by repeated halving, using an appropriate partitioning strategy. An example of challenging equal sharing problem was, "6 children want to share 14 pancakes so that each child gets the same amount. How much can each child have?" They reasoned that each child got two whole pancakes and then they either portioned the remaining 2 pancakes into thirds or partitioned each pancake into sixths. This shows that equal-sharing tasks help children to use informal knowledge of partitioning to think about fractions mathematically. Similarly, interactions through discussion enhanced re-contextualizing children's fractional concepts. The result showed that building on their informal knowledge of equal sharing developed children's overall fractional thinking. A future study needs to investigate functional classroom activities that could lead to children's cognitive changes in fraction learning and how these activities could be implemented to assist teachers to build on learners' mathematical thinking.

Biddlecomb (2002) conducted a case study of 3rd grader, Jerry, to examine how initial numerical sequence knowledge influenced the learner's fractional knowledge in the specific domain of part-whole relationships. The aim of this study was to provide validity of Steffe's reorganization hypothesis (Steffe, 2001). The hypothesis claims that when a new scheme is constructed by using an old scheme in a different way, the new scheme could be emerged as a form of reorganized prior scheme and the new scheme supersedes the old scheme. The superseding scheme assists learners to solve the problems better than the old scheme did and also assists to solve new problems the old scheme did not.

This study was a 3-year teaching experiment and the computer program, microworlds was used in this study. During the first year, teaching focused on Jerry's numerical knowledge and the last 2 years, it focused on his fractional knowledge. The interviews with Jerry showed that his numerical concepts not only assisted him to construct but also constrained him to

construct fractional concepts. For instance, he was able to see that 1/N implies one of N equally divided parts and iterating the piece N times create the whole. The result shows that the Steffe's reorganization hypothesis was intact in construction of fractional knowledge.

According to the results of these studies, students possess fragile informal knowledge of fraction such as equal sharing and partitioning, and a teacher needs to know that they bring the informal knowledge to fraction instructions (Steffe & Olive, 1990) and provide a chance for learners to use the informal knowledge to build a new fraction concept during the instruction so that they can construct deeper fractional concepts meaningfully.

Manipulative and Visual Models

According to Fazio and Siegler (2011), there is a positive correlation between students' fractional conceptual understanding and their success in using procedures to solve problems. For instance, children who understand the reason to have a common denominator are more likely to retain the correct procedures than those who do not have the understanding. Because of this, teachers need to aim at developing conceptual understanding aligned with procedural fluency. One of the effective ways to improve students' fractional understanding is the usage of manipulatives and visual models. O'Shea (1993) states that manipulative usage during the class can assist students of all grades to understand mathematical processes, express their mathematical thinking and enhance their mathematical ideas to higher cognitive levels.

In the interviews of Burns (2004), 5th grade students put fraction kits together and played different games, using the strips and the fraction kit aids students to understand the relationship between the whole and fractions such as three 1/3 strips are equal to 1. The study by Harrison, Brindley and Bye (1989) investigated two different instructional methods toward 435 students who were age 12. Half of them used concrete materials during the instruction and the other half

were in the traditional way of instruction in learning ratio, fractions and its operations. Before the study, participants took a pre-test regarding ratio, proportion, and fractional number thinking and at the end of treatment instructions, they took the attitude test toward ratio and fractions, in addition to the pre-test. The results indicated that those who used the concrete materials during the instruction did much better not only in understanding fraction and the concepts of ratio but also gained positive attitude toward the subject. From the results, it can be said that using concrete materials assists students to organize the fraction and ratio concepts.

Similarly, in a study conducted by Butler, Miller, Crehan, Babitt and Pierce (2003), 50 students participated who were 6^{th}, 7^{th}, 8^{th} graders with mild to moderate mathematical disabilities and were placed in one of the instructional groups: Concrete Representational Abstract (CRA) instruction or Representational Abstract (RA) instruction. Those who were placed in the CRA instruction group used concrete manipulative devices while RA instructional group used representational drawings. The study conducted 10 lessons and the participants took the pre and post tests to measure their increment of fractional conceptual understanding. These tests measured the participants' knowledge and conceptual understanding of fractions such as ratio, proportion, equivalent fractions, and improper fractions.

The students who were in both instruction groups improved the score at the post-test significantly. At the pre-test, they had very vague understanding of equivalencies and many of them did not even attempt the regarding problems. However at post-test, they were able to choose appropriate methods to solve problems by drawing graphic representations. Although the result was not statistically significant, CRA group scored higher than the other group. Because of the time constraint, the study was limited to 10 lessons and the students who were in the treatment group had some learning disabilities such as attention disorder and emotional disorder.

Hence, this could affect the generalization of the result. It is valuable to conduct a future study that investigates how students can come up with their own solution by drawing fraction representations when they use this intervention.

In fraction addition and subtraction, visual representations can be utilized to illustrate common denominators for fractional additions and subtractions (Fazio & Siegler, 2011). Concrete models are essential forms of representations and are necessary to support students' understanding of and operations with fractions. Other critical representations include pictures, contexts, their language and symbols (Cramer, Wyberg, & Leavitt, 2008).

In a study conducted by Cramer and Henry (2002), the researchers compared two curriculums, commercial curricula (CC) and Rational Number Project (RNP), in the development of students' fractional understanding. CC focused on more computational symbolic skills and did not use models and manipulatives and RNP aimed at developing conceptual understanding of fractions by using variety of teaching methods using different manipulatives and models such as fraction circles, chips, paper folding, Cuisenaire rods and real world contexts, to build fractional meaning to examine. The study integrated quantitative analysis of the pre and retention test and qualitative interviews with students and teachers. About 2700 fourth and fifth graders and 200 teachers in the same school district participated in the study. The researchers found that the fraction circle model is the most effective manipulative to build mental images of fractions. The mental images of fractions enhance the ability to judge magnitude of fractions, which is necessary for fractional understanding. On the contrary, the CC students relied on their procedures and did not develop fractional concepts. Since those who were in RNP were able to develop the understanding of fraction symbols, by wrestling with fraction ideas using variety of models and manipulatives.

In another study (Cramer, Wyberg, & Leavitt 2008), continued using the circle model based on the previous study's result to investigate how learners could deepen conceptual understanding of fraction addition and subtraction. They chose 6^{th} graders from a large urban school district and they used the circle models instructions to develop meaning of common denominators for fractional additions and subtractions symbolically. Fraction circles models help learners to construct the inverse relationship between the magnitude of the denominator and the magnitude of entire fraction. They were able to observe that the more a circle was divided into equal size parts, the smaller each portion gets both fractions are one piece away from the whole circle, the fifths are larger pieces than the sixths and the fifths are missing the larger piece. This ability to compare fractions is the foundation for estimations of fractional additions and subtractions and by using the circle model, students gained clear mental image of size of fractions.

The researchers concluded that the 6^{th} graders were able to construct strong mental images of sizes of fractions and this ability is critical because the ability to estimate sum of different fractions will help them to detect their work when finding exact fractional addition and subtraction. They also believe that the understanding of fractional size students gained is a strong foundation to build conceptual understanding of a common denominator for addition and subtraction. Being able to finding a common denominator is the key to understand how to add and subtract with symbols and they were provided extended periods of time to explore connections between exchanging circle pieces and find common denominators symbolically on their own with the circle model. They suggested that it is necessary for learners to experience addition and subtraction concretely with appropriate manipulatives and models before operating symbols.

These findings of the studies aligned with Bray and Abreu-Sanchez also (2010) indicating that the circle model builds a helpful foundation to develop mental images for comparing fractions. The researchers suggested that math teachers are encouraged to use the circle model more to assist students to compare fractions, because they can see the relationship between a fractional part and its representation, unlike other manipulatives.

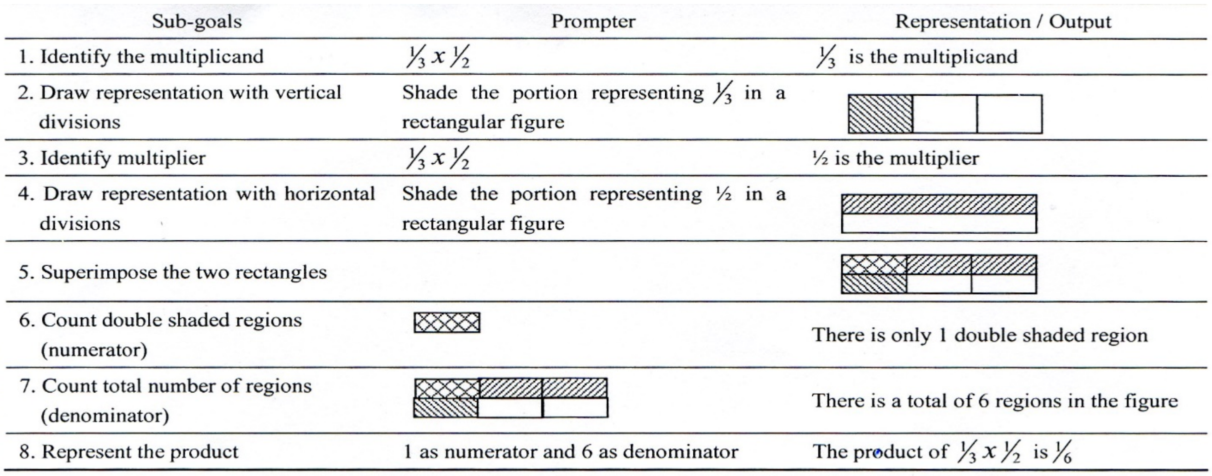
To compute fraction multiplication, students traditionally learn the cancellation algorithm (cancel-and- multiply) and in fraction division, they are taught to follow the division computational procedure by inverting the divisor and changing the division to multiplication, invert-multiply algorithm (de Castro, 2008). However, students are incapable of finding their errors and clear up the errors during the procedures because they lack understanding of the concepts underlying fractions and division (NCTM, 2000; de Castro, 2004; Tirosh, 2000).

A few studies show major problems with ongoing teaching methods in the area of fraction multiplication and division. The main problem is focusing on rules rather than meaning in teaching rational numbers. Teachers often emphasize algorithms in fraction operations at the cost of developing children's conceptual understanding of rational numbers (Moss & Case, 1999; Lubinski & Fox, 1998). This way of learning leads learners to memorize meaningless series of steps and often they forget some of the steps or change the steps during operations, which leads to errors (Tirosh, 2000; Freiman & Volkov, 2004). Secondly, teachers tend to assume that younger learners already have formed a solid conception of rational numbers, by not considering their informal knowledge of fractions (Moss & Case, 1999). This is why children misinterpret the mathematical concept of fractions and tend to apply the natural number principles to fractions (Stafylidou & Vosniadou, 2004). Tirosh (2000) indicates that this misconception plants the idea that it is impossible to solve division expression with a dividend smaller than the divisor.

The study conducted by de Castro (2008) utilized a rectangular visual model so that students can gain understanding of fractional divisions and multiplications with an explanation of why the product of fraction multiplications can be smaller than the factors and why the quotient of fraction division can be greater than the divisor or the dividend so that they can replace the misconception, which is multiplication makes larger and a division makes smaller, originated from the natural number thinking. Two sections in a public school were selected as control and experimental group for this study. Those who were in the control group received the traditional procedural way of learning multiplication and division of fractions. On the other hand, the experimental group utilized a rectangular model to understand the concept of fractional multiplication and division. Pre and post- tests were administered and results showed that the experimental group that used the rectangle model had a significant impact statistically on their conceptual understanding on fractional multiplication and division as well as discarded misconceptions they had in the operations. The figures below are the models used in this study.

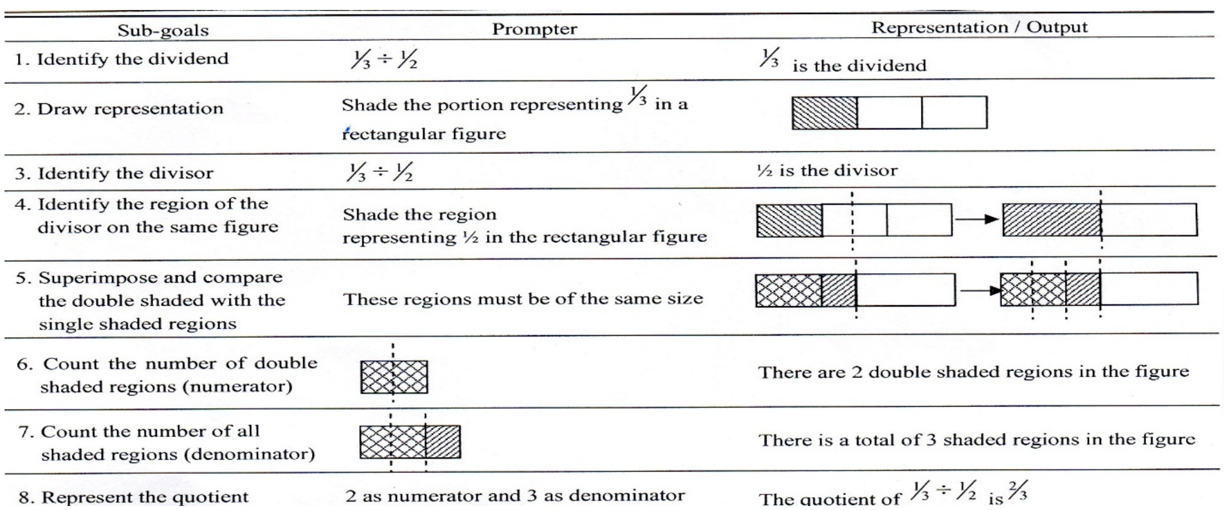
In figure 2, the learners are able to understand what the multiplicand and the multiplier means in the fraction multiplication and it is related to the area divided by the fractions. By using the visual model during the instructions, the learners were able to reduce the natural number misconception and understand the concept behind the fraction operation procedures.

In figure 3, the model helps them to understand what it means for fraction division visually and why the answer of the fraction division could be greater than its dividend and divisor. Using this model for understanding the concept of fraction multiplication and division can be very useful.



Sub-goals	Prompter	Representation / Output
1. Identify the multiplicand	$\frac{1}{3} x \frac{1}{2}$	$\frac{1}{3}$ is the multiplicand
2. Draw representation with vertical divisions	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify multiplier	$\frac{1}{3} x \frac{1}{2}$	½ is the multiplier
4. Draw representation with horizontal divisions	Shade the portion representing ½ in a rectangular figure	
5. Superimpose the two rectangles		
6. Count double shaded regions (numerator)		There is only 1 double shaded region
7. Count total number of regions (denominator)		There is a total of 6 regions in the figure
8. Represent the product	1 as numerator and 6 as denominator	The product of $\frac{1}{3} x \frac{1}{2}$ is $\frac{1}{6}$

Figure 2. Multiplication Model



Sub-goals	Prompter	Representation / Output
1. Identify the dividend	$\frac{1}{3} \div \frac{1}{2}$	$\frac{1}{3}$ is the dividend
2. Draw representation	Shade the portion representing $\frac{1}{3}$ in a rectangular figure	
3. Identify the divisor	$\frac{1}{3} \div \frac{1}{2}$	½ is the divisor
4. Identify the region of the divisor on the same figure	Shade the region representing ½ in the rectangular figure	
5. Superimpose and compare the double shaded with the single shaded regions	These regions must be of the same size	
6. Count the number of double shaded regions (numerator)		There are 2 double shaded regions in the figure
7. Count the number of all shaded regions (denominator)		There is a total of 3 shaded regions in the figure
8. Represent the quotient	2 as numerator and 3 as denominator	The quotient of $\frac{1}{3} \div \frac{1}{2}$ is $\frac{2}{3}$

Figure 3. Division Model

From these studies, the usage of manipulatives and visual models assists learners to connect visual representation to its fraction symbol and also can reduce the natural number bias, especially the concept of fraction size. This understanding is indispensable to tie fraction operation procedures with their concepts.

Intervention with Multiple Embodiments

As described above, usage of the circle (area) model is a quite effective way to understand the size of fractions and gain conceptual understanding in fractional addition and subtraction. However, there is a drawback to a heavy emphasis on the area model. According to

the study conducted by Samsiah (2002), the emphasis on area model approaches did not appear to be impressive in helping elementary school learners to build strong and flexible conceptual understanding of fractions. The interviews in the study showed that the 6th graders in her study were able to answer questions regarding the shading of areas, but they were unable to transfer the knowledge from the area model to the real-life situations that did not involve areas.

Similarly, Moss and Case (1999) claimed that overemphasizing area models inadvertently reinforced the whole number thinking such as discreetness, learners getting confused with rational and whole numbers, and increased the difficulties in learning concepts of fraction. It is difficult for learners to reconcile the natural number bias with the area model, which causes learners to think fractions are discrete and are not infinitely divisible as the natural numbers are (Behr, Lesh, Post & Silver, 1983; Hibert & Tonnessen, 1978). Other studies also express the concern that is fractions are almost always associated with area-related contexts and children are less likely to connect fraction concepts with non-area situations (Clements & Lean, 1988; 1994; Gould, 2008).

Researchers suggest the importance of introducing fraction concepts through a variety of embodiment approaches. Dienes claimed that the key quality for conceptual learning is to be maximized when learners are exposed to a concept by a multiple of different situations through different embodiments (1960). Similarly, in the study of Lesh, Post and Behr, they provided a summary of five distinct types of mathematical representation systems including manipulative and pictures and they emphasized that it is important for learners to construct mathematical concepts through these distinct types of representation systems, but for them to understand how the concepts are transformed within the systems is also important (1987). This implies that if students are to construct a deeper conceptual understanding of fractions, it is necessary to

provide them opportunities to become acquainted with a variety of models (Clarke, Roche, & MItchell, 2008) and to be able to have access to “some alternative form of representation” of fractions besides the area models (Moss & Case, 1999, p.144).

Zhang, Clements & Ellerton (2015) conducted a study to investigate the effect of the teaching intervention that incorporates multiple embodiments, on improvement of students’ understanding of fractions and forty fifth grade students were randomly selected from a public school in the Midwest of the United States. Twenty students were assigned in each group (control and treatment). The treatment, group 1, took the intervention first, while the control, group 2, was not receiving the intervention. After group 1 finished the intervention lessons, the control group 2 took the intervention. The five fraction lessons contain 6 activities embedded in the dynamic principle of Dienes (1960, 2007), which states mathematics is both abstract and general, and learning the subject includes the process of abstraction and generalization.

According to Dienes, to construct abstract structure of mathematics, children need to see it in a multiple of different situations. To generalize a mathematical concept, the essential variables embedded in the concept need to be varied and it is necessary to provide opportunities to construct and experience mathematical ideas before they are expected to think them abstractly and generalize them (1960, 2007).

Before the instruction, both groups took the pre-teaching test and had the interviews and they did best on the area model tasks on the pre-teaching test. However, more than 30% of them were unable to provide correct answers for the problems where fractions were not associated with the area-models such as capacities, lengths of ribbons, number lines and discrete sets of apples or marbles. After the instruction intervention, the post-test and the interviews were administered and they performed much better on the problems, which are not related to the area

model, and about 76% of the answers were correct and were able to explain their rational for their solutions. More than 3 months after this instruction, a retention test was conducted to investigate how much the students could retain the fraction knowledge they learned through the instruction with multiple embodiments. Although, their performance went down slightly, comparing to the post-teaching stage, the result was still very impressive, considering they were in summer vacation in these three months and did not receive any formal mathematics lesson.

The group 1 scored 83% and the group 2 scored 74% on the retention test. After the intervention lessons, there were more students who were able to perform fraction operations correctly and justify their answers. This indicates that they were applying procedures with conceptual understanding to the operations and were developing structural conceptions of unit fractions (Sfard, 1991). Since this study had only 40 students and was conducted with only 5 sessions, to draw general conclusion, a future study need to adjust these shortcomings.

Conceptual Based Instruction (CBI) directed by Hiebert and Wearne (1989) focused on problem solving in the content of decimal numbers by using a variety of methods including pictures, words, symbols, and manipulatives. CBI moved away from practicing procedures for symbol manipulation and emphasizes on developing conceptual understanding of decimals. The researchers used study lessons that aimed at assisting to develop connections between physical models and symbols, procedures for addition and subtraction of decimals and translate between decimal and fraction form in two fourth grade classrooms. The study results from interviews and assessments indicated that this instruction developed conceptual understanding of decimals and their procedures. Although, the improvement in translating between decimal and fraction froms was not prominent, the improvement was there.

The study of Tchoshanov (1997) conducted a pilot quasi experiment study with 70 high

school pre-calculus students to investigate two instructional methods on students' inverse trigonometric identity. One utilized only one visual manipulative aid and the other one used different representational modes. Those who were in the single representation instruction were stuck to one specific representation and were not comfortable of using different representations and this showed that usage of only particular representation does not enhance students' conceptual understanding of trigonometry. On the contrary, those who were in the multiple representational group were more versatile changing from one representation to another to gain better understanding. Although this study was not conducted on fraction learning, the result suggests the importance of learners getting exposed to different representations and models to deepen their mathematical understanding.

These studies results clearly show that in the intervention lessons, the students were exposed to a variety of embodiments and were provided opportunities to construct conceptual understanding of not only the concept of fractions and rational numbers but also other mathematical concepts. The pedagogy, which provides learners opportunities to be exposed with a multiple of models and activities, certainly deepens their fractional conceptual understanding and reinforces the concepts that cannot always be covered by usage of the area-model alone.

Technology Usage

There has been an exponential increase in the creation of educational apps in the last decade to address learning challenges in different areas (Habgood & Ainsworth, 2011). The emergence of new technologies, especially mobile devices such as iPad, and iPhone, offers more opportunities to create effective learning experiences that overcome the elements of the traditional teacher-centered teaching such as dry presentation and limited interaction in the confined classroom setting (Riconscenete, 2013). Some researchers have agreed the capability of

the new technology to provide expressive learning experiences. This idea is built on the insight, which cognitive processes are deeply rooted in the body's interaction with the world (Wilson, 2002). In other words, knowledge is rooted in our physical interactions with the world (Clark, 1999; Wilson, 2002).

When a child constructs conceptual structures, there is always some sort of interaction and mental stimulation (Steffe, 1995). If any knowledge is to be built, the learner must have some interactions among currently available constructs (Tzur, 1999). For instance, Ramani and Siegler indicated that preschool children gained mathematical knowledge from playing board games and theorized that chances to physically interact with a number line by moving tokens on the linear board games assisted them to develop a mental image of number line by touching clues about the order and magnitude of numbers (2008).

Although utilizing touch screen apps has been getting attention in children's mathematical learning, there have been a few studies conducted regarding how interaction with the mathematics apps can support or hinder children's learning (Moyer-Packenham et al., 2015). They conducted a study to investigate how the interactions with the apps on touch screen devices can change children's mathematics learning performance and efficiency. This study utilized Piaget's cognitive development idea, which is when children learn mathematics using virtual manipulatives, their cognitive restructure process can be observed by investigating changes in their activities on the apps and changes in what the activity depends on (Piaget, 1970).

The researcher used a mixed method design so that they can answer the question quantitatively and qualitatively. One-hundred children between 3 and 8 years old participated in this study and pre and post- tests and interviews were given and they used 18 different iPad mathematics-learning apps. The result indicates that there were changes in their learning

performance and efficiency. This implies that young children perhaps could gain numerical performance significantly in a short term by interacting with the iPad apps.

The study of Seffe and Olive (2002) investigated how children engaged in cognitive play and how they transformed the play into independent mathematical activity. The researchers created computer tools called Tools for Interactive Mathematical Activity (TIMA) to provide computer environments where children can engage in cognitive play activity and learn to play mathematically. This medium was created based on the Papert's idea of microworld (1980), that children can learn mathematics and express what they have learned creatively through a dynamic medium. A microworld is a self-possessed world where children "learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction" (Papert, 1980, p.117).

The study was conducted under the context of a constructivist teaching experiment with 12 children over a three-year span. One 4th grade child partitioned a stick on TIMA into 9 equal parts and she was asked to make half of the ninth's stick. Then her teacher posed a question that what fractional part of the whole stick one half of the ninth could make. She counted the nine-ninths by twos and answered that it would be one- eighteenth. After this, she was asked to make 1/27 from the 1/9 stick. She was able to partition the 1/9 into three equal parts on TIMA and called it 1/27 and was able to produce the repeating partitioning procedure and this concept of recursive partitioning of parts serves children to reorganize fraction concepts (Pitkethly & Hunting, 1996). The result indicated that TIMA provided children the opportunity to engage in cognitive play and they turned it into mathematical play.

Tzur (1999, 2004) conducted a study on how students come up with unit fractions utilizing the same computer program, TIMA:Sticks. The author played a role as a researcher and

a teacher and studied 2 fourth grade children and the framework of the study followed a constructivist scheme theory. The researcher videotaped a teaching episode with the two children. The tasks the children took on were close to their daily life experience such as sharing a candy bar or birthday cake in a party or slicing pizza so that they could construct mathematical concept through doing the tasks and it could become more natural for them to relate the stick on the screen to familiar things such as pizza, cake or candy. The children constructed the equipartitioning and the partitive scheme while they were using TIMA:Sticks. For instance, students were asked to create what the whole look like for 5/8, given the piece 5/8. Then they had to break the whole into five pieces, coping one of those pieces, 1/8, and repeat it 8 times to produce the original whole. Once they were able to understand this initial concept, they moved on to more complicated fractions.

Understanding of a unit fraction is beneficial for students developing some core fractional ideas such as: they can keep the part to the whole relationship, producing the whole, and given a fractional segment, because understanding the whole assists children to reorganize fractional concepts (Pitkethly & Hunting, 1996). The study also revealed that although the children's fraction scheme was prompted by the technology usage, teacher-learner interactions played a vital role as well because the teacher's role is fostering the concept by communicating with them in the usage of TIMA:sticks.

Another important aspect in learning fraction is learners' motivation and attitudes (Ashcraft, 2002; Stipek et al., 1998). Children construct their reality through playing. In other words, playing in a mathematical context would help children to construct mathematical reality and become their motivation to do mathematics (Steffe & Weigel, 1994). Many studies indicate that learners' motivation influences their academic achievement strongly and they are more

likely to develop interest in materials they understand and see as relevant (Hilgard & Bower, 1966; Von Glasersfeld, 1983; Hidi & Renninger, 2006; Riconscente, 2010). Technology usage can certainly provide the learning opportunity with motivation since that is the main aim of designing apps because players of the games are challenged constantly to overcome difficult obstacles to attain a goal and the players need to discover the rules of the games to win (Riconscenete, 2013).

The study of Riconscenete (2013) aimed at determining whether Motion Math increased students' fraction understanding and their attitudes toward fractions. Motion Math aims at helping children to strengthen their understanding of the relations between fractions, proportions and percentage on the number line by the player tilting the device to direct a falling star to the right place on the number line on the bottom of the screen. This study did not aim at comparing two different instructions, one with Motion Math and the other one without Motion Math, because the purpose of this study was to investigate whether usage of Motion Math itself gains students' fraction learning and their attitude toward fraction learning. It was critical to minimize the effect of classroom instruction that might have on their fraction knowledge. Because of the nature of the study, teachers did not provide any fractional lecture during the study.

The participants were 122 fifth graders in two schools in southern California and the students were assigned at random to classes and since there has not been a study that used an experimental design regarding mathematical apps, this study conducted an experimental design. Those who were in the control group had normal mathematics instruction without fractions and the students in the treatment group played Motion Math 20 minutes for 5 straight days. The three tests, pre, mid and post- tests, were administered to measure their fraction understanding and attitude.

Motion Math increased students' fraction understanding and their positive attitudes toward fractions. The interactive learning by technology can not only enhance students' fractional concepts but make the learning more interesting so that learners can get motivated. Since the students played the game only for 5 days, the results does not indicate how deeper students' fraction conceptual understanding can be developed over a longer period. Similarly, another question is if Motion Math actually teaches students fractional concept or assists to activate and reinforce their previous knowledge of numbers.

Technology such as computer software and iPad apps are very beneficial not only for deepening the fraction concepts but also for motivating students to learn because it promotes interactions with technology and other students and a teacher as well. A few studies indicate that their motivation and negative attitude toward the subject has a heavy influence on their fraction understanding (Aschcraft, 2002; Stipek et al., 1998). Hence technology usage can be one of the possible solutions to improve learners' fraction understanding at the same time changing their negative attitude toward the subject to a positive attitude (Riconscenete, 2013).

Explicit Links between Concepts and Procedures

Mathematics teaching need to focus on assisting students to acquire key main concepts and principles along with the ability to use procedures flexibly while solving complex problems (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2000). A main feature of mathematical proficiency involves both conceptual knowledge and procedural knowledge (Baroody, Feil & Johnson, 2007), but it has also become obvious that a critical aspect of mathematics learning involves making connections between concepts and procedures (Hiebert, 1986). Mathematical understanding also involves using concepts as rationales for symbolic representations and solutions. Skemp (1971) claimed that making such links between

concepts and procedures, knowing what symbols mean and why procedures work the way they do, is the heart beat of learning mathematics. Therefore, it is obvious that students need to develop an understanding of how concepts can be utilized to understand procedures (Hiebert, 1992; Kilpatrick et al., 2001). The ability that connects understanding of the meaning of fraction symbols to knowing the conceptual rationale for fraction procedures assists learners to recognize and correct errors and transfer knowledge to new and unfamiliar tasks and problems (Osana & Pitsolantis, 2013).

It is encouraged for teachers to use manipulatives and pictorial models to teach important meaning of mathematical symbols (Clements & McMillen, 1996; Uttal, Liu, & DeLoache, 2006), and making connection between the model and the symbol is the most effective teaching, if it is clearly understood for students (Sarama & Clements, 2004; Uttal, Scudder, & DeLoache, 1997; Wearne & Hiebert, 1988). For example, to communicate the meaning of the fraction symbol 1/4, a teacher could use a rectangular area-model that is divided into four equal parts and shade in one of the parts to show the meaning of 1/4.

An effective teaching goes beyond introducing the two symbols, considering the rectangle itself as a symbol also, are equal and makes connections between the rectangle model and the symbol 1/4 meaningfully to learners by referring the concept of one quarter (Osna & Pitsolantis, 2013). In other words, to understand the symbol 1/4, it takes more than making an association between the external representation of 1/4 and another external representation, such as model or diagram. It takes connecting the concept of 1/4 (internal representation) and the area-model (external representation).

To assure the importance of making connections between models and fractional symbols, a handful of studies claim that simple usage of manipulatives and models are not always

effective in strengthening learners' mathematical understanding (Kaminski, Sloutsky, & Heckler, 2008; Sloutsky, Kaminski, & Hecker, 2005). Uttal, Liu and DeLoache (2006) suggested that certain conditions have to be met for models to be effective. For instance, pictures and concrete objects must not be utilized as precursors to symbolic representations but need to be used as visualizations of the relationship between concepts and symbols. In other words, manipulatives and their referents need to be linked at the same time. Similarly, Scudder and DeLoache (1997) warned that it cannot be expected for children to make connections between fractional symbols and their concepts built by manipulative and models on their own. Therefore, the effective instruction has to support them to make conceptual link between models and symbols explicitly. When the learners do not see the relationship between the manipulatives and the symbols, they could use manipulatives without really thinking as they use procedures (Ambrose, 2002).

Wearne and Hibert (1988) provided an explicit instruction to $4^{th,}$ 5^{th} and 6^{th} graders on how to make connections between the symbolic decimal fractions representations and base 10 blocks as their concrete referents. During the instruction, students were also encouraged to use the concrete referent to act out putting them together and separating them to link these actions to corresponding symbols. The study results showed that students were able to learn how to make correct links between the symbols and the concrete referents and their understanding of the connections helped them to transfer their knowledge of the key concepts to novel problems. Similarly, the study of Fuson and Briars (1990) on multidigit addition and subtraction found the similar result. Students' ability to use place value concepts to explain the steps taken in the multidigit addition and subtraction algorithms increased dramatically after they had completed the instruction which explicitly connected written representations with place value concepts embedded in concrete materials. These results indicate that the instruction, which makes explicit

connection between mathematical concept and procedural knowledge, is a very effective way of strengthening learners' conceptual understanding and procedural knowledge

The study conducted by Osna and Pitsolantis (2013) tested the effectiveness of an instructional intervention (site instruction) which explicitly links fractions concepts and procedures in fractions toward 72 Canadian 5th and 6th grade students, translating the site theory posed by Hiebert (1984) into the instruction. The theory claims that during mathematical learning meaningful relationship can be constructed between conceptual understanding and knowledge of symbols and procedures. The site instruction focused on fractional concepts and procedures, such as comparison, addition, and subtraction of fractions and made clear links between concepts in concrete and pictorial models and fraction symbols. The authors compared the sites condition to the students in the control group who received instruction on fraction concepts and procedures but made no clear connections between the two. The pre and post- tests and interviews were administered.

Although students in both control and treatment groups gained conceptual understanding of fractions, those who were in the sites instruction improved more because the students in the site instruction were better at connecting fractions symbols with concrete and real world situation and with the fraction concepts. Since the study was a 3-week intervention and conducted with the small sample size, the generalizability of the result can be marginal. These results indicate that the instruction, which makes explicit connection between mathematical concept and procedural knowledge, is a very effective way of strengthening learners' conceptual understanding and procedural knowledge. However, when learners are left in the situation where learners do not see clear connections between conceptual understanding and procedural knowledge, they tend to rely on prior knowledge of whole numbers and this interferes with understanding and using the new

symbol (Mack, 1993, 1995; Mix, Levine, & Huttenlocher, 1999). Hence, it is necessary for a teacher to bridge the gap between the symbolic representations of fractions and their concepts while learners use manipulatives and visual models.

Teacher's Role

It is commonly known that teachers and teaching are highly correlated with students' learning of mathematics in the classroom. Because of this, educational research has rapidly increased its emphasis on teachers and their practices (Sikula, 1996; Townsend & Bates, 2007). The lack of adequate preparation in mathematics usually leads teachers to difficult experiences in teaching that requires conceptual understanding (Borko et al., 1992). For instance, the topic of fraction division is challenging in school mathematics not only for students, but also for teachers (Carpenter et al., 1980).

The study of Ma (1999) compared the fractional understanding of American elementary teachers to Chinese elementary teachers. This study was conducted with 21 American teachers and in this study, they were asked to calculate $1\frac{3}{4} \div \frac{1}{2}$ and provide a mathematical sentence for the representation. Although only 43% of the American teachers were able to answer completely, all the Chinese teachers provided correct and complete answers. Additionally, when they were asked if the computational procedure made sense, the Chinese teachers, but not the American teachers, were able to explain details of the procedure. While all the U.S. teachers mentioned the invert and multiply algorithm, the Chinese teachers were able to derive additional approaches besides the algorithms. It was revealed that 16 teachers had misconceptions in their story problems and 6 of them were not able to come up with a story and only one American teacher was able to come up with conceptually sound representation. Based on the study result, the American teachers lacked conceptual understanding of fractional division and they regarded the

subject as a collection of facts and procedures and think that doing mathematics is following the procedures to get to solutions.

A study conducted by Li and Kulm (2008) aimed at pre-service teachers' mathematical knowledge of fraction division to answer two research questions: (1) How do pre-service middle school teachers perceive their knowledge preparation in curriculum and instruction for their future teaching career? And (2) How is the pre-service teachers' mathematics knowledge to teach fraction division? Forty-six participants were pre-service teachers who were enrolled in a mathematics and science interdisciplinary teacher education program at Texas A&M University. The survey and the mathematics test on the topic of fractions division were administered in the last class of the senior method course. The result showed that there was a wide gap between pre-service middle school teachers' general perceptions/confidence and their limited mathematics knowledge for teaching fraction division conceptually. These pre-service teachers needed to develop conceptual understanding for teaching to build their confidence for classroom teaching.

Because of lack of teachers' conceptual understanding, this topic is usually taught as an algorithmic procedure that can be easily taught and learned such as "invert and multiply." Students often perform calculations without knowing why (Kerslake, 1986) and this causes a misunderstanding of mathematical symbols (Byrnes & Wasik, 1991). As a result, many computational errors happen due to a lack of conceptual understanding. However, this topic is rich in concept and difficult and the meaning requires explanation through connections with other mathematical knowledge, different representations, and/or real world contexts (Li & Kulm, 2008; Ma, 1999).

Although dealing with fractions is a part of our everyday life and is used in situations such as estimation, cooking recipe or reading map, students have a difficulty of learning this

topic. Moreover, fractions play a major role in mathematics, because the other areas of the subject use the concept of fractions, such as probability, proportion and algebraic reasoning. Therefore, it is indispensable for teachers to have the ability to clarify the misconceptions originated from the natural number bias and to aim at deepening learners' fraction conceptual understanding by utilizing interventions, which could eliminate misconceptions and improve students' understanding of decimal fractions (Rittle-Johnson, Siegler, & Alibali, 2001).

Dorward (2002) reflected on the different ways teachers implemented for fraction concepts to discern what is working from what is not working. This information certainly can be helpful to improve fraction lessons. According to Dorward, the key element is teachers engaging students as active participants in their fractions lessons as it provides positive energy to students' learning. Similarly, Burrill (1997) claimed that good teaching is not making learning easy but rather making it active and engaging for learners. Making fractions interesting and exciting can improve students' learning. The NCTM claims that effective math teaching has to understand what students know and need to learn and then challenge and support them (NCTM, 2000).

The study of Capraro, Naiser and Wright (2003) aimed at identifying effective teaching strategies to improve middle school fraction instructions toward 178 sixth graders and 92 seventh graders from five different middle schools. The study aimed at fractional instructional methods more than at the content. The result based on the observation notes, interviews and analysis of video-taped instructions, suggest these elements listed below could make fraction teaching effective. First, when students were given an opportunity to think about how fraction concepts are related to their personal lives, the fractional lesson made more rigorous to them. Relating fractions to realistic situations can also lead them to use their informal knowledge to construct meaning for themselves. However, many teachers in this study did not make the connections and

the lessons were not engaging nor were students involved. Finally, manipulative usage was another way to make the lesson more engaging because manipulative can provide opportunities to express their thinking and it also helps teachers to understand what learners are thinking by observing their actions with manipulatives. Therefore, combining the study result, manipulatives with real- life problems could motivate students in solving problems and this helps them to construct fraction meaning on their own. Lastly, when the type of instruction gives students a chance to construct fraction knowledge and ideas on their own through the real-life situations, it becomes more meaningful. Using authentic problems can lead them to demonstrate how the concept of fractions may be interpreted and applied in real-life situations (Tzur, 1999). By this way, teachers can build fraction concept on learners' prior knowledge, and discover misconceptions students have (Tirosh, 2000).

Moyer and Mailley (2004) discussed about an activity using visual models and physical representations, originated from the book Inchworm and a Half (Princzes, 2001) to assist the children to understand fraction parts and equivalent fractions. The children were to measure vegetables by using different length of the worms such as ¼ and 1/3. Then they needed to find equivalent fractions using the different lengths of worms. After this, they were to represent the fractions symbolically. The children deepened the foundation for meanings and operations with rational numbers. This study indicates that when learners have fun, they are more engaging and the learning becomes meaningful.

However, as Hatfield described usage of manipulative is very low, compared to the familiarity and availability that teachers report regarding manipulatives (1994). The main reason of not using manipulatives is teachers do not feel competent in using them during the instruction. Therefore, one of the responsibilities of college teacher education programs is informing pre-

service teachers how the usage of manipulatives and models during instruction can enhance students' conceptual understanding of fractions and provide pre-service teachers opportunities to use manipulatives and models during method courses so that they can be more comfortable using them during instructions. When teachers view their mathematics instruction through the eyes of their students, they can gain a deeper understanding of the subject and are more willing to use manipulatives (Putnam & Borka, 1999).

Interaction with a teacher could influence mathematics learning as well. Steffe and Tzur (1994) suggested a model of mathematical learning where they created the concept of learning in terms of learner's mathematical interaction. In this view, a teacher can be involved in student's learning by interacting with them with a purpose of bringing experiences, which could create perturbation (challenged and inquiring frame of mind), while the learners try to figure out the perturbation. As the studies (Steffe & Olive, 2002; Tzur, 1999, 2004) described in the technology section earlier indicate that the interactions between learners and the teacher was a vital part of the children constructing the concept of a unit fraction. Therefore, the teacher aiming at assisting the learners to learn fraction can lead the interaction with children and that could influence the learners' experiences and the perturbation. But perturbation is a double edged sword and the teacher always need to remember that it can enhance mathematical activity and learning, but it also can hinder children's learning (Steffe & Tzur, 1994).

According to Tzur (1996), when children are satisfied with the experiences of perturbation and its neutralization, the perturbation is meaningful but if children cannot neutralize the perturbation, this could lead them to frustration and discourage their willingness to be interested in future learning. Therefore, the teacher has the responsibility to plan and implement teaching-learning situations and tasks that will help them to modify their current

mathematics. Steffe and Tzur claimed that although teachers can be a part of children's mathematics learning, they cannot simply make the teachers' knowledge their own. They need to interiorize the knowledge through their activities (1994).

College Remedial Mathematics Students

Adults enrolled in community college have difficulties with fractions (Stigler, Givvin, & Thompson, 2010) and the topic of fractions is the main focus of study in community colleges since many students are unable to master the fraction concepts in their previous schooling (Gal, 2000). A recent study with community college students shows that the students do not possess a strong conceptual understanding of fractions since the topic was introduced to them in the manner of a rote memorization of procedures without building strong connections among understanding, symbols and rules (Hiebert et al., 2003; Pesek & Kirshner, 2000). Because of their lack of fractional concepts, their mathematical ability to learn advanced mathematics that require the concepts such as algebra (Stigler, Givvin , & Thompson, 2010).

A study (Bonato, Fabbri, Umilta, & Zorzi, 2007) conducted with 19 year old college freshmen indicated that the students were not able to represent fraction magnitude mentally and when comparing the magnitude of fractions, they compared each fraction component instead of considering the fraction as a number. This attributes to the whole number thinking. However the study of Schneider and Siegler (2010) provides a contradicting result. In the study, they compared students of a selective university to students of a community college in fraction magnitude problems. The result found that their strategy selections were not strictly based on the natural number reasoning but depended on how complicating the fractions appeared.

Hoyte (2012) described responses from adult mathematics students who were from different international locations. These participants answered a survey to identify their ability

and experiences in remedial algebra courses. According to their survey, their mathematical confidence and performance were aligned with personal intervention and material related to the real world. Hence, the key elements for success for remedial mathematics students are them knowing what they are doing, real life applications, instructors' patience and sufficient time for the material.

Spatial Temporal Mathematics

It has been found that Computer based instruction (CBI) has a positive impact on students' learning (Tran et al., 2012). CBI programs that use the spatial contiguity principle, which is aligning words to corresponding graphics (Clark & Mayer, 2011), are found to have more of positive effect on students' mathematical achievement compared to the CBI programs that do not utilize the principle (Harter & Ku, 2008). Spatial Temporal Mathematics (ST Math) uses images to assist students to develop spatial temporal reasoning, which can lead to deeper mathematical conceptual understanding in mathematics content of fractions, proportions, symmetry and other arithmetic operations (Tran et al., 2012).

ST Math is built on the concept that students can gain conceptual understanding with procedures and algebraic skills by learning the meaning behind the procedures through intuitive spatial relationships (Rutherford et al., 2014). This learning process could ultimately enhance students' mathematical competency and retention (Shaw & Peterson, 2000). The relation between spatial representations and an implicit ability to understand quantity to non-symbolic representations could be an effective path to enhance learners' numerical magnitudes and support connections between symbolic and non-symbolic representations (Geary, 1995, 2011). As studies show in fraction learning, visual representations can develop conceptual understanding of both the fraction magnitudes and its manipulation (Siegler et al., 2010; Siegler, Thompson, &

Schneider, 2011). ST Math was designed to have the system of visual representations to assist learners' conceptual understanding of the magnitude and its relation as the difficulty level increases in the games (Rutherford et al., 2014). Repeating spatial representations such as the number line, could assist students' acquiring simple mathematical understanding to deeper mathematical understanding such as fractions (Siegler et al., 2010; Wu, 2005).

Mathematics is continuous and one concept is built on another concept and it is critical for students to understand foundational mathematical concepts as they advance to higher level of mathematics because there is a strong relation between early mathematical skills and later achievement (Duncan et al., 2007; Ramani & Siegler, 2008). ST Math's individualized curriculum is aimed at gradual progression; that is students can only move onto new concepts once they have mastered fundamental concepts. The designers believe that the software facilitates greater gains for struggling students because they can spend enough time and review the fundamental concepts at their own pace (Rutherford et al., 2014). Research has shown that lower performing students can benefit more from computer assisted instruction (CAI) to improve their mathematical achievement (Edwards et al., 1975).

Prompt feedback is an important feature of CAI. According to Fraij (2010), effective ways of feedback can influence learners using CAI positively and feedback helps learners to reconstruct their knowledge and support their understanding process. ST Math gives prompt feedback, average twice in a minute, and this provides the thorough introduction to mathematical concepts within lessons (Rutherford et al., 2014). Hence the software can increase the learners' engagement (Breuninger, 2015). Because of these qualities ST Math can offer effective mathematical learning.

Rutherford and researchers (2014) conducted a longitudinal study to examine correlation

between ST Math and students' mathematical achievement measured by the California Standards Test (CST). One thousand three hundred and eight students who were from 2^{nd} to 5^{th} grades in 52 low-performing schools from 10 school districts in Southern California participated in the study. The students were assigned to the control group and the treatment group. Those who were in the treatment group attended ST Math lab sessions twice a week for 45 minutes for one year and those who were in the control group did not utilize the software. The result indicates that ST Math produced modest gains in the CST score between 3^{rd} and 5^{th} graders. Although the effectiveness of year 2 was stronger than the effectiveness of year 1, it was not significant statistically.

The effect of the software on low-performing students was supported little after 1-year usage or 2-year usage within this study. The most benefited group from ST Math was the students with below basic mathematics skills, which was the second lowest group between below basic and far below basic. The lowest and highest achieving students had the smallest effects. One possible reason for this is that the lowest-performing students might have needed to be proficient in below grade level content first. Another reason could be teachers failed to connect the concept the students have learned playing ST Math games to classroom instructions. Thirty-eight percent of treatment teachers mentioned the software during non ST Math lab time and only 21% of the teachers made connections between ST Math and the mathematics contents during instructions. A future study needs to investigate which mathematics skills are strengthened by the software when connections are made by the teacher to the ST Math lab time. ST Math content also might not have correlated with questions on the CST.

The study of Graziano, Peterson and Shaw (1999) investigated the influence of ST Math video game on students' conceptual understanding of fractions and ratios with music training.

Five 2nd grade classrooms, a total of 136 students, in one school in Los Angeles took part in this study. The first class received both ST Math video game training and piano keyboard training. The second class received the ST Math training and regular mathematics instruction; while the third class assigned students randomly to either the piano training and ST Math video game or a regular mathematics instruction and ST Math video game. The last two classes received regular mathematics instruction. Pre- and post-tests were administered to assess students' fraction, ratios, and spatial understanding. Since this study conducted a quasi-experimental study, it did not use true random assignment and some students were assigned to the different levels of instruction because they participated in this study later. Therefore, the result provides marginal evidence of the effectiveness of ST Math. A future study that utilizes random assignment and selection is necessary. The result show that the children, who received both ST Math game training and piano keyboard training or regular mathematics instructions outscored those who did not receive ST Math video game training by a significant margin.

Breuninger (2015) focused on how ST Math usage can close the gender gap in mathematics achievement of elementary school children. Seven classrooms of 150 second graders, 75 male and 75 female, in a school located in San Diego participated in this study. The participants took district benchmark math exams before and after the ST Math usage to determine if there was a gender difference in achievement after the usage of ST Math. The results indicated that there was no significant impact of gender. Both male and female students achieved the same amount of increment from the first benchmark exam to the second benchmark exam by 8%.

However, the finding that needs to be noted here is that the class who were below-average on the initial benchmark exam recorded greater gain at the second benchmark exam after

the usage of ST Math, compared to the results of the classes who were considered as average or above average. When the study data were collected, about 52% of the academic year had been completed. However, male students only completed 30% of the ST Math curriculum and female students completed only 26% of it. This could really hinder the result of this study. Similarly, the sample size was relatively small and the time frame of this study was short. Hence, future studies needs to address these limitations of this research.

Quick (2013) conducted her study to answer the following two research questions: 1) How does ST Math impact students who have difficulties with or have failed a pre-algebra or algebra course in their previous academic year? 2) How does ST Math impact students' who are receiving special education service through an Individual Education Plans (IEPs) mathematical achievement? Twenty seven 9^{th}, and 10^{th} students, and two 11^{th} grade students in a high school in Denver, Colorado took part in this research. They were selected based on their previous academic year's math grade and all the participants scored at or below 65% in the final grade. The students were assigned to two types of Algebra Readiness interventions. Those who were assigned to the tier 2 intervention, which is students are below grade level academic proficiency and they need specialized instructions and interventions, were considered to be ready for Algebra I and were enrolled in both Algebra I and the readiness intervention class simultaneously. Seven students without IEPs were in the tier 2 intervention. The students who were assigned to the tier 3 intervention, which is students display low academic achievement even with the tier 2 intervention and they need more intensive intervention and includes students with special needs with IEPs, were enrolled only in Algebra Readiness intervention as their mathematics course. For each intervention, they played ST Math individually in the computer lab for at least 90 minutes a week. The researcher used a pre-experimental design with two high school classrooms

without randomization and a control group and the pre-test and three different post-tests were administered. The study lasted for one academic year (36 weeks).

The result shows that no significant improvement was revealed on all the post-tests. Any of the measurements indicated there was a significant difference between students with IEPs and those without IEPs. This result could be affected by the limitations of the study, such as not including a control group, small sample size, the timing of one of the assessment tests and relying heavily on visual tools that could not be a right intervention for some learners. A future study needs to adjust these limitations and repeat this study to draw a more reliable result.

The studies above indicate inconsistent results. One of them shows that ST Math usage influences students' mathematical learning positively and other results claim that the software was not much of a support in learners' mathematical achievement. However, all of these studies have critical limitations as described. Hence at this moment, it is not fair to draw a conclusion without having more studies that adjust the previous studies limitations. By reviewing these studies, one main thing is seen; that ST Math is the most beneficial for those who have slightly below average mathematical competency.

Research Findings

Students' natural number bias is persistent and the bias affects understanding the fractional concepts negatively. This is the main reason that students tend to learn fractions procedurally so that they do not need to go through frustration of understanding the concept because the fractional concept is very different from natural numbers and the concepts are foreign to them. Hence, memorizing different rules is the easy way out to learners. However, this way of fraction learning is harmful. Since they do not understand the fractional concepts that underlie procedures, they cannot apply the procedures to novel situations. Once they get lost in

the concept of fractions, this is going to affect not only their mathematical achievement, but also their academic career as a student since fractions are the foundation for algebra and algebra is the cornerstone in mathematics learning. This is one of the main reasons that many students are enrolled in college remedial mathematics courses and the main focus of the remedial mathematics classes is to understand the fraction concepts. Hence, it is necessary to assist students' fraction conceptual understanding so that they can be successful in their academic career.

In order to strengthen their fraction concept, as the studies suggest, utilizing manipulatives and models is one of the most effective ways because this provides learners to make sense the fundamental fraction concept on their own. For instance, by using an area model, learners can gain the concept of fraction size and this concept is the foundation of having a common denominator in addition and subtraction of fractions. Similarly, connecting the fraction concept to the real life situation is another effective way because they can actually relate the concept to their daily life. By combining different manipulatives, models and the real life situation, they can deepen their fractional understanding and the understanding can be more flexible, instead of relying on a particular model such as the area model. Better yet, this does not only enhance learners' understanding but also creates an active learning environment and makes fraction learning more fun. Hence, it is critical for leaners to experience a variety of models and manipulatives in fraction learning.

Technology is another form of manipulative and technology usage provides an engaging learning opportunity as well. Since technology plays one of the major roles in current mathematics education and in these days, students are exposed to a digital device such as iPad and iPhone, using fraction apps that aim at fractional conceptual understanding is another

effective way of instruction, because they can learn the fraction concepts as if they played a game. Technology usage certainly offers more exciting and fun learning opportunities to students.

Among different technology based learning programs, a program using spatial contiguity principle is considered more effective in mathematical learning. ST Math utilizes the principle to develop learners' mathematical understanding through computer games. The software can aim at through progression in acquiring mathematical concepts and they can only move onto the next concepts upon mastering the previous concepts. Hence, low achieving students can benefit from this sequential learning. Also it provides language free learning environment and this would work favor to those who are confused with different mathematical technical terms. Lastly, ST Math offers immediate feedback to users and this can certainly enhance the learners' engagement.

Although there have been studies conducted to examine the effectiveness of this software, these studies appeared to be inconsistent in their results because of their research limitations. Hence, it is still uncertain how effective ST Math is and a new study that could adjust the limitations is definitely necessary so that educators can be sure that the software can provide students a meaningful fraction learning opportunity.

Lastly, teacher's quality plays a major part of effective fraction learning. Teachers need to have deeper conceptual understanding of fractions so that their students do not settle for the traditional procedural way of learning and also need to be comfortable with using manipulatives, models and technology to provide learners more engaging and meaningful learning opportunities. Although teachers know the interventions can be very effective for strengthening learners' fractional concept, they tend not to use them because of their unfamiliarity. Therefore,

the challenge for university teacher preparation program is assisting pre-service teachers to be familiar with the interventions so that they can serve as an effective teacher.

Therefore, in the current mathematics education, the usage of mathematical software and apps for effective learning, not only the topic of fractions but also the entire mathematics content, has been the major focus. Particularly, software and apps utilizing spatial temporal concepts, such as ST Math, have been found more effective in mathematical learning. However, studies regarding the software have shown the inconsistent results and the effectiveness of the software is not certain, yet. Also, these studies targeted mainly from elementary school to high school students. Hence, a study is needed to adjust the previous studies limitations and target at college students so that it can be proven that whether the software can be effective even for adults.

Summary

The natural number bias students constructed during their first few years of elementary education is persistent even throughout high school education and this affects them understanding fractions negatively. To reduce the bias and deepen their fraction concepts, there are many different elements that are necessary for an effective fraction instruction, such as usage of visual models, manipulatives, and technology. Similarly, teacher quality is equally important because teachers are the ones who interact with their students daily and decide and deliver their pedagogies. Aside from this, in the recent years, technology has developed rapidly and utilizing technology such as a touch screen device and its apps, in mathematical learning has been a focal point in mathematics education. Among the software and apps, the ones utilizing spatial temporal concepts such as ST Math, have more positive effect on learners' mathematical learning. However, the results of the studies conducted to examine the effectiveness of ST Math

have been inconsistent and more studies in this software need to be conducted to investigate the effectiveness.

CHAPTER THREE

Research Method

Purpose of the Study

Understanding the concept of fractions is one of the major obstacles students face in mathematics (Anthony & Walshaw, 2007; Verschaffel, Greer &Torbeyns, 2006; Young-Loveridge, Taylor, Hawera & Sharma, 2007). One of the reasons for this is the natural number bias, which is that learners tend to apply the natural number properties to fractions (Fazio & Siegler, 2011). This bias is persistent and even secondary school students struggle with the bias when they learn fractions (Vamvakpussi & Vosniadou, 2004). Because they hold this bias, misconceptions occur when they add new information of fractions to their initial concept of natural numbers (DeWolf & Vosniadou, 2011). For instance, learners tend to think that a fraction multiplication gives a larger value as the multiplication of natural number does, since the natural number bias is too strong to overcome when they are in process of constructing the new concepts of fractions (Inagaki & Hatano, 2008).

Fractions are one of the primary mathematical contents many studies have been done with, especially in the past fourteen years (Biesenthal, 2006). Mack (1990) indicated that children have very limited conceptual understanding of fractions. This is mainly because fractions are taught as a set of procedures and rules to be followed without emphasizing the concepts behind algorithms (Aksu, 1997; Lammon, 2001; Moss & Case, 1999). Although the extensive study of Mack proves that students do not understand fractions from the procedural way of learning, many educators still teach this content, using the traditional procedural methods.

This is very harmful for students' mathematical understanding for a few reasons. First, they do not have an opportunity to learn fractions conceptually. Secondly, students are not given

a chance to discover the meaning of procedures to make sense why the procedures they use to solve fraction problems are reasonable (Biesenthal, 2006). The only thing they have done is applying the procedures to the problem solving and follow the steps with little attention to the reasonableness of their answer (Hasemann, 1981). Therefore, as Lamon (2001) states, since learners do not learn rules and procedures with meaning, they tend to forget the procedures and are not able to apply them to problems that look different in their appearance but have the same core concept underlying. Studies claim that it is essential for students to have a chance to explain their own reasoning for their answers so that they can strengthen their conceptual understanding, while they solve problems, before procedures and algorithms are introduced (Bulgar, 2003; Kamii & Warrington, 1999; Lappan & Bouck, 1998; Warrington & Kamii, 1998). The widened understanding can assist them to be more flexible and adaptive to a variety of different fractional problem situations (Lamon, 2001; Warrington & Kamii, 1998). Hence, as the Common Core State Standard (2010) states, teaching fractions conceptually is the aim of a fractional instruction.

To teach fractions for understanding, a teacher should be able to implement a variety of effective pedagogies, utilizing different visual models, manipulatives, and technology such as apps for a touch screen device to assist learners to gain the fractional conceptual understanding and reduce the persistent natural number bias simultaneously. As the study of Riconscenete (2013) shows, usage of iPad apps provides more fun and engaging learning opportunity to students. Although mathematical apps usage has been a major topic in mathematical education, there have not been enough studies conducted to investigate how the usage can deepen students' mathematical understanding, especially this particular content of fraction. Riconscenete aimed at examining the effectiveness of the fraction-learning app called Motion Math, but the study focused more on children's motivational aspect than their conceptual understanding.

Among a variety of technology based learning programs, one utilizing spatial contiguity principle is considered more efficient in mathematical learning (Tran et al., 2012). Spatial Temporal Math (ST Math) is built based on this principle and it offers a learning opportunity to develop learners' conceptual understanding through visual games. Although there have been studies conducted to examine the effectiveness of this software (Graziano, Peterson, & Shaw, 1999; Rutherford et al., 2010; Quick, 2013; Breuninger, 2015), these results have been inconsistent and it is not still clear that the software really influences students' mathematical achievement positively because of the research limitations. Therefore, more studies that adjust the limitations are indispensable so that mathematical educators can know the effectiveness of the software and utilize it during their instruction.

Similarly, the studies regarding understanding the fractional concepts have targeted mainly elementary school children and it is a very rare occasion for a study to be conducted toward secondary students or college students. Hence, a study that targets college remedial math students' fraction understanding is particularly useful.

The number of students enrolled in college remedial mathematics courses has been increasing (Ramussen et al., 2011). This happens because students struggle with understanding the fractional concepts that are one of the most important concepts and the foundation for mathematical skills (Siegler, Scherider, & Thmopson, 2011; Siegler et al., 2012). It is obvious that understanding the concept is the key element for college students to be successful in college level mathematics courses and the instruction of a remedial course should focus on assisting students to understand the concepts. However, this is not the case in the instruction. College mathematics courses are usually taught procedurally since remedial classes have many topics to be covered and instructors are obligated to teach all the topics quickly and shallowly in a

procedural manner (Hinds, 2009). This implies that students are not given opportunity to understand the concepts on their own by utilizing visual models, manipulatives and technologies.

This is why even if they move to a subsequent college level math course such as college algebra that is a requirement for a non-science and engineering degree, they usually will not succeed because of a lack of fraction understanding. Hence, it is not exaggerating to say that not understanding fraction holds them back to move forward in their college academic career.

Therefore, it is crucial for college mathematics departments and its instructors to identify how they can help the students to understand the concepts so that they can not only be successful in college level mathematics courses but also move forward to pursue their degree. Since this research utilizes mathematical learning software, ST Math, to examine its effectiveness in fraction learning for college remedial students, the result could certainly provide a college mathematics department one possible solution. Similarly, the results will encourage secondary school mathematics educators who have never considered the technology usage during the fraction instruction to take advantage of the software to provide an engaging and a meaningful fraction learning opportunity. Hence, the study strictly focuses on how ST Math usage deepens college remedial math students' fraction conceptual understanding.

The aim of this study is to answers two research questions:

1. *To what extent do college remedial mathematics students possess whole number misconceptions? In addition, to what extent can each class, ST Math and non- ST Math, express fractions and their operations, pictorially, realistically, symbolically and verbally?*
2. *To what extent does the usage of ST Math eliminate the natural number misconceptions and deepen college remedial mathematics students' conceptual understanding in*

learning fractions compared to a non-ST Math class of remedial mathematics students taught traditionally without technology?

The purpose of this chapter is to describe the procedures used to decide the effectiveness of ST Math in understanding the fraction concepts by reducing the natural number bias. It consists of importance of the study, research design, target population, procedures, data analysis, expected findings, about ST Math, and summary.

Significance of the Study

Examining how ST Math strengthens college remedial mathematics course students' conceptual understanding of fractions and how the technology reduces the whole number bias were the goals of this study. Many studies conducted previously show that technology usage such as virtual manipulatives, mathematical software and iPad apps, is beneficial in learning mathematical concepts (Tzur, 1999, 2000, Reimer & Moyer, 2005; Suh & Moyer, 2007; Riconscenete, 2013). However, these studies were mainly done with elementary school students and very few studies were conducted with secondary students and above. It was necessary to study the effectiveness of technology usage with college students.

In college remedial mathematics courses, these students have fragile fractional conceptual understanding and understanding the concept is the key to be successful in subsequent college level mathematics courses required for earning a degree. However, many courses are taught by the lecture centered procedural manner without utilizing manipulatives, models or technology. Because of the nature of pedagogy, the students struggle with mastering the concept. This study used a iPad app called ST Math to investigate how the app could deepen students' fraction understanding. The results could inform college mathematics departments that technology usage in a college remedial course provided students an effective fraction learning

opportunity, instead of relying on the traditional teacher-centered procedural instruction.

It has been found that computer based instruction using the spatial contiguity principal, which utilizes graphics has a positive effect on students' mathematical learning (Clark & Mayer, 2011). ST Math is built on this principle and many school districts in the U.S. use it (Rutherford et al., 2011; Tran et al., 2012). However, there was not a study done to investigate the effectiveness of this technology in the particular content area of fractions. This reason alone made this study very unique. Although there had been studies conducted regarding the effectiveness of ST Math (Graziano, 1999; Rutherford et al., 2014; Quick, 2013; Breuninger, 2015), the results were not consistent. For instance, one of the studies did not implement a pre-test. Hence, utilizing pre and post- tests adjusted the limitation of the study and could identify the areas where students had troubles, common errors they tended to make and/or misconceptions they had in learning fractions, and the results of this study could inform that the areas of conceptual understanding they had gained by using ST Math. Most importantly, this study can demonstrate how effective ST Math could be for understanding fractional concepts and could assist those educators who are not completely sold on the technology usage during instruction yet, to consider taking advantage of ST Math in the near future.

Research Design

To answer the research questions of this study, the researcher compared two remedial mathematics courses using a quasi-experimental non randomized control group, pretest-posttest design because the study used already existing classes and the students could not be assigned into these two classes randomly. The class had two 75 minutes meetings in a week. Those who were in a treatment group used ST Math 30 minutes during every class meeting and the students in non ST Math class did not use any technology.

Since this study did not compare different instruction methods, the ST Math class played the ST Math fraction games during the regular instruction after they received a lecture that covered the required topics for the course and non ST Math class received the same instruction but did not use ST Math. The researcher taught both classes. The entire study lasted for 12 weeks including the pre and post-tests and the pre and post-interviews. Before the study started, a pre-test was administered to examine students' initial fraction understanding and at the end of the study, a post-test was administered to examine how the usage of ST Math strengthened the participants' fractional conceptual understanding statistically by comparing the scores of the pre and post-tests of both classes.

Because this study used a quasi-experiment nonrandomized control group-pretest-posttest design, there were potential threats to its internal and external validities. The possible internal validity threats were experimenter effect, diffusion, subject effects and interaction of selection and regression. The experimenter effect included unintentional effects such as bias the researcher had on the study. Diffusion could happen if participants in the ST Math class provided information of the experiment to the participants in the non ST Math class and this could then influence the dependent variable that was the post-test result.

The subject effect was also another threat since the attitude and motivation of the subjects could not be controlled and these factors would affect the study result. The interaction of selection and regression was likely to occur when the study used classes from different populations. The possible external validity threats that influenced the generalizability of this study would be the differential selection and the subject effect.

To identify the effectiveness of ST Math statistically, by comparing the pre and post-tests scores of both groups, independent two-sample t-test was conducted. To investigate how

learners' fraction concept had changed throughout the study qualitatively, the interviews also was conducted with selected six students, three from the ST Math class and three from the non ST Math class, at the completion of the pre and post- tests. Students could have a right solution, but their rationale behind their solution had been erroneous and finding out the discrepancy was an aim for the interviews.

Target Population and Sample

Participants

The target population of this study was students who were enrolled in college remedial mathematics courses in a southwestern research institute in the United States. These students were from diverse populations and had a wide range of academic background, age, ethnicity, native language and socioeconomic status (Pusser & Levin, 2009; Erickson & Robertshaw, 1982). They included young adults, new graduates of high school, adults who decided to come back to get their college degree, and those who took classes for fun (Alexander, 2013) and they lacked adequate understanding for college level mathematics courses (Green, 2012; Hagedon, Lester, & Cypers, 2010). Hence, most of them were majoring in something other than mathematics and sciences. The sample of this study was two sections of an elementary algebra course at a large southwestern research university during the 2016 fall semester. The ST Math class had fifteen students, 9 male students and 6 female students, and the non ST Math class had 21 students, 6 male and 15 female. Of these 35 students, 3 students from the ST Math class and 3 students from the non ST Math class were selected to be interviewed as part of the study. It was necessary to conduct a post hoc power analysis to examine the authenticity of this study under the given sample size (Keppel & Wickens, 2004). The selection process will be explained in a subsequent section.

Setting

The setting for this study is a large research institution in the southwestern part of the United States. The class was worth three credits and meet twice a week for 75 minutes. The course covered simple algebraic expressions and computations, fractions, decimals, linear equations and inequalities, systems and systems of linear equations. Students received instruction of the required content and had online homework every week that was aligned with the material they were going to learn in class.

The students in the ST Math class got to play the ST Math fraction games for 30 minutes every class meeting for 8 weeks and the students in the non ST Math class did not use any technology intervention. Instead, they were given a worksheet that contains fraction computation problems and also problems they have learned for the class meeting. Each week, those who are in the ST Math class were going to finish one sequence of the eight sequences of fraction games from the ST Math High School Intervention section.

To make sure they finish one sequence every week, each student who was unable to finish the sequence for the week during the class play, was to play outside of the classroom for another 1 hour. When the fraction content was covered, the students in both groups received the same instruction. While they played ST Math, they were free to discuss fraction games they did not understand so that they could get a chance to connect between the technology usage and communication in fraction understanding.

Instructor

The researcher taught both classes. He was a doctoral student at a research institution and his academic discipline was secondary mathematics education. The researcher's academic mathematics background concentration was in statistics and he had earned his Master's degree in

this concentration from the university. He had taught both elementary college level mathematics and remedial mathematics courses at this institution as a teaching assistant for 5 years and this was the researcher's third time of teaching the remedial course. Through the experience, although initially he taught these classes procedurally, he always had been zealous for his students to understand not just how but also the why behind mathematics. He aimed at teaching mathematics for understanding. Because of this, the researcher was pursuing a doctoral degree to be a better-equipped mathematics teacher educator.

Procedures

Sample Selection and Assignment

The study sample was students who were enrolled in two different sections of an elementary algebra course at the research institution during the fall 2016 semester. Since these two sections were assigned to the researcher from the department, the students were not randomly assigned to the ST Math class and the non ST Math class. The researcher assigned one section to the ST Math class that played the fraction games on ST Math and the other section to the non ST Math class that did not play the fraction games randomly. The researcher intended to select 3 students from each class based on their responses on the interview procedure consent form—below average, average, and above average— for the interview from the ST Math class and non ST Math class. However, the researcher was only able to get 3 students from each class who would agree to be interviewed. The total of 6 students from both classes ended up with 2 average students and 1 high student from the ST Math class and 2 low students and 1 above average student from the non ST Math class.

About ST Math

According to Mind Research Institute (2016), the developer of ST Math, ST Math is a game-based instructional software for K-12. It is aimed at boosting mathematics comprehension

and proficiency through visual learning. Integrated with classroom instruction, ST Math enhances mastery-based learning and mathematical understanding. The software uses interactive, graphically- rich animations that visually represent mathematical concepts to improve conceptual understanding and problem-solving skills.

Whether in the classroom, computer lab, or at home, learning always occurs with ST Math and when ST Math is utilized in the classroom, the games help students make connections between the visual representations from ST Math and symbolic representations found in the classroom instruction. With the touch functions, students experience greater level of interactions (Mind Research Institute, 2016).

Below are a few examples of ST Math game in fraction learning (Andrew, 2014). In figure 4, the penguin named JiJi sits on top of 2 whole circles at the position marked 0 on the number line. The student is to move the balloon apparatus to the point where JiJi will end, on the number line. In this figure, the student needs to click on the 2. When the student clicks 2, JiJi will move the distance represented by the number of circles. Then JiJi will fly away on the balloons. This level of game helps a learner to get comfortable with the idea of whole circles to match units on the number line.

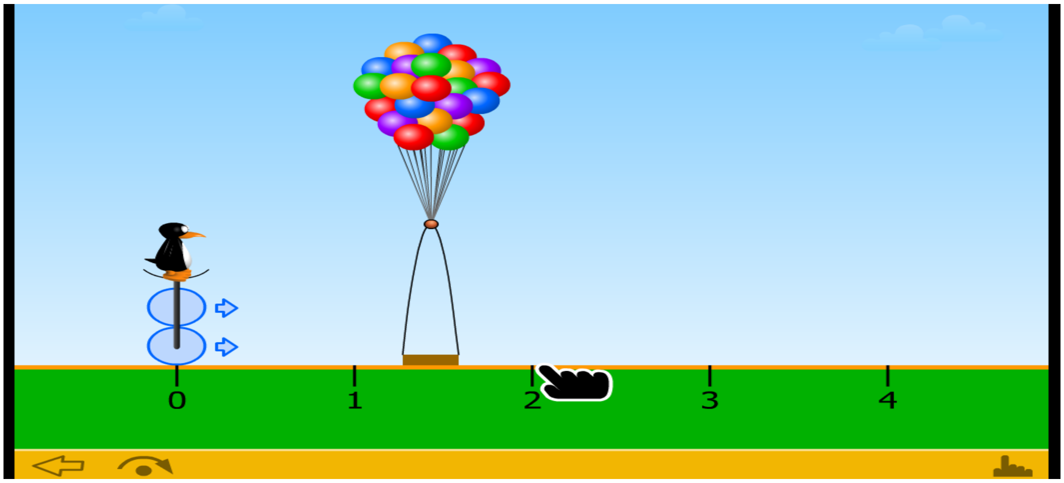


Figure 4. Two Circles and Distance on the Number Line

Once students learn fundamental concepts from a problem such as above, they will face more challenging problems that require them to incorporate fractions of a circle such as Figure 4.

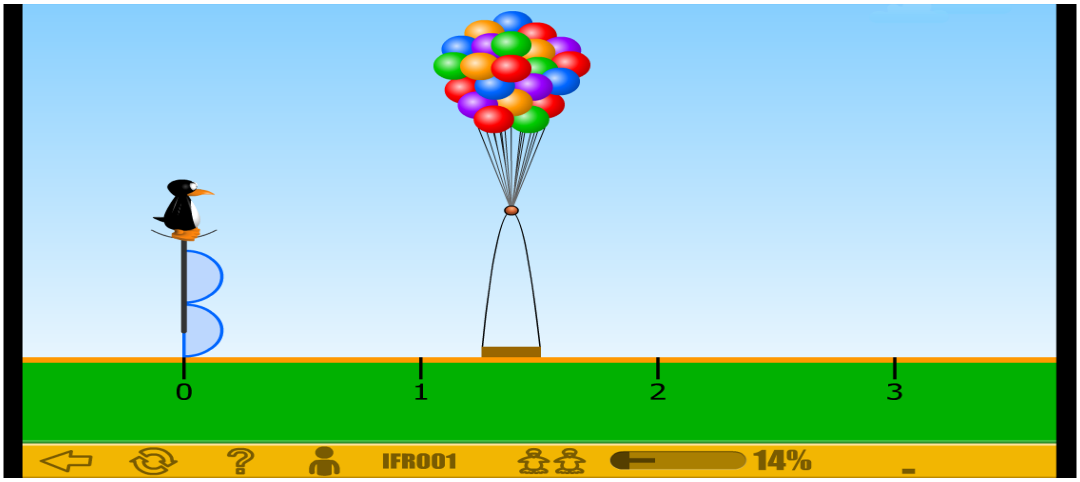


Figure 5. Half-Circles and Distance on the Number Line

In this problem (Figure 5), JiJi is to move 1 unit since JiJi is on two half-circles. Learners need to understand that JiJi will move only one half of single unit with the one half circle. This is a great learning opportunity to understand pieces of a whole and how the pieces correspond to actual numbers.

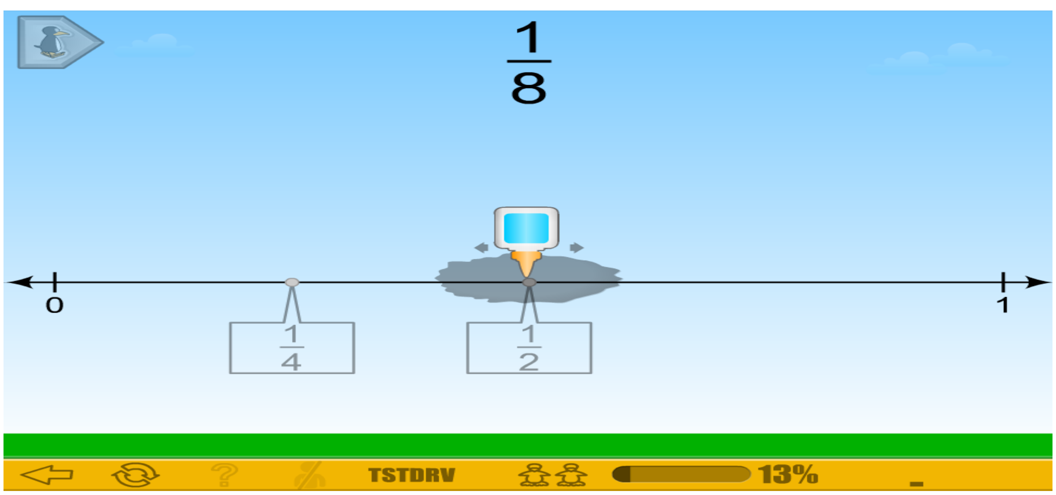


Figure 6. A Fraction and Its Magnitude

In Figure 6, this particular game helps students to understand the density concept and the magnitude concept of fraction visually because they can see there are infinitely many fractions between 0 and 1. They also can explore that the magnitude of fractions depends on the denominator and realize that a larger denominator does not mean having a larger value in fractions, which is originated from the natural number bias. Similarly, ST Math provides an opportunity to explore a mixed number, which is one of the most difficult fraction content, as well (Figure 7).

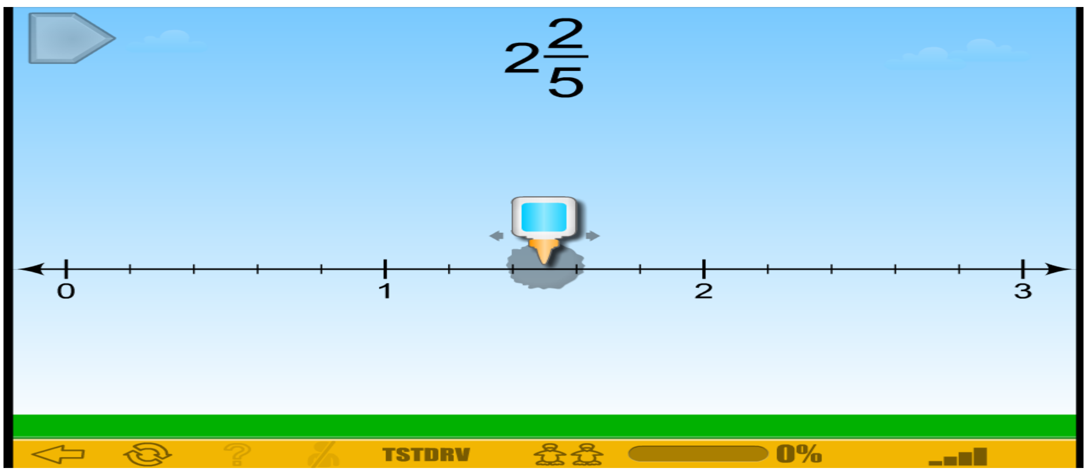


Figure 7. A Mixed Number and Its Magnitude

In Figure 8, the problem does not use a fraction of a circle any more. It only provides the fractions and the learner needs to add the fractions by himself/herself and move the balloon apparatus to the right place on the number line. At this point, students are connecting visual representations of fractions to the symbolic (numerical) representation of fractions.

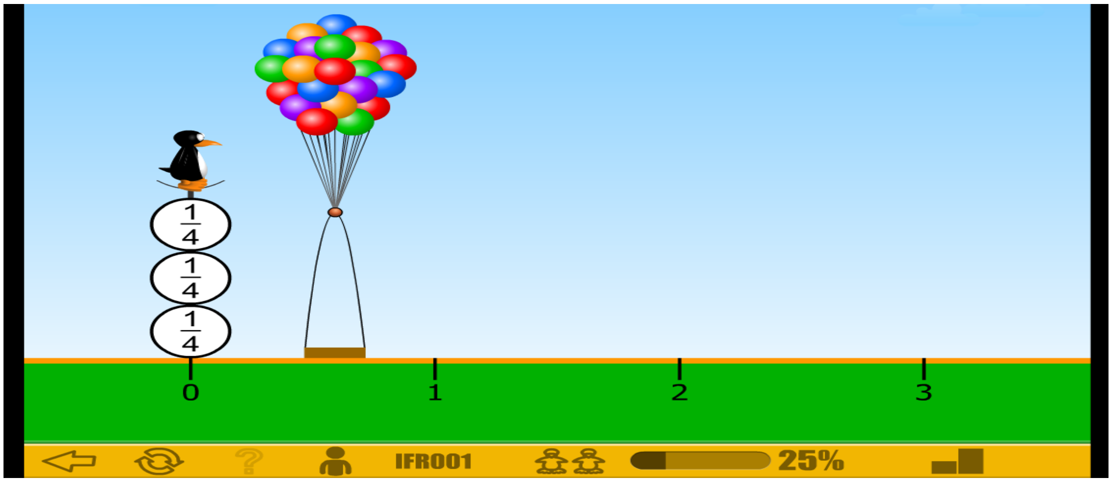


Figure 8. Addition of Fraction Circles and Distance on the Number Line

In fraction addition, ST Math guides learners to explore equivalent fractions. Once the learners understand the concept of equivalent fractions, it helps them to connect the concept of equivalent fractions to finding a common denominator (Figure 9). Through understanding the connection by exploring ST Math game, they can understand why it is necessary to have a common denominator in addition and subtraction of fractions.

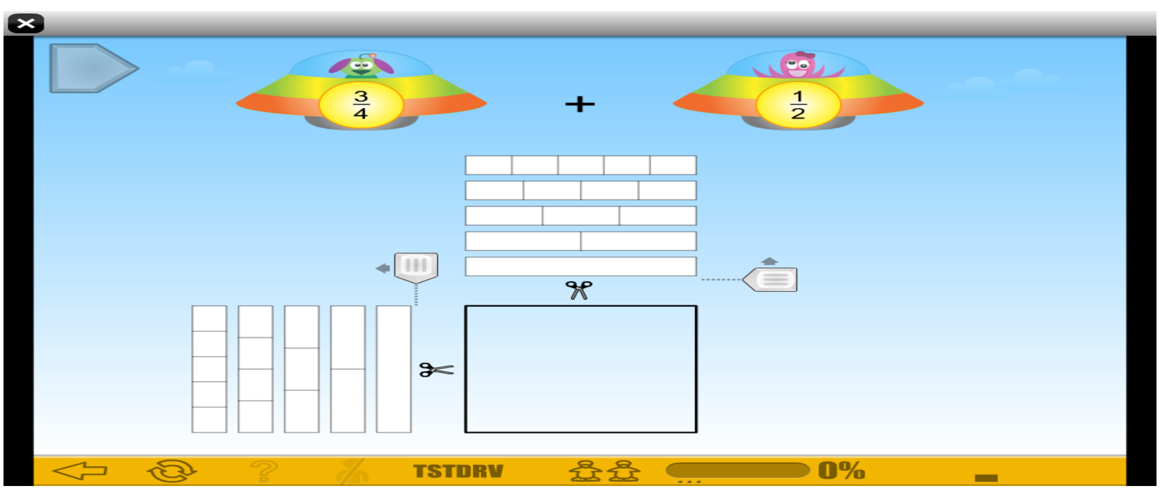


Figure 9. Adding Fractions

In St Math, learning fractions is focused not on procedures but conceptual understanding. Figure 10 shows the setup of one of the games in the app. By completing this game, students have to understand how 3/5 of the peanut can be fed equally to the two divided body parts of the

elephant (Figure 10). It truly makes them think and strengthen conceptual understanding of fractional division. Moreover, by gaining the understanding, the learners' natural number bias that is division always leads to a smaller answer will be challenged and can be diminished.

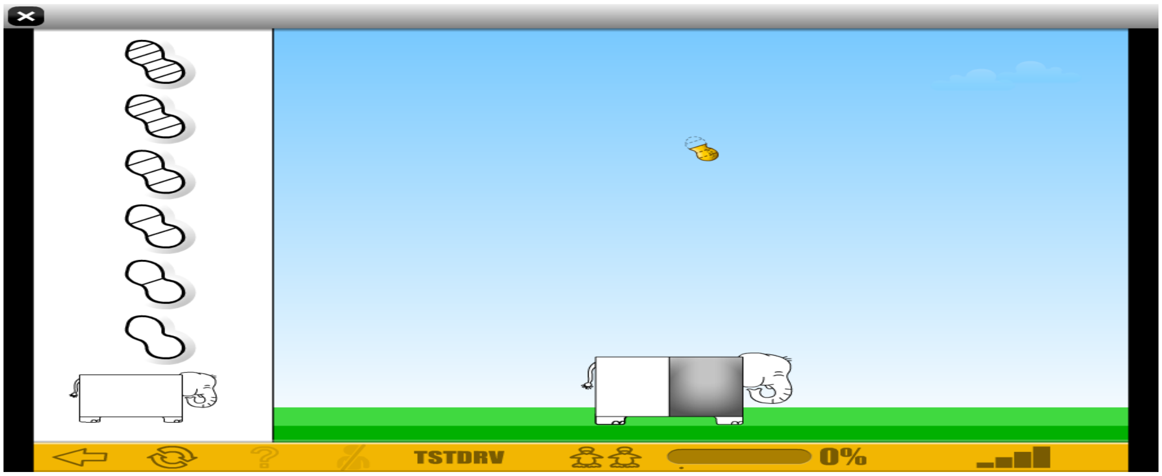


Figure 10. Fraction Division

As described above, ST Math provides a great learning opportunity to strengthen conceptual understanding of fractions and to reduce the natural number bias students have.

Data Collection

Date collection took place during the fall 2016 semester in two different sections of Math 095 from the last week of August until the second week of November (Table 1). Students met 75 minutes twice a week. During the first week of the class meeting, the pre-test was administered to examine their initial fraction understanding quantitatively. The selected 6 students had interviews with the researcher regarding the selected pre-test problems (Q1, Q2, Q5:(3), Q7:(2), Q8:(1), Q8:(4), Application Problem 2, Application Problem 5 and Application Problem 7) so that the researcher could identify their misconceptions of the fraction concepts deeper during the second week.

Table 1

Time Frame of Data Collection

Data types	Dates	Aim for data collection
Pre-Test	August 29/ 2016	To examine students' fraction understanding.
Pre-interview	August 31-September 7,/2016	To investigate students' ability to explain reasoning for their answers of the pre-test problems verbally.
ST Math Play and classroom discussion	September 12- November 3/2016	The data of classroom discussion will be collected to identify students' development of fraction concepts while playing ST Math.
Post-test	November 7/2016	To examine students' fraction understanding after the intervention.
Post-interview	November 14,16/2016	To investigate if students can make connections between fraction concepts they have learned on ST Math and the problems on the post-test and explain the connections verbally.

Those who were in the ST Math class got to play the ST Math fraction games from the software's High School Interventions section (Table 2) for 30 minutes every class meeting and outside of the classroom for 8 weeks. Since there were eight sequences of fraction games to be played, the students were to finish each sequence of the game each week. To make sure they were on the schedule, those who were not able to finish the sequence for the week were to play the game outside of the classroom for another 1 hour. The researcher kept track of their progress through the ST Math teacher view section so that he could collect the accurate data regarding usage of the software. At the completion of the 8 weeks, both classes took the post-test and the 6

students from the ST Math class and the non ST Math class had the final interviews to examine their growth in understanding fractions on the same selected problems as the pre-test interview

Table 2

Fraction Content in ST Math

Game 1	Visual Fraction Concepts
Game 2	Fractions on the Number Line
Game 3	Comparing and Equivalent Fractions
Game 4	Fraction Addition and Subtraction
Game 5	Fraction Multiplication
Game 6	Unlike Denominator Concepts and Strategies
Game 7	Unlike Denominator Addition and Subtraction
Game 8	Fraction Division

Content Validity of the Pre and Post- Tests Problems

The problems on the pre and post-tests were selected from different sources to investigate both students' procedural knowledge and conceptual understanding of fractions (Appendix C). Problem 1 was to examine students' concept of fraction magnitude (Van Hoof et al., 2015) and also was to examine the concept of the distance to 1 by comparing the magnitude of fractions, the understanding of the magnitude, symbolic representation and its size (Meert, Gregoire, & Noel, 2010). Problem 2 was to test students' natural number bias in the dense structure (Van Hoof et al., 2015). Problem 3 was used in the study of Clarke, Sullivan and McDonough regarding the Early Numeracy Project (Clarke, Sullivan, & McDonough, 2002) and the study of Biesenthal (2006) to examine the participants' equivalent fractions concept. Problem 4 regarding comparing fractions was used in Mack's (1990) and Biesenthal's studies (2006) and to investigate her students' initial fraction knowledge regarding the fraction magnitude concept that could be influenced by the natural number bias. Problems 5 and 6, which were fraction additions and subtractions with unlike denominators, were used to examine the students' initial fraction knowledge that could be influenced by the natural number thinking such as adding and

subtracting without having a common denominator (Mack, 1990; Biesenthal, 2006).

Similarly, the result from the National Assessment of Educational Progress (Carpenter et al., 1980) claims that students tend to calculate as 12/13 + 7/8 = 19/21. Problem 7, a true or false question, was used in the study of Van Hoof and her colleagues (Van Hoof et al., 2014) to investigate the natural number of operation effect.

The study of Siegler, Fazio, Bailey, and Zhou (2013) states that students have misunderstandings in fraction arithmetic operations and showed mistakes they make such as 2/5 * 3/5 = 6/5. This was one of the reasons that the multiplication questions in problem 8 were used. The division problems were used to assess pre-service teachers' common content knowledge (Li & Kulum, 2008). Another reason for these multiplication and division problems was that during the interview, they were used to examine students' natural number biases in operations, which are multiplication makes bigger and division makes smaller (Christou, 2015).

For the Application problems, problems 1, 2, and 3 came from Biesenthal (2006). The researcher used these problems for investigating students' conceptual understanding of fractions. The purpose of these problems was to see how students could solve problems when they cannot rely on procedures. Problem 4 was modified based on the Mack's study (2001) regarding understanding the concept of fraction multiplication. In her study, this type of problem was used to assist students to develop deeper understanding of fraction multiplication from the perspective of equal sharing and partitioning. Application problem 5 was selected from the Philipps' study (2000) of division fraction concepts and this problem aims at the meaning of fractions divisions and concept of the unit changes. Problem 6 was created based on the study of Ball (1990), which investigated the pre-service teachers' fraction division. In the study, the participants were asked to choose the story problem that represented ¼ ÷ ½. Ervin (2015) created a similar task to test

pre-service teachers' understanding of a model in fraction division. Problem 7 was from the study of Marchionda (2006) and was used to construct understanding of why a common denominator is necessary for fraction addition and subtraction.

Pre and Post- Tests

To determine what fraction knowledge the remedial mathematics students had, the pre-test and the post-test was administered on August 29 and November 7. The problems were non-routine simple procedural problems and consisted of real life related, fraction symbols (fraction computation), and pictures. These problems were aligned with the Lesh Translation Model so that the researcher was able to discern the students' conceptual understanding of fractions. At the end of this study, the post-test was given. The problems on the post-test were the same problems as the pre-test so that the researcher could know students' development from the pre-test stage to the post-test stage numerically.

Interviews

Interviews were conducted with the 6 students who were in the ST Math class and the non ST Math class at the completion of the pre and post- tests. For the ST Math class, 2 were average and 1 was high on their pre-test score and for the non ST Math class, 2 were low and 1 was high. Since these 6 interviewed students were the only ones who agreed to be interviewed, the selection process did not follow a random selection. In the pre-interviews, the students explained their reasoning for their answers of the selected problems on the pre-test so that the researcher could examine their conceptual understanding and how their fraction understanding was influenced by the whole number biased misconceptions. Although learners could have a right answer for a mathematics problem, their reasoning for the answer could be incoherent. Hence, the purpose of the interviews was to further elaborate on their understanding. For the

post-interviews, it was investigated how their conceptual understanding had been strengthened and how much the whole number bias misconceptions had also been reduced, compared to the pre-interview, after playing ST Math. All interviews were recorded and transcribed.

Instruction

Since the researcher was an instructor for both sections, he used the most comfortable method for him to teach fractions with a textbook used for the course. Although the researcher focused on strengthening conceptual understanding, it was a lecture centered teaching style. In the fraction lessons, the instructor introduced conceptual understanding of fractions but the practice problems given to the learners focused more on fraction calculation procedures. For instance, when adding two fractions, such as 1/2 + 1/5, he showed the two fractions by drawing pictures of two pieces of pizzas representing each fraction and told the students that these could not be simply added without having a common denominator because the sizes of the two pieces were different, showing 2/7 is not equal to 7/10 on the pictures. However, he did not connect the pictorial representation with the symbolic representation of the fractions in a deeper manner.

After introducing the reason of needing a common denominator, students were introduced the procedural way of finding it, which was the least common multiple of both numbers. Similarly, whenever he saws students' misconceptions caused by the natural number bias, he did not address why the rules of natural numbers did not apply to fractions conceptually. Instead, he told his students not to follow the natural number rules because fractions were not natural numbers. The researcher was observed and evaluated by his colleague four times during the study based on The Reformed Teaching Observation Protocol (RTOP) (Appendix A) so that the consistency of the instructions could be analyzed more precisely. The reason for implementing the RTOP in this study was to make sure that the instructions for both the ST Math

class and the non ST Math class were consistent. Also, the protocol was used to explain the teaching conditions under which the study was conducted. Studies have implemented the RTOP to investigate instructional practices between courses and monitor instructional differences in science and mathematics (Amrein-Beardsley, & Popp, 2012; Lund et al., 2015; Sawada et al., 2002). The only difference between the ST Math class and the non ST Math class was that those who were in the ST Math class had the set amount of time to play the ST Math fraction games after instruction and the others did not. Instead of playing with ST Math, they were be given fraction computation practice problems and problems aligned with each mathematical content covered for the day.

Research Questions and Hypotheses

1. *To what extent do college remedial mathematics students possess whole number misconceptions? In addition, to what extent can each class, ST Math and non-ST Math, express fractions and their operations, pictorially, realistically, symbolically and verbally?*
2. *To what extent does the usage of ST Math eliminate the natural number misconceptions and deepen college remedial mathematics students' conceptual understanding in learning fractions compared to a non-ST Math class of remedial mathematics students taught traditionally without technology?*

The researcher hypothesized at the pre-test stage, that there would be no statistical significant difference in the fractional understanding of the students in both classes. However, after the intervention, the ST Math students would show statistical significance in their understanding on the assessment test. Similarly, through the pre-test interviews, the selected 6 students from both the ST Math class and non ST Math class would not explain reasons for their

answers verbally and show their fraction concept was biased by the natural number concepts. This would hold for the non ST Math class students in the post-interview. In the post-test interview, the ST Math class students would connect the fraction concepts they had gained on ST Math play with the problems on the assessment and would be able to explain the reasons for their answers verbally more effectively. They would be able to display their understanding pictorially and would demonstrate the misconceptions caused by the natural number bias were diminished.

Data Analysis

In order to address the research questions, the data was analyzed qualitatively and quantitatively.

Organizing the Interview Data

Because the aim of this study ascertains how the iPad app assisted the learners to strengthen their fractional concepts, students were to express the understanding they had gained through playing with the app verbally.

In the interview questions (Appendix B), the students explained the reasons for their answers for the selected problems on the pre and the post-tests so that the researcher could examine that their fraction understanding had been strengthened and their misconceptions had been diminished. The pre-test interviews to the post-test interviews were compared by connecting the fractional understanding students gained on ST Math with real life situations, symbolic representations, and pictorial representations and also making connections among these categories, based on the Lesh Translation Model. The interviews were videotaped and transcribed and analyzed. The researcher developed the coding scheme based on the Lesh Translation Model (Table 3).

Table 3

Fraction Understanding Code

Code	Description
Q1: Order +	In ordering ½,1/3, ¼, he/she orders correctly and can explain why the order is correct
Q1: Order -	In ordering ½, 1/3, ¼, he/she orders based on the magnitude of the denominator such as ½ <1/3<1/4
Q2:Density +	Between 1/3 and ¼, he/she can explain they there are infinitely many numbers between the fractions such as 1/5 is between 1/3 and 1/4 . Hence 1/6 is between 1/5 and ¼.
Q2:Density -	Between 1/3 and ¼, he/she answers there is only one number exists such as 1/5
Q5(3):Add/Sub+	In 1/15 + 1/12, he/she knows the procedure and can explain why the common denominator is necessary
Q5(3): Add/Sub 0	In 1/15 +1/12, he/she knows the procedure but cannot explain why the common denominator is necessary
Q5(3): Add/Sub -	In 1/15 +1/12, he/she does neither know the procedure nor know the reason why the necessity of a common denominator
Q7(2): Mag +	In 3 < 3/x, he/she can justify the inequality can be true when x is not a natural number
Q7(2): Mag -	In 3 < 3/x, he/she assumes that x is a natural number and think that the inequality is wrong
Q8(1): Multi +	In 2/5 * 3/5, he/she can execute the correct procedure and can explain why the multiplication can be smaller than 3/5
Q8(1): Multi 0	In 2/5 *3/5, he/she can execute the correct procedure but cannot explain why the multiplication can be smaller than 3/5
Q8(1): Multi -	In 2/5*3/5, he/she cannot execute the procedure and thinks multiplication always get larger value then 3/5
Q8(4): Div +	In 4/5 ÷3/7, he/she can execute the procedure and explain why the division gets larger than 3/7
Q8(4): Div 0	In 4/5 ÷ 3/7, he/she can execute the procedure but cannot explain why the division gets larger than 3/7
Q8(4): Div -	In 4/5 ÷ 3/7, he/she cannot execute the procedure nor explain why the division gets larger
Application P2: Pic +	In Application problem 2, he/she can draw the representation pictorially and explain why this represents a multiplication
Application P2: Pic 0	In Application problem 2, he/she can draw the representation but cannot explain why this represents a multiplication
Application P2: Pic -	In Application problem 2, he/she can do neither
Application P5: Story Div +	In Application problem 5, he/she can come up with a problem that represents 1/3 ÷ 1/6, explain why that is a right representation and can connect the concept to why a fraction division can be larger than its dividend, which is 1/6 here.
Application P5:	In Application problem 5, he/she can come up with a problem that

Story Div 0	represents 1/3 ÷ 1/6, explain why that is a right representation but cannot connect the concept to why a fraction division can be larger than its dividend, which is 1/6 here.
Application P5: Story Div -	In Application problem 5, he/she cannot come up with a problem that represents 1/3 ÷ 1/6, cannot explain why that is a right representation and cannot connect the concept to why a fraction division can be larger than its dividend, which is 1/6 here.
Application 7: Story Add +	In Application problem 7, he/she can explain the difference between Kevin and James views verbally and pictorially.
Application 7: Story Add 0	In Application problem 7, he/she can explain the view of Kevin is wrong based on his/her procedural knowledge but cannot explain why Kevin came up with the view.
Application 7: Story Add -	In Application Problem 7, he/she cannot explain at all.

The interviews were evaluated by both the researcher and another doctoral student in mathematics education. The Cohen's Kappa coefficient inter-rater agreement was 0.96 and was within an acceptable range (Fleiss, 1981; Landis & Koch, 1977). Once coding differences were calculated, the raters came to agreement on the discrepancies so that full agreement was reached.

Organizing the Pre and Post- Tests

For both pre and post-tests (Appendix C), the computational section was scored as 1 for a correct answer and 0 for an incorrect answer and the other problem were scored as 0 for incorrect, ½ for understanding partially and 1 for understanding completely. When the tests were graded, the names of the students were hidden from the researcher to protect against bias. These quantitative measures were utilized to answer the research questions statistically and the scores of both sections were summed up to measure up overall understanding. To prove statistical significance of the ST Math usage, independent two-sample t-test was conducted. To test the statistical significance of the results of the post-test scores between the ST Math class and the non ST Math class, independent two-sample t-test was conducted. To reduce the type I error, the confidence level was set 0.05. Based on the result, the study questions were to be answered statistically.

Expected Findings

In the statistical test, the researcher hypothesized that at the pre-test, there would be no significant difference in the score when their fraction understanding between the ST Math class and the non ST Math class was compared. For the post-test, it would be expected that the ST Math class students would score higher than the non ST Math class students. Within the ST Math class, they would score higher on the post-test than on the pre-test.

Similarly, the researcher expected that the interviews with the selected 6 students from both groups would not be able to explain their rationale for their solutions verbally and show very little fractional conceptual understanding and show misconceptions caused by the natural number bias. On the contrary to the pre-test result, the 3 students from the ST Math class would be able to explain their rationale for the answers pictorially, realistically and verbally and would be able to connect the fractional concepts they gained on ST Math with the problems on the tests and the natural number misconceptions would be decreased, comparing to at the point of the post-test. On the contrary, the 3 students from the non ST Math class would display a little improvement in their fraction understanding because the instruction covered fractions during the study, compared to the pre-test interview. However, the improvement could be more on the procedural aspects of fractions and they would still have the natural number bias in their reasoning and have difficulties with explaining their rationale efficiently by using pictures and words. Therefore, the usage of the iPad app would deepen their fractional understanding and at the same time weakened the natural number bias.

CHAPTER FOUR

Data Analysis and Results

Introduction

This study was conducted to examine the influence of the ST Math fraction games on college remedial mathematics students to determine if the ST Math games diminish the natural number bias in fraction learning and strengthen fraction conceptual understanding and knowledge. Both quantitative and qualitative methods were used to collect and analyze data.

The chapter represents the results of the data analysis and consists of three main sections along with the research questions to be answered with this study. The first section of this chapter presents the initial assessment results; the pre-test and pre-interviews to examine the ST Math students and non ST Math students' natural number bias, knowledge and understanding. The second section discusses the post assessment and post interviews to examine how the natural number way of thinking of the ST Math students diminished after using ST Math, which is compared to the Non ST Math students based on their pre and post interview responses. The third section describes how the fraction conceptual understanding of the ST Math students has changed from the pre to the post interviews compared to the non ST Math students' fraction conceptual understanding.

Research Question 1

To what extent do college remedial mathematics students possess whole number misconceptions? In addition, to what extent can each class, ST Math and non-ST Math, express fractions and their operations, pictorially, realistically, symbolically and verbally?

This section would present the extent to which students held the natural number bias, as well as the initial fraction conceptual understanding of students in the ST Math class and non ST Math class from the fraction assessment pre-test. The students' pre assessment tests scores and

post assessment tests (APPENDIX C) scores were analyzed quantitatively to partially answer the questions above. The statistical results of the pre assessment test for both classes will be discussed. Then a summary of the findings from the pre-test result will be presented. Subsequently, the pre-interview responses will be examined.

In general from both the pre-test and the pre-interview responses, the students in the ST Math class and non ST Math class strongly possessed the natural number bias in all three aspects: fraction magnitude, density and operation. Also, the students' fraction conceptual understanding was limited and many students in both groups could not execute the fraction computation procedures accurately.

Participants

The sample of this study was students who were enrolled in two different sections of Elementary Algebra (Math 095) at a large Southwestern research institution during the fall 2016 semester. These students represented diverse populations and had a wide range of academic backgrounds, age, ethnicity, native language and socioeconomic status. The quantitative data collection of the fraction pre-test occurred in the second week of the semester and aimed at both students' initial fraction procedural knowledge and conceptual understanding of fractions, there were twenty-one students from the non ST Math class who took the fraction pre-test. This class received traditional class instruction covering the required materials including the fraction concepts for the course, but did not play the ST Math fraction games. Fifteen students in the ST Math class took part in the pre-test. These students received the same instruction as the non-ST Math students but played ST Math fraction games during and outside of class. A summary of the participants' classification is listed in Table 4. The rate of underclassmen for the non ST Math group was over 90% and 80% for the ST Math group. Math 095 is the most fundamental

remedial mathematics course offered by this institution and the students who were enrolled in this course did not meet the prerequisite test score, which could be a score from the SAT, ACT, or the university's mathematics placement test to take the required college level mathematics courses.

Table 4

Participant's Classification

	non ST Math	ST Math
Freshman	17	9
Sophomore	2	3
Junior	1	1
Senior	1	1
Non-degree seeking	0	1

Comparison of the Pedagogies in ST Math Class and non ST Math Class

To evaluate the pedagogical consistency for both groups, a doctoral student in the area of pure mathematics, observed the lectures four times; twice for each group in the first two weeks of the semester. The observer used the RTOP (Reformed Teaching Observation Protocol: Appendix A). During these four observations, the same mathematical content was covered for both classes. The total and average scores of the RTOP for these observations was calculated and compared to show the consistency of the pedagogies in both classes (Table 5). The RTOP has 25 categories to evaluate pedagogy and each category is worth 4 points with a total possible of 100 points. The total points for the ST Math class was 154 points and the average was 77 points. The total points for the non ST Math class was 162 and the average was 82 points. Hence the researcher's pedagogies for both classes showed consistency.

Table 5

RTOP Scores for the ST Math and the Non ST Math Classes

	ST Math	non ST Math
Observation 1	70/100	82/100
Observation 2	84/100	80/100
Total Points	154/200	162/200
Average	77/100	81/100

Statistical Result Fraction Pre-Test

At the second week of the semester, the participants took the fraction pre-test, which consisted of two parts: a procedural section with questions also targeted at the natural number bias and a conceptual understanding part. The pre-assessment has a total of 27 questions and the students were given 40 minutes to complete it. However, everyone was finished in 20 to 25 minutes. The first part had a total of 20 questions and the second part of the assessment focused on the participants' fraction conceptual understanding consisted of 7 application problems. These application problems were intended to examine how strong students' conceptual fraction understanding was by solving real life problems pictorially, realistically, symbolically and through written language. All these questions on the pre-test were selected from previous studies focused on students' fraction conceptual understanding from middle school students to college remedial mathematics students. The descriptive statistics result is shown in Table 6.

Two independent samples t-test were conducted to investigate if there were statistically significant mean differences in the initial fraction conceptual understanding and procedural knowledge between the Non ST Math participants and the ST Math participants. To check that the sample was normally distributed, the nonparametric Kolmogorov Smirnov test was conducted on the dependent variable X, the fraction assessment pre-test score, and the percentage of X for the non ST Math class was, $D(21) = .161$, $p = .165 > .05$. Hence, the test result indicated that the sample was normally distributed. The percentage of X for the ST Math

class was, D (15) = .193, p = .140 > .05; so it can be concluded that the data was normally distributed. Similarly, for equal variances assumption, the Leven's test was used. F (14, 21) = 2.103, p = .156 > .05 and this showed equal variances for both samples.

Two samples t-test indicated that scores of the pre-tests for both classes were not significantly different. Non ST Math (M = 11.095, SD = 5.983) and ST Math (M = 10.633, SD = 5.0124), t (34) = -.244, p = .809 > .05. The power under the sample size was 0.057. Based on the statistical test result, at the fraction pre-test stage, the non ST Math class and ST Math class had equivalent fractional knowledge.

Table 6

Comparison of Pre-Fraction Assessment Descriptive Statistics

ST Math					non ST Math				
n	M	Min	Max	SD	n	M	Min	Max	SD
15	10.633	4.0	23.0	5.0124	21	11.0952	1.0	21.0	5.9825

Initial Fraction Procedural Knowledge and the Natural Number Bias

Problems 1 through 8 were aimed at students' fraction procedural knowledge and the existence of the natural number bias. At the pre-test stage, as a whole, the students from both groups exhibited the natural number bias in their fraction knowledge in all three areas: fraction magnitude, fraction density, and fraction computation. Likewise, they exhibited weak procedural knowledge. For instance, more than 50% of the students who were in the both classes displayed fragile fraction computation procedures for fraction multiplication/division problems. When the students computed problem 8:(1), 2/5 * 3/5, they got an answer of 10/15 since they applied the fraction division computation procedure. Since they did not have conceptual understanding of fraction multiplication and division, they did not know the correct procedure, why it works, or how to determine if their answer was reasonable. Although they memorized the procedure as "change flip"; because of the lack of understanding, they were confused with when to use the

procedure. Similarly, in the division problem 8:(3), 2/5 ÷ 3, one student in the ST Math class conducted a cross multiplication as 2* 1 =3 and 3 *5 = 15, but this student did not know which one was going to be the denominator and which one was going to be the numerator. This also indicates procedural steps without conceptual understanding. Because of the fragile conceptual understanding, the procedure was not retained correctly. Table 7 breaks down the distribution of students' answer for each problem. The following sections describe the purpose of each problem and the precise details of the results of each problem.

Table 7

Initial Fraction Knowledge Result of the ST Math vs. the Non ST Math Classes

ST Math				non ST Math			
Problem	Correct	Incorrect	No Answer	Problem	Correct	Incorrect	No Answer
1	11	4	0	1	15	6	0
2	0	9	6	2	0	8	13
3:(1)	9	3	3	3:(1)	11	2	8
3:(2)	10	2	3	3:(2)	14	1	6
3:(3)	8	0	7	3:(3)	12	4	5
4:(1)	7	6	2	4:(1)	10	8	3
4:(2)	12	1	2	4:(2)	17	1	3
4:(3)	7	6	2	4:(3)	11	7	3
5:(1)	10	4	1	5:(1)	18	3	0
5:(2)	4	8	3	5:(2)	10	9	3
5:(3)	3	8	4	5:(3)	9	9	3
6:(1)	13	1	1	6:(1)	20	0	1
6:(2)	5	6	4	6:(2)	14	5	2
6:(3)	5	6	4	6:(3)	14	5	2
7:(1)	9	3	3	7:(1)	11	4	6
7:(2)	0	13	2	7:(2)	0	14	7
8:(1)	7	5	3	8:(1)	12	5	4
8:(2)	8	1	6	8:(2)	9	5	7
8:(3)	4	2	9	8:(3)	7	3	11
8:(4)	3	2	10	8:(4)	7	4	10

Problem 1: Fraction magnitude and its representation.

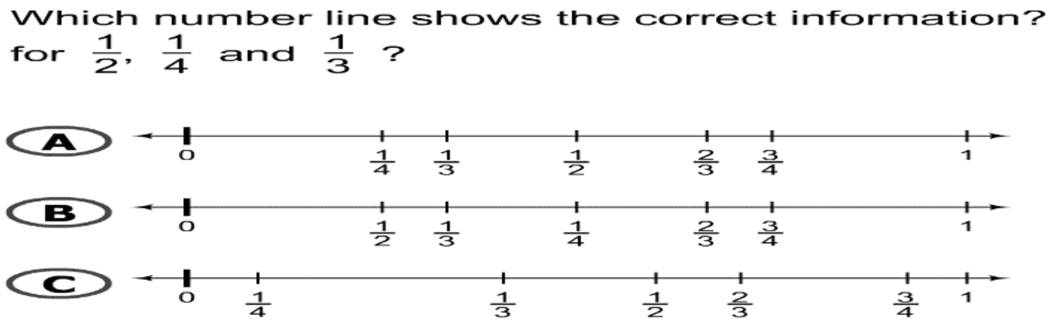


Figure 11. Problem 1: Display of a number line and a fraction magnitude

This problem (Figure 11) examined the existence of the natural number bias and the students' understanding of fraction representation and its magnitude. Overall the participants performed well for both classes but there was an existence of the natural number bias in the fraction magnitude concept and its representation. Five students from the non ST Math group and one student from the ST Math group selected answer choice B, which orders the fraction as 1/2, 1/3 and 1/4 on the number line. These students had the natural number bias in the fraction magnitude concept because they thought that the larger the denominator is, the larger the fraction magnitude. Only one student from the non ST Math class and three from the ST Math class selected the choice C, which although the order of each fraction is correct, the distance of each fraction is not represented correctly. This indicates that these four students did not have the natural number bias in the magnitude concept, but they did not possess a strong understanding in the representation and its magnitude. At the pre-test stage, while the ST Math group had only one student who had the natural number bias, compared to the non ST Math group which had five, the ST Math group had more students who had the weak understanding of the fraction magnitude and its representation, compared to the non ST Math group.

Problem 2: Fraction density.

2) How many fractions are there between 1/3 and ¼?

The nature of this problem aimed at the existence of the natural number bias. No participants from either class were able to answer this problem correctly. The majority of the students answered either none or that one fraction existed between ¼ and 1/3. This clearly indicated that they had the natural number bias in fraction density and they thought that a fraction is discrete as is a natural number. For the students in the non ST Math class, thirteen did not even answer, and the ST Math group result followed the same result. Six students did not answer and nine stated incorrectly. Among these nine students, seven of them stated either zero fractions or one fraction existed between ¼ and 1/3 and two stated that there were either two or four fractions in the interval.

Problem 3: Equivalent fractions.

3) Find equivalent fractions.

(1) 3/10 = /20   (2) 4/8 = 8/   (3) 3/5 =

This measured the students' equivalent fraction understanding. For the non ST Math group, they tended to come up with their own rules to find the equivalent fractions. For instance, 3:(1) to find an equivalent fraction of 3/10, they performed it as 3/10 = (10+3)/(10+10) = 13/20, for 3:(2) to find an equivalent fraction of 4/8, they did 4/8 = (4+4)/(8+4) = 8/12 and for 3:(3), to find an equivalent fraction of 3/5, they came up with 3/5 = 5/3 and 3/5 = (3+3)/(5+3) = 6/8. They possessed the natural number way of thinking in fraction magnitude and did not consider a fraction as one number. Instead, they thought of a numerator and a denominator as a different entity.

The students in the ST Math group showed the natural number way of thinking in this concept as well although they did not use the same procedures as the students in the non ST Math class. For instance, in 3:(1), they did 3/10 = 3/20, for 3:(2), they did 4/8 = 8/4 and for 3:(3),

they stated it as 3/5 = 5/3.

Specifically, for the non ST Math class, eleven got a right equivalent fraction for 3:(1), two got incorrect and eight did not try. For 3:(2), fourteen got a right equivalent fraction for 3-(2), one was incorrect, and six did not try. For 3:(3), twelve got a right equivalent fraction, four were incorrect, and five did not try. For the ST Math group, nine got a correct equivalent fraction for 3:(1), six got this incorrect, and three did not try. For 3:(2), ten gave a correct equivalent fraction, two got this incorrect, and three did not try. For 3:(3), eight were able to calculate an equivalent fraction, two got it incorrect, and five did not try.

Problem 4: Fraction Magnitude.

4) ***Circle a larger fraction.***

(1) 4/9, 1/5 (2) ½, 1/12 (3) 3/8, 6/11.

The existence of the natural number bias in fraction magnitude was tested through this problem, which had students compare fractions. Problem 4:(1) was to compare: 4/9 and 1/5 and 4:(3) was to compare 3/8 and 6/11. These problems examined the bias that the value of the fraction increases as the denominator alone decreases. Problem 4:(2) was to compare ½ and 1/12. This investigated the natural number bias that the value of fraction increases as the denominator increases. The results certainly showed that there existed the natural number way of thinking, especially the bias tested by 4:(1) and 4:(3). Many students from both classes answered that 1/5 > 4/9 in 4:(1) and 3/8 > 6/11 in 4:(3). However, in 4:(2) only a few stated that ½ < 1/12. Specifically, in the non ST Math group, ten students stated that 4/9 > 1/5, eight students got it wrong, and three did not try; seventeen students stated that ½ > 1/12, one got it wrong, and three did not try; elven students stated 3/8 < 6/11, seven got it wrong, and three did not try. For the ST Math class, seven stated that 4/9 > 1/5, and eight got it wrong; twelve stated that ½ > 1/12 and

three got it wrong; seven stated 3/8 < 6/11, and eight got it wrong.

Problem 5: Addition.

5) Add the fractions.

(1) 3/5 + 2/5 (2) 7/12 + ½ (3) 1/15 + 1/12.

In the fraction addition problems, the students showed the natural number bias in fraction magnitude although they did much better on the problem with a common denominator 5:(1),3/5 + 2/5. They did not look at a fraction as one number, instead, they tended to look at each natural number, a numerator and a denominator separately and added or subtracted without having a common denominator. For 5:(1), 3/5 + 2/5, they computed 3/5 + 2/5 = 5/10, for 5:(2), 7/12 + ½ = 8/14 and for 5:(3), 1/15 + 1/12 = 2/27. Since they did not possess the understanding of fraction magnitude nor considered a fraction as one number, they did not think that the same base was necessary in adding or subtracting fractions. Instead, they added across. The other common mistake was getting a common denominator but not correctly changing the numerator: 7/12 + ½ = 7/24 + 1/24 = 8/24 and 1/15 + 1/12 = 1/60 + 1/60 = 2/60. These also indicated lack of understanding of an equivalent fraction concept.

For the non ST Math group, eighteen got the computation problem 5:(1), 3/5 + 2/5 right and three got it incorrect by adding both the numerators and the denominators. For the problem 5:(2), 7/12 + ½, ten got it right, nine got it wrong, and two did not try. For 5:(3), 1/15+ 1/12, only nine got the right answer, nine got it wrong, and three did not try. For the ST Math group, for problem 5:(1), 3/5 + 2/5, ten students got the correct answer, four got an incorrect answer by making the mistakes mentioned above, and one student did not try. For 5:(2), 7/12 + 1/2, only four students got the right answer, eight students got an incorrect answer by making mistakes mentioned above, and three students did not answer. For 5:(3), 1/15 + 1/12, only three students

got it right, eight students calculated wrong as mentioned above, and four students did not try.

Problem 6: Subtraction.

6) Subtract the fractions.

(1) 9/11 – 5/11   (2) 2/5 – 1/3   (3) 3/10 – 2/5.

This problem aimed at the same bias as the problem above. For the subtraction problem 6, the common denominator problem 6:(1), 9/11 -5/11, had the highest correct scores for both classes. However, for 6:(2), 2/5 – 1/3 and 6:(3), 3/10 – 2/5, the students showed evidence of the natural number bias in the fraction magnitude concept by subtracting across without having a common denominator as 2/5 – 1/3 = ½ and 3/10 – 2/5 = 1/5. The other common mistake was getting a common denominator but not changing the numerator accordingly: 2/5 – 1/3 = 2/15 – 1/15 = 1/15 and 3/10 – 2/5 = 3/20 – 2/20 = 1/20. As it was seen in Problem 5, this indicates lack of understanding of an equivalent fraction concept.

For the non ST Math group, in 6:(1), 9/11 – 5/11, twenty students got it right and only one did not try. Fourteen got it right for 6:(2), 2/5 – 1/3, five got incorrect, and two did not try. Those who got a wrong answer showed the natural number bias by subtracting across without having a common denominator as 2/5 – 1/3 = ½. Fifteen got the right answer for 6:(3), 3/10 – 2/5, five got incorrect, and two did not try. Those who did the problem incorrectly exhibited the natural number bias as 3/10 – 2/5 = 1/5.

Specifically, in the ST Math group, for 6:(1), thirteen got the right answer, one got an incorrect answer which was 9/11 – 5/11 = 4/11 = 4, and one did not try. However, only five got the right answer for both 6:(2) and 6:(3), six got incorrect, and four did not try both. Those who got an incorrect answer indicated the existence of the natural number bias with the fraction magnitude concept by subtracting across as mentioned above. This was a combination of a lack

of the fraction magnitude concept and the natural number way of thinking, which is a fraction consist of two natural numbers instead of thinking a fraction as one number. Hence, the student did not really understand why it is necessary to have a common denominator. Instead, the student just memorized the fraction addition procedure but since the fraction magnitude concept was fragile, as a result, the procedure was not retained correctly.

Problem 7: Natural Number Bias in Operation.

7) State true or false.

(1) X/4 < X   (2) 3 < 3/X.

For question 7:(1), X/4 < X , which aimed at detecting the natural number bias in fraction operation, the majority of the students stated it was true since when they substituted a natural number in X, the inequality was true. However, for 7:(2), 3 < 3/X, only one student in the non ST Math group was able to state this as undetermined based on the information given. All the rest from both groups stated false by substituting a natural number in X. The most prominent case was that they substituted X =1 and 3 < 3. Hence the statement was false. They had this strong natural number bias in fraction computation.

In the non ST Math group, for 7:(1), X/4 < X, elven students stated this is true and ten stated it was false. For 7:(2), 3 < 3/X, three stated true, seventeen stated false, and only one stated undetermined based on the information given. In the ST Math group, for problem 7:(1), X/4 < X, nine students stated true, three stated false, and three did not try. For problem 7-(2): 3 < 3/X, four stated true, nine students stated false, and two did not try it.

Problem 8: Multiplication & Division.

8) Multiply or divide the fractions.

(1) 2/5 * 3/5 (2) 2/7* 3/10 (3) 2/5 ÷ 3 (4) 4/5 ÷ 3/7.

More than half of the students who were in both classes displayed weak fraction computation procedures with the fraction multiplication and division problems. For instance, when they computed in problem 8:(1), the common mistakes were 2/5*3/5 = 6/5 and 2/5 * 3/5 = 6/10. The other mistake in this problem was 2/5 * 3/5 = 10/15 since they applied the fraction division computation procedure in this multiplication problem by cross-multiplying. For 8:(2), 2/7 * 3/10, some of the students did this as 2/7 * 3/10 = 20/70 *21/70. This showed that they confused a fraction addition/subtraction procedure with a fraction multiplication procedure. Similarly, in the division problem 8:(3), 2/5 ÷ 3 and 8:(4), 4/5 ÷ 3/7, a few students indicated a cross multiplication as 2* 1 and 3 *5, but they stopped or they got a wrong answer because they did not know which one was going to be the denominator and which one was going to be the numerator. The other common mistakes among the non ST Math students in 8:(3) and 8:(4) were they tended to compute 2/5 ÷3 = 2/5 ÷ 15/5 = 30/25, or 2/5 ÷ 3 = 5/2 * 3 = 15/2 and 4/5 ÷ 3/7 = 28/35 ÷ 15/35. This tendency was not observed in the ST Math group.

For the non ST Math students, twelve got 8:(1), 2/5*3/5 right, nine got 8:(2), 2/7 * 3/10 correct, seven got both 8:(3), 2/5 ÷ 3, and 8:(4), 4/5 ÷ 3/7 correct. For the ST Math class, seven got 8:(1) right, eight got 8:(2) correct, only four got 8:(3) right, and three did 8:(4) correctly. For the ST Math group, five students got 8:(1),2/5 * 3/5 incorrect. These who got a wrong answer performed either 2/5 * 3/5 = 6/5 or 2/5 * 3/5 = 6/10. Eight students got 8:(2): 2/7 * 3/10 right, one got incorrect and six did not try. For 8:(3): 2/5 ÷3, only four got the correct answer and nine did not try. Those who got an incorrect answer did the cross multiplication procedure mentioned above but they did not know which one was going to be a denominator. Only three executed 8-(4): 4/5 ÷ 3/7 right, ten did not try, and two did cross multiplication getting an answer of 12/35.

The students in both classes indicated that the strong natural number bias in all three

areas based on the pre-test results. The bias in fraction density concept, question 2, and fraction computation, 7:(2), were the strongest biases and no one was able to state that there are infinitely many fraction in the specified interval for question 2 and only one student in the non-ST Math student was able to state the inequality was undetermined based on the information given.

Similarly, the students in both classes exhibited the natural number way of thinking in the fraction computations such as adding or subtracting without a common denominator. Similarly, in fraction multiplication and division, some of them came up with their own procedures as 2/5 * 3/5 = 6/5 and 2/5 ÷ 3 = 5/2 * 3 = 15/2. This clearly exhibited fragile procedural memorization without conceptual understanding of division. Overall they executed the fraction division procedure much poorly than the other fraction computational procedures.

Initial Conceptual Understanding of Fraction

Table 8

Initial Fraction Application Problem Result of the ST Math vs. the non ST Math Classes

ST Math					Non ST Math				
Problem	Correct	Incorrect	Partial	No Answer	Problem	Correct	Incorrect	Partial	No Answer
1	2	10	1	2	1	0	9	5	7
2	3	8	0	4	2	3	10	1	7
3	5	4	1	5	3	9	2	0	10
4	0	8	0	7	4	0	5	0	16
5	0	5	0	10	5	0	5	0	16
6	4	8	0	3	6	6	8	0	7
7	2	2	6	5	7	0	1	8	12

The application problems aimed at investigating the students' conceptual understating. I focused on analyzing the student work on the pre-test for representations; which are pictorial, realistic, symbolical and verbal (Figure 1). Table 8 displays a brief summary of the results. A detailed description of each problem follows this. In short, the students in both classes were unable to express their fraction understanding with the different representations of the Lesh

Translation Model. Therefore, all together their conceptual understanding was weak.

Application Problem 1: Pictorial Representation of a Realistic fraction multiplication or division problem.

1) Paul has 7/8 of a piece of a chocolate bar. He eats 1/2 of it. How much does he have left?

This problem investigated the students' fraction concept, by solving a realistic story problem using a pictorial representation when they cannot simply rely on procedures. For the non ST Math group, no one was able to come up with the correct answer. Nine students got it wrong by calculating 7/8 – ½ = 3/8. Although five students got it partially right by stating either 3 ½ or 3.5 for their answer, they did not consider that the chocolate bar was initially divided into 8 pieces in their answer. Seven students did not try.

For the ST Math group, two students were able to come up with the right answer. Ten described the expression incorrectly as 7/8 – ½ = 3/8. One got it partially correct because this student did not finalize the answer and miswrote the denominator as 3.5/7. Two students did not try. Those who stated the answer was 7/8 – ½ = 3/8 drew a pictorial representation. From the results, the majority of students in both classes were unable to correctly express the realistic representation pictorially or symbolically.

Application Problem 2: Pictorial Representation of a Realistic fraction multiplication or division problem.

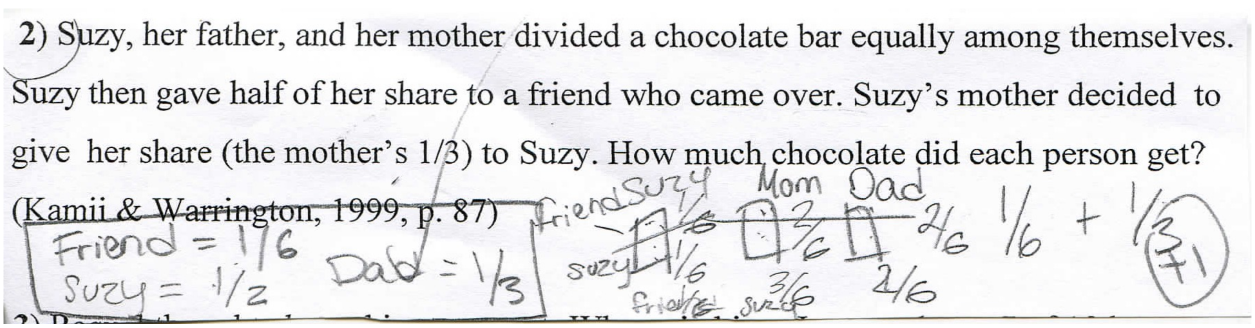
2) Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?

This problem also dealt with chocolate bars, but was more complex than Problem 1, because more people were involved. However, the core aim of the problem investigated if the students can connect the pictorial representation of parts of a part, which translates division as

splitting an existing quantity into a certain number of subsets (Koichu, Harel & Manaster, 2012) to the fraction multiplication concept.

Among the non ST Math students, three were able to connect the pictorial representation with the concept. Two got it partially correct since they only stated that Suzy's friend got 1/6, not stating the other three people's share. Ten students got it incorrect by stating that all of the people but Suzy's dad got 1/3, and seven did not try. For the ST Math group, three were able to answer correctly, eight got the wrong answer, by stating all of the people but Suzy's dad got 1/3 of a share or Suzy's friend got 1/3 ÷ ½ = 2/3, and four did not try.

A total of six students answered correctly. Five of the six students who answered correctly indicated that the parts of part was division, writing 1/3 ÷ 2 (Figure 12). Therefore, they possessed the notion of 'parts of a part' implying division. Most students did not connect the pictorial representation to a fraction multiplication concept. Figure 12 is the typical pictorial representation of those who connected the realistic representation to fraction division but did not connect it to fraction multiplication.


2) Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get? (Kamii & Warrington, 1999, p. 87)

Friend = 1/6
Suzy = 1/2
Dad = 1/3

Figure 12. Student's Pictorial Representation

Application Problem 3: Concept of fraction magnitude and equivalent fractions.

3) Raquel thought about this statement: When pitching, Joe struck out 7 of 18 batters. She said that it was better to say that Joe struck out about 1/3 of the batters than to say that Joe struck out about ½ of the batters. "I think that seven-eighteenths is closer to one-third than one-half," she said. Do you agree or disagree with Raquel? Explain your reasoning.

This problem probed the students' conceptual understanding of fraction magnitude and equivalent fraction symbolically and verbally. For the students in the non ST Math class, nine showed understanding of the magnitude of a fraction and an equivalent fraction symbolically. They stated that Raquel's reasoning is correct because 1/3 is equivalent to 6/18 and this is closer to 7/18 than ½, which is equivalent to 9/18. Two students stated otherwise by reasoning, 6/18 was closer to ½ than to 1/3, and ten did not try.

Among the ST Math students, four students were able to display their understanding symbolically and verbally with the same reasoning as those who got it correct in the non ST Math group. One student got it partially correct, five students got it incorrect, and five did not try. The student who got it partially correct made a calculation mistake as the 1/3 * 18 = 7 and responded the equivalence of 1/3 was 7/18 and ½ was equivalent to 9/18. Hence this student concluded that the Raquel's view was correct. Those who disagreed with Raquel, did so because they stated that 1/3 of 18 was 3. The results showed that the students who did not agree with Raquel and did not try did not have a solid conceptual understanding of fraction magnitude and of equivalent fractions symbolically.

Application Problem 4: Representing the concept of fraction multiplication pictorially.

4) Draw a picture to represent 2/3 × 4/5.

This problem discerned the students' conceptual understanding to express the symbolical representation of a fraction multiplication problem pictorially. No one in either group was able to express 2/3 * 4/5 pictorially. Many of those who tried drew either two circles or rectangles and divided each circle or rectangle into three parts and five parts as a part of a whole. Then they shaded two and four divided parts. One student represented her realistic model as below in

Figure 13. She expressed 2/3 * 4/5 = 10/15 * 12/15 and this representation mirrored the unknown fraction multiplication procedure they showed on Fraction Knowledge problem 8, 2/7 * 3/10 = 20/70 *21/70. Seventeen did not try in the non ST Math group and seven did not try in the ST Math group. The students were unable to represent fraction multiplication pictorially and this indicated a deficiency in their conceptual understanding of fraction multiplication.

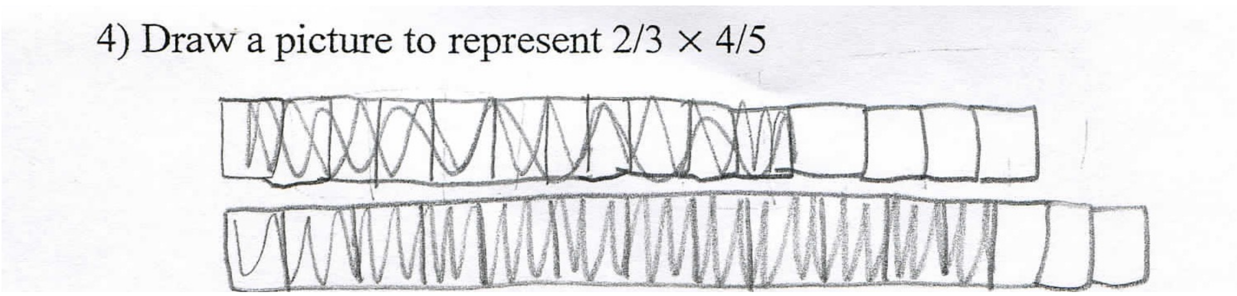


Figure 13. Student's Pictorial Representation of Fraction

Application Problem 5: Express fraction division realistically.

5) Come up with a problem that indicates the expression 1/3 ÷ 1/6.

The ability to connect the concept of fraction division realistically was measured through this problem. As in the previous problem, no one in either group was able to represent the division problem 1/3 ÷ 1/6 realistically. For both the non ST Math and ST Math classes, five students tried. The five students in the non ST Math group just computed the division problem procedurally. However, the five students in the ST Math class who tried came up with a very similar story problem as Application Problem 2. All of these five were able to come up with a real life story problem. For instance,

James has 1/3 of a chocolate bar and he divided it into 6 friends. How much of the chocolate were left?

However, this expressed 1/3 ÷ 6, instead of 1/3 ÷ 1/6. They misunderstood ÷ 6 as equivalent to ÷ 1/6. Hence this revealed their weak conceptual understanding of fraction division. Likewise, their story problems confirmed that they carried the idea of 'parts of a part'

meaning division, shown in Problem 2. This probably was the reason that all those who tried came up with this type of realistic problem, representing parts of a part, instead of approaching from the fundamental measurement concept of division, which is 'how many 1/6 can go into 1/3?' Sixteen students in the non ST Math class did not try and ten did not try in the ST Math class.

Application Problem 6: Express fraction division realistically.

6) Choose a story problem that represents 1/3 ÷ 8.
a. How much chocolate will each person get if 8 people share 1/3 lb of chocolate equally?
b. One of three people wants a piece of chocolate. There are 8 pieces. How many pieces will the other two people get?
c. Eight friends get 1/3 of a chocolate bar. How many chocolate bars will we need to buy?
d. Both (a) and (c).

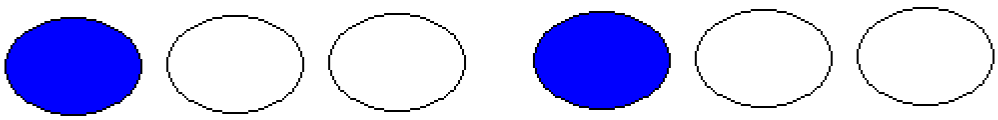
Although the main purpose of this problem was the same as the previous problem, it also was designed to see if students could discern the differences between fraction multiplication and fraction division realistically by selecting the right story which represents 1/3 ÷ 8. This problem was placed to lead the students to discern the differences between the expression of the previous problem 1/3 ÷ 1/6 and the expression 1/3 ÷ 6 so that their understanding of fraction division could have been strengthened by pondering the difference of the two different division algebraic expressions.

For the non ST Math class, only six students were able to select the right choice (a) that represents 1/3 ÷ 8. Three selected (b), five chose (d) and seven did not try. Among the ST Math students, only four students selected the right choice (a). Two chose (b) and eight selected (d). In both groups, those who selected (d) showed that they could not discern the difference between 1/3 ÷ 8 and 1/3 * 8, since (c) expressed the latter expression. Consequently, including the students who selected (b) and did not try in both classes, the majority of students did not

demonstrate the realistic representation of fraction division on this problem.

Application Problem 7: Conceptual understanding of fraction magnitude and its pictorial representation.

7) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"
1. Explain how Kevin is viewing this problem.
2. Explain how James is viewing this problem.

The conceptual understanding of fraction magnitude was examined through the pictorial representation of the ovals and its symbolical representation. If students had the fraction magnitude conceptual understanding, they would have been able to discern what the difference is between Kevin's view and James' view and explain which one has the correct view. Among the non ST Math students, no one was able to state which one had the right view conceptually. One student got it incorrect by stating that the James' view was correct, which is 1/3 + 1/3 = 2/6 = 1/3. Eight students got it partially correct because they stated that Kevin viewed two sets of three ovals and James viewed the six ovals as one set and chose two of them, but they did not describe why James's view was not correct from the concept of fraction magnitude. Twelve students did not answer.

For the ST Math group, two students were able to state the differences of the views of Kevin and James and why James's view was incorrect conceptually. They described that Kevin did not add the denominators as James did 1/3 + 1/3 = 2/6 = 1/3. Hence what James did was incorrect because he added the denominators. One student stated that both views were correct.

Another student stated the Kevin's view was correct. Six students got it partially correct

by stating the same as what the non ST Math students stated above. Five students did not try. From this result, the students were unable to connect the pictorial representation of the ovals to the correct symbolical representation to explain why (language representation) James' view was incorrect conceptually. This revealed that their conceptual understanding of fraction magnitude was weak on this question.

Overall, the students in the non-ST Math class and ST Math class were not able to express the fraction concepts pictorially, realistically, symbolically or verbally. They revealed that they possessed that the notion, which was 'parts of a part' is division. Hence no one was able to connect Application Problem 2 to fraction multiplication from their pictorial representations. Likewise, about half of the students in both classes interpreted Application Problem 1 as a subtraction problem because the problem said Paul ate half of his share. This clearly indicated that they had the weak fraction multiplication concept.

No one was able to express Problem 5: 1/3 ÷ 1/6 realistically. None of the students approached this problem from the fundamental measurement division concept as "To make one gallon of Gatorade solution, 1/6 ounce of Gatorade powder is needed. Now I have 1/3 ounce of the powder. How many gallons of the solution can be made?" Consequently, their tendency of 'parts of a part' equalizing division shown in Application Problem 2, was displayed in this problems as well. Similarly, their representations indicated 1/3 ÷ 6 instead and they misunderstood ÷ 1/6 with ÷ 6. Lastly, in Application Problem 7, more ST Math students showed their fraction magnitude concept, compared to the non-ST Math class. Regardless, most of them were unable to connect the pictorial representations to the symbolical representation of fraction addition to explain the magnitude concept.

Pre-Test Interviews

Interviews with six students, three students from each class, were conducted. These six students agreed with the interview consent form on the first day of the class. The main purpose of this interview was to investigate their mathematical conceptual reasoning behind their answers for the selected problems from the fraction assessment (Appendix B) because often they could state a correct answer for mathematical problems, but the correct answer could overshadow weak conceptual understanding (Ron, Dreyfus, & Hershkowitz, 2010). Secondly, through the interview, the students could express and explain their thought process on the paper verbally so that the researcher was able to discern the depth of their fraction conceptual understanding through how they expressed the understanding based on the Lesh Translation Model, pictorially, realistically, symbolically and verbally.

For the non ST Math class, one student scored the highest of 20/28 in the class and two scored, 5/28 and 7/28, much below class average of 11.1/28. For the ST Math class, one student had the highest score of 19/28 in the class and two scored, 11/28 and 12/28 around the class average of 10.7/28. Each student's response was analyzed individually first before comparing and contrasting within each class. Finally, a summary was written based on any differences and similarities of the students between classes. Pseudonyms were used for the privacy protection of the participants.

Interviews with the ST Math students.

Question 1: Fraction magnitude and its representation.

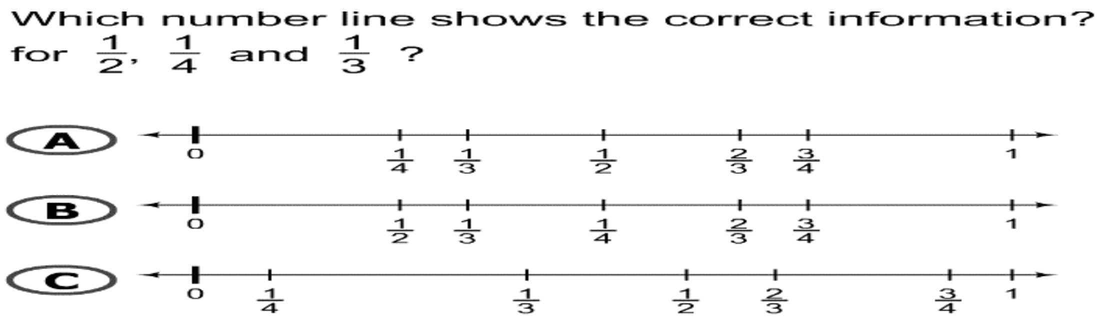


Figure 14. Interview Question 1 with ST Math: Display of a number line and a fraction magnitude

The students were asked the reason why they chose their answer selection. This question aimed (Figure 14) at investigating the existence of the natural number bias in the concept of fraction magnitude and also examined the depth of their conceptual understanding in fraction magnitude by connecting the symbolical representation of the fractions to the pictorial representation of the number line.

Two of the students Craig (Average) and Kylie (High) were able to explain why they selected the correct answer (A) by connecting the representations to the pictorial representation of the number line and they did not show the existence of the natural number bias in the magnitude concept. When Craig (Average) and Kylie (High) were asked why they selected (A). Craig (Average) stated, “Because it ½ is in the middle and ¼ is the quarter way of the number line.” And Kylie (High) responded “I drew the circles and divided them into ½, 1/3 and 1/4. Then I knew which one is the closer to 1.” However, Jordan (Average) selected (C), instead of (A) and was asked what the reason for selecting (C), he replied “Because ¼ is the smallest and the ¼, 1/3 and ½ are in the correct order. (A) and (C) are the same. The difference of (A) and (C) is only the space isn’t it?” Although his rationale for (C) was partially correct, he was not able to grasp the magnitude of the fraction symbolically and pictorially.

Question 2: Fraction density.

2) How many fractions are there between 1/3 and ¼? Explain your rationale for your

answer.

This question examined if the students had the natural number bias with fraction density. Although Jordan (Average) knew that the answer for the question was there are infinitely many fractions between ¼ and 1/3, he did not write this down as an answer on the pre-test . In the pre-interview he stated:

The way I thought about it is it could be like infinitely many decimal numbers in the interval, but I did not know how to write it down as the answer. That is why I did not write down anything.

From the response, he was not completely sure how to write his reasoning, but, he did not indicate the natural number way of thinking in the concept of fraction density since he did not say that there was only one number in the interval. Craig (Average) and Kylie (High) had the natural number bias in the concept. Craig (Average) answered no fraction existed between ¼ and 1/3 and when he was asked why he thought there was no fraction in the interval changed his answer to 1 fraction:

Researcher: You said there was no fraction existed between ¼ and 1/3. Can you tell me why?

Craig: I don't know. I don't know why I said none. I was guessing.

Researcher: Okay. What about now? If I ask you this question, what do you think? What is the answer?

Craig: Ah. There in one answer in the interval.

Researcher: Okay. That one should be in the middle?

Craig: Yes.

Researcher: What would the number be? Can you think of it?

Craig: Ah. I forgot how.. sum of. It is like.

Researcher: Are you going to add ¼ and 1/3 and divide the sum by 2?

Craig: Yes

Researcher: So just one number in the interval?

Craig: Yeah just one.

Kylie (High) also wrote down there was only one fraction in the interval but could not remember her reasoning in the pre-interview.

I don't know why I said there was one fraction between and I tried to remember why I said 1. I was not guessing. Is the answer wrong?

Kylie (High) did not remember why she stated there was 1 fraction in the interval, so it is hard to discern if the natural number bias interfered with her thinking or not. Two students did not have developed reasoning and one did not show the natural number bias in the fraction density concept.

Question 3: Fraction Magnitude.

3) *Compute 1/15 + 1/12. Why do you think a common denominator is unnecessary or necessary to compute it?*

The researcher investigated if the students could explain the necessity of a common denominator in fraction addition from the concept of fraction magnitude pictorially, realistically or any other way. The researcher posed the following question: "Do you think we need a common denominator to calculate 1/15 + 1/12? Why do you think you have to have a common denominator?"

Craig (Average) responded, "Yeah. I need a common denominator to add them but I cannot tell you why (you) need a common denominator but I know I have been taught that way" Although he was able to execute the procedure, it was clear that he did not have the conceptual understanding of equivalent fraction. He knew how to calculate it because he was taught to have a common denominator in fraction addition.

Jordan (Average) did not even remember how to compute the addition when he took the pre-test and he left this question blank. The reason for this was he had not taken any mathematics class for ten years. Hence he was not unable to explain the necessity of a common denominator although he was able to execute the procedure during the interview. On the contrary to the previous two, Kylie (High) was able to explain the fraction magnitude concept by connecting the symbolical representation of the addition to the pictorial representation of it. She explained:

Researcher: Do you think we need to have a common denominator to add 1/15 + 1/12?

Kylie: Yes.

Researcher: Why? Can you tell me why we need a common denominator?

Kylie: Because you cannot do 15 + 12 to get 27. That does not make sense.

Researcher: Okay. That is true but my question is why we cannot add 15 and 12?

Kylie: Why can't you? They are not the same circles. If I cut this circle in 12 pieces and cut the other one in 15 pieces, yah they are not the same sizes. That is why you cannot add. 1/12 is not the same size as 1/15.

Her response clearly indicated that she was able to explain the concept of fraction magnitude by connecting the symbolical representation to the pictorial representation. Hence, Craig (Average) and Jordan (Average) had the procedural knowledge but they were unable to explain the concept of fraction magnitude symbolically nor pictorially. However, Kylie (High) revealed her understanding by explaining the concept pictorially and symbolically.

Question 4: Natural Number Bias in Fraction Operation.

4) In 3 < 3/X, is the statement true or false? If you think it is true, explain why you think it is true. If you think it is false, explain why you think it is false.

All of the students stated the statement 3 < 3/X was false and all of them had the same reasoning for their statement. They assumed the X represented a natural number, especially 1,

and this was why the inequality was false. For instance, Jordan (Average) had the response "I guess I assumed the X was 1." No one even suspected that the X could represent any real number. Hence the natural number bias in fraction operation was shown by the three students.

Question 5: Fraction Multiplication.

*5) In 2/5 * 3/5, guess if your answer is going to be larger or smaller than 3/5 without calculating. If you guess the answer is going to be larger than 3/5, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.*

The researcher intended to discern the existence of the natural number bias thinking, which is multiplication makes bigger. If the students had understood the concept, they would have been able to explain the concept verbally, pictorially, realistically, and justify why the natural number thinking is not always true for this concept. The researcher posed the question "Do you think multiplication makes bigger?" during the interview.

Initially all of them answered yes, because the problem was multiplication. Hence every single one of them possessed the natural number way of thinking. Then the researcher started posing further questions to Craig (Average):

Researcher: Do you think that the answer of 2/5 * 3/5 makes (it) bigger than 3/5?

Craig: No

Researcher: Why did you say no?

Craig: I don't know

Researcher: Do you think multiplication makes bigger?

Craig: Yes

Researcher: Do you think that happens all the time?

Craig: Ah not all the time.

Researcher: Why did you say no?

Craig: I was guessing.

Craig (Average) initially responded that the answer of the multiplication did not make bigger but he was unable to provide the reason. Although, he did not think that is the always the case for multiplication, he could not provide a rationale because he was guessing. He was unable to explain the fraction multiplication concept, indicating a possible reliance on the natural number bias.

Jordan (Average) was able to explain why fraction multiplication could make smaller in the interview, although initially he had the natural number way of thinking:

Researcher: 2/5 * 3/5 is a fraction multiplication, as you know. Do you think multiplication makes bigger?

Jordan: Yeah. It should make (it) bigger as far as I remember

Researcher: My question is in this multiplication 2/5 * 3/5, do you think the answer of this multiplication makes bigger than 3/5?

Jordan: Oh. I know guess not, because like you multiply by a decimal which is a fraction. That makes smaller.

Although he did not explain the concept by connecting to a pictorial representation, he was able to connect the symbolical representation to the equivalent relationship between fraction and decimal. Through the interview, his natural number bias of "multiplication makes bigger" could have diminished and he was able to understand why the answer became smaller from the fraction multiplication concept.

Kylie (High) was unable to explain why the answer is going to be smaller by looking without actually calculating it, but after calculating it, she explained why the answer of the

fraction was smaller than 3/5 by connecting the symbolic representation to the pictorial representation:

Researcher: This answer of 2/5 * 3/5 is going to be greater than 3/5? What do you think?

Kylie: Without doing mathematics?

Researcher: Yes. Exactly just seeing it. Or maybe I can ask this question. Do you think multiplication makes bigger?

Kylie: I did.

Researcher: You did but now it is different? Why do you think multiplication does not always make it bigger?

Kylie: Ah okay let me think about this. 2/5 * 3/5 is 6/25. A circle cut into 25 is not the same as a circle cut into 5. So you have circles (drawing two circles). Wait. I don't know how to cut it in 5.

Researcher: It is not easy to cut a circle in 5 though.

Kylie: So if I shade in 3/5, it is going to be bigger than 6/25.

Although she indicated the partial understanding of the fraction multiplication by stating the answer was smaller than the multiplicand from the pictorial representation, she did not explain the reason from the concept fraction multiplication. However, this partial understanding might have assisted her to make sense that multiplication does not always make bigger.

Jordan (Average) was able to explain why the answer of the fraction multiplication makes smaller than 3/5 by connecting fraction to decimal, although initially he had the natural number way of thinking during the interview. Although Kylie (High) was able to explain why the answer was smaller by comparing the pictorial representation of 3/5 to its 6/25, she did not connect it to the concept entirely. Craig (Average) was not able to explain why the answer of

fraction multiplication makes bigger and possessed the bias in his response.

Question 6: Fraction Division.

6) In 4/5 ÷ 3/7, guess if your answer is going to be larger or smaller than 3/7 without calculating. If you guess the answer is going to be larger than 3/7, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.

The purpose of this question was to examine if the students possessed the natural number bias of "division makes smaller". Also if they could explain the concept of fraction division pictorially or realistically why the answer of a division problem may be bigger. The first question that the researcher asked in the interview was "Do you think division makes smaller?" Craig (Average) and Jordan (Average) responded that that was not always the case and Kylie (High) stated that it was. Then the researcher posed the question "What about 4/5 ÷ 3/7? Do you think the answer makes greater than 3/7 without calculating and why?" Craig (Average) answered "Yes, because it is a cross multiplication" This indicated that although he did not actually calculate the division to state his answer for the question, his reasoning was from the computational procedure, mentioning cross multiplication. His response did not exhibit the natural number bias of 'division makes smaller' but he really did not provide any conceptual reasoning why the answer is going to be greater than the divisor.

Kylie (High) responded for the same question, "I always thought division makes smaller but I am not still sure on this one." After she computed the division, she realized that the answer is greater than 3/5 but Kylie (High) could not explain why the answer becomes greater conceptually. Despite the fact that she had the natural number way of thinking initially, once she was asked this question, it certainly disturbed the natural number way of thinking, expressing she was not sure. Jordan (Average) responded:

Researcher: Do you think division makes smaller?

Jordan: Not necessarily

Researcher: So do you think the answer of 4/5 ÷ 3/7 makes smaller than 3/7. What do you think?

Jordan: Ah I guess I would imagines the answer creates smaller.

Researcher: Why do you think that? Because it's division?

Jordan: Yeah because it is division.

Initially he responded that he did not think division makes smaller, but when asked why he thought that the answer was going to be smaller, he said because it was division.

Consequently, his reasoning was influenced by the natural number bias. Jordan (Average) was unable to execute the division procedure during the interview and was unable to compare the answer to the divisor. Craig (Average) and Kylie (High) were able to execute the division procedure but they were unable explain the reason why the answer of the fraction makes greater than the divisor 3/7 from the fraction division concept after the comparison between the answer and the divisor. Similarly, the existence of the natural number bias was displayed in Kylie (High) and Jordan (Average)

Question 7: Pictorial representation of fraction multiplication or fraction division.

7) In the story problem, "Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?"
Explain your answer pictorially? Why do you think this problem represent multiplication?

The researcher investigated the depth of their understanding in fraction multiplication by how they interpreted the realistic story problem and how they connected the problem to fraction multiplication, pictorially, symbolically and verbally. It was also examined how they can

discover the relationship between fraction division and multiplication.

The students lacked the ability to connect the story problem to the symbolical representation of fraction multiplication or correctly to fraction division. Asked if they thought this problem indicated a multiplication problem, when they read "Suzy gave half of her share (1/3) to her friend." Two of them answered that "half of 1/3" implied division. Jordan (Average) reasoned that it indicated division because the picture he drew initially was divided into a third and he said that half of 1/3 is 0.5/3 mathematically by dividing the Suzy's share into half pictorially. When the researcher asked Jordan (Average), "From the picture, can we say that half of 1/3 as ½*1/3?" Jordan responded "1/3 divided by 2 more makes sense. But it is the same both of them are the same." He explained why he thought the story problem represented division pictorially. Although he affirmed the relationship between multiplication and division, he was not able to explain this by connecting the symbolic representation of the division and multiplication through the pictorial representation.

Kylie (High) stated that it was a division problem since Suzy cut her share into half from the picture and she divided 1/3 by 1/2, instead of dividing it by 2. This showed her fragile understanding of division because she thought "dividing the Suzy's share by half" indicated 1/3 ÷ ½, instead of 1/3 ÷ 2. It is certainly true that this story problem could indicate the division as well, but she was unable to represent the pictorial representation in the correct division symbolical representation form. On the contrary to these two, Craig (Average) responded:

> <u>Researcher:</u> Suzy gave half of her share 1/3. When you hear half of 1/3, what do you think?
>
> <u>Craig:</u> 1/6 right?
>
> <u>Researcher:</u> Okay. Show me your calculation. Is it multiplication or division, when you

hear half of 1/3?

Craig: Multiplication.

Researcher: Why do you think this is multiplication? Can you tell me why?

Craig: Ah because multiplication makes smaller.

He indicated that this story problem implies multiplication and he was able to state the correct answer. However, his explanation lacked further reasoning that the story was multiplication. He did not connect the pictorial representation to the symbolical representation of multiplication to explain his reasoning. Therefore, Jordan (Average) and Kylie (High) tended to think that parts of a part is division and did not think that could also represent this as multiplication as well and this is the indication of the marginal understanding of fraction multiplication. In brief, the notion of parts of part indicating division was influential and the notion hindered their understanding of fraction multiplication. They were unable to explain the parts of a part problem could express multiplication from the pictorial representation. Hence, none of them indicated their deeper conceptual understanding here.

Question 8: Realistic representation of fraction division.

8) *In the problem, "Come up with a problem that indicates the expression 1/3 ÷ 1/6." Explain why the problem that you came up with is correct expression of the division pictorially. By using the pictorial expression, explain why the answer of the division is larger than 1/6?*

The question was intended to investigate the depth of the students' fraction division conceptual understanding. The researcher ascertained their ability to connect the fraction division symbolic representation to a pictorial representation, language, and realistic representation to explain why the "change flip" procedure is used and the answer is greater than the divisor in the division, conceptually.

None of the students came up with a correct realistic story problem that represents 1/3 ÷

1/6. The common misunderstanding was when they saw the division sign, as if it was the second nature, they tended to think that a division sign implies a part of a whole. They did not go back to the fundamental measurement concept of division, which is “how many 1/6 can go into 1/3?” Instead, all three decided to represent the division to a similar story problem as Question 7, which is parts of a part realistic representation.

Craig (Average) came up with this story problem:

There is 1/3 of a chocolate bar. Only one person out of the 6 people gets the chocolate bar. How much chocolate bar is left?

Craig’s problem represented 1/3 and 1/6 independently and he was unable to connect these two quantities as a realistic division problem.

Jordan (Average) answered the following:

Researcher: Can you come up with any story problem representing 1/3 ÷ 1/6?

Jordan: A story problem? I guess you can do a similar thing as the previous one. 3 people get a chocolate bar divided into a third. Each one of them is sharing the bar and each one of them gave a half away.

Researcher: Meaning dividing the chocolate bar in half?

Jordan: Yeah.

Researcher: That means each one get 1/6?

Jordan: Guess not.

His story expressed 1/3 ÷ 2 not 1/3 ÷ 1/6. Hence the answer represented 1/6. As Craig (Average) did, in Jordan’s first sentence represented 1/3 and the second sentence represented 1/6, but he could not express these two numbers in a realistic story problem.

Kylie’s (High) realistic representation was: “Suzy had 1/3 of a chocolate bar and there are 6 children. She divides the bar equally.” Her story represented 1/3 ÷ 6 instead of 1/3 ÷ 1/6. Since

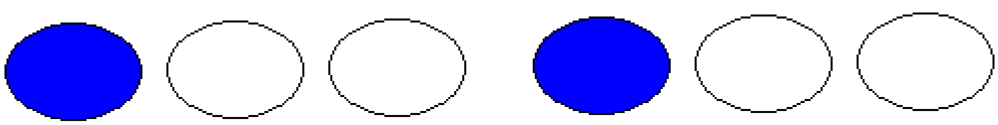
she misunderstood ÷ 1/6 was equivalent to ÷ 6. Hence no one was able to express the symbolical representation, pictorially, realistically and verbally and this showed that they lacked conceptual understanding of the fraction division.

All three sampled Question 7 to come up with a realistic problem to represent the division problem and this phenomenon certainly agreed with the result of previous question. Since they had the strong notion that part of a whole or parts of a part indicates division, when they saw the symbolic representation of division, they did not even think about the concept of division, "how many 1/6 can go into 1/3?", but tweaked Question 7, representing parts of a part.

In conclusion, the notion of 'a part of a whole' indicating division seemed very strong and they tried to force a partitive realistic representation, instead of approaching this problem from the fundamental measurement concept of division. Hence, this notion held them back from understanding not only the multiplication deeper but also the division concept.

Question 9: Fraction magnitude concept.

9) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"

1. *Explain how Kevin is viewing this problem.*
2. *Explain how James is viewing this problem.*

In Question 3, the researcher investigated their conceptual understanding of the fraction magnitude two dimensionally with the symbolic representation of 1/15 + 1/12 along with the language representation. However, this question investigated the depth of their understanding of the concept three dimensionally with the pictorial representation of ovals and the symbolic representation of 1/3 + 1/3. The aim of this was to see if they could connect the pictorial

representation of ovals to the symbolical representation of the additions to explain (language representation) which view is correct and why a common denominator cannot be added as 1/3 + 1/3 = 2/6 = 1/3 from the concept of fraction magnitude.

When the researcher asked them to explain James's view and Kevin's view Craig (Average) responded "James is viewing the problem incorrectly by adding the entire fractions and Kevin is viewing it the correct way by adding across the top numbers." Although his response was correct by explaining the differences in their views, his reasoning strictly came from procedural knowledge. When asked why the denominators cannot be added, he was unable to provide a conceptual reason. His procedural reasoning agreed with his response of Question 3, which is how he had been taught to have a common denominator for fraction addition.

Jordan (Average) answered, "James claims that 2 out of 6. James views the two 1/3s representations are two separate objects. Kevin looks at 1/3 of 1 object." He did not mention which one viewed the problem correctly nor explain why. From his response, Jordan (Average) did not connect the pictorial representation of ovals to the symbolical representation of addition.

Kylie (High) showed more developed reasoning and responded:

> James is wrong because you have the same denominator and you do not add because the denominator is not to be added, once a circle is divided into 3 parts, you do not add the 3. Kevin is right because he has the same denominator and the denominator does not change.

Her reasoning for the fraction magnitude concept was consistent with the response of Question 3.

Kylie (High) was the only one who showed the deeper conceptual understanding of magnitude by connecting the pictorial representation of a circle to the symbolic representation to explain the concept of fraction magnitude and common denominator. The other two students did not show the depth of understanding in fraction magnitude.

Pre-test interviews with the non ST Math students.

Question 1: Fraction Magnitude and Its Representation.

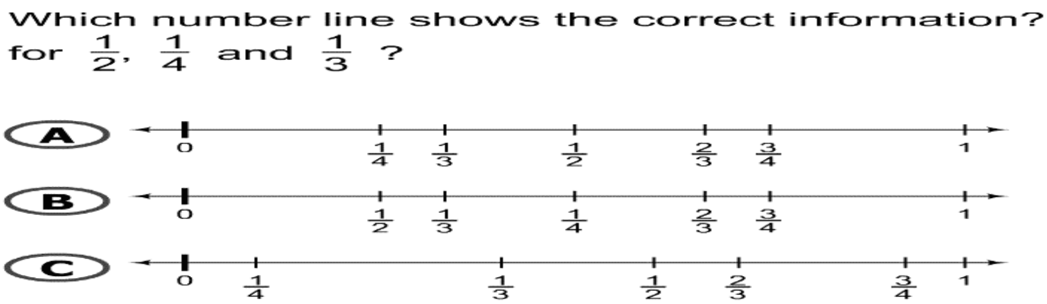


Figure 15. Interview Question 1 with non ST Math: Display of a number line and a fraction magnitude

Only one student indicated the natural number bias by choosing the selection (B). The researcher asked why Alexis (Low) thought that (B) was a correct representation of these fractions magnitude on the number line:

Researcher: Why did you choose (B)?

Alexis: I thought B was the right answer because (B) was more in order.

Researcher: Okay. So you did not think about ½ being greater than ¼?

Alexis: No. I thought about that and then 1/3 was going to be in between ¼ and 1/2.

Researcher: Okay. Right.

Alexis: This is why I chose (B) because 1/3 in the middle.

Researcher: You have never thought about choosing (A)?

Alexis: No.

Researcher: When you chose (B), did you have a concept of which one is greater, greatest and which one is the smallest?

Alexis: No. I just thought about having a number from smallest to greatest or from greatest to smallest.

Researcher: So you looked at the denominators to decide which one is the greatest and that is why you chose (B)?

Alexis: Yes

She decided the magnitude of the fractions based on the magnitude of the denominator and that was why she reasoned that ¼ > 1/3 > ½ and selected (B). She possessed the natural number bias in the concept of magnitude. However, the researcher asked more questions in depth, she eventually recognized that her reasoning for selecting (B) was incorrect. She showed partial understanding during the interview to selecting the right choice:

Researcher: Okay. What about now? Do you understand which one is the right answer?

Alexis: I know that ½ is always going to be bigger even though the denominator is the smallest one.

Researcher: Sure.

Alexis: But the correct answer here is I am not still sure.

Researcher: As you said ½ is the biggest. Which one do you think is the smallest?

Alexis: The smallest is going to be ¼.

Researcher: Which one do you think is the right answer?

Alexis: I think it would be (A).

Researcher: Because?

Alexis: Because ah the numbers on the other side of ½ are bigger. From 0 to 1, 1 down has to be whatever is smaller.

The other two students Morgan (High) and Caleigh (Low) did not show the bias in the concept by stating the correct rationale for the right selection. Morgan's (High) rationale was "1/4 and 1/3 are disproportionally far from each other in (C) and (B) is in the incorrect order."

Caleigh (Low) reasoned, “1/2 is the half of 1 and ¼ is 25% of 1 and it is a quarter way. 1/3 is just a third way.”

Question 2: Fraction Density.

2) How many fractions are there between 1/3 and ¼? Explain your rationale for your answer.

No one got the correct answer on the pre-test and two of the students possessed the natural number bias in the density concept by stating there was no or one fraction in the interval. One student left it blank on the pre-test. Morgan (High) indicated that she was trying to use the number line on the previous question and she wrote down zero fractions in between. During the interview she was able to provide why there are infinitely many fractions. Hence she did not have the natural number bias in the density concept:

Researcher: You said zero fractions are between ¼ and 1/3. Can you explain why you said none?

Morgan: Well I guess I was looking at the number line.

Researcher: This number line of the previous question?

Morgan: Yeah. That is probably why I said zero. Because I knew there are values in between 33% and 25%.

Researcher: Yes exactly. So if you think now, how many fractions are there between ¼ and 1/3?

Morgan: Probably a lot. But I did not know answer it in that way. I must have caught up on the number line which looks like there is no number between the two.

Caleigh (Low) left it blank on the pre-test and when the researcher asked the question during the interview, she stated that there was one fraction. Therefore, she showed the natural number bias in the concept initially. However, as the researcher got deeper in the interview

questions, she started to understand the concept:

Researcher: How many fractions are between ¼ and 1/3? You did not answer. Can you think of it now?

Caleigh: Is it just between like I can say like 1? I don't really know.

Researcher: Okay. So ¼ is 0.25 and 1/3 is about 0.33 right?

Caleigh: So I can say anything between that?

Researcher: If you think about it, there are so many fractions in them?

Caleigh: Yes. I know decimal point but I don't know how to put it into a fraction.

Researcher: Okay. Can I ask this? How many fractions exist in between?

Caleigh: Actual number?

Researcher: Yes like 1, 2 or infinitely many numbers.

Caleigh: Not infinity. Ah or could be infinity.

Researcher: Why do you think that?

Caleigh: Because always like 0.000.

Alexis (Low) answered there was 0 fraction between ¼ and 1/3 because "there was no ½ or 1/3 to be between." This clearly displayed not only her fragile fraction understanding but also her treating these fractions as natural numbers that have the characteristic of discreteness.

Morgan (High) was the only student who did not have the natural number bias in this concept and was able to explain why by connecting the fractions to percentages.

Question 3: Fraction Magnitude.

3) Compute 1/15 + 1/12. Why do you think a common denominator is unnecessary or necessary to compute it?

Morgan (High) was the only student who explained the necessity of a common denominator conceptually from the concept of fraction size the when calculating 1/15 + 1/12.

She responded “We need to have a common denominator, because make all the fractions out of the same number like to equalize them. We need to make them in the same proportion.” The other two students stated the necessity of a common denominator from procedural knowledge because having a common denominator is a part of the process and they have learned in that way. Hence Alexis (Low) and Caleigh (Low) were unable to connect the necessity of a common denominator to the concept of fraction magnitude.

Question 4: Natural Number Bias in Fraction Operation.

4) In 3 < 3/X, is the statement true or false? If you think it is true, explain why you think it is true. If you think it is false, explain why you think it is false.

Alexis (Low) and Caleigh (Low) stated the statement 3 < 3/X is false and Morgan (High) stated the statement could not be determined. Both Alexis (Low) and Caleigh (Low) thought the X was 1 and they substituted 1 in for X. Since 3 < 3 is false, they concluded the statement as false. The existence of the natural number bias was displayed in their reasoning. However, Morgan (High) explained:

Researcher: You stated it is unknown. Can you explain?

Morgan: Because you do not know what the X is.

Researcher: Sure.

Morgan: So you can’t just say that 3 is greater than 3/X.

Researcher: Right

Morgan: Because the X could be 1, that still is wrong. X could be 2 and the inequality could be right. So you do not know.

Researcher: Your reasoning is since you do not know the X, you cannot say true or false?

Morgan: Right. You cannot make statement about it.

Researcher: So when you saw this, did you think that the X was 1 or 2?

Morgan: I don't think the X was anything.

Researcher: So you thought the X could be anything like 0.5 or a negative number?

Morgan: Right. Anything. Could be a hundred.

From the response, although Morgan (High) did not automatically assume that the X represented a natural number alone, she displayed her misunderstanding of the inequality concept, by stating that when the X =2, $3 < 3/2$ is true. This exhibited that she comprehend anything on the left side of the inequality sign was greater than the right hand side of the inequality sign. For her the direction of the sign $<$ or $>$ did not mean anything. This was why she said the inequality was undetermined based on the given information although she stated that the X could be anything. Hence, even though on the pre-test, she stated that the inequality is cannot be determined, in this interview, her responses verified that the statement was originated from her misconception of the inequality. It was very hard to discern that she did not have the natural number bias in fraction computation.

Question 5: Fraction Multiplication.

> *5) In 2/5 * 3/5, guess if your answer is going to be larger or smaller than 3/5 without calculating. If you guess the answer is going to be larger than 3/5, explain your rationale. If **you** guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.*

Alexis (Low) and Caleigh (Low) displayed the natural number bias in the fraction multiplication concept. Although Alexis (Low) initially showed the natural number thinking of multiplication makes bigger, toward the end of the interview, she was able to state the reason why the answer of the fraction maybe smaller after she compared the answer 6/25 to the multiplicand 3/5. She reasoned:

Researcher: 6/25 and 3/5. Which one do you think is greater?

Alexis: Oh. That's fraction. So bigger the denominator, the smaller the fraction is.

Researcher: So which one?

Alexis: 3/5 is greater than 6/25.

Researcher: You got it right. Now you understand why though?

Alexis: Because you are dividing the number into smaller pieces.

Alexis's (Low) rationale was from her knowledge without understanding by stating "the bigger the denominator is, the smaller the fraction is". Consequently, Alexis (Low) did not explain why the multiplication answer is smaller than 3/5 conceptually and this indicated her weak understanding of the concept.

Caleigh (Low) also had the bias initially by stating the answer was going to be smaller than the multiplicand. However, after she compared the answer to 3/5 by visualizing two pies divided into fifths and twenty fifths, she was able to say the answer is going to be smaller than 3/5. Although she exhibited the partial understanding of fraction multiplication, she could not explain the reason from the concept of fraction multiplication fully. The partial understanding could have eliminated the natural number bias she possessed initially by comparing the fractions pictorially on her own.

On the contrary to that, Morgan (High) explained "The answer will be smaller, because it is like saying I want to know 40% of 3/5 is, which is like smaller number than it is going to take a small portion to give you answer." She connected a fraction multiplication to a decimal multiplication to explain why the answer is going to be smaller than 3/5. Consequently, Morgan (High) did not possess the natural number bias and explained the reason conceptually.

Question 6: Fraction Division.

6) In 4/5 ÷ 3/7, guess if your answer is going to be larger or smaller than 3/7 without calculating. If you guess the answer is going to be larger than 3/7, explain your

rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.

Two of the students, Alexis (Low) and Caleigh (Low), possessed the natural number bias of "fraction makes smaller". Alexis (Low) stated, "I think the answer is going to be smaller than 3/7 because we are dividing." After she compared the divisor to the answer 28/15, she recognized that 28/15 is greater but she could not explain why from the concept of the fraction division. Similarly, Caleigh (Low) thought division always makes smaller, "because division is a fast way of subtraction." Based on the response, she only thought about division with natural numbers. After she compared the answer 28/15 to 3/7 and she realized that the answer is greater than the divisor, the researcher asked her why the answer of 4/5 ÷ 3/7 is greater than the divisor. She explained:

> Because division is flip right? So 4/5 ÷ 3/7 is going to be 4/5 * 7/3 and it is going to be l, like another whole thing. So it is going to be bigger than 3/7.

Caleigh's (Low) reasoning came straight from the division procedure and did not explain from the concept of fraction division. On the contrary to these two, Morgan (High) did not think the answer was going to be greater than the divisor. She reasoned:

An opposite reason to a multiplication. When you are dividing you know how much like. So when you divide with a whole number, it makes smaller because you are cutting into small pieces. But when you are dividing with fractions, you are still cutting into small pieces but you want to know how many do you need to multiply to get that answer. That is why it will be bigger.

Morgan (High) revealed her understanding of the fundamental measurement concept of division, "how many 3/7 can go into 4/5?" Hence, her rationale did not indicate the natural number bias, and she was able to explain why the answer was going to be greater from the concept of fraction division. Hence Morgan (High) showed her conceptual understanding of the division.

Question 7: Pictorial representation of fraction multiplication.

7) In the story problem, "Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?"
Explain your answer pictorially? Why do you think this problem represent multiplication?

All three considered this story problem as a division problem because they had the parts of a part notion. Since the notion was persistent, no one was able to connect the pictorial representation of the story problem to multiplication. The researcher asked Alexis (Low) to try this question during the interview. She drew the picture, while reading the story. Initially she thought this story represented either subtraction or division:

<u>Alexis:</u> We have to divide or subtract because we are subtracting 1/3 from 1/2. Do you subtract in this situation?

<u>Researcher:</u> What do you think of ½ of 1/3? When you said ½ of 1/3, what does that mean?

<u>Alexis:</u> Multiplication. When it is 'of', that means multiplication.

<u>Researcher:</u> Do you know why though? Why it is multiplication? What do you do when you heard half of 50?

<u>Alexis:</u> You multiply.

Although initially she thought that the problem indicated subtraction or division, once the researcher asked the specific question asking half of 50, she was able to connect the idea to ½ of 1/3 to assure this indicated multiplication. However, her reasoning came from her knowledge as when she heard 'of', it represents multiplication. Alexis (Low) did not connect the pictorial representation to the symbolical representation of fraction multiplication to explain her rationale.

On the other hand, Morgan (High) thought this represented division:

Researcher: When you read this story problem, did you think this was a multiplication or division problem?

Morgan: I thought this was a division problem because the chocolate bar was divided. I have to draw it (Figure 16).

Researcher: Okay. Let me ask you this question. Suzy got 1/3 of chocolate and gave half of her share to her friend. When you heard half of 1/3, do you think it was multiplication?

Morgan: It is a division.

Researcher: Okay. So you did 1/3 divide by 2?

Morgan: Yes.

Researcher: Okay. 1/3 divided by 2. Do you see this is equivalent to 1/3 * ½?

Morgan: Yes. Cross multiply and I could have done that way. Fraction is always opposite anyway.

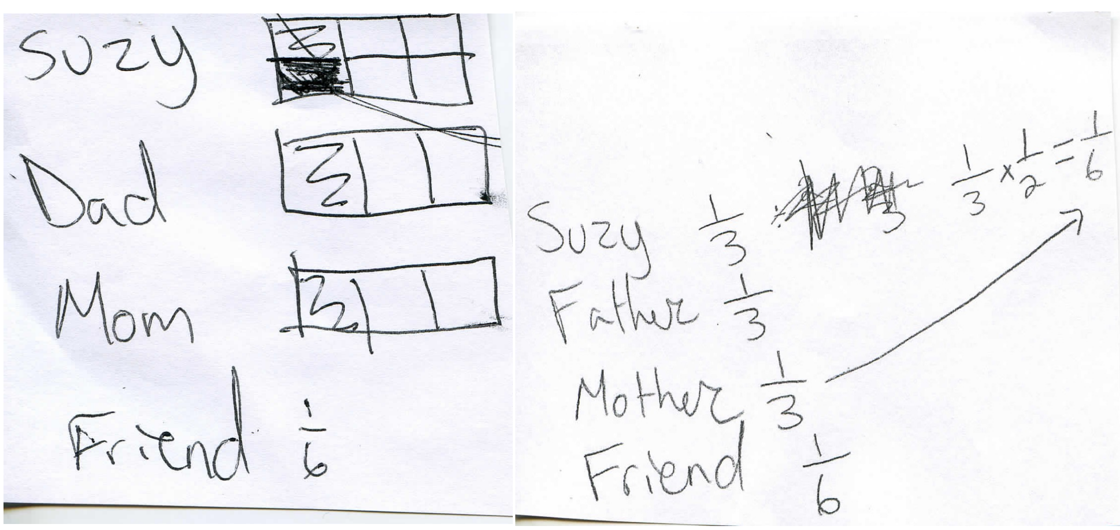


Figure 16. Morgan's Pictorial Representation

Morgan (High) explained why this story represented a division problem because the pictorial representation was divided. She did not think that the representation could express

multiplication as well. Although she saw the relationship between fraction division and fraction multiplication from the procedural view, stating “cross multiply”. She did not explain the relationship conceptually from the pictorial representation.

Initially, Caleigh (Low) was unable to derive the correct symbolical representation of the story problem from her picture because she did not really comprehend the problem. However, when she was asked if the realistic representation indicated division or multiplication, she thought that this story indicated division because the problem asked who got how much after they divided the chocolate bar. She asked:

<u>Caleigh:</u> Do I have to do a multiplication thing? Flip it or no?

<u>Researcher:</u> What do you think?

<u>Caleigh:</u> That is what I am going for. 1/3 ÷ ½ is 2/3. That does not make sense.

<u>Researcher:</u> When you saw this problem, did you think that this question is a multiplication problem or division problem?

<u>Caleigh:</u> Division because you are figuring out which one gets how much.

Caleigh (Low) were neither able to connect the pictorial representation to the correct symbolical division representation nor to the correct symbolical multiplication representation.

All three had thought dividing a chocolate bar implied division and they did not think that parts of a part could also describe multiplication from the pictorial representation. Hence their conceptual understanding of fraction multiplication was weak.

Question 8: Realistic representation of fraction division.

8) In the problem, “Come up with a problem that indicates the expression 1/3 ÷ 1/6.” Explain why the problem that you came up with is correct expression of the division pictorially. By using the pictorial expression, explain why the answer of the division is larger than 1/6?

No one was able to come up with a realistic story problem that represents 1/3 ÷ 1/6. All

three tweaked the previous question of Suzy sharing a chocolate bar. Alexis' (Low) representation was, "Suzy got a chocolate bar and cut the bar into 3 pieces. Then cut the bar into 6 pieces. Her representation expresses 1/3 ÷ 6. She misunderstood that ÷ 1/6 was equivalent to ÷ 6 and could not express ÷ 1/6, realistically.

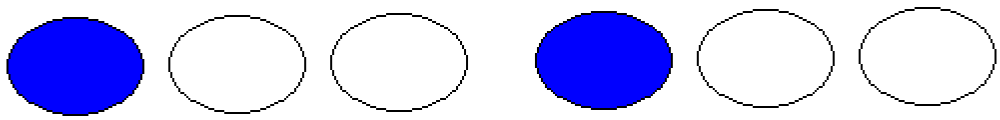
Morgan (High) presented her problem, but could not express the part of ÷ 1/6 realistically. "Suzy gets a chocolate bar and splits between friends. Two other friends including Suzy. Then divided it by 1/6."

Caleigh (Low) in her problem indicated 1/3 ÷ 6, not 1/3 ÷ 1/6 and showed the misconception in division as Alexis (Low) showed. Caleigh (Low) thought that ÷ 1/6 was the same as ÷ 6. "Suzy has a chocolate bar and she divides it into 3 pieces and Suzy divide the 1/3 of bar into 6 people."

As shown in their realistic problems, their notion of division as a part of a whole was very strong and this notion prevented them from going back to the fundamental measurement concept of division, which is "how many 1/6 can go into 1/3?" Alexis (Low) and Caleigh (Low) misunderstood ÷ 1/6 as ÷ 6 by stating dividing something into 6 people. Their approach to a symbolic representation of division was aligned with the result of Question 7, since they tended to think division implies part of a whole or parts of a part, and this strong notion had them to come up with the similar realistic story problem as the previous question for the division. In short, no one was able to connect the symbolic representation of division to a pictorial representation and a realistic problem.

Question 9: Fraction magnitude concept.

9) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"
1. Explain how Kevin is viewing this problem.
2. Explain how James is viewing this problem.

All three did not explain the concept of fraction magnitude by connecting the pictorial representation of the ovals to the symbolic representation. Morgan's (High) reasoning was close to explain why James's view was incorrect by adding the common denominator conceptually. Morgan (High) responded:

Researcher: What do you think the view of James?

Morgan: 1/3 + 1/3 = 2/6

Researcher: Does it make sense?

Morgan: No it doesn't.

Researcher: Which one do you think is right James or Kevin?

Morgan: Definitely, James is not right.

Researcher: Right. Because?

Morgan: Because James did 1/3 + 1/3 was equivalent to 2/6, but it is not equal. He was doing wrong because he would have got 2/3, if he had understood the meaning of a common denominator.

Although Morgan (High) showed her conceptual understanding of the fraction magnitude in Question 3 to explain her understanding of the fraction magnitude from adding fractions that are the same size of pieces, she did not explicitly reference the understanding here:

Alexis (Low) initially thought that James' view was correct:

Researcher: Can you explain the view of James and Kevin? Kevin said 1/3 + 1/3 = 2/3. James said 1/3 + 1/3 = 2/6.

Alexis: When add two fractions, you add the denominators. So 1/3 + 1/3 = 2/6.

Researcher: So do you think James is right?

Alexis: Yah.

Researcher: According to what you have said in Question 3, you have to have a common denominator, adding two fractions.

Alexis: Oh No. I take that back.

Researcher: This one, 1/3 + 1/3 = 2/3, is Kevin's view. Do you think he is doing right?

Alexis: Yes he is right.

Researcher: What about James? James is doing 1/3 + 1/3 = 2/6. Adding the numerators 1 + 1 and also the denominators 3 + 3.

Alexis: You have to have a common denominator.

The researcher provided the questions that assisted her to realize that the James' view was incorrect. From the guided questions, Alexis (Low) was able to state that the James' view was incorrect but she was unable to explain why the view was wrong from the concept of magnitude by connecting the ovals representation to the symbolic representation of addition.

Since Caleigh (Low) responded that both James and Kevin were the same initially, the researcher led her to see the difference between James's view and Kevin's view:

Researcher: Can you explain the James and Kevin's views? James did 1/3 + 1/3 = 2/6 and Kevin did 1/3 + 1/3 = 2/3.

Caleigh: Either one is kind of the same. One is looking at a whole and the other one is

looking at a half.

Researcher: Which one do you think is right?

Caleigh: Kevin, because he kept the same denominator.

Researcher: Yes, you are not supposed to add a denominator. But why?

Caleigh: That's the process.

Although Caleigh (Low) knew that James' view was wrong because he added the denominator, Caleigh was unable to explain why conceptually. Therefore, Alexis (Low) and Caleigh (Low) could not connect the pictorial representation of the ovals to the symbolical representation of the fraction addition to explain why James's view was incorrect. This result was aligned with the result of Question 3. In Question 3, Alexis and Caleigh did not indicate the understanding of fraction magnitude, by stating having a common denominator is a mathematical process to add two fractions. On the contrary to that, Morgan (High) displayed her understanding.

Summary of the Initial Interviews

Overall, the interviewed students in both classes demonstrated similar understanding. There was no significant difference between the ST Math students and non ST Math students as the pre-test scores revealed. The natural number bias in the three areas, fraction density in Question 2, magnitude in Question 1, 3, 5, and 6, and computation in Question 4, were revealed in the interviews. Likewise, while they executed their procedural knowledge in computations, they did not hold conceptual understanding behind the algorithms, especially in multiplication and division. Specifically in Question 7, the students laid out parts of a part thinking in fraction division. Since they had the strong reliance on the procedural notion of division, they automatically concluded that the parts of a part pictorial representation implied division and

could not see the representation also represented multiplication. Consequently, their fraction division and multiplication concepts were weak. Likewise in Question 8, no one was able to express the fraction division 1/3 ÷ 1/6 realistically. Seeing the symbolical expression of division, they presented similar story problems as Question 7 represented parts of a part, instead of approaching from the measurement concept of division, which "how many 1/6 can go into 1/3?"

This trend reaffirmed that they had the influential idea, which is parts of a part represents division. Likewise, their partitive representations actually indicated 1/3 ÷ 6, instead of 1/3 ÷ 1/6. Consequently, Kylie (ST Math, High), Caleigh (non-ST Math, low) and Alexis (non-ST Math, low) possessed the misconception. In Question 9, many of them knew the fraction addition procedure, but they were unable to explain the necessity of a common denominator from the concept of fraction magnitude.

Research Question 2

To what extent does the usage of ST Math eliminate the natural number misconceptions and deepen college remedial mathematics students' conceptual understanding in learning fractions compared to a non-ST Math class of remedial mathematics students taught traditionally without technology?

This section would present the statistical result of the post fraction assessment and the post interviews. The statistical result of the post assessment test for both classes will be discussed. Then the summary of the test will be presented. Subsequently, the result of the post interviews for the same six students will be analyzed.

Generally speaking based on the statistical result of the post assessment, although the natural number bias in fraction magnitude was less, the fraction density and its operation were persistent. The students exhibited much more precise computational skills and their score

increased significantly, compared to the initial assessment. However, the students in both classes did not show significant score increases on the application problems, which examined the depth of their understanding of fractions. Consequently the majority was unable to express their fraction understanding based on the Lesh Translation Model, pictorially, realistically, symbolically and verbally. This indicated that their conceptual understanding of fractions was not deeply rooted.

According to the interview responses, despite the fact that the quantitative result did not show statistically significant differences between the two classes, the ST Math students actually revealed conceptual gain in fraction magnitude in addition and multiplication and similarly diminished the natural number misconception in the density concept.

The ST Math high school fraction intervention games had a series of 8 sequential games (Table 8). The class average completion was 58% and this indicates that in average the students finished up to game 4, Fraction Addition, and played some of game 5, Fraction Multiplication. Seven students completed less than half of the games and their completion average was 24% and eight students completed more than half of the games and their average completion was 88%, which is equivalent of completing game 7, Unlike Denominator Subtraction, and playing some of game 9, Fraction division.

Table 9

List of the ST Math Fraction Games Played by the ST Math Students

Game 1	Visual Fraction Concepts
Game 2	Fractions on the Number Line
Game 3	Comparing and Equivalent Fractions
Game 4	Fraction Addition and Subtraction
Game 5	Fraction Multiplication
Game 6	Unlike Denominator Concepts and Strategies
Game 7	Unlike Denominator Addition and Subtraction
Game 8	Fraction Division

Statistical Result of Post Fraction Assessment

At the eleventh week of the semester, the students who participated in this study took the post fraction assessment. All of the students who took the pre-assessment took part in the assessment. The problems on the assessment consisted of the same problems as the pre fraction assessment and aimed at investigating their fraction computational procedural knowledge, the existence of the natural number bias, and their fraction conceptual understanding. The assessment was conducted under the same procedures as the initial assessment and many of the students finished the assessment within 20 – 25 minutes.

An independent two -sample t-test was conducted to investigate if there was a statistically significant mean difference in the post fraction conceptual understanding and procedural knowledge between the non ST Math participants and the ST Math participants. To check that the sample was normally distributed, the nonparametric Kolmogorov Smirnov test was conducted on the dependent variable X, the post fraction assessment test score. The percentage of X for the non ST Math class was, D (21) = .137, p = .200 > .05. Hence, the test result indicated that the sample is normally distributed. The percentage of X for the ST Math class was, D (15) = .200, p = .097 > .05 and it can be concluded that the data was normally distributed. Similarly, for the equal variances assumption, Leven's test was used. F (14, 21) = .476, p = .495 > .05 and this showed the equal variances.

A two- sample t-test indicated that the scores of the post assessment tests for both classes were not significantly different. Non ST Math class (M = 15.7143, SD = 4.795) and ST Math class (M = 16.033, SD = 5.514), t (34) = .185, p = .854 > .05. The power under the sample size was 0.098.

Consequently, from the statistical test result, at the post fraction assessment stage, the

participants who were in both the non ST Math class and the ST Math class, had similar fraction understanding. Although it can be clearly seen from the descriptive statistics (Table 9) that the students in both classes gained overall mean scores on the post assessment from the initial fraction assessment, the overall mean scores for the ST Math students was not statistically significant, compared to the non ST Math students after the 8 weeks of the ST Math fraction games intervention. The ST Math group did gain 5.4 points from the initial assessment to the post assessment on average, while the non ST Math group gained 4.6 points in average. As a result, the ST Math group increased their mean score by 50.79%, compared to the non ST Math group that gained 41.6 %. The difference is 9.15 %. Although there was no statistical significance on the post assessment between the non ST Math students and the ST Math students after they had received the intervention, the mean score increase of the ST Math group was more than the non ST Math group.

Table 10

Comparison of Descriptive Statistics of Post Fraction Assessment

ST Math					non ST Math				
n	M	Min	Max	SD	n	M	Min	Max	SD
21	15.714	8.0	23.50	4.795	15	16.033	6.5	24.0	5.514

Post Fraction Procedural Knowledge and the Natural Number Bias

Overall the participating students for both groups performed significantly better than the initial fraction assessment and they were able to weaken certain natural number biases, such as fraction magnitude and magnitude in fraction addition/subtraction computation. However, the results indicated that the bias had not been completely diminished although some biases, fraction density and the operation, were stronger and more persistent than the other bias. Table 10 has a summary of the post-test performance of each class.

The students reduced the natural number bias of the fraction magnitude knowledge on

problem 1 and 3. However, in the fraction density concept, problem 2, and the fraction computation, problem 7:(2), the students showed that there still existed the persistent natural number bias in these areas. For the fraction procedural knowledge, students executed the fraction addition (Problem 5), subtraction (Problem 6), and multiplication and division (Problem 8) computational procedures with much higher accuracy, compared to the pre-test Specifically, they struggled with the division problems the most on the pre-test such as 8:(1), 2/5 * 3/5, and 8:(3), 2/5 ÷ 3. However, the majority of the students in both classes applied the correct procedures to get the right answers in the division problems on the post assessment.

In conclusion, the majority of the students in both groups became much more proficient on the post assessment. Although the natural number bias in the density concept was significant even after the ST Math intervention, more ST Math students diminished the bias. Consequently, the fraction games promoted those students to acquire the concept. The descriptions below are the detailed analysis of each problem (Table 11).

Table 11

Post Fraction Knowledge Result of the ST Math vs. the non ST Math Classes

ST Math				non ST Math			
Problem	Correct	Incorrect	No Answer	Problem	Correct	Incorrect	No Answer
1	13	2	0	1	19	2	0
2	3	6	6	2	1	10	10
3:(1)	13	1	1	3:(1)	17	2	2
3:(2)	15	0	0	3:(2)	15	4	2
3:(3)	12	2	1	3:(3)	16	1	4
4:(1)	10	3	2	4:(1)	15	3	3
4:(2)	12	1	2	4:(2)	17	1	3
4:(3)	9	4	2	4:(3)	14	5	2
5:(1)	14	1	0	5:(1)	17	3	1
5:(2)	12	2	1	5:(2)	19	1	1
5:(3)	11	3	1	5:(3)	16	2	3
6:(1)	15	0	0	6:(1)	19	0	2
6:(2)	13	2	0	6:(2)	18	0	3
6:(3)	12	3	0	6:(3)	15	3	3
7:(1)	12	2	1	7:(1)	9	3	9
7:(2)	0	14	1	7:(2)	1	11	9

8:(1)	11	4	0	8:(1)	19	1	1
8:(2)	10	3	2	8:(2)	20	0	1
8:(3)	8	2	5	8:(3)	13	6	2
8:(4)	8	3	4	8:(4)	16	4	1

Problem 1: Fraction magnitude and its representation.

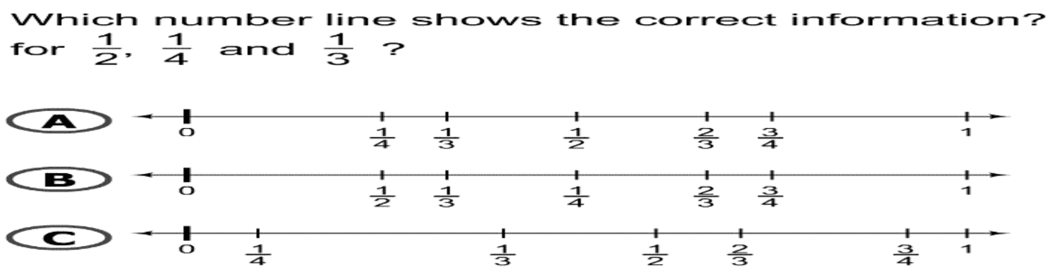


Figure 17. Post Fraction Knowledge Problem 1: Display of a number line and a fraction magnitude

For the non ST Math class, nineteen students selected the correct choice A, no one chose (B), and only two chose (C). For the ST Math group, the same trend appeared. Thirteen students chose the right choice (A), no one picked (B), and two chose (C). This signified that no one in both classes judged the magnitude of the fraction based on the natural number way of thinking, as some of them did on the initial stage. However, the students who selected (C) were unable to connect the magnitude to its representation precisely and possessed the weak understanding in the fraction magnitude concept. Therefore, although the natural number bias in the fraction magnitude concept completely diminished in both classes, there still existed not fully developed understanding of fraction magnitude for a few students who selected (C).

Problem 2: Fraction density.

2) How many fractions are there between 1/3 and ¼?

This problem investigated the natural number bias of the fraction density concept, as mentioned in the initial assessment section, none of the participants got the correct answer and the result showed the clear existence of the natural number bias in this concept. However, at the post assessment stage, one student stated that there are infinitely many fractions between ¼ and

1/3 in non ST Math group. Ten students did not try this problem and thirteen students answered incorrectly. Among these thirteen students, nine of them stated there was zero fractions in the interval and one stated there was one fraction in the interval.

Three students in ST Math group stated there existed infinitely many fractions between ¼ and 1/3. Six students did not try the problem and five students stated either there was none, one, two or three fractions in the interval. One student stated there are more than one fractions between the fractions but did not state infinitely many. Since this student answered that there was one fraction between them at the initial fraction assessment stage, she indicated her conceptual change with partial understanding. Consequently, the ST Math group had more students who understood the density concept, diminishing the natural number bias. As the result revealed, as a whole, the natural number bias in the fraction density concept was very strong and persistent. However, the ST Math games assisted some of the students to understand the concept better, diminishing the bias.

Problem 3: Equivalent fraction.

3) Equivalent Fractions.

(1) 3/10 = /20   (2) 4/8 = 8/   (3) 3/5 =.

For both groups, the participants' performance increased, however some students still possessed the natural number bias for the equivalent fraction concept. In the non ST Math group, seventeen students answered correctly for 3:(1), 3/10 = /20, fifteen for 3:(2), 4/8 = 8/ and sixteen for 3:(3), 3/5 = . In the ST Math group, for 3:(1), only one student got an incorrect answer and one did not try. All fifteen students got 3:(2) right and twelve students were able to come up with an equivalent fraction for 3:(3). For both classes, these who got incorrect answers did the following: 3:(1), 3/10 = 3/20, 3:(2), 4/8 = (4+4)/(8+4) = 8/12 and 3:(3), 3/5 = (3+2)/(5+2) = 5/7.

These indicated the natural number way of thinking to consider a numerator and a denominator separately along with the lack of the fraction magnitude concept. Hence very few students in both classes still possessed the bias in the magnitude concept.

Problem 4: Comparing fractions.

4) Circle a larger fraction

(1) 4/9, 1/5 (2) ½, 1/12 (3) 3/8, 6/11

Altogether, the natural number way of thinking was diminished in the fraction magnitude concept compared to the pre-test. However, some still had the natural number bias in this concept. Among the non ST Math group, fifteen students stated 4:(1), 4/9 > 1/5, seventeen students stated 4:(2), ½ > 1/12 and fourteen students stated 3/8 < 6/11. For the ST Math group, ten students stated 4/9 > 1/5, thirteen stated ½ > 1/12 and nine stated 3/8 < 6/11. From the result, the students in both classes performed better than the initial assessment. Therefore, there was not enough evidence to make a conclusive statement that the fraction games were a significant factor in diminishing the natural number bias in the concept.

Problem 5: Addition.

5) *Add the fractions.*
(1) 3/5 + 2/5 (2) 7/12 + ½ (3) 1/15 + 1/12.

For the fraction addition and subtraction computation problems, the students in both groups computed the problems much better than on the initial assessment. The number of students who added the numerators and the denominators without having a common denominator dramatically decreased from the initial assessment stage. The students in the non ST Math class executed better on 5:(2) and 5:(3) than on the initial assessment, although one less student got the correct answer on 5:(1). On the contrary to that, the students in ST Math performed the calculations much better than they did on the initial assessment and there was no decrease in the

number of students who executed the addition problems correctly.

For the non ST Math group, there was no one who added the numerators and the denominators for problem 5:(1), 3/5 + 2/5. Three students committed computational mistakes such as 3/5 + 2/5 = 6/5 or 3/5 + 2/5 = 7/5 and there was no one who did not try the problem. Nineteen students got problem 5:(2), 7/12 + ½ right, only one did not try, and one got the incorrect answer by making a computation mistake as 7/12 + ½ = 7/12 + 6/12 = 14/12 = 7/6. Sixteen students got the right answer for 5:(3), 1/15 + 1/12. Two students made computation mistakes as 1/15 + 1/12 = 12/170 + 15/170 = 27/170 and 1/15 + 1/12 = 5/60 + 4/60 = 9/12, and three did not try.

In the ST Math class, for 5:(1), 3/5 + 2/5, fourteen of them got the right answer and only one added the denominator and the numerator. For 5:(2), 7/12 + 1/2, twelve students had the common denominator of 12 and the right answer. One added the denominators and the numerators as 7/12 + ½ = 8/14, and two student did not try. On 5:(3), 1/15 + 1/12, eleven students got it right, two added the numerators and the denominators as 1/15 + 1/12 = 2/27, one student used a common denominator as 45 which is not a common multiple of 15 and 12, and two did not try. In short in both classes, the students were able to diminish the natural number bias in fraction magnitude with the addition problems. However, the ST Math students did not show any decrease in the number of students who got the correct answers in these addition problems, contrary to the non ST Math class. Hence the intervention could have been beneficial in fraction addition.

Problem 6: Subtraction.

6) *Subtract the fractions*
(1) 9/11 – 5/11 (2) 2/5 – 1/3 (3) 3/10 – 2/5.

On the initial assessment, the non ST Math group performed better than the ST Math

group on the problem. On the post-test nineteen non ST Math students got 6:(1), 9/11 – 5/11 right and two did not try. For 6:(2), 2/5 – 1/3, eighteen students got it right and two did not try. For 6:(3), 3/10 – 2/5, fifteen students got it right, three got it incorrect but these three made the same calculation mistake as 3/10 – 2/5 = 3/10 – 4/10 = -2/10, and no one who answered the problems showed the natural number bias by subtracting across without having a common denominator as shown on the initial fraction assessment. Three students did not try 6:(3) though. All fifteen ST Math group students got the right answer for problem 6:(1), 9/11 – 5/11. For problem 6:(2), 2/5 – 1/3, thirteen got it right and two exhibited the natural number bias by subtracting across without having a common denominator. For 6:(3), 3/10 – 2/5, twelve students got it right and three did not get the right answer. Among these three, one made a computation mistake despite of having the right common denominator and numerators and two still exhibited the natural number bias by subtracting across without having a common denominator. Although the result indicated that the students in both group performed better on the post assessment, the ST Math group dramatically executed better on this problem by diminishing the natural number bias in the fraction magnitude concept, compared to the initial assessment. Hence, the students playing the ST Math games positively affected their performance on the fraction subtraction problems.

Problem 7: Natural number bias in operation.

7) State True or False.
(1) X/4 < X   (2) 3 < 3/X.

For this problem the natural number way of thinking was very strong and persistent in both groups. For the non ST Math group, nine students stated the problem 7:(1), X/4 < X as true, three stated it as false, and nine did not try. For 7:(2), 3 < 3/X, the same student as the pre-test was able to state undetermined based on the given information.

Compared to the pre-test more students in the ST Math group tried these questions. Twelve students stated 7:(1) as true, two stated it as false, and one did not try. For problem 7:(2), no student was able to state that it is undetermined based on the information they had, fourteen students stated it as false, and one did not try. As described in the interview sections, the students tended to substitute X = 1 and conclude 3 < 3 is false or reasoned since 3/X is division, division is smaller than the whole number 3, the natural number bias in fraction operation did not diminish and was persistent in the both groups. In this problem, there was no advantage of playing the ST Math games to diminish the natural number bias.

Problem 8: Fraction multiplication and division.

8) *Multiplication and Division.*
*(1) 2/5 * 3/5 (2) 2/7* 3/10 (3) 2/5 ÷ 3 (4) 4/5 ÷ 3/7.*

The students in both groups displayed significant improvement for the fraction multiplication and division procedures comparing to the initial fraction assessment and those who did not try the questions decreased dramatically. In the non ST Math class, for 8:(1), 2/5 * 3/5, nineteen performed correctly, for 8:(2), 2/7 * 3/10, twenty obtained the right answer, for 8:(3), 2/5÷ 3, thirteen executed it right and for 8:(4), 4/5 ÷ 3/7, sixteen got the correct answer.

For the students who did not perform the computation correctly, they made procedural mistakes as 8:(1), 2/5 * 3/5 = 6/5, 8:(3), 2/5 ÷ 3 = 5/2 * 3/1 = 15/2, or 2/5 ÷ 3 = 2/5 * 3/1 = 6/5 and 8:(3), 4/5 ÷ 3/7 = 5/4 * 3/7 = 15/28.

For the ST Math group, eleven computed 8:(1), 2/5 * 3/5 correctly, ten got the right answer for 8:(2), 2/7* 3/10, eight got 8:(3), 2/5 ÷ 3 correct, and also eight got 8:(4), 4/5 ÷ 3/7 right. Those who computed these problems incorrectly computed as 8:(1), 2/5 * 3/5 = 6/5, 8:(2), 2/7 * 3/10 = 20/70* 21/70 and stopped or 2/7 * 3/7 = 5/70, 8:(3), 4/5 ÷ 3 = 12/35 or 4/5 ÷ 3/7 = 28/35 ÷ 15/35 and stopped and 8:(4), 4/5 ÷ 3/7 = 12/35 or 4/5 ÷ 3/7 = 4/35 ÷ 21/35 and stopped.

Although some non ST Math group students performed the wrong procedure for 2/5 ÷ 3 as 2/5 ÷ 3 = 2/5 * 3/1 = 6/5 or 2/5 ÷ 3/1 = 6/15 ÷ 5/15 and 4/5 ÷ 3/7 and 4/5 ÷ 3/7 = 12/35 or 4/5 ÷ 3/7 = 4/35 ÷ 21/35 mentioned above, none of the ST Math students did the same wrong procedural mistakes. Similarly, ST Math group had twice as many students who executed the fraction division problems 8:(3) and 8:(4) correctly as the initial assessment stage.

Consequently, the ST Math students had the edge over the non ST Math students in executing the fraction division problems accurately. Although the entire performance of these computations got much higher accuracy as a whole, some students in both groups still had difficulty in the computations.

As it can be seen in the results comparison between the initial assessment and the post assessment, both groups improved significantly on the fraction knowledge problems, except Problem 2 and Problem 7:(2). However, in Problem 2, the ST Math group had more students who were able to answer correctly. The problem is a non-routine procedural problem and aimed at investigating students' fraction density and magnitude concepts along with the natural number bias.

Therefore, although overall the statistical result was not significant in the improvement of their fraction knowledge between the two groups, at the microscopic level, a greater percentage of ST Math students revealed their conceptual gain and diminished the natural number bias in the fraction density and magnitude.

Post Fraction Conceptual Understanding

The same application problems on the pre-test were used for the post-test. The main focus of these problems was to examine how the students in ST Math group were able to gain conceptual understanding of fractions after they played the ST Math fraction games for 8 weeks,

compared to the non ST Math students who did not play the games. If the students in the ST Math class had gained understanding, they would have been able to express their fraction understanding pictorially, realistically, symbolically and verbally.

Overall, the ST Math games did not really make significant differences in the students' fraction understanding between the two classes based on the statistical result shown earlier. From Table 12, the ST Math class had more students who got correct answers only on Problem 6 (the fraction division concept) and Problem 7 (the fraction magnitude concept). While both classes showed improvement in conceptual understanding the ST Math group did not gain more conceptual understanding compared to the non ST Math students. The descriptions below are the details of the post fraction conceptual understanding result (Table 12).

Table 12

Post Fraction Application Problem Results of the ST Math vs. the non ST Math Classes

ST Math					non ST Math				
Problem	Correct	Incorrect	Partial	No Answer	Problem	Correct	Incorrect	Partial	No Answer
1	0	11	2	2	1	2	16	2	1
2	5	5	1	4	2	5	9	2	5
3	7	5	0	3	3	9	4	2	7
4	0	13	0	2	4	0	15	1	5
5	0	7	0	8	5	0	11	0	10
6	7	8	0	0	6	6	10	0	5
7	5	3	7	2	7	3	1	9	8

Application Problem 1: Pictorial representation of fraction multiplication.

1) Paul has 7/8 of a piece of a chocolate bar. He eats 1/2 of it. How much does he have left?

For the non ST Math class, two students were able to state the correct answer, expressing it as either 3.5/8 or 7/8* ½ = 7/16 and two got partially correct because although they were able to express the pieces of the chocolate as either 3.5 or 3 ½. They did not express it based on the original pieces of the chocolate bar as 3.5/8 or 3 ½ /8 though. Seventeen students were unable to

come up with the right answer. Although two students got the correct answer, only one of them expressed this story problem as multiplication and the other one expressed it as division. Consequently, only one student was able to connect this realistic problem to multiplication. Among those who were in the ST Math group, no one was able to state the right answer and two got it partially correct, because of the same reason as the students in non ST Math group, described above. Eleven stated wrong answers, expressing the problem either as 7/8 – ½ or 7/8 ÷ ½ and two did not try. For both classes, some of them drew the right pictorial representation, but they stated the wrong answers because they were unable to connect the representation to the fraction multiplication or division concept. In short, the game play did not deepen the students' conceptual understanding for this problem.

Application Problem 2: Pictorial representation of fraction multiplication.

2) ***Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?***

For the non ST Math group, five students came up with the correct answer, one got it partially correct by stating only the Suzy's friend's share 1/6, and ten got it incorrect by answering that all of them but Suzy's mom got 1/3.

In the ST Math class, five got the correct answer, one got it partially correct, only stating Suzy's friend got 1/6, five got an incorrect answer, and four did not try. Those who performed incorrectly answered either everyone but Suzy's father had 1/3 or that Suzy's friend share was 1/3 ÷ ½ = 2/3. Many students in both classes struggled with the connection between the pictorial representation and the fraction multiplication concept symbolically. As for the ST Math students, although more students were able to connect the pictorial representation to the fraction multiplication symbolically than on the initial assessment, still more than a half of the students

were not able to express the connection. Hence, ST Math usage was not a significant defining factor for this problem.

Application Problem 3: Concept of fraction magnitude and its equivalent fraction.

3) Raquel thought about this statement: When pitching, Joe struck out 7 of 18 batters. She said that it was better to say that Joe struck out about 1/3 of the batters than to say that Joe struck out about ½ of the batters. "I think that seven-eighteenths is closer to one-third than one-half," she said. Do you agree or disagree with Raquel? Explain your reasoning.

Nine of the non ST Math students and seven of the ST Math students were able to state that Raquel's reasoning was correct by writing since 6/18 = 1/3 and 9/18 = ½, 7/18 is closer to 1/3 than ½. Four students in the non ST Math class and five in ST Math class disagreed with the reasoning. Two students in the non ST Math got it partially correct since they only showed 6/18 = 1/3 and 9/18 = ½. Seven students in the non ST Math and three in ST Math did not try. In the non ST Math class, the number of students who got the correct answer was the same as at the initial assessment stage but the ST Math class showed improvement, with two more students answering correctly. Hence ST Math slightly assisted the students to understand the fraction magnitude concept and its equivalent fraction better.

Application Problem 4: Representing the concept of fraction multiplication pictorially.

4) Draw a picture to represent 2/3 × 4/5.

In both classes, no one was able to express 2/3*4/5 pictorially. Only one student in the non ST Math class got it partially correct by drawing a rectangle whose width is 2/3 and length is 4/5 and indicating the area of the figure. Although this is correct conceptually, the student did not express what 2/3 and 4/5 indicate clearly. Fifteen of the non ST Math students tried but were unable to express the multiplication pictorially and five did not try.

For the ST Math class, thirteen students tried but could not express it pictorially and two did not try. Many students who tried but could not come up with the right pictorial representation in both groups had the same tendency as they showed on the initial assessment. They drew two circles and divided them into three pieces and five pieces and shaded two of the three and four of the five as a part of a whole. The others drew eight ovals and separated them into three ovals and five ovals and shaded two ovals and four ovals to express the multiplication. This indicated that the students in ST Math did not have any advantage in expressing the fraction multiplication pictorially and the fraction games did not assist them to have a deeper conceptual understanding of the fraction multiplication.

Application Problem 5: Express fraction division realistically and pictorially.

5) Come up with a problem that indicates the expression 1/3 ÷ 1/6.

No one in both classes was able to express 1/3 ÷ 1/6 realistically and pictorially. Eleven students of the non ST Math class tried but they either computed it algebraically or stated a story problem that represented 1/3 ÷ 6 as "James had 1/3 of a chocolate bar and he divided the bar into 6 friends." They misunderstood ÷ 6 as equivalent to ÷ 1/6. Likewise, their story problems confirmed that they carried the idea of 'parts of a part' meaning of division. This revealed their fragile conceptual understanding of a fraction division. Ten students did not try.

For the ST Math students, seven students tried. These seven students expressed the division in the same way as the non ST Math students expressed or expressed it as "1/3 of pizza was made and 1/6 of them were taken. How much pizza is left?" However, that indicates 1/3 * 1/6. Therefore there was no evidence that ST Math assisted the students to deepen their fraction division concept.

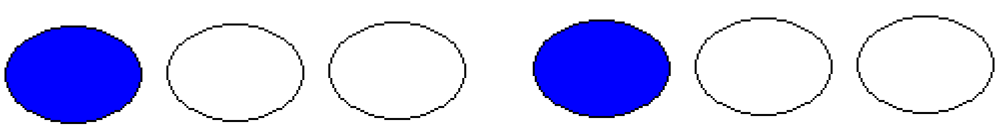
Application Problem 6: Express fraction division realistically.

6) Choose a story problem that represents 1/3÷ 8.
a. How much chocolate will each person get if 8 people share 1/3 lb of chocolate equally?
b. One of three people wants a piece of chocolate. There are 8 pieces. How many pieces will the other two people get?
c. Eight friends get 1/3 of a chocolate bar. How many chocolate bars will we need to buy?
d. Both (a) and (c).

In the non ST Math class, six students selected the correct answer choice (a), two selected (b), ten chose (d), and five did not try. For the ST Math class, seven got the right choice (a), one chose (b), and seven selected (d). These who selected (d) were not able to discern the difference between 1/3 ÷ 1/8 and 1/3 ÷ 8. According to the result, since more students in the ST Math class were able to select the correct answer choice, the fraction games assisted them to choose the correct realistic representation of the symbolic representation of the fraction division. However, as seen in Problem 5, which aimed at the same concept as Problem 6 aimed at, the students were unable to express the fraction division concept pictorially and realistically by creating their own story problem, which represents 1/3 ÷ 1/6, although they were able to select the correct story problem here, representing 1/3 ÷ 8. Consequently, this indicated that the students gained their fraction division concept marginally through the game play.

Application Problem 7: Conceptual understanding of fraction magnitude and its pictorial representation.

7) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"
1. Explain how Kevin is viewing this problem.
2. Explain how James is viewing this problem.

Among the non ST Math students, three students were able to discern the differences of Kevin's view and James's views by stating the view of James was wrong since he added the denominators. Nine students got it partially correct because they only stated that Kevin's view was looking at the two sets of three ovals and added one shaded oval from each group, expressing 1/3 + 1/3 = 2/3. They also stated that James's view looked at the six ovals as one set and shaded two ovals out of the six, expressing 1/3 + 1/3 = 2/6, but they did not state James view was incorrect, changing the unit of the fraction. One student stated the both views were correct and eight students did not try.

In the ST Math class, five students stated the Kevin's view was correct because he did not change the denominator to express 1/3 + 1/3 = 2/3, contrary to the James' view. Three stated otherwise. Seven got it partially correct because they only stated how each person looked at the representation of 1/3 + 1/3, but did not point out that James's view was not correct. The ST Math class had more students who were able to discern the view of Kevin and James correctly. Hence, playing the ST Math fraction games strengthened their fraction magnitude concept by connecting the pictorial representation to the symbolical representation of fraction addition.

Overall the interventions did not indicate a significant conceptual difference in fraction understanding, compared to the non ST Math students. However, some of the ST Math students revealed a glimpse of conceptual gain thorough a few problems such as Problem 3, 6 and 7. In Problem 3, more ST Math students got the correct answer than their counterparts. For Problem 6, regardless of the fact that no one was able to represent the previous problem realistically on their own from scratch, more ST Math students were able to select the correct realistic representation of 1/3 ÷ 8. Consequently, the ST Math games have assisted their division concept partially. Lastly, in Problem 7, more ST Math students could explain the concept of fraction magnitude

pictorially.

Post Assessment Interviews

The researcher conducted the post assessment interviews with the same six students who participated in the initial fraction assessment interviews. The purpose of the interviews was for the researcher to examine how the natural number bias the students in the ST Math class possessed in the initial interview was reduced and how their fraction conceptual understanding had changed based on the Lesh Translation Model, after they played the ST Math fraction games for eight weeks, compared to the students in the non ST Math class. All six students in both the non ST Math and ST Math classes showed improvement on their post assessment scores. Overall through the interviews, the natural number bias in fraction computation was persistent and the ST Math students indicated more conceptual understanding in the concept of magnitude, fraction addition and multiplication, compared to the non ST Math students.

For the ST Math students, Craig (Average) completed up to Game 1: Visual Fraction Concepts and Game 2: Fraction on the Number Line and played Game 3: Comparing and Equivalent Fractions, partially. Jordan (Average) and Kylie (High) have played through the entire sequence of the games.

Post Interviews with the ST Math Students.

Question 1: Fraction magnitude and its representation.

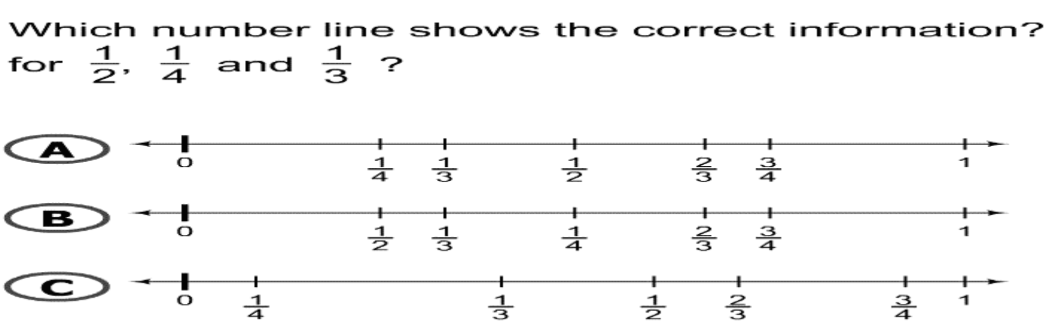


Figure 18. Post Interview Question 1 with ST Math: Display of a number line and a fraction magnitude

All three students got the right reasoning for selecting (A). Jordan (Average) selected (C) on the initial assessment but he selected the right answer on the post assessment and provided the right reasoning as well by stating, “The half marker seems directly in the middle and ¼ marker is before the 1/3 marker. Hence that more makes sense than (C).” At the initial interview, he lacked the fraction magnitude concept, not being able to connect the number line representation to the symbolic representation of the fractions correctly. The students showed their understanding in the magnitude concept.

Question 2: Fraction density.

2) How many fractions are there between 1/3 and ¼? Explain your rationale for your answer.

Craig (Average) responded there was no fraction between ¼ and 1/3 and he was unable to explain why. Jordan (Average) and Kylie (High) provided the right reasoning why there are infinitely many fractions in the intervals. On the initial assessment, Jordan (Average) did not write down any answer and during the initial interview, he revealed that he thought there are countless numbers in the interval, but he did not write it down because he was not completely sure. However, he answered “Infinitely many number exist because endless amounts of fractions can fit between.” Comparing to the previous interview, he showed his confidence in his answer.

In the initial interview, Kylie (High) showed the natural number way of thinking in the fraction density concept by stating there was one fraction that existed in the interval. However, in this post interview, she answered, “There are infinitely many numbers that exist between because there are so many tiny points in there.” This indicates that her natural number bias in the concept had diminished. Only one student still showed the bias in this problem.

Question 3: Fraction magnitude.

3) Compute 1/15 + 1/12. Why do you think a common denominator is unnecessary or

necessary to compute it?

Craig's (Average) reasoning for the necessity of a common denominator for adding 1/12 + 1/15 did not show any conceptual improvement. Although he knew the right procedure to compute the addition, when he was asked why a common denominator is necessary, he responded:

We cannot simply add across. All I know is that. That is what I have learned. A denominator has to match to add.

His response clearly showed that he did not have the fraction magnitude concept. On the contrary, Jordan (Average) showed his conceptual gain. During the initial interview, he was unable to answer the same question because he had not taken any mathematics for ten years. On the post-interview he showed more developed understanding:

<u>Jordan:</u> You cannot just add straight across as 1+1 and 15+ 12.

<u>Researcher:</u> Right. It is not a right procedure.

<u>Jordan:</u> Yes. Adding a common denominator is not a right way to do.

<u>Researcher:</u> You are cutting a circle in 12 pieces and 15 pieces.

<u>Jordan:</u> You want to cut them in the same pieces.

<u>Researcher:</u> Right.

<u>Jordan:</u> Each circle has to have the same size to put them together.

He showed the conceptual gain in the magnitude concept by connecting the symbolical representation to a pictorial representation to explain the necessity of a common denominator.

Kylie (High) was not able to answer the question conceptually, although she was able to explain conceptually in the first interview. She answered "A common denominator is necessary to add them because you just cannot add as that" and did not provide the reason from the conceptual view as she did in the previous interview. Hence, only Jordan (Average) revealed he

gained his conceptual understanding of fraction magnitude.

Question 4: Natural number bias in fraction operation.

4) In 3 < 3/X, is the statement true or false? If you think it is true, explain why you think it is true. If you think it is false, explain why you think it is false.

All three stated that 3 < 3/X was a false statement. The reason why Craig (Average) thought the inequality was false was because the fraction 3/X was smaller than the whole number 3. This clearly indicates his natural number bias in fraction operation and he assumed the X represented a natural number. Jordan (Average) had the same reasoning as Craig (Average), initially. However, during the interview, his view changed:

<u>Jordan:</u> I guess I assumed because you are dividing the X, the answer is going to be smaller.

<u>Researcher:</u> Okay. So your initial response was because it is a division. That is why you thought this was going to be smaller than 3?

<u>Jordan:</u> Yes. But I guess it depends on the value of the X.

Though he stated that X could be any number, he could not justify the statement for different X values.

Kylie (High) substituted a natural number in for X and concluded the inequality was false:

<u>Kylie:</u> No matter what number I put it in the X, it did not. If I put 10, 3/10 and still going to be less.

<u>Researcher:</u> You thought the X was a positive number such as 1,2,3,4?

<u>Kylie:</u> Yah.

<u>Researcher:</u> So you have never thought the X was going to be like ½, fractions or decimals?

Kylie: No.

The natural number bias for this problem was not diminished. Although Jordan (Average) was able to see the X could be any number, he did not mention the X could be a decimal or a fraction.

Question 5: Fraction multiplication.

5) *In 2/5 * 3/5, guess if your answer is going to be larger or smaller than 3/5 without calculating. If you guess the answer is going to be larger than 3/5, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.*

Every student computed the multiplication correctly. In the interview, when Craig (Average) was asked if he thought the answer of 2/5*3/5 makes bigger and multiplication makes bigger, he answered "The answer is going to be smaller than 3/5 because the denominator is going to be 5*5 = 25. No, multiplication does not make bigger because multiplying two fractions could make it smaller." Although his explanation was not fully developed conceptually, he explained why the answer was going to be smaller than the multiplicand from the procedure of fraction multiplication. For the same question, initially Jordan (Average) responded the answer was going to be larger than 3/5, but he recognized that multiplying two fractions makes smaller:

Researcher: When you guess your answer of 2/5*3/5, your answer will be smaller or larger than 3/5?

Jordan: Larger.

Researcher: Why do you think it is larger than 3/5?

Jordan: Do you mean 2/5*3/5 is larger than 3/5?

Researcher: Right.

Jordan: I guess it seems like it would be larger because you are multiplying. But I guess

since these are two fractions, it creates a smaller piece.

As can be seen the excerpt, he initially showed the natural number bias “multiplication makes bigger”, but he recognized that multiplying two fractions creates smaller pieces.

The case of Kylie (High) was very similar as Jordan’s case. She initially thought that the answer of the multiplication was going to be greater than 3/5 because it is a multiplication. However, once she compared 3/5 to the answer 6/25, she reassessed that the multiplication does not always make bigger.

The researcher asked Kylie (High) why the multiplication makes smaller and she answered, “Because a denominator. Since you are multiplying, a denominator, the pieces get smaller.” Despite the fact that she indicated the existence of the natural number bias initially, she was able to see that is not always the case by explaining the reason from the concept of fraction multiplication. Therefore, all of them showed their conceptual gain in fraction multiplication and made sense that fraction multiplication made smaller throughout the interview, although two showed the bias of “multiplication makes bigger” initially.

Question 6: Question 6: Fraction division.

6) In 4/5 ÷ 3/7, guess if your answer is going to be larger or smaller than 3/7 without calculating. If you guess the answer is going to be larger than 3/7, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.

No one was able to explain why the answer of 4/5 ÷ 3/7 is going to be larger than the divisor conceptually. When the researcher asked them if the answer was going to be smaller or greater than the divisor, Jordan (Average) and Kylie (High) stated the answer of the division was going to be smaller and this showed the indication of the natural number bias. Craig (Average) thought otherwise though. However, his reason for his statement is attributed to the division

procedure 'keep, change, flip', not to the concept of the division. Craig (Average) stated "In the process of the division, you do keep change flip and you multiply and that makes it bigger." He did not explain his reasoning conceptually by stating why the division becomes multiplication and the divisor is changed to its reciprocal, pictorially or realistically. Jordan (Average) had the natural number way of thinking and Jordan's reasoning for the answer to be smaller was originated from the wrong memorization of the computational algorithm:

Researcher: Can you guess that the answer of 4/5 ÷ 3/7 is going to be smaller or larger than 3/7?

Jordan: Smaller. The answer is going to be a little less than 3/7.

Researcher: Why do you think this is going to be smaller?

Jordan: I guess could do 5/4 times 3/7 and two fractions multiplying together makes it smaller number.

As seen in the response, he memorized the procedure wrong and this itself revealed that his memorization was not rooted in the division concept.

Kylie (High) implied that the answer was going to be smaller because this is division. After she calculated the division and compared the answer 28/15 to the divisor 3/7, she recognized that 28/15 is greater than 3/7:

Researcher: You said the answer is going to be smaller. So you think it is a division problem, you think the answer is going to be smaller?

Kylie: Yes. Is it not?

Researcher: Let's see. You got 28/15 for 4/5 ÷ 3/7. Which one is greater 28/15 or 3/7?

Kylie: 28/15.

Researcher: Now you saw a fraction division could make bigger. Let me ask you this.

Why do you think that a fraction division could make bigger?

<u>Kylie:</u> Why a fraction division could make bigger? I really do not know why you even flip it. It is something happening in there I guess.

<u>Researcher:</u> Okay. So because we flip, you think a fraction division could make bigger? But you don't know why you need to flip?

<u>Kylie:</u> Yeah.

Although Kylie (High) still held the bias, she suspected that the reason why the divisions answer became greater than the divisor had something to do with the division algorithm. During the first interview, Kylie could not even have this suspicion, not answering anything for the same question. This certainly disclosed thinking happening in her reasoning between the initial assessment and the post assessment when asked her reasoning. All three did not show the conceptual gain in fraction division and the natural number way of thinking "division makes smaller" was persistent in Jordan (Average) and Kylie (High).

Question 7: Pictorial representation of fraction multiplication.

7) In the story problem, "Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?" Explain your answer pictorially? Why do you think this problem represent multiplication?

They saw the story problem implied a division problem when they read "Suzy gave half of her share to her friend." No one thought that this story problem could indicate the multiplication problem and did not see the relationship between fraction multiplication and division from the pictorial representation of the story. Craig (Average) responded:

<u>Researcher:</u> Suzy gave half of her share 1/3 of chocolate bar. What do you think? How much did her friend have?

Craig: 1/6

Researcher: How did you calculate?

Craig: 1/2, cut in half of 1/3 and 2 pieces insides of 1/3 portion. So 1/6.

Researcher: So you are doing 1/3 divided by 2?

Craig: Yeah. 1/3 divided by 2.

Researcher: Good. My question is when you read this question, did you think that this problem was a multiplication problem or a division problem?

Craig: When I read it, I just thought she was giving it away some stuff. So felt like dividing it.

Researcher: So you thought it was a division problem?

Craig: Yeah. But as soon as you started asking further, I realized hey this is a multiplication problem.

He connected his reasoning why the story problem represented a division problem to the pictorial representation but did not see that the equivalent relationship from the pictorial representation.

Jordan's (Average) reasoning for this as a division problem was the same as Craig's reasoning. Jordan reasoned, "Suzy gave her half of her share to her friend. So I divide it into 2 parts which is 1/6." From the pictorial representation he drew, he was unable to see the relationship between fraction multiplication and division. The researcher asked him if he knew "half of Suzy's share 1/3" could imply multiplication, he stated "No I did not know that. So like ½ times?" This response confirmed that he was not aware of the relationship.

Since Kylie (High) wrote down ½*1/3 on the post- assessment, the researcher asked her questions on why she multiplied:

Researcher: Okay. Why did you multiply?

Kylie: Because if I divide, it does not give me the right answer.

Researcher: So you wanted to divide it first?

Kylie: Yeah I did it first. And I was like.

Researcher: So when you divided, you did 1/3 divided by ½ or 1/3 divided by 2? Which one you did?

Kylie: 1/3 divided by ½.

Researcher: Okay.

Kylie: Then I swapped it to 2.

Researcher: When you saw this problem, did you think that this was a multiplication problem?

Kylie: I thought it was a division. Then I thought divided it by ½.

Researcher: Let me ask you a question. What is 10% of 100?

Kylie: Oh now I get it. I should divide it by 2 then have me 2/1 and flip it and multiply it.

Kylie thought this represented a division problem but since she derived 1/3 ÷ ½ instead of 1/3 ÷ 2, form the pictorial representation and it did not have her the right answer, she changed it to 1/3 *1/2. Therefore, Kylie could not connect the pictorial representation to symbolic representations of the fraction multiplication and division and this indicated her fragile conceptual understanding in fraction multiplication. However, once the researcher aided her to see the relationship between fraction division and multiplication, she was able to connect the relationship from the procedural angle.

Question 8: Realistic representation of fraction division.

8) In the problem, "Come up with a problem that indicates the expression 1/3 ÷ 1/6." Explain why the problem that you came up with is correct expression of the division

pictorially. By using the pictorial expression, explain why the answer of the division is larger than 1/6?

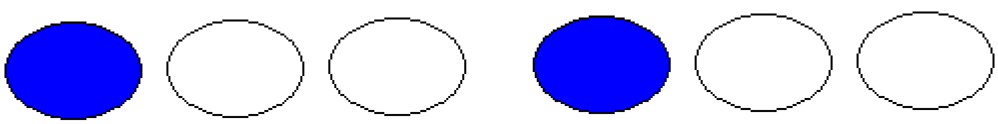
All three were unable to come up with a realistic problem which represents 1/3 ÷ 1/6. Jordan (Average) could not come up with any realistic problem representing the division. As in the case of the initial interview, their story problems came from parts of a part notion. This aligned with the result of the previous question, where they showed the tendency of parts of a part implying division. In this question, they revealed their division concept was heavily influenced by the notion, by presenting the parts of a part for the division symbolic representation.

Craig's (Average) story problem was, "My mom made 1/3 batch of cookies. And 1/6 of them are taken. How many cookies remain?" and Kylie's (High) story problem was, "Carl shares his 1/3 bar of chocolate with 6 people. How much does each individual have?"

Craig's problem represented 1/3* 1/6 instead of 1/3÷ 1/6 and Kylie's realistic problem represented 1/3 ÷ 6 instead of 1/3 ÷ 1/6. She had the same misconception, which ÷ 6 was equal to ÷ 1/6. The common finding between them is that they tend to represent fraction division as partitive model, instead of thinking about the fundamental measurement concept of "how many 1/6 can go into 1/3?" Since the notion of a part of a whole is so strong, they would not have room to connect the symbolical expression to a realistic representation from the fundamental measurement concept. They did not show the deep conceptual gain in fraction division.

Question 9: Fraction magnitude concept.

9) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"

1. *Explain how Kevin is viewing this problem.*
2. *Explain how James is viewing this problem.*

Craig (Average) explained the difference of James' view and Kevin's view from the procedural aspect of fraction addition by responding "Kevin is doing right because he was not adding a denominator and James looked at them altogether and got 2/6, but he did wrong by adding a denominator." When the researcher asked why a denominator cannot be added. Craig answered "I do not know. I was taught as don't add." Hence his reasoning for a fraction magnitude was purely procedural.

Jordan (Average) responded the same as Craig (Average) did. Jordan stated, "Kevin sees this as two separate entities. These can be added together because they have a common denominator 3. And James sees this as a whole." When the researcher asked that James' view was correct or not by doing $1/3 + 1/3 = 2/6$, he responded he could add as James did. Because of that, the researcher went back to Question 3 to have him recall, and Jordan recognized that a denominator cannot be added. Consequently, Jordan was not able to connect the picture of ovals to the symbolical representation of the addition to see that James' view was conceptually incorrect. Jordan seemed he did not really understand the meaning of this question because he indicated the interesting response, "Yeah James had a wrong concept but it worked because this is special case." Not understanding the problem correctly could have certainly affected him not showing the understanding shown in Question 3. Or it also could have indicated that his understanding was not rooted in depth because he was able to explain the same concept in Question 3 but in this different representation of the addition, he could not discern what James did was incorrect.

Kylie (High) answered that James did wrong and Kevin did right. The researcher asked why James did wrong and Kylie replied:

Kylie: Because he added the denominator.

Researcher: Right. Do you know why?

Kylie: Because the way I see is like there are imaginary pies. Then only thing is a size again. I can have like one pie (drawing picture of a circle and cutting one into 3rd). I don't know how to draw but one 1/3 piece and then if I have another 1/3 piece in the pie. That does not mean I have two whole pies cut into 3rd and 2/6.

Researcher: Of the same size? So basically you are saying the base does not change?

Kylie: Yah. I am only about the pieces.

Kylie was the only one who was able to connect the pictorial representation to the symbolical representation to explain the concept of fraction magnitude. The other two students did not show conceptual gain in this concept.

Among these three students, Jordan (Average) showed the most conceptual improvement, compared to the initial interview stage. Jordan was able to explain the fraction magnitude pictorially and symbolically in Question 1 selecting (A), instead of selecting (c) as he did on the pre-test. He also indicated his confidence in answering the fraction density reasoning conceptually. Likewise, he revealed his conceptual gain in fraction magnitude in addition in Question 3, from the concept of fraction base. Jordan (Average) and Kylie (High) were able to explain why fraction multiplication could make smaller conceptually and diminished the bias of "fraction makes bigger". On the contrary to the multiplication concept, all three revealed the persistency of the natural number thinking in fraction division, which is "division makes smaller" and no one exhibited their conceptual gain in Question 6.

In Question 7, all of them thought the story problem implied division because a chocolate is divided into different pieces and no one was able to see this partitive realistic representation

could also indicate multiplication. Hence their fraction multiplication concept was weak and the notion, which is a partitive model indicates division, was strong. This agreed with their representations in Question 8 because Kylie (High), Jordan (Average) and Craig (Average), all three forced to represent 1/3 ÷ 1/6 in a partitive model. Consequently, there was no evidence of conceptual gain in fraction division concept. Although Jordan (Average) indicated his conceptual gain in fraction magnitude in addition (Question 3), he was not necessarily able to reveal the understanding, connecting the pictorial representation to the symbolic representations in Question 9. This could be either 1) Jordan did not comprehend the meaning of question or 2) his understanding was partial.

Post Interviews with the non ST Math Students.

Question 1: Fraction magnitude and its representation.

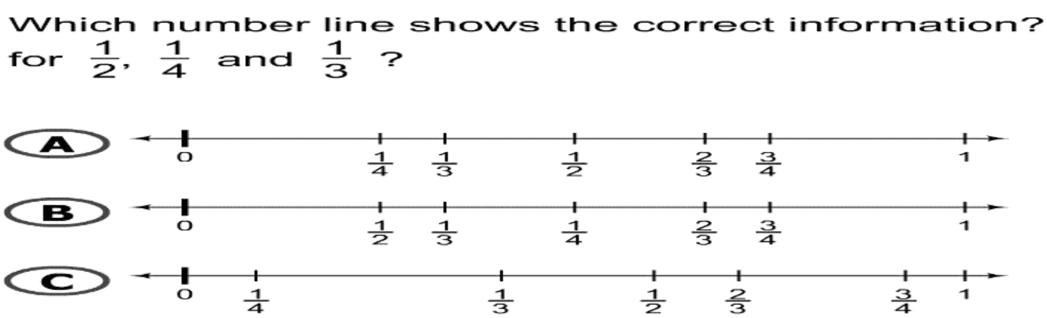


Figure 19. Post Interview Question 1 with non ST Math: Display of a number line and a fraction magnitude

All three students chose the correct choice (A) and their rationale for their selection did not indicate the natural number bias for fraction magnitude. At the initial interview, Alexis (Low) displayed the natural number bias, evaluating the fraction magnitude based solely on the numbers of the denominators, instead of evaluating a fraction as a whole and concluded ¼ > 1/3 > ½. Alexis explained, “1/2 is a half of 1. So 1/4 is a quarter of it and so forth” To investigate, the researcher posed why she did not select (C) because the choice has the fraction in the right

order on the number line. Alexis replied, “No I did not think about choosing (B) because ½ is a half of 1 and should be in the middle.” She certainly diminished the natural number bias for fraction magnitude. Morgan (High) and Caleigh (Low) had the same reasoning as they indicated in the initial interview. Morgan stated that the location of 1/4 and 1/3 are disproportional in (C) and (B) is in the incorrect order. Caleigh reasoned “½ is in the middle, ¼ is 25% of the line. 1/3 is in the between ¼ and ½. C does not have ½ in the middle.”

Question 2: Fraction density.

2) *How many fractions are there between 1/3 and ¼? Explain your rationale for your answer**.***

Contrary to the initial assessment, Morgan (High) and Caleigh (Low) were able to state that there are infinitely many fractions between ¼ and 1/3 with the correct rationale. Caleigh (Low) certainly exhibited her conceptual gain in the density concept. She reasoned, “Because you don’t know how many points are between. So I put infinitely many numbers between ¼ and 1/3.” Hence, Caleigh’s natural number bias in this concept was diminished.

Morgan’s (High) rationale for the existence of infinitely many fractions in the interval was the same as her response in the initial interview, stating “I think about it in a percentage. 33% and 25%. There are many numbers between.” On the contrary to them, Alexis (Low) exhibited the natural number bias, stating there was 0 fractions in between. She could not provide a reason why her answer was 0. Her natural number bias did not diminish. It is important to note that Morgan did not write her answer on the post assessment and Caleigh was the only student who answered it correctly on the post assessment paper. This is why the number of the student who got it correct is listed as 1 (Table 9).

Question 3: Fraction magnitude.

3) Compute 1/15 + 1/12. Why do you think a common denominator is unnecessary or

necessary to compute it?

Morgan (High) was the only student who was able to explain the necessity of a common denominator conceptually. The other two knew the procedure to compute 1/15 + 1/12, but they were unable to state the concept behind the procedure. As seen in the initial interview, Morgan's rationale was from the concept of the fraction proportion stating "the answer remains proportional because adding without having a common denominator does not represent the same". Alexis (Low) and Caleigh (Low) had the very similar responses. Alexis mentioned "In math, this is one of these rules in subtraction and addition. Otherwise you cannot make it. It is a rule." Caleigh reasoned, "You just need a common denominator to make it work in addition and subtraction." Therefore, there was no indication of the conceptual gain in Alexis and Caleigh.

Question 4: Natural number bias in fraction operation.

4) In 3 < 3/X, is the statement true or false? If you think it is true, explain why you think it is true. If you think it is false, explain why you think it is false.

Morgan (High) was the only one who did not display the natural number bias in fraction operation. The other two students displayed the bias. On the initial assessment, Alexis (Low) stated 3 < 3/X was false, but she answered otherwise on the post assessment. Hence the researcher asked her to provide a reason for her answer. Alexis replied that she was guessing. Then the researcher had her think about the statement during the interview and she responded:

Researcher: When you just saw 3 < 3/X, did you think that the X is a natural number, meaning 1,2,3,4 and so like a positive whole number?

Alexis: Yah. I was thinking maybe the X was 1.

Researcher: Okay. So you were assuming that the X was 1?

Alexis: Yah.

Researcher: You never thought about the X was 0.5 or any?

Alexis: No. I didn't know.

Researcher: You assumed the X was a positive whole number?

Alexis: Yah.

Her responses displayed her natural number bias in this operation. Although Caleigh (Low) stated that the X could be any number, she stated that the inequality was false, because she implied the X was any natural number and she did not mean the X was any real number. She responded, "The X can be anything. It could be 1, 2 or 100." When the researcher asked that she thought about the X being 0.5 or any fraction as 1/2, she replied that she did not think that the X was going to be a fraction or a decimal. Therefore the natural number bias was persistent in Alexis (Low) and Caleigh (Low).

Morgan's (High) response was the consistent as the initial interview. She said that the X could be anything in the world and she did not think that the X represented only the natural numbers. However, Morgan had the misunderstanding with the inequality sign as she indicated in the initial interview, which she thought anything on the right side of the inequality sign was greater than the left side. Hence the direction of the sign either $<$ or $>$ did not make any difference in her comprehension of the inequality. Likewise, in this response, she did not discern $<$ from $\leq$. Because of the weak concept of the inequality sign, although she stated that the X could be anything, it is very difficult to discern that she did not have the natural number bias in fraction computation or not. Morgan reasoned:

Researcher: Can you give me the reason? Why we cannot define the inequality?

Morgan: Because X can be 1 and that would make the statement $3 < 3$ true.

Researcher: When you see this, you assume the X is 1 or?

Morgan: No, it assumes the X is anything in the world.

Researcher: What about the X is 3?

Morgan: Still makes. Oh makes true because 3 would be greater than 3/3 = 1.

Researcher: Okay what about X is 0.5 which is ½?

Morgan: 3 is greater than 6.

This is why Morgan thought the statement could not be determined, but she could not justify the inequality with correct understanding. Consequently her reasoning was originated from the wrong reason.

Question 5: Fraction multiplication.

*5) In 2/5 * 3/5, guess if your answer is going to be larger or smaller than 3/5 without calculating. If you guess the answer is going to be larger than 3/5, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.*

Although Alexis (Low) was able to explain why the answer of 2/5 * 3/5 is going to be smaller than 3/5 in the initial interview, she could not explain the why in the post interview. When the researcher asked her that the answer was going to be smaller or greater than 3/5, she stated it was going to be greater because in her head she computed as 3/5*2/5 = 6/5. She executed her wrong procedure in this calculation. Then the researcher prompted her with the correct answer:

Researcher: Okay you said it is going to be greater. Do you think multiplication makes bigger always?

Alexis: For the most part.

Researcher: Okay let's see 2/5 * 3/5 is 6/25.

Alexis: Oh wow. I forgot.

Researcher: Which one is greater 3/5 or 6/25?

Alexis: 3/5 is greater.

Researcher: Do you know why though?

Alexis: Because we are trying to find a common denominator in this fraction multiplication.

The natural number bias she exhibited in the initial interview still existed in her and her response certainly indicated her confusion in her fraction knowledge, by referencing a common denominator. Alexis (Low) did not show conceptual gain in the concept.

Morgan (High) did not display the bias of "multiplication makes bigger". She said because, "fraction, and percentage multiplication could make smaller and a fraction multiplication could make smaller because it is not a whole number." It is necessary note that she excluded the case of an improper fraction in the response. As she showed her conceptual understanding of fraction multiplication in the initial interview, Morgan displayed the understanding in here likewise.

Caleigh (Low) provide her rationale when she was asked what she thought of multiplication makes bigger:

Researcher: Do you think multiplication makes bigger?

Caleigh: Could be smaller or bigger. It depends.

Researcher: What about this 2/5*3/5?

Caleigh: It depends. It would be smaller?

Researcher: Why do you think it would be smaller?

Caleigh: Wait. Hold on. I think it would be smaller.

Researcher: Smaller?

Caleigh: Yeah because if we do like a part of a pie, shade 6/25 pieces of the pie, it is

smaller than 3/5.

Caleigh did not have the natural number bias and she revealed her partial understanding of the concept, comparing the answer 6/25 to 3/5 pictorially to decide which one is greater, but she did not fully explain why the multiplication led to the smaller answer than the multiplicand, pictorially. Hence the natural number bias was persistent in Alexis (Low) and only Morgan (High) was able to explain why the answer becomes smaller, conceptually.

Question 6: Fraction division.

6) In 4/5 ÷ 3/7, guess if your answer is going to be larger or smaller than 3/7 without calculating. If you guess the answer is going to be larger than 3/7, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain why your guess was right. If it is not, explain why your guess was wrong.

There was no indication of conceptual gain in fraction division for Alexis (Low) and Caleigh (Low). Morgan (High) did not hold the natural number bias and revealed her understanding of the concept. On the other hand, the natural number bias of "division makes smaller" did not diminish in Caleigh (Low). Alexis (Low) showed confusion in the division procedure although she was able to execute it in the initial interview. When the researcher asked her to guess if the answer of 4/5 ÷ 3/7 was going to be smaller or greater than 3/7, Alexis answered "Finding a least common denominator of 5 and 7." This clearly shows that she mixed the procedure of the fraction addition/subtraction with the procedure of the division:

Researcher: Can you guess the answer of the 4/5 ÷ 3/7 is going to be greater or smaller than 3/7?

Alexis: Yah. Finding a least common denominator. Multiplying both by. Multiplying this one by 5.

Hence she did not possess the conceptual understanding of both fraction magnitude and fraction

division. Because of the confusion, the researcher guided her to the correct division procedure. With the guidance, Alexis (Low) calculated the division and was asked:

Researcher: You got 28/15 as the answer. Which one is greater 3/7 or 28/15?

Alexis: 28/15.

Researcher: When you were computing the division, did you think that division makes smaller?

Alexis: Do I think division makes smaller? I think that is not necessarily because sometimes you get a whole as an answer and a whole number is bigger than a fraction. So it depends.

In her response, although Alexis did not think that division makes always smaller, her reasoning was influenced by the natural number bias because she thought a whole number was always greater than a fraction. This indicated Alexis possessed the fragile fraction conceptual understanding.

Alexis's case was difficult to examine because her division computation skill was not intact and we had to go over the procedure during the interview. Hence the review process influenced her response to the question "Do you think that division makes smaller?" Because the question was asked after she computed the division.

Contrary to this, Morgan (High) did not have the natural number bias of 'division makes smaller'. She reasoned:

Researcher: Do you think division makes smaller?

Morgan: Yes except in the cases of fraction and decimals.

Researcher: Why?

Morgan: Because if you have one chocolate bar and you give it away to 6 people equally,

now it has 6 chocolate pieces of a whole chocolate bar. They are still one chocolate bar and it does not make more chocolate, but more people can enjoy it because it is divided.

Researcher: Because of the reason, do you think the fraction division makes bigger?

Morgan: Yes.

Because of the rationale, she did not think that the answer of 4/5 ÷ 3/7 was going to be smaller than the divisor. The fundamental concept of fraction, which is how many 1/6 pieces of chocolate can go into 1 chocolate bar, was displayed in the response. Consequently, Morgan displayed the depth of her understanding in the concept.

Caleigh's (Low) response was identical to the initial interview. She said, "I would assume that the answer of 4/5 ÷ 3/7 is going to be smaller because dividing is a fast way of subtraction." She had the natural number bias in division. After she computed the division, she saw that the answer was greater:

Researcher: Which one is greater 3/7 or 28/15?

Caleigh: 28/15 makes bigger.

Researcher: Right. Do you know why though?

Caleigh: Because it becomes a multiplication?

Caleigh was not completely sure about her reasoning, but clearly the response indicates her rationale came from the fraction division procedure. Therefore, Caleigh could not explain why the answer is going to be greater than the divisor, along with the concept of fraction division.

Question 7: Pictorial representation of fraction multiplication.

7) In the story problem, "Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get?" Explain your answer pictorially? Why do you think

this problem represent multiplication?

When Alexis (Low) read the story “Suzy share a half of her share (1/3) to her friend”, she wrote down ½ *1/3. The researcher asked her why she wrote this down:

<u>Researcher</u>: How did you get this 1/2 * 1/3?

<u>Alexis</u>: The word ‘of’ does not mean always multiplication?

Her response was identical to the initial interview and was originated from her knowledge, which was ‘of’ meant multiplication and could not explain why the story could indicate multiplication from the pictorial representation she drew.

Morgan (High) thought the story represented division because “it divided the Suzy’s share into half.” Then she continued, “it could be whether division or multiplication because ‘of” means multiplication and divided the Suzy’s share by 2 is like 0.5 times.” Morgan thought this represented division initially, even though she correctly indicated that the story problem could be either division or multiplication. She did not explain why this problem could be a multiplication problem conceptually from the pictorial representation.

Although Caleigh (Low) got the correct answer on the post assessment, she could not explain how she got ½*1/3 = 1/6 for Suzy’s friend’s portion:

<u>Researcher:</u> Your answer said Suzy’s friend got 1/6. How did you get 1/6?

<u>Caleigh:</u> Basically I did Suzy shared it of her friend ½ of 1/3.

<u>Researcher:</u> did you divide it or multiply?

<u>Caleigh:</u> Divided it.

<u>Researcher:</u> Did you divide 1/3 by 2?

<u>Caleigh:</u> I did divide 1/3 by ½.

Since she could not explain her reasoning, the researcher started asking questions to assist her.

Researcher: Okay. Let me ask you this. You said Suzy gave ½ of her share. When you saw the problem, did you see that this was a multiplication or division problem?

Caleigh: I thought it was divided.

Researcher: You did not really know why it is 1/3 * ½ although you wrote it down as that?

Caleigh: No.

Researcher: So go back to the question again. Half of her share 1/3. So when you see half of 1/3, do you think this implies a division or multiplication?

Caleigh: It is a multiplication! Because of the 'of'. I had somebody tell me that.

Toward the end, Caleigh saw that 'half of Suzy's share' could indicate multiplication but that came from knowledge without understanding. Therefore, there was no conceptual gain in the fraction multiplication concept In short, their fraction multiplication understanding did not show any development from the initial interview.

Question 8: Realistic representation of fraction division.

8) In the problem, "Come up with a problem that indicates the expression 1/3 ÷ 1/6." Explain why the problem that you came up with is correct expression of the division pictorially. By using the pictorial expression, explain why the answer of the division is larger than 1/6?

No one was able to come up with a realistic story problem that represents 1/3 ÷ 1/6. Alexis (Low) intended to come up with a story problem, tweaking the previous question. Her story problem was:

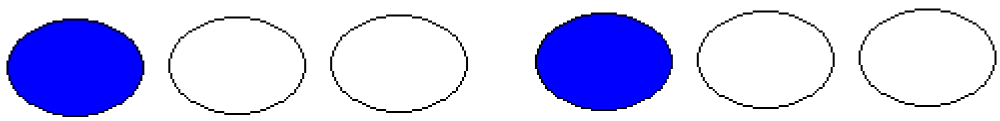
Suzy got one chocolate bar and she divided the bar into 3 pieces. Then Suzy gave her half of her piece to her friend. How much did Suzy have after giving the portion to her friend?

Alexis' representation represented 1/3* ½ instead of the division. Neither Morgan (High) nor Caleigh (Low) came up with any realistic representation, although they tried during the

interview. Therefore, there was no deeper gain in the realistic representation of fraction division for this problem. As Alexis' representation exhibited, the notion of division implying a part of a whole was strong as the result of the previous question indicated. They possessed the division computation knowledge, but did not hold deeper understanding in fraction division.

Question 9: Fraction magnitude concept.

9) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?"
1. Explain how Kevin is viewing this problem.
2. Explain how James is viewing this problem.

Compared to the initial interview responses, Morgan (High) and Caleigh (Low) showed greater conceptual understanding in the magnitude concept. However, Alexis (Low) showed that she still had the difficulty in understanding the concept since her initial response was the same as the response in the post-interview that James's view, which was 1/3 + 1/3 = 2/6, was correct:

Researcher: Why do you think James is right?

Alexis: Because he has a common denominator. So.

Researcher: Can we add a common denominator though as 1/3 + 1/3 = 2/6?

Alexis: I think they stay the same.

Researcher: Right.

Alexis: Means he is wrong. Kevin is right.

Researcher: Why do you think Kevin is right?

Alexis: Because we have a common denominator, meaning we can add but a common denominator stays the same.

Researcher: Why does the common denominator stay the same?

Alexis: You have 3 pieces of chocolates and those 3 pieces stay the same. I am just trying to make sense a little bit. It is a rule. That's just a way it is.

After the researcher assisted her to discern which had the correct view, the researcher asked her to explain why Kevin is correct. Alexis' initial response was originated from the procedural view but certainly the question made her question her reasoning to have her think more conceptually, by her responding the 3 pieces of chocolate is not going to change. Although Alexis displayed the glimpse of the conceptual approach for her explanation, she ended up settling the procedural view, mentioning "it is a rule."

Morgan (High) explained what James did wrong in adding the denominators:

James didn't put it in proper proportions. 1/3 + 1/3 = 2/6 = 1/3, does not make any sense, because it take out of the realm of equal. Like you can't compare an apple to an orange without having the same base.

Morgan explained the reason why the unit stays the same in fraction addition by referencing comparison of two different objects, apple and orange.

Caleigh (Low) explained what James did wrong, "Because when you are dealing with pie charts, when you add two pieces who were cut into different sizes, the size of the added pieces do not make any sense." Caleigh connected the symbolical representation to the imaginary pieces of pies to explain why Kevin's view is wrong. Although her words used to explain her rationale were not mathematically sound, Caleigh exhibited some sort of understanding of fraction magnitude but her understanding was not completely sound because she did not exhibit the equivalent fraction concept. The noteworthy finding here was that Caleigh did not indicate the conceptual understanding of the magnitude in Question 3 by stating having a common denominator is a mathematical process. Because of the fact, the researcher explained the concept

of the fraction magnitude in the question. This could have influenced Caleigh's response here.

Overall, Alexis (Low) did not show any indication of conceptual gain and exhibited the natural number bias throughout the interviews. During the post interview, Alexis exhibited her confusion she did not display during the pre-test interview, with the fraction computation procedures, constantly stating common denominator in fraction multiplication and division. Caleigh (Low) certainly diminished the bias in density and fraction multiplication, but she was unable to explain why the answer of fraction multiplication 2/5 * 3/5 becomes smaller conceptually. Hence she also did not indicate strong conceptual gain during the post-interviews, although she revealed the glimpse of conceptual approach in her explanation in fraction magnitude (Question 9). Morgan (High) explained Question 9 conceptually, compared to the pre-test interview, but besides this, her explanations for each question was the same as the initial interview response. No one was able to connect the pictorial representation to multiplication in Question 6 and no one was able to come up with a correct realistic representation of 1/3 ÷ 1/6, in Question 8.

Summary of the Post Interviews

Conceptual change in fractions.

Students of both classes diminished the natural number bias in fraction density related to question 2. However, the bias in fraction computation, in Question 4, was strong and persistent in all six students. As for fraction magnitude, in Question 3, Jordan (Average), ST Math student, displayed conceptual gain. For the non ST Math class Morgan (High) was only one who showed the same conceptual understanding in both interviews. In Question 5, two of the ST Math students, Jordan (Average) and Kylie (High) diminished the natural number bias of 'multiplication makes bigger' by making sense from the fraction multiplication concept.

However, the Non ST Math students did not show their understanding in this concept. Alexis (Low) had the bias and did not show conceptual gain. Although Caleigh (Low) diminished the bias, she was unable to explain the reason why the answer of 2/5*3/5 was going to be smaller than the multiplicand. Morgan (High) was the only one who did not have the bias and exhibited her conceptual understanding as she did in the initial interview.

Regarding Question 6, Jordan (Average) and Kylie (High) of ST Math students exhibited the bias of 'division makes smaller' as they did in the initial interview and they could not explain why the answer of 4/5 ÷ 3/7 is going to be greater than the divisor conceptually. However, thinking was revealed in Kylie's response. During the initial interview, she simply stated she did not know in her response, but during the post interview, she suspected that something happens in the division procedure 'keep, change, flip' and that is going to have the answer bigger than the divisor, but she could not explain it. Although Craig (Average) did not reveal the natural number bias in his interview response, his reasoning was attributed to the division algorithm 'keep, change, flip'. Among the Non ST Math students, as the case of the initial interview, Morgan (High) was the only student who did not have the bias and explained the reason from the conceptual aspect of division. Alexis (Low) could not execute the division computation correctly and did not show conceptual gain. Caleigh (Low) thought the answer was going to be greater than the divisor because it becomes multiplication. Consequently, she diminished the natural number way of thinking from her initial interview, but her reasoning was originated form the division procedure. Hence, the bias persistent in ST Math students and there was not conceptual gain in both classes in division.

The bias in fraction operation in Question 4, was significantly strong and did not diminish in all of the students in both classes but one. They reasoned either a fraction is always

smaller than a whole number or substituting a natural number in the X makes the inequality 3 < 3/X false. Even though Caleigh (Low) of the non ST Math student stated that the X was going to be any number, when the researcher asked what numbers she substituted in the X, she substituted natural numbers in the X. Despite the fact that Morgan (High) stated the correct answer on both pre and post- tests, her reasoning came from the misunderstanding of inequality. Hence, it was difficult to conclude that she did not possess the bias.

For the application problem, Question 7, all three ST Math students initially thought the story problem represented division because they possessed the strong notion of 'parts of a part' describing division. None of them comprehended the problem could also express multiplication from the pictorial representation they drew and none could connect the symbolical representation of the division to the multiplication. This trend was seen in the non ST Math students as well. Two of them replied that the story was division, because their pictorial representation displayed parts of a part. Although one student indicated that the story indicated division, her rationale came from her knowledge without understanding, stating that the word 'of' means multiplication. Hence their rule-based understanding in fraction division and multiplication remained significant.

No one in both classes was able to come up with a realistic story problem representing 1/3 ÷ 1/6 in Question 8. All of them who shared their story problems exhibited a very similar realistic problem to Question 7, which displays the partitive model. The way of thinking they revealed in the question, which is 'parts of a part' is division, was confirmed. Since they were influenced by the thinking heavily, they did not even approach the symbolical representation from the fundamental measurement concept of division, which is 'how many 1/6 can go into 1/3?' As they exposed their fragile conceptual understanding of division in Question 6, they did

not have the concept in depth.

In Question 9, for the ST Math students, Kylie (High) was the only student who was able to connect the pictorial representation of the ovals to the symbolical representation of the addition to explain why James's view was incorrect. Craig (Average) and Jordan (Average) were able to discern what James did was not appropriate but they did not explain conceptually by connecting the pictorial representation symbolically. Their reasoning was attributed from their procedural knowledge of not adding a common denominator. On the contrary, two of the Non ST Math students indicated conceptual gain in this question. Morgan (High) was able to able to explain the reason the James' view was incorrect pictorially. Although Caleigh (Low)'s explanation was not mathematically sound and did not display complete understanding, her explanation indicated partial understanding in the fraction magnitude concept. Hence more non ST Math students, revealed their conceptual depth in this concept.

Conclusions

Pre-Test and Post-Test

On the pre-test, the students in both classes displayed the natural number bias in all three areas: fraction magnitude, fraction density, and fraction computation. Similarly, their fraction understanding and their computational knowledge were influenced by the bias. Because of this fact, the students did not execute the computational problems accurately nor were they able to express their fraction knowledge, pictorially, realistically, symbolically or verbally to indicate the depth in conceptual understanding. There was no statistically significant difference in the fraction competency between the two classes on the pre-test. This indicated that the students in the ST Math class and the non ST Math class possessed the same fragile fraction conceptual understanding before the study was taken place.

On the post-test the students in both classes were able to execute the computation problems with much higher accuracy, compared to the pre-test while diminishing the natural number bias in the fraction magnitude concept. For instance, although many tended to add the numerators and denominators of two fractions without having a common denominator initially, they were able to determine common denominators and add correctly on the post-test. As a whole, the students in both classes improved procedurally. Specifically, the ST Math students improved dramatically in Problem 6:(2) and 6:(3), fraction subtraction. Therefore, the ST Math game play may have strengthened their computational skills in this area.

The bias in the fraction density concept and fraction magnitude concept were persistent regardless of the class. The students' answers for these problems, Problem 2 and Problem 7:(2), were heavily influenced by the natural number bias on the post assessment as well. Consequently, the ST Math game play did not necessarily provide an advantage on problem 7:(2), though this problem did not align with the problems in the ST Math game. As for Problem 2, regardless of the persistency of the bias, more students in ST Math answered correctly. Consequently, the game play assisted some of the students to acquire the density concept.

For the problems testing students' conceptual understanding, the statistical result did not show the significance of the ST Math intervention, but both classes did show improvement. This implied that ST Math did not allow the students to deepen their fraction conceptual understanding more compared to the non ST Math class. For problems 6 and 7 though the ST Math class had more students who got the correct answer on these questions. Specifically, they were able to select the right realistic representation of the symbolic representation of the division in Problem 6. They also were able to connect the pictorial representation to the symbolical representation of the fraction addition problem to explain why the Kevin's view was incorrect.

Consequently, although overall the statistical result was not significant in the improvement of the students' conceptual understanding on the assessment between the two groups, at the microscopic level, a greater percentage of the class of ST Math students revealed their conceptual gain in fraction division and fraction magnitude and diminished the natural number bias in the fraction density and magnitude.

Interview Responses

In the pre-interviews, the natural number bias in magnitude, density and computation was clearly presented. Most of the students who participated in this interview showed the bias in magnitude by fraction multiplication and division. All but one from the non ST Math class thought that multiplication makes bigger and division makes smaller. Similarly, they were unable to explain why fraction multiplication does not always make bigger and why fraction division does not always make smaller conceptually. For fraction density, only two, one from each class, was able to exhibit their understanding, although these two students were influenced by the natural number bias on their initial assessment.

For the inequality $3 < 3/X$, the students indicated the existence of the bias on both pre and the post- tests. Although Morgan (High) of the non ST Math class provided the correct answer, stating the inequality cannot be determined on the tests, her rationale for the statement was attributed to her misunderstanding of the inequality sign. Hence it is very hard to tell she did not have the bias or not. All the other students possessed the natural number bias in this computation, substituting a natural number in for X, especially $X = 1$, stating false.

Concerning their initial fraction conceptual understanding, as a whole, the students in both classes could not indicate their understanding in fraction addition, multiplication and division, although they executed the computational procedures. Consequently, their fraction

learning relied on memorization of the procedures without understanding the concept underlying. In a like manner, they were unable to explain the concepts of fraction addition, multiplication and division, pictorially, realistically, symbolically and verbally. Hence they did not reveal strong understanding during the initial interviews.

On the post assessment, more interviewed students of both classes had diminished in the bias in fraction density. Also, all three ST Math students diminished the natural number way of thinking “multiplication makes bigger” and they were able to explain the reason from the multiplication concept. While the non ST Math students did not show the conceptual gain to explain it conceptually. However, in fraction division, the bias of “division makes smaller” was persistent in some students, one in the ST Math and two in the non ST Math. The two students in the ST Math class did not demonstrate the bias, but they could not explain why from the concept of fraction division.

In essence, the intervention did not assist the students to deepen their fraction division conceptual understanding. This is mainly because most of the ST Math students did not play the game on fraction division. Though the ST Math program looks at fraction division from a partitive perspective instead of a measurement perspective which is easier to connect from a realistic representation to a pictorial representation. For the bias in fraction computation, it was persistent and the result was the same as the initial interview. Morgan (High) from the non ST Math was the only one who showed that she did not have the bias, stating the inequality $3 < 3/X$ was undetermined, but even her case, it is hard to make a conclusion that she did not possess the bias because her rationale came from her misconception of the inequality. Consequently, the students had the natural number bias stating it was false.

The student Jordan (Average) in ST Math showed gain in conceptual understanding in

fraction magnitude in Question 3, explaining the necessity of a common denominator pictorially and verbally, compared to the initial stage. Likewise, two of the, Kylie (High) and Jordan (Average), ST Math students revealed conceptual gain in fraction multiplication in Question 5, rationalizing pictorially and verbally and diminished the natural number way of thinking ‘multiplication makes bigger’. However, Kylie and Jordan were not able to represent the symbolic representation 2/3 * 4/5 pictorially on Application Problem 5. Hence, it is fair to say that their conceptual gain in this concept was partial. On the other hand, the non ST Math students showed the conceptual gain in fraction magnitude and fraction division and diminish the natural number way of thinking.

Students in both classes struggled with the concept of division in Question 6. Many still had the bias ‘division makes smaller’ and could not explain why that is not always the case conceptually. However, Kylie (High) of the ST Math student, despite of the fact that she did not show her conceptual gain fully, revealed her disturbance in her response, considering that she did not make any response in the initial interview. She questioned the legitimacy of the connection between the division procedure and the concept, instead of settling for the procedural rationale such as ‘keep, change, flip’ in her response during the post interview. Hence, the ST Math intervention could have caused the perturbation.

The students in both groups revealed the notion ‘parts of a part indicates division’ they had at the initial interviews. They did not consider that Question 7 could express multiplication. Consequently, they were neither able to connect the pictorial representation of the realistic problem to the symbolic representation of multiplication nor see the relationship between multiplication and division form the pictorial representation. Similarly, none of them were able to indicate their conceptual understanding in depth by representing the division 1/3 ÷ 1/6,

pictorially, realistically and verbally. In Question 9, the ST Math student Jordan (Average) did not show any advantage in their conceptual gain although in Question 3, which also aimed at the concept of magnitude, he showed the gain.

Overall it appears that, the ST Math intervention reduced the students' natural number bias in density, and fraction magnitude in addition and multiplication. Conceptually, the game play contributed to the students' conceptual gain in fraction addition and partial conceptual gain in fraction multiplication.

CHAPTER FIVE

Discussion of Findings

Introduction

Understanding the meaning of fractions is one of the greatest difficulties in mathematical learning for many students (Anthony & Walshaw, 2007; Verschaffel, Greer & Torbrynes, 2006; Young-Loveridge, Taylor, Hawera & Sharma, 2007), since it requires a deeper understanding of numbers than whole numbers. Understanding the fraction concepts is a key because it can be a strong indicator of students' mathematical achievement in their academic careers (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012). The concepts are a cornerstone for learning algebra, geometry and other higher mathematics (Fazio & Siegler, 2011). However, mastering fractions has been a prominent hardship for many students (Pantziara & Philippou, 2012).

As the number of students who are enrolled in remedial mathematics course have been increasing, many students are not ready for college level mathematics courses (La Joy, 2013). Moreover, those students in remedial courses tend to fail the courses (Wisely, 2011). In remedial courses, the core focus is mastering the fraction concepts because students have not been proficient in the concepts (Gal, 2000). Consequently, finding a solution to assist college remedial mathematics students to master fractions should be a main endeavor of college mathematics departments so that students can succeed in college level mathematics courses to meet the graduation requirement.

However, mathematics courses offered in college are taught procedurally because remedial mathematics classes have to cover many topics in the limited amount of time and instructors may present the topics quickly and shallowly in a procedural manner (Hinds, 2009).

As a result, the instructors do not utilize interventions to deepen learners' conceptual understanding such as manipulatives, models and computer based mathematical software.

The purpose of this study was to investigate how mathematical software, ST Math, could assist students to better understand the fraction concepts that are a foundation for other mathematical concepts and college level mathematics courses. The chapter consists of five parts. Initially a concise summary and discussion of the results are presented. Then, limitations of this study are provided, followed by discussion of implications of the study. Fourth, recommendations for future study will be presented. Lastly, a conclusion will be provided.

Summary and Discussion of the Results

Research Question 1: *To what extent do college remedial mathematics students possess whole number misconceptions? To what extent can each class, ST Math and non-ST Math, express fractions and their operations, pictorially, realistically, symbolically and verbally?*

According to the initial fraction assessment statistical result of the two sample independent t-test, the students who were in the ST Math class and non ST Math class had equivalent fraction competency. Through a closer examination of the responses on the assessment and the pre-interviews the students revealed they possessed the strong existence of the natural number bias in all three areas: fraction magnitude, density and computation.

In the bias in fraction magnitude, ten students were unable to select the correct pictorial representation of the number line, which ordered the symbolical representation of the fractions ½, 1/3 and ¼. They tended to judge the magnitude of fractions based on the magnitude of the natural number in the symbolic representation. This result is aligned with the result of the study of De Wolf and Vosniadou (2011). Their study revealed that college undergraduate students' natural number ordering rationale was persistent for ordering fractions. Similarly, those who

were unable to compute 7/12 + ½ incorrectly performed the computation as 7/12 + ½ = 8/14. This was another clear indication of the natural number way of thinking in the size of fractions as researchers have stated students tend to think of each fraction components, a numerator and a denominator, differently, instead of looking at a fraction as a number (Pitkethly & Hunting, 1996; Clarke &Roche, 2009). Therefore, they added across, as the fractions were natural numbers, without considering the magnitude of each fraction.

Likewise, this bias was revealed in the equivalent fraction task as 3/10 = (10+3)/ (10+10) = 13/20. This way of thinking was revealed in the study of Hart (1987). The study was aimed at students age 10-12 regarding equivalent fraction tasks. Twelve students were taught how to find equivalent fractions by multiplying the numerator and denominator by the same number. Even after they were taught the rule, seven students did not use this method. Instead, they tended to use a subtract model in finding an equivalent fraction: 9/19 = 6/ ( ) as (9-3)/ (19-3) = 6/16. Consequently, the natural number way of thinking in fraction magnitude has shown to be persistent from this age group even through college students.

For the fraction magnitude comparison task about half of the students in both classes indicated the possession of the bias, stating 4/9 < 1/5 and 3/8 > 6/11. These students thought that the value of the fraction increased when either the numerator or the denominator gets smaller. This result is aligned with the research of Stafylidou and Vosniadou (2004). They investigated how students between fifth grade and high school understood the fraction magnitude concept and the result revealed the same pattern. The students in their study thought the value of a fraction gets larger when the numerator or the denominator get smaller. The researchers claimed that this way of thinking could be attributed to their fraction understanding with reference to a part of a whole model. However, the students in my study were able to select the larger fraction between

½ and 1/12. This agrees with previous research that students have better a feel for fraction magnitude when the fraction is a more familiar benchmark fraction (Sowder et al., 1988). Fractions such as 1/2, ¼ and ¾ are much more familiar to students than smaller fractions as 1/29 and larger ones as 28/29 (Sigler et al., 2011).

Regarding the density concept, the students applied the characteristic of discreetness in density (Van Hoof, Verschaffel & Van Dooren, 2015). All students but one applied the discreetness characteristics of the natural number by stating that there was either zero or one fraction between ¼ and 1/3. This result coincides with the study of Vamvakoussi and Vosniadou (2010). They examined secondary students' fraction density concept by asking them to identify the number of fractions between 1/8 and 1/7. The students in their study revealed they tended to apply the discrete characteristic of a natural number and they answered that there was a finite amount of numbers in the interval.

In the fraction computational bias, all but one stated that the statement $3 < 3/X$ was false. Through the interview responses, they either substituted 1 for the X and stated $3 < 3$ was false or they thought the whole number 3 was always greater than the fraction 3/X. Even the one student who thought that X could be any real number was not able to explain this problem thoroughly. The study of Van Hoof and colleagues indicated the same result. Their study was conducted to investigate the bias with 8^{th}, 10^{th} and 12^{th} graders using the same problem, $3 < 3/X$. The students who held the bias stated that the inequality is false since a fraction 3/X is always smaller than a whole number (2014).

Through the pre-interviews, the existence of the bias in the operations of fraction multiplication and division was clearly displayed by students stating fraction multiplication makes something bigger and fraction division makes something smaller. Similarly,

Vamvakoussi, Van Dooren and Verschaffel (2013) conducted a study regarding the computational bias in educated adults between 18 and 28 years old who were studying Psychology and Educational Science at university in Belgium. The result indicated that these adults possessed the bias of ‘division makes smaller’.

Besides the natural number biases described above, the students had difficulty expressing application problems pictorially and symbolically and also were unable to connect the pictorial representation to the symbolic representation of fraction multiplication because of the possession of a strong parts of a part thinking connected to fraction division. The study result of Toluk-Ucar (2009) aligns with this also. In his study, ninety-five pre-service teachers were asked to pose a story problem involving fraction division. They had difficulty representing it correctly with only 5% of the pre-service teachers being successful. Likewise in my study no students in either class were able to express the symbolical representation of fraction division 1/3 ÷ 1/6 realistically. The students that tried to come up with a story problem, exhibited a common misconception, which caused them to think ÷ 6 was equivalent to ÷ 1/6. Consequently, the students in both classes revealed weak fraction division concepts. This finding is aligned with the study of Ma (1999). In the study, many US teachers confused dividing by ½ with dividing by 2 when they were asked to express 1 ¾ ÷ ½ realistically.

As for the execution of fraction computations, the remedial mathematics students also exhibited fragile procedural knowledge on the pre-assessment by not executing fraction computational algorithms correctly. This indicates they were taught fractions operations without developing conceptual understanding that leads to longer retention (Moss & Case, 1999; Lubinski & Fox, 1998). Consequently, they were susceptible to incorrect execution of the procedures (Tirosh, 2000; Freiman & Volkov, 2004). For instance, in a fraction multiplication

problem, they computed 2/5 * 3/5 as 2/5 * 3/5 = 6/5 and 2/7 * 3/10 = 20/70 * 21/70 = 41/70. Likewise, in a division task, it was seen that they executed the division task incorrectly as 2/5 ÷3 = 2/5 ÷ 15/5 = 30/25, or 2/5 ÷ 3 = 5/2 * 3 = 15/2. The study of Sigler, Fazio, Bailey, and Zhou (2013) revealed the same incorrect fraction operation strategy. They aimed at investigating how the natural number thinking influences students from 6^{th} graders to 8^{th} graders in acquiring fraction concepts. The study showed that the students applied incorrect fraction operation procedures on multiplication and division problems. They tended to have a common denominator thinking in the procedures such as 4/5 * 2/5 = 8/5 and 3/5* ½ = 6/10 * 5/10 = 30/10 or came up with unknown strategies they created. Consequently, the weak conceptual understanding of fractions influenced very negatively the execution of the computational procedures and this trend continues up through to college level.

At the pre-assessment stage, the students in both classes exposed their natural number way of thinking in fraction concepts and they did not indicate the depth in their fraction concepts based on the Lesh Translation model. In addition, their fraction procedural knowledge was weak because of lack of the conceptual understanding and the influence of the bias.

Research Question 2: *To what extent does the usage of ST Math eliminate the natural number misconceptions and deepen college remedial mathematics students' conceptual understanding in learning fractions compared to a non-ST Math class of remedial mathematics students taught traditionally without technology?*

The post assessment results of the ST Math intervention class was compared to the results of the non ST Math students at the completion of the 8 weeks intervention. Based on the post-test results there was not a significant difference between the two classes. The students in both groups executed their procedural knowledge much more accurately than on the initial assessment

and diminished the natural number bias in fraction magnitude. For instance, in fraction addition and subtraction computations, the number of students who added across without having a common denominator dramatically decreased. Likewise in fraction multiplication and division tasks, the high accuracy of executing the computations was revealed. Also, in the fraction magnitude comparison task, the tendency they showed on the pre-test, which was if the natural numbers in the symbolic representation are greater, the magnitude of fraction is larger accordingly (Gabriel et al., 2013) was reduced. Many students did not add the identical number to both numerator and denominator in equivalent fraction task such as 3/10 = (10+3)/ (10+10) = 13/20 as seen in the study of Hart (1987). Precisely, the ST Math class improved their execution in subtraction with different denominators, compared to the pre-test. Hence, the ST Math game play may have assisted their computational skills while reducing the bias in fraction magnitude. However, the other areas of natural number bias, density and operation, were still persistent even after the 8-week intervention. In the fraction density problem, many students in the ST Math class responded that there was only a limited amount of fractions between ¼ and 1/3. Only three students were able to state that there were infinitely many fractions in the interval.

In the operation task, no one in the ST Math intervention group was able to state that the algebraic expression $3 < 3/X$ is only true for numbers between 0 and 1. Instead, they stated that the inequality was false. Based on the interview responses, the rationale for the statement was either students substituted a natural number for the X, particularly $X = 1$, and concluded the failure of the inequality or they thought that the division 3/X makes a smaller number. Consequently, the bias on this task was as strong as on the density task, followed by the magnitude task. Hence this confirms that the study result of Van Hoof, Janssen, Verschaffel and Van Dooren (2015). The researchers examined the persistence of the natural number bias in all

areas, magnitude, density and operation, toward 4th through 12th graders and the result indicated that although the higher the grade was, the less the natural number bias was exhibited in all three areas, this did not mean that the bias was completely diminished. Many students still applied the natural number way of reasoning to these tasks and that lead them to incorrect solutions. In 12th graders, the bias in density and operation items were still revealed. However, more ST Math students showed the understanding of the density concept, compared to the non ST Math students on the post assessment test and in the interview. They were able to explain why there are infinitely many numbers in the interval conceptually. Hence, the ST Math game play diminished the bias in density and assisted the students to acquire the density concept.

Besides those who gained the density concept, one student in the ST Math class revealed her partial understanding. On the initial assessment, she stated that there existed only one fraction in the interval. However, her statement changed to "more than one fraction in the interval" on the post assessment. This certainly indicates her conceptual change in density. Her existing framework theory (Vamvakoussi & Vosniadou, 2010) of the natural number way of thinking in density was changed while she played the fraction games, aiming at the density concept. Even though she was able to answer correctly regarding the density concept on the specific platform of ST Math game play, a version of Papert's Microworld (1980), she could not necessarily translate what she gained through the game play to the different form of task on the assessment. In other words, the new fraction density concept, which superseded her initial density, concept influenced by the natural number way of thinking, did not completely emerge through the game play, Microworld, based on the reorganization hypothesis. Consequently, the reorganization hypothesis (Steffe & Olive, 1990) did not completely follow through in her case.

The application problems, which examined students' depth of fraction conceptual

understanding, did not reveal a statistical significance for the ST Math intervention. However, the ST Math students showed the edge on Problem 6 and 7. In Problem 6, more students were able to connect the symbolic representation of 1/3 ÷ 8 to the realistic representation of the division. For Problem 7, the ST Math class had more students who were able to explain why James' view was incorrect by connecting the pictorial representation of the ovals to the symbolic representation of 1/3 + 1/3.

In essence, the game play influenced conceptual understanding in fraction division partially. Additionally, more ST Math students diminished the bias in fraction density. The game play certainly assisted some of them to reorganize their initial fraction concept with the bias, to facilitate the superseding fraction concept so that they could make connections between different representations.

As for the interview responses of the fraction computation, the bias in computation was still persistent. The possible reason for this is that the ST Math fraction games used in this study did not have a similar format of the assessment questions. Because of that, the students were unable to make a connection with what they played on the division concept interventions to the particular question. Hence, the reorganization of their initial fraction concept did not happen and they revealed the existence of the bias by stating 3 < 3/X is false, reasoning division makes smaller, a fraction is smaller than a whole number, or substituting a natural number for the X.

All three ST Math students interviewed were able to diminish the bias in multiplication computation, which is 'multiplication makes bigger', and two of them were able to explain why that is not always the case conceptually. For instance, Jordan (Average) responded that since it is multiplication, it seems like getting larger, but you are multiplying two fractions, and it creates a smaller piece. The response revealed the new superseding concept in fraction multiplication

occurred in him, reorganizing the initial fraction multiplication concept influenced by the bias. He was able to connect the multiplication representation of 2/5 * 3/5 to the visualized mental representation to explain the concept.

Likewise, in addition, Jordan indicated his conceptual gain by explain why a common denominator is necessary to add fractions. During the initial interview, he could not explain the necessity of a common denominator in the computation. However, he connected the symbolic representation of 1/15 + 1/12 to his mental image of different fraction magnitude to explain the necessity of a common denominator. Wherefore, the game play certainly deepened his fraction addition and multiplication understanding. However, although both Kylie (High) and Jordan (Average) displayed their conceptual gain in fraction multiplication, they were unable to represent 2/3 *4/5 pictorially on the post-test. Hence their conceptual gain was considered partial.

On the contrary to these conceptual gains displayed, they did not diminish the bias and show any conceptual gain in fraction division. The bias in division, which is division makes smaller, was persistent and they could not explain why that is not the case always. There could be a few reasons for this. First of all, there were not many students who actually completed the entire series of the ST Math fraction game. Since the fraction division concept game was placed at the end, those who did not finish or stopped playing the game before the fraction concept did not experience it. Secondly, although those who completed the games, they had difficulty of understanding the intention of the fraction division games. Jordan (Average) revealed his frustration of not understanding the meaning of the division games. Consequently, although they played the games, some were playing aimlessly and the reorganization of the initial division concept did not occur through the game play alone.

For the interview responses with Application Problem 2 and 5, the notion of 'parts of a part is division' thinking was as strong as the initial interview in both classes. Hence, the ST Math games did not reinforce the relationship between the fraction multiplication concept and division concept. One of the possible reasons for this is the lack of conceptual understanding of fraction division. Another reason could be that although they revealed their conceptual gain in multiplication, the reorganization of their fraction concept was not fully emerged because they could not translate the superseding concept to the application problem. Along with the multiplication concept, no one was able to express the division representation of 1/3 ÷ 1/6 realistically. In this question, the confusion of ÷ 1/6 with ÷ 6 was revealed strongly as the studies of Ma (1999) and Toluk-Ucar (2009) described earlier.

In Question 9, although Jordan (Average) showed his conceptual gain to explain the necessity of a common denominator described earlier on fraction addition 1/15 + 1/12, he was unable to illustrate the understanding on question 9. He could not explain the fraction magnitude, connecting the representation of 1/3 + 1/3 to the pictorial representation of the ovals. It seemed that he did not really understand the meaning of this question because he stated an ambiguous comment in his response during the interview, "James had a wrong concept but it worked because this is a special case." As a result, his response was not consistent with the earlier response. The reorganization did not fully emerge in him and consequently he could not translate the understanding to the different form of a question.

From the study result, the intervention usage alone did not contribute the students to deepen their fraction conceptual understanding as a whole although it showed the positive influence in some areas. The primary reason for the result could be the technology alone was used for their conceptual understanding gain in this study without reinforcement or connections

from the class instruction. When learners constructs conceptual structures, there exist some type of mental stimulation and interaction (Steffe, 1995) and ST Math offers the stimulation and interaction, aiming at the developing learners' mathematical understanding to connect between the mathematical concepts and problems through visual learning (Rutherford et al., 2014).

Consequently, regardless of the design of the software, powerful technology usage has to be aligned with appropriate instruction (Lim, 2011). Likewise, technology usage cannot be replacement for an instruction instead it needs to be used to foster learners' mathematical understanding and intuitions (Knuth & Harmann, 2005). In this study, the researcher let the students play the game and did not support them to link the concept they have learned on the games to the different representations of fractions. Hence, it is possible that the learners did not see the relationship and they played the games without really thinking as they use procedures (Ambrose, 2002). Because of this, the concept, which supersedes the fragile initial fraction concept influenced by the natural number bias was not reorganized through the intervention usage alone. The results of the interviews agree with previous studies' results stating that the students with high mathematical competency benefited less from the ST Math games (Rutherford et al., 2014; Breuninger, 2015).

Limitations

There are several limitations of this study that should be stated that provide further details for interpretation. The sample sizes in this study were small and the power of the study was not strong as demonstrated by the post hoc power analysis result. Consequently, the generalizability of the study result is marginal.

Secondly, a campus event one of the weeks of the ST Math intervention reduced a week of classes not to be held as scheduled. This was attributed to less in-class time for the students to

play the ST Math games and more out of class ST Math game play responsibilities for the students. The researcher had to cover the content for the missed week of classes the following week along with the regularly scheduled content.

Thirdly two students, one of them was Caleigh (Low), in the non-ST Math class insisted to take the post assessment home because they wanted to spend more time solving the problems. As a result, their assessment results were probably increased because of the extra time. Their scores were both higher scores than the class average of 15.7; Caleigh scored 20.5/28 and the other student scored the highest 23.5/28. Lastly, the students' ST Math game completion percentage could have influence on their fraction conceptual understanding gain.

Implications

The natural number bias in all three areas—density, magnitude, and computation—was persistent. The result of this study align with the study of Van Hoof, Janssen, Verschaffel and Van Dooren (2015), which investigated the bias in all three areas with students from elementary school to high school and the biases were persistent even with secondary students. Students construct an intuitive idea of what a number is based on their experience with natural numbers through their elementary education and this intuitive notion of natural numbers can be established fully. Once the bias is solidified in students, it becomes a major barrier to break through the belief, which is the natural number rules and principles are applied to all numbers (Vamvakoussi & Vosniadou, 2010). For instance, the undergraduate students in De Wolf and Vosniadou's study (2011) exhibited they had the bias in judging a numerical size of fraction because the bias they constructed in their childhood has lasting effects into adulthood. In this study, the concept of magnitude in fraction computational tasks and the bias in fraction density were the strongest ones and the weakest one was fraction magnitude.

The results of this study indicate that the natural number bias in fraction is originated from a lack of conceptual understanding. For this reason, mathematics educators of all levels need to aim at discerning fractions from the natural numbers and strengthen students' fraction understanding, instead of simply focusing on the procedural aspects. Hence providing students chances to make sense of natural numbers on their own to experience the differences between natural numbers and rational numbers, instead of teachers transmitting their knowledge of these two kinds of numbers to the learners (Moss & Case, 1999; Steffe, 2003; Tzur, 2004).

Besides the natural number bias, students had a strong fixed notion toward fraction division, which they predetermined that division means 'parts of a part'. Consequently, when they represented the symbolic representation of division realistically, they approached it from this notion. Because of the influence of the notion, it forced them to express the symbolic representation of the division representation $1/3 \div 1/6$ as a parts of a part model, instead of thinking about this problem from the fundamental measurement concept of division.

Likewise, this fixed notion of fraction division influenced their fraction multiplication understanding. In this study, when the students were asked if the parts of a part pictorial representation implies multiplication in the interviews, they did not think that the representation indicates multiplication. As a result, they were unable to make connections between different representations for fraction division and fraction multiplication. This shows a lack of conceptual understanding and really understanding what the symbolic representation indicates. Conceptual understanding leads to longer retention, which could help these students in future mathematics classes. The fact that the students confused the procedures for different fraction operations demonstrates the need to focus on conceptual understanding, which takes time to develop. Hence, since it takes a variety of experiences for learners to reorganize their initial fraction

knowledge with more complex concepts, mathematics educators are highly encouraged to implement different models in this matter to enhance learners' flexible understanding, which is enable them to connect the concept with different representations (Samsiah, 2002).

To provide students learning opportunities to make sense of different kinds of numbers and enhance their conceptual understanding of fractions, it is important for mathematics educators to give students a variety of platforms to experience fractions, besides the area model (Clark et al., 2008). Technology such as mobile devices can offer more opportunities to create effective learning experience through lively representation and interactions (Riconscenete, 2013). Similarly, Steff stated that when a learner builds conceptual structures, there are some interactions and mental stimulation (1995).

Through this study, the students certainly experienced the interaction and the stimulation with the fraction games. Certainly, through the games play, the ST Math students reorganized their initial fraction concept influenced by the natural number way of thinking to perceive the superseding fraction concept in their fraction addition and multiplication concept and the density concept, compared to the non ST Math class.

Contrary to the result mentioned above, the intervention did not reveal positive influence on the application problems, which examined the students' depth in fraction conceptual gain after the 8-weeks of game play. The main reason for this could be that the researcher did not interact with the students to make connections between what the students learned through the intervention to their initial fraction concepts. The study of Tzur (1999, 2004) regarding the students conceptual gain utilizing technology called Tools for Interactive Mathematical Activity (TIMA):sticks, revealed that while the learners' fraction scheme was enhanced by the technology, the teacher's interactions with the learners were also influential as well because the

role of the teacher was fostering the concept under the usage of TIMA;sticks.

Therefore, even when mathematics educators implement the mathematical learning software that aims at students' mathematical understanding through the spatial contiguity principle, students cannot be left alone to make connections between the fraction concepts they have learned through the game play and their initial fraction knowledge. This aligns with the study result of Osna and Pitsolantis (2013). When the teacher made explicit connections between mathematical concepts and procedural knowledge in fraction learning, the students strengthened their fraction concept by connecting fraction symbols to concrete and realistic situations. In conclusion, by making the connection, the students might be able to express the fraction concepts, pictorially, realistically, symbolically and verbally more effectively through the intervention.

Lastly, the motivation of the students influenced the game completion rate and the results of the effectiveness of the ST Math games. According to the study of Rutherford and the colleagues, it indicated that ST Math usage was the least beneficial to those who were considered the highest and the lowest achieving students (2014). The study certainly confirmed the case of the lowest achieving students. The students who were considered the lowest competency based on the initial assessment result had difficulty of moving forward with the game on their own. As many researchers claim, motivation influences students' academic achievement positively and students tend to develop interest in materials they comprehend and see as relevant (Hilgard & Bower, 1966; Von Glaserfeld, 1983; Hidi & Renninger, 2006; Riconscenete, 2010). Therefore, even with this perspective, the teacher-student interaction to provide necessary support is vital while using technology or any sort of manipulatives.

Recommendations for Future Research

There are a handful of areas of future research that can be constructed from this study or to replicate this study. Future studies could aim at a longitudinal study of the fraction conceptual change because reducing the misconceptions attributed to the natural number way of thinking and re-conceptualize their fraction concepts could be a long gradual process (Mack, 2001; Vamvakousi & Vosniadou, 2004). The duration of the study was eleven weeks and certainly the ST Math students indicated a glimpse of conceptual gain in the certain areas. If the study had been conducted for a longer period, more conceptual gain could have been revealed through the intervention.

A future study could integrate the usage of the ST Math fraction games with instruction which makes the explicit connection between what the students have learned through the game play and their initial fraction knowledge. By comparing to an instruction without the intervention to an instruction with the connection and the intervention, the study would be able to inform the how to best structure the usage of the games in fraction learning. Future research could provide the opportunity to have student-student interactions during the game play. Through the conversations, a researcher could examine how their understanding is going to be deepened such as students asking what they do not understand or what they have discovered on the games during the course of the study among themselves. Likewise the content of the interactions themselves for mathematical learning could shed light.

Also, a future study is encouraged to have more interviewed students so that the study would have students who represent the different levels of their fraction competency such as lowest, below average, average, above average and highest so that the study could indicate which competency category could benefit the most from the intervention.

Future research could focus on the fraction division concept since this study revealed that students possessed the dominant bias 'division makes smaller'. The interviewed students revealed the persistency of the bias and they could not even explain why it is not always the case conceptually. The students also could not see the relationship between fraction division and multiplication. Therefore, the future study could examine possible effective pedagogies to diminish the bias and the notion to deepen learners' division concept.

Lastly, future research certain should include fraction contents which are aligned with the ST Math fraction games' content, since in this study, although the researcher aimed at conceptual understanding of fraction through the fraction instructions, the fraction content covered in the course was more toward acquisition of the fraction skills such as executing the fraction computations precisely. Therefore, there was a certain gap between the ST Math games and what they have learned in the course and the gap could have led the students' marginal fraction conceptual gain through the game play.

Conclusions

Understanding the fraction concepts is the heart-beat of a college remedial mathematics course because the concepts are the foundation for college level algebra and above. In recent years, the number of students who pursue higher education degrees has been increasing. However, along with the increase of university enrollment, unfortunately more students have been placed in college remedial mathematics courses because they do not possess the readiness to take college level mathematics courses required for their degree. Without mastering the concepts, the chance of students being fruitful in college education is very slim. Consequently, it is significant for college mathematics departments to figure out effective pedagogies to facilitate students' flexible fraction conceptual understanding. In this study, the implementation of the

technology usage in college remedial mathematics course did not reveal significant difference numerically, compared to the class without the implementation. However, the students who were in the intervention class indicated more conceptual gain in certain areas of fraction concepts. Therefore, implementing the software with mathematical instruction which connects students' initial fraction knowledge with the fraction game play could reorganize their fragile initial fraction concept to more sophisticated ones.

APPENDIX A

The Reformed Teaching Observation Protocol (RTOP)

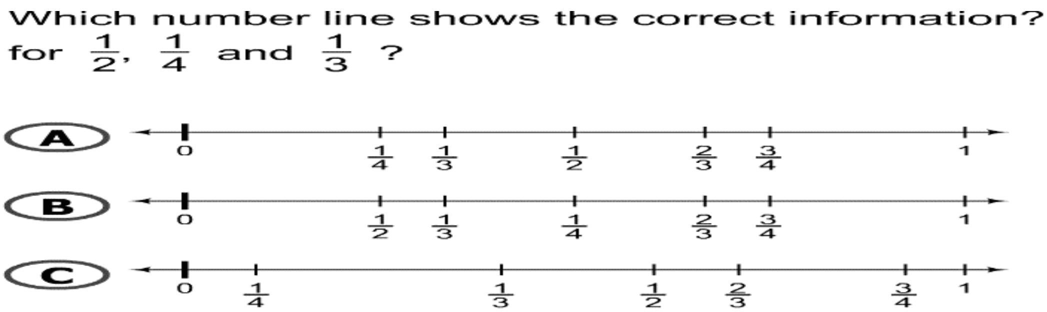
RTOP Item
1. Instructional strategies and activities respected students' prior knowledge and the preconceptions inherent therein.
2. The lesson was designed to engage students as members of a learning community.
3. In this lesson, student exploration preceded formal presentation.
4. The lesson encouraged students to seek and value alternative modes of investigation or of problem solving.
5. The focus and direction of the lesson was often determined by ideas of originating with students.
6. The lesson involved fundamental concepts of the subject.
7. The lesson promoted strongly coherent conceptual understanding.
8. The teacher had a solid grasp of the subject matter content inherent in the lesson.
9. Elements of abstraction (i.e., symbolic representations, theory building) were encouraged where it was important to do so.
10. Connections with other content disciplines and/or real world phenomena were explored and valued.
11. Students used a variety of means (models, drawings, graphs, concrete materials, manipulatives, etc) to represent phenomena.
12. Students made predictions, estimations and/or hypotheses and devised means for critical assessment of procedures.
13. Students were actively engaged in thought-provoking activity that often involved the critical assessment of procedure.
14. Students were reflective about their learning.

15. Intellectual rigor, constructive criticism, and the challenging of ideas were valued.
16. Students were involved in the communication of their ideas to others using a variety of means and media.
17. The teacher's questions triggered divergent modes of thinking.
18. There were high proportion of student talk and a significant amount of it occurred between and among students.
19. Student questions and comments often determined the focus and direction of classroom disclosure.
20. There was a climate of respect for what others had to say.
21. Active participation of students was encouraged and valued.
22. Students were encouraged to generate conjectures, alternative solution strategies, and ways of interpreting evidence.
23. In general the teacher was patient with students.
24. The teacher acted as a resource person, working to support and enhance student investigations.
25. The metaphor "teacher as listener" was very characteristic of this classroom.

APPENDIX B

The Interview Questions

1.



Explain why you ordered these fraction in the way you ordered.

2. How many fractions are there between 1/3 and ¼? Explain your rationale for your answer.

3. Compute 1/15 + 1/12. Why do you think a common denominator is unnecessary or necessary to compute it?

4. In 3 < 3/X, is the statement true or false? If you think it is true, explain why you think it is true. If you think it is false, explain why you think it is false.

5. In 2/5 * 3/5, guess your answer is going to be larger or smaller than 3/5 without calculating. If you guess the answer is going to be larger than 3/5, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain your guess was right. If it is not, explain your guess was wrong.

6. In 4/5 ÷ 3/7, guess your answer is going to be larger or smaller than 3/7 without calculating. If you guess the answer is going to be larger than 3/7, explain your rationale. If you guess the answer is going to be smaller, explain your rationale. Then compute it. Is your answer same as your guess? If it is, explain your guess was right. If it is not, explain your guess was wrong.

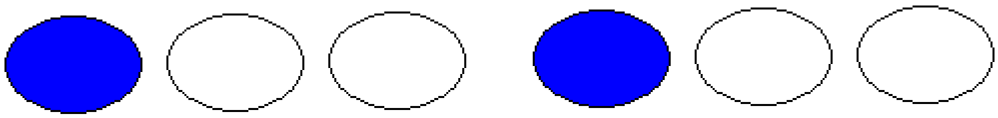
7. In the story problem, "Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get? (Kamii & Warrington, 1999, p. 87)"

   Explain your answer pictorially? Why do you think this problem represent multiplication?

8. In the problem, "Come up with a problem that indicates the expression 1/3 ÷ 1/6."

Explain why the problem that you came up with is correct expression of the division pictorially. By using the pictorial expression, explain why the answer of the division is larger than 1/6?

9. Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed "the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?" (Schifter, et al., 1999)

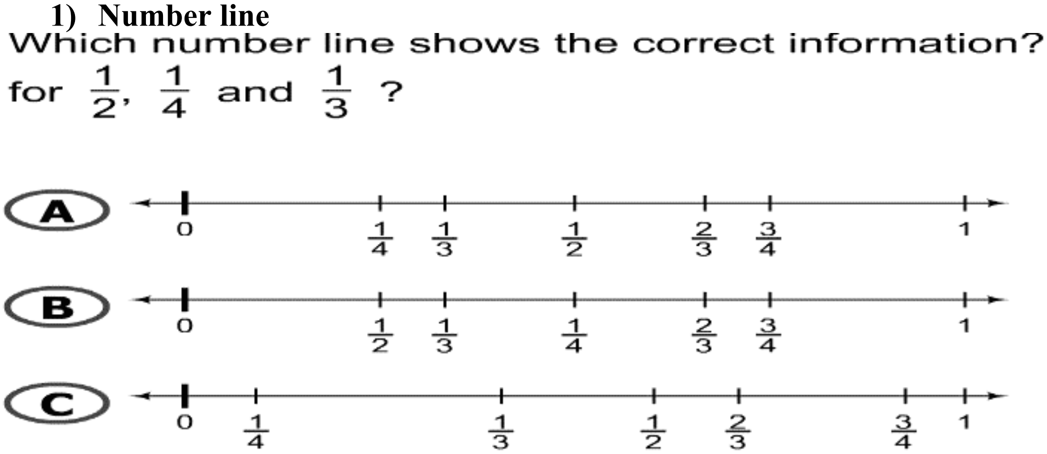
3. Explain how Kevin is viewing this problem.

4. Explain how James is viewing this problem.

APPENDIX C

Pre and Post-Test Problems

1) Number line



2) How many fractions are there between 1/3 and ¼?

3) Equivalent Fractions

(1) 3/10 = /20 (2) 4/8 = 8/ (3) 3/5 =

4) Comparing Fractions

Circle a larger fraction

(1) 4/9, 1/5 (2) ½, 1/12 (3) 3/8, 6/11

5) Addition

(1) 3/5 + 2/5 (2) 7/12 + ½ (3) 1/15 + 1/12

6) Subtraction

(1) 9/11 – 5/11 (2) 2/5 – 1/3 (3) 3/10 – 2/5

7) True or False

State true or false

(1) X/4 < X (2) 3 < 3/X

8) Multiplication and Division

(1) 2/5 * 3/5 (2) 2/7* 3/10 (3) 2/5 ÷ 3 (4) 4/5 ÷ 3/7

Application Problem

Answer the following word problems. Be sure to show all your work and use pictures, numbers and words to help explain your answer.

1) Paul has 7/8 of a piece of a chocolate bar. He eats 1/2 of it. How much does he have left? (adapted from Van de Walle & Folk, 2005, p. 245)

2) Suzy, her father, and her mother divided a chocolate bar equally among themselves. Suzy then gave half of her share to a friend who came over. Suzy's mother decided to give her share (the mother's 1/3) to Suzy. How much chocolate did each person get? (Kamii & Warrington, 1999, p. 87)

3) Raquel thought about this statement: When pitching, Joe struck out 7 of 18 batters.

She said that it was better to say that Joe struck out about 1/3 of the batters than to say that Joe struck out about ½ of the batters. "I think that seven-eighteenths is closer to one-third than one-half," she said. Do you agree or disagree with Raquel? Explain your reasoning. (Bums, 2001, p. 152)

4) Draw a picture to represent 2/3 × 4/5.

5) Come up with a problem that indicates the expression 1/3 ÷ 1/6.

(To make one gallon of Gatorade liquid, 1/6 pound of Gatorade powder is necessary. How many gallons of Gatorade can be made from 1/3 pounds of the powder?)

6) Choose a story problem that represents 1/3 ÷ 8.

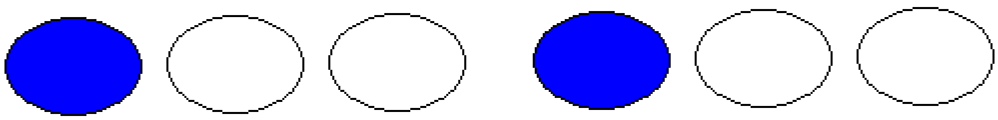
a. How much chocolate will each person get if 8 people share 1/3 lb of chocolate equally?

b. One of three people wants a piece of chocolate. There are 8 pieces. How many pieces will the other two people get?

c. Eight friends get 1/3 of a chocolate bar. How many chocolate bars will we need to buy?

d. Both (a) and (c).

7) Kevin represented 1/3 + 1/3 in this manner.



This representation made the student to believe that 1/3 + 1/3 = 2/3. However, James claimed “the picture represents 2 out of 6 or 2/6? And how can 1/3 +1/3 = 1/3?” (Schifter, et al., 1999)

1. Explain how Kevin is viewing this problem.
2. Explain how James is viewing this problem.

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Curriculum Vitae

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TARO ITO

OBJECTIVE

To obtain a teaching position in mathematics and statistics.

EDUCATION

May 2017- Conferred — **University of Nevada Las Vegas, Las Vegas, Nevada**
Ph.D. in Math Education

May 2012 — **University of Nevada Las Vegas, Las Vegas, Nevada**
Master of Education in Secondary Mathematics

September 2007 — **California State University Long Beach, Long Beach, California**
Teaching Credential in Mathematics

December 2006 — **University of Nevada Las Vegas, Las Vegas, Nevada**
Master of Science- Mathematical Science-Statistics

May 2003 — **University of Nevada Las Vegas, Las Vegas, Nevada**
Bachelor of Science- Mathematical Science-Statistics

TEACHING EXPERIENCE

University of Nevada Las Vegas — Las Vegas, Nevada
August 2003- 2007 & Currently
Part-Time Math Instructor

- Designed, administered and graded examinations to assess achievement of course objectives as identified in the syllabus
- Planed course instruction based upon approved syllabus/outline provided, to assure that course content and objectives were met
- Utilized a variety of teaching styles and methods to accommodate diverse learning styles of students
- Advised students, tutored and provided other assistance as needed
- Held open office hours for students to answer their questions
- Utilized Mathematical Learning App such as ST Math
- Have taught courses:

MATH 095 Elementary Algebra
MATH 120 College Mathematics
MATH 124 College Algebra
MATH 126 Pre-Calculus I
MATH 127 Pre-Calculus II
MATH 132 Finite Mathematics

Foothill High School — Henderson, Nevada
August 2011- December 2011

Student Teaching

- Observe and discuss various teaching strategies with mentor teacher
- Develop and use self instructional materials and use creative teaching strategies
- Prepare written lesson plans
- Take fulltime teaching and classroom management responsibilities
- Supervise the work of students during study period
- Help students study after school
- Attend staff develop meeting

WORK EXPERIENCE

University of Nevada Las Vegas   Las Vegas, Nevada
August 2008 - Current
Professor assistant / Math Learning Center

- Assist professors' research
- Coach at Math Lab

California State University Long Beach   Long Beach, California
September 2007- December 2007
Math Tutor

- Worked at the math tutoring center on campus and tutored different level of college math courses

Club Z Tutoring Service   Las Vegas, Nevada
February 2007- July 2007
Math Tutor

- Tutored high school students and elementary school student in mathematics

University of Nevada Las Vegas   Las Vegas, Nevada
May 2001- August 2003
Recreational   Facility Attendant

- Contributed and worked toward building collective and harmonious working relationships with other staff and students under the multicultural environment
- Assisted and supervised the participants of the Fitness Center by providing usage knowledge of the cardiovascular and strength training equipment; and by promoting safety and proper use of the equipment through appropriate supervision. Maintain the cleanliness of the facilities and equipment
- Provided service to perspective and current members of the facility, sold

memberships, locker rentals and assisted in program registration. Conducted facility tour

- Took responsibility for all recreational activities equipment and facilities on campus needed to implement scheduled activities
- Assisted recreational events in the facility

University of Nevada Las Vegas Las Vegas, Nevada
August 1999- May 2001
Mathematical Science Professor Assistant

- Graded homework, quizzes and tests assigned by math professors
- Supervised testing center
- Assisted students with homework, class assignments

University of Nevada Las Vegas Las Vegas, Nevada
August 1998- August 1999
UNLV Law Library Circulation Desk

- Assisted patrons at the circulation desk and use of online circulation system to check out books and other Library materials to patrons
- Processed newspapers, helped patrons locate materials and maintained the Department in an orderly manner. Restocked Shelves
- Increased customer service and interpersonal skills
- Gained an experience of working in a multicultural environment.

TECHNICAL SKILLS

- Ability to define, gather, analyze data, and draw conclusions
- Capable of building and implementing behavior models utilizing databases and other information source
- Can extract data from data bases and summarize the data for meaningful data comparison
- Competence to effectively communicate results using analytical knowledge and techniques with an understanding of business practices
- Can use statistical softwares such as S-Plus, MiniTab, SPSS and R-Project
- Ability of using Microsoft Offices, HTML, WebCT and

WebCampus

LANGUAGES

- Speak and write Japanese fluently

PROFESSIONAL MEMBERSHIP

- National Scholars Honor Society
- American Statistical Association
- National Council of Teachers of Mathematics

CONFERENCE PRESENTATIONS

- Shih, J., Adkins, A., DeVaul, L., Ito, T. & Allen, C. (2015). *Early childhood gender differences in number sense when learning with iPads.* The Research Council on Mathematics Learning 42nd Annual Conference. Las Vegas, NV: RCML
- Shih, J., Adkins, A., DeVaul, L., Ito, T. & Allen, C. (2015). *Early childhood gender differences in number sense when learning with iPads*. National Council of Teachers of Marthematics Annual Conference. Boston, MA: NCTM

- Sanogo, A., Adkins, A., Cassel, D. & Ito, T. (2015) *Examining Cognitive Demand and Content of Early Number and Fraction iPad Apps*. School Science Mathematics Association Annual Convention: Oklahoma City, OK: SSMA

- Sanogo., A, Adkins, A., DeVaul, L., & Ito, T. (2016). *iPad Apps for Early Math Learning.* The Research Council on Mathematics earning 43rd Annual Conference. Orlando, FL: RCML

- Adkins, A., DeVaul, L., Lockett, D. & Ito, T. (2016). *iPad Statistic's Apps.* School Science Mathematics Association Annual Convention. Phoenix, AZ: SSMA

- Ito, T (2017). *Effectiveness of ST Math in College Remedial Mathematics Students learning Fraction Concepts*. The Research Council on Mathematics Leaning 44th Annual Conference. Fort Worth, TX : RCML

PUBLICATIONS

- Shih, J., Adkins, A., DeVaul, L., Ito, T. & Allen, C. (2015). Early childhood gender differences in number sense when learning with iPads. *Proceedings of the 42nd Annual Meeting of the Research Council on Mathematics*

- Schacter, J, Shih, J., Allen, C., DeVaul, L., Adkins, B., Ito, T., & Jo, B (2015). Maeth Shelf: A Randomized Trial of a Prekindergarten Tablet Number Sense Curriculum. *Early Education and Development,* 27 (1), 74-88.

- Stohlmann, M., DeVaul, L., Allen, C., Adkins, A., Ito, T., Lockett, D., & Wong, N. (2016). What is known about secondary grades mathematical modelling- a review. *Journal of Mathematics Research,* 8(5), 12-28.

REVIEWER

- 43rd RCML conference in Orlando proposal reviewer

SERVICE

- Was a member of the 42nd RCML in Las Vegas conference organizers

RELEVANT COURSES

- STAT 663 (Applied Statistics For Engineer)
- STAT 715 (Multivariate Statistics)
- STAT761(Analysis of Variance)
- STAT762 (Generalized Linear Statistical Models)
- STAT763 (Regression Analysis)
- STAT 765(Statistics Decision Theory)
- STAT767 (Advanced Statistical Mathematics)

REFERENCES

1. Dr. Micah Stohlmann: stohlman@unlv.nevada.edu, *702 – 895 - 0836*
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3. Dr. Travis Olson : travis.olson@unlv.edu, *702 - 895 - 0471*
4. Dr. Howard Gordon : howard.gordon@unlv.edu, *702 - 895 -2729*
5. Dr. William Speer: william.speer@unlv.edu,*702- 895 - 4885*