Fraction instruction for students with disabilities: Comparing two teaching sequences

Frances Mary Butler

University of Nevada, Las Vegas

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FRACTION INSTRUCTION FOR STUDENTS
WITH DISABILITIES: COMPARING
TWO TEACHING SEQUENCES

by

Frances Mary Butler

Bachelor of Arts
University of California, Berkeley
1968

Master of Education
University of Southern Mississippi
1996

A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Education Degree
Department of Special Education
College of Education

Graduate College
University of Nevada, Las Vegas
August 1999
The Thesis prepared by

FRANCES MARY BUTLER

Entitled

FRACTION INSTRUCTION FOR STUDENTS WITH DISABILITIES: COMPARING TWO TEACHING SEQUENCES

is approved in partial fulfillment of the requirements for the degree of

DOCTOR OF EDUCATION IN SPECIAL EDUCATION

Susa O'Malley
Examination Committee Chair

Dean of the Graduate College

Graduate College Faculty Representative
ABSTRACT

Fraction Instruction for Students with Disabilities: Comparing Two Teaching Sequences

by

Frances Mary Butler

Dr. Susan P. Miller, Examination Committee Chair
Associate Professor of Special Education
University of Nevada, Las Vegas

This study investigated the effects of teaching middle-school students with mild to moderate disabilities equivalent fraction concepts and procedures using the concrete-representational-abstract (CRA) instructional sequence or the representational-abstract (RA) instructional sequence. Twenty-six students formed the CRA group, and twenty-four students formed the RA group, while sixty-five eighth-grade students without disabilities served as a contrast group. The two treatment groups received carefully sequenced instruction over ten lessons. The only difference between the two treatment groups was that the CRA group used concrete manipulative devices for the first three lessons while the RA group used representational drawings. The eighth-grade contrast group received traditional instruction using a basal text.

Analyses of the data indicated that students in the treatment groups scored significantly higher than did students in the contrast group on items demonstrating conceptual knowledge, had higher scores on the attitude measure, and overall improved...
their understanding of fraction equivalency from pretest to posttest. Students in the treatment groups performed as well as did contrast group students on abstract problems. On word problems containing embedded fraction equivalencies, students in the CRA group had significantly higher scores than did contrast group students. On all achievement measures, students in the CRA group had overall higher mean scores than did students in the RA group although the results were not statistically significant.

Some conclusions were drawn as a result of this study. First, students who used manipulative devices had a better understanding of fraction equivalency than those who did not. Second, training in the use of graphic representations had a positive effect on students’ abilities to solve abstract problems and word problems. Students in both treatment groups used graphic representations to solve problems, while students in the contrast group did not. Even though students in the contrast group solved problems correctly when they were presented abstractly, they appeared not to transfer their knowledge to problems presented graphically or to word problems. Implications for classroom instruction and suggestions for further research are discussed in the last chapter.
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ACKNOWLEDGEMENTS

The writer would like to thank her dissertation committee chair, Dr. Susan P. Miller, and committee members, Dr. Beatrice Babbitt, Dr. Thomas Pierce, and Dr. Kevin Crehan, for their thoughtful guidance and encouragement throughout this endeavor. Above all, I wish to thank Dr. Miller for her enormous patience, not only for the dissertation project but throughout my doctoral studies. Her constant support and wise advise allowed me to devote the time and energy necessary to see this project through successfully. Special thanks is given for all of the opportunities she gave me to grow professionally. Experiences such as presenting at national conferences, teaching college-level classes, and conducting teacher workshops will prove invaluable in the years to come.

I would also like to thank Dr. Babbitt for her learned insights into the teaching of mathematics, Dr. Pierce for interjecting his sensitivity to students, and Dr. Crehan for his voluminous knowledge of statistics. Each member provided his own special expertise, and I am all the stronger for that. In addition, I would like to express my gratitude to the Clark County School District and the faculty, parents, and students of Lied Middle School.

Finally, this goal could not have been reached without the love and support of my husband Wayne. You have always been my best fan and were there to pick me up and get me started again. Thank you for all the years of cooked meals, clean dishes, and folded laundry.
CHAPTER 1

INTRODUCTION

Students with learning disabilities make unacceptably poor progress in mathematics as they progress from elementary through secondary school (McLeod & Armstrong, 1982; Baroody & Hume, 1991; Engelmann, Carone, & Steely, 1991; Mercer & Miller, 1992). According to Mercer and Miller, students who lack an adequate math foundation have as much of a disability as students who have poor reading skills because daily living requires basic math skills. McLeod and Armstrong found that two-thirds of adolescents with learning disabilities receive special education in math. Authorities have implied a variety of causes for math difficulties. Batchelor, Gray, and Dean (1990) proposed that difficulties with mathematics may come from problems within the student, such as poor attention span, visual-spatial processing deficits, and verbal-auditory discrimination weaknesses. Mercer and Miller suggested that poor instructional practices and inappropriate curriculum and textbooks may contribute to students' difficulties in learning math.

Although it is well documented that students with disabilities have difficulty with math computation and problem solving, it appears that both students with and without disabilities have problems understanding fractions (Tourniare & Pulos, 1985; McLeod &
According to Hiebert (1985), students have difficulty with fractions because they often fail to connect form and understanding. Hiebert defined form as the syntax, e.g., symbols, numerals, and algorithms, while understanding is defined as the ability to relate mathematical ideas to real-world situations. Hiebert identified three specific areas where students should connect form and understanding. First, symbolic representation should connect to real-life cases. This requires students to visualize the fraction 3/4, for example, as a pizza with three of four equal-sized pieces remaining. Second, students should connect the algorithm or procedure to the fundamental concept so that they understand why the algorithm or rule works. For example, when we add fractions with like denominators, the denominator does not change. Students should reason that the numerator has increased because we have more pieces, but the denominator has remained constant because we have not changed the size of the pieces. Third, students should connect the answer to a problem with real-life experience. This requires the development of number sense and estimation skills. If we add 3/4 and 11/12, the answer should be close to 2 because both fractions are close to 1 and 1 + 1 is 2. According to Hiebert, most mathematics curricula and textbooks spend little time on making connections in these areas. He suggested that teachers make much greater use of diagrams and stories to help students visualize mathematical concepts.

Heller, Post, Behr, and Lesh (1990) found that seventh-grade and eighth-grade students failed to apply rational number concepts to higher level proportional reasoning problems. In a study involving 467 seventh graders and 522 eighth graders, students were asked to solve rate and proportion word-problems and context-free fraction problems numerically identical to the word problems. An analysis of the data revealed a low
correlation between the fraction test and the word-problem test. The researchers proposed that students use different reasoning processes in solving the numerically identical problems. There was a small difference between scores of the seventh-graders and scores of the eighth-graders suggesting a developmental effect in acquiring proportional reasoning ability. However, the researchers commented that inappropriate instruction in fraction-equivalence and missing-value problems may also contribute to students' difficulty in transferring rational number skills to proportional reasoning problems. They concluded that the concepts of proportion and rate should be taught within the context of rational number skills.

Citing the need for effective, validated practices for teaching mathematics to adolescents with learning disabilities, Maccini and Hughes (1997) reviewed the literature from 1988 to 1995. They found 19 articles (20 studies) that met their criteria for selection, i.e., an experimental or quasi-experimental design examining the effects of an intervention on students from sixth through twelfth grade. Of the included studies, thirteen concentrated on mathematical procedures such as learning rules and facts, four included procedures and concepts (mathematical relationships and ideas), and three of the studies did not provide enough information about the task to enable the reviewer to categorize it. Thus, little information is available to teachers wishing to incorporate conceptual knowledge in their lessons. However, several effective design practices emerged from the review. These included exposing students to a wide variety of examples, recognizing and separating potentially confusing items, and requiring students to achieve a preset criterion before moving on to more advanced topics. Furthermore, Maccini and Hughes noted that concept development was improved through cognitive
and metacognitive strategy instruction and the use of the concrete-representational-abstract (CRA) sequence.

Baroody and Hume (1991) emphasized that effective instruction requires active student participation in activities that are meaningful to the student. Teachers should help students link information previously learned (informal knowledge) to new information relating to daily life. When teaching fractions, teachers should start with verbal problems using manipulative devices and then proceed to teaching the algorithms after students have demonstrated an understanding of the concept. This is the reverse of the traditional method of teaching facts first and problem solving last. Since the ability to communicate math ideas and concepts is an area frequently overlooked in traditional instruction, Baroody and Hume further emphasized that students should be encouraged to devise their own strategies and then to reflect upon, compare, and discuss their strategies with their peers.

Statement of the Problem

The present study is designed to investigate the effects of teaching middle-school students with mild to moderate disabilities equivalent fraction concepts and procedures using the concrete-representational-abstract (CRA) sequence or the representational-abstract (RA) sequence. Specifically, the following questions will be addressed:

1. Is concrete-representational-abstract (CRA) instruction more effective than representational-abstract (RA) instruction for teaching students to compute equivalent fractions?

2. Is CRA instruction more effective than RA instruction for teaching students to solve word problems involving equivalent fractions?
3. Are CRA and RA instructional methods more effective than traditional instructional methods for teaching students to solve problems involving equivalent fractions?

4. Do students who receive CRA or RA instruction have a more positive attitude toward math and fractions than students who receive traditional instruction?

5. How much improvement do students make after receiving CRA or RA instruction?

Rationale of the Study

Engelmann, Carnine, and Steely (1991) listed four problems with mathematics curricula. First, too much time was spent on teaching computational skills at the expense of problem solving and understanding of concepts. Second, at least 70% of math topics typically received limited coverage (i.e., 30 minutes or less of instructional time). Third, most teachers varied greatly in the actual amount of time spent in teaching math. Fourth, and perhaps most important, the sequential and spiral nature of most math curricula ensured that students repeated topics year after year, often not having mastered concepts begun years before. These poor instructional practices often result in students’ feelings of frustration, anxiety, and, ultimately, failure. Students’ attitudes toward mathematics can have lasting consequences that affect the decisions they make concerning their future vocations. The National Council of Teachers of Mathematics (1989) noted that failure to study mathematics can lead to loss of opportunities for entering vocational-technical schools, college majors, and careers. Therefore, teachers must promote a positive disposition toward mathematics.
Zigmond and Baker (1994) and Zigmond et al. (1995) suggested that fraction instruction for students with learning disabilities may not be sufficiently intense to promote adequate learning (as cited in Brigham, Wilson, Jones, & Moisio, 1996). Furthermore, fraction instruction should be organized around the “Big Idea” of division (Camine, Jitendra, & Silbert, 1997). Camine et. al contended that using the Big Idea helps students to form links between new skills to be learned and skills already mastered. Brigham et al. offered the following suggestions and guidelines for teaching fractions to students with learning disabilities.

- Ensure that students have mastered prerequisite skills.
- Explicitly demonstrate the Big Idea and its links to concepts such as division, fractions, decimals, and percents.
- Demonstrate new skills clearly and succinctly.
- Use the CRA sequence in introducing new concepts.
- Include a variety of teaching examples to avoid students making incorrect generalizations.
- Provide ample guided and independent practice.
- Link instruction to life-skills.
- Include evaluations to detect and remediate student error patterns.

Thus, many authorities have noted the need for improved math curricula and instruction. The National Council of Teachers of Mathematics has taken a leadership role in advocating such improvements.

In 1989, the National Council of Teachers of Mathematics (NCTM) published the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). This
document addressed mathematics as it relates to our increasingly technological society. The NCTM standards presented mathematics instruction as an active process in which students construct meaning rather than relying on rote memorization. Five general goals for all students were proposed: to learn to value mathematics, to learn to reason mathematically, to become confident in their mathematical ability, to solve problems, and to communicate in the language of mathematics.

Payne and Towsley (1990) discussed the significance of the new standards as they apply to fraction instruction. The NCTM standards recommended that teachers approach math instruction from a problem-solving perspective that relates math to real-life situations. In this way, students could develop a conceptual framework in the early grades and build computational procedures in the middle and upper grades. Payne and Towsley offered six general guidelines for math instruction that were based upon the NCTM standards. These are as follows:

- Devote more instructional time to the development of number concepts. They suggested that concrete materials should be used at all levels of instruction. Teachers should spend at least two weeks per year on fraction and decimal instruction in grades K-4 and at least one week per year in grades 5-8.

- Make clear connections between models and algorithms when teaching operations with fractions and decimals. The authors suggested several days for teaching each operation.

- Adapt math textbooks to provide adequate conceptual development. Since most middle and upper-grade texts emphasize computational procedures,
they cautioned that teachers should expect to spend at least half of their instructional time in working outside the textbook developing concepts.

- Include estimation and the use of calculators for computation activities and problem-solving.
- Connect problem-solving with real-life events to help students transfer knowledge to their own experiences and to encourage students to understand the practical utility of math instruction.
- Avoid complex computations, especially those involving addition and subtraction of fractions and mixed numbers. The authors counseled that such computations are neither practical nor useful for developing future math reasoning.

With these general guidelines in mind, Payne and Towsley (1990) recommended that teachers in grades K-4 should present fraction instruction using concrete models and real-life problems. The authors recommended that students share items such as apples fairly to develop the concept of equal-sized pieces. They also suggested that students construct fraction strips for use in comparison and equivalence problems and to explore operations with fractions.

In grades 5-8, teachers should continue to stress concrete models, realistic problems and oral language. The authors advised that realistic problems should be presented before introducing algorithms. After algorithms are taught, students should prove their answers using concrete models or drawings. At this stage, teachers can introduce decimals as another form of fraction.
One of the criticisms leveled at the NCTM standards is an apparent lack of sensitivity to students with special needs, including those with mild to moderate disabilities as well as those at risk for school failure. The November/December, 1993 issue of *Remedial and Special Education* included a discussion of the implications of the NCTM standards as they relate to students with disabilities. In that issue, several authors noted that little reference was made to students with special needs, and the standards did not incorporate validated, research-based instructional methods for students with mild to moderate disabilities (Giordano, 1993; Hofmeister, 1993; Hutchinson, 1993; Mercer, Harris, & Miller, 1993; Rivera, 1993).

Mercer, Jordan, and Miller (1994) explored the implications of constructivist theory as outlined in the NCTM standards as they apply to students with mild to moderate disabilities. They reported that the standards promoted an endogenous constructivist approach in which the teacher is viewed as a facilitator for student discovery. However, this approach has been shown to be inadequate or even damaging to students with disabilities (Hofmeister, 1993). Nevertheless, Mercer, Jordan, and Miller contended that validated research-based instruction could be integrated with the NCTM standards in an exogenous constructivist approach in which the teacher performs an active instructional role by modeling and guiding students to develop effective cognitive and metacognitive strategies. They examined 14 articles from the constructivist and learning strategy literature and identified 19 specific instructional components that were validated and effective for math instruction. These components included the use of teacher modeling, self-regulation, mnemonics, verbal rehearsal, and the use of the CRA sequence.
Woodward and Baxter (1997) examined the effects of an innovative NCTM standards-based mathematics program on the math achievement of 205 third-grade students. Of the 205 participants, twelve were classified as having a learning disability and received special education services in the general education classroom. Results of the Iowa Test of Basic Skills (ITBS) indicated that experimental students had overall stable performance and achieved higher scores on each subtest than did comparison group students. Experimental group students showed improvement in the area of concepts, but showed declines in computation and problem-solving. The comparison group declined in all areas over the year-long study. In both groups low-ability students, including those with learning disabilities, showed nonsignificant differences in the total test and the three subtests. During interviews, these students exhibited confusion and indecision when solving problems. The authors concluded that the innovative curriculum assisted most of the students and that low-ability students failed to benefit due to other factors, such as availability of resources and teacher time. They recommended that low-ability students may require small, homogeneous groupings for some instruction to increase their opportunities for teacher feedback and student-teacher discussion.

In summary, many authorities have identified components of effective math instruction. However, as noted by Maccini and Hughes (1997), little research focused on math concept development has been done with adolescents with disabilities. Thus, teachers have limited guidance for incorporating the NCTM standards into math curricula for students with disabilities. This is important when teachers consider including students with disabilities into general education math classes and when they guide students in planning for transition from school to work.
Definition of Terms

**Mathematical concepts.** Mathematical concepts include mathematical relationships and ideas. These enable us to decide on the appropriate procedure needed to solve a problem.

**Algorithms:** Mathematical procedures or formulas used to compute math problems.

**Problem-solving.** Math problems that are embedded in a real-life context. They may be written or verbal and require students to select and apply appropriate procedures.

**National Council of Teachers of Mathematics (NCTM).** This is a professional association for mathematics educators.

**Equivalent fractions.** Two fractions that represent the same numerical value, i.e. 2/3 and 4/6 are equivalent fractions. Computing equivalent fractions requires the student to rewrite a fraction in a different form without changing its value. When students reduce fractions to lowest terms, they are practicing fraction equivalency.

**Ratio and proportion problems.** Ratio and proportion problems involve comparing the number in one group to that of another. The comparison may be between either dissimilar or similar groups. Ratio and proportion problems are an application of the concept of fraction equivalency.

**CRA.** An instructional sequence in which concepts are developed first through the use of concrete manipulative devices. After an understanding of the concept is demonstrated at the concrete level, instruction proceeds to the representational level. The student uses representational drawings to further develop the concept. Finally, the abstract level is introduced when the student has demonstrated understanding at the
concrete and representational levels. At this level, numerical symbols are used to solve math problems.

Mild to moderate disabilities. This is a broad term to include learning disabilities, behavioral disorders, emotional disabilities, physical disabilities, or mental retardation. Students with mild to moderate disabilities may receive instruction in the general education setting or in a resource room for part of the day. The disabilities are not severe enough to warrant a separate class placement for most of the day.

Delimitations of the Study

This study was delimited geographically to Las Vegas, Nevada, a large, urban center in southern Nevada. It was further delimited to include only subjects attending Lied Middle School, a public middle school in northwestern Las Vegas. Moreover, only subjects receiving math instruction in resource rooms were included in the treatment groups.

Limitations of the Study

Because this study investigated only fraction equivalence, the results cannot be generalized to other mathematical skills or concepts. The subjects included middle-school students with mild to moderate disabilities; therefore, the findings should not be generalized to students with severe or profound disabilities, or students in elementary or high school. Finally, caution should be used in generalizing the results of this study to students outside the Las Vegas area or to students living in rural or semi-rural settings.

Summary and Overview of Remaining Chapters

Research has shown that most students have difficulty with developing fraction concepts. Students incorrectly apply whole number concepts to fractions and fail to
transfer informal knowledge to abstract procedures. Few empirical studies exist that investigate math instruction for adolescents with disabilities, and most interventions continue to deal with procedures rather than concepts. This study is intended to provide new information related to teaching fraction equivalence concepts. Specifically, comparisons will be made between the concrete-representational-abstract sequence to representational-abstract instruction in the solving of word problems, representational problems, and abstract problems with fraction equivalencies. Moreover, the CRA and RA instructional sequences will be compared to traditional instruction of these three types of problems. Additionally, student attitudes toward the instruction and procedures will be explored. The results of this study have important implications for special educators who teach mathematics.

In Chapter 2, a review of literature pertinent to this study is presented. Chapter 3 contains a discussion of the methodology used in the study. The results of the study and a discussion of implications are stated in Chapters 4 and 5.
Teaching fractions has been one of the most difficult areas of mathematics instruction. Even typically-achieving students experience difficulty in understanding fractions, and learning is slow and complex (Tourniaire & Pulos, 1985). It is no surprise that students with learning disabilities experience even more problems in this area. According to McLeod and Armstrong (1982), secondary mathematics students with learning disabilities have the greatest amount of difficulty with terminology and operations with fractions.

A review of the literature on fraction studies revealed that many researchers focused on the proportional nature of fractions. A fundamental comprehension of equivalent fractions is essential if students are to understand the algorithms for solving problems involving fractions. The equivalent fraction concept is also essential in ratio and proportion problems which lead to more complex mathematical concepts such as probability, rates, and functions (Carnine, Jitendra, & Silbert, 1997).

Literature Review Procedures

Studies included in this review were selected through a comprehensive search through Education Resources Information Center (ERIC), a manual search through
selected journals, and an ancestral search through the reference lists of obtained articles. Included in the manual journal search were *Journal for Research in Mathematics, Journal of Learning Disabilities, Learning Disabilities Research & Practice, Focus on Learning Problems in Mathematics, Remedial and Special Education, and Exceptional Children*. Studies were included if they examined the effectiveness of an instructional method used to teach fractions to students in elementary or secondary school. Studies that examined the effectiveness of the CRA sequence in teaching math were also included in this review.

As a result of this search and selection procedure, sixteen studies were identified. Nine studies involved fraction investigations involving students without disabilities, two studies involved fraction studies involving students with disabilities, and five studies investigated the effectiveness of the CRA sequence in math instruction. Tables 1, 2, and 3 provide summaries of the studies included in this review.

**General Education Studies**

Larson (1980) observed that students had greater difficulty associating a fraction number with a point on a number line than with a shaded region of a geometric figure. She conducted a study with 382 seventh-grade students to assess their ability to place or identify fractions on a number line. Students were given a 16-item multiple-choice test designed by the investigator to measure their ability to locate fractions on a number line. The test was divided into four subtests, each assessing a separate subconcept. A 2 x 2 repeated measures ANOVA was used to analyze the students' scores on the subtests. The investigator found that students were significantly more successful when the number...
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<td>Armstrong &amp; Larson, (1995)</td>
<td>36</td>
<td>4&lt;sup&gt;th&lt;/sup&gt;, 6&lt;sup&gt;th&lt;/sup&gt;, and 8&lt;sup&gt;th&lt;/sup&gt; grades; 1 elementary &amp; 1 junior high school</td>
<td>Students tended to use direct comparison instead of part-whole comparison in tasks, even when fractional notation was used. Students were taught with representational drawings.</td>
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<td>Behr, Wachsmuth, &amp; Post, (1985)</td>
<td>16</td>
<td>4&lt;sup&gt;th&lt;/sup&gt; grade; 2 elementary school sites</td>
<td>Abstract level task. High-scoring students demonstrated an understanding of order and equivalence and had good estimation skills.</td>
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<td>Behr, Wachsmuth, &amp; Post, (1984)</td>
<td>12</td>
<td>4&lt;sup&gt;th&lt;/sup&gt; grade; 2 elementary school sites</td>
<td>After instruction with manipulatives, most students developed strategies, although many did not generalize them. Manipulatives helped visualization of problems.</td>
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<td>Bright, Behr, Post, &amp; Wachsmuth (1988)</td>
<td>47</td>
<td>4\textsuperscript{th} and 5\textsuperscript{th} grades; 2 elementary school sites</td>
<td>Study 1: Little difference in pre- and posttest scores over 4-day intervention with number lines. Studies 2 &amp; 3: Eight days with more instruction with number lines. Students improved in pre- to posttest scores.</td>
</tr>
<tr>
<td>Confrey &amp; Scarano, (1995)</td>
<td>Not stated</td>
<td>3\textsuperscript{rd} -5\textsuperscript{th} grade elementary class</td>
<td>3-year study using project-based curriculum with representational drawings to teach ratio and proportion. Students compared favorably to older students in traditional curriculum.</td>
</tr>
<tr>
<td>Larson, (1980)</td>
<td>32</td>
<td>7\textsuperscript{th} grade, 1 junior high school</td>
<td>No intervention used. Students were assessed in beginning of 7\textsuperscript{th} grade. Students had not developed the concept of equivalence using a number line.</td>
</tr>
<tr>
<td>Citation</td>
<td>Subjects</td>
<td>Setting</td>
<td>Description</td>
</tr>
<tr>
<td>------------------</td>
<td>----------</td>
<td>-------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Mack, (1990)</td>
<td>8</td>
<td>6th grade, 1</td>
<td>Rote knowledge interfered with students' ability to solve problems.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>middle school</td>
<td>Students did not transfer informal knowledge to classroom</td>
</tr>
<tr>
<td>Mack, (1995)</td>
<td>7</td>
<td>3rd and 4th grade</td>
<td>Students applied whole-number notation to fractions and fractional notation to whole numbers in abstract problems. Students could solve real-life problems accurately.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>at 1 elementary site</td>
<td></td>
</tr>
<tr>
<td>Morris, (1995)</td>
<td>31</td>
<td>6th grade; 1</td>
<td>Students in experimental group used manipulatives and drawings.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>middle school</td>
<td>They had better understanding of concepts than control group students.</td>
</tr>
</tbody>
</table>
Table 2

Summary of Special Education Fraction Studies

<table>
<thead>
<tr>
<th>Citation</th>
<th>Subjects</th>
<th>Setting</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kelly, Gersten,</td>
<td>28 total, 9(^{th}), 10(^{th}), 11(^{th})</td>
<td>9(^{th}), 10(^{th}), 11(^{th})</td>
<td>Compared two curricula both using active teaching techniques for fraction operations. Experimental group was taught specific strategies. The experimental group did better on posttest than control group. No significant difference on maintenance test.</td>
</tr>
<tr>
<td>&amp; Carnine, (1990)</td>
<td>including 17 in special education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moore &amp; Carnine,</td>
<td>29 total, 9(^{th}), 10(^{th}), 11(^{th})</td>
<td></td>
<td>Compared two curricula both using active teaching techniques for ratio and proportion. Experimental group was taught explicit strategies. Both groups improved, although videodisc group was better on posttest. No significant differences on follow-up test.</td>
</tr>
<tr>
<td>(1989)</td>
<td>including 6 in special education</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 3

**Summary of CRA Studies**

<table>
<thead>
<tr>
<th>Citation</th>
<th>Subjects</th>
<th>Setting</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Funkhouser,</td>
<td>12 students with learning disabilities</td>
<td>Kindergarten &amp; 1&lt;sup&gt;st&lt;/sup&gt; grade</td>
<td>All students achieved over 90% mastery in counting and adding sums from 0 through 5. Concrete manipulatives only.</td>
</tr>
<tr>
<td>(1995)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Harris, Miller &amp; Mercer,</td>
<td>112 students, including 12 education with learning disabilities</td>
<td>2&lt;sup&gt;nd&lt;/sup&gt; grade general classes</td>
<td>All students learned multiplication facts using CRA sequence. The improvement ranged from 25 to 85 percentage points.</td>
</tr>
<tr>
<td>(1995)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hiebert,</td>
<td>25 students without disabilities</td>
<td>4&lt;sup&gt;th&lt;/sup&gt; grade; elementary school</td>
<td>Base-10 blocks and number lines were used successfully to teach decimal concepts and addition and subtraction of decimals.</td>
</tr>
<tr>
<td>Citation</td>
<td>Subjects</td>
<td>Setting</td>
<td>Description</td>
</tr>
<tr>
<td>---------------------</td>
<td>----------------</td>
<td>--------------------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Miller &amp; Mercer, (1993)</td>
<td>5 students</td>
<td>Elementary</td>
<td>CRA was used to determine the “crossover” from C to R to A.</td>
</tr>
<tr>
<td></td>
<td>with learning</td>
<td>resource room</td>
<td>Students mastered addition facts and coin sums.</td>
</tr>
<tr>
<td></td>
<td>disabilities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Peterson, (1988)</td>
<td>24 students</td>
<td>Elementary and</td>
<td>CRA was more effective than abstract-only in teaching place</td>
</tr>
<tr>
<td></td>
<td>with learning</td>
<td>middle-school,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>disabilities</td>
<td>resource and</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>self-contained</td>
<td></td>
</tr>
</tbody>
</table>
lines were equivalent to one unit in length than when the number lines were longer than one unit. This implied that students could directly apply the procedures learned with shaded regions of geometric figures representing one whole unit to number lines representing one whole unit. However, they could not transfer their knowledge when more than one unit was represented. In addition, students were unable to place a fraction on a number line when the number of subdivisions on the number line did not correspond directly to the denominator of the fractions. For example, students could identify the fraction $1/3$ when the number line was divided into three parts but not when the number line was divided into six or nine parts. She concluded that students lacked conceptual understanding, perhaps due to teaching methods that encouraged algorithm and rule formation over concept development.

Behr, Wachsmuth, Post, and Lesh (1984) explored students’ reasoning as they compared unequal fractions. The researchers were particularly interested in the effect of manipulative devices on students’ ability to transition from concrete to abstract reasoning. In this study, 12 fourth-graders participated in the 18-week intervention. The students came from two geographically separated school sites with 6 students at each site. Each group received instruction using the same methods and materials. The students were interviewed as they worked on problems, and their answers were coded according to the strategy the child employed. Results of this study showed that students initially tried to apply whole-number concept to fraction problems. However, after instruction, most students developed appropriate strategies, and many times students were able to visualize the manipulative devices to help them solve the problems. The authors observed that even after extensive instruction many students were not able to generalize
their learning to new situations. Although the authors identified several commonly used strategies, they did not address the efficacy of the strategies through a comparative analysis of accuracy or length of time required to solve the problems. The authors also did not address the way in which children make the transition from one strategy to another strategy, nor did they determine when, on average, children were ready to move from manipulative devices to abstract computation.

In another study exploring the concept of fraction size, Behr, Wachsmuth, and Post (1985) investigated how the cognitive processes of high-achieving students differed from those of low-achieving students. The authors cited research with whole numbers that indicated good estimating skills were related to a firm concept of number size. Behr, Wachsmuth, and Post devised a test to determine whether or not students' understanding of fraction size could be demonstrated by estimation activities. Sixteen students from two different school sites were interviewed for this study. As in the previous study, the students were interviewed as they worked on problems and their answers were coded. Students were given cards with numerals and instructed to use the cards to determine two fractions such that when added, their sum was as close to 1 as possible without equaling or exceeding 1. Data were collected at the end of 20 weeks of instruction and again at the end of 27 weeks of instruction. Average deviation scores were used to categorize students' responses as high, medium, or low without regard to any theoretical criteria for what constituted high, medium, or low scores. Then the responses were sorted into six categories. Students with high scores used concepts of fraction equivalence, estimation, and fraction order, while low scorers did not. The researchers concluded that the development of a good quantitative concept of rational numbers depended on both skills.
in estimation and concepts of fraction order and equivalence. This study was useful in providing insights into the conceptual reasoning of students, although the lack of a theoretical basis for characterization of scores should be viewed with caution. In addition, the task that was selected, construct-a-sum, does not bear a close resemblance to any tasks that students are regularly asked to perform in math classes.

Bright, Behr, Post, and Wachsmuth (1988) explored the connections between students’ understanding of fractions and the representations of fractions on a number line. They conducted three experiments with fourth and fifth grade students using the number-line test developed by Larson (1980). In the first experiment with 5 students over a four-day period, the researchers found little difference between students’ scores on the pretest and posttest. Therefore, in the second and third experiments, researchers included more instruction on equivalence and on translations between number line representations and shaded areas of geometric figures. In the second experiment lasting 8 days, the eight students had significant improvements in posttest scores over pretest scores. Although students were able to solve equivalence problems using algorithmic procedures (i.e., multiplication or division), they continued to have difficulty in solving problems using the number line. The researchers noted that even when students were able to correctly solve a problem presented in symbols, they were not able to demonstrate the answer with the number line. The third experiment, also conducted over 8 days, was a large-group (34 students) replication of the second experiment. Again, posttest scores were significantly higher than pretest scores, indicating that the instruction was successful. This series of experiments demonstrated that students did not spontaneously make the transition from geometric representation of fractions to number line representation of fractions. This is
an important observation for several reasons. First, fraction instruction typically moves from work with manipulative devices to work with representations using shaded areas of geometric figures. The representations closely resemble the manipulative devices. Number lines, on the other hand, do not look like the manipulative devices. Second, students used the number line most successfully when they did not have to repartition the number lines or reduce fractions. However, partitioning and repartitioning number lines is useful for developing the notion that a fraction may be associated with many different equivalent fractions. Increased work with number lines could enhance students’ concept development and better prepare them for subsequent work with fraction operations.

Mack (1990) investigated students’ informal knowledge about fractions and the extent to which the learning of rote procedures interfered with the application of informal knowledge in problem-solving. Eight sixth-grade students were treated as individual case studies, and instruction was based upon each student’s existing store of informal knowledge. Students were allowed to use manipulative devices such as fraction circles to help solve problems as long as the student felt they were needed. However, if a student seemed dependent on the use of manipulative devices by the beginning of the fifth week, the teacher gradually encouraged the use of pencil-and-paper symbolic representations. Mack found that students’ initial understanding of fractions was limited to partitioning of areas. Even though most students came to instruction with informal knowledge of fractions, they failed to connect that knowledge to fraction symbols and algorithms. Students devised their own alternative algorithms for solving fraction problems and often persisted in using these alternative algorithms even after the traditional algorithms were taught. Mack noted that students tended to explain faulty
algorithms with rote answers instead of applying their informal knowledge. While this study provided some insights into students' conceptual understanding of fractions, its individualized case-study nature makes it difficult to generalize it to a larger group of students or to systematically replicate the study. Although students' use of informal knowledge and rote memory was detailed, the investigator did not address the process by which students develop conceptual understanding of fractions.

In a subsequent study, Mack (1995) investigated the ways that students used informal knowledge to develop meanings and representations for fraction symbols during instruction. Four third-grade and three fourth-grade students of average ability who had not previously received formal fraction study were selected for this study. Using the same method as her previous study (Mack, 1990), each of the seven students was considered an independent case study. Instruction consisted of six 30-minute individual lessons over a 3-week period. Specifically, the lessons were designed to facilitate and explore the process that students used to construct meaning using prior knowledge. The investigator guided students to solve problems by asking leading questions that attempted to connect the lesson to preliminary informal knowledge. Throughout the verbal instruction students were encouraged to use concrete materials such as fraction circles and fraction strips to develop solutions. Paper and pencil were not introduced until students solved the problems successfully using concrete materials. As in the earlier study, the lessons differed in their scope and content depending on the knowledge base of the student. Students were not expected to reach any predetermined mastery criterion before moving on. Rather, the investigator was flexible regarding the lesson topics based on her understanding of how students develop relationships between concepts and procedures.
Assessment tasks also varied among students, and they were based upon that individual’s responses to previous questions. Mack concluded that students in the study overgeneralized the previously learned whole-number concepts to abstractly presented fraction problems although they were able to solve successfully closely-matched problems that related to real-life situations. There appeared to be little transference of knowledge gained using concrete objects to abstract problems. In addition, students overgeneralized newly learned fraction procedures to problems involving both whole numbers and fractions. This study is limited in its generalization to other students because of the lack of consistency in the lessons and assessments.

Morris (1995) examined the ways that low-achieving students constructed meaning for fraction symbols. Specifically, she focused on categorizing the difficulties students encountered as they built connections between symbols and concrete manipulative devices and on pinpointing variables affecting students’ development of meaning for fraction symbols. Nineteen low-ability sixth-grade students were included in the group that received the experimental treatment while twelve sixth-grade students in another class formed the comparison group. Students in the experimental group received instruction on a variety of fraction topics including naming fractions, partitioning, equivalence, ordering fractions, and addition and subtraction of fractions. These students used concrete manipulative devices, representative drawings, and spoken language to model and solve the problems. In contrast, the comparison group was taught the same topics on the abstract level using procedures and algorithms for solving the problems through a conventional textbook. Data were collected through interviews with students, and responses were scored for accuracy and for quantitative reasoning. Results indicated...
that students in the experimental group solved more problems correctly and used more appropriate reasoning strategies than students in the comparison group. Students in the experimental group successfully linked written symbols to appropriate referents and developed their own rules or strategies for adding and subtracting fractions. Morris discovered nine variables that seemed to affect students' development of meaning for fraction symbols. The variables were as follows:

- Semantic processes. Students in the experimental group discussed and debated problems. Morris noted that “the conceptually-based instruction allowed children with poor memories to bypass/reconstruct procedures” (p. 27).

- Familiarity with referents. The experimental group spent a great deal of class time using concrete manipulative devices and representative drawings. During the interviews, these students either recalled mental images or drew representations to help them solve problems.

- Translation skills. Students in the experimental group were taught to move from symbolic representation to referents to solve problems, as well as from referents to symbolic representation to prove answers.

- Difficulty with the notion of area. Although students frequently used area models to construct meaning for symbols, they had difficulty partitioning the areas correctly. These students drew unequal parts or used wholes of different sizes to compare fractions.
• Interaction of the subconstructs. Students had difficulty connecting fractions expressed as part of a whole to fractions represented as a place on a number line.

• Reliance on a single strategy. Once students acquired a strategy using referents, they tended to continue using it regardless of its appropriateness to the task at hand.

• Quickly formulated rules and attention to a single attribute. Like the problem noted above, once students formulated a rule for solving abstract problems, they tended to use it indiscriminately.

• Cognitive processing demands. When problems were presented symbolically, students seemed overwhelmed. However, they were able to make connections and reason out the solution when they discussed the problem.

• Extended reasoning. Students gradually gave lengthier explanations for problems through oral prompting and peer discussions. Students began to use self-monitoring procedures orally, but their written work took longer before showing improvement.

Confrey and Scarano (1995) conducted a 3-year study that investigated how students’ understanding of proportion and ratio could be affected by altering the traditional curriculum’s scope and sequence. The project was based on the theory that partitioning or splitting was as fundamental a concept as counting and that it could be employed to facilitate students’ learning. The authors introduced a project-based curriculum to a heterogenous class of third-graders and collected data on their progress.
through the end of their fifth-grade year. Data were collected at the end of fourth-grade and again at the end of fifth-grade. The curriculum used a variety of representational forms such as Venn diagrams, daisy chains, contingency tables, tables of values, dot drawings, two-dimensional graphing, and ratio boxes. The following three concepts about ratio and proportion provided the framework for the curriculum:

- Ratio and proportion are closely connected to multiplication and division.
- Addition and subtraction of fractions is secondary to multiplication and division.
- Connections to geometry are more important than additive relationships.

Students were given written assessments at the end of the fourth- and fifth-grade years. Results indicated that the 10 and 11 year old students in the study had higher scores than did 14 and 15 year old students taught with a traditional curriculum. The authors concluded that students given a context-rich curriculum that used a wide variety of problem-solving and representational activities could understand and apply the concepts of ratio and proportion at a younger age than was previously believed. Although this study provided comparative data, the results are difficult to interpret or to generalize because the data on the older students was taken from a study conducted in 1988 by Hart. Thus, the results may be confounded due to lack of experimental control over the comparison group's treatment.

Armstrong and Larson (1995) examined the types of strategies used by students in comparing fractions using representational diagrams. Previous researchers (Morris, 1995, and Mack, 1990 and 1995) noted that students often attempted to integrate informal knowledge in solving comparison problems by partitioning geometric figures, but they
often drew inaccurate representations. Armstrong and Larson conducted an observational study incorporating geometric area models for comparing fractions. Thirty-six students, 12 each in fourth, sixth, and eighth grades, volunteered to participate in the study. Data on students' responses to problems, their behaviors, and their explanations were collected through audiotaped interviews. Students were presented with a variety of comparison tasks represented by partitioned rectangles. Perceptual distractors were imbedded in the tasks; in some cases the sizes of the rectangles were different, and in other cases the rectangles were partitioned in opposite directions. The problems, which asked students to compare the sizes of cakes, were stated verbally, and symbols were included in only the last eight tasks. Results of the study were categorized into three types of strategy: direct comparison, part-whole, and combination direct comparison and part-whole. The authors concluded that younger students preferred to use a direct comparison strategy, even after fraction symbols were introduced. As the grade level increased, students increasingly relied on a part-whole strategy for comparing the fractions. The authors observed that most textbooks use same-size wholes in representational drawings for comparing fractions. This practice appeared to reinforce students' use of direct comparison and may inhibit development of part-whole relationships. The authors concluded by recommending that teachers recognize the developmental aspects of rational number concepts and give students many opportunities to connect fractional models to fractional symbols from third grade to middle school.

Many of the studies cited in this section demonstrated that students have a weak understanding of fraction concepts. Teachers enhanced students' conceptual development through the use of manipulative devices, number lines, and explicit strategy
instruction. Moreover, students did not readily connect informal knowledge of fractions to fraction concepts as they were taught in school. Teachers were advised to anchor their instruction in real-life situations that are meaningful to students. These studies provided important insights into how students develop concepts related to fraction equivalence. However, they did not isolate critical variables or compare instructional methods. Thus, it remains unclear how teachers are to implement these suggestions in their classrooms.

Special Education Studies

Moore and Carnine (1989) compared two curricula for teaching ratio and proportion problems. The experimental curriculum consisted of videodisc instruction following empirically validated design principles. The comparison group received instruction developed from basal textbooks and taught by a teacher. Both groups were taught using research-validated active teaching methods, and the instructional components were similar in both groups. However, the experimental group received explicit strategy instruction while the comparison group did not. Also, the experimental group was taught one procedure that could be applied to a variety of problem types, while the comparison group was taught several procedures for solving a single problem type. The researchers designed a 21-item criterion-referenced test to evaluate student achievement; two forms of the test were developed; one was used as a posttest and the other was used as a maintenance check 10 days after the completion of the unit. A shorter, 10-item version was used as a screening pretest. In addition, data were collected on student time-on-task using eight 1-minute observations in each class period. Students and teachers also completed attitude questionnaires at the end of the study. A 2 x 2 ANOVA was used to analyze test results. A significant difference favoring the
experimental group was found on the posttest, although the difference between groups on the maintenance test was not significant. In the experimental group, 77% of the students reached the acceptable criterion of 80%, while only 44% of the comparison group reached the acceptable criterion level. The experimental group’s time-on-task exceeded that of the comparison group, and both student and teacher satisfaction were higher in the experimental group than in the comparison group. The authors established that students in both groups made considerable progress from the pretest to the posttest and that students with disabilities could learn in a general education setting and achieve comparable progress as their peers without disabilities. The experimental group’s superior performance on the posttest was attributed to three curriculum design features: 1) explicit strategies that were composed of small steps; 2) regular use of strategies; and, 3) many varied examples. With regard to the method of presentation, i.e. videodisc or teacher, the authors concluded that a single teacher would not be able provide the pacing, monitoring, and variety of examples that could be provided with the videodisc instruction. They noted that the preparation of the twenty teacher-directed lessons took two weeks and would be unreasonable to expect of classroom teachers.

Kelly, Gersten, and Carnine (1990) also compared curricular designs for fraction instruction. Forty low-performing high-school students, including 17 students with learning disabilities, participated in the study. The experimental condition consisted of videodisc instruction that incorporated three design variables: 1) systematic practice in discriminating among similar problem types, 2) separation of easily confused concepts and terminology, and 3) a wide variety of examples of each concept. The comparison group was taught by a teacher using a basal curriculum. Both groups received guided
practice, daily feedback, and regular review activities according to effective instruction principles. A 12-item curriculum-referenced test was used to evaluate the effectiveness of the instructional programs and was administered immediately following the unit's completion. Both groups showed improved scores from pretest to posttest, and both groups had high on-task behavior. A t test was performed on the group means, and results indicated a significant difference between group means favoring the experimental group. There were also significant differences favoring the experimental group between group means on each of the three design variables. Error analyses performed on the test items suggested that errors could be directly traced to the curriculum design. Students in the comparison group made more errors when they had to choose an appropriate strategy or when the problem was presented with the unknown on the left instead of the right. Students with learning disabilities in both groups made gains comparable to their peers without disabilities. There were no significant differences between students with and without disabilities in either group. Thus, the researchers concluded that students with disabilities, given an effective curriculum, can benefit from instruction in general math classes.

These two studies utilized videodisc instruction to teach fraction concepts to students. In both studies, the researchers found that students with disabilities progressed as well as students without disabilities, and that instructional design variables, rather than delivery method, were the key differences between the experimental and comparison groups. These variables were instruction in explicit strategies, practice in discriminating among similar problem types, separation of easily confused concepts and terminology, and a wide variety of examples of each concept. One study noted that videodisc
instruction provided more intense, efficient practice than could be provided by a teacher alone. Neither study emphasized the use of manipulative devices or drawings to enhance concept development.

Concrete-Representational-Abstract Instruction

Several studies have concentrated on the effectiveness of the concrete-representational-abstract (CRA) teaching sequence for teaching various math skills (Funkhouser, 1995; Harris, Miller, & Mercer, 1995; Hiebert, Wearne, & Taber, 1991; Miller & Mercer, 1993, Peterson, Mercer, & O’Shea, 1988). Each of these studies involved carefully scripted lessons combined with direct instruction methods.

Peterson et al. (1988) compared the effectiveness of two teaching methods – CRA and abstract-only – in teaching place value to students with learning disabilities. In this study, 24 elementary and middle-school students with learning disabilities were divided into an experimental and control group. Each group of students received instruction in identifying place value through carefully scripted lessons. The only difference between the two groups was that the experimental group received three lessons using manipulative devices (popsicle sticks), three lessons using representational instruction (drawings), and three lessons at the abstract level (numbers only). The control group received all nine lessons at the abstract level. Statistically significant differences, favoring the experimental group, were noted for instructional method on three acquisition measures: posttest, maintenance, and retention. No group differences were found in generalizing to untaught place value skills.

Hiebert, Wearne, and Taber (1991) investigated the ways in which various concrete materials shaped and influenced students’ conceptions of decimals. They
pointed out that decimals have both a discrete nature similar to whole numbers and a continuous nature similar to fractions. In other words, operations involving decimals resemble operations involving whole numbers, yet decimals are continuous because between any two decimals a third decimal can be inserted. One fourth-grade class of 25 students received decimal instruction for eleven consecutive days. Of these students, eight were selected for case-study observation to enable researchers to gather more detailed information about students’ understanding. The first four lessons introduced the discrete nature of decimals with a concrete representation, base-10 blocks. In the next three lessons, students explored the continuous nature of decimals by using number lines and a circular stopwatch. In the final three lessons, students used the base-10 blocks to perform addition and subtraction operations on decimals. Students were given four assessments: a pretest, a test between phase 1 and phase 2, a test between phase 2 and phase 3, and a posttest. Results indicated that the types of representations used affected students’ performance on the tests. Students’ scores improved on discrete-type problems directly after instruction with the base-10 blocks, and students’ scores also improved on continuous-type problems directly following instruction with the number lines and stopwatch. Interestingly, students’ scores on abstract problems only improved significantly after the first, discrete phase. The researchers suggested that students had more difficulty conceptualizing the continuous nature of decimals, even after instruction. The case-study data revealed that some students were able to perceive patterns and apply those patterns successfully to solving problems. The researchers also found that the development and use of appropriate language facilitated students’ learning, but that language may play different roles for different students.
Miller and Mercer (1993) examined the effectiveness of the CRA procedure in teaching addition facts and coin sums to students with learning disabilities. In addition, they determined how many lessons were needed at each level before students were able to transfer skills to abstract problems. Five students with learning disabilities participated in the multiple baseline-across-subjects intervention. During the baseline phase, daily 1-minute probes were administered with no teacher feedback. The treatment phase consisted of 20-minute scripted lessons including an advance organizer, demonstration and modeling, guided practice, and independent practice. Each lesson was followed by a 1-minute assessment probe at the abstract level. Students progressed to the next lesson if they achieved 80% accuracy on independent practice problems.

All students in this investigation reached the 80% criterion on their first attempt. Results indicated that the CRA sequence was effective for acquiring math skills after five 20-minute lessons at each stage. During the concrete stage, three of the students with learning disabilities answered more problems correctly than incorrectly on the 1-minute probes, indicating a "crossover effect," or ability to generalize from concrete instruction to abstract problems. The other two students achieved the crossover effect during representational instruction. Thus, for some students, fewer lessons at the concrete stage may be effective, whereas other students may need five at each stage to achieve mastery. It is not clear from the results whether or not the students who were able to transfer from the concrete to the abstract stage needed representational instruction.

In a similar study, Harris, Miller, and Mercer (1995) explored the use of the CRA procedure to teach multiplication facts to students with learning disabilities in general education settings. Twelve students with learning disabilities, along with 99 students
without disabilities and one student with an emotional disability, participated in the single-subject multiple baseline-across-classrooms investigation. Students were also taught a mnemonic device to facilitate the transition from representational to abstract instruction. All 12 students with learning disabilities improved from pretest to posttest, with the extent of improvement ranging from 25 to 85 percentage points.

The use of manipulative devices was applied successfully to teach basic number facts to 12 kindergarten and first-grade children with learning disabilities (Funkhouser, 1995). In this intervention, children were taught to glue jellybeans within five-cell frames (vertical rectangle divided into five equal-sized boxes). By the end of the 4-week intervention, all 12 students achieved over 90% mastery in recognizing and matching numbers from 0 to 5 and in adding sums to 5. This investigation demonstrated that the use of manipulative devices, without a representational stage, was an effective teaching method.

The above five studies revealed that the CRA sequence can be used successfully to teach a variety of math concepts and skills. The CRA interventions were used to promote conceptual understanding and to provide students with a method for solving problems independently. All of these studies used carefully scripted instructional formats and emphasized a high degree of proficiency before criterion was met. These studies also included the successful instructional design components discussed in the previous sections.

Several conclusions can be drawn from this review of literature. First, there have been relatively few studies that have examined effective methods for teaching fractions. Second, the general education studies emphasized qualitative research related to concept
development but did not provide quantitative data comparing instructional methods. Although they used concrete manipulative devices and representational drawings, the researchers did not provide empirical data to support their suggestions. Third, the special education studies compared instructional methods, but the researchers did not examine the effectiveness of the CRA model in teaching fractions. And fourth, while the CRA studies also compared instructional methods, they did not specifically address fraction instruction. Thus, there appears to be a need for studies that combine the use of CRA with fraction instruction and that incorporate effective curriculum design variables.
CHAPTER 3

METHOD

This study was conducted to investigate the effects of the concrete-representational-abstract instructional sequence and the representational-abstract instructional sequence upon students' understanding of fraction equivalence. Methods and procedures used in this study are detailed in this chapter. The chapter is organized into five sections: statement of the null hypotheses, description of subjects and setting, description of the research instrumentation, procedures, and treatment of the data.

Statement of the Null Hypotheses

The following null hypotheses were tested at the .05 level of confidence:

H1. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on the computation of equivalent fractions.

H2. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on solving word problems with embedded equivalent fractions.

H3. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on attitudes toward mathematics.
H4. There will be no statistically significant differences between pretest and posttest measures for the CRA and RA groups.

Description of the Subjects and Setting

The participants in this study were two middle school teachers and their students in a public middle school located in Clark County, Nevada. Each teacher taught two 50-minute periods of math daily. One of each teacher's classes was randomly designated the CRA group and the other became the RA group. Thus, the sample consisted of two CRA classes and two RA classes.

A total of 50 students enrolled in grades 6, 7, and 8 formed the two treatment groups. The students ranged in age between 11 and 15 years. Each student had been identified as having specific learning disabilities, mental retardation, attention deficit disorder, or emotional disabilities and was placed in a special education resource room for mathematics instruction. Table 4 gives details on the age, sex, grade placement, disability, and IQ test scores. Pretest measures indicated no significant differences between groups for age, sex, grade, disability, or IQ.

The pretest-posttest measure was also administered to 65 eighth-grade students enrolled in three general education math classes. These classes were designed for students of average ability levels. In addition to these general math classes, the school provided both an algebra class and an advanced math class for eighth-grade students of higher ability levels. The students in these three classes ranged in age from 13 to 15 years and included 31 boys and 34 girls. In October 1998, these students took the TerraNova Comprehensive Test of Basic Skills, 5th ed. (CTB/McGraw-Hill, 1996-1997). When
Table 4

Characteristics of the Two Treatment Groups

<table>
<thead>
<tr>
<th></th>
<th>Treatment Group</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CRA (n = 26)</td>
</tr>
<tr>
<td><strong>Age</strong></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td><strong>Sex</strong></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>15</td>
</tr>
<tr>
<td>Female</td>
<td>11</td>
</tr>
<tr>
<td><strong>Grade Level</strong></td>
<td></td>
</tr>
<tr>
<td>Sixth Grade</td>
<td>8</td>
</tr>
<tr>
<td>Seventh Grade</td>
<td>11</td>
</tr>
<tr>
<td>Eighth Grade</td>
<td>7</td>
</tr>
<tr>
<td><strong>Disability</strong></td>
<td></td>
</tr>
<tr>
<td>Learning disabilities</td>
<td>22</td>
</tr>
<tr>
<td>Mental retardation</td>
<td>1</td>
</tr>
<tr>
<td>Attention Deficit Disorder</td>
<td>3</td>
</tr>
<tr>
<td>Emotional Disorder</td>
<td>0</td>
</tr>
<tr>
<td><strong>IQ</strong></td>
<td></td>
</tr>
<tr>
<td>Mean scores</td>
<td>86.19</td>
</tr>
<tr>
<td>Range</td>
<td>56 - 113</td>
</tr>
</tbody>
</table>

Note: A t-test revealed no significant differences between groups on any of these variables.
compared to a national group of eighth-grade students, they achieved a mean stanine score of 4.75 on the mathematics portion, indicating math ability in the average range.

Immediately before taking the pretest-posttest measure, these students had just completed approximately three weeks instruction on rational numbers and ratio and proportion. Two of the students in the general education contrast group were identified with specific learning disabilities, but their areas of deficit did not include mathematics, and they received all of their math instruction in the general education classroom without modifications or adaptations. Thus, data were collected on 115 students.

Description of the Research Instrumentation

Pretest - Posttest

The pretest-posttest consisted of five parts and was the primary dependent measure. Two subtests measured students’ knowledge of prerequisite skills, namely recognition and naming of fractional parts of sets and geometric figures (Stein, Silbert, & Carnine, 1997). The Brigance Comprehensive Inventory of Basic Skills-Revised (Brigance, 1999) subtests Q3, Understands Complex Fractions Related to Quantity, and Q4, Understands Fractions Related to Area, were used for this purpose. Subtest Q3 consisted of 24 tasks requiring students to circle the appropriate fractional part of each set and measured students’ understanding of ratio and proportion. Subtest Q4 had 18 tasks requiring students to write the fraction that was indicated by the shaded portion of a geometric shape and measured part-whole discrimination. The third subtest was the Brigance Comprehensive Inventory of Basic Skills-Revised subtest Q6, Converts Fractions and Mixed Numbers. This subtest contained 16 fraction-related items at the abstract level. The first four items required students to convert a given fraction to a larger
equivalent fraction, e.g., 1/3 = ?/12. The next four items required students to convert a
given fraction to a smaller equivalent fraction, e.g., 3/6 = ?/2. The final eight items
required students to convert mixed numbers to equivalent improper fractions and
improper fractions to equivalent mixed numbers, e.g., 8/5 = ?, or 2 1/2 = ?/2. The fourth
subtest measured the same four areas with fractions embedded in word problems. For
example, "John ate one-third of the pizza, and the pizza had twelve slices. How many
slices did John eat?" The investigator composed the problem-solving subtest using the
same fraction equivalencies as the abstract subtest (see Appendix A). The investigator
designed the fifth subtest to measure students' knowledge of mixed numbers and
improper fractions. Students were required to shade in appropriate parts of geometric
shapes or to write a fraction representing shaded in portions of geometric shapes (see
Appendix A). Students were encouraged to try their best, and directions were reread or
rephrased as necessary. No other prompting was given. Students worked independently and
were allowed one 50-minute class period to complete the pretest or posttest.

Attitude Measurement

Students' attitude toward mathematics activities was measured using an
investigator-made 10-item questionnaire (see Appendix A). Students rated their
responses as (1) Don't Agree, (2) OK, or (3) Agree. Students were asked to rate how well
they liked working fraction problems, their perception of their own ability to work
fraction problems, and how well they liked the activities in math class. The attitude
questionnaire was administered prior to beginning the fraction unit and again
immediately after the fraction unit. Guidelines discussed by Linn and Gronlund (1995)
were used to develop this questionnaire.
Procedures

The study took place in a middle-school in a large urban area of the Southwest. Two teachers, one of whom was the investigator, taught the four instructional groups in special education resource rooms. Each teacher taught one class period using the CRA sequence and one period using RA instruction. Each lesson was carefully scripted to minimize possible teacher effects (see Appendix B). Both teachers were trained in the specific requirements of each instructional method.

Before beginning this study, explanatory letters and consent forms were sent home with students (see Appendix C). Only data from students whose parents signed and returned the consent forms were included in the study. Permission for the study was also granted by the school district and Human Subjects committees at the University.

Teacher Training Sessions

The investigator conducted a training session for the teachers involved in the study. The purpose of the training session was to promote consistency among the groups and to minimize any possible teacher effects. The session began with an overview of the instructional unit detailing the purposes and objectives, teaching methods, and lesson formats. Then, each teacher taught a 30-minute demonstration lesson while being observed. Specifically, teachers were observed for the following five criteria: 1) using explanations that were consistent with the strategies, 2) providing appropriate and adequate student feedback, 3) monitoring students during guided and independent practice, and 4) following the sequential order of the lesson, and 5) allowing adequate time for each activity. To obtain an estimate of fidelity of treatment, two observers independently rated the demonstration lessons. The observers sat in the back of the
classroom while they completed a checklist of the above five criteria. After the demonstration lesson, the observers compared checklists. There was 100% interobserver agreement that both teachers performed all five items adequately. This feedback was given to each teacher at the conclusion of the training session. During the study, two additional observations of each teacher were held, and feedback was given to each teacher according to the criteria listed above. Again, 100% interobserver agreement was reached in each of these two observations. The first observation resulted in one teacher being directed to include more feedback and practice at the conclusion of the lesson, while the second observation noted that both teachers performed all five criteria adequately.

**Lesson Format**

Each daily lesson followed a predictable format based upon the methodology of the *Strategic Math Series* (Mercer & Miller, 1991-1994). This program was selected as a model since it uses the CRA instructional sequence and has been validated with students who have learning difficulties in math. Each lesson consisted of six parts:

- **Advance Organizer**—This step linked the current lesson to previous instruction, identified daily objective, and gave a rationale for learning the skill.

- **Describe and Model**—The teacher first demonstrated the skill while describing aloud the steps, then the teacher and students solved problems together through a question-and-answer format.
• Guided Practice — The teacher gave prompts and cues as students solved problems together. As students gained independence, the teacher monitored students and assisted only as needed.

• Independent Practice — Students solved problems independently using the skills that had been taught. The teacher did not provide assistance.

• Problem-Solving Practice — The problem-solving activities were intended to allow students to apply their skills to real-life problems. Students and teacher constructed word problems using items taught earlier in the lesson.

• Feedback - Independent practice was collected and scored daily, and students who achieved a score of 80% or more were deemed to be at criterion level for that day’s lesson. Students who did not achieve criterion were given the opportunity to correct any errors before being allowed to move to the next lesson.

Lesson Sequence

For the CRA group, the first three lessons were designed to introduce the concept of fraction equivalence through the use of concrete manipulative devices. Concept development continued in lessons four through six. These lessons involved the use of representational drawings to represent fraction equivalence. In lesson seven, students were introduced to the abstract algorithm for computing equivalent fractions. In lessons eight through ten, students were provided practice in applying the algorithm to symbolic and word problems. Students practiced with mixed problems at the abstract level after lesson ten. The posttest was given the second day after completion of lesson ten.
Students in the RA group received the same instruction in the same sequence as students in the CRA group. However, instead of using manipulative devices during lessons one through three, students in the RA group drew representational drawings demonstrating the concepts being taught. Lessons four through ten were the same as the lessons for the CRA group.

Materials

Materials for both groups included an instructional guide for the teacher (see Appendix B), student worksheets and answer keys, and overhead projector. The instructional guide contained 10 daily lessons, student reference sheets, the pretest-posttest, and the practice worksheets used after lesson ten. The guide included scripted lessons for each day's activities including specific items to model and specific questions to ask students as they worked through the problems. Problems for guided, independent, and problem-solving practice were also provided.

In the CRA group, students used a variety of concrete manipulative devices during the first three lessons. These included commercially available fraction circles, small white dried beans, and student-made fraction squares using construction paper. For all remaining lessons for the CRA group and for all lessons for the RA group, students used paper-and-pencil or small white boards with dry-erase markers to compute problems. In all of the lessons for both groups, the teacher used an overhead projector and transparency of the day's lesson.

Eighth-grade Contrast Group

The eighth-grade students in the contrast group received instruction based on the curriculum guide produced by the school district and the textbook Mathematics:
Applications and Connections, Course 3, (Collins et al., 1995). The text presented each unit in a real-life context. For example, the rational numbers chapter was organized around the theme of baseball statistics. Each concept was described in words, illustrated with graphic representations, and then an algorithm was developed. Students then completed exercises at the abstract level applying the new algorithm. They were not given the opportunity to use concrete manipulative devices or to draw their own graphic representations to test the algorithms. Classwork and homework were collected regularly and scored. Students wishing additional help had the opportunity to stay after school with a math teacher, but they were not required to correct mistakes or attend remedial sessions.

Treatment of the Data

Data from all three groups (two treatment groups and 8th grade contrast group) were analyzed using a multivariate analysis of variance (MANOVA). To compare the two treatment groups, data from the pretests were used as a covariate in a multivariate analysis of covariance (MANCOVA). One-way analysis of variance (ANOVA) was used to analyze the data from the attitude questionnaire and the word-problem subtest, and a paired samples t-test was use to compare pretest scores to posttest scores for the two treatment groups. Level of confidence for rejection of the null hypotheses was set at .05 for all statistical tests.
CHAPTER 4

RESULTS

The purpose of this chapter is to present the results of statistical analyses of the data obtained in this investigation. Four null hypotheses were presented for analysis, and each hypothesis will be analyzed in turn.

Interrater Reliability

An assessment was performed on the implementation of treatment across groups. Two observers independently rated each teacher’s lesson presentation. Three observations of each teacher were held, once during the training session and twice during the implementation phase. During each observation, the observer completed an implementation checklist of five criteria described in Chapter 3. For all three observations, 100% interobserver agreement was attained. Any deviations from the scripted lessons were immediately brought to the teacher’s attention and corrected. As a result, one teacher was directed to provide more immediate feedback and remedial practice.

To obtain an estimate of scoring reliability, two raters independently scored a random sample of 33 student tests (20% of the total tests). The scorers agreed on 32 of the 33 tests for an interrater agreement of about 97%.
H1. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on the computation of equivalent fractions.

Data were collected from posttest scores for 115 students (26 in the CRA treatment group, 24 in the RA treatment group, and 65 eighth-graders who received traditional instruction). The subtests used for this analysis were subtests Q3 and Q6 of the Brigance (1999) and the investigator-made improper fraction-mixed number subtest because these tests measured students’ computation of fraction equivalences. Subtest Q3 required students to circle the appropriate fraction of a set, i.e., 5/6 of 18 squares. The investigator-made improper fraction-mixed number subtest required students to identify shaded portions of geometric figures when more than one unit was represented. These partitioning exercises provided a measure of students’ conceptual understanding of fraction equivalency. Subtest Q6 required students to solve abstract problems involving fraction equivalency, i.e., 6/8 = ? This test measured students’ ability to apply the algorithms for solving fraction equivalencies.

The set of subtests was designed to measure students’ understanding of fraction equivalency at both the conceptual and abstract levels, using graphic representations, abstract problems, and word problems. A high degree of correlation would be expected among the four subtests since similar items appeared in different forms on the various subtests. Table 5 gives the correlations among subtests for the treatment group posttest using the Pearson correlation.
Table 5
Correlations between posttest means for the CRA and RA groups (n = 50)

<table>
<thead>
<tr>
<th></th>
<th>Q3</th>
<th>Q6</th>
<th>Improper fractions</th>
<th>Word problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q3</td>
<td>–</td>
<td>.284*</td>
<td>.359*</td>
<td>.179</td>
</tr>
<tr>
<td>Q6</td>
<td>.284*</td>
<td>–</td>
<td>.590**</td>
<td>.568**</td>
</tr>
<tr>
<td>Improper fractions</td>
<td>.359*</td>
<td>.590**</td>
<td>–</td>
<td>.646**</td>
</tr>
<tr>
<td>Word problems</td>
<td>.179</td>
<td>.568**</td>
<td>.646**</td>
<td>–</td>
</tr>
</tbody>
</table>

Note. *p < .05, two-tailed. **p < .01, two-tailed.

Mean and standard deviation scores on the posttest were computed for each group (see Table 6). A MANOVA revealed that there was a statistically significant difference for the set of dependent variables among the groups, approximate $F_{(6, 220)} = 7.023$, $p < .0005$, using Wilk’s criterion. Eta squared was .161, indicating a relatively weak association between the independent variable and the set of dependent variables.

Univariate follow-up tests of between-subjects effects revealed that the difference among the groups was found in the improper fraction-mixed number subtest ($F_{(2, 112)} = 9.782$, $p < .0005$). The alpha level for the univariate tests was corrected according to Bonferroni’s correction, yielding an adjusted $\alpha$ of .017. In addition, Tukey HSD was used to analyze differences between groups on specific subtests. There were significant differences in the mixed number-improper fraction test between the CRA group and the comparison group, $p < .005$, favoring the CRA group, and between the RA group and the comparison group, $p = .016$, favoring the RA group.
Table 6

Group mean and standard deviation scores for computation.

<table>
<thead>
<tr>
<th>Measure</th>
<th>CRA group (n = 26)</th>
<th>RA group (n = 24)</th>
<th>Traditional instruction (n = 65)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
</tr>
<tr>
<td>Q3</td>
<td>77.31</td>
<td>20.63</td>
<td>55.33</td>
</tr>
<tr>
<td>Q6</td>
<td>79.46</td>
<td>21.54</td>
<td>70.54</td>
</tr>
<tr>
<td>Improper fractions</td>
<td>70.96</td>
<td>18.69</td>
<td>63.42</td>
</tr>
</tbody>
</table>

Because this study was specifically designed to determine differences between the two treatment groups, a MANCOVA was used to test treatment effects. Pretest scores were used as a covariate (see Table 9). There was a statistically significant difference between the two treatment groups on the set of dependent variables, approximate $F_{(3, 43)} = 5.439$, $p = .003$, using Wilk’s criterion. Eta squared was .275. Univariate follow-up tests revealed a statistically significant difference only in subtest Q3, $F_{(1, 45)} = 16.128$, $p < .0005$, favoring the CRA group.

H2. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on solving word problems with embedded equivalent fractions.

A one-way ANOVA was used to analyze data obtained from the word problem posttest. This analysis revealed a statistically significant difference among groups, $F_{(2, 112)} = 3.583$, $p < .05$. Using the Tukey method, a significant difference was found between the CRA group and the comparison group favoring the CRA group, $p < .05$. There was no
significant difference between the two treatment groups. Table 7 lists the means and standard deviation scores for each group.

Table 7

<table>
<thead>
<tr>
<th>Measure</th>
<th>CRA group (n = 26)</th>
<th>RA group (n = 24)</th>
<th>Traditional instruction (n = 65)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
</tr>
<tr>
<td>Word problems</td>
<td>69.12</td>
<td>30.68</td>
<td>63.42</td>
</tr>
</tbody>
</table>

H3. There will be no statistically significant differences among the CRA, RA, and traditional instruction groups on attitudes toward mathematics.

A one-way ANOVA was used to analyze data obtained from the attitude posttest. This analysis revealed a statistically significant difference among groups, $F_{(2,112)} = 19.435$, $p < .0005$. Post-hoc Tukey HSD indicated significant differences between the CRA group and traditional group favoring the CRA group, $p < .005$, and between the RA group and traditional group favoring the RA group, $p < .005$. There was no significant difference between the CRA and RA groups. Table 8 lists group means and standard deviation scores.
Table 8

<table>
<thead>
<tr>
<th>Measure</th>
<th>CRA group (n = 26)</th>
<th>RA group (n = 24)</th>
<th>Traditional instruction (n = 65)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attitude</td>
<td>Mean 23.42, SD 2.3</td>
<td>Mean 24.21, SD 3.28</td>
<td>Mean 20.08, SD 3.47</td>
</tr>
</tbody>
</table>

H4. There will be no statistically significant differences between pretest and posttest measures for the CRA and RA groups.

A paired samples t-test was used to test the differences between the pretest and posttest measures for the CRA and RA groups. Table 9 lists means, standard deviations, and correlations for the pretest-posttest measures for each group.

Students in both treatment groups improved in achievement. The paired samples t-test revealed statistically significant differences between all pairs of achievement subtests for both groups, p < .0005. There were also statistically significant differences in attitude between pretest and posttest measures. For the CRA group, p = .010 on attitude, and for the RA group, p = .032 on attitude.

Summary

Statistical analyses of data led to rejection of all four null hypotheses. The CRA group had significantly higher scores than did the contrast group on conceptual understanding of fraction equivalency and on solving word problems with embedded fraction equivalencies. The CRA group also had significantly higher scores than did the RA group on subtest Q3, partitioning sets. Both treatment groups had significantly higher scores on the improper fraction-mixed numbers subtest and on the attitude measure than
did the contrast group, although there was no difference between the two treatment
groups on these tests. Finally, both treatment groups made significant gains from pretest
scores to posttest scores.

Table 9

Pretest-Posttest comparison for treatment groups

<table>
<thead>
<tr>
<th>Measure</th>
<th>CRA (n = 26)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>RA (n = 24)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pretest</td>
<td>Posttest</td>
<td>Corr.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>SD</td>
<td></td>
<td>Mean</td>
<td>SD</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Attitude</td>
<td>20.81</td>
<td>5.15</td>
<td>23.42</td>
<td>2.30</td>
<td>.365</td>
<td>21.92</td>
<td>4.14</td>
<td>24.21</td>
<td>3.28</td>
</tr>
<tr>
<td>Q3</td>
<td>30.54</td>
<td>23.83</td>
<td>77.31</td>
<td>20.63</td>
<td>-.085</td>
<td>28.42</td>
<td>17.20</td>
<td>55.33</td>
<td>21.95</td>
</tr>
<tr>
<td>Q6</td>
<td>16.08</td>
<td>22.46</td>
<td>79.46</td>
<td>21.54</td>
<td>.246</td>
<td>8.21</td>
<td>9.56</td>
<td>70.54</td>
<td>32.59</td>
</tr>
<tr>
<td>Improper fractions</td>
<td>15.31</td>
<td>22.44</td>
<td>70.96</td>
<td>18.69</td>
<td>.103</td>
<td>18.79</td>
<td>19.05</td>
<td>63.42</td>
<td>28.14</td>
</tr>
<tr>
<td>Word problems</td>
<td>6.69</td>
<td>19.37</td>
<td>69.12</td>
<td>30.68</td>
<td>.140</td>
<td>3.46</td>
<td>9.52</td>
<td>63.42</td>
<td>37.30</td>
</tr>
</tbody>
</table>

Note. Significant correlations are starred (p < .05).
CHAPTER 5

DISCUSSION

The purpose of this study was to investigate the effects of the CRA and RA instructional sequences on the learning of equivalent fraction concepts and procedures by students with mild to moderate disabilities. This chapter is organized into three parts. First, a brief review of the study will be presented. Second, four research hypotheses were formulated and tested, and each hypothesis will be addressed in this chapter. Finally, a summary is provided giving limitations of the study, classroom implications, and suggestions for further research.

Review of the Purpose and Objectives

Research has shown that most students have difficulty with learning fraction concepts and procedures. Some researchers have discovered that students did not connect their informal knowledge of fractions to the procedures taught in the classroom, or that students applied whole-number concepts to fraction problems. These difficulties suggested that students failed to develop conceptual understanding before applying algorithms. Other researchers have focused on the instructional methods used to teach mathematics and have commented that instructional design is often the source of students’ problems with fractions. Few studies exist that investigated math instruction
for students with disabilities, and the interventions studied often dealt with procedures and algorithms rather than concepts. Three categories of empirical study were reviewed in Chapter 2: general education studies, special education studies, and CRA studies.

The general education studies emphasized qualitative research related to concept development but did not provide quantitative data comparing instructional methods. These studies revealed that students have a poor understanding of fraction concepts. Although the investigators used concrete manipulative devices and representational drawings as well as explicit strategy instruction, the instructional sequences were not well defined. Further, because of the case-study methods employed, teachers and future researchers may have difficulty replicating these studies in classroom situations. These studies did not isolate critical variables or compare instructional methods.

Only two special education studies were found that addressed teaching fractions. Both studies compared videodisc instruction to teacher-directed instruction. The investigators concluded that the delivery method was less important than instructional design variables such as strategy instruction, ample practice, and separation of easily confused concepts or terminology. The researchers in these studies did not examine the effectiveness of the CRA model in teaching fractions.

The CRA studies revealed that the CRA sequence can be used successfully to teach a variety of math concepts and skills. The CRA interventions were used to promote conceptual understanding and to provide students with a method for solving problems independently. These studies used carefully scripted instructional formats and emphasized a high degree of proficiency before criterion was met. These studies also included the successful instructional design components discussed in the previous
sections. However, while the CRA studies compared instructional methods, they did not specifically address fraction instruction.

The present study was intended to address some of the areas of concern noted above. Specifically, this study targeted math instruction for adolescents with disabilities. The investigator employed instructional design principles that were empirically validated for students with disabilities, and the instruction concentrated on the development of conceptual understanding before introducing mathematical procedures. Data from students with mild to moderate disabilities were compared to data from eighth-grade students without disabilities who had received traditional math instruction. Further, students’ attitudes toward math and fractions were compared, both between groups and within groups.

Fifty students identified with mild to moderate disabilities and enrolled in grades 6, 7, and 8 formed the two treatment groups. These students received all of their math instruction in a math resource room and were taught by teachers licensed in the area of mild to moderate disabilities. Twenty-six students received the CRA instructional sequence while twenty-four students received the RA instructional sequence. Pretest measures indicated no significant differences between the two treatment groups for age, sex, disability, grade, or IQ. Sixty-five eighth-grade students enrolled in general education math classes formed the contrast group. These students received traditional instruction according to the curriculum guide designed by the school district for rational number instruction. The information obtained from these students gives an estimate of what a typical student should understand about fractions by the end of the eighth-grade year.
Pretest-posttest measures consisted of a battery of subtests designed to evaluate students' acquisition of concepts and procedures for computing fraction equivalencies. The same instruments were used for the pretest and the posttest. The subjects were tested by their classroom teachers in regular 50-minute class periods. Each of the treatment groups took a pretest before beginning the treatment and a posttest upon completion of the treatment. The contrast group took only a posttest.

Review of the Hypotheses

Hypothesis 1

This hypothesis dealt with the effects of treatment upon students' computation of fraction equivalencies. The null hypothesis stated that there would be no differences among the three groups. An analysis of data led to rejection of the null hypothesis. Students in the two treatment groups performed significantly better than did students in the contrast group on improper fraction-mixed number conversion when this task was presented graphically. Interestingly, there were no significant differences among the groups at the abstract level (subtest Q6). Students in the contrast group were able to compute improper fraction-mixed number conversions when they were presented numerically but not when they were presented graphically.

An examination of students' papers revealed that many students in the contrast group did not connect the denominator of a fraction to the number of parts of each whole unit. In one test item, students were asked to shade 6/3 rectangles. The item contained two rectangles divided into 6 equal parts each. In a typical incorrect response, one whole rectangle was shaded, or 6/6. Students appeared to understand that the numerator should indicate the number of parts shaded, but they failed to connect the denominator, 3, with
the number of parts each whole should contain. And, even though students knew how to convert improper fractions to mixed numbers on subtest Q6, they did not apply the algorithm to derive the answer $6/3 = 2$, so 2 whole rectangles should be shaded, regardless of the number of partitions in each rectangle. This type of error may be due to the instructional method used in the traditional curriculum. Although students were exposed to representative diagrams in their basal textbook, they were not taught specifically how to draw their own representations as an aid to analyzing problems. Moreover, when fraction concepts were taught, students were given illustrations of proper fractions, and few of the practice problems involved improper fractions or mixed numbers.

When only the two treatment groups were considered, post hoc tests revealed a significant difference in subtest Q3. This finding led the writer to conclude that the CRA group demonstrated better conceptual understanding of fraction equivalency than did the RA group. In addition, the mean scores for the CRA group on all three subtests were higher than were those of the RA group. This finding is interesting because the only difference between the two treatment groups was the use of concrete manipulative devices for the initial three lessons in the CRA group.

**Hypothesis 2**

This hypothesis dealt with the effects of treatment upon students’ solving of word problems with embedded equivalent fractions. The null hypothesis stated that there would be no differences among the three groups. An analysis of data led to rejection of the null hypothesis. The CRA group had significantly higher scores on solving word problems than did the contrast group, although both treatment groups had higher mean
scores than did the contrast group. This finding is interesting because the problems presented were identical to those presented in subtest Q6, and there were no significant differences among groups for subtest Q6. Whereas the contrast group had the highest mean scores for subtest Q6, they had the lowest mean scores for the word problem subtest. An examination of individual student papers showed that some students in the treatment groups solved the word problems using graphic representations, while students in the contrast group appeared to rely on application of an algorithm. The use of the graphic representations may have enabled students who had forgotten the algorithm to solve the problem by reasoning it out with a drawing. As noted in the discussion above, treatment group students were explicitly taught how to draw representational drawings while contrast group students' instruction focused on application of algorithms.

**Hypothesis 3**

This hypothesis dealt with the effects of treatment upon students' attitudes toward math. The null hypothesis stated that there would be no differences among the three groups. An analysis of data led to rejection of the null hypothesis. Post-hoc analyses revealed that there were significant differences between each of the treatment groups and the contrast group. There were no significant differences in attitude between the two treatment groups. In general, both treatment groups had moderately favorable attitudes toward math while the contrast group had a neutral attitude toward math.

**Hypothesis 4**

This hypothesis dealt with the amount of improvement in students' scores after treatment. The null hypothesis stated that there would be no differences between pretest and posttest measures for the treatment groups. An analysis of data led to a rejection of
the null hypothesis. Students in both groups improved in achievement and in attitude after the 10-lesson intervention, although students in the CRA group had higher achievement subtest scores than did students in the RA group on all achievement measures. An examination of the pretests revealed that most students could recognize and name fractional parts of geometric figures, but they had no clear understanding of equivalencies, abstract problems, or word problems. In many cases, these problems were not even attempted on the pretests. In comparison, after instruction students seemed able to decide on appropriate methods of solving problems, and they often drew graphic representations. All of the students attempted all of the posttest questions.

Discussion and Implications

The analysis of the data relative to the principle objectives of the study indicated that students in the treatment groups scored significantly higher than did students in the contrast group on items demonstrating conceptual knowledge, had higher scores on the attitude measure, and overall improved their understanding of fraction equivalency from pretest to posttest. Students in the treatment groups did as well as students in the contrast group on abstract problems. On word problems containing embedded fraction equivalencies, students in the CRA group had significantly higher scores than did contrast group students. On all achievement measures, students in the CRA treatment group had higher scores than did students in the RA treatment group, although the differences were not statistically significant.

The data that were analyzed allowed the writer to conclude that students who worked with manipulative devices had a better understanding of fraction equivalency than those who did not. Subtest Q3 measured students' understanding of ratio and
proportion by asking them questions such as, "Circle 5/6 of the 18 squares." In the CRA group, this task was first introduced with small white beans in Lesson 2. Students were taught to repartition the group of 18 beans into 6 subgroups, representing the denominator of the new fraction. Then, 5 of the groups were separated, yielding an answer of 15/18. Students in the RA group performed the same tasks, but they used drawings of small squares arranged in arrays instead of beans. They were taught to lightly draw the subgroups, then circle darkly the number of groups called for in the numerator of the fraction. This was difficult for some students with minor motor problems. In later lessons, students in both treatment groups used these drawings, or arrays, to solve problems converting improper fractions to mixed numbers, converting mixed numbers to improper fractions, and reducing fractions to simplest terms. Students in the treatment groups used this strategy for solving the problems in subtest Q3 and in the improper fraction subtest, while students in the contrast group did not. Even though contrast group students knew how to solve the same problems when they were presented abstractly, they appeared not to transfer that knowledge to the graphic representations. These findings are consistent with those of Armstrong and Larson (1995), Behr, Wachsmuth, Post, and Lesh (1984), Confrey and Scarano (1995), Mack (1990), and Morris (1995).

Several conclusions may be drawn concerning the higher attitude scores and achievement scores of students in the treatment groups. First, students appeared to enjoy the variety of activities presented in the two treatment groups. The daily lessons used a variety of concrete manipulative devices and representational drawings. In lessons 5 and 6, students were encouraged to use the representations that they felt helped them best.
Second, several instructional variables may have contributed to the success of the two treatment groups. The lessons were planned sequentially, so the students did not feel that the tasks were too difficult. The guided practice segments of the lessons provided ample time for students to practice the new skills being taught and to receive immediate feedback from the teacher. Thus, students were not allowed to practice error patterns. In contrast, students in the traditional curriculum were presented with daily lessons, but most of the practice was independent seatwork or homework and did not result in immediate feedback. Thus, it is possible that the feedback received was the determining factor rather than instructional method. Further research would help clarify this point.

In addition, students in both treatment groups were provided with typewritten notes from the teacher that they could use in case they became confused or forgot the steps for solving a problem. The traditional curriculum provided for students to take their own daily notes. However, some students do not take accurate or complete notes and thus have little to guide them when they become confused. Further research could help clarify the role of this instructional variable.

Finally, students in the treatment groups were evaluated daily on their independent practice. This information was used to reteach skills as necessary, and students were not allowed to move to the next lesson until they reached a criterion of 80%. Students in the traditional curriculum were also evaluated on daily practice and homework, but they moved on to the next lesson regardless of their understanding of the previous lesson. This may have resulted in more frustration for students in the contrast group. Therefore, the above instructional design variables may have contributed to student achievement, and they also appear to have increased student satisfaction. These
findings are in accordance with those of Kelly, Gersten, and Carnine (1990) relative to achievement. None of the studies reviewed considered student attitude, however.

Discussion of the Problems and Limitations.

There were problems and limitations encountered in the process of implementing this study that should be considered when interpreting these data. This study was conducted at the end of the school year. Due to a shortage of time, the study was limited to 10 lessons and no maintenance probes were performed. Thus, follow-up data on generalization and maintenance could not be gathered. Most of the students in the treatment groups were identified with specific learning disabilities, although a few students had attention deficit disorder, emotional disabilities, or mild mental retardation. Therefore, caution should be used in generalizing the data to students with disabilities other than learning disabilities. The results of this study were obtained with group instruction in a resource room setting. Students spanned three grade levels, and placement was determined by deficits in math rather than by diagnostic category or grade level. This study did not address instruction of students with disabilities in general education settings when placement was determined by grade level regardless of math ability.

Discussion of the Practical Implications.

Both the CRA and RA instructional sequences can be easily implemented in the classroom. The materials used can be obtained at low cost or can be made by the teacher or students. In addition, several validated instructional variables were included in the lessons for the two treatment groups. These included careful sequencing of skills, selection of appropriate examples, immediate feedback, and mastery learning.
Developing the lessons was time consuming. First, a task analysis of the required skills was needed so that the lessons could be properly sequenced. This information is available in the scope and sequence sections of most math textbooks. For this purpose, this investigator relied on information found in Designing Effective Mathematics Instruction: A Direct Instruction Approach (Stein, Silbert, & Carnine, 1997). Second, each lesson was developed with an appropriate range of examples, adequate practice, and clear explanations (Baroody & Hume, 1991; Brigham, Wilson, Jones, & Moisio, 1996; Maccini & Hughes, 1997). Finally, materials appropriate to each day’s objectives were assembled so that the teacher could practice the skills being asked of the students. This step was important because the teachers had been trained as special educators and did not have advanced training in math. Therefore, the teachers had to develop their own conceptual understanding of the daily lessons in order to effectively present them to students.

Ongoing assessment was an important part of this instructional sequence because it allowed the teacher to make changes as needed in direct response to the needs of the learners. In a few cases, students failed to reach the 80% criterion in a single lesson, and this required the teacher to make time to reteach the needed skills and provide further explanation and practice. However, since formative assessment is an integral part of any effective teaching method, this did not result in any more time than normal.

Suggestions for Further Research.

The results of this study indicated that students with disabilities who participated in the CRA and RA instructional methods had a better conceptual understanding of fraction equivalency than did students without disabilities enrolled in the traditional
Students with disabilities did equally well as students without disabilities on word problems and abstract problems. Moreover, students in the two treatment groups improved from pretest (mean overall score 15.94) to posttest (mean overall score of 68.97) over the ten lessons.

Future research is needed to determine the stability of these results over time. The skills that were taught in this intervention are used in operations with fractions and in computing ratio and proportion problems. Further research is needed to determine whether students would be able to draw representations to solve problems even if they forgot the abstract algorithms.

This study involved students with disabilities who received their math instruction in resource rooms containing students of differing ages and grade levels. Future research would be useful in determining the applicability of the CRA instructional sequence to students with disabilities in general education classrooms alongside their same-grade peers. Such a study could also address the applicability of this instructional method to students without disabilities.

Finally, this study yielded information relative to the acquisition of concepts of fraction equivalency. Future research would be useful in exploring how the CRA method incorporating validated instructional design principles might be used in teaching operations with fractions or algebra concepts. Such a study might compare students who participated in CRA instruction for fraction equivalency to students who had traditional instruction in fraction equivalency. Both groups would be given CRA instruction for the new concepts. This could help determine the role of conceptual understanding in developing higher level mathematics skills.
APPENDIX A

PRETEST-POSTTEST

Name: ________________________________

DIRECTIONS: Read each problem carefully. Solve the problem using rules about fractions or by drawing in the fractions.

Example: My brother brought 15 cupcakes to school for his birthday party. At the end of the day, 2/5 were left. How many cupcakes were left?

\[
\frac{2}{5} = \frac{6}{15}
\]

1. Joan had 15 hair ribbons. 1/3 of the ribbons were red. How many of the ribbons were red? Write your answer as a fraction.

2. In our math class, 3/5 of the students are boys. There are 15 students altogether. How many of the students are boys? Write your answer as a fraction.

3. Ricardo bought a pizza with 12 slices. He ate 1/3 of the pizza. How many slices did Ricardo eat? Write your answer as a fraction.

4. The cafeteria served french fries to 6 of the 8 students in line A. What fraction of the students in line A ate french fries? Write your answer in simplest terms.

5. I have 10 coins in my pocket, and 4 of them are dimes. What fraction of my coins are dimes? Write your answer in simplest terms.
6. A baseball team has 9 players. 6 players are girls. What fraction of the players are girls? Write your answer in simplest terms.

7. We ordered some pizza for our party. Each pizza was cut into 5 slices. How much pizza will 8 people eat if each person gets one slice of pizza? Write your answer as a mixed number.

8. A carton of eggs holds 12 eggs. My mom has 15 eggs left. How many cartons is this? Write your answer as a mixed number.

9. There are 3 feet in a yard. My room is 10 feet long. How many yards long is my room? Write the answer as a mixed number.

10. Two and one-half apples is the same as how many half apples? Solve the problem using an improper fraction.

11. If one quarter is the same as 1/4 of a dollar, how many quarters are in 3 1/4 dollars? Solve the problem using an improper fraction.

12. My crackers are divided into 4 pieces each. If I have 1 3/4 crackers, how many pieces do I have? Solve the problem using an improper fraction.
ATTITUDE QUESTIONNAIRE

Name: ________________________________

DIRECTIONS: Read each question carefully. Circle the number that you think best matches your own feelings.

😊 1 = Don’t Agree
😊 2 = OK
😊 3 = Agree

😊 0 😊 0 😊

1 2 3 Math classes are interesting.
1 2 3 It is easy to get tired of math.
1 2 3 It is fun working on math problems.
1 2 3 Fractions are easy.
1 2 3 I like the math class activities.
1 2 3 Learning fractions is a waste of time.
1 2 3 Math class is dull and boring.
1 2 3 I want to study math in high school.
1 2 3 It is hard to understand fractions.
1 2 3 Knowing math is not helpful when you get out of school.
Directions: Shade in the fraction in each box as the directions tell you.

Look at the EXAMPLE.

**EXAMPLE:** Shade in $\frac{2}{3}$

<table>
<thead>
<tr>
<th>1. Shade $\frac{2}{6}$</th>
<th>2. Shade $\frac{1}{2}$</th>
<th>3. Shade $\frac{3}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Shaded Boxes" /></td>
<td><img src="image2" alt="Shaded Boxes" /></td>
<td><img src="image3" alt="Shaded Boxes" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4. Shade $\frac{10}{3}$</th>
<th>5. Shade $\frac{6}{3}$</th>
<th>6. Shade $\frac{9}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image4" alt="Shaded Boxes" /></td>
<td><img src="image5" alt="Shaded Boxes" /></td>
<td><img src="image6" alt="Shaded Boxes" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>7. What fraction of the shape is shaded?</th>
<th>8. What fraction of the shape is shaded?</th>
<th>9. What fraction of the shape is shaded?</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image7" alt="Shaded Boxes" /></td>
<td><img src="image8" alt="Shaded Boxes" /></td>
<td><img src="image9" alt="Shaded Boxes" /></td>
</tr>
</tbody>
</table>

Name: ________________________________
Goal: Introduce Fraction Equivalence Using Concrete Objects

Lesson 1

**Goals:**

- To visualize and comprehend the relative values of fractions by using physical representations.
- To recognize relative sizes of fractions and equivalent fractions.
- To use the terms numerator and denominator in describing fractions.

**Materials:**

- Fraction circles
- Overhead fraction circles
- Learning Sheet 1
- Transparency of Learning Sheet 1
- Overhead projector and screen

**Advance Organizer:**

1. Review and tell the students what they will be doing and why.
Sample dialogue: Yesterday we discussed our pretest scores, and everyone agreed to learn more about fractions. Today we are going to learn how fractions are shown using fraction circles. We will be making models of such fractions as \( \frac{1}{2} \), \( \frac{4}{12} \), and \( \frac{1}{3} \). We will also learn that fractions for the same amount are often shown by different numbers.

Describe and Model:

1. Distribute sets of fraction circles to each student along with one copy of Learning Sheet 1.

2. Demonstrate how to compute Problem 1. You will also introduce the term “denominator” to indicate the total number of pieces in a given fraction circle and the term “numerator” to indicate the number of pieces being used in a given problem. You will introduce the term “equivalent fractions” to describe two fractions that show the same amounts.

Sample Dialogue: I’m going to show you how to work these problems using the fraction circles. First, I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. The problem says, “One-half equals how many fourths?” The 2 in one-half tells us how many pieces there are in the whole fraction circle. The 2 is called the denominator of the fraction. The 1 in the fraction one-half tells us that we need to use one of the two pieces to show the fraction one-half. So, I’m going to find the fraction circle that has 2 parts and place one of the parts on my desk. Now, I need to make a fraction that is the same size as one-half, but this time the fraction has a different denominator. The
denominator is 4. So, I find the fraction circle that has 4 parts. How many parts of this fraction circle do I need to cover the same amount as the one-half circle?

That's right, I need 2 pieces. So, \( \frac{1}{2} \) equals 2/4. We can also say that \( \frac{1}{2} \) is equivalent to 2/4. Two fractions are equivalent when they describe the same amount. Let's fill in the answer on the blank.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to select the correct fraction circle by identifying the denominator of the given fraction. Then, ask students to count the correct number of pieces to demonstrate the numerator of the fraction. In problem 2, explain that the 4 pieces could be covered by 2 of the pieces from the sixths set or by 1 of the pieces from the thirds set. Either answer is correct. Point out that fractions often have different names to show the same amount. So, 4/12, 2/6, and 1/3 are all equivalent fractions. Repeat the demonstration with problem 3.

Guided Practice:

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

Sample dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the first fraction we need to show? Yes, it is two-fifths. Get out the fraction circle that is divided into fifths. How many pieces do we need? Yes, we need 2 pieces. Which pieces from the other sets of fraction circles will exactly cover these 2 pieces? Great, 4 of the pieces from the tenths set will cover those 2 pieces. So, 2/5 is equivalent to 4/10.
2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

**Independent Practice:**

1. Instruct students to solve problems 6 - 12 independently.

   **Sample dialogue:** Now do problems 6 through 12 on your learning sheet.

   Remember to use the fraction circles to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

**Problem-Solving Practice:**


   **Sample dialogue:** Look at problem 13 on your Learning Sheet. Problem 13 is a word problem, but we will be using fraction circles just as we have been doing. We have been using fraction circles to help us solve our problems, so we will write the words “fraction circle” on the blanks. Now we can read the problem: “Four-twelfths of a fraction circle is the same as blank thirds of a fraction circle.” Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

**Feedback:**

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.
2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 2

Begin Generalization of Fraction Equivalence Using Concrete Objects

Goals:

• To visualize and comprehend the relative values of fractions by using physical representations.
• To express fractions related to quantity.

Materials:

• At least 30 small objects such as paper clips, M & M’s, or game chips
• Learning Sheet 2
• Transparency of Learning Sheet 2
• Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discovered how to represent numbers such as 2/3, 3/4, and 1/6 using fraction circles. We also learned that fractions sometimes can be named in more than one way. For example, we learned that 2/5 is the same fraction as 4/10. We learned the terms “numerator”, “denominator”, and “equivalent fractions.” Today, we will learn how to express parts of a group as fractions, such as 5/8 of 8 M & M’s, or 7/10 of 30 paper clips. Instead of using fraction circles, today we will use small objects to show fractions.
Describe and Model:

1. Distribute small objects to each student along with one copy of Learning Sheet 2.

2. Demonstrate how to compute Problem 1 and demonstrate how to group the objects to represent fractions. Divide the objects into two groups. Explain that the objects are divided into two equal parts, so each group is one-half of the total number of objects. Reinforce the idea that fractional parts must be the same size. There is no such thing as “the bigger half.”

Sample Dialogue: I’m going to show you how to work these problems using the objects in front of you. First, I want you to watch me as I work the first problem on your sheet. Please do not use your objects yet. You will have time to use them in a moment.

Look at problem 1 on your sheet. The problem says, “Circle one-third of the 3 squares.” The denominator of the fraction is 3, so that means we have to have three equal parts. Since we only have three objects, there will be only one object in each group. The numerator of the fraction is 1, so that means we have to circle one of the three groups. So, one-third of three squares is one square. Let’s fill in the answer on the learning sheet. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to group the objects by identifying the denominator of the given fraction. Then, ask students to count the correct
number of objects to demonstrate the numerator of the fraction. For problem 2, separate the objects into four groups of two each. Explain that the objects are now divided into fourths. The numerator asks us to circle three fourths, so we need to count out three groups, or six objects. In problem 3, allow students to explain the steps as they work through the problem.

**Guided Practice:**

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

   **Sample dialogue:** Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the fraction we need to show? Yes, it is five-sixths. How many groups do we need? Yes, we need 6 groups. Now, let’s count to see how many objects are in each group. So, how many objects do we have in all? Yes, five-sixths equals fifteen-eighteenths.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

**Independent Practice:**

1. Instruct students to solve problems 6 - 12 independently.

   **Sample dialogue:** Now do problems 6 through 12 on your learning sheet. Remember to use the small objects to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.
Problem-Solving Practice:


Sample dialogue: Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. We have been using small objects to help us solve our problems, so we will use the objects to help us solve these problems, too.

Let's read the problem to see how many objects Jose had. Now, how many groups did he have? How many objects were in each group? Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student's paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 3

Continue Generalization of Fraction Equivalence Using Concrete Objects

Goals:

- To visualize and comprehend the relative values of fractions by using physical representations.
- To express mixed numbers as improper fractions.

Materials:

- Precut square pieces of paper
- Learning Sheet 3
- Transparency for Learning Sheet 3
- Overhead projector and screen

Advance Organizer:

- Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discovered how to represent numbers such as 2/6, 3/8, and 9/10 using small objects. We also reviewed that fractions can be named in more than one way. For example, we showed that 1/4 is the same as 3/12.

Today, we will learn to express numbers such as 2 1/2, 3 3/8, and 1 7/10 as fractions. Instead of using fraction circles or small objects, today we will fold squares of paper to show fractions.

Describe and Model:

1. Distribute precut squares of paper to each student along with one copy of Learning Sheet 3.
2. Demonstrate how to compute Problem 1 and demonstrate how to fold the square of paper to represent fractions. Fold the paper in half. Explain that the paper is divided into two equal parts, so each part is one-half of the paper. Reinforce the idea that fractional parts must be the same size.

Sample Dialogue: I’m going to show you how to work these problems using the squares of paper. First, I want you to watch me as I work the first problem. Do not fold your paper yet; you will have time to do that after I am finished explaining Problem 1.

Look at problem 1 on your sheet. The problem says, “Two and one-half equals blank.” The number 2 ½ is a mixed number. That means that the 2 represents two whole units and the ½ represents one-half of a whole unit. I need to have two whole squares of paper plus another square of paper that is divided into halves. I’ll fold each square in half. How many halves will be equal to two and one-half squares of paper? That’s right, five halves will be the same as two and one-half. So, 2 ½ is equivalent to 5/2. Let’s fill in the answer on the learning sheet. You should notice that the denominator in our fractions did not change. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to fold the paper squares by identifying the denominator of the given fraction. Then, ask students to count the correct number of pieces to demonstrate the numerator of the fraction. For problem 2, fold the paper in half crosswise, then lengthwise. Explain that
the paper is now folded into fourths. Now, fold the paper in half again. Now, we have folded the paper into eighths. We need to shade in two-eighths. Demonstrate that the two-eighths we have shaded in also represents one-fourth of the paper square. In problem 3, show students that after the fraction 2/3 is represented, we can fold the paper in half crosswise to demonstrate sixths.

Guided Practice:

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

   Sample dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the mixed number we need to show? Yes, it is three and three-eighths. How many whole squares do we need? Yes, we need 3 squares. We also need another square to show the fraction three-eighths. Fold your squares in the same way as we folded the square for problem 2. Now, let’s count to see how many eighths are in three and three-eighths. Yes, there are twenty-seven eighths in three and three-eighths.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.
Sample dialogue: Now do problems 6 through 12 on your learning sheet. Remember to use the paper squares to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

Problem-Solving Practice:


Sample dialogue: Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. We have been using paper squares to help us solve our problems, so we can use the paper squares to help us solve these problems, too. Let’s read the problem. The paper will need to be folded into how many parts to show the denominator of the fraction? Yes, four parts. How many crackers do I have? Yes, two and one-fourth crackers. So, two and one-fourths is the same as how many fourths? Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 1

Introduce Fraction Equivalence Using Representational Drawings

Goals:

- To visualize and comprehend the relative values of fractions by using graphic representations.
- To recognize relative sizes of fractions and equivalent fractions.
- To use the terms numerator and denominator in describing fractions.

Materials:

- Learning Sheet 1
- Transparency of Learning Sheet 1
- Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discussed our pretest scores, and everyone agreed to learn more about fractions. Today we are going to learn how fractions are shown using fraction circle drawings. We will be shading in drawings of such fractions as \( \frac{1}{4} \), \( \frac{4}{12} \), and \( \frac{1}{3} \). We will also learn that fractions for the same amount are often shown by different numbers.

Describe and Model:

1. Distribute one copy of Learning Sheet 1 to each student.

2. Demonstrate how to compute Problem 1. You will also introduce the term “denominator” to indicate the total number of pieces in a given fraction circle and the term “numerator” to indicate the number of pieces being
shaded in a given problem. You will introduce the term “equivalent fractions” to describe two fractions that show the same amounts.

**Sample Dialogue:** I’m going to show you how to work these problems using the fraction circle drawings. First, I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. The problem says, “One-half equals how many fourths?” The 2 in one-half tells us how many pieces there are in the whole fraction circle. The 2 is called the denominator of the fraction. The 1 in the fraction one-half tells us that we need to shade one of the two pieces to show the fraction one-half. So, I’m going to shade in one of the two parts of the fraction circle drawing. Now, I need to make a fraction that is the same size as one-half, but this time the fraction has a different denominator. The denominator is 4. So, I find the fraction circle that is divided into 4 parts. How many parts of this fraction circle do I need to shade to show the same amount as the one-half circle? That’s right, I need to shade in 2 pieces. So, \( \frac{1}{2} \) equals \( \frac{2}{4} \). We can also say that \( \frac{1}{2} \) is equivalent to \( \frac{2}{4} \). Two fractions are equivalent when they describe the same amount. Let’s fill in the answer on the blank.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to select the correct fraction circle by identifying the denominator of the given fraction. Then, ask students to shade the correct number of pieces to indicate the numerator of the fraction. Repeat the demonstration with problem 3. Point out that fractions often have different names to show the same amount. In problem
2, explain that the four-twelfths is the same as two-sixths. This is the same as the amount shaded in problem 3, so two-sixths is also equal to one-third. So, $\frac{4}{12}$, $\frac{2}{6}$, and $\frac{1}{3}$ are equivalent fractions.

**Guided Practice:**

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

   **Sample dialogue:** Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the first fraction we need to show? Yes, it is two-fifths. Find the fraction circle that is divided into fifths. How many pieces do we need to shade in? Yes, we need to shade in 2 pieces. Which pieces from the other sets of fraction circles will show the exact same amount as these 2 pieces? Great, 4 of the pieces of the tenths circle will be the same as 2 pieces of the fifths circle. So, $\frac{2}{5}$ is equivalent to $\frac{4}{10}$.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

**Independent Practice:**

1. Instruct students to solve problems 6 - 12 independently.

   **Sample dialogue:** Now do problems 6 through 12 on your learning sheet. Remember to use the fraction circle drawings to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

**Problem-Solving Practice:**

Sample dialogue: Look at problem 13 on your Learning Sheet. Problem 13 is a word problem, but we will be using fraction circle drawings just as we have been doing. We have been using fraction circles to help us solve our problems, so we will write the words “fraction circle” on the blanks. Now we can read the problem: “Four-twelfths of a fraction circle is the same as blank thirds of a fraction circle.” Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 2

Begin Generalization of Fraction Equivalence Using Representational Drawings

Goals:

• To visualize and comprehend the relative values of fractions by using graphic representations.
• To express fractions related to quantity.

Materials:

• Learning Sheet 2
• Transparency of Learning Sheet 2
• Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discovered how to represent fractions such as 2/3, 3/4, and 1/6 using drawings of fraction circles. We also learned that fractions sometimes can be named in more than one way. For example, we learned that 2/5 is the same fraction as 4/10. We learned the terms “numerator”, “denominator”, and “equivalent fractions.” Today, we will learn how to express parts of a group as fractions, such as 5/8 of 8 squares or 7/10 of 30 squares. Instead of shading in fraction circles, today we will circle groups of squares to show fractions.

Describe and Model:

1. Distribute one copy of Learning Sheet 2 to each student.

2. Demonstrate how to compute Problem 1 and demonstrate how to group the squares to represent fractions. First, draw eight small squares on the
Divide the squares into two groups. Explain that the squares are divided into two equal parts, so each group is one-half of the total number of squares. Reinforce the idea that fractional parts must be the same size. There is no such thing as “the bigger half.”

Sample Dialogue: Now, I’m going to show you how to work these problems using the squares on your paper. First, I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. The problem says, “Circle one-third of the 3 squares.” The denominator of the fraction is 3, so that means we have to have three equal parts. Since we only have three squares, there will be only one square in each group. The numerator of the fraction is 1, so that means we have to circle one of the three groups. So, one-third of three squares is one square. Let’s fill in the answer on the learning sheet. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to group the squares by identifying the denominator of the given fraction. Then, ask students to count the correct number of squares to demonstrate the numerator of the fraction. For problem 2, separate the squares into four groups of two each. Explain that the squares are now divided into fourths, and lightly draw a circle around each group. The numerator asks us to circle three-fourths, so we need to
circle three groups, or six squares. In problem 3, allow students to explain
the steps as they work through the problem.

Guided Practice:

1. Guide the students through problem 4. Do not demonstrate the procedure
   unless students are having difficulty.

   Sample dialogue: Now you can do problem 4 by yourself at your desk. Look at
   the problem on your Learning Sheet. What is the fraction we need to show? Yes,
   it is five-sixths. How many groups do we need? Yes, we need 6 groups, so let’s
   lightly draw in the six circles. Now, let’s count to see how many squares are in
   each group. Make sure that you draw a darker circle around five groups of six. So,
   how many squares do we have in all? Yes, five-sixths equals fifteen-eighteenths.

2. Guide the students through problem 5, but do not ask for the answer. This
   problem will be used in assessing the student’s comprehension.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.

   Sample dialogue: Now do problems 6 through 12 on your learning sheet.
   Remember to draw the small squares to help you find the answer. When you are
   through, put down your pencils.

2. Circulate through the room and monitor students as they work
   independently.

Problem-Solving Practice:

Sample dialogue: Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. We have been using small squares to help us solve our problems, so we will use the squares to help us solve these problems, too. Let’s read the problem to see how many objects Jose had. Now, how many groups did he have? How many objects were in each group? Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 3

Continue Generalization of Fraction Equivalence Using Representational Drawings

Goals:

• To visualize and comprehend the relative values of fractions by using representational drawings.
• To express mixed numbers as improper fractions.

Materials:

• Learning Sheet 3
• Transparency for Learning Sheet 3
• Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discovered how to represent numbers such as 2/6, 3/8, and 9/10 using small squares. We also reviewed that fractions can be named in more than one way. For example, we showed that 1/4 is the same as 3/12. Today, we will learn to express numbers such as 2 ½, 3 3/8, and 1 7/10 as fractions. Instead of using circles or small squares, today we will use large squares to show fractions.

Describe and Model:

1. Distribute one copy of Learning Sheet 3 to each student.

2. Demonstrate how to compute Problem 1 and demonstrate how to divide the square to represent fractions. Divide a square into two parts by drawing a line down the middle. Explain that the square is divided into
two equal parts, so each part is one-half of the square. Reinforce the idea that fractional parts must be the same size.

**Sample Dialogue:** I’m going to show you how to work these problems using the large squares. First, I want you to watch me as I work the first problem. Do not draw lines on your paper yet; you will have time to do that after I am finished explaining Problem 1.

Look at problem 1 on your sheet. The problem says, “Two and one-half equals blank.” The number 2½ is a **mixed number**. That means that the 2 represents two whole units and the ½ represents one-half of a whole unit. I need to shade in two whole squares plus one-half square. Notice that the squares are divided into pieces that are exactly the same size. We can’t combine pieces that are different sizes. How many pieces have I shaded in? That’s right. I’ve shaded in five pieces. So, two and one-half is equivalent to five halves. Let’s fill in the answer on the blank. You should notice that the denominator in our fractions did not change. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to shade the correct fraction on each square.

**Guided Practice:**

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

**Sample dialogue:** Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the mixed number we need to show?
Yes, it is three and three-eighths. How many whole squares do we need to shade in? Yes, we need to shade in three whole squares. How many pieces do we need to shade in on the last square? Great, 3 pieces should be shaded in. Now let's see how many eighths pieces we have shaded. So, three and three eighths is the same as twenty-seven eighths.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

**Independent Practice:**

1. Instruct students to solve problems 6 - 12 independently.

    **Sample dialogue:** Now do problems 6 through 12 on your learning sheet. Remember to use the large square drawings to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

**Problem-Solving Practice:**


    **Sample dialogue:** Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. We have been using large squares to help us solve our problems, so we can use the squares to help us solve these problems, too.

    Let’s read the problem. Each square will need to be divided into how many parts to show the denominator of the fraction? Yes, four parts. How many crackers do I have? Yes, two and one-fourth crackers. So, two and one-fourth crackers is the same as how many fourths? Fill in the answer on your paper.
2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student's paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 4

Continue Generalization of Fraction Equivalence Using Representational Drawings

Goals:

- To visualize and comprehend the relative values of fractions by using graphic representations.
- To express improper fractions as mixed numbers.

Materials:

- Learning Sheet 4
- Transparency of Learning Sheet 4
- Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

   Sample dialogue: Yesterday we discovered how to express numbers such as 2 1/2, 3 3/8, and 1 7/10 as fractions. We also learned that the term "mixed number" means a number that is made up of a whole number and a fraction. Today, we will learn how to express fractions such as 12/8, 9/3, or 7/2 as mixed numbers. Today we will use rectangles to show fractions.

Describe and Model:

1. Distribute one copy of Learning Sheet 4 to each student.

   2. Demonstrate how to compute Problem 1.

   Sample Dialogue: I'm going to show you how to work these problems using the rectangle drawings on your papers. First, I want you to watch me as I work the first problem on your sheet.
Look at problem 1 on your sheet. The problem says, "Thirty-nine twelfths equals blank." The number 39/12 is an improper fraction. That means that the numerator is larger than the denominator. I will need to use more than one whole rectangle to show this fraction. The first rectangle is already divided into twelve parts. I need to divide the other rectangles in the same way for this problem. Now I need to count thirty-nine twelfths. That means I'll shade in three whole rectangles and three-twelfths of the next rectangle. So, 39/12 is equivalent to the mixed number 3 3/12. Let's fill in the answer on the blank. You should notice that the denominator in our fractions did not change. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

Now, I'll show you a shorter way to solve this problem. Each whole rectangle in this problem has how many parts? Yes, each whole rectangle has twelve parts. We can solve this problem by counting by twelves. Let's put 12 by the first rectangle, 24 by the second rectangle, and 36 by the third rectangle. Now it will be easy to decide that we need only 3 more rectangles to solve the problem.

3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to shade the correct fraction on each rectangle.

Guided Practice:

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.
Sample dialogue: Now you can do problem 4 by yourself at your desk.

Look at the problem on your Learning Sheet. What is the mixed number we need to show? Yes, it is one and two-tenths. How many whole rectangles do we need to shade in? Yes, we need to shade in one whole rectangle. The denominator of the fraction tells us that this rectangle is divided into 10 parts. How many pieces do we need to divide the next rectangle into? Great, we should divide the rectangle into ten pieces also. Now let's see how many pieces we should shade. So, $\frac{12}{10}$ is equivalent to $\frac{12}{10}$.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student's comprehension.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.

Sample dialogue: Now do problems 6 through 12 on your learning sheet.

Remember to use the rectangles to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

Problem-Solving Practice:


Sample dialogue: Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. We have been using rectangles to help us solve our problems, so we can use rectangles to solve these problems, too. Lets read problem 13 to decide how many parts each rectangle needs. A pound of butter has
four cubes, so each rectangle needs to be divided into four parts. There are seven cubes of butter in the refrigerator, so how much butter do we have? Fill in the answer on your paper.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student's paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 5

Practice Generalization of Fraction Equivalence Using Representational Drawings

Goals:

- To visualize and comprehend the relative values of fractions by using graphic representations.
- To express proper, improper, and mixed number fractions as equivalent fractions.
- To introduce the term “simplest terms”

Materials:

- Learning Sheet 5
- Transparency of Learning Sheet 5
- Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we discovered how to express improper fractions such as 12/8, 9/3, and 7/2 as mixed numbers. We also learned that the term “improper fraction” means a fraction for a number greater than one whole whose numerator is greater than the denominator. So far, we have used circles, small squares, rectangles, and large squares to represent fractions. Today, you may choose to draw any shape to represent the fractions. We will review making equivalent fractions for proper, improper, and mixed number fractions. We will also be learning how to write fractions in simplest terms.

Describe and Model:
1. Distribute one copy of Learning Sheet 5 to each student.

2. Demonstrate how to compute Problem 1.

Sample Dialogue: I’m going to show you how to work these problems. There are no fraction drawings on your papers, so we’ll have to decide which shape we want to use. First, I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. The problem says, “One and one-third equals blank.” The number 1 1/3 is a mixed number. That means that the fraction will be more than one whole. I’m going to use rectangles for this problem. I will need to draw two rectangles, one whole plus another rectangle for the fractional part. The denominator of this fraction is 3, so I’ll have to divide my rectangles into three equal parts. Now, I’ll shade in one whole plus one-third of the other rectangle. So, 1 1/3 is equivalent to the improper fraction 4/3. Let’s fill in the answer on the blank. You should notice that the denominator in our fractions did not change. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem.

Do you remember the shorter way to solve this problem. Each whole rectangle in this problem has how many parts? Yes, each whole rectangle has three parts. We can solve this problem by counting by threes. Let’s put 3 by the first rectangle. Now it will be easy to decide that we need only 1 more rectangles to solve the problem.
3. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to shade the correct fraction on each rectangle.

Guided Practice:

1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

   Sample dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the improper fraction we need to show? Yes, it is twenty-five tenths. How many whole rectangles do we need to shade in? Yes, we need to shade in two whole rectangle. The denominator of the fraction tells us that this rectangle is divided into 10 parts. How many pieces do we need to divide the next rectangle into? Great, we should divide the rectangle into ten pieces also. Now let's see how many pieces we should shade. So, 25/10 is equivalent to 2 5/10. Who knows another name for 5/10? (Elicit other names for one-half). Yes, we could say that 2 5/10 is equivalent to 2 1/2. When we change 2 5/10 to 2 1/2, we have put the fraction in simplest terms. That means that only one half covers the same amount of the rectangle as 5 of the tenths pieces. From now on, whenever possible write your answers in simplest terms.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student's comprehension.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.
**Sample dialogue:** Now do problems 6 through 12 on your learning sheet. Remember to use the drawings to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

**Problem-Solving Practice:**


**Sample dialogue:** Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. Remember to use drawings to help you solve these problems, too. Read problem 13 to yourself. How many blocks make up the whole distance? Yes, six blocks make up the whole distance. How far has Francisco walked? Fill in the answer on your paper by writing a proper fraction.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

**Feedback:**

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 6

Complete Generalization of Fraction Equivalence Using Representational Drawings

Goals:

• To visualize and comprehend the relative values of fractions by using graphic representations.
• To express proper, improper, and mixed number fractions as equivalent fractions.
• To practice expressing fractions in simplest terms.

Materials:

• Learning Sheet 6
• Transparency of Learning Sheet 6
• Overhead projector and screen

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample dialogue: Yesterday we learned that simplest terms means to write an equivalent fraction with the smallest possible number in the denominator. For example, we can express the fraction 10/12 as the equivalent fraction 5/6. We say that 5/6 is written in simplest terms. Today, we will be reviewing everything we have learned so far, and you may choose to draw any shape to represent the fractions.

Describe and Model:

1. Distribute one copy of Learning Sheet 6 to each student.
2. Demonstrate how to compute Problem 1.
Sample Dialogue: I’m going to show you how to work these problems. Just like yesterday, there are no fraction drawings on your papers, so we’ll have to decide which shape we want to use. First, I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. The problem says, “Four and six-elevenths equals blank.” The number 4 6/11 is a mixed number. That means that the fraction will be more than one whole. I’m going to use rectangles for this problem. I will need to draw five rectangles, four whole rectangles plus another rectangle for the fractional part. The denominator of this fraction is 11, so I’ll have to divide my rectangles into eleven equal parts. Now, I’ll shade in four whole rectangles plus six-elevenths of the other rectangle. Remember the shortcut trick that I showed you a few days ago? We can just write 11 by each of the whole rectangles and then count by 11 to figure out the problem. Remember, the denominator tells us how many parts each whole has. The numerator tells us how many parts we need in our problem. So, we have 11, 22, 33, 44 plus 6 more. That’s a total of 50, so 4 6/11 is equivalent to 50/11. Let’s write that on our sheets.

Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to decide on a geometric shape and shade the correct fraction on each shape. Use a variety of shapes to promote generalization.

Guided Practice:
1. Guide the students through problem 4. Do not demonstrate the procedure unless students are having difficulty.

   **Sample dialogue:** Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. What is the proper fraction we need to show? Yes, it is two-fourths. We are being asked to find an equivalent fraction expressed in eighths. We will learn later that sometimes we will need to express fractions as equivalent fractions with larger denominators when we are adding and subtracting fractions with different denominators. Go ahead and draw your fractions and solve the problem by yourself.

2. Guide the students through problem 5, but do not ask for the answer. This problem will be used in assessing the student’s comprehension.

**Independent Practice:**

1. Instruct students to solve problems 6 - 12 independently.

   **Sample dialogue:** Now do problems 6 through 12 on your learning sheet. Remember to make drawings to help you find the answer. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

**Problem-Solving Practice:**


   **Sample dialogue:** Look at problem 13 on your Learning Sheet. Problem 13 is a word problem using fractions. Remember to use drawings to help you solve these problems, too. Read problem 13 to yourself. How many eggs make up the whole
carton? Yes, twelve eggs make up the whole carton. How many eggs does Mom have left? Fill in the answer on your paper by writing a mixed number.

2. Instruct students to complete problem 14 in the same manner.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student's paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 7

Introduce Fraction Equivalence at the Abstract Level

Goals:

- To write equivalent fractions using the Fundamental Law of Fractions
- To identify equivalent fractions

Materials:

- Learning Sheet 7
- Transparency of Learning Sheet 7
- Overhead projector and screen
- Cue card for the Fundamental Law of Fractions

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample Dialogue: So far, we have learned to represent fractions using concrete objects or drawings. We have learned the terms numerator, denominator, equivalent fractions, improper fraction, mixed number, and simplest terms. Today, we are going to learn how to figure out equivalent fractions a new way. We will learn a rule to help us solve problems without having to use objects or drawings.

Describe and Model:

1. Distribute one copy of Learning Sheet 7 and one cue card to each student.
2. Explain the Fundamental Law of Fractions.
Sample Dialogue: Remember that an equivalent fraction is a fraction that has the same value as another fraction. We need to figure out a way to change fractions but keep the value the same. When we learned to multiply, we learned that any number times itself is the same number. We can use the same rule to work fraction problems. Let's think of some ways to write the number 1 as a fraction. (Elicit responses such as 2/2, 3/3, etc.) If we multiply the fraction ½ by 1, does the value of the fraction change? No, it doesn’t. What if we multiply the fraction ½ by 2/2? Have we changed the value of the fraction? No, we haven’t, because 2/2 is another name for the number 1. Let’s multiply ½ by 2/2. One times one is two, and two times two is four, so we get 2/4. We’ve already learned using objects and drawings that 2/4 is equivalent to ½, so we have just figured out a new way to do equivalent fractions. Let’s use this way to figure out more names for ½. (Elicit some more equivalent fractions and list them on the overhead).

Let’s read the Cue Card about the Fundamental Law of Fractions. The Fundamental Law of Fractions says that the value of a fraction does not change if its numerator and denominator are multiplied by the same number. For example, the fractions 3/5 and 9/15 are equivalent fractions. We just multiplied the numerator and denominator by 3, but we could have used a different number if we wanted to. Let’s think of some other fractions equivalent to 3/5. (Elicit student responses).

3. Demonstrate how to compute Problem 1.
Sample Dialogue: I’m going to show you how to solve these problems using the Fundamental Law of Fractions. First I want you to watch me as I work the first problem on your sheet.

Look at problem 1 on your sheet. We need to find three equivalent fractions for 4/5. What are some of the fractions for 1 that we can use to solve this problem? (Elicit 2/2, 3/3, 4/4, etc.). Let’s multiply the numerator and the denominator by 2. We get 8/10. Now, let’s multiply by the other fractions for 1. Let’s write these answers on the blanks.

4. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to multiply to find equivalent fractions.

Guided Practice:
1. Guide the students to solve problem 4. Do not demonstrate unless students are having difficulty.

Sample Dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. We need to find equivalent fractions for which fraction? Yes, 1/4. Who can find some different equivalent fractions for 1/4? (List some of their suggestions). You’re doing great, go ahead and finish the problem on your sheet.

2. Guide students through problem 5, but do not ask for the answer. This problem will be used in assessing student’s comprehension.

Independent Practice:
1. Instruct students to solve problems 6 - 12 independently.
Sample Dialogue: Now do problems 6-12 on your learning sheet. Remember to use the Fundamental Law of Fractions to help you find the answers. When you are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

Problem-Solving Practice:


Sample Dialogue: Look at Problem 13 on your Learning Sheet. Problem 13 is a word problem that is asking us to find an equivalent fraction. We can use the Fundamental Law of Fractions to help us solve this problem, too. Let's read the problem. How many cupcakes did my brother bring to school? Yes, he brought 18 cupcakes to school. The class ate 2/3 of the cupcakes. So, we have to find a fraction that is equivalent to 2/3 but that has a denominator of 18 (the total number of cupcakes brought to school). What can we multiply by 2/3 to get a fraction with 18 in the denominator? (Elicit the answer). Go ahead and finish the problem. Be sure to multiply the numerator and denominator by the same number to get your answer.

2. Instruct students to complete problem 14 independently.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.
2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 8

Begin Generalization of Fraction Equivalence at the Abstract Level

Goals:

- To write fractions in simplest terms
- To find the greatest common factor of two whole numbers

Materials:

- Learning Sheet 8
- Transparency of Learning Sheet 8
- Overhead projector and screen
- Cue card for the Writing Fractions in Simplest Terms

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample Dialogue: Yesterday we learned how to write equivalent fractions using the Fundamental Law of Fractions. We learned that the value of a fraction does not change when we multiply it by another fraction equal to 1. Today, we are going to learn how to write fractions in simplest terms by finding the greatest common factor or GCF of two numbers. We will be writing fractions like 4/8, 9/12, and 14/16 in simplest terms without having to use fraction drawings to help us.

Describe and Model:

1. Distribute one copy of Learning Sheet 8 and one cue card to each student.

2. Explain how to find the GCF.
Sample Dialogue:

Factors are numbers that are being multiplied together. So, since \(5 \times 3 = 15\), we can say that 5 and 3 are factors of 15. The numbers 1 and 15 are also factors of 15. What are some factors of 18? (Elicit the answers 1, 2, 3, 6, 9, 18.)

Sometimes, two numbers have some of the same factors. These are called common factors. Do the numbers 6 and 9 have any common factors? Yes, 3 is a common factor of 6 and 9. What about 12 and 16? Yes, 2 and 4 are common factors of 12 and 16. We call 4 the greatest common factor or GCF of 12 and 16. (Ask students to find the GCF of some other pairs of numbers, such as 10 and 5, 4 and 8, and 12 and 15.)

We learned yesterday that the value of a fraction does not change when we multiply it by a fraction equal to 1. When we divide a number by 1, the number does not change, so if we divide a fraction by a fraction equal to 1, do you think that the value of the fraction will change? No, the value of the fraction stays the same when we divide it by a fraction equivalent to 1.

Let’s read the Cue Card about writing fractions in simplest terms. There are two steps in writing fractions in simplest terms. Step 1 says to find the GCF of the numerator and denominator of the fraction. Step 2 says to divide the fraction by a fraction equivalent to 1 that has the GCF as its numerator and denominator. We are going to use this rule to solve our problems today.

3. Demonstrate how to compute problem 1.

Sample Dialogue: Let’s look at problem 1 on our Learning Sheet. Watch me and listen as I explain how to work the problem. The problem asks us to write the
fraction 10/16 in simplest terms. Step 1 says that I have to find the GCF of 10 and 16. The factors of 10 are 1, 2, 5, and 10. The factors of 16 are 1, 2, 4, 8, and 16. Do 10 and 16 have any common factors? Yes, 2 is the only common factor, so it is the GCF of 10 and 16. That means that we have to divide 10/16 by a fraction equivalent to 1 that has the GCF in the numerator and denominator. What fraction is that? Yes, it is 2/2. Now we have to divide 10/16 by 2/2. Ten divided by two equals 5 and sixteen divided by two equals 8, so our new fraction is 5/8. So, 10/16 written in simplest terms is 5/8. Let’s fill that in on our sheet.

4. Demonstrate problems 2 and 3 with student participation. Using the same format, show students how to find the GCF and divide to write fractions in simplest terms.

Guided Practice:

1. Guide the students to solve problem 4. Do not demonstrate unless students are having difficulty.

Sample Dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. We need to write the fraction 4/6 in simplest terms. Who can tell me the GCF for 4 and 6? Yes, it’s 2. Go ahead and divide by 2/2. You’re doing great, go ahead and finish the problem on your sheet.

2. Guide students through problem 5, but do not ask for the answer. This problem will be used in assessing student’s comprehension.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.
Sample Dialogue: Now do problems 6-12 on your learning sheet. Remember to use the Cue Card and the GCF to help you find the answers. When your are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

Problem-Solving Practice:


Sample Dialogue: Look at Problem 13 on your Learning Sheet. Problem 13 is a word problem that is asking us to write the answer in simplest terms. Today we’ve learned how to use the GCF to write fractions in simplest terms, so we can use the GCF to help us solve this problem, too. Let’s read the problem. How many students are in Mr. Wong’s class? Yes, there are 24 students in the class, so that is the denominator of the fraction. How many like rap music? Yes, 18 like rap music, so that is the numerator of our fraction. Now we need to find the GCF of 18 and 24. (Elicit the answer). Go ahead and finish the problem by following the steps on your Cue Card.

2. Instruct students to complete problem 14 independently.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers.
Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 9

Complete Generalization of Fraction Equivalence at the Abstract Level

Goals:

• To express mixed numbers as improper fractions

Materials:

• Learning Sheet 9
• Transparency of Learning Sheet
• Overhead projector and screen
• Cue card “Changing Mixed Numbers to Improper Fractions”

Advance Organizer:

1. Review and tell the students what they will be doing and why.

Sample Dialogue: Yesterday, we learned how to find the greatest common factor or GCF of two numbers by listing all of the factors of each number. We also learned to write fractions in simplest terms by finding the GCF of two numbers. We wrote fractions like 4/8, 9/12, and 14/16 in simplest terms without having to use fraction drawings to help us. We have already learned how to show mixed numbers by shading in parts of circles or squares and how to change mixed numbers to improper fractions using drawings. Today, we will learn how to write mixed numbers such as 1 3/4 and 3 1/3 as improper fractions without using fraction drawings. We will be learning a new rule and using what we have already learned about equivalent fractions and simplest terms to help us get the answers.

Describe and Model:
1. Distribute one copy of Learning Sheet 9 and one copy of the cue card to each student.

2. Review the terms proper fraction, mixed number, and improper fraction.

Sample Dialogue:

We have already learned how to show proper fractions, mixed numbers, and improper fractions using fraction drawings. We have already learned that a proper fraction is a fraction that is less than one whole unit. For example, the fraction 2/3 is a proper fraction. The numerator, 2, tells us that we are shading in or circling two parts. The denominator, 3, tells us that there are three equal parts in the whole unit. So, 2/3 is a proper fraction and names a number less than one whole unit.

We have also learned that an improper fraction is a fraction that is one or more whole units. For example, 9/6 is an improper fraction. The numerator, 9, tells us that we are shading in or circling nine parts. The denominator, 6, tells us that there are six equal parts in each whole unit. So, we have to draw more than one whole unit to express the fraction 9/6. We have to shade in 6/6 of one unit and 3/6 of another unit. We have also talked about improper fractions that are equivalent to one whole unit. These are fractions like 3/3 or 5/5.

We have also learned that a mixed number is made up of a whole number and a proper fraction. So the mixed number 3 1/4 means that we are talking about three complete units and one-fourth of another unit. We learned that the mixed number 3 1/4 is equivalent to the improper fraction 13/4 because we have to
shade in four parts each of three whole units plus one more part of a unit.

(Demonstrate these concepts for students using the overhead or white board).

Today, we will learn how to change mixed numbers to improper fractions using some math rules without having to draw diagrams.

3. Explain how to change a mixed number to an improper fraction.

**Sample Dialogue:**

Let’s read the cue card for changing mixed numbers to improper fractions.

Step 1 says to multiply the whole number by the denominator of the fraction. We do this because we know that the denominator of any fraction tells us how many parts the whole unit has. So, for the fraction 10 2/3, each whole unit would have how many parts? Yes, each whole unit would have three parts, or thirds. So 10 whole units with 3 thirds each would equal how many parts in all? Yes, there would be 30 thirds, or 10 x 3 thirds.

Step 2 says to add the numerator of the fraction to the answer we get in Step 1. We do this because we have just counted the 10 whole units, but we still have 2/3 of a unit left. So, 30 thirds plus 2 more thirds equals how many thirds? Yes, our answer will be 32 thirds.

Step 3 reminds us that the denominator of our fraction does not change. Each whole unit is still divided into the same number of parts. So, now we know that 10 2/3 is equivalent to 32/3.

4. Demonstrate how to compute problem 1.

**Sample Dialogue:** Let’s look at problem 1 on our Learning Sheet. Watch me and listen as I explain how to work the problem. The problem asks us to write the
mixed number 2 1/15 as an improper fraction. Step 1 says that I have to multiply my whole number by the denominator of my fraction. So, 2 times 15 equals what? Yes, 2 times 15 equals 30. Step 2 says I have to add my answer, 30, to the numerator of the fraction in the mixed number. So, 30 plus 1 equals 31. Step 3 says that my improper fraction will have the same denominator as my mixed number, which is 15. So, 2 1/15 equals 31/15. Let’s fill that in on our sheet.

5. Demonstrate problems 2 and 3 with student participation. These problems are a review of Lessons 7 and 8. Ask students to use their Cue Cards to help them solve these problems along with you.

Guided Practice:

1. Guide the students to solve problem 4. Do not demonstrate unless students are having difficulty.

Sample Dialogue: Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. We need to write the mixed number 4 7/9 as an improper fraction. What is the first thing we need to do? Yes, we need to multiply the whole number by the denominator. Follow the directions on your Cue Card to solve this problem. (Elicit answers from the students for each step). You’re doing great, go ahead and finish the problem on your sheet.

2. Guide students through problem 5, but do not ask for the answer. This problem will be used in assessing students’ understanding.

Independent Practice:

1. Instruct students to solve problems 6 - 12 independently.
Sample Dialogue: Now do problems 6-12 on your learning sheet. Remember to use the Cue Cards to help you find the answers. When your are through, put down your pencils.

2. Circulate through the room and monitor students as they work independently.

Problem-Solving Practice:


Sample Dialogue: Look at Problem 13 on your Learning Sheet. Problem 13 is a word problem that is asking us to solve the problem using an improper fraction. Today we’ve learned how to solve these problems using a math rule, so we can use the same math rule to help us solve this problem, too. Let’s read the problem. How many packages of tennis balls does Angela have? Yes, she has 4 2/3 packages of tennis balls. How many tennis balls are in each package? Yes, there are 3 tennis balls in each package because the denominator of our fraction tells us how many are in each whole unit. So, how many tennis balls does Angela have? Yes, 4 time 3 plus 2 equals 14, so that is the numerator of our fraction. What is the denominator of our fraction? (Elicit the answer). Go ahead and finish the problem by following the steps on your Cue Card.

2. Instruct students to complete problem 14 independently.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.
2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
Lesson 10

Complete Generalization of Fraction Equivalence at the Abstract Level

Goals:

• To express improper fractions as mixed numbers
• To recognize the fraction bar as a symbol for division

Materials:

• Learning Sheet 10
• Transparency of Learning Sheet 10
• Overhead projector and screen
• Cue card “Changing Improper Fractions to Mixed Numbers”

Advance Organizer:

1. Review and tell the students what they will be doing and why.

   **Sample Dialogue:** Yesterday, we learned how to write mixed numbers such as $1 \frac{3}{4}$ and $3 \frac{1}{3}$ as improper fractions. We learned the steps to change any mixed number to an equivalent improper fraction. Today, we will learn how to write improper fractions such as $\frac{20}{6}$ and $\frac{15}{12}$ as mixed numbers. We will be using what we have already learned about equivalent fractions and simplest terms to help us get the answers.

Describe and Model:

1. Distribute one copy of Learning Sheet 10 and one copy of the cue card to each student.

   **Sample Dialogue:**
Today, we will learn how to change improper fractions to mixed numbers using some math rules without having to draw diagrams.

Let's read the Cue Card about writing improper fractions as mixed numbers. There are two steps in writing improper fractions as mixed numbers. Step 1 says that the fraction bar means divide. The fraction bar is just another symbol for division, just as the division sign and division box tell us to divide. So we need to divide the numerator of the fraction into the denominator of the fraction. We do this because the denominator of a fraction tells us how many parts each whole unit is divided into. Step 2 says that if there is a remainder, we make it the numerator and the divisor is the denominator. We do this because the remainder tells us how many parts are left over after we figure out how many whole units we have. We are going to use this rule to solve our problems today.

2. Demonstrate how to compute problem 1.

Sample Dialogue: Let's look at problem 1 on our Learning Sheet. Watch me and listen as I explain how to work the problem. The problem asks us to write the improper fraction 62/8 as a mixed number. Step 1 says that I have to divide the numerator by the denominator. How many times can 8 go into 62? Yes, 8 goes into 62 seven times with how many left over? What fraction is that? Yes, it is 7 6/8. But, we're not quite finished. Can we write 6/8 in simplest terms? Yes, 6/8 equals 3/4. So, our answer is 7 3/4. Let's fill that in on our sheet.

3. Demonstrate problems 2 and 3 with student participation. These problems are reviews of lessons 7 and 9. Remind students to use their cue cards to help them solve these problems.

Guided Practice:
1. Guide the students to solve problem 4. This is a review of lesson 8. Do not demonstrate unless students are having difficulty.

   **Sample Dialogue:** Now you can do problem 4 by yourself at your desk. Look at the problem on your Learning Sheet. We need to write the fraction 6/14 in simplest terms. Who can tell me the GCF for 6 and 14? Yes, it's 2. Go ahead and divide by 2/2. You're doing great, go ahead and finish the problem on your sheet.

2. Guide students through problem 5, but do not ask for the answer. This problem will be used in assessing student's comprehension.

   **Independent Practice:**

   1. Instruct students to solve problems 6 - 12 independently.

      **Sample Dialogue:** Now do problems 6-12 on your learning sheet. Remember to use your Cue Cards to help you find the answers. When your are through, put down your pencils.

   2. Circulate through the room and monitor students as they work independently.

   **Problem-Solving Practice:**


      **Sample Dialogue:** Look at Problem 13 on your Learning Sheet. Problem 13 is a word problem that is asking us to write the answer as a mixed number. Today we’ve learned how to divide to change improper fractions into mixed numbers, so we can use that rule to help us solve this problem, too. Let’s read the problem. How many photographs does Carminda have? Yes, she has 29 photographs, so that is the numerator of the fraction. How many photographs fit on a page? Yes, 6 photographs fit on a page,
so that is the denominator of our fraction. Now we divide 29 by 6. (Elicit the answer).

Go ahead and finish the problem by following the steps on your Cue Card.

2. Instruct students to complete problem 14 independently.

3. Collect papers when students have finished working.

Feedback:

1. Score problems 5 through 14 on each student’s paper. Determine percentage answered correctly.

2. Meet with each student individually to discuss his/her performance. Praise students for correct answers and ask students to explain incorrect answers. Show students how to solve any incorrect problems. Provide additional practice as needed until students demonstrate understanding.
March 16, 1999

Dear Parents:

My name is Frances Butler, and I am a doctoral candidate at the University of Nevada, Las Vegas as well as being a resource teacher at Lied Middle School. Your child's math class will be involved in a special learning project that is designed to help students improve their math skills. The project will start in May, and it will be part of the regular math program for the class. The entire study should take about three weeks, or fifteen days.

The purpose of the project is to design a better way to teach fractions to middle-school students. In the first three lessons, one group of students will use concrete manipulative devices such as fraction circles or fraction strips to help them understand fraction equivalence, while the other group of students will use representative drawings instead. The lessons will be the same for both groups of students. The information gained from this project will help other teachers in designing fraction instruction.

The project will be part of my dissertation study for the doctoral degree from the University of Nevada, Las Vegas. The project will be supervised by Dr. Susan Miller of
the Special Education Department at UNLV. While the results will be reported, your child's name will not appear, and he/she will remain anonymous. Your child's inclusion in the study is voluntary. There will be no monetary compensation for participation in the study.

Your child may benefit from the project in several ways. First, I believe that his/her ability to understand fractions and to solve problems with fractions will improve. Second, I have designed the lessons to be fun and to help each student develop a positive attitude toward math. This is important for developing skills in more complicated areas of math such as ratio, proportion, probability, and functions.

If you have any questions about this study, you may contact me at school at 799-4620 or at home at 655-7545. You may also contact the UNLV Office of Sponsored Projects, 895-1357, for information regarding the rights of research subjects. You may keep a copy of this informational letter for your records. If you agree to have your child participate in this study, please return the attached consent form to Lied Middle School. I will provide you with a copy of the consent form for your records.

I am looking forward to this project and the many benefits for students and teachers. Thank you for your support of the project.

Sincerely,

Frances M. Butler
Parent Consent Form

I agree to allow my child to participate as a volunteer in a research study that investigates the learning of fraction equivalence. The study will be conducted in the classroom by my child’s math teacher, under the direction of Frances M. Butler and Dr. Susan Miller. This study will take approximately 45 minutes each day for about 3 weeks.

I have been advised that my child’s identity will not be revealed in any publication or document related to the study and that I may withdraw my consent for my child’s participation at any time during the study.

__________________________________________
Parent/Guardian Date

__________________________________________
Frances M. Butler Date
Teacher Consent Form

I agree to participate as a volunteer in a research study that investigates the learning of fraction equivalence. The study will be conducted in my classroom under the direction of Fran Butler and Dr. Susan Miller. This study will take approximately 45 minutes each day for about 3 weeks. I have been advised by the researcher that I will attend a teacher-training session and that all materials and lesson plans will be provided to me at no cost to me.

I have also been advised that I may discontinue my participation at any time and withdraw any information gathered in my classroom for the study. Neither my name nor the names of my students will be revealed in any publication or document related to the study.

Name

Date

Frances M. Butler

Date
REFERENCES


Graduate College
University of Nevada, Las Vegas

Frances Mary Butler

Home Address:
7269 Golden Star Avenue
Las Vegas, Nevada 89130

Degrees:
Bachelor of Arts, Anthropology, 1968
University of California, Berkeley

Master of Education, Special Education, 1996
University of Southern Mississippi, Hattiesburg

Special Honors and Awards:
National Merit Scholarship, 1964 - 1968
Gamma Beta Phi Society, 1995 - 1996

Publications:
Butler, F. M. (1999). Reading partners: Students can teach each other to read! Education and Treatment of Children.


Dissertation Title: Fraction Instruction for Students with Disabilities: Comparing Two Teaching Sequences.

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Dissertation Examining Committee:
  Chairperson, Dr. Susan P. Miller, Ph. D.
  Committee Member, Dr. Beatrice Babbitt, Ph. D.
  Committee Member, Dr. Thomas Pierce, Ph. D.
  Graduate Faculty Representative, Dr. Kevin Crehan, Ph. D.