Numerical Methods for Option Pricing under the Two-Factor Models

Jiacheng Cai
University of Nevada, Las Vegas, caij@unlv.nevada.edu

Follow this and additional works at: https://digitalscholarship.unlv.edu/thesesdissertations
Part of the Applied Mathematics Commons, Corporate Finance Commons, Finance Commons, Finance and Financial Management Commons, and the Mathematics Commons

Repository Citation
https://digitalscholarship.unlv.edu/thesesdissertations/3072

This Dissertation is brought to you for free and open access by Digital Scholarship@UNLV. It has been accepted for inclusion in UNLV Theses, Dissertations, Professional Papers, and Capstones by an authorized administrator of Digital Scholarship@UNLV. For more information, please contact digitalscholarship@unlv.edu.
NUMERICAL METHODS FOR OPTION PRICING UNDER THE TWO-FACTOR MODELS

by

Jiacheng Cai

Bachelor of Science - Mathematics
University of Science and Technology of China, China
2011

A dissertation submitted in partial fulfillment of the requirements for the

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
August 2017
This dissertation prepared by

Jiacheng Cai

entitled

Numerical Methods for Option Pricing under the Two-Factor Models

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy - Mathematical Sciences
Department of Mathematical Sciences

Hongtao Yang, Ph.D.  
Examination Committee Chair

Michael Marcozzi, Ph.D.  
Examination Committee Member

Monika Neda, Ph.D.  
Examination Committee Member

Pengtao Sun, Ph.D.  
Examination Committee Member

Jianzhong Zhang, Ph.D.  
Graduate College Faculty Representative

Kathryn Hausbeck Korgan, Ph.D.  
Graduate College Interim Dean
ABSTRACT

NUMERICAL METHODS FOR OPTION PRICING UNDER THE TWO-FACTOR MODELS

by

Jiacheng Cai

Dr. Hongtao Yang, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas, USA

Pricing options under multi-factor models are challenging and important problems for financial applications. In particular, the closed form solutions are not available for the American options and some European options, and the correlations between factors increase the complexity and difficulty for the formulations and implements of the numerical methods.

In this dissertation, we first introduce a general transformation to decouple correlated stochastic processes governed by a system of stochastic differential equations. Then we apply the transformation to the popular two-factor models: the two-asset model, the stochastic volatility model, and the stochastic interest rate models. Based on our new formulations, we develop a mixed Monte Carlo method, a lattice method, and a finite volume-alternating direction implicit method for pricing the European and American options under these models. The proposed methods can be easily implemented and need less memory. Numerical results are also presented to validate our C++ programs and to examine our methods. It shows that our methods are very accurate and efficient.
ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my advisor, Dr. Hongtao Yang for his guidance, support, and expertise. I would never have been able to finish my dissertation without his help. I am also very grateful to my committee members Dr. Michael Marcozzi, Dr. Monika Neda, Dr. Pengtao Sun, and Dr. Jianzhong Zhang for their time, expertise and support. I want to thank Dr. Jichun Li for his help.

I would also like to thank my friends Robert Ain, Moinak Bhaduri, Sean Breckling, Sharang Chaudhry, Min Chen, Daniel Corral, Zhiwei Fang, Zhanman He, Edward Huynh, Rihui Lan, Bowen Liu, John Nguyen, Wenxin Peng, Gary Phelps, Sidney Shield, Xudong Sun, Yuzhou Sun, Zhou Wang, Jianbo Xu, Lanxuan Yu, and Libo Zhou for their helps during this journey.

Finally I would like to show my gratitude and appreciation to my parents Mr. Jianxin Cai and Mrs. Lixuan Lin, my brother Jiarun Cai, and my entire family for their love. Their constant support and encouragement over the years motivated me to work more effectively and efficiently towards my goal.
# TABLE OF CONTENTS

## ABSTRACT iii

## ACKNOWLEDGEMENTS iv

## LIST OF TABLES vii

## LIST OF FIGURES ix

## 1 INTRODUCTION 1

1.1 Option Pricing under Two-Factor Models .................................. 2
   1.1.1 The Two-Asset Model ........................................... 2
   1.1.2 The Stochastic Volatility Model .............................. 5
   1.1.3 The Stochastic Interest Rate Models .......................... 6
1.2 Summary and Organization of this Dissertation .......................... 7

## 2 DECOUPLING MULTI-FACTOR MODELS 10

2.1 Decoupling the Correlated Stochastic Processes .......................... 10
2.2 Two-Factor Models .................................................................. 12
   2.2.1 The Two-Asset Model ............................................ 12
   2.2.2 The Stochastic Volatility Model .............................. 14
   2.2.3 The Stochastic Interest Rate Model .......................... 15

## 3 MIXED MC METHODS WITH CONTROL VARIATES 17

3.1 The Two-Asset Model .................................................................. 17
3.2 The Stochastic Volatility Model .............................................. 21
3.3 The Stochastic Interest Rate Model .......................................... 24
3.4 Numerical Results .................................................................... 25

## 4 LATTICE METHODS 36

4.1 The Two-Asset Model .................................................................. 36
4.2 The Stochastic Interest Rate Model .......................................... 41
4.3 Numerical Results .................................................................... 46

## 5 A FINITE VOLUME - ADI METHOD 78

5.1 The Partial DifferentialVariational Inequalities ........................... 79
5.2 The Boundary Conditions ....................................................... 81
5.3 The Semi-discretization by a Finite Volume Method ..................... 83
| 5.3.1 | The Interior Nodes | 84 |
| 5.3.2 | The Nodes on the Boundary \( x = X_{\text{min}} \) | 87 |
| 5.3.3 | The Nodes on the Boundary \( x = X_{\text{max}} \) | 87 |
| 5.3.4 | The Nodes on the Boundary \( y = 0 \) | 88 |
| 5.3.5 | The Nodes on the Boundary \( y = Y_{\text{max}} \) | 90 |
| 5.3.6 | Discretization of the Operator \( \mathcal{L} \) | 92 |
| 5.4  | Time Discretization: an ADI method | 94 |
| 5.5  | Numerical Results | 98 |

6 CONCLUSION 114

APPENDIX: THE ANALYTIC FORMULAS FOR \( \tilde{V} \) 117

BIBLIOGRAPHY 120

CURRICULUM VITAE 125
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Popular two-asset options</td>
</tr>
<tr>
<td>3.1</td>
<td>Notations</td>
</tr>
<tr>
<td>3.2</td>
<td>The parameters for the spread option</td>
</tr>
<tr>
<td>3.3</td>
<td>The European spread option prices: $S_1 = 100$, $N = 2000$, $M = 1000$</td>
</tr>
<tr>
<td>3.4</td>
<td>The parameters for the Heston model</td>
</tr>
<tr>
<td>3.5</td>
<td>The European option prices (Heston): $v = 0.09$, $N = 2000$, $M = 1000$</td>
</tr>
<tr>
<td>3.6</td>
<td>The parameters for the Vasicek model</td>
</tr>
<tr>
<td>3.7</td>
<td>The European option prices (Vasicek): $r = 0.11$, $N = 2000$, $M = 1000$</td>
</tr>
<tr>
<td>4.1</td>
<td>Notations</td>
</tr>
<tr>
<td>4.2</td>
<td>Parameters for the European spread option</td>
</tr>
<tr>
<td>4.3</td>
<td>The European spread option prices: $\rho = -0.8$</td>
</tr>
<tr>
<td>4.4</td>
<td>The European spread option prices: $\rho = -0.4$</td>
</tr>
<tr>
<td>4.5</td>
<td>The European spread option prices: $\rho = 0.4$</td>
</tr>
<tr>
<td>4.6</td>
<td>The European spread option prices: $\rho = 0.8$</td>
</tr>
<tr>
<td>4.7</td>
<td>Parameters for the American options with two-asset</td>
</tr>
<tr>
<td>4.8</td>
<td>Parameters for the European call option: the Vasicek model</td>
</tr>
<tr>
<td>4.9</td>
<td>The European call prices (Vasicek): $\theta = 0.05$, $\rho = -0.8$</td>
</tr>
<tr>
<td>4.10</td>
<td>The European call prices (Vasicek): $\theta = 0.05$, $\rho = -0.4$</td>
</tr>
<tr>
<td>4.11</td>
<td>The European call prices (Vasicek): $\theta = 0.05$, $\rho = 0.4$</td>
</tr>
<tr>
<td>4.12</td>
<td>The European call prices (Vasicek): $\theta = 0.05$, $\rho = 0.8$</td>
</tr>
<tr>
<td>4.13</td>
<td>Parameters for the European call option: the CIR model</td>
</tr>
<tr>
<td>4.14</td>
<td>The European call prices (CIR): $\theta = 0.05$, $\rho = -0.8$</td>
</tr>
<tr>
<td>4.15</td>
<td>The European call prices (CIR): $\theta = 0.05$, $\rho = -0.4$</td>
</tr>
<tr>
<td>4.16</td>
<td>The European call prices (CIR): $\theta = 0.05$, $\rho = 0.4$</td>
</tr>
<tr>
<td>4.17</td>
<td>The European call prices (CIR): $\theta = 0.05$, $\rho = 0.8$</td>
</tr>
<tr>
<td>5.1</td>
<td>Parameters for the Heston model for put option</td>
</tr>
<tr>
<td>5.2</td>
<td>The CPU times</td>
</tr>
<tr>
<td>5.3</td>
<td>The absolute errors: Case A</td>
</tr>
<tr>
<td>5.4</td>
<td>The absolute errors: Case B</td>
</tr>
<tr>
<td>5.5</td>
<td>The absolute errors: Case C</td>
</tr>
<tr>
<td>5.6</td>
<td>The absolute errors: Case D</td>
</tr>
<tr>
<td>5.7</td>
<td>The absolute errors: Case E</td>
</tr>
<tr>
<td>5.8</td>
<td>The absolute errors: Case F</td>
</tr>
<tr>
<td>5.9</td>
<td>Parameters for the Heston model for comparison</td>
</tr>
</tbody>
</table>
5.10 The American option prices for Case G: \( v = 0.0625 \) . . . . . . . . . . . . . . 113
5.11 The American option prices for Case G: \( v = 0.25 \) . . . . . . . . . . . . . . . 113
5.12 The American option prices for Case H: \( v = 0.0348 \) . . . . . . . . . . . . . . . 113
LIST OF FIGURES

3.1 MAE vs $N$ for the European spread option ........................................... 27
3.2 RMSE vs $N$ for the European spread option ........................................... 28
3.3 The variance ratios vs $S_2$ for the European spread option......................... 28
3.4 MAE vs $N$ for the Heston model ........................................................... 30
3.5 RMSE vs $N$ for the Heston model ........................................................... 31
3.6 The variance ratios vs $S$ for the Heston model ......................................... 31
3.7 MAE vs $N$ for the Vasicek model ........................................................... 33
3.8 RMSE vs $N$ for the Vasicek model ........................................................... 34
3.9 The variance ratios vs $S$ for the Vasicek model ......................................... 34
4.1 The maximum absolute errors of the European spread options ......................... 47
4.2 The early exercise boundaries of the spread options .................................. 51
4.3 The early exercise boundaries of the call option on the maximum .................. 53
4.4 The early exercise boundaries of the maximum call option ......................... 55
4.5 The early exercise boundaries of the put option on the minimum ................... 57
4.6 The early exercise boundaries of the maximum put option ......................... 59
4.7 The maximum absolute errors of the European nall of Vasicek ....................... 61
4.8 The early exercise boundaries of call option (Vasicek, $\rho$, $\theta$) ................. 64
4.9 The early exercise boundaries of call option (Vasicek, $\rho$, $q$) ..................... 65
4.10 The early exercise boundaries of call option with Vasicek (extreme) ............. 66
4.11 The early exercise boundaries of put option (Vasicek, $\rho$, $\theta$) ................. 67
4.12 The early exercise boundaries of put option (Vasicek, $\rho$, $q$) ..................... 68
4.13 The maximum absolute errors of the European Call of CIR ......................... 70
4.14 The early exercise boundaries of call option (CIR, $\rho$, $\theta$) ..................... 73
4.15 The early exercise boundaries of call option (CIR, $\rho$, $q$) ....................... 74
4.16 The early exercise boundaries of call option with CIR (extreme) ................. 75
4.17 The early exercise boundaries of put option (CIR, $\rho$, $\theta$) ..................... 76
4.18 The early exercise boundaries of put option (CIR, $\rho$, $q$) ....................... 77
5.1 An interior node ......................................................................................... 85
5.2 A node on the boundary $y = 0$ ............................................................... 88
5.3 A node on the boundary $y = Y_{\text{max}}$ ................................................... 90
5.4 Discretization mesh .................................................................................. 99
5.5 The rates of convergence for maximum absolute errors (validation) ............... 103
5.6 The rates of convergence for maximum absolute errors ................................ 105
5.7 The rates of convergence for the root mean square errors ......................... 106
5.8 The rates of convergence for the $L_2$ errors ........................................... 107
CHAPTER 1

INTRODUCTION

In finance, an option is a contract that gives the buyer (owner) the right (no obligation) to buy or to sell an underlying asset at a specified strike price on or before a specified date. Whenever the buyer exercises the option, the seller of the option has the obligation to fulfill the transaction. The buyer pays a premium, which is the value of the option, to the seller for the right. The options that gives the right to buy the underlying asset is referred to as a call option, whereas a put option gives the right to sell the asset. There are two standard styles of options: the European and American options. The European options can only be exercised at the option expiration date, while the American options allow the owner to exercise at any time up to the option expiration date. The other styles such as the Asian options, Bermuda options, look-back options, etc. are referred as the exotic options.

The valuation problem of the options has been widely studied. The classical model (Black-Scholes Model) for stock options was first introduced by Fischer Black and Myron Scholes in 1973 ([5]). Since then, the extensions of their model to other financial derivatives have been investigated ([34][40]), for example, bonds and their options, futures, swaps, etc. Besides one factor models, various multi-factor models have been proposed in order to fit the real markets more accurately. These models include the jump diffusion models (the Merton
model and Kuo model), stochastic volatility models (the Heston model), stochastic interest rate models (the Vasicek model and the CIR model for interest rate processes), the stochastic volatility with jump (the Bates model), etc.

Most of the European options under the multi-factor models cannot be evaluated analytically or efficiently, and the American options have to be evaluated numerically. Pricing of these options becomes one of the most challenging and important problems for financial applications. The difficulty is either due to the nonlinearity (the American option problems) or the correlation between the factors. Various numerical methods have been extensively studied such as Monte Carlo method, lattice method, finite difference/element/volume methods, and semi-analytic methods. We are referred to [28][34][40][54] and the references cites therein.

The objective of this dissertation is to develop several numerical methods to approximate option prices under popular two-factor models by decoupling the correlated two factors.

1.1 Option Pricing under Two-Factor Models

In this section, we review the existing works on option pricing under the two-asset model, the Heston model and the stochastic interest rate models.

1.1.1 The Two-Asset Model

The two-asset model is the extension of the Black-Scholes model from one asset to two. The price processes of the assets are governed by the following stochastic differential equations

\[
\frac{dS_i(t)}{S_i(t)} = (r_i - q_i)dt + \sigma_i dB_i(t), \quad i = 1, 2,
\]
where \( r \) is the risk-free interest rate, \( q_i \) is the dividend rate and \( \sigma_i \) is the volatility for the \( i \)-th asset, and the Wiener processes \( B_1(t) \) and \( B_2(t) \) are correlated with the correlation \( dB_1(t)dB_2(t) = \rho dt \).

Let \( V(s_1, s_2, t) \) be the value of the option when \( S_1(t) = s_1 \) and \( S_2(t) = s_2 \) at time \( t \in [0, T) \). According to the no-arbitrage pricing theory, the rational price of the European option with the payoff \( \Phi(s_1, s_2) \) (see Table 1.1) is given by

\[
V(s_1, s_2, t) = \mathbb{E}\left[ e^{-r(T-t)}\Phi(S_1(T), S_2(T)) \right| S_1(t) = s_1, S_2(t) = s_2],
\]

For the American option, we have

\[
V(s_1, s_2, t) = \sup_{t \leq \tau \leq T} \mathbb{E}\left[ e^{-r(T-t)}\Phi(S_1(\tau), S_2(\tau)) \right| S_1(t) = s_1, S_2(t) = s_2],
\]

where \( \mathbb{E} \) is the expectation under the risk neutral measure and the \( \tau \) is a stopping time. The most used payoff functions are

<table>
<thead>
<tr>
<th>Type</th>
<th>Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spread</td>
<td>( \max(S_2 - S_1 - K, 0) )</td>
</tr>
<tr>
<td>Call on maximum</td>
<td>( \max(\max(S_1, S_2) - K, 0) )</td>
</tr>
<tr>
<td>Maximum call</td>
<td>( \max(\max(S_1 - K_1, 0), \max(S_2 - K_2, 0)) )</td>
</tr>
<tr>
<td>Put on minimum</td>
<td>( \max(K - \min(S_1, S_2), 0) )</td>
</tr>
<tr>
<td>Maximum put</td>
<td>( \max(\max(K_1 - S_1, 0), \max(K_2 - S_2, 0)) )</td>
</tr>
</tbody>
</table>

Table 1.1: Popular two-asset options

It is known that the European option price is the solution of the following partial differential equation

\[
\frac{\partial V}{\partial t} + \mathcal{L}V = 0,
\]

where

\[
\mathcal{L}V = \frac{1}{2} \sum_{ij} \rho_{ij} \sigma_i \sigma_j s_i s_j \frac{\partial^2 V}{\partial s_i \partial s_j} + \sum_i (r - q_i) s_i \frac{\partial V}{\partial s_i} - rV.
\]
For the American option, we have the following variational inequality problem

$$\frac{\partial V}{\partial t} + \mathcal{L}V \leq 0, \quad V \geq \Phi, \quad (V - \Phi)\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right) = 0.$$ 

The expectation for the European option price can be expressed in term of the CDF of the normal and multi-variate normal distribution for the exchange option ([46]) and the options on the maximum or minimum ([56, 37]), respectively. For the spread option, we reformulate the expectation as the one with respect to two independent processes so that it can be also computed by numerical integration (see (3.14) of Example 3.1).

There are several numerical methods developed for the American options. Boyle, Evnine and Gibbs [7] applied the binomial tree methods in two and three underlying assets. Gamba and Trigeorgis [27] improved the lattice method by using a transformation to obtain uncorrelated processes. Monte Carlo methods are applied to the high dimensional European options valuation [8] and the upper and lower boundaries of the American option value. Details of Monte Carlo application are shown in Glasserman’s book [28]. Kovalov, Linetsky and Marcozzi [41] developed a computational method for the valuation of multi-asset American-style options based on approximating partial differential variational inequality. We are also referred to [25] for a comprehensive survey of numerical methods in high dimensional American options.
1.1.2 The Stochastic Volatility Model

The Heston model, a commonly used stochastic volatility model, was proposed by Heston in 1993 [33]. It assumes the stochastic volatility $v$ and underlying asset price $S$ follow

$$
\begin{align*}
    dv(t) &= \kappa [\eta - v(t)] \, du + \sigma \sqrt{v(t)} \, dB_1(t), \\
    \frac{dS(t)}{S(t)} &= (r - q) \, dt + \sqrt{v(t)} \, dB_2(t),
\end{align*}
$$

where $\eta$ is the long-term expectation of variance, $\kappa > 0$ is the speed of mean reversion, $\sigma$ is the volatility of volatility, and the Wiener processes $B_1(t)$ and $B_2(t)$ are correlated with correlations $dB_1(t) dB_2(t) = \rho dt$. If $2\kappa \eta > \sigma^2$ (Feller condition), then $v(t)$ is strictly positive ([13]).

Let $V(s, v, t)$ be the value of the option when $v(t) = v$ and $S(t) = s$ at time $t$. It is known that the European option price is the solution of the following partial differential equation

$$
\frac{\partial V}{\partial t} + \mathcal{L} V = 0,
$$

where

$$
\mathcal{L} V = \frac{1}{2} s^2 v \frac{\partial^2 V}{\partial s^2} + \rho \sigma sv \frac{\partial^2 V}{\partial s \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 V}{\partial v^2} + (r - q) s \frac{\partial V}{\partial s} + \kappa (\eta - v) \frac{\partial V}{\partial v} - r V.
$$

For the American option, we have the following variational inequality problem:

$$
\frac{\partial V}{\partial t} + \mathcal{L} V \leq 0, \quad V \geq \Phi, \quad (V - \Phi) \left( \frac{\partial V}{\partial t} + \mathcal{L} V \right) = 0.
$$

The European option price can be computed by numerical integration with the characteristic function of $S(T)$ ([33]). Since there is no closed form solution for the American option problem under the Heston model, various numerical methods have been considered. For the Monte Carlo methods, we are referred to [28] and references cited therein. Loeper and
Pironneau [44] introduced a mixed PDE/Monte Carlo method for the European options with stochastic volatility. Longstaff and Schwartz [45] provided a Monte Carlo regression method for the American option. As to the lattice method, several papers [42][53] studied the application in this type of models. Beliaeva and Nawalkha [4] proposed a lattice scheme by using a transformation to generate path-independent tree. A detailed survey of lattice method application in the Heston model is presented in [4]. For the finite difference approaching, we are referred to Zvan, Forsyth and Vetzal[62], Oosterlee [50], Ikonen and Toivanen [36]. Haenetjens and in’t Hout [31] presented a summary of ADI schemes for pricing the American option under the Heston model. However, the above works didn’t avoid the mixed partial derivative terms raising from the correlation. Detail survey of finite difference scheme can be found in [31].

1.1.3 The Stochastic Interest Rate Models

Stochastic interest rate models assume that the asset price follows

\[ \frac{dS(t)}{S(t)} = (r(t) - q)dt + \sigma dB_1(t). \]

The stochastic interest rate \( r(t) \) follows the Vasicek model ([58])

\[ dr(t) = \kappa (\theta - r(t)) dt + \nu dB_2(t) \]

or the CIR model ([13])

\[ dr(t) = \kappa (\theta - r(t)) du + \nu \sqrt{r(t)} dB_2(t). \]

Here \( \theta \) is the long-term expectation of interest rate, \( \kappa > 0 \) is the speed of mean reversion, \( \nu \) is the volatility of the interest rate, and the Wiener processes \( B_1(t) \) and \( B_2(t) \) are correlated with the correlation \( dB_1(t)dB_2(t) = \rho dt \).
Let $V(s, r, t)$ be the value of the option when $S(t) = s$ and $r(t) = r$ at time $t$. It is known that the European option price is the solution of the following partial differential equation

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0,$$

where for the Vasicek model,

$$\mathcal{L}V = \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2} + \rho \sigma sv \frac{\partial^2 V}{\partial s \partial r} + \frac{1}{2} v^2 \frac{\partial^2 V}{\partial r^2} + (r - q)s \frac{\partial V}{\partial s} + \kappa(\theta - r) \frac{\partial V}{\partial r} - rV,$$

and for the CIR model,

$$\mathcal{L}V = \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 V}{\partial s^2} + \rho \sigma sv \sqrt{r} \frac{\partial^2 V}{\partial s \partial r} + \frac{1}{2} v^2 \frac{\partial^2 V}{\partial v^2} + (r - q)s \frac{\partial V}{\partial s} + \kappa(\theta - r) \frac{\partial V}{\partial r} - rV.$$

For the American option, we have the following variational inequality problem

$$\frac{\partial V}{\partial t} + \mathcal{L}V \leq 0, \quad V \geq \Phi, \quad (V - \Phi) \left(\frac{\partial V}{\partial t} + \mathcal{L}V\right) = 0.$$

For the European options, we are referred to Kim and Kunitomo [38] for an analytic approximation for the CIR model and Fang [23] for an analytic formula for the Vasicek model. However, there are few papers about the numerical methods for the American options under the stochastic interest rate models.

### 1.2 Summary and Organization of this Dissertation

In this dissertation, we propose a transformation to decouple correlated stochastic processes governed by a system of stochastic differential equations. Then we apply the new transformation to the popular two-factor models: the two-asset model, the stochastic volatility model, and the stochastic interest rate models. Based on our new formulations, we develop
a mixed Monte Carlo method, a lattice method, and a finite volume-alternating direction implicit method for pricing the European and American options under these models. The proposed methods can be easily implemented and need less memory. Numerical results are also presented to validate our C++ programs and to examine our methods. The numerical experiments show our methods are highly accurate and efficient.

The outlines of the remaining chapters are as follows:

- **Chapter 2: Decoupling Multi-factor models.**

  A transformation to decouple correlated stochastic processes is introduced and applied to the popular two-factor models.

- **Chapter 3: Mixed Monte Carlo Methods with Control Variates.**

  With the uncorrelated new processes, we are able to express the rational prices of the European contingent claims under the two-factor models as the nested expectations. The inner expectation is the price of the European contingent claim for an artificial asset and can be analytically evaluated by the Black-Scholes formula. Then we use the Monte Carlo method to estimate the outer expectation. We also employ the control variates technique to reduce the variances. Numerical results are presented to examine our methods.

- **Chapter 4: Lattice Methods.**

  We propose a new lattice method for the European and American options under the two-asset model and the stochastic interest rate models. Since our schemes are based on the uncorrelated stochastic differential equations, they need less nodes to generate
the lattice and thus can be easily implemented. Numerical results are also presented to examine our methods and the early exercise boundaries for the American options.

• **Chapter 5: A Finite Volume - Alternative Direction Implicit Method.**

We develop a finite volume - alternating direction implicit method for the transformed American option problem under the stochastic volatility model (the Heston model). Numerical results show that the method provides fast and accurate approximations of option prices for all the possible combinations of the model parameters.

• **Chapter 6: Conclusion.**

We summarize the dissertation and propose several future research topics.
CHAPTER 2

DECOUPLING MULTI-FACTOR MODELS

In this chapter, we shall introduce a transformation to decouple correlated stochastic processes governed by a system of stochastic differential equations. Hence, the option prices can be evaluated by the nested expectations and the partial differential equations without the mixed terms.

2.1 Decoupling the Correlated Stochastic Processes

Consider the following system of stochastic differential equations

\[ dX_i(t) = \phi_i(t, X(t)) \, dt + \psi_i(t, X(t)) \, dB_i(t), \quad i = 1, \ldots, n, \] (2.1)

where \( X(t) = (X_1(t), \ldots, X_n(t))^T \) and \( B_1(t), \ldots, B_n(t) \) are Wiener processes. Let

\[
\Sigma = \begin{bmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\
\rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2n} \\
\rho_{31} & \rho_{32} & 1 & \cdots & \rho_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & 1
\end{bmatrix},
\]

where

\[ dB_i(t) dB_j(t) = \rho_{ij} \, dt. \]
It is known that $\Sigma$ is a positive definite matrix. Thus it admits the following Cholesky decomposition

$$\Sigma = AA^T,$$

where $A = (a_{ij})$ is a lower triangular matrix with positive diagonal entries. For $n = 2$ and $n = 3$, we have

$$A = \begin{bmatrix} 1 & 0 \\
\rho_{12} & \sqrt{1 - \rho_{12}^2} \end{bmatrix}, \quad (2.2)$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 \\
\rho_{12} & \sqrt{1 - \rho_{12}^2} & 0 \\
\rho_{13} & \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}} & \sqrt{1 - \rho_{13}^2} - \left( \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}} \right)^2 \end{bmatrix}. \quad (2.3)$$

Let

$$W(t) = \begin{bmatrix} W_1(t) \\
\vdots \\
W_n(t) \end{bmatrix} = A^{-1} \begin{bmatrix} B_1(t) \\
\vdots \\
B_n(t) \end{bmatrix}.$$  

It is easy to verify that $W$ is a $n$-dimensional Brownian motion. Assume that

$$\psi_i (t, X(t)) = \lambda_i \psi (t, X(t)) , \quad i = 1, \ldots, n, \quad (2.4)$$

where $\lambda_1, \ldots, \lambda_n$ are constants. Then the system (2.1) can be rewritten to

$$dX(t) = \Phi (t, X(t)) dt + \psi(t, X(t)) \Lambda AdW(t), \quad (2.5)$$

where $\Phi(t, X(t)) = (\phi_1(t, X(t)), \ldots, \phi_n(t, X(t)))^T$ and $\Lambda = \text{diag} (\lambda_1, \ldots, \lambda_n)$. Let

$$Y(t) = \begin{bmatrix} Y_1(t) \\
\vdots \\
Y_n(t) \end{bmatrix} = BX(t), \quad (2.6)$$

where $B = \Lambda D(\Lambda A)^{-1}$ and $D = \text{diag} (a_{11}, \ldots, a_{nn})$. Then we have by (2.5)

$$dY(t) = \Lambda D(\Lambda A)^{-1} \Phi (t, B^{-1}Y(t)) dt + \psi (t, B^{-1}Y(t)) \Lambda DdW(t). \quad (2.7)$$

It is apparent that the new processes $Y_1(t), \ldots, Y_n(t)$ are mutually uncorrelated.
Remark 2.1. when $n = 2$, we may assume that $\lambda_1$ is a function of $X_1$ instead of a constant.

Let

$$Y_1(t) = X_1(t), \quad Y_2(t) = X_2(t) - F(X_1(t)),$$

where

$$F(x) = a_{21} \int_0^x \lambda_1(u) du.$$ 

Then we have by (2.5)

$$dY_1(t) = \phi_1(t, X(t)) dt + \psi_1(t, X(t)) dW_1(t),$$

$$dY_2(t) = a_{21} \lambda_1(X_1(t)) dX_1(t) - \frac{1}{2} a_{21} (\lambda'_1(X_1(t)))^2 (\psi_1(t, X(t)))^2 dt$$

$$= \tilde{\phi}_2(t, X(t)) dt + a_{22} \sqrt{1 - \rho_{12}^2} \psi_2(t, X(t)) dW_2(t),$$

where

$$\tilde{\phi}_2(t, X(t)) = \phi_2(t, X(t)) - a_{21} \lambda_1(X_1(t)) \phi_1(t, X(t)) - \frac{1}{2} (\lambda'_1(X_1(t)))^2 (\psi_1(t, X(t)))^2.$$

2.2 Two-Factor Models

In this section, we will apply the decoupling transformation in the previous section to the various popular two-factor models in asset pricing.

2.2.1 The Two-Asset Model

The two-asset model reads as follows

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i) dt + \sigma_i dB_i(t), \quad i = 1, 2,$$

(2.8)

where $r$ is the risk-free interest rate, $q_i$ is the dividend rate for the $i$-th asset, $\sigma_i$ is the volatility, and the Wiener processes $B_1(t)$ and $B_2(t)$ are correlated with the correlation
$dB_1(t)dB_2(t) = \rho dt$. Let

$$X_i(t) = \ln \left( \frac{S_i(t)}{S_i(0)} \right), \quad i = 1, 2.$$  \hfill (2.9)

where $S_i(0)$ is a given asset price. Then we have by Ito’s Lemma

$$dX_i(t) = \left( r - q_i - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i dB_i(t), \quad i = 1, 2.$$  

Notice that

$$\psi(t, X(t)) \equiv 1, \quad \lambda_i = \sigma_i, \quad i = 1, 2.$$  

Let

$$\tilde{\sigma}_2 = \sqrt{1 - \rho^2 \sigma_2}, \quad \tilde{q}_2 = q_2 + \frac{1}{2} \rho^2 \sigma_2^2 + \frac{\rho \sigma_2}{\sigma_1} \left( r - q_1 - \frac{1}{2} \sigma_1^2 \right),$$  

$$\mu_1 = r - q_1 - \frac{1}{2} \sigma_1^2, \quad \mu_2 = r - \tilde{q}_2 - \frac{1}{2} \tilde{\sigma}_2^2.$$  

Then we have by (2.2), (2.6) and (2.7)

$$Y_1(t) = X_1(t),$$  \hfill (2.10)

$$Y_2(t) = -\frac{\rho \sigma_2}{\sigma_1} X_1(t) + X_2(t),$$  \hfill (2.11)

and

$$dY_1(t) = \mu_1 dt + \sigma_1 dW_1(t),$$  \hfill (2.12)

$$dY_2(t) = \mu_2 dt + \tilde{\sigma}_2 dW_2(t).$$  \hfill (2.13)

**Remark 2.2.** We are referred to [27] for a similar transformation.
2.2.2 The Stochastic Volatility Model

The most popular stochastic volatility model is also known as the Heston model ([33]). It models one asset price process and its variance as follows

\[
\begin{align*}
    dv(t) &= \kappa (\eta - v(t)) \, du + \sigma \sqrt{v(t)} dB_1(t), \\
    \frac{dS(t)}{S(t)} &= (r - q) \, dt + \sqrt{v(t)} dB_2(t),
\end{align*}
\]

where \( r \) is the risk-free interest rate, \( q \) is the dividend rate for the asset, \( \sigma \) is the volatility of volatility, and the Wiener processes \( B_1(t) \) and \( B_2(t) \) are correlated with the correlation \( dB_1(t) dB_2(t) = \rho dt \). Let

\[
X_1(t) = v(t), \quad X_2(t) = \ln \left( \frac{S(t)}{K} \right),
\]

where \( K \) is the strike price. Then we have by Ito’s Lemma

\[
\begin{align*}
    dX_1(t) &= \kappa (\eta - X_1(t)) \, du + \sigma \sqrt{X_1(t)} dB_1(t), \\
    dX_2(t) &= \left( r - q - \frac{1}{2} X_1(t) \right) \, dt + \sqrt{X_1(t)} dB_2(t).
\end{align*}
\]

Notice that

\[
\psi(t, X(t)) = \sqrt{X_1(t)}, \quad \lambda_1 = \sigma, \quad \lambda_2 = 1.
\]

We have by (2.6) and (2.7)

\[
\begin{align*}
    Y_1(t) &= X_1(t), \quad Y_2(t) = -\rho \frac{\sigma}{\sigma} X_1(t) + X_2(t),
\end{align*}
\]

and

\[
\begin{align*}
    dY_1(t) &= (a_1 + b_1 Y_1(t)) \, dt + \sigma \sqrt{Y_1(t)} dW_1(t), \\
    dY_2(t) &= (a_2 + b_2 Y_1(t)) \, dt + \sqrt{(1 - \rho^2) Y_1(t)} dW_2(t),
\end{align*}
\]
where
\[ \rho = \rho_{12}, \quad a_1 = \kappa \eta, \quad b_1 = -\kappa, \quad a_2 = r - q - \frac{\rho}{\sigma} \kappa \eta, \quad b_2 = \frac{\rho}{\sigma} \kappa - \frac{1}{2}. \]

**Remark 2.3.** We are referred to [4] for a similar transformation.

### 2.2.3 The Stochastic Interest Rate Model

The interest rates may be assumed to follow a stochastic process. Here we consider two popular interest rate models: the CIR model ([13]) and Vasicek model ([58]). The coupled stochastic differential equations for the asset price and interest rate are as follows

\[
\begin{align*}
  dr(t) &= \kappa(\theta - r(t))dt + v(r(t))^p dB_1(t), \\
  \frac{dS(t)}{S(t)} &= (r(t) - q)dt + \sigma dB_2(t),
\end{align*}
\]  

(2.20) (2.21)

where \(q\) is the dividend rate, \(\theta\) is the long-term expectation of interest rate, \(\kappa > 0\) is the speed of mean reversion, \(\sigma\) is the volatility of the stock price, and \(v > 0\). It is the Vasicek model and the CIR model when \(p = 0\) and \(p = \frac{1}{2}\), respectively.

Let

\[
X_1(t) = (r(t))^{1-p}, \quad X_2(t) = \ln \left( \frac{S(t)}{K} \right),
\]  

(2.22)

where \(K\) is the strike price. Then we have by Ito’s Lemma

\[
\begin{align*}
  dX_1(t) &= \mu_1(X_1(t))dt + \sigma_1 dB_1(t), \\
  dX_2(t) &= \left( (X_1(t))^{1-p} - \frac{1}{2} \sigma^2 \right) dt + \sigma dB_2(t),
\end{align*}
\]

where

\[
\mu_1(x) = (1 - p)\kappa \left( \frac{\theta}{x^{1-p}} - x \right) - \frac{p(1 - p)v^2}{2x}, \quad \sigma_1 = (1 - p)v.
\]
Since
\[ \psi(t, X(t)) = 1, \quad \lambda_1 = (1 - p)v, \quad \lambda_2 = \sigma, \]
we have by (2.6) and (2.7)
\[ Y_1(t) = X_1(t), \quad Y_2(t) = -\frac{\rho \sigma}{(1 - p)v} X_1(t) + X_2(t), \quad (2.23) \]
and
\[ dY_1(t) = \mu_1 (Y_1(t)) dt + \sigma_1 dW_1(t), \quad (2.24) \]
\[ dY_2(t) = \mu_2 (Y_1(t)) dt + \sigma_2 dW_2(t). \quad (2.25) \]
where
\[ \mu_2(y) = y^{1-p} - q - \frac{1}{2} \sigma^2 - \frac{\rho \sigma}{(1 - p)v} \mu_1(y). \quad \sigma_2 = \sigma \sqrt{1 - \rho^2}, \]

**Remark 2.4.** When \( p = \frac{1}{2} \), using the transformation in Remark 2.1, we obtain
\[ Y_1(t) = X_1(t), \quad Y_2(t) = X_2(t) - \frac{2\rho \sigma}{v} \sqrt{X_1(t)}, \quad (2.26) \]
and
\[ dY_1(t) = \kappa (\theta - Y_1(t)) dt + v \sqrt{Y_1(t)} dW_1(t), \quad (2.27) \]
\[ dY_2(t) = g(Y_1(t)) dt + \sigma \sqrt{1 - \rho^2} dW_2(t), \quad (2.28) \]
where
\[ g(Y_1(t)) = \left( Y_1(t) - q - \frac{1}{2} \sigma^2 - \frac{2\rho \sigma \kappa}{v \sqrt{Y_1(t)}} (\theta - Y_1(t)) + \frac{\rho \sigma v}{4 \sqrt{Y_1(t)}} \right). \]
CHAPTER 3

MIXED MC METHODS WITH CONTROL VARIATES

The mixed Monte Carlo method was first introduced by Loeper and Pironneau[44] for stochastic volatility model. Cozma and Reisinger extended the method into Heston-CIR model [16]. In their paper, the stochastic volatility/interest rate process are simulated using Monte Carlo method, while the option values based on the the asset prices are computed via PDE/Analytic method. However, their simulation processes and asset price process are in fact not independent. In this chapter, the mixed Monte Carlo method is developed based on our decoupled stochastic processes in Chapter 2. We shall show that the rational prices of the European contingent claims under various two-factor models can be expressed as the nested expectations. The inner expectation is the price of the European contingent claim for an artificial asset and can be analytically evaluated by the Black-Scholes formula. We also use the method of control variates to reduce the variance.

3.1 The Two-Asset Model

Solving the stochastic differential equations (2.12) and (2.13), we get
\[ Y_1(t, T) = \left( r - q_1 - \frac{\sigma_1^2}{2} \right) (T - t) + \sigma_1 (W_1(T) - W_1(t)), \]
\[ Y_2(t, T) = \left( r - \tilde{q}_2 - \frac{\tilde{\sigma}_2^2}{2} \right) (T - t) + \tilde{\sigma}_2 (W_2(T) - W_2(t)). \]

Then we have by (2.9), (2.10) and (2.11)

\[ S_1(T) = S_1(t) e^{Y_1(t, T)}, \quad S_2(T) = S_2(t) e^{\alpha Y_1(t, T) + Y_2(t, T)}, \quad (3.1) \]

where \( \alpha = \frac{\sigma_2}{\sigma_1} \). We introduce the artificial asset price process

\[ \tilde{S}_2(t) = \tilde{S}_2(0) e^{Y_2(0, t)}, \]

where

\[ \tilde{S}_2(0) = \frac{S_2(0)}{S_1(0)}, \]

which satisfies the SDE

\[ \frac{d\tilde{S}_2(t)}{\tilde{S}_2(t)} = (r - \tilde{q}_2) dt + \tilde{\sigma}_2 dW_2(t). \]

Then we have

\[ S_2(t) = S_1^\alpha(t) \tilde{S}_2(t), \quad t \in [0, T]. \]

Let \( \Phi(S_1(T), S_2(T)) \) be the payoff of a European contingent claim. Then its price at time \( t \) is given by

\[ V(S_1, S_2, t; T) = \mathbb{E} \left[ e^{-(T-t)r} \Phi(S_1(T), S_2(T)) \bigg| S_1(t) = S_1, S_2(t) = S_2 \right] \quad (3.2) \]

\[ = \mathbb{E} \left[ e^{-(T-t)r} \Phi \left( S_1(T), S_1^\alpha(T) \tilde{S}_2(T) \right) \bigg| S_1(t) = S_1, \tilde{S}_2(t) = S_1^{-\alpha} S_2 \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ e^{-(T-t)r} \Phi \left( S_1(T), S_1^\alpha(T) \tilde{S}_2(T) \right) \bigg| \tilde{S}_2(t) = S_1^{-\alpha} S_2 \right] \bigg| S_1(t) = S_1 \right] \]

\[ = \mathbb{E} \left[ \tilde{V} \left( S_1^{-\alpha} S_2, S_1(T), t, T \right) \bigg| S_1(t) = S_1 \right], \quad (3.3) \]
where

\[ \tilde{V}(\tilde{S}_2, z, t, T) = \mathbb{E} \left[ e^{-(T-t)r} \Phi \left( z, z^\alpha \tilde{S}_2(T) \right) \right] \tilde{S}_2(t) = \tilde{S}_2 . \] (3.4)

In the appendix, we shall work out the analytic formulas for \( \tilde{V} \) for various payoff functions \( \Phi \).

Our mixed Monte Carlo method (MMC) is based on evaluating the expectations in (3.3) while \( \tilde{V} \) is computed by using the analytic formula (see the appendix). The first algorithm is the crude Monte Carlo method:

**Algorithm 1.** A MMC method for the two-asset European contingent claim

1. Initialize positive integer \( N \) as the number of simulations. Set \( V = 0 \).

2. For \( n = 1, 2, \ldots, N \), do
   
   - Simulate \( Y_1 = Y_1(t, T) \) and compute \( \tilde{S}_1 = S_1 e^{Y_1} \).
   
   - Compute \( P = \tilde{V} \left( S_1^{-\alpha} S_2, \tilde{S}_1, t, T \right) \).
   
   - Let \( V = V + P \).

   End do.

3. The approximate value of the price is \( \bar{V} = V/N \).

To speed up the crude Monte Carlo method, we use the technique of control variates to reduce the variance of the random variable \( \tilde{V} \left( S_1^{-\alpha} S_2, S_1(T), t, T \right) \). It is apparent that \( \tilde{V} \) is highly correlated to \( Y_1(T) \). Hence we naturally take it for the control variates. Our algorithm for the mixed Monte Carlo method with the control variates (MMCCV) is as follows:
Algorithm 2. A MMCCV method for the two-asset European contingent claim

1. Initialize positive integers $N$ as the number of simulations. Set

$$V = b_1 = b_2 = b_3 = 0$$

and

$$\bar{Y}_1 = \mathbb{E}[Y_1(t, T)] = \left(r - q_1 - \frac{1}{2}\sigma_1\right)(T-t).$$

2. For $n = 1, 2, \ldots, N$, do
   - Simulate $Y_1 = Y_1(t, T)$ and let $\bar{S}_1 = S_1e^{Y_1}$.
   - Compute $P = \tilde{V}\left(S_1^{-a}S_2, \bar{S}_1, t, T\right)$.
   - Let
     $$b_1 = b_1 + P(Y_1 - \bar{Y}_1),$$
     $$b_2 = b_2 + (Y_1 - \bar{Y}_1),$$
     $$b_3 = b_3 + (Y_1 - \bar{Y}_1)^2,$$
     $$V = V + P.$$ 
   End do.

3. Let $\bar{V} = V/N$, $b = (b_1 - \bar{V}b_2)/b_3$. The approximate value of the option price is $V^* = \bar{V} - bb_3/N$.

Remark 3.1. We use control variates technique for variance reduction according to Glasserman’s book [28]. There are other techniques for variance reduction, like stratified sampling and Latin hypercube sampling. It shall be pointed out that variance reduction techniques usually introduce dependence across replications, but the dependence from control variate technique becomes negligible as the number of simulation increases, compared with other techniques.

Remark 3.2. According to the central limit theorem, we expect that the error from Monte
Carlo simulations is $O\left(N^{-\frac{1}{2}}\right)$.

### 3.2 The Stochastic Volatility Model

We only consider the European call option since the European put options can be treated similarly. Let $K$ be the strike price and $T$ be the expiration date of the option.

Using (2.16), (2.17), (2.18), and (2.19), we have

$$S(T) = S(t)e^{X(t,T)+Y(t,T)}, \quad (3.5)$$

where

$$X(t, T) = (r - q)(T - t) - \frac{1}{2}\lambda^2 \int_t^T v(s) ds + \lambda \int_t^T \sqrt{v(s)} dW_2(s), \quad (3.6)$$

$$Y(t, T) = -\frac{1}{2}\rho^2 \int_t^T v(s) ds + \rho \int_t^T \sqrt{v(s)} dW_1(s), \quad (3.7)$$

where $\lambda = \sqrt{1 - \rho^2}$.

Consider the artificial asset price process $\tilde{S}(t) = \tilde{S}(0)e^{X(0,t)}$ which is the solution to the following stochastic differential equation

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = (r - q)dt + \lambda \sqrt{v(t)} dW_2(t).$$

If a path $\{v(s) : t \leq s \leq T\}$ of the volatility is given, then the European call option price for this stock is given by

$$\tilde{V} \left(\tilde{S}, t, T\right) = \mathbb{E} \left[ e^{-(T-t)r} \left( \tilde{S}(t)e^{X(t,T)} - K \right)^+ \Big| \tilde{S}(t) = \tilde{S} \right]$$

$$= e^{-q(T-t)}\tilde{S}N(d_1) - e^{-r(T-t)}KN(d_2), \quad (3.8)$$
where

\[
d_1 = \ln\left(\frac{\tilde{S}}{K}\right) + (r - q + \frac{1}{2}\sigma^2)(T - t)\tilde{\sigma} \sqrt{T - t},
\]

\[
d_2 = d_1 - \tilde{\sigma} \sqrt{T - t}, \quad \tilde{\sigma} = \left(\frac{\lambda}{T - t} \int_t^T v(s) ds\right)^{\frac{1}{2}}.
\]

Then the European call option price for the Heston model is given by

\[
V(S, v, t, T) = \mathbb{E}\left[e^{-r(T-t)} (S(t)e^{X(t,T)} + Y(t,T) - K)^+ \mid S(t) = S, v(t) = v\right]
\]

\[
= \mathbb{E}\left[\mathbb{E}\left[e^{-(T-t)r} (\tilde{S}(t)e^{X(t,T)} + Y(t,T) - K)^+ \mid \tilde{S}(t) = Se^{Y(t,T)}\right] v(t) = v\right]
\]

\[
= \mathbb{E}\left[\tilde{V} (Se^{Y(t,T)} , t, T) \mid v(t) = v\right].
\] (3.9)

As in the previous section, we have the following crude Monte Carlo algorithm:

---

**Algorithm 3.** A MMC method for the Heston European call option

1. Initialize positive integer \( N \) as the number of simulations. Set \( V = 0 \).

2. For \( n = 1, 2, \ldots, N \), do
   
   - Simulate \( Y = Y(t, T) \) and compute \( \tilde{S} = Se^{Y} \tilde{\sigma} \).
   
   - Compute \( P = \tilde{V} (\tilde{S}, t, T) \).
   
   - Let \( V = V + P \).

   End do.

3. The approximate value of the option price is \( \bar{V} = V/N \).

---

In order to reduce the variance, we use \( Y(t, T) \) for the control variates. To this purpose, we need \( \bar{V} = \mathbb{E}[Y(t, T)|v(t) = v] \). Taking the expected value on both sides of (3.7) and solving the resulting differential equation gives

\[
\mathbb{E}[v(s)|v(t) = v] = \eta + (v - \eta)e^{-rt}.
\]
Then we have by (3.7)

\[
\bar{Y} = \mathbb{E} \left[ -\frac{1}{2} \rho^2 \int_t^T v(s) ds + \rho \int_t^T \sqrt{v(s)} dW_1(s) \right] = -\frac{1}{2} \rho^2 \int_t^T \mathbb{E} [v(s)] ds
\]

\[
= -\frac{1}{2} \left( \eta(T - t) + \frac{v - \eta}{\kappa} (1 - e^{-\kappa(T-t)}) \right).
\]

Our algorithm for the mixed Monte Carlo method with the control variates (MMCCV) is as follows:

---

**Algorithm 4.** A MMCCV method for the Heston European call option

1. Initialize positive integer \( N \) as the number of simulations. Set \( V = 0, \ b_1 = 0, \ b_2 = 0, \) and \( b_3 = 0. \)

2. For \( n = 1, 2, \ldots, N, \) do
   - Simulate \( Y = Y(t, T) \) and compute \( \tilde{S} = Se^{Y} \tilde{\sigma}. \)
   - Compute \( P = \tilde{V} \left( \tilde{S}, t, T \right). \)
   - Let
     \[
     b_1 = b_1 + P(Y - \bar{Y}),
     \]
     \[
     b_2 = b_2 + (Y - \bar{Y}),
     \]
     \[
     b_3 = b_3 + (Y - \bar{Y})^2,
     \]
     \[
     V = V + P.
     \]

   End do.

3. Let \( \bar{V} = V/N, \ b = (b_1 - \bar{V}b_2)/b_3. \) The approximate value of the option price is \( V^* = \bar{V} - bb_3/N. \)

---

**Remark 3.3.** Consider the stochastic volatility model with jumps introduced by Bates in
1996 ([3])

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r - q - \lambda \zeta) dt + \sqrt{v(t)} dB_1(t) + dZ(t), \\
\quad \text{where } Z(t) = \sum_{n=1}^{N(t)} \left(e^{J_n} - 1\right), \\
\quad N(t) \text{ is a Poisson process with intensity } \lambda \text{ and independent of} \\
\quad \text{the Brownian motions } B_1(t) \text{ and } B_2(t), \\
\quad \{J_n\}_1^\infty \text{ is a sequence of independent and identically} \\
\quad \text{distributed normal random variables with the mean } \ln(1 + \zeta) \text{ and variance } \delta^2, \quad \text{and } \mu_J = \mathbb{E} \left[e^{\eta J} - 1\right] \text{ is the expected jump percentage. By Itô’s formula, we get} \\
S(T) &= S(t)e^{X(t,T)+Y(t,T)+Z(t,T)},
\end{align*}
\]

(3.10)

where \(X(t, T)\) and \(Y(t, T)\) are defined in (3.7) and (3.7), and \(Z(t, T) = \sum_{n=N(t)}^{N(T)} J_n - \lambda J \mu_J (T - t)\). Hence, we can apply the above two algorithms to the call option under the Bates model while \(Y(t, T)\) is replaced by \(Y(t, T) + Z(t, T)\).

### 3.3 The Stochastic Interest Rate Model

We only consider the Vasicek model and the CIR model can be treated similarly. Using (2.22), (2.23), (2.24), and (2.25), we have

\[
S(T) = S(t)e^{X(t,T)+Y(t,T)},
\]

where

\[
\begin{align*}
X(t, T) &= \int_t^T (r(s) - q) ds - \frac{1}{2} \lambda^2 \sigma^2 (T - t) + \lambda \sigma (W_2(T) - W_2(t)), \\
Y(t, T) &= -\frac{1}{2} \rho^2 \sigma^2 (T - t) + \rho \sigma (W_1(T) - W_1(t)).
\end{align*}
\]

(3.11) (3.12)
Define artificial asset price process \( \tilde{S}(t) = S(0)e^{X(0,t)} \) that follows the SDE

\[
\frac{d\tilde{S}(t)}{\tilde{S}(t)} = (r(t) - q)dt + \lambda\sigma dW_2(t).
\]

If a path \( \{ r(s) : t \leq s \leq T \} \) of the interest rate is given, then the European call option price for this asset is given by

\[
\tilde{V}(\tilde{S}, t, T) = \mathbb{E}\left[ e^{-\int_t^T r(s)ds} \left( \tilde{S}(t)e^{X(t,T)} - K \right)^+ \left| \tilde{S}(T) = \tilde{S} \right. \right],
\]

which can be computed by a closed form formula. Then the European call option price for the stochastic interest rate model is given by

\[
V(S, r, t) = \mathbb{E}\left[ e^{-\int_t^T r(s)ds} \left( S(t)e^{X(t,T)+Y(t,T)} - K \right)^+ \left| S(t) = S, r(t) = r \right. \right]
\]

\[
= \mathbb{E}\left[ \mathbb{E}\left[ e^{-\int_t^T r(s)ds} \left( \tilde{S}(t)e^{X(t,T)} - K \right)^+ \left| \tilde{S}(t) = S e^{Y(t,T)} \right. \right] \left| r(t) = r \right. \right]
\]

\[
= \mathbb{E}\left[ \tilde{V}(S e^{Y(t,T)}, t, T) \left| r(t) = r \right. \right].
\]

Hence, we have the algorithms similar to Algorithms 3 – 4 to compute the above expectation for the call price.

### 3.4 Numerical Results

In this section, we present numerical examples to examine the convergence and accuracy of the proposed MMC and MMCCV methods in the previous sections. We only consider the European call options due to the put-call parity. For convenience, we introduce the following notations.
<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>REF</td>
<td>Reference value</td>
</tr>
<tr>
<td>CV</td>
<td>Numerical result using MMCCV</td>
</tr>
<tr>
<td>MMC</td>
<td>Numerical result using MMC</td>
</tr>
<tr>
<td>RMSE</td>
<td>Root mean square error</td>
</tr>
<tr>
<td>AE</td>
<td>Absolute error</td>
</tr>
<tr>
<td>MAE</td>
<td>Maximum absolute error</td>
</tr>
<tr>
<td>N</td>
<td>Number of MC simulations</td>
</tr>
<tr>
<td>M</td>
<td>Number of time steps for each simulation</td>
</tr>
</tbody>
</table>

Table 3.1: Notations

**Example 3.1.** *(The two-asset model)* In this example, we consider the European spread option, a popular two-asset option with payoff $\Phi(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+$. The parameters are given in Table 3.2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>$15$</td>
</tr>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$T$</td>
<td>1.0 year</td>
</tr>
<tr>
<td>$q_1$</td>
<td>0.03</td>
</tr>
<tr>
<td>$q_2$</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.10</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Table 3.2: The parameters for the spread option

Using the processes $Y_1(t)$ and $Y_2(t)$, we have for the value of the European contingent claim in (3.2)

$$V(S_1, S_2, t; T) = \mathbb{E} \left[ e^{-(T-t)r} \Phi(Y_1(T), Y_2(T)) \bigg| Y_1(t) = Y_1, Y_2(t) = Y_2 \right]. \tag{3.14}$$

Since $Y_1$ and $Y_2$ follow independent normal distributions, the above expectation can be computed by numerical integration and will be taken as the reference values.

We first examine the rate of convergence with respect to the number of simulations $N$. 26
We display the maximum absolute errors and root mean square errors for $S_1 = 100$ and $S_2 = 50 : 5 : 120$ in Figs. 3.1–3.2, respectively. We can observe that the rate of convergence is about $\frac{1}{2}$ as expected. It is also shown that the MMCCV method is about 10 times accurate as the MMC method, which is due to the variances have been reduced significantly (see Fig. 3.3 for $S = 100$, $N = 2000$, $M = 1000$). We display the option prices and their errors in Table 3.3, which shows that the MMCCV method provides very accurate approximations of option prices even with a small number of simulations.

![Figure 3.1: MAE vs N for the European spread option](image-url)
Figure 3.2: RMSE vs $N$ for the European spread option

Figure 3.3: The variance ratios vs $S_2$ for the European spread option.
Table 3.3: The European spread option prices: $S_1 = 100$, $N = 2000$, $M = 1000$.

Example 3.2. (The stochastic volatility model) In this example, we consider the European call options under the Heston model with the parameters in Table 3.4. We use the option prices computed by the Bates’ formula in [3] with numerical integration as the reference values.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>100</td>
</tr>
<tr>
<td>$r$</td>
<td>0.05</td>
</tr>
<tr>
<td>$t_0$</td>
<td>0.0</td>
</tr>
<tr>
<td>$T$</td>
<td>1.0</td>
</tr>
<tr>
<td>$q$</td>
<td>0.00</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.09</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Table 3.4: The parameters for the Heston model

We display the maximum absolute errors and root mean square errors for $v = 0.09$ and
$S = 50 : 5 : 150$ in Figs. 3.4–3.5, respectively. Again, we can observe that the rate of convergence is about $\frac{1}{2}$ as expected. It is also shown that the MMCCV method is about 10 times accurate as the MMC method due to the variance reduction (see Fig. 3.6 for $\nu = 0.09$, $N = 2000$, $M = 1000$). The option prices and their errors in Table 3.5 show that the MMCCV method provides very accurate approximations of option prices even with a small number of simulations.

![Figure 3.4: MAE vs $N$ for the Heston model](image-url)
Figure 3.5: RMSE vs $N$ for the Heston model

Figure 3.6: The variance ratios vs $S$ for the Heston model
Table 3.5: The European option prices (Heston): $v = 0.09$, $N = 2000$, $M = 1000$.

<table>
<thead>
<tr>
<th>S</th>
<th>REF</th>
<th>CV</th>
<th>MMC</th>
<th>AE-CV</th>
<th>AE-MMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>1.935153</td>
<td>1.882274</td>
<td>1.822402</td>
<td>0.052879</td>
<td>0.112751</td>
</tr>
<tr>
<td>70</td>
<td>2.604112</td>
<td>2.579329</td>
<td>2.539507</td>
<td>0.024783</td>
<td>0.064605</td>
</tr>
<tr>
<td>75</td>
<td>3.458298</td>
<td>3.438213</td>
<td>3.409161</td>
<td>0.020085</td>
<td>0.049137</td>
</tr>
<tr>
<td>80</td>
<td>4.547084</td>
<td>4.520669</td>
<td>4.684157</td>
<td>0.026415</td>
<td>0.137073</td>
</tr>
<tr>
<td>85</td>
<td>5.934010</td>
<td>6.014273</td>
<td>5.946144</td>
<td>0.080263</td>
<td>0.012134</td>
</tr>
<tr>
<td>90</td>
<td>7.695986</td>
<td>7.680517</td>
<td>7.366249</td>
<td>0.015469</td>
<td>0.329737</td>
</tr>
<tr>
<td>95</td>
<td>9.912327</td>
<td>9.894834</td>
<td>9.548330</td>
<td>0.017493</td>
<td>0.363997</td>
</tr>
<tr>
<td>100</td>
<td>12.637215</td>
<td>12.662328</td>
<td>12.543154</td>
<td>0.025113</td>
<td>0.094061</td>
</tr>
<tr>
<td>105</td>
<td>15.866950</td>
<td>15.898040</td>
<td>16.087151</td>
<td>0.031090</td>
<td>0.220201</td>
</tr>
<tr>
<td>110</td>
<td>19.535526</td>
<td>19.493760</td>
<td>19.365479</td>
<td>0.041766</td>
<td>0.170047</td>
</tr>
<tr>
<td>115</td>
<td>23.546932</td>
<td>23.595580</td>
<td>23.958395</td>
<td>0.048648</td>
<td>0.411463</td>
</tr>
<tr>
<td>120</td>
<td>27.810157</td>
<td>27.885419</td>
<td>27.509502</td>
<td>0.075262</td>
<td>0.300655</td>
</tr>
<tr>
<td>125</td>
<td>32.254179</td>
<td>32.266008</td>
<td>31.715033</td>
<td>0.011829</td>
<td>0.539146</td>
</tr>
<tr>
<td>130</td>
<td>36.828075</td>
<td>36.790434</td>
<td>37.161237</td>
<td>0.037641</td>
<td>0.333162</td>
</tr>
<tr>
<td>135</td>
<td>41.496362</td>
<td>41.442971</td>
<td>41.244130</td>
<td>0.053391</td>
<td>0.252232</td>
</tr>
<tr>
<td>MAX</td>
<td></td>
<td></td>
<td>0.080263</td>
<td>0.539146</td>
<td></td>
</tr>
</tbody>
</table>

Example 3.3. (The stochastic interest rate model) In this example, we consider the European call options under the stochastic interest rate model. We shall assume that the interest rate follows the Vasicek process and use the option prices computed by the analytic formula in Fang's paper [23] as the reference values. The parameters are given in Table 3.6. We display the maximum absolute errors and root mean square errors for $r = 0.11$ and $S = 50 : 5 : 150$ in Figs. 3.7–3.8, respectively. The variance ratios are displayed in Fig. 3.9 for $r = 0.11$, $N = 2000$, $M = 1000$. The option prices and their errors are presented in Table 3.7. Again, we have the same observations as in Examples 3.1–3.2.
### Table 3.6: The parameters for the Vasicek model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>Changing</td>
</tr>
<tr>
<td>$K$</td>
<td>100</td>
</tr>
<tr>
<td>$r_0$</td>
<td>0.11</td>
</tr>
<tr>
<td>$t_0$</td>
<td>0.0</td>
</tr>
<tr>
<td>$T$</td>
<td>1.0</td>
</tr>
<tr>
<td>$q$</td>
<td>0.00</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.20</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>2.00</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.07</td>
</tr>
<tr>
<td>$v$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Figure 3.7: MAE vs $N$ for the Vasicek model
Figure 3.8: RMSE vs $N$ for the Vasicek model

Figure 3.9: The variance ratios vs $S$ for the Vasicek model
<table>
<thead>
<tr>
<th>S</th>
<th>REF</th>
<th>CV</th>
<th>MMC</th>
<th>AE-CV</th>
<th>AE-MMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>0.190539</td>
<td>0.193729</td>
<td>0.197276</td>
<td>0.003190</td>
<td>0.006737</td>
</tr>
<tr>
<td>70</td>
<td>0.506896</td>
<td>0.512861</td>
<td>0.521067</td>
<td>0.005965</td>
<td>0.014171</td>
</tr>
<tr>
<td>75</td>
<td>1.131576</td>
<td>1.141477</td>
<td>1.157507</td>
<td>0.009901</td>
<td>0.025931</td>
</tr>
<tr>
<td>80</td>
<td>2.193787</td>
<td>2.207401</td>
<td>2.178983</td>
<td>0.013614</td>
<td>0.014804</td>
</tr>
<tr>
<td>85</td>
<td>3.795243</td>
<td>3.812772</td>
<td>3.769229</td>
<td>0.017529</td>
<td>0.026014</td>
</tr>
<tr>
<td>90</td>
<td>5.987420</td>
<td>6.008064</td>
<td>5.947071</td>
<td>0.020644</td>
<td>0.040349</td>
</tr>
<tr>
<td>95</td>
<td>8.765632</td>
<td>8.788288</td>
<td>8.708736</td>
<td>0.022656</td>
<td>0.056896</td>
</tr>
<tr>
<td>100</td>
<td>12.078444</td>
<td>12.022113</td>
<td>12.172700</td>
<td>0.056331</td>
<td>0.094256</td>
</tr>
<tr>
<td>105</td>
<td>15.845353</td>
<td>15.788699</td>
<td>15.966250</td>
<td>0.056654</td>
<td>0.120897</td>
</tr>
<tr>
<td>110</td>
<td>19.975172</td>
<td>19.919495</td>
<td>20.121899</td>
<td>0.055677</td>
<td>0.146727</td>
</tr>
<tr>
<td>115</td>
<td>24.380248</td>
<td>24.326120</td>
<td>24.551069</td>
<td>0.054128</td>
<td>0.170821</td>
</tr>
<tr>
<td>120</td>
<td>28.984916</td>
<td>28.932356</td>
<td>29.177691</td>
<td>0.052560</td>
<td>0.192775</td>
</tr>
<tr>
<td>125</td>
<td>33.728793</td>
<td>33.683015</td>
<td>33.411174</td>
<td>0.045778</td>
<td>0.317619</td>
</tr>
<tr>
<td>130</td>
<td>38.566558</td>
<td>38.521009</td>
<td>38.231306</td>
<td>0.045549</td>
<td>0.335252</td>
</tr>
<tr>
<td>135</td>
<td>43.465823</td>
<td>43.420008</td>
<td>43.113595</td>
<td>0.045815</td>
<td>0.352228</td>
</tr>
<tr>
<td>MAX</td>
<td></td>
<td></td>
<td></td>
<td>0.056654</td>
<td>0.352228</td>
</tr>
</tbody>
</table>

Table 3.7: The European option prices (Vasicek): $r = 0.11$, $N = 2000$, $M = 1000$
CHAPTER 4
LATTICE METHODS

In this chapter, we shall develop lattice methods for the European and American options under the two-asset model and the stochastic interest rate models. As usual, our lattice methods are based on simulating the solutions of the stochastic differential equations by lattice trees. Numerical results will be given to show the efficiency and accuracy of our methods.

4.1 The Two-Asset Model

Consider an option with expiration date $T$ under the two-asset model (2.8). We want to compute the option price when $S_1(t_0) = S_{1,0}$ and $S_2(t_0) = S_{2,0}$.

We have from (2.9)–(2.11)

\begin{align*}
S_1(t) &= S_1(t_0)e^{Y_1(t)}, \quad S_2(t) = (S_1(t))^\alpha \tilde{S}_2(t), \tag{4.1} \\
\tilde{S}_2(t) &= \tilde{S}_2(t_0)e^{Y_2(t)}, \quad \tilde{S}_2(t_0) = S_2(t_0)(S_1(t_0))^{-\alpha}, \tag{4.2}
\end{align*}

where $\alpha = \frac{\sigma_2}{\sigma_1}$, and $Y_1(t)$ and $Y_2(t)$ are determined by the stochastic differential equations (2.12) and (2.13). Without loss of generality, we may assume that $\sigma_1 \geq \sigma_2$.

For a given positive integer $M$, let $t_m = t_0 + m\Delta t$ for $m = 0, 1, \ldots, M$, where $\Delta t = \frac{T-t_0}{M}$ is the step size in time. The tree for simulating the solution $Y_1(t)$ of the stochastic differential
equation (2.12) consists of the following nodes

\[(y_{1,i}, t_{m}): i = -m, \ldots, 0, \ldots, m, \ m = 0, 1, \ldots, M,\]

where \(y_{1,i} = ih_1\) and \(h_1\) is the step size for the values of \(Y_1(t)\). Let \(p_1\) and \(q_1\) be the probabilities by which the tree branches from \((y_{1,i}, t_{m})\) to \((y_{1,i+1}, t_{m+1})\) and \((y_{1,i-1}, t_{m+1})\), respectively.

It follows from the stochastic differential equation (2.12) that

\[Y_1(t_{m+1}) = Y_1(t_{m}) + \mu_1 \Delta t + \sigma_1 \sqrt{\Delta t} Z_m,\]

where \(Z_m\) is a standard normal random variable. Then we have

\[\mathbb{E}[Y_1(t_{m+1})|Y_1(t_{m}) = y_{1,i}] = y_{1,i} + \mu_1 \Delta t,\]

and

\[\mathbb{V}[Y_1(t_{m+1})|Y_1(t_{m}) = y_{1,i}] = \sigma_1^2 \Delta t.\]

Matching the means and variances, we get

\[p_1 (y_{1,i} + h_1) + q_1 (y_{1,i} - h_1) = y_{1,i} + \mu_1 \Delta t,\]

\[p_1 (h_1 - \mu_1 \Delta t)^2 + q_1 (-h_1 - \mu_1 \Delta t)^2 = \sigma_1^2 \Delta t.\]

Solving the above equations together with \(p_1 + q_1 = 1\), we obtain

\[h_1 = \left(\sigma_1^2 \Delta t + (\mu_1 \Delta t)^2\right)^{\frac{1}{2}}, \quad p_1 = \frac{1}{2} \left(1 + \frac{\mu_1 \Delta t}{h_1}\right), \quad q_1 = 1 - p_1.\]

Similarly, we can build the tree for simulating the solution \(Y_2(t)\) of the stochastic differential equation (2.13) consists of the following nodes

\[(y_{2,j}, t_{m}): j = -m, \ldots, 0, \ldots, m, \ m = 0, 1, \ldots, M,\]
where

\[ y_{2,j} = jh_2, \quad h_2 = \left( \tilde{\sigma}_2^2 \Delta t + (\mu_2 \Delta t)^2 \right)^{\frac{1}{2}}. \]

The probabilities by which the tree branches from \((y_{2,j}, t_m)\) to \((y_{2,j+1}, t_{m+1})\) and \((y_{2,j-1}, t_{m+1})\) are

\[ p_2 = \frac{1}{2} \left( 1 + \frac{\mu_2 \Delta t}{h_2} \right), \quad q_2 = 1 - p_2, \]

respectively.

Now the tree for simulating the two-dimensional process \((Y_1(t), Y_2(t))\) consists of the nodes

\[(y_{1,i}, y_{2,j}, t_m), \quad i, j = -m, \ldots, 0, \ldots, m, \quad m = 0, \ldots, M.\]

The tree for the process \((Y_1(t), Y_2(t))\) naturally branches from \((y_{1,i}, y_{2,j}, t_{m-1})\) to \((y_{1,i-1}, y_{2,j-1}, t_m)\), \((y_{1,i+1}, y_{2,j-1}, t_m)\), \((y_{1,i+1}, y_{2,j+1}, t_m)\), and \((y_{1,i-1}, y_{2,j+1}, t_m)\) with the probabilities

\[ P_1 = q_1 q_2, \quad P_2 = p_1 q_2, \quad P_3 = p_1 p_2, \quad P_4 = q_1 p_2, \]

respectively. According to (4.1) and (4.2), the corresponding tree for the process \((S_1(t), \tilde{S}_2(t))\) consisting of the nodes

\[ (S_{1,0} e^{ih_1}, \tilde{S}_{2,0} e^{jh_2}, t_m), \quad i, j = -m, \ldots, 0, \ldots, m, \quad m = 0, \ldots, M. \]

For a given payoff \(\Phi(S_1, S_2)\), we have the following algorithms to compute the prices of the European and American options.
Algorithm 5. A lattice method for the European option on two assets

1. Compute
   \[ S_{1,i} = S_{1,0}e^{(2.0^i-M)h_1}, \quad \tilde{S}_{2,i} = \tilde{S}_{2,0}e^{(2.0^i-M)h_2}, \quad i = 0, 1, \ldots, M. \]

2. Compute the payoff at the option expiration:
   \[ V_{i,j} = \Phi \left( S_{1,i}, (S_{1,i})^\alpha \tilde{S}_{2,j} \right), \quad i, j = 0, 1, \ldots, M. \]

3. For \( m = M-1, M-2, \ldots, 0, \) do
   
   For \( i = 0, \ldots, m, \) do
       
   For \( j = 0, \ldots, m, \) do
       
       \[ V_{i,j} = e^{-r\Delta t} \left( P_1 V_{i,j} + P_2 V_{i+1,j} + P_3 V_{i+1,j+1} + P_4 V_{i,j+1} \right). \]
       
   End do.
       
   End do.
       
   End do.

4. Rerun \( V_{0,0} \) for the approximate value of the option price.
Algorithm 6. A lattice method for the American option on two assets

1. Compute

\[ S_{1,i} = S_{1,0} e^{(i-M)h_1}, \quad \tilde{S}_{2,i} = \tilde{S}_{2,0} e^{(i-M)h_2}, \quad i = 0, 1, \ldots, 2M. \]

2. Compute the payoff at the option expiration:

\[ V_{i,j} = \Phi \left( S_{1,i}, (S_{1,2i})^\alpha \tilde{S}_{2,2j} \right), \quad i, j = 0, 1, \ldots, M. \]

3. For \( m = M - 1, M - 2, \ldots, 0 \), do

   For \( i = 0, \ldots, m \), do

   - Set \( S_1 = S_{1,M-m+2i} \).
   - For \( j = 0, \ldots, m \), do

     * Compute

     \[ V_{i,j} = e^{-r\Delta t} \left( P_1 V_{i,j} + P_2 V_{i+1,j} + P_3 V_{i+1,j+1} + P_4 V_{i,j+1} \right). \]

     * Check for early exercise:

     \[ V_{i,j} = \max \left( V_{i,j}, \Phi \left( S_1, S_2 \right) \right), \]

     where \( S_2 = (S_1)^\alpha \tilde{S}_{2,M-m+2j} \).

   End do.

End do.

End do.

4. Return \( V_{0,0} \) for the approximate value of the option price.
4.2 The Stochastic Interest Rate Model

Consider an option with expiration date $T$ under the stochastic interest rate model \((2.20) – (2.21)\). We want to compute the option price $V$ when $S(t_0) = S_0$ and $r(t_0) = r_0$.

We have from \((2.22)\) and \((2.23)\)

$$S(t) = Ke^{\alpha Y_1(t) + Y_2(t)}, \quad r(t) = (Y_1(t))^{1/(1-p)},$$

(4.3)

where $\alpha = \frac{\rho a}{(1-p)v}$, and $Y_1(t)$ and $Y_2(t)$ are determined by the stochastic differential equations \((2.24)\) and \((2.25)\) with the initial conditions $Y_1(t_0) = y_{1,0}$ and $Y_2(t_0) = y_{2,0}$, where

$$y_{1,0} = r_0^{1-p}, \quad y_{2,0} = \log(S_0/K) - \alpha r_0^{1-p}.$$

For a given positive integer $M$, let $t_m = t_0 + m\Delta t$ for $m = 0, 1, \ldots, M$, where $\Delta t = \frac{T-t_0}{M}$ is the step size in time. The tree for simulating the solution $Y_1(t)$ of the stochastic differential equation \((2.24)\) consists of the following nodes

$$(y_{1,i}, t_m) : i = -m, \ldots, 0, \ldots, m, \quad m = 0, 1, \ldots, M,$$

where $y_{1,i} = y_{1,0} + ih_1$ and $h_1$ is the step size for the values of $Y_1(t)$. Let $p_1$ and $q_1$ be the probabilities by which the tree branches from $(y_{1,i}, t_m)$ to $(y_{1,i+1}, t_{m+1})$ and $(y_{1,i-1}, t_{m+1})$, respectively.

We can approximate the stochastic differential equation \((2.24)\) by the Euler scheme:

$$Y_1(t_{j+1}) \approx Y_1(t_j) + \mu_1(Y_1(t_j)) \Delta t + \sigma_1 \sqrt{\Delta t} Z_m,$$

where $Z_m$ is a standard normal random variable. Then we have

$$\mathbb{E}[Y_1(t_{m+1}) | Y_1(t_m) = y_{1,i}] \approx y_{1,i} + \mu_1(y_{1,i}) \Delta t.$$
and

\[ \mathbb{V} [Y_1 (t_{m+1}) | Y_1 (t_m) = y_{1,i}] \approx \sigma_1^2 \Delta t. \]

Matching the variances, we get

\[ p_1 (h_1 - \mu_1 (y_{1,i}) \Delta t)^2 + q_1 (-h_1 - \mu_1 (y_{1,i}) \Delta t)^2 = \sigma_1^2 \Delta t. \]

Since \( p_1 + q_1 = 1 \), we have from the above equation

\[ h_1 = \left( \sigma_1^2 \Delta t + (\mu_1 (y_{1,i}) \Delta t)^2 \right)^{\frac{1}{2}}, \]

which depends on \( y_{1,i} \). After dropping the higher order term \( (\mu_1 (y_{1,i}) \Delta t)^2 \), we get

\[ h_1 = \sigma_1 \sqrt{\Delta t}. \]

We have by matching the means

\[ p_1 (y_i + h_1) + q_1 (y_j - h_1) = y_{1,i} + \mu_1 (y_{1,i}) \Delta t, \]

which implies

\[ p - q = \frac{\mu_1 (y_{1,i})}{h_1}. \]

Thus we have

\[ p_1 = \frac{1}{2} \left( 1 + \frac{\mu_1 (y_{1,i}) \Delta t}{h_1} \right), \quad q_1 = 1 - p_1. \]

Notice that \( p_1 \) may not be between 0 and 1. We shall artificially set

\[ p_1 = \begin{cases} 
0, & \text{if } p_1 < 0; \\
1, & \text{if } p_1 > 1.
\end{cases} \]

Since \( Y_1(t) \) should be always positive for the CIR model, we also set

\[ p_1 = 1.0, \quad \text{if } y_{1,i} \leq 0 \]
to make the value of $Y_1(t_{m+1})$ positive.

Similarly, we can build the tree for simulating the solution $Y_2(t)$ of the stochastic differential equation (2.25) when $Y_1(t) = y_{1,i}$ is given. It consists of the following nodes

$$(y_{2,j}, t_m) : j = -m, \ldots, 0, \ldots, m, \ m = 0, 1, \ldots, M,$$

where

$$y_{2,j} = y_{2,0} + jh_2, \ h_2 = \sigma_2\sqrt{\Delta t}.$$

The probabilities by which the tree branches from $(y_{2,j}, t_m)$ to $(y_{2,j+1}, t_{m+1})$ and $(y_{2,j-1}, t_{m+1})$ are

$$p_2 = \frac{1}{2} \left( 1 + \frac{\mu_2(y_{1,i}) \Delta t}{h_2} \right), \ q_2 = \frac{1}{2} \left( 1 - \frac{\mu_2(y_{1,i}) \Delta t}{h_2} \right),$$

respectively. As for probability $p_1$, we shall artificially set

$$p_2 = \begin{cases} 0, & \text{if } p_2 < 0; \\ 1, & \text{if } p_2 > 1. \end{cases}$$

Now the tree for simulating the two-dimensional process $(Y_1(t), Y_2(t))$ consists of the nodes

$$(y_{1,i}, y_{2,j}, t_m), \ i, j = -m, \ldots, 0, \ldots, m, \ m = 0, \ldots, M.$$

The tree for the process $(Y_1(t), Y_2(t))$ naturally branches from from $(y_{1,i}, y_{2,j}, t_m)$ to $(y_{1,i-1}, y_{2,j-1}, t_m)$, $(y_{1,i+1}, y_{2,j-1}, t_m)$, $(y_{1,i+1}, y_{2,j+1}, t_m)$, and $(y_{1,i-1}, y_{2,j+1}, t_m)$ with the probabilities

$$P_1 = q_1q_2, \ P_2 = p_1q_2, \ P_3 = p_1p_2, \ P_4 = q_1p_2,$$

respectively. It follows from (4.3) that the corresponding tree for the process $(S(t), r(t))$ consisting of the nodes

$$\left(K e^{\alpha_1 i + y_{2,j}}(y_{1,i})^{1/(1-p)}, t_m \right), \ i, j = -m, \ldots, 0, \ldots, m, \ m = 0, \ldots, M.$$
For a given payoff $\Phi(S)$, we have the following algorithms to compute the prices of the European and American options.

**Algorithm 7.** A lattice method for the European option (stochastic interest rate)

1. Compute

   $$y_{1,i} = y_{1,0} + (i - M)h_1, \quad y_{2,i} = y_{2,0} + (i - M)h_2, \quad i = 0, 1, \ldots, 2M.$$ 

2. Compute the payoff at the option expiration:

   $$V_{i,j} = \Phi(K \exp(y_{2,2j} + \alpha y_{1,2i})), \quad i, j = 0, 1, \ldots, M.$$ 

3. For $m = M - 1, M - 2, \ldots, 0$, do

   For $i = 0, \ldots, m$, do

   - Compute $p_1 = \frac{1}{2} \left(1 + \mu_1 \frac{(y_{1,M-m+2i})\Delta t}{dz}\right).$
     - If $p_1 < 0$, set $p_1 = 0$; if $p_1 > 1$, set $p_1 = 1$; if $y_{1,M-m+2i} \leq 0$, set $p_1 = 1$.
     - Set $q_1 = 1 - p_1$.
   - Compute $p_2 = \frac{1}{2} \left(1 + \mu_2 \frac{(y_{1,M-m+2i})\Delta t}{dz}\right).$
     - If $p_2 < 0$, set $p_2 = 0$; if $p_2 > 1$, set $p_2 = 1$.
     - Set $q_2 = 1 - p_2$.
   - Compute the discount factor $D = \exp\left(-\frac{(y_{1,M-m+2i})^{1/(1-p)}}{(1-p)} \Delta t\right)$.
     - For $j = 0, \ldots, m$, do

       $$V_{i,j} = D(P_1V_{i,j} + P_2V_{i+1,j} + P_3V_{i+1,j+1} + P_4V_{i,j+1}).$$

     End do.

   End do.

End do.

4. Rerun $V_{0,0}$ for the approximate value of the option price.
Algorithm 8. A lattice method for the American option (stochastic interest rate)

1. Compute

\[ y_{1,i} = y_{1,0} + (i - M)h_1, \quad y_{2,i} = y_{2,0} + (i - M)h_2, \quad i = 0, 1, \ldots, 2M. \]

2. Compute the payoff at the option expiration:

\[ V_{i,j} = \Phi(K \exp(y_{2,2j} + \alpha y_{1,2i})), \quad i, j = 0, 1, \ldots, M. \]

3. For \( m = M - 1, M - 2, \ldots, 0 \), do

   For \( i = 0, \ldots, m \), do

   \(-\) Compute \( p_1 = \frac{1}{2} \left( 1 + \frac{\mu_1(y_{1,m+2i})}{dz} \Delta t \right) \);

   \(-\) If \( p_1 < 0 \), set \( p_1 = 0 \); if \( p_1 > 1 \), set \( p_1 = 1 \).

   \(-\) Set \( q_1 = 1 - p_1 \).

   \(-\) Compute \( p_2 = \frac{1}{2} \left( 1 + \frac{\mu_2(y_{1,m+2i})}{dz} \Delta t \right) \);

   \(-\) If \( p_2 < 0 \), set \( p_2 = 0 \); if \( p_2 > 1 \), set \( p_2 = 1 \).

   \(-\) Set \( q_2 = 1 - p_2 \).

   \(-\) Compute the discount factor \( D = \exp\left(- (y_{1,m+2i})^{1/(1-p)} \Delta t \right) \).

   \(-\) For \( j = 0, \ldots, m \), do

   \(-\) \( V_{i,j} = D(P_1V_{i,j} + P_2V_{i+1,j} + P_3V_{i+1,j+1} + P_4V_{i,j+1}) \).

   \(-\) \( S = K \exp(y_{2,m+2j} + \alpha y_{1,m+2i}) \).

   \(-\) \( V_{i,j} = \max(V_{i,j}, \Phi(S)) \).

   End do.

   End do.

End do.

4. Rerun \( V_{0,0} \) for the approximate value of the option price.
4.3 Numerical Results

In this section, we present numerical examples to examine the convergence and accuracy of the proposed lattice methods in the previous sections. We will focus on the accuracy of the method, especially for the European options. We will also examine the early exercise boundaries for the American options. For convenience, we introduce the following abbreviations.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>REF</td>
<td>Reference value</td>
</tr>
<tr>
<td>LAT</td>
<td>Numerical result using lattice</td>
</tr>
<tr>
<td>MMC</td>
<td>Numerical result using mixed Monte Carlo method</td>
</tr>
<tr>
<td>AE</td>
<td>Absolute error between the numerical results of LAT and MMC</td>
</tr>
<tr>
<td>N</td>
<td>Number of MC simulations</td>
</tr>
<tr>
<td>M</td>
<td>Number of time steps</td>
</tr>
</tbody>
</table>

Table 4.1: Notations

Example 4.1. (The European Spread Option) In this example, we consider the European spread options whose payoff is \( \Phi(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+ \). The reference values are computed the same as in example 3.1. The parameters for the two-asset model are given in Table 4.2.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>$10</td>
</tr>
<tr>
<td>( T )</td>
<td>1.0 year</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>0.0</td>
</tr>
<tr>
<td>( q_1 )</td>
<td>0.05</td>
</tr>
<tr>
<td>( q_2 )</td>
<td>0.07</td>
</tr>
<tr>
<td>( r )</td>
<td>0.08</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 4.2: Parameters for the European spread option

In order to examine the rate of convergence of our lattice method, we plot the maximum
absolute errors (MAE) against the time step sizes in Figs. 4.1. The MAEs are computed at the points $S_1 = 60 : 5 : 120 \times S_2 = 60 : 5 : 120$ with different correlations $\rho = -0.8, -0.4, 0.4, 0.8$ and different interest rates $r = 0.04, 0.05, 0.06, 0.07, 0.08$. We can observe that the rate of convergence of our lattice method is about 1. For the accuracy of the lattice scheme, we display the option prices and the absolute errors (AE) in Tables 4.3–4.6. We can see that the AEs is $O(10^{-3})$ when the number of time steps $M = 1000$. All of these numerical results are as expected since the theoretical rate of convergence of the lattice method is $O(M^{-1})$.

Figure 4.1: The maximum absolute errors of the European spread options
<table>
<thead>
<tr>
<th>$S_2$</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>22.2686</td>
<td>22.2679</td>
<td>0.0007</td>
</tr>
<tr>
<td>90</td>
<td>25.9780</td>
<td>25.9775</td>
<td>0.0005</td>
</tr>
<tr>
<td>95</td>
<td>29.8525</td>
<td>29.8521</td>
<td>0.0004</td>
</tr>
<tr>
<td>100</td>
<td>33.8675</td>
<td>33.8672</td>
<td>0.0002</td>
</tr>
<tr>
<td>105</td>
<td>38.0013</td>
<td>38.0012</td>
<td>0.0001</td>
</tr>
<tr>
<td>110</td>
<td>42.2353</td>
<td>42.2353</td>
<td>0.0000</td>
</tr>
<tr>
<td>115</td>
<td>46.5535</td>
<td>46.5536</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Table 4.3: The European spread option prices: $\rho = -0.8$

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>21.0588</td>
<td>21.0583</td>
<td>0.0005</td>
</tr>
<tr>
<td>90</td>
<td>24.8238</td>
<td>24.8234</td>
<td>0.0004</td>
</tr>
<tr>
<td>95</td>
<td>28.7665</td>
<td>28.7663</td>
<td>0.0003</td>
</tr>
<tr>
<td>100</td>
<td>32.8575</td>
<td>32.8573</td>
<td>0.0001</td>
</tr>
<tr>
<td>105</td>
<td>37.0710</td>
<td>37.0710</td>
<td>0.0000</td>
</tr>
<tr>
<td>110</td>
<td>41.3855</td>
<td>41.3856</td>
<td>0.0001</td>
</tr>
<tr>
<td>115</td>
<td>45.7825</td>
<td>45.7827</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Table 4.4: The European spread option prices: $\rho = -0.4$

<table>
<thead>
<tr>
<th>$S_2$</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>18.2373</td>
<td>18.2372</td>
<td>0.0002</td>
</tr>
<tr>
<td>90</td>
<td>22.2524</td>
<td>22.2523</td>
<td>0.0001</td>
</tr>
<tr>
<td>95</td>
<td>26.4707</td>
<td>26.4707</td>
<td>0.0000</td>
</tr>
<tr>
<td>100</td>
<td>30.8415</td>
<td>30.8416</td>
<td>0.0001</td>
</tr>
<tr>
<td>105</td>
<td>35.3246</td>
<td>35.3247</td>
<td>0.0001</td>
</tr>
<tr>
<td>110</td>
<td>39.8890</td>
<td>39.8891</td>
<td>0.0002</td>
</tr>
<tr>
<td>115</td>
<td>44.5115</td>
<td>44.5116</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Table 4.5: The European spread option prices: $\rho = 0.4$
Table 4.6: The European spread option prices: \( \rho = 0.8 \)

<table>
<thead>
<tr>
<th>( S_2 )</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>16.6045</td>
<td>16.6045</td>
<td>0.0000</td>
</tr>
<tr>
<td>90</td>
<td>20.9474</td>
<td>20.9474</td>
<td>0.0001</td>
</tr>
<tr>
<td>95</td>
<td>25.4668</td>
<td>25.4669</td>
<td>0.0001</td>
</tr>
<tr>
<td>100</td>
<td>30.0922</td>
<td>30.0923</td>
<td>0.0001</td>
</tr>
<tr>
<td>105</td>
<td>34.7782</td>
<td>34.7783</td>
<td>0.0001</td>
</tr>
<tr>
<td>110</td>
<td>39.4977</td>
<td>39.4978</td>
<td>0.0001</td>
</tr>
<tr>
<td>115</td>
<td>44.2351</td>
<td>44.2352</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Example 4.2. (The American options on two assets) In this example, we examine the early exercise boundaries of the 1-year American options with the popular payoffs as listed in Table 1.1. The volatilities for the two-asset model and the parameters for the payoffs are given in Table 4.7. The other parameters (the interest rate \( r \), correlation \( \rho \), and dividend rates \( q_1, q_2 \)) will be specified later. All the early exercise boundaries are computed via bisection method while the option prices are computed by using Algorithm 6 in section 4.1 with the number of time steps \( M = 500 \).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Spread</th>
<th>Call on max</th>
<th>Max call</th>
<th>Put on min</th>
<th>Max put</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>$10</td>
<td>$100</td>
<td>N/A</td>
<td>$50</td>
<td>N/A</td>
</tr>
<tr>
<td>( K_1 )</td>
<td>N/A</td>
<td>N/A</td>
<td>$80</td>
<td>N/A</td>
<td>$5</td>
</tr>
<tr>
<td>( K_2 )</td>
<td>N/A</td>
<td>N/A</td>
<td>$120</td>
<td>N/A</td>
<td>$12</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 4.7: Parameters for the American options with two-asset

In the following, we will use \( S_i^t \) \((i = 1, 2)\) for the spot price of the \( i \)-th asset at time \( t \in [0, T) \), \( E \) for the immediate exercise region, and \( B_i^t \) for the exercise boundary for a standard American option on the \( i \)-th asset.

We first consider the spread options and list the properties of the exercise region proved
in [10] as follows:

1. \((S^1_t, S^2_t, t) \in E\) implies \(S^2_t > S^1_t + K\).

2. \((S^1_t, S^2_t, t) \in E\) implies \((S^1_s, S^2_s, s) \in E\) for all \(t \leq s \leq T\).

3. \((S^1_t, S^2_t, t) \in E\) implies \((S^1_t, \lambda S^2_t, t) \in E\) for all \(\lambda \geq 1\).

4. \((S^1_t, S^2_t, t) \in E\) implies \((\lambda S^1_t, S^2_t, t) \in E\) for all \(0 \leq \lambda \leq 1\).

5. \((0, S^2_t, t) \in E\) implies \(S^2_t \geq B^2_t\); \(S^2_t \geq B^2_t\) and \(S^1_t = 0\) implies \((0, S^2_t, t) \in E\).

6. \((S^1_t, S^2_t, t) \in E\) and \((\tilde{S}^1_t, \tilde{S}^2_t, t) \in E\) implies \((S^1_t(\lambda), S^2_t(\lambda), t) \in E\) for all \(0 \leq \lambda \leq 1\),
   
   where \(S^i_t(\lambda) = \lambda S^i_t + (1 - \lambda) \tilde{S}^i_t\) for \(i = 1, 2\).

7. The \(S_2\) intercept of the early exercise boundary at time \(t\) is \((0, B^2_t)\). And \(\lim_{t \to T^-} B^2_t = \max \left( \frac{r}{q^2} K, K \right)\).

8. When \(t \to T^-\), the early exercise boundary is given by
   
   \[
   S^2_T = \max \left( \frac{q_1}{q_2} S^1_T + \frac{r}{q_2} K, S^1_T + K \right). 
   \]

Our numerical results in Fig. 4.2 agree with the above theoretical properties. In addition, we can observe that the early exercise boundaries of different times are more dispersive when the correlation is negative.
(a) $\rho = 0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$
(b) $\rho = -0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$
(c) $\rho = 0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$
(d) $\rho = -0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$
(e) $\rho = 0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$
(f) $\rho = -0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

Figure 4.2: The early exercise boundaries of the spread options
Secondly, we consider the call option on the maximum. Define \( E^i = E \cap G^i, i = 1, 2 \) where \( G^i \equiv \{(S^1_t, S^2_t, t) : S^i_t = \max(S^1_t, S^2_t)\} \). We list the properties of the exercise region proved in [10]:

1. \((S^1_t, S^2_t, t) \in E \) implies \((S^1_s, S^2_s, t) \in E \) for all \( t \leq s \leq T \).

2. \((S^1_t, S^2_t, t) \in E^1 \) implies \((\lambda S^1_t, S^2_t, t) \in E^1 \) for all \( \lambda \geq 1 \). \((S^1_t, S^2_t, t) \in E^2 \) implies \((S^1_t, \lambda S^2_t, t) \in E^2 \) for all \( \lambda \geq 1 \).

3. \((S^1_t, S^2_t, t) \in E^1 \) implies \((S^1_t, \lambda S^2_t, t) \in E^1 \) for all \( 0 \leq \lambda \leq 1 \). \((S^1_t, S^2_t, t) \in E^2 \) implies \((\lambda S^1_t, S^2_t, t) \in E^2 \) for all \( 0 \leq \lambda \leq 1 \).

4. \((S^1_t, 0, t) \in E^1 \) implies \( S^1_t \geq B^1_t \). \((0, S^2_t, t) \in E^2 \) implies \( S^2_t \geq B^2_t \).

5. \((S^1_t, S^2_t, t) \in E^j \) and \((\tilde{S}^1_t, \tilde{S}^2_t, t) \in E^j \) implies \((S^1_t(\lambda), S^2_t(\lambda), t) \in E^j \) for \( j = 1, 2 \) and all \( 0 \leq \lambda \leq 1 \), where \( S^i_t(\lambda) = \lambda S^i_t + (1 - \lambda)\tilde{S}^i_t \) for \( i = 1, 2 \).

6. The \( S_1 \) intercept of the early exercise boundary at time \( t \) is \((B^1_t, 0)\). The \( S_2 \) intercept of the early exercise boundary at time \( t \) is \((0, B^2_t)\). And \( \lim_{t \to T^-} B^i_t = \max \left( \frac{r}{q}, K, K \right) \).

7. When \( t \to T^- \), the early exercise boundary is given by

\[
S^1_T = \max \left( \max \left( \frac{r}{q_1}, K, K \right), S^2_T \right) \text{ for } E^1,
\]

\[
S^2_T = \max \left( \max \left( \frac{r}{q_2}, K, K \right), S^1_T \right) \text{ for } E^2.
\]

Our numerical results in Fig. 4.3 agree with the above theoretical properties. In addition, we can observe that the early exercise boundaries of different times are more dispersive when the correlation is negative.
(a) $\rho = 0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(b) $\rho = -0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(c) $\rho = 0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(d) $\rho = -0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(e) $\rho = 0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

(f) $\rho = -0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

Figure 4.3: The early exercise boundaries of the call option on the maximum
Thirdly, we consider the maximum call option. Define $E^i = E \cap G^i, i = 1, 2$ where $G^i \equiv \{(S^i_1, S^i_2, t) : S^i_t - K_i = \max(S^i_1 - K_1, S^i_2 - K_2)\}$. We list the properties of the exercise region proved in [10]:

(1) $(S^1_t, S^2_t, t) \in E$ implies $(S^1_t, S^2_t, s) \in E$ for all $t \leq s \leq T$.

(2) $(S^1_t, S^2_t, t) \in E^1$ implies $(\lambda S^1_t, S^2_t, t) \in E^1$ for all $\lambda \geq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies $(S^1_t, \lambda S^2_t, t) \in E^2$ for all $\lambda \geq 1$.

(3) $(S^1_t, S^2_t, t) \in E^1$ implies $(S^1_t, \lambda S^2_t, t) \in E^1$ for all $0 \leq \lambda \leq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies $(\lambda S^1_t, S^2_t, t) \in E^2$ for all $0 \leq \lambda \leq 1$.

(4) $(S^1_t, 0, t) \in E^1$ implies $S^1_t \geq B^1_t$. $(0, S^2_t, t) \in E^2$ implies $S^2_t \geq B^2_t$.

(5) $(S^1_t, S^2_t, t) \in E^j$ and $(\tilde{S}^1_t, \tilde{S}^2_t, t) \in E^j$ implies $(S^1_t(\lambda), S^2_t(\lambda), t) \in E^j$ for $j = 1, 2$ and all $0 \leq \lambda \leq 1$, where $S^i_t(\lambda) = \lambda S^i_t + (1 - \lambda)\tilde{S}^i_t$ for $i = 1, 2$.

(6) The $S_1$ intercept of the early exercise boundary at time $t$ is $(B^1_t, 0)$. The $S_2$ intercept of the early exercise boundary at time $t$ is $(0, B^2_t)$. And $\lim_{t \to T^-} B^i_t = \max(\frac{v}{q_i} K_i, K_i)$.

(7) When $t \to T^-$, the early exercise boundary is given by

$$S^1_T = \max \left( \max \left( \frac{r}{q_1} K_1, K_1 \right), S^2_T \right) \text{for } E^1,$$

$$S^2_T = \max \left( \max \left( \frac{r}{q_2} K_2, K_2 \right), S^1_T \right) \text{for } E^2.$$
(a) $\rho = 0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(b) $\rho = -0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(c) $\rho = 0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(d) $\rho = -0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(e) $\rho = 0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

(f) $\rho = -0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

Figure 4.4: The early exercise boundaries of the maximum call option
Fourthly, we consider the put option on the minimum. Define $E^i = E \cap G^i$, $i = 1, 2$ where

$G^i \equiv \{(S^1_t, S^2_t, t) : S^i_t = \min(S^1_t, S^2_t)\}$. We have the following observation from the Fig. 4.5:

1. $(S^1_t, S^2_t, t) \in E$ implies $(S^1_t, S^2_t, s) \in E$ for all $t \leq s \leq T$.

2. $(S^1_t, S^2_t, t) \in E^1$ implies $(\lambda S^1_t, S^2_t, t) \in E^1$ for all $0 \leq \lambda \leq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies

$(\lambda S^1_t, S^2_t, t) \in E^2$ for all $0 \leq \lambda \leq 1$.

3. $(S^1_t, S^2_t, t) \in E^1$ implies $(S^1_t, \lambda S^2_t, t) \in E^1$ for all $\lambda \geq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies

$(\lambda S^1_t, S^2_t, t) \in E^2$ for all $\lambda \geq 1$.

4. $(S^1_t, S^2_t, t) \in E^j$ and $(\tilde{S}^1_t, \tilde{S}^2_t, t) \in E^j$ implies $(S^1_t(\lambda), S^2_t(\lambda), t) \in E^j$ for $j=1,2$ and all

$0 \leq \lambda \leq 1$, where $S^i_t(\lambda) = \lambda S^i_t + (1 - \lambda) \tilde{S}^i_t$ for $i = 1, 2$.

5. When $t \to T^-$, the early exercise boundary is given by

$$S^1_T = \min \left( \min \left( \frac{r}{q_1} K, K \right), S^2_T \right) \text{ for } E^1,$$

$$S^2_T = \min \left( \min \left( \frac{r}{q_2} K, K \right), S^1_T \right) \text{ for } E^2.$$

6. The early exercise boundaries of different times are more dispersive when the correlation is negative.
Figure 4.5: The early exercise boundaries of the put option on the minimum

(a) $\rho = 0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(b) $\rho = -0.5, q_1 = 0.02, q_2 = 0.04, r = 0.06$

(c) $\rho = 0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(d) $\rho = -0.5, q_1 = 0.02, q_2 = 0.06, r = 0.04$

(e) $\rho = 0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$

(f) $\rho = -0.5, q_1 = 0.04, q_2 = 0.06, r = 0.02$
Finally, we consider the maximum put option. Without lose of generality, assume that $K_1 < K_2$. Define $E^i = E \cap G^i, i = 1, 2$ where $G^i \equiv \{(S^1_t, S^2_t, t) : K_i - S^i_t = \max(K_1 - S^1_t, K_2 - S^2_t)\}$. We have the following observation form the Fig. 4.6:

(1) $(S^1_t, S^2_t, t) \in E$ implies $(S^1_t, S^2_t, s) \in E$ for all $t \leq s \leq T$.

(2) $(S^1_t, S^2_t, t) \in E^1$ implies $(\lambda S^1_t, S^2_t, t) \in E^1$ for all $0 \leq \lambda \leq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies $(S^1_t, \lambda S^2_t, t) \in E^2$ for all $0 \leq \lambda \leq 1$.

(3) $(S^1_t, S^2_t, t) \in E^1$ implies $(S^1_t, \lambda S^2_t, t) \in E^1$ for all $\lambda \geq 1$. $(S^1_t, S^2_t, t) \in E^2$ implies $(\lambda S^1_t, S^2_t, t) \in E^2$ for all $\lambda \geq 1$.

(4) $(S^1_t, S^2_t, t) \in E^j$ and $(\tilde{S}^1_t, \tilde{S}^2_t, t) \in E^j$ implies $(S^1_t(\lambda), S^2_t(\lambda), t) \in E^j$ for $j = 1, 2$ and all $0 \leq \lambda \leq 1$, where $S^i_t(\lambda) = \lambda S^i_t + (1 - \lambda)\tilde{S}^i_t$ for $i = 1, 2$.

(5) When $t \to T^-$, the early exercise boundary is given by

\[ S^1_T = \min \left( \min \left( \frac{r}{q_1} K_1, K_1 \right), \max(S^2_T - K_2 + K_1, 0) \right) \text{ for } E^1, \]

\[ S^2_T = \min \left( \min \left( \frac{r}{q_2} K_2, K_2 \right), S^1_T + K_2 - K_1 \right) \text{ for } E^2. \]

(6) The early exercise boundaries of different times are more dispersive when the correlation is negative.
Figure 4.6: The early exercise boundaries of the maximum put option
**Example 4.3.** *(The European option under the Vasicek model)* In this example, we consider the European call option under the Vasicek model using lattice method. The reference values are computed using analytic formula in Fang’ paper [23]. The parameters are given in Table 4.8. Other parameters \((\rho, \theta)\) will be specified later.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K)</td>
<td>100</td>
</tr>
<tr>
<td>(t_0)</td>
<td>0.0</td>
</tr>
<tr>
<td>(T)</td>
<td>1.0</td>
</tr>
<tr>
<td>(q)</td>
<td>0.05</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.2</td>
</tr>
<tr>
<td>(v)</td>
<td>0.2</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 4.8: Parameters for the European call option: the Vasicek model

In order to examine the rate of convergence of our lattice method, we plot the maximum absolute errors (MAE) against the time step sizes in Figs. 4.7. The MAEs are computed at the points \(S = 60 : 5 : 120 \times r = 0.01 : 0.01 : 0.2\) with different correlations \(\rho = -0.8, -0.4, 0.4, 0.8\) and different long-term expectations of interest rates \(\theta = 0.04, 0.05, 0.06, 0.07, 0.08\). We can observe that the rate of convergence of our lattice method is about 1. For the accuracy of the lattice scheme, we display the option prices and the absolute errors (AE) in Tables 4.9–4.12. We can see that the AEs is \(O(10^{-3})\) when the number of time steps \(M = 1000\). All of these numerical results are as expected since the theoretical rate of convergence of the lattice method is \(O(M^{-1})\).
Figure 4.7: The maximum absolute errors of the European call of Vasicek

Table 4.9: The European call prices (Vasicek): $\theta = 0.05$, $\rho = -0.8$
Table 4.10: The European call prices (Vasicek): $\theta = 0.05$, $\rho = -0.4$

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>1.4073</td>
<td>1.4074</td>
<td>0.0001</td>
</tr>
<tr>
<td>90</td>
<td>2.5025</td>
<td>2.5025</td>
<td>0.0000</td>
</tr>
<tr>
<td>95</td>
<td>4.0684</td>
<td>4.0683</td>
<td>0.0001</td>
</tr>
<tr>
<td>100</td>
<td>6.1414</td>
<td>6.1414</td>
<td>0.0000</td>
</tr>
<tr>
<td>105</td>
<td>8.7188</td>
<td>8.7189</td>
<td>0.0000</td>
</tr>
<tr>
<td>110</td>
<td>11.7644</td>
<td>11.7646</td>
<td>0.0003</td>
</tr>
<tr>
<td>115</td>
<td>15.2194</td>
<td>15.2199</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

Table 4.11: The European call prices (Vasicek): $\theta = 0.05$, $\rho = 0.4$

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>2.7313</td>
<td>2.7331</td>
<td>0.0018</td>
</tr>
<tr>
<td>90</td>
<td>4.1606</td>
<td>4.1625</td>
<td>0.0019</td>
</tr>
<tr>
<td>95</td>
<td>5.9893</td>
<td>5.9914</td>
<td>0.0021</td>
</tr>
<tr>
<td>100</td>
<td>8.2224</td>
<td>8.2247</td>
<td>0.0023</td>
</tr>
<tr>
<td>105</td>
<td>10.8463</td>
<td>10.8489</td>
<td>0.0026</td>
</tr>
<tr>
<td>110</td>
<td>13.8333</td>
<td>13.8361</td>
<td>0.0029</td>
</tr>
<tr>
<td>115</td>
<td>17.1459</td>
<td>17.1491</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

Table 4.12: The European call prices (Vasicek): $\theta = 0.05$, $\rho = 0.8$

<table>
<thead>
<tr>
<th>$r$</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>LAT</td>
<td>REF</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>3.3544</td>
<td>3.3575</td>
<td>0.0031</td>
</tr>
<tr>
<td>90</td>
<td>4.8977</td>
<td>4.9010</td>
<td>0.0033</td>
</tr>
<tr>
<td>95</td>
<td>6.8137</td>
<td>6.8173</td>
<td>0.0036</td>
</tr>
<tr>
<td>100</td>
<td>9.1012</td>
<td>9.1053</td>
<td>0.0040</td>
</tr>
<tr>
<td>105</td>
<td>11.7455</td>
<td>11.7498</td>
<td>0.0043</td>
</tr>
<tr>
<td>110</td>
<td>14.7208</td>
<td>14.7255</td>
<td>0.0047</td>
</tr>
<tr>
<td>115</td>
<td>17.9953</td>
<td>18.0004</td>
<td>0.0051</td>
</tr>
</tbody>
</table>
Example 4.4. (The early exercise boundaries of the American options under the Vasicek model) In this example, we consider the early exercise boundaries of the American call and put options under the Vasicek stochastic interest rate model with the following parameters:

$$K = 100, t = 0, T = 1, \sigma = 0.3, \kappa = 2.0, v = 0.2.$$  

The other parameters (correlation $\rho$, dividend rate $q$, and long term mean $\theta$) will be specified later. All the early exercise boundaries are computed via bisection methods using Algorithm 4 in section 4.2 with number of time steps $M = 500$.

Firstly, we consider the call option. Figures 4.8 show the early exercise boundaries with fixed $q$ and changing $\theta$, while figures 4.9 are with changing $q$ and fixed $\theta$. Figures 4.10 are plotted with extreme case ($r$ up to 1.0). Denote $S_t, r_t$ as the spot asset price and interest rate at time $t$, and let $E$ be the immediate exercise region. From the above figures, we have the following observations:

1. $(S_t, r_t, t) \in E$ implies $S_t > K$.
2. $(S_t, r_t, t) \in E$ does not implies $(S_t, r_t, s) \in E$ for all $t \leq s \leq T$.
3. $(S_t, r_t, t) \in E$ implies $(\lambda S_t, r_t, t) \in E$ for all $\lambda \geq 1$.
4. $(S_t, r_t, t) \in E$ implies $(S_t, \lambda r_t, t) \in E$ for all $0 \leq \lambda \leq 1$.
5. $(S_t, r_t, t) \in E$ and $(\tilde{S}_t, \tilde{r}_t, t) \in E$ does not implies $(S_t(\lambda), r_t(\lambda), t) \in E$ for some $0 \leq \lambda \leq 1$, where $S_t(\lambda) = \lambda S_t + (1 - \lambda)\tilde{S}_t, r_t(\lambda) = \lambda r_t + (1 - \lambda)\tilde{r}_t$ for $i = 1, 2$.
6. When $t \to T^-$, the early exercise boundary is given by $S_T = \max\left(\frac{e^r}{q}K, K\right)$.
7. $\theta$ does not significantly change the shape of early exercise region.
Figure 4.8: The early exercise boundaries of call option (Vasicek, \( \rho, \theta \))

(a) \( \rho = 0.5, q = 0.04, \theta = 0.02 \)

(b) \( \rho = -0.5, q = 0.04, \theta = 0.02 \)

(c) \( \rho = 0.5, q = 0.04, \theta = 0.04 \)

(d) \( \rho = -0.5, q = 0.04, \theta = 0.04 \)

(e) \( \rho = 0.5, q = 0.04, \theta = 0.06 \)

(f) \( \rho = -0.5, q = 0.04, \theta = 0.06 \)
Figure 4.9: The early exercise boundaries of call option (Vasicek, \( \rho, q \))
Next, we consider the put option. Fig. 4.11 show the early exercise boundaries with fixed $q$ and changing $\theta$, while Fig. 4.12 are with changing $q$ and fixed $\theta$. Denote $S_t, r_t$ as the spot asset price and interest rate at time $t$, and let $E$ be the immediate exercise region. From the above figures, we have the following observations:

1. $(S_t, r_t, t) \in E$ implies $S_t < K$.

2. $(S_t, r_t, t) \in E$ implies $(S_t, r_t, s) \in E$ for all $t \leq s \leq T$.

3. $(S_t, r_t, t) \in E$ implies $(S_t, \lambda r_t, t) \in E$ for all $\lambda \geq 1$.

4. $(S_t, r_t, t) \in E$ implies $(\lambda S_t, r_t, t) \in E$ for all $0 \leq \lambda \leq 1$.

5. $(S_t, r_t, t) \in E$ and $(\tilde{S}_t, \tilde{r}_t, t) \in E$ implies $(S_t(\lambda), r_t(\lambda), t) \in E$ for all $0 \leq \lambda \leq 1$, where

$$S_t(\lambda) = \lambda S_t + (1 - \lambda) \tilde{S}_t, r_t(\lambda) = \lambda r_t + (1 - \lambda) \tilde{r}_t$$

for $i = 1, 2$.

6. When $t \to T^-$, the early exercise boundary is given by $S_T = \min \left( \frac{e^{rt}}{q} K, K \right)$.
(7) \( \theta \) does not significantly change the shape of early exercise region.

Figure 4.11: The early exercise boundaries of put option (Vasicek, \( \rho, \theta \))
Figure 4.12: The early exercise boundaries of put option (Vasicek, ρ, q)
Example 4.5. *(The European options under the CIR model)* In this example, we consider the European call options under the CIR model using lattice method with the parameters in table 4.13. Other parameters \((\rho, \theta)\) will be specified later. The parameters are chosen to satisfy the Feller condition \(2\kappa\theta > v^2\). Notice that there is no analytic solution for the European options, we compute the result using mixed Monte Carlo method with the control variates from previous chapter with \(M = 1000, N = 1000000\) as reference values.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K)</td>
<td>100</td>
</tr>
<tr>
<td>(t_0)</td>
<td>0.0</td>
</tr>
<tr>
<td>(T)</td>
<td>1.0</td>
</tr>
<tr>
<td>(q)</td>
<td>0.05</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.2</td>
</tr>
<tr>
<td>(v)</td>
<td>0.2</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 4.13: Parameters for the European call option: the CIR model

In order to examine the rate of convergence of our lattice method, we plot the maximum absolute errors (MAE) against the time step sizes in Figs. 4.13. Since there is no analytic solutions, we investigating the rate of convergence by examining the errors between the approximate solutions for numbers of steps \(M\) and \(2M\). The MAEs are computed at the points \(S = 60 : 5 : 120 \times r = 0.01 : 0.01 : 0.2\) with different correlations \(\rho = -0.8, -0.4, 0.4, 0.8\) and different long-term means of interest rates \(\theta = 0.04, 0.05, 0.06, 0.07, 0.08\). We can observe that the rate of convergence of our lattice method is about 1. For the accuracy, we display the option prices and the absolute errors (AE) in Tables 4.14–4.17. We can see that the AEs is \(O(10^{-3})\) when the number of time steps \(M = 1000\). All of these numerical results are as expected since the theoretical rate of convergence of the lattice method is \(O(M^{-1})\).
Figure 4.13: The maximum absolute errors of the European Call of CIR

Table 4.14: The European call prices (CIR): $\theta = 0.05$, $\rho = -0.8$
<table>
<thead>
<tr>
<th>r</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LAT</td>
<td>MMC</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>1.6515</td>
<td>1.6526</td>
<td>0.0010</td>
</tr>
<tr>
<td>90</td>
<td>2.8314</td>
<td>2.8325</td>
<td>0.0011</td>
</tr>
<tr>
<td>95</td>
<td>4.4737</td>
<td>4.4731</td>
<td>0.0006</td>
</tr>
<tr>
<td>100</td>
<td>6.6056</td>
<td>6.6053</td>
<td>0.0003</td>
</tr>
<tr>
<td>105</td>
<td>9.2198</td>
<td>9.2207</td>
<td>0.0010</td>
</tr>
<tr>
<td>110</td>
<td>12.2800</td>
<td>12.2801</td>
<td>0.0001</td>
</tr>
<tr>
<td>115</td>
<td>15.7314</td>
<td>15.7322</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

Table 4.15: The European call prices (CIR): $\theta = 0.05$, $\rho = -0.4$

<table>
<thead>
<tr>
<th>r</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LAT</td>
<td>MMC</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>1.8771</td>
<td>1.8763</td>
<td>0.0009</td>
</tr>
<tr>
<td>90</td>
<td>3.1180</td>
<td>3.1199</td>
<td>0.0019</td>
</tr>
<tr>
<td>95</td>
<td>4.8073</td>
<td>4.8090</td>
<td>0.0017</td>
</tr>
<tr>
<td>100</td>
<td>6.9659</td>
<td>6.9668</td>
<td>0.0009</td>
</tr>
<tr>
<td>105</td>
<td>9.5846</td>
<td>9.5876</td>
<td>0.0030</td>
</tr>
<tr>
<td>110</td>
<td>12.6294</td>
<td>12.6312</td>
<td>0.0018</td>
</tr>
<tr>
<td>115</td>
<td>16.0501</td>
<td>16.0537</td>
<td>0.0035</td>
</tr>
</tbody>
</table>

Table 4.16: The European call prices (CIR): $\theta = 0.05$, $\rho = 0.4$

<table>
<thead>
<tr>
<th>r</th>
<th>0.02</th>
<th>0.06</th>
<th>0.10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LAT</td>
<td>MMC</td>
<td>AE</td>
</tr>
<tr>
<td>85</td>
<td>2.0066</td>
<td>2.0153</td>
<td>0.0087</td>
</tr>
<tr>
<td>90</td>
<td>3.2731</td>
<td>3.2832</td>
<td>0.0101</td>
</tr>
<tr>
<td>95</td>
<td>4.9786</td>
<td>4.9766</td>
<td>0.0020</td>
</tr>
<tr>
<td>100</td>
<td>7.1423</td>
<td>7.1439</td>
<td>0.0016</td>
</tr>
<tr>
<td>105</td>
<td>9.7559</td>
<td>9.7610</td>
<td>0.0051</td>
</tr>
<tr>
<td>110</td>
<td>12.7871</td>
<td>12.7976</td>
<td>0.0105</td>
</tr>
<tr>
<td>115</td>
<td>16.1887</td>
<td>16.1944</td>
<td>0.0057</td>
</tr>
</tbody>
</table>

Table 4.17: The European call prices (CIR): $\theta = 0.05$, $\rho = 0.8$

**Example 4.6.** *(The American options under the CIR model)* In this example, we consider the early exercise boundaries (with respect to $S$ and $r$) of the American call and put options
under the CIR stochastic interest rate model with the following parameters:

\[ K = 100, t = 0, T = 1, \sigma = 0.3, \kappa = 2.0, v = 0.2. \]

The other parameters (correlation \( \rho \), dividend rate \( q \), and long term mean \( \theta \)) will be specified later. All the early exercise boundaries are computed via bisection methods using Algorithm 4 in section 4.2 with number of time steps \( M = 500 \).

Firstly, we consider the call option. Fig. 4.14 show the early exercise boundaries with fixed \( q \) and changing \( \theta \), while Fig. 4.15 are with changing \( q \) and fixed \( \theta \). Fig. 4.16 are plotted with extreme case (\( r \) up to 1.0). Denote \( S_t, r_t \) as the spot asset price and interest rate at time \( t \), and let \( E \) be the immediate exercise region. We have the following observations:

1. \((S_t, r_t, t) \in E\) implies \( S_t > K \).
2. \((S_t, r_t, t) \in E\) does not implies \((S_t, r_t, s) \in E\) for all \( t \leq s \leq T \).
3. \((S_t, r_t, t) \in E\) implies \((\lambda S_t, r_t, t) \in E\) for all \( \lambda \geq 1 \).
4. \((S_t, r_t, t) \in E\) implies \((S_t, \lambda r_t, t) \in E\) for all \( 0 \leq \lambda \leq 1 \).
5. \((S_t, r_t, t) \in E\) and \((\tilde{S}_t, \tilde{r}_t, t) \in E\) does not implies \((S_t(\lambda), r_t(\lambda), t) \in E\) for some \( 0 \leq \lambda \leq 1 \), where \( S_t(\lambda) = \lambda S_t + (1 - \lambda)\tilde{S}_t, r_t(\lambda) = \lambda r_t + (1 - \lambda)\tilde{r}_t \) for \( i = 1, 2 \).
6. When \( t \to T^- \), the early exercise boundary is given by \( S_T = \max\left(\frac{K}{q}, K \right) \).
7. \( \theta \) does not significantly change the shape of early exercise region.
8. The early exercise boundaries at different times in each case are closer to each other, compare with Vasicek model.
Figure 4.14: The early exercise boundaries of call option (CIR, $\rho$, $\theta$)

(a) $\rho = 0.5, q = 0.04, \theta = 0.02$

(b) $\rho = -0.5, q = 0.04, \theta = 0.02$

(c) $\rho = 0.5, q = 0.04, \theta = 0.04$

(d) $\rho = -0.5, q = 0.04, \theta = 0.04$

(e) $\rho = 0.5, q = 0.04, \theta = 0.06$

(f) $\rho = -0.5, q = 0.04, \theta = 0.06$
Figure 4.15: The early exercise boundaries of call option (CIR, $\rho$, $q$)
Next, we consider the put option. Fig. 4.17 show the early exercise boundaries with fixed $q$ and changing $\theta$, while Fig. 4.18 are with changing $q$ and fixed $\theta$. We have the following observations:

1. $(S_t, r_t, t) \in E$ implies $S_t < K$.
2. $(S_t, r_t, t) \in E$ implies $(S_t, r_t, s) \in E$ for all $t \leq s \leq T$.
3. $(S_t, r_t, t) \in E$ implies $(S_t, \lambda r_t, t) \in E$ for all $\lambda \geq 1$.
4. $(S_t, r_t, t) \in E$ implies $(\lambda S_t, r_t, t) \in E$ for all $0 \leq \lambda \leq 1$.
5. $(S_t, r_t, t) \in E$ and $(\tilde{S}_t, \tilde{r}_t, t) \in E$ implies $(S_t(\lambda), r_t(\lambda), t) \in E$ for all $0 \leq \lambda \leq 1$, where $S_t(\lambda) = \lambda S_t + (1 - \lambda) \tilde{S}_t, r_t(\lambda) = \lambda r_t + (1 - \lambda) \tilde{r}_t$ for $i = 1, 2$.
6. When $t \to T^-$, the early exercise boundary is given by $S_T = \min \left( \frac{e^r K}{q}, K \right)$.
7. $\theta$ does not significantly change the shape of early exercise region.
(8) The early exercise boundaries at different times in each case are closer to each other, compare with Vasicek model.

Figure 4.17: The early exercise boundaries of put option (CIR, ρ, θ)
Figure 4.18: The early exercise boundaries of put option (CIR, $\rho$, $q$)
The alternating direction implicit (ADI) method was first introduced by Peaceman and Rachford in 1955[51]. It uses the idea of splitting the finite difference equations into two, one taking $x$-derivative implicitly and the other taking $y$-derivative implicitly. We are referred to the book from W. Hundsdorfer and J. Verwer [35].

For multi-asset options, Villeneuve and Zanette [61] performed a coordinate transformation to get an operator that is essentially the standard two-dimensional Laplacian, then Peaceman-Rachford ADI scheme can be applied. Dang, Christara, and Jackson developed an Alternating Direction Implicit Approximate Factorization (ADI-AF) techniques based on Graphics Processing Units (GPUs) in 2010[18]. They used a combination of an efficient GPU-based parallelization of ADI-AF techniques with a penalty approach for the pricing of multi-asset American options in the Black-Scholes framework. For Stochastic Volatility models (especially Heston model), Haentjens and in’t Hout [31] developed an effective adaptation of ADI time discretization schemes to the semi-discretized Heston partial differential complementarity problem for American-style options in 2015. The method is applied to the PDE directly without transformation of variables. For models with both stochastic volatility and stochastic interest rate, Haentjens and in’t Hout [30] applied ADI for the Heston
Hull White model. Grzelak and Oosterlee [29] applied ADI for the Heston-Hull-White and Heston-CIR models, Donnelly, Jaimungal, and Rubisov [22] valued guaranteed withdrawal benefits with stochastic interest rates and volatility. Since the the mixed partial derivatives are not removed in these papers, the schemes are complicated and thus inefficient.

In this chapter, we applied the ADI method to the partial differential equations derived under the uncorrelated processes in Chapter 2. The partial differential operator will be discretized by a finite volume method and thus the Neumann boundary conditions can be treated more accurately. Since there is not mixed partial derivatives, our scheme is numerically simplest and very fast. Here we only consider the stochastic volatility model (the Heston model) and it is not difficult to extend our method to the other models in Chapter 2.

5.1 The Partial Differential Variational Inequalities

Consider the European contingent claim with expiration date $T$ and payoff function $\Phi(S)$ under the the Heston model (2.14)–(2.15). Its rational price at time $t$ is given by

$$p(s, v, t) = \mathbb{E} \left[ e^{-r(T-t)} \Phi(S(T)) | S(t) = s, v(t) = v \right].$$

Since $e^{-rt}p(S(t), v(t), t)$ is a martingale, it follows from Ito’s Lemma that function $p(s, v, t)$ solves the parabolic partial differential equation

$$\frac{\partial p}{\partial t} + K_p = 0, \quad (5.1)$$

where

$$K_p = \frac{1}{2} s^2 v p_{ss} + \rho \sigma s v p_{sv} + \frac{1}{2} \sigma^2 v p_{vv} + (r - q) s p_s + \kappa (\eta - v) p_v - r V. \quad (5.2)$$
The rational price of the American contingent claim is given by

\[ q(s, v, t) = \sup_{t \leq t^* \leq T} \mathbb{E} \left [ e^{-r(t^*-t)} \Phi(S(t^*)) \mid S(t) = s, v(t) = v \right ], \]

where \( t^* \) is the stopping time. It is known that function \( q(S, v, t) \) solves the parabolic partial differential variational inequalities (PDVIs)

\[ \frac{\partial q}{\partial t} + \mathcal{K}q \leq 0, \quad q \geq \Phi, \quad (q - \Phi) \left ( \frac{\partial q}{\partial t} + \mathcal{K}q \right ) = 0. \]

It should be pointed out that the operator \( \mathcal{K} \) has a mixed partial derivative term. These PDVIs are widely used by many papers, like Ikonen and Toivanen [36], Haentjens and in’t Hout [31].

Under the transformed processes \( Y_1(t) \) and \( Y_2(t) \) defined by the SDEs (2.18)–(2.19), the rational price of the European contingent claim is given by

\[ P(x, y, t) = \mathbb{E} \left [ e^{-r(T-t)} \Psi (Y_1(T), Y_2(T)) \mid Y_1(t) = x, Y_2(t) = y \right ], \]

where

\[ \Psi(x, y) = \Phi \left ( Ke^{x+\frac{r}{2}y} \right ). \]

Since \( e^{-rt}P(Y_1(t), Y_2(t), t) \) is also a martingale, we have by Ito’s Lemma

\[ \frac{\partial P}{\partial t} + \mathcal{L}P = 0, \quad (5.3) \]

where

\[ \mathcal{L}P = \frac{1}{2} \lambda^2 y P_{xx} + \frac{1}{2} \sigma^2 y P_{yy} + (a_1 + b_1 y) P_x + (a_2 + b_2 y) P_y - r P. \quad (5.4) \]

For the American option, the price is given by

\[ Q(x, y, t) = \sup_{t \leq t^* \leq T} \mathbb{E} \left [ e^{-r(t^*-t)} \Psi (Y_1(T), Y_2(T)) \mid Y_1(t) = x, Y_2(t) = y \right ], \]
which satisfies the parabolic partial differential variational inequalities

\[
\frac{\partial Q}{\partial t} + \mathcal{L}Q \leq 0, \quad Q \geq \Psi, \quad (Q - \Psi) \left( \frac{\partial Q}{\partial t} + \mathcal{L}Q \right) = 0. \tag{5.5}
\]

Now there is no mixed partial derivative term in the operator \( \mathcal{L} \).

Let

\[ U(x, y, \tau) = \begin{cases} 
P(x, y, T - \tau), & \text{for the European contingent claim,} \\
Q(x, y, T - \tau), & \text{for the American contingent claim.}
\end{cases} \]

Then the backward partial differential equation (5.3) and variational inequalities (5.5) becomes

\[
\frac{\partial U}{\partial \tau} - \mathcal{L}U = 0, \tag{5.6}
\]

and

\[
\frac{\partial U}{\partial \tau} - \mathcal{L}U \geq 0, \quad U \geq \Psi, \quad (U - \Psi) \left( \frac{\partial U}{\partial \tau} - \mathcal{L}U \right) = 0. \tag{5.7}
\]

The boundary conditions will be specified in the next section in order to solve the problems on a bounded domain.

### 5.2 The Boundary Conditions

From now on, we only consider the put option with expiration date \( T \) and strike price \( K \).

The call option can be evaluated according to the put-call parity or the put-call symmetry (\([1][11][19][55]\)). The payoff function of the put is

\[
\Phi(S) = (K - S)^+. 
\]

The partial differential equation (5.6) and the partial differential variational inequalities (5.7) are posed on the unbounded domain \((-\infty, +\infty) \times (0, +\infty) \times (0, T)\). We need to solve them
numerically on a bounded domain \((X_{\text{min}}, X_{\text{max}}) \times (0, Y_{\text{max}}) \times (0, T)\) for sufficiently small negative number \(X_{\text{min}}\) and sufficiently large positive numbers \(X_{\text{max}}\) and \(Y_{\text{max}}\). 

For the European put, when \(S = 0\), we have from (5.1)
\[
\frac{\partial p}{\partial t} - rp = 0,
\]
which means that
\[
p(0, v, t) = p(0, v, T)e^{-r(T-t)}.
\]
It is apparent that \(p(0, v, T) = \Phi(0) = K\). Thus the boundary condition at \(S = 0\) is
\[
p(0, v, t) = Ke^{-r(T-t)}.
\]
Notice that
\[
U(x, y, \tau) = p \left( Ke^{x+\frac{\sigma}{2}y}, y \right) \rightarrow p(0, y, t) = Ke^{-r(T-t)}, \quad x \rightarrow -\infty.
\]
We set the boundary condition at \(x = X_{\text{min}}\) for the European put option as
\[
U(X_{\text{min}}, y, \tau) = Ke^{-r\tau}.
\]
(5.8)
It is known that the American put price is equal to its payoff when \(S\) is sufficiently small. The boundary condition at \(x = X_{\text{min}}\) for the American put option is naturally set as
\[
U(X_{\text{min}}, y, \tau) = K - K \exp \left( X_{\text{min}} + \frac{D}{\sigma} y \right).
\]
(5.9)
Since the put price goes to zero as \(S \rightarrow \infty\), we can set the boundary condition at \(x = X_{\text{max}}\) as follows:
\[
U(X_{\text{max}}, y, \tau) = 0.
\]
(5.10)
Since we have degenerate partial differential operator $L$ with respect to $v$, the boundary condition at $v = 0$ is the partial differential equation for the European put option and the partial differential variational inequalities for the American put option obtained from (5.6) and (5.7) by letting $v = 0$: for the European option

$$\frac{dU}{d\tau} - a_1 U_x - a_2 U_y + rU = 0,$$

and for the American option

$$\begin{cases} 
\frac{dU}{d\tau} - a_1 U_x - a_2 U_y + rU \geq 0, \quad U \geq \Psi, \\
(U - \Psi) \left( \frac{dU}{d\tau} - a_1 U_x - a_2 U_y + rU \right) = 0.
\end{cases}$$

(5.11)

As pointed in [12], [36] and [62], the put price would be expected be insensitive to volatility change as $v \to \infty$. Hence, we use the following artificial Neumann boundary conditions:

$$\frac{\partial p}{\partial v} = 0 \quad \text{and} \quad \frac{\partial q}{\partial v} = 0.$$

The above Neumann boundary conditions for $U$ become

$$U_y - \frac{\rho}{\sigma} U_x = 0.$$  

(5.12)

### 5.3 The Semi-discretization by a Finite Volume Method

Let $X_{\min} = x_0 < x_1 < x_2 < \cdots < x_{N_1-1} < x_{N_1} = X_{\max}$ and $0 = y_0 < y_1 < y_2 < \cdots < y_{N_2-1} < y_{N_2} = Y_{\max}$ be the partitions of intervals $[X_{\min}, X_{\max}]$ and $[0, Y_{\max}]$, respectively. The dual nodal points for the partitions are

$$x_{i-\frac{1}{2}} = \frac{1}{2} (x_{i-1} + x_i), \quad i = 1, \ldots, N_1,$$
and
\[ y_{j - \frac{1}{2}} = \frac{1}{2} (y_{j - 1} + y_{j}), \quad j = 1, \ldots, N_2. \]

For a given positive integer \( M \), let
\[ \tau_m = m \Delta \tau, \quad m = 0, 1, \ldots, M, \]
where \( \Delta \tau = \frac{T}{M} \) is the step size in time. The approximation of \( U(x, y, \tau_m) \) will be denoted by \( U_{i,j}^m \). For simplicity, we shall drop the superscript \( m \). In the following, we consider a finite volume method to discretize the spatial differential operator \( \mathcal{L} \). The full discretization will be given in the next section.

We first rewrite the spatial differential operator \( \mathcal{L} \) in divergent form
\[
\mathcal{L} U = \frac{\partial}{\partial x} ((a_1 + b_1 y)U) + \frac{\partial}{\partial y} \left( \left( a_2 - \frac{1}{2}\sigma^2 + b_2 y \right) U \right) \\
+ \frac{\partial}{\partial x} \left( \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{2} \sigma^2 y \frac{\partial U}{\partial y} \right) - (b_2 + r)U \\
= \frac{\partial M_1}{\partial x} - \frac{\partial L_1}{\partial y} + \frac{\partial M_2}{\partial x} - \frac{\partial L_2}{\partial y} - (b_2 + r)U,
\]
where
\[
M_1 = (a_1 + b_1 y)U, \quad L_1 = - \left( a_2 - \frac{1}{2}\sigma^2 + b_2 y \right) U, \\
M_2 = \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x}, \quad L_2 = - \frac{1}{2} \sigma^2 y \frac{\partial U}{\partial y}.
\]

### 5.3.1 The Interior Nodes

For an interior node \((x, y)\) \((1 \leq i \leq N_1 - 1 \text{ and } 1 \leq j \leq N_2 - 1)\), let \( R \) be the dual rectangle \( ABCD \) as shown in Fig. 5.1.
By the Green’s formula, we have

$$\iint_R \mathcal{L}U \, dx \, dy = \iint_R \frac{\partial M_1}{\partial x} - \frac{\partial L_1}{\partial y} + \frac{\partial M_2}{\partial x} - \frac{\partial L_2}{\partial y} - (b_2 + r)U$$

$$= \oint_{\partial R} (L_1 \, dx + M_1 \, dy) + \oint_{\partial R} (L_2 \, dx + M_2 \, dy) - \iint_R (b_2 + r)U \, dx \, dy$$

$$= \int_{AB} (L_1 + L_2) \, dx + \int_{BC} (M_1 + M_2) \, dy + \int_{CD} (L_1 + L_2) \, dx$$

$$+ \int_{DA} (M_1 + M_2) \, dy - \int_R (b_2 + r)U \, dx \, dy.$$
For the integrals in $x$ direction, we have

$$\int_{BC} M_1 dy = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} (a_1 + b_1 y) U dy \approx \frac{1}{2} (y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}) \left( a_1 + b_1 y_j \right) \left( U_{i,j} + U_{i+1,j} \right),$$

$$\int_{DA} M_1 dy = \int_{y_{j+\frac{1}{2}}}^{y_{j-\frac{1}{2}}} (a_1 + b_1 y) U dy \approx -\frac{1}{2} \left( y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right) \left( a_1 + b_1 y_j \right) \left( U_{i-1,j} + U_{i,j} \right),$$

$$\int_{BC} M_2 dy = \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx \frac{1}{2} \lambda^2 y_j \left( y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right) \frac{U_{i+1,j} - U_{i,j}}{x_{i+1} - x_i},$$

$$\int_{DA} M_2 dy = \int_{y_{j+\frac{1}{2}}}^{y_{j-\frac{1}{2}}} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx -\frac{1}{2} \lambda^2 y_j \left( y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right) \frac{U_{i,j} - U_{i-1,j}}{x_i - x_{i-1}}.$$

For the integrals in $y$ direction, we have

$$\int_{AB} L_1 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{j-\frac{1}{2}} \right) U dx$$

$$\approx -\frac{1}{2} \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{j-\frac{1}{2}} \right) \left( U_{i,j-1} + U_{i,j} \right),$$

$$\int_{CD} L_1 dx = -\int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{j+\frac{1}{2}} \right) U dx$$

$$\approx \frac{1}{2} \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{j+\frac{1}{2}} \right) \left( U_{i,j} + U_{i,j+1} \right),$$

$$\int_{AB} L_2 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{1}{2} \sigma^2 y_j \frac{\partial U}{\partial y} dx \approx -\frac{1}{2} \sigma^2 y_j \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \frac{U_{i,j} - U_{i,j-1}}{y_j - y_{j-1}},$$

$$\int_{CD} L_2 dx = -\int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} \frac{1}{2} \sigma^2 y_j \frac{\partial U}{\partial y} dx \approx \frac{1}{2} \sigma^2 y_j \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \frac{U_{i,j+1} - U_{i,j}}{y_{j+1} - y_j}.$$

For the last integral, we have

$$\iint_{R} (b_2 + r) U dx dy \approx \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right) (b_2 + r) U_{i,j}.$$

To sum up, we have for $1 \leq i \leq N_1 - 1$, $1 \leq j \leq N_2 - 1$,

$$LU(x_i, y_j, \tau_m) \approx \frac{1}{m(R)} \iint_{R} LU(x, y, \tau_m) dx dy \approx AU_{i,j} + BU_{i,j} - rU_{i,j},$$

86
where \( m(R) = \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \right) \) is the area of the rectangle \( ABCD \) and

\[
AU_{i,j} = a_{i-1,j}U_{i-1,j} - a_{i,j}U_{i,j} + a_{i+1,j}U_{i+1,j},
\]

\[
BU_{i,j} = b_{i,j-1}U_{i,j-1} - b_{i,j}U_{i,j} + b_{i,j+1}U_{i,j+1},
\]

\[
a_{i-1,j} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \frac{\lambda^2 y_j}{\Delta x_i} - (a_1 + b_1 y_j) \right],
\]

\[
a_{i,j} = \frac{\lambda^2 y_j}{\Delta x_i \Delta x_{i+1}},
\]

\[
a_{i+1,j} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \frac{\lambda^2 y_j}{\Delta x_{i+1}} + (a_1 + b_1 y_j) \right],
\]

\[
b_{i,j-1} = \frac{1}{\Delta y_j + \Delta y_{j+1}} \left[ \frac{\sigma^2 y_j}{\Delta y_j} - (a_2 + b_2 y_j) + \frac{1}{2} b_2 \Delta y_j \right],
\]

\[
b_{i,j} = \frac{\sigma^2 y_j}{\Delta y_j \Delta y_{j+1}} + \frac{1}{2} b_2,
\]

\[
b_{i,j+1} = \frac{1}{\Delta y_j + \Delta y_{j+1}} \left[ \frac{\sigma^2 y_j}{\Delta y_{j+1}} + (a_2 + b_2 y_j) + \frac{1}{2} b_2 \Delta y_{j+1} \right].
\]

### 5.3.2 The Nodes on the Boundary \( x = X_{\min} \)

For the boundary node \((0, y_j), 0 \leq j \leq N_2\), we have by the boundary conditions (5.8) and (5.9)

\[
U_{0,j}^m = \begin{cases} 
    Ke^{-rt} & \text{for European put options,} \\
    K - K \exp \left( X_{\min} + \frac{\rho y_j}{\sigma} \right) & \text{for American put options.}
\end{cases}
\]  

(5.14)

### 5.3.3 The Nodes on the Boundary \( x = X_{\max} \)

For the boundary node \((x_{N_1}, j), 1 \leq j \leq N_2 - 1\), we have by the boundary conditions (5.10)

\[
U_{N_1,j}^m = 0.
\]  

(5.15)
5.3.4 The Nodes on the Boundary $y = 0$

For the boundary node $(x_i, 0), 1 \leq i \leq N_1 - 1$, consider the dual rectangle $ABCD$, denoted by $R$, as shown in Figure 5.2. We recall the boundary condition (5.11).

Using the same idea as section 5.3.1

$$\int \int_{R} \mathcal{L} U dx dy = \oint_{\partial R} (L_1 dx + M_1 dy) + \oint_{\partial R} (L_2 dx + M_2 dy) - \int \int_{R} (b_2 + r) Udxdy$$

$$= \int_{AB} (L_1 + L_2) dx + \int_{BC} (M_1 + M_2) dy + \int_{CD} (L_1 + L_2) dx + \int_{DA} (M_1 + M_2) dy$$

$$- \int \int_{R} (b_2 + r) Udxdy.$$  

For the integrals in $x$ direction, we have

$$\int_{BC} M_1 dy = \int_{y_0}^{y_1} (a_1 + b_1 y) U dy \approx \frac{1}{2} \left( y_1 - y_0 \right) \left( a_1 + b_1 y_{\frac{1}{4}} \right) \left( U_{i,0} + U_{i+1,0} \right),$$

$$\int_{DA} M_1 dy = \int_{y_{\frac{1}{2}}}^{y_0} (a_1 + b_1 y) U dy \approx -\frac{1}{2} \left( y_1 - y_0 \right) \left( a_1 + b_1 y_{\frac{1}{4}} \right) \left( U_{i-1,0} + U_{i,0} \right),$$

$$\int_{BC} M_2 dy = \int_{y_0}^{y_1} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx \frac{1}{2} \lambda^2 y_{\frac{1}{4}} \left( y_1 - y_0 \right) \frac{U_{i+1,0} - U_{i,0}}{x_{i+1} - x_i},$$

$$\int_{DA} M_2 dy = \int_{y_{\frac{1}{2}}}^{y_1} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx -\frac{1}{2} \lambda^2 y_{\frac{1}{4}} \left( y_1 - y_0 \right) \frac{U_{i,0} - U_{i-1,0}}{x_i - x_{i-1}}.$$
For the integrals in $y$ direction, we have

$$\int_{AB} L_1 dx = - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_0 \right) dx \approx - \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 \right) U_{i,0},$$

$$\int_{CD} L_1 dx = - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_\frac{1}{2} \right) dx \approx \frac{1}{2} \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_\frac{1}{2} \right) (U_{i,0} + U_{i,1}),$$

$$\int_{AB} L_2 dx = - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \frac{1}{2} \sigma^2 y_0 \right) \frac{\partial U}{\partial y} dx = 0,$$

$$\int_{CD} L_2 dx = - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \frac{1}{2} \sigma^2 y_\frac{1}{2} \right) \frac{\partial U}{\partial y} dx \approx \frac{1}{2} \sigma^2 y_\frac{1}{2} \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \frac{U_{i,1} - U_{i,0}}{y_1 - y_0}.$$

For the last integral, we have

$$\iint_R (b_2 + r) U dx dy \approx \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( y_\frac{1}{2} - y_0 \right) (b_2 + r) U_{i,0}.$$

To sum up, we have for $1 \leq i \leq N_1 - 1$,

$$LU(x_i, 0, \tau_m) \approx \frac{1}{m(R)} \iint_R LU(x, y, \tau_m) dx dy \approx AU_{i,0} + BU_{i,0} - r U_{i,0},$$

where $m(R) = \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( y_\frac{1}{2} - y_0 \right)$ is the area of the rectangle $ABCD$ and

$$AU_{i,0} = a_{i-1,0} U_{i-1,0} - a_{i,0} U_{i,0} + a_{i+1,0} U_{i+1,0},$$

$$BU_{i,0} = b_{i,0} U_{i,0} + b_{i,1} U_{i,1},$$

$$a_{i-1,0} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \lambda^2 \Delta y_1 \frac{4 \Delta x_i}{\Delta x_i - \left( a_1 + b_1 \frac{\Delta y_1}{4} \right)} \right],$$

$$a_{i,0} = \frac{\lambda^2 \Delta y_1}{4 \Delta x_i \Delta x_{i+1}},$$

$$a_{i+1,0} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \lambda^2 \Delta y_1 \frac{4 \Delta x_i}{\Delta x_i + \left( a_1 + b_1 \frac{\Delta y_1}{4} \right)} \right],$$

$$b_{i,0} = \frac{1}{\Delta y_1} \left[ -a_2 - \frac{1}{2} a_2 \Delta y_1 \right],$$

$$b_{i,1} = \frac{1}{\Delta y_1} \left[ a_2 + \frac{1}{2} a_2 \Delta y_1 \right].$$
5.3.5 The Nodes on the Boundary \( y = Y_{\text{max}} \)

\[
\frac{\partial U}{\partial y} = \frac{\rho}{\sigma} \frac{\partial U}{\partial x}.
\]

For boundary node \((i, N_2), 1 \leq i \leq N_1 - 1\), consider the dual rectangle \(ABCD\), denoted by \(R\), as shown in Figure 5.3. By the boundary condition (5.12), on the side \(\overrightarrow{CD}\), we have

\[
\int \int_R \mathcal{L}U dx dy = \int \int_R (L_1 dx + M_1 dy) + \int \int_R (L_2 dx + M_2 dy) - \int \int_R (b_2 + r)U dx dy
\]

\[
= \int_{\overrightarrow{AB}} (L_1 + L_2) dx + \int_{\overrightarrow{BC}} (M_1 + M_2) dy + \int_{\overrightarrow{CD}} (L_1 + L_2) dx + \int_{\overrightarrow{DA}} (M_1 + M_2) dy - \int \int_R (b_2 + r)U dx dy.
\]
For the integrals in $x$ direction, we have

$$\int_{BC} M_1 dy = \int_{y_{N_2} - \frac{1}{2}}^{y_{N_2}} (a_1 + b_1 y) U dy \approx \frac{1}{2} \left( y_{N_2} - y_{N_2 - \frac{1}{2}} \right) (a_1 + b_1 y_{N_2}) (U_{i,N_2} + U_{i+1,N_2}),$$

$$\int_{DA} M_1 dy = \int_{y_{N_2}}^{y_{N_2} - \frac{1}{2}} (a_1 + b_1 y) U dy \approx -\frac{1}{2} \left( y_{N_2} - y_{N_2 - \frac{1}{2}} \right) (a_1 + b_1 y_{N_2}) (U_{i-1,N_2} + U_{i,N_2}),$$

$$\int_{BC} M_2 dy = \int_{y_{N_2} - \frac{1}{2}}^{y_{N_2}} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx \frac{1}{2} \lambda^2 y_{N_2} \left( y_{N_2} - y_{N_2 - \frac{1}{2}} \right) \frac{U_{i+1,N_2} - U_{i,N_2}}{x_{i+1} - x_i},$$

$$\int_{DA} M_2 dy = \int_{y_{N_2}}^{y_{N_2} - \frac{1}{2}} \frac{1}{2} \lambda^2 y \frac{\partial U}{\partial x} dy \approx -\frac{1}{2} \lambda^2 y_{N_2} \left( y_{N_2} - y_{N_2 - \frac{1}{2}} \right) \frac{U_{i,N_2} - U_{i-1,N_2}}{x_i - x_{i-1}}.$$

For the integrals in $y$ direction, we have

$$\int_{AB} L_1 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{N_2 - \frac{1}{2}} \right) U dx$$

$$\approx -\frac{1}{2} \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{N_2 - \frac{1}{2}} \right) (U_{i,N_2-1} + U_{i,N_2}),$$

$$\int_{CD} L_1 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{N_2} \right) U dx \approx \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( a_2 - \frac{1}{2} \sigma^2 + b_2 y_{N_2} \right) U_{i,N_2},$$

$$\int_{AB} L_2 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \frac{1}{2} \sigma^2 y_{N_2 - \frac{1}{2}} \right) \frac{\partial U}{\partial x} dx \approx -\left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( \frac{1}{2} \sigma^2 y_{N_2 - \frac{1}{2}} \right) \frac{U_{i,N_2} - U_{i,N_2-1}}{y_{N_2} - y_{N_2-1}},$$

$$\int_{CD} L_2 dx = -\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \frac{1}{2} \sigma^2 y_{N_2} \right) \frac{\partial U}{\partial y} dx = -\int_{x_{i+\frac{1}{2}}}^{x_{i-\frac{1}{2}}} \left( \frac{1}{2} \sigma^2 y_{N_2} \right) \frac{\partial U}{\partial y} dx$$

$$\approx \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( \frac{1}{2} \rho \sigma y_{N_2} \right) \frac{U_{i+1,N_2} - U_{i-1,N_2}}{x_{i+1} - x_{i-1}}.$$
where \( m(R) = \left( x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \right) \left( y_{N_2} - y_{N_2 - \frac{1}{2}} \right) \) is the area of the rectangle \( ABCD \) and

\[
\mathcal{A}U_{i,N_2} = a_{i-1,N_2} U_{i-1,N_2} - a_{i,N_2} U_{i,N_2} + a_{i+1,N_2} U_{i+1,N_2},
\]

\[
\mathcal{B}U_{i,N_2} = b_{i,N_2-1} U_{i,N_2-1} + b_{i,N_2} U_{i,N_2},
\]

\[
a_{i-1,N_2} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \frac{\lambda^2 y_{N_2}}{\Delta x_i} - (a_1 + b_1 y_{N_2}) - \frac{\rho \sigma y_{N_2}}{\Delta y_{N_2}} \right],
\]

\[
a_{i,N_2} = \frac{\lambda^2 y_{N_2}}{\Delta x_i \Delta x_{i+1}},
\]

\[
a_{i+1,N_2} = \frac{1}{\Delta x_i + \Delta x_{i+1}} \left[ \frac{\lambda^2 y_{N_2}}{\Delta x_{i+1}} + (a_1 + b_1 y_{N_2}) + \frac{\rho \sigma y_{N_2}}{\Delta y_{N_2}} \right],
\]

\[
b_{i,N_2-1} = \frac{1}{\Delta y_{N_2}} \left[ \frac{\sigma^2 y_{N_2}}{\Delta y_{N_2}} - (a_2 + b_2 y_{N_2}) + \frac{1}{2} b_2 \Delta y_{N_2} \right],
\]

\[
b_{i,N_2} = \frac{1}{\Delta y_{N_2}} \left[ -\frac{\sigma^2 y_{N_2}}{\Delta y_{N_2}} + (a_2 + b_2 y_{N_2}) - \frac{1}{2} b_2 \Delta y_{N_2} \right].
\]

### 5.3.6 Discretization of the Operator \( \mathcal{L} \)

Combining the discretizations in the previous subsections, we have

\[
\mathcal{L}U(x_i, y_j) \approx \mathcal{L}^* U_{i,j} = \mathcal{A}U_{i,j} + \mathcal{B}U_{i,j} - r U_{i,j}
\]

for \( 1 \leq i \leq N_1, 0 \leq j \leq N_2, \) and

\[
\mathcal{L}^* U_{i,j} = \begin{cases} -r U_{i,j} & \text{for European option,} \\ 0 & \text{for American option.} \end{cases}
\]

for \( i = 0, N_1 \) and \( 0 \leq j \leq N_2, \) where \( U_{0,j} \) and \( U_{N_1,j} \) are given by (5.14) and (5.15). Let

\[
\mathcal{A}^* = \mathcal{A} - \frac{1}{2} r \mathcal{I}, \quad \mathcal{B}^* = \mathcal{B} - \frac{1}{2} r \mathcal{I},
\]

where \( \mathcal{I} \) is the identity operator. Then

\[
\mathcal{L}^* = \mathcal{A}^* + \mathcal{B}^*.
\]
For fixed $0 \leq j \leq N_2$, let $U_{.,j} = [U_{0,j}, U_{1,j}, U_{2,j}, \ldots, U_{N_1,j}]^T$ and $A_j^*$ be the matrix such that $A^* U_{.,j} = A_j^* U_{.,j}$. Then we have

$$A_j^* = \begin{bmatrix} a^* & 0 \\ a_{0,j}^{-1} & a_{1,j}^0 & a_{2,j}^0 & a_{3,j}^0 & \cdots \\ & \ddots & \ddots & \ddots & \ddots \\ & & & a_{i-1,j}^0 & a_{i,j}^0 & a_{i+1,j}^0 & \cdots \\ & & & & \ddots & \ddots & \ddots \\ & & & & & a_{N_1-2,j}^0 & a_{N_1-1,j}^0 & a_{N_1,j}^0 \end{bmatrix}, \tag{5.17}$$

where

$$a^* = \begin{cases} -\frac{1}{2}r & \text{for European option,} \\ 0 & \text{for American option.} \end{cases}$$

For fixed $0 \leq i \leq N_1$, let $U_{i,.} = [U_{i,0}, U_{i,1}, U_{i,2}, \ldots, U_{i,N_2}]^T$ and $B_i^*$ be the matrix such that $B^* U_{i,.} = B_i^* U_{i,.}$. Then we have

$$B_i^* = \begin{bmatrix} b_{i,0}^0 & b_{i,1}^0 & b_{i,2}^0 & b_{i,3}^0 & \cdots & b_{i,j-2}^0 & b_{i,j-1}^0 & b_{i,j}^1 & b_{i,j+1}^1 & \cdots & b_{i,N_1-2}^0 & b_{i,N_1-1}^0 & b_{i,N_1}^1 & b_{i,N_2}^1 \\ b_{i+1,0}^{-1} & b_{i+1,1}^{-1} & b_{i+1,2}^{-1} & b_{i+1,3}^{-1} & \cdots & b_{i+1,j-2}^{-1} & b_{i+1,j-1}^{-1} & b_{i+1,j}^{+1} & b_{i+1,j+1}^{+1} & \cdots & b_{i+1,N_1-2}^{-1} & b_{i+1,N_1-1}^{-1} & b_{i+1,N_1}^{+1} & b_{i+1,N_2}^{+1} \end{bmatrix}, \tag{5.18}$$

$$(N_1+1) \times (N_1+1)$$
where

\[
\begin{align*}
   b_{i,0}^0 &= \begin{cases} 
   b^* & \text{for } i = 0, N_1, \\
   -\frac{1}{\Delta y_i} \left[ a_2 + \frac{1}{2} b_2 \Delta y_i \right] - \frac{1}{2} r & \text{for } 1 \leq i \leq N_1 - 1.
   \end{cases} \\
   b_{i,1}^+ &= \begin{cases} 
   0 & \text{for } i = 0, N_1, \\
   \frac{1}{\Delta y_i} \left[ a_2 + \frac{1}{2} b_2 \Delta y_i \right] & \text{for } 1 \leq i \leq N_1 - 1.
   \end{cases} \\
   b_{i,j}^{-1} &= \begin{cases} 
   0 & \text{for } i = 0, N_1, \\
   \frac{1}{\Delta y_j + \Delta y_{j+1}} \left[ \frac{\Delta^2 y_{j+1}}{\Delta y_j} - (a_2 + b_2 y_j) + \frac{1}{2} b_2 \Delta y_j \right] & \text{for } 1 \leq i \leq N_1 - 1, 1 \leq j \leq N_2 - 1.
   \end{cases} \\
   b_{i,j}^0 &= \begin{cases} 
   b^* & \text{for } i = 0, N_1, \\
   -\left[ \frac{\Delta^2 y_{j+1}}{\Delta y_j \Delta y_{j+1}} + \frac{1}{2} b_2 \right] - \frac{1}{2} r & \text{for } 1 \leq i \leq N_1 - 1, 1 \leq j \leq N_2 - 1.
   \end{cases} \\
   b_{i,j+1}^+ &= \begin{cases} 
   0 & \text{for } i = 0, N_1, \\
   \frac{1}{\Delta y_{j+1} + \Delta y_{j+2}} \left[ \frac{\Delta^2 y_{j+1}}{\Delta y_{j+2}} + (a_2 + b_2 y_{j+1}) + \frac{1}{2} b_2 \Delta y_{j+1} \right] & \text{for } 1 \leq i \leq N_1 - 1, 1 \leq j \leq N_2 - 1.
   \end{cases} \\
   b_{i,N_2-1}^{-1} &= \begin{cases} 
   0 & \text{for } i = 0, N_1, \\
   \frac{1}{\Delta y_{N_2}} \left[ \frac{\Delta^2 y_{N_2}}{\Delta y_{N_2}} - (a_2 + b_2 y_{N_2}) + \frac{1}{2} b_2 \Delta y_{N_2} \right] & \text{for } 1 \leq i \leq N_1 - 1.
   \end{cases} \\
   b_{i,N_2}^0 &= \begin{cases} 
   b^* & \text{for } i = 0, N_1, \\
   \frac{1}{\Delta y_{N_2}} \left[ -\frac{\Delta^2 y_{N_2}}{\Delta y_{N_2}} + (a_2 + b_2 y_{N_2}) - \frac{1}{2} b_2 \Delta y_{N_2} \right] - \frac{1}{2} r & \text{for } 1 \leq i \leq N_1 - 1.
   \end{cases} \\
   b^* &= \begin{cases} 
   -\frac{1}{2} r & \text{for European option,} \\
   0 & \text{for American option.}
   \end{cases}
\end{align*}
\]

\[
5.4 \text{ Time Discretization: an ADI method}
\]

Applying the Crank-Nicolson scheme and introducing an auxiliary vector \( \lambda^m \) as in [31], we have the following full-discretization scheme

\[
\begin{align*}
   \frac{U^m - U^{m-1}}{\Delta \tau} &= \frac{1}{2} \left( \mathcal{L}^* U^m + \mathcal{L}^* U^{m-1} \right) + \lambda^m, \\
   \lambda^m \geq 0, & \quad U^m \geq G, & (U^m - G) \lambda^m = 0,
\end{align*}
\]

where \( U^m \) is the approximation of \( U(\tau_m) \) and \( G_{i,j} = \Psi(x_i, y_j) \). The equation (5.19) can be rewritten as

\[
\left( \mathcal{I} - \frac{1}{2} \Delta \tau \mathcal{L}^* \right) U^m = \left( \mathcal{I} + \frac{1}{2} \Delta \tau \mathcal{L}^* \right) U^{m-1} + \Delta \tau \lambda^m,
\]

where
where $I$ is the identity operator. In order to solve the above linear complementarity problem (LCP) by an ADI method, we have the following approximate LCP by using the technique introduced by Ikonen and Toivanen [36]

$$
\left( I - \frac{1}{2} \Delta \tau L^* \right) \bar{U}^m = \left( I + \frac{1}{2} \Delta \tau L^* \right) U^{m-1} + \Delta \tau \lambda^{m-1}, \tag{5.22}
$$

$$
U^m - \bar{U}^m - \Delta \tau (\lambda^m - \lambda^{m-1}) = 0, \tag{5.23}
$$

$$
\lambda^m \geq 0, \quad U^m \geq G, \quad (U^m - G) \lambda^m = 0, \tag{5.24}
$$

where $\bar{U}^m$ can be regarded as the prediction of the approximation to $U(t_m)$ and $\lambda^0 = 0$.

Once $\bar{U}^m$ is obtained by solving (5.22), we can easily solve (5.23)–(5.24) to get

$$
U^m = \max \left\{ \bar{U}^m - \Delta \tau \lambda^{m-1}, G \right\},
$$

$$
\lambda^m = \max \left\{ \lambda^{m-1} + \frac{G - \bar{U}^m}{\Delta \tau}, 0 \right\}.
$$

By Theorem 1 in [36], the truncation errors for schemes (5.22)–(5.24) and (5.19)–(5.20) are of the same order, which is at most $O((\Delta \tau)^2)$. The possible irregularity of the solution with respect to time might reduce the order of accuracy.

Next we consider how to solve (5.22) by an ADI method. Since $L^* = A^* + B^*$, equation (5.22) becomes as

$$
\left( I - \frac{1}{2} \Delta \tau (A^* + B^*) \right) \bar{U}^m = \left( I + \frac{1}{2} \Delta \tau (A^* + B^*) \right) U^{m-1} + \Delta \tau \lambda^{m-1}.
$$

By adding and subtracting the corresponding $\frac{1}{4}(\Delta \tau)^2 A^* B^*$ term, we have

$$
\left( I - \frac{1}{2} \Delta \tau A^* \right) \left( I - \frac{1}{2} \Delta \tau B^* \right) \bar{U}^m = \left( I + \frac{1}{2} \Delta \tau A^* \right) \left( I + \frac{1}{2} \Delta \tau B^* \right) U^{m-1}
$$

$$
+ \Delta \tau \lambda^{m-1} + \frac{1}{4}(\Delta \tau)^2 A^* B^* (\bar{U}^m - U^{m-1}).
$$
After dropping the last term, we get
\[
\left( I - \frac{1}{2} \Delta \tau A^* \right) \left( I - \frac{1}{2} \Delta \tau B^* \right) U^m = \left( I + \frac{1}{2} \Delta \tau A^* \right) \left( I + \frac{1}{2} \Delta \tau B^* \right) U^{m-1} + \Delta \tau \lambda^{m-1}.
\]
The above equation can be rewritten as
\[
\left( I - \frac{1}{2} \Delta \tau B^* \right) U^m = \left( I + \frac{1}{2} \Delta \tau A^* \right) \left( I - \frac{1}{2} \Delta \tau A^* \right)^{-1} \left( I + \frac{1}{2} \Delta \tau B^* \right) U^{m-1} \\
+ \Delta \tau \left( I - \frac{1}{2} \Delta \tau A^* \right)^{-1} \lambda^{m-1}.
\]
Letting
\[
U^{m-\frac{1}{2}} = \left( I - \frac{1}{2} \Delta \tau A^* \right)^{-1} \left( I + \frac{1}{2} \Delta \tau B^* \right) U^{m-1},
\]
we have the following ADI scheme to compute \( U^m \):
\[
\left( I - \frac{1}{2} \Delta \tau A^* \right) U^{m-\frac{1}{2}} = \left( I + \frac{1}{2} \Delta \tau B^* \right) U^{m-1}, \tag{5.25}
\]
\[
\left( I - \frac{1}{2} \Delta \tau B^* \right) U^m = \left( I + \frac{1}{2} \Delta \tau A^* \right) U^{m-\frac{1}{2}} + \Delta \tau \left( I - \frac{1}{2} \Delta \tau A^* \right)^{-1} \lambda^{m-1}. \tag{5.26}
\]
To conclude this section, we formulate the following algorithm for our FV-ADI method, which has been implemented by writing a C++ package.
Algorithm 9. A FV-ADI method for the American put option under Heston model

1. Let

\[ U_{i,j}^0 = \Psi(x_i, y_j), \quad i = 0, \ldots, N_1, \quad j = 0, \ldots, N_2. \]

\[ U_{0,j}^m = K - K \exp\left( X_{\min} + \frac{\rho}{\sigma} y_j \right), \quad m = 0, \ldots, M, \quad j = 0, \ldots, N_2. \]

\[ U_{N_1,j}^m = 0, \quad m = 0, \ldots, M, \quad j = 0, \ldots, N_2. \]

\[ \lambda^0 = 0 \]

2. For \( m = 1, \ldots, M \), do

   For \( i = 0, \ldots, N_1 \), do
       - compute \( f_{i,*} = U_{i,*}^{m-1}(I + \frac{1}{2}\Delta \tau B^*)^T \) to get the \( i \)-th row of \( f \)
   End do

   For \( j = 0, \ldots, N_2 \), do
       - solve \( (I - \frac{1}{2}\Delta \tau A^*_j) \overline{U}_{.,j}^{m-\frac{1}{2}} = f_{.,j} \) to get \( \overline{U}_{.,j}^{m-\frac{1}{2}} \)
       - solve \( (I - \frac{1}{2}\Delta \tau A^*_j) \gamma_{.,j} = \lambda_{.,j}^{m-1} \) to get the \( j \)-th column of \( \gamma \)
       - compute \( g_{.,j} = (I + \frac{1}{2}\Delta \tau A^*)\overline{U}_{.,j}^{m-\frac{1}{2}} + \Delta \tau\gamma_{.,j} \) to get the \( j \)-th column of \( g \)
   End do

   For \( i = 1, \ldots, N_1 \), do
       - solve \( (I - \frac{1}{2}\Delta \tau B^*_i) (U_{i,*}^m)^T = g_{i,*}^T \) to get \( U^m \)
   End do

Let

\[ U_{i,j}^m = \max\{U_{i,j}^m - \Delta \tau \lambda_{i,j}^{m-1}, U^0\}, \quad \lambda_{i,j}^m = \max\{\lambda_{i,j}^{m-1} + \frac{U_{i,j}^0 - U_{i,j}^m}{\Delta \tau}, 0\}. \]

for \( i = 0, \ldots, N_1, \quad j = 0, \ldots, N_2. \)

End do
5.5 Numerical Results

In this section, we present numerical results to validate our C++ codes to implement Algorithm 9 and examine the rate of convergence and efficiency of our FV-ADI method.

We shall consider the options with strike price $K = $100 and expiration date $T = 1$ year. The values of the parameters for the Heston model are specified in Table 5.1. The six cases can be grouped according to $2\kappa\eta > \sigma^2$ (Cases A and B), $2\kappa\eta = \sigma^2$ (Cases C and D), and $2\kappa\eta < \sigma^2$ (Cases E and F). It should be pointed out that $2\kappa\eta > \sigma^2$ is called the Feller condition under which the volatility (the solution of the SDE (2.14)) is always positive ([13]). In addition, the correlations are positive for Cases A, C and E and negative for the other cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\sqrt{0.4}$</td>
<td>$\sqrt{0.4}$</td>
<td>$\sqrt{0.9}$</td>
<td>$\sqrt{0.9}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
<td>0.5</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters for the Heston model for put option

We want to obtain the approximate option prices on the domain that contains $[S_{\text{min}}, S_{\text{max}}] \times [0, v_{\text{max}}]$ for stock price $S$ and volatility $v$, where $S_{\text{min}}$, $S_{\text{max}}$ and $v_{\text{max}}$ will be set according to the actual needs. It follows from the non-linear transformations in (2.17) that the computational domain for the variational inequality problem (5.7) is $[X_{\text{min}}, X_{\text{max}}] \times [0, y_{\text{max}}]$, 

98
where \( y_{\text{max}} = v_{\text{max}} \) and

\[
X_{\text{min}} = \text{floor} \left( \ln \left( \frac{S_{\text{min}}}{K} \right) - \max \left( 0, \frac{\rho}{\sigma} v_{\text{max}} \right) \right),
\]

\[
X_{\text{max}} = \text{ceil} \left( \ln \left( \frac{S_{\text{max}}}{K} \right) - \min \left( 0, \frac{\rho}{\sigma} v_{\text{max}} \right) \right).
\]

In all numerical examples, we shall set \( S_{\text{min}} = 1, S_{\text{max}} = 1000 \) and \( v_{\text{max}} = 5 \).

The uniform partition is used in \( x \) by setting \( \Delta x = \Delta t \). The graded mesh is employed in \( y \) for interval \([0, 1]\) with 32% of the total number of nodes and the uniform partition for interval \([1, 5]\) with 68% of the total number of nodes. We illustrate the partition of the computational domain for Case A in Fig. 5.4.

![Figure 5.4: Discretization mesh](image)

We run our programs on a PC with Intel(R) Core(TM) i7-5820K CPU @3.3GHz and
16G memory. The CPU times for Case A in Example 5.2 are given in the following table.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$N_1$</th>
<th>$N_2$</th>
<th>step size</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>525</td>
<td>125</td>
<td>0.04000</td>
<td>0m1.858s</td>
</tr>
<tr>
<td>50</td>
<td>1050</td>
<td>250</td>
<td>0.02000</td>
<td>0m8.015s</td>
</tr>
<tr>
<td>100</td>
<td>2100</td>
<td>500</td>
<td>0.01000</td>
<td>0m41.500s</td>
</tr>
<tr>
<td>200</td>
<td>4200</td>
<td>1000</td>
<td>0.00500</td>
<td>4m9.328s</td>
</tr>
<tr>
<td>400</td>
<td>8400</td>
<td>2000</td>
<td>0.00250</td>
<td>29m4.656s</td>
</tr>
<tr>
<td>800</td>
<td>16800</td>
<td>4000</td>
<td>0.00125</td>
<td>260m27.656s</td>
</tr>
</tbody>
</table>

Table 5.2: The CPU times

**Example 5.1. (Validation)** In this example, we validate our C++ codes for our FV-ADI method. It is well known that the values of American put option and European put option are equal when the interest rate is zero ($r = 0$). We use our ADI method to compute the American put prices while the European put prices are computed by using numerical integration based on the Heston’s formula in [33].

We display the absolute errors between the reference values and the values computed by our program for today’s option prices in Tables 5.3–5.8 for asset prices $S = 80 : 5 : 120$ while the volatility is a typical value $v = 0.16$. We also plot the maximum absolute errors (MAE) for $S = 10 : 1 : 200$ and $v = [0 : 0.01 : 1, 1.25 : 0.25 : 2]$ in Fig. 5.5. It is shown that our program produces convergent sequences of the approximate option prices as the time step size decreases. We can also observe that the rate of convergence is 2 as expected and that the super-convergence occurs at some points.
<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>29.85054</td>
<td>0.05120955</td>
<td>0.01269067</td>
<td>0.00318701</td>
<td>0.00079893</td>
<td>0.00019892</td>
</tr>
<tr>
<td>85</td>
<td>26.74502</td>
<td>0.04183880</td>
<td>0.01081631</td>
<td>0.00274977</td>
<td>0.00070083</td>
<td>0.00016184</td>
</tr>
<tr>
<td>90</td>
<td>23.87882</td>
<td>0.03817732</td>
<td>0.00950394</td>
<td>0.00242347</td>
<td>0.00054080</td>
<td>0.00014087</td>
</tr>
<tr>
<td>95</td>
<td>21.24947</td>
<td>0.03110983</td>
<td>0.00836335</td>
<td>0.00181905</td>
<td>0.00054080</td>
<td>0.00014087</td>
</tr>
<tr>
<td>100</td>
<td>18.85105</td>
<td>0.01865687</td>
<td>0.00493423</td>
<td>0.00120392</td>
<td>0.00029581</td>
<td>0.00007514</td>
</tr>
<tr>
<td>105</td>
<td>16.67483</td>
<td>0.01976479</td>
<td>0.00594802</td>
<td>0.00110563</td>
<td>0.00031791</td>
<td>0.00009123</td>
</tr>
<tr>
<td>110</td>
<td>14.70994</td>
<td>0.0181217</td>
<td>0.00594802</td>
<td>0.00110563</td>
<td>0.00031791</td>
<td>0.00009123</td>
</tr>
<tr>
<td>115</td>
<td>12.94395</td>
<td>0.01587990</td>
<td>0.00364697</td>
<td>0.0009805</td>
<td>0.00003384</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>11.36342</td>
<td>0.01331012</td>
<td>0.00067097</td>
<td>0.00021192</td>
<td>0.00001892</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3: The absolute errors: Case A

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>28.99942</td>
<td>0.08435909</td>
<td>0.02134354</td>
<td>0.00525510</td>
<td>0.00132428</td>
<td>0.00033168</td>
</tr>
<tr>
<td>85</td>
<td>25.94948</td>
<td>0.09337602</td>
<td>0.02274683</td>
<td>0.00557729</td>
<td>0.00138627</td>
<td>0.00036714</td>
</tr>
<tr>
<td>90</td>
<td>23.17441</td>
<td>0.09252513</td>
<td>0.02313850</td>
<td>0.00569280</td>
<td>0.00152574</td>
<td>0.00037244</td>
</tr>
<tr>
<td>95</td>
<td>20.66417</td>
<td>0.09627934</td>
<td>0.02331302</td>
<td>0.00614986</td>
<td>0.00150639</td>
<td>0.00036621</td>
</tr>
<tr>
<td>100</td>
<td>18.40459</td>
<td>0.10600380</td>
<td>0.02614307</td>
<td>0.00655431</td>
<td>0.00165116</td>
<td>0.00041078</td>
</tr>
<tr>
<td>105</td>
<td>16.37883</td>
<td>0.09946856</td>
<td>0.02370950</td>
<td>0.00633546</td>
<td>0.00154205</td>
<td>0.00037224</td>
</tr>
<tr>
<td>110</td>
<td>14.56866</td>
<td>0.09605189</td>
<td>0.02434278</td>
<td>0.00590746</td>
<td>0.00162415</td>
<td>0.00039523</td>
</tr>
<tr>
<td>115</td>
<td>12.95537</td>
<td>0.09400179</td>
<td>0.02600756</td>
<td>0.00649274</td>
<td>0.00161499</td>
<td>0.00039462</td>
</tr>
<tr>
<td>120</td>
<td>11.52050</td>
<td>0.09203324</td>
<td>0.02442909</td>
<td>0.00590676</td>
<td>0.00143069</td>
<td>0.00039053</td>
</tr>
</tbody>
</table>

Table 5.4: The absolute errors: Case B

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>30.35873</td>
<td>0.01146552</td>
<td>0.00286257</td>
<td>0.00074755</td>
<td>0.00018511</td>
<td>0.00004609</td>
</tr>
<tr>
<td>85</td>
<td>27.19850</td>
<td>0.00958386</td>
<td>0.00261187</td>
<td>0.00060518</td>
<td>0.00015687</td>
<td>0.00004129</td>
</tr>
<tr>
<td>90</td>
<td>24.06991</td>
<td>0.00807667</td>
<td>0.00234213</td>
<td>0.00050648</td>
<td>0.00014516</td>
<td>0.00003424</td>
</tr>
<tr>
<td>95</td>
<td>21.25083</td>
<td>0.00799582</td>
<td>0.00123010</td>
<td>0.00042663</td>
<td>0.00013264</td>
<td>0.00001971</td>
</tr>
<tr>
<td>100</td>
<td>18.65650</td>
<td>0.00326452</td>
<td>0.00161903</td>
<td>0.00044220</td>
<td>0.00009912</td>
<td>0.00002918</td>
</tr>
<tr>
<td>105</td>
<td>16.28917</td>
<td>0.00143776</td>
<td>0.00022331</td>
<td>0.00026266</td>
<td>0.00011091</td>
<td>0.0000071</td>
</tr>
<tr>
<td>110</td>
<td>14.17482</td>
<td>0.00282990</td>
<td>0.00168531</td>
<td>0.00051915</td>
<td>0.00005473</td>
<td>0.00002749</td>
</tr>
<tr>
<td>115</td>
<td>12.22805</td>
<td>0.0054611</td>
<td>0.00141071</td>
<td>0.00035889</td>
<td>0.00009391</td>
<td>0.00002089</td>
</tr>
<tr>
<td>120</td>
<td>10.52209</td>
<td>0.00723739</td>
<td>0.00067983</td>
<td>0.00049088</td>
<td>0.00002014</td>
<td>0.00002566</td>
</tr>
</tbody>
</table>

Table 5.5: The absolute errors: Case C
<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>27.85898</td>
<td>0.0277764</td>
<td>0.00756279</td>
<td>0.00174577</td>
<td>0.00043790</td>
<td>0.00010893</td>
</tr>
<tr>
<td>85</td>
<td>24.72579</td>
<td>0.03333746</td>
<td>0.00761086</td>
<td>0.00170611</td>
<td>0.00047349</td>
<td>0.00010618</td>
</tr>
<tr>
<td>90</td>
<td>21.93416</td>
<td>0.02665749</td>
<td>0.00890817</td>
<td>0.00208209</td>
<td>0.00046546</td>
<td>0.00010291</td>
</tr>
<tr>
<td>95</td>
<td>19.46937</td>
<td>0.03327126</td>
<td>0.00724237</td>
<td>0.00158146</td>
<td>0.00056782</td>
<td>0.00013470</td>
</tr>
<tr>
<td>100</td>
<td>17.30640</td>
<td>0.03056834</td>
<td>0.00644426</td>
<td>0.00157497</td>
<td>0.00052111</td>
<td>0.00012007</td>
</tr>
<tr>
<td>105</td>
<td>15.41978</td>
<td>0.02366643</td>
<td>0.0061857</td>
<td>0.00140930</td>
<td>0.00051111</td>
<td>0.00012007</td>
</tr>
<tr>
<td>110</td>
<td>13.76349</td>
<td>0.02148121</td>
<td>0.00755193</td>
<td>0.00165662</td>
<td>0.00034488</td>
<td>0.00009214</td>
</tr>
<tr>
<td>115</td>
<td>12.32153</td>
<td>0.02112664</td>
<td>0.00537457</td>
<td>0.00126829</td>
<td>0.00035586</td>
<td>0.00007612</td>
</tr>
<tr>
<td>120</td>
<td>11.06117</td>
<td>0.02083894</td>
<td>0.00460881</td>
<td>0.00157878</td>
<td>0.00033658</td>
<td>0.00007078</td>
</tr>
</tbody>
</table>

Table 5.6: The absolute errors: Case D

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>30.45936</td>
<td>0.00392613</td>
<td>0.00099924</td>
<td>0.00029043</td>
<td>0.00006599</td>
<td>0.00002427</td>
</tr>
<tr>
<td>85</td>
<td>27.07738</td>
<td>0.00221526</td>
<td>0.00069900</td>
<td>0.00019699</td>
<td>0.00005042</td>
<td>0.00001316</td>
</tr>
<tr>
<td>90</td>
<td>23.89063</td>
<td>0.00072293</td>
<td>0.00039320</td>
<td>0.00002659</td>
<td>0.00000052</td>
<td>0.00000282</td>
</tr>
<tr>
<td>95</td>
<td>20.91583</td>
<td>0.00086218</td>
<td>0.00036702</td>
<td>0.00002590</td>
<td>0.00003908</td>
<td>0.00002828</td>
</tr>
<tr>
<td>100</td>
<td>18.16891</td>
<td>0.00757869</td>
<td>0.00105941</td>
<td>0.00006331</td>
<td>0.00008200</td>
<td>0.0000860</td>
</tr>
<tr>
<td>105</td>
<td>15.66366</td>
<td>0.00959582</td>
<td>0.00141597</td>
<td>0.00013887</td>
<td>0.0001972</td>
<td>0.00001972</td>
</tr>
<tr>
<td>110</td>
<td>13.41003</td>
<td>0.00562558</td>
<td>0.00054272</td>
<td>0.000071235</td>
<td>0.00010537</td>
<td>0.00000580</td>
</tr>
<tr>
<td>115</td>
<td>11.41225</td>
<td>0.00207529</td>
<td>0.00150489</td>
<td>0.00000527</td>
<td>0.00018626</td>
<td>0.00002700</td>
</tr>
<tr>
<td>120</td>
<td>9.66736</td>
<td>0.00158195</td>
<td>0.00263193</td>
<td>0.00026566</td>
<td>0.00004216</td>
<td>0.00001374</td>
</tr>
</tbody>
</table>

Table 5.7: The absolute errors: Case E

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>Ref</th>
<th>M=25</th>
<th>M=50</th>
<th>M=100</th>
<th>M=200</th>
<th>M=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>27.15882</td>
<td>0.02023512</td>
<td>0.00641775</td>
<td>0.00149440</td>
<td>0.00033288</td>
<td>0.00007425</td>
</tr>
<tr>
<td>85</td>
<td>23.86274</td>
<td>0.03010901</td>
<td>0.00692433</td>
<td>0.00150614</td>
<td>0.00030779</td>
<td>0.00008200</td>
</tr>
<tr>
<td>90</td>
<td>20.96071</td>
<td>0.01963132</td>
<td>0.00796429</td>
<td>0.00179218</td>
<td>0.00037148</td>
<td>0.00007228</td>
</tr>
<tr>
<td>95</td>
<td>18.44960</td>
<td>0.02618924</td>
<td>0.00494498</td>
<td>0.00127605</td>
<td>0.00028369</td>
<td>0.00008891</td>
</tr>
<tr>
<td>100</td>
<td>16.30881</td>
<td>0.02794540</td>
<td>0.00556526</td>
<td>0.00103173</td>
<td>0.00039462</td>
<td>0.00007543</td>
</tr>
<tr>
<td>105</td>
<td>14.47135</td>
<td>0.01719641</td>
<td>0.00407220</td>
<td>0.00104849</td>
<td>0.00023544</td>
<td>0.00007080</td>
</tr>
<tr>
<td>110</td>
<td>12.91428</td>
<td>0.01323280</td>
<td>0.00685797</td>
<td>0.00162911</td>
<td>0.00037148</td>
<td>0.00007805</td>
</tr>
<tr>
<td>115</td>
<td>11.58531</td>
<td>0.01208901</td>
<td>0.00408051</td>
<td>0.00071725</td>
<td>0.00025836</td>
<td>0.00004534</td>
</tr>
<tr>
<td>120</td>
<td>10.44577</td>
<td>0.01151331</td>
<td>0.00285281</td>
<td>0.00069075</td>
<td>0.00017204</td>
<td>0.00004149</td>
</tr>
</tbody>
</table>

Table 5.8: The absolute errors: Case F
Figure 5.5: The rates of convergence for maximum absolute errors (validation)
Example 5.2. \textit{(The American put options)} In this example, we consider the American put option problems when \( r = 0.1 > 0 \). Since there is no analytic solutions, we investigate the rate of convergence by examining the errors between the approximate solutions to the variational inequality problem (5.7) when the number of time steps is \( M \) and \( 2M \). The errors are computed on the domain \([X_{\min} + 1, X_{\max} - 1] \times [0, 1]\). We plot the maximum absolute errors (MAE), root mean square errors (RMSE) and \( L_2 \) errors (L2E) in Figs. 5.6–5.8 when time \( t = T \). We can observe that the RMSE and L2E rates of convergence are almost 2 for Cases A and B, while the MAE rates of convergence are 1.24 and 2.13, respectively. By checking the data, we have found that the maximum absolute errors occur near the early exercise boundary for Case A. Since the initial value as well as the constrain function (the payoff function) is only in \( H^1 \), we should not expect that the exact solution to the variational inequality problem (5.7) would have the desired regularity for the MAE and RMSE estimates. The same observations can be made for Cases C, D, E, and F when Feller condition is not satisfied. We also notice that the convergency is better when the correlation is negative. In fact, the correlation between stock price \( S \) and volatility \( v \) is also negative in the real financial market ([14][43]).

We also plot the option prices as the functions of \( S \) and \( v \) when time \( t = 0, 0.5T, 0.75T, 0.99T \) in Figs. 5.9–5.14. The payoff function is also plotted for comparison. Their joint part is the exercise region. As shown in these figures, the surfaces of option prices will converge to the payoff surface as time is approaching to \( T \). In addition, the projections of the boundary lines of the two surfaces on the \( Sv \)-plain are the early exercise boundaries, which are displayed in Fig. 5.15.
<table>
<thead>
<tr>
<th>Case</th>
<th>Step size in time</th>
<th>max error</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>B</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>C</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>D</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>E</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>F</td>
<td>$10^{-3}$</td>
<td>$10^{-1}$</td>
</tr>
</tbody>
</table>

Figure 5.6: The rates of convergence for maximum absolute errors
Figure 5.7: The rates of convergence for the root mean square errors
Figure 5.8: The rates of convergence for the $L_2$ errors
Figure 5.9: The American put option prices: Case A

(a) $t = 0$
(b) $t = 0.5T$
(c) $t = 0.75T$
(d) $t = 0.99T$

Figure 5.10: The American put option prices: Case B

(a) $t = 0$
(b) $t = 0.5T$
(c) $t = 0.75T$
(d) $t = 0.99T$
Figure 5.11: The American put option prices: Case C

Figure 5.12: The American put option prices: Case D
Figure 5.13: The American put option prices: Case E

Figure 5.14: The American put option prices: Case F
Figure 5.15: The early exercise boundaries for the American put options
Example 5.3. *(Comparison)* In this example, we compare the approximate option prices computed by our method and the others in the literatures. We consider two sets of the parameters in Table 5.9 as in [31]. The parameters of Case G satisfies the Feller condition \((2\kappa \eta > \sigma^2)\) while Case H fails the condition.

<table>
<thead>
<tr>
<th>Case</th>
<th>G</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K)</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>(r)</td>
<td>0.10</td>
<td>0.04</td>
</tr>
<tr>
<td>(q)</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>(T)</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>(\kappa)</td>
<td>5.00</td>
<td>1.15</td>
</tr>
<tr>
<td>(\eta)</td>
<td>0.16</td>
<td>0.0348</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.9</td>
<td>0.39</td>
</tr>
<tr>
<td>(\rho)</td>
<td>0.1</td>
<td>-0.64</td>
</tr>
</tbody>
</table>

Table 5.9: Parameters for the Heston model for comparison

Here we choose the number of the steps such that the mesh size is 0.01 which is compatible to those in the reference papers. The approximate option prices are given in Tables 5.10–5.11 and Tables 5.12. The values in the first row are obtained by our method (FVADI). The other values are from Table 2-4 in [31] and the references therein. All the approximate option prices are compatible. However, the validations of the schemes are not carried out in the other papers as in our Example 5.1. Hence, we believe that our C++ codes should provide more accurate results.
### Table 5.10: The American option prices for Case G: $v = 0.0625$

<table>
<thead>
<tr>
<th>S</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>FVADI</td>
<td>2.0000</td>
<td>1.1078</td>
<td>0.5200</td>
<td>0.2136</td>
<td>0.0821</td>
</tr>
<tr>
<td>Haentjens &amp; Hout [31]</td>
<td>2.0000</td>
<td>1.1081</td>
<td>0.5204</td>
<td>0.2143</td>
<td>0.0827</td>
</tr>
<tr>
<td>Zvan, Forsyth &amp; Vetzal [62]</td>
<td>2.0000</td>
<td>1.1076</td>
<td>0.5202</td>
<td>0.2138</td>
<td>0.0821</td>
</tr>
<tr>
<td>Ikonen &amp; Toivanen [36]</td>
<td>2.0000</td>
<td>1.1076</td>
<td>0.5199</td>
<td>0.2135</td>
<td>0.0820</td>
</tr>
<tr>
<td>Persson &amp; Von Sydow [52]</td>
<td>1.9998</td>
<td>1.1085</td>
<td>0.5195</td>
<td>0.2150</td>
<td>0.0822</td>
</tr>
<tr>
<td>Oosterlee [50]</td>
<td>2.00</td>
<td>1.107</td>
<td>0.517</td>
<td>0.212</td>
<td>0.0815</td>
</tr>
<tr>
<td>Clarke &amp; Parrott [12]</td>
<td>2.0000</td>
<td>1.1080</td>
<td>0.5316</td>
<td>0.2261</td>
<td>0.0907</td>
</tr>
<tr>
<td>Vellekoop &amp; Nieuwenhuis [59]</td>
<td>1.9968</td>
<td>1.1076</td>
<td>0.5202</td>
<td>0.2134</td>
<td>0.0815</td>
</tr>
</tbody>
</table>

### Table 5.11: The American option prices for Case G: $v = 0.25$

<table>
<thead>
<tr>
<th>S</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>FVADI</td>
<td>2.0787</td>
<td>1.3339</td>
<td>0.7960</td>
<td>0.4483</td>
<td>0.2428</td>
</tr>
<tr>
<td>Haentjens &amp; Hout [31]</td>
<td>2.0788</td>
<td>1.3339</td>
<td>0.7962</td>
<td>0.4486</td>
<td>0.2433</td>
</tr>
<tr>
<td>Zvan, Forsyth &amp; Vetzal [62]</td>
<td>2.0784</td>
<td>1.3337</td>
<td>0.7961</td>
<td>0.4483</td>
<td>0.2428</td>
</tr>
<tr>
<td>Ikonen &amp; Toivanen [36]</td>
<td>2.0785</td>
<td>1.3336</td>
<td>0.7959</td>
<td>0.4482</td>
<td>0.2427</td>
</tr>
<tr>
<td>Persson &amp; Von Sydow [52]</td>
<td>2.0784</td>
<td>1.3333</td>
<td>0.7955</td>
<td>0.4479</td>
<td>0.2426</td>
</tr>
<tr>
<td>Oosterlee [50]</td>
<td>2.0790</td>
<td>1.3340</td>
<td>0.7960</td>
<td>0.4490</td>
<td>0.2430</td>
</tr>
<tr>
<td>Clarke &amp; Parrott [12]</td>
<td>2.0733</td>
<td>1.3290</td>
<td>0.7992</td>
<td>0.4536</td>
<td>0.2502</td>
</tr>
</tbody>
</table>

### Table 5.12: The American option prices for Case H: $v = 0.0348$

<table>
<thead>
<tr>
<th>S</th>
<th>90</th>
<th>100</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>FVADI</td>
<td>10.0042</td>
<td>3.2096</td>
<td>0.9288</td>
</tr>
<tr>
<td>Haentjens &amp; Hout [31]</td>
<td>10.0039</td>
<td>3.2126</td>
<td>0.9305</td>
</tr>
<tr>
<td>Fang &amp; Oosterlee [24]</td>
<td>9.9958</td>
<td>3.2079</td>
<td>0.9280</td>
</tr>
</tbody>
</table>

113
CHAPTER 6

CONCLUSION

In this dissertation, we introduce a general transformation to decouple correlated stochastic processes governed by a system of stochastic differential equations and apply the new transformation to some popular two-factor models such as the two-asset model, the stochastic volatility model, and the stochastic interest rate models. The transformed stochastic processes are uncorrelated and result in simpler and more effective numerical implements. This transformation can extended to higher dimensional cases.

In Chapter 3, we develop a mixed Monte Carlo/analytic method for the European options under two-factor models (two-asset model, stochastic volatility model, stochastic interest rate models). A control variates technique based on the formulation is also applied. Numerical results show that the new method is very accurate and efficient.

In Chapter 4, we develop a lattice method for the European and American options under the two-asset model and the stochastic interest rate models. The numerical results show that the lattice method is convergent linearly as expected. We also examine the properties of the early exercise regions for the American options numerically.

In chapter 5, we develop a finite volume - alternating direction implicit method for American option under the Heston model (stochastic volatility). We validate the scheme,
check the convergence and accuracy with several sets of parameters, and compare our result with other researches. We can conclude that our method is accurate and efficient.

For the future work, we will focus on the following topics:

1. Study the finite volume-ADI method to the option problems under the other two-factor models.

2. Apply the proposed methods in Chapters 3 and 4 to the stochastic volatility jump model (the Bates model [3]):

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r - q)dt + \sqrt{v(t)}dW_2(t) + dZ(t), \\
dv(t) &= \kappa [\eta - v(t)] dt + \sigma \sqrt{v(t)}dW_1(t).
\end{align*}
\]

3. Consider the following popular three-factor models:

- The Fong-Vasicek model:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r(t) - q)dt + \sigma dW_1(t), \\
dr(t) &= \kappa_r (\theta_r - r(t)) dt + \sqrt{v(t)}dW_2(t), \\
dv(t) &= \kappa_v (\theta_v - v(t)) dt + \delta \sqrt{v(t)}dW_3(t).
\end{align*}
\]

- The stochastic interest rate and volatility model:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= (r(t) - q)dt + \sqrt{v(t)}dW_1(t), \\
dv(t) &= \kappa [\eta - v(t)] dt + \sigma \sqrt{v(t)}dW_2(t), \\
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]

\[
\begin{align*}
\end{align*}
\]
The interest rate swap model:

\[ dr_d = \lambda_d (\theta_d - r_d)dt + \sigma_d r_d^\alpha dW_1(t), \]
\[ dr_f = \lambda_f (\theta_f - r_f)dt + \sigma_d r_f^\beta dW_2(t), \]
\[ \frac{dX}{X} = (r_d - r_f)dt + \sigma_X dW_3(t). \]
APPENDIX: THE ANALYTIC FORMULAS FOR $\tilde{V}$

We first consider the spread option. Its payoff function is

$$\Phi(S_1(T), S_2(T)) = (S_1(T) - S_2(T) - K)^+.$$ 

Then (3.4) becomes

$$\tilde{V}(\tilde{S}_2, z, t, T) = \mathbb{E}\left[e^{-(T-t)r} \left(z - z^{\alpha} \tilde{S}_2(T) - K\right)^+ \Big| \tilde{S}_2(t) = \tilde{S}_2\right]$$

$$= z^{\alpha} \mathbb{E}\left[e^{-(T-t)r} \left(z^{\alpha} (z - K) - \tilde{S}_2(T)\right)^+ \Big| \tilde{S}_2(t) = \tilde{S}_2\right]$$

$$= \begin{cases} 
0, & \text{if } z - K < 0, \\
\mathbb{E}\left[e^{-(T-t)r} (\tilde{K} - \tilde{S}_2(T))^+ \Big| Y(t, T), F_t\right], & \text{if } z - K \geq 0, \\
\end{cases}$$

$$= \begin{cases} 
0, & \text{if } z - K < 0, \\
z^{\alpha} e^{-(T-t)\tilde{K} N(-d_2)} - z^{\alpha} e^{-q(T-t)} \tilde{S}_2 N(-d_1), & \text{if } z - K \geq 0, \\
\end{cases}$$

(6.1)

where

$$\tilde{K} = z^{-\alpha} (z - K),$$

$$d_1 = \frac{\ln(\frac{\tilde{S}_2}{K}) + (r - \tilde{q} + \frac{1}{2} \tilde{\sigma}^2)(T - t)}{\tilde{\sigma} \sqrt{T - t}},$$

$$d_2 = d_1 - \tilde{\sigma} \sqrt{T - t}.$$ 

It should be pointed out that the spread option becomes the exchange option when $K = 0$, which can be evaluated by the Margrabe’s formula [46].

Next, we consider the classic two-assets call options. For the corresponding put options,
the put-call parity can be applied. We have the following four payoff functions

Call on max: \( \Phi_1(S_1, S_2) = \max(S_1, S_2) - K \).

Call on min: \( \Phi_2(S_1, S_2) = \min(S_1, S_2) - K \).

Maximum call: \( \Phi_3(S_1, S_2) = \max((S_1 - K_1), (S_2 - K_2)) \).

Minimum call: \( \Phi_4(S_1, S_2) = \min((S_1 - K_1), (S_2 - K_2)) \).

The first two options are just the special cases of the last two options. Indeed, we have

\[
\Phi_1(S_1, S_2) = \max(S_1, S_2) - K = \max((S_1 - K), (S_2 - K)),
\]

\[
\Phi_2(S_1, S_2) = \min(S_1, S_2) - K = \min((S_1 - K), (S_2 - K)).
\]

Notice that

\[
\Phi_3(S_1, S_2) = \max((S_1 - K_1), (S_2 - K_2)) = (S_2 - K_2 - (S_1 - K_1)) + (S_1 - K_1).
\]

We have for the maximum call option

\[
\tilde{V}(\tilde{S}_2, z, t, T) = E\left[ e^{-(T-t)r} \left( (z^\alpha \tilde{S}_2(T) - K_2 - (z-K_1)^+) + (z-K_1)^+ \right) \bigg| \tilde{S}_2(t) = \tilde{S}_2 \right]
\]

\[
= z^\alpha E\left[ e^{-(T-t)r} (\tilde{S}_2(T) - \tilde{K})^+ \bigg| \tilde{S}_2(t) = \tilde{S}_2 \right] + e^{-r(T-t)}(z-K_1)^+,
\]

where

\[
\tilde{K} = z^{-\alpha} (K_2 + (z-K_1)^+).
\]

Hence, we have by the Black-Scholes formula

\[
\tilde{V}(\tilde{S}_2, z, t, T) = z^\alpha e^{-q(T-t)}\tilde{S}_2 N(d_1) - z^\alpha e^{-r(T-t)}\tilde{K} N(d_2) + e^{-r(T-t)}(z-K_1)^+,
\]

where

\[
d_1 = \frac{\ln(\frac{\tilde{S}_2}{K}) + (r - \tilde{q} + \frac{1}{2} \tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}},
\]

\[
d_2 = d_1 - \tilde{\sigma}\sqrt{T-t}.
\]
Notice that

\[ \Phi_4(S_1, S_2) = (S_1 - K_1)^+ + (S_2 - K_2)^+ - h_3(S_1, S_2) = (S_2 - K_2)^+ - (S_2 - K_2 - (S_1 - K_1)^+)^+. \]

We have for the minimum call option

\[
\tilde{V} \left( \tilde{S}_2, z, t, T \right) = \mathbb{E} \left[ e^{-(T-t)r} \left( z^n \tilde{S}_2(T) - K_2 \right)^+ - \left( z^n \tilde{S}_2(T) - K_2 - (z - K_1)^+ \right)^+ \right| \tilde{S}_2(t) = \tilde{S}_2 
\]

\[
= z^n \left( \mathbb{E} \left[ e^{-(T-t)r} \left( \tilde{S}_2(T) - \tilde{K}_2 \right)^+ \left| \tilde{S}_2(t) = \tilde{S}_2 \right. \right) - \mathbb{E} \left[ e^{-(T-t)r} \left( \tilde{S}_2(T) - \tilde{K}_1 \right)^+ \left| \tilde{S}_2(t) = \tilde{S}_2 \right. \right) \right],
\]

where

\[ \tilde{K}_1 = z^{-n} \left( K_2 + (z - K_1)^+ \right), \quad \tilde{K}_2 = z^{-n} K_2. \]

Hence, we have by the Black-Scholes formula

\[
\tilde{V} \left( \tilde{S}_2, z, t, T \right) = z^n \left( e^{-q(T-t)} \tilde{S}_2 \left( N(d_1) - N(d_3) \right) - e^{-r(T-t)} \left( \tilde{K}_2 N(d_2) - \tilde{K}_1 N(d_4) \right) \right),
\]

where

\[
d_1 = \frac{\ln \left( \frac{\tilde{S}_2}{K_2} \right) + \left( r - \tilde{q} + \frac{1}{2} \tilde{\sigma}^2 \right) (T - t)}{\tilde{\sigma} \sqrt{T - t}},
\]

\[
d_2 = d_1 - \tilde{\sigma} \sqrt{T - t},
\]

\[
d_3 = \frac{\ln \left( \frac{\tilde{S}_2}{K_1} \right) + \left( r - \tilde{q} + \frac{1}{2} \tilde{\sigma}^2 \right) (T - t)}{\tilde{\sigma} \sqrt{T - t}},
\]

\[
d_4 = d_3 - \tilde{\sigma} \sqrt{T - t}.
\]


CURRICULUM VITAE

Graduate College
University of Nevada, Las Vegas

Jiacheng Cai

Degrees:
   Bachelor of Science - Mathematics, 2011
   University of Science and Technology of China, China

Dissertation Title:
   Numerical Methods for Option Pricing under the Two-Factor Models

Dissertation Examination Committee:
   Chairperson, Hongtao Yang, Ph.D.
   Committee Member, Michael Marcozzi, Ph.D.
   Committee Member, Monika Neda, Ph.D.
   Committee Member, Pengtao Sun, Ph.D.
   Graduate Faculty Representative, Jianzhong Zhang, Ph.D.