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A variational principle and its connection with fixed point theory

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A VARIATIONAL PRINCIPLE AND ITS CONNECTION WITH FIXED POINT THEORY

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ABSTRACT

In this thesis, the following topics will be discussed.

In chapter 2, a variational principle due to I.Ekeland (EVP) will be considered which deals with minimization of functions on a **complete** metric space.

In chapter 3, the notion of completeness of a metric space will be characterized by means of various approaches. Several different statements will be given and shown to be all equivalent. One of these will be considered separately in chapter 4.

In chapter 4, a direct approach to finding a fixed point of a self-mapping T on a complete metric space will be discussed. A transfinite induction argument will be used.

Chapter 5 deals with an application. We will present a new proof of a **Minimum/Maximum Principle at Infinity** using Ekeland's Variational Principle.

Finally, in chapter 6, we will give an informal explanation (through an **iteration** process) of how transfinite induction works in finding fixed points of T. Several illustrative examples will be presented.

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CHAPTER 1

INTRODUCTION

In this thesis, we are going to focus on a complete metric space M and properties of real valued functions on M which are bounded from below as well as fixed points of a mapping from M to M

In Chapter 1, we will mainly prove Ekeland's Variational Principle (EVP) which will be used to prove other results and will play an important role in showing completeness of a metric space.

Roughly speaking, given a lower semi-continuous function f on a complete metric space M, which is bounded from below (say, by 0), and given u in M such that f(u) is smaller than a given $\varepsilon > 0$, we can always find an element v in an arbitrary small neighborhood of u such that f(v) is less that f(u). In other words, we can obtain a better approximation than f(u). Moreover, if f is a differentiable function, we can arrange that $|f'(v)|$ be also small. The main reason for (EVP) to hold true is the fact that M is complete.

In Chapter 3, we will characterize completeness through various statements, namely: Let (M,d) be a metric space. Then, the following statements are all equivalent.

(1) M is complete.

(2) For any non-increasing sequence $\{S_n\}_{n=1}^{\infty}$ of nonempty closed subsets of M, i.e., $S_1 \supset S_2 \supset S_3 \supset \cdots$, such that $\lim_{n\to\infty}(\operatorname{diam} S_n) = 0$, one has $\bigcap_{n=1}^{\infty} S_n = \{x\}$.

(3) For every function $F: M \to \mathbf{R} \cup \{+\infty\}$, $F \neq +\infty$, which is continuous and bounded from below, and for every $\varepsilon > 0$, there exists $v \in M$ such that

(a) $F(v) \le \inf_M F + \varepsilon$

(b) $F(w) + \varepsilon d(v, w) > F(v)$ for all $w \neq v$ in M.

(4) For every function $\varphi: M \to \mathbf{R}$, $\varphi \ge 0$, which is l.s.c. and for every mapping $T: M \to M$ satisfying $d(u, Tu) \le \varphi(u) - \varphi(Tu)$ for all $u \in M$, then T has a fixed point.

Statement (3) is a weak version of (EVP) (Ekeland [1], Sullivan [1]), while (4) is a useful fixed point theorem due to Caristi (Caristi [1], Mawhin and Willem [1]).

In Chapter 4, a direct approach to finding a fixed point of $T: M \to M$ in (4) will be discussed (Caristi [1]). One should notice that in (4) T is not assumed to be a contraction and nor even a continuous mapping. The main role is played by the lower semi-continuous real valued "entropy function" [see page 55] φ in (4), which is non-increasing along the orbits $\{T^n u\}$ of elements u in M. A transfinite induction argument will be the formal mathematical tool in obtaining a fixed point of T.

In Chapter 5, we will discuss an application. Specifically, we will present a new proof of the Minimum/Maximum Principle at Infinity (Chen and Xin [1]) using Ekeland's Variational Principle. It concerns functions $f \in C^2(\mathbf{R})$ which are bounded from below, without loss of generality (WLOG) say $f(x) \geq 0$ for all $x \in \mathbf{R}$. Suppose an arbitrary $\varepsilon > 0$ is given. Let $u \in \mathbf{R}$ be such that $f(u) \leq \varepsilon$. Then, we can always find $v \in \mathbf{R}$ satisfying

(i) $f(v) \leq \varepsilon$

(ii) $|f'(v)| \leq \varepsilon$

(iii) $f''(v) \geq 0$.

Notice that indeed the above conclusions are reminiscent of EVP.

Finally, in Chapter 6, we will give an informal explanation (through an iteration process)of how transfinite induction works in finding fixed points of T. The main idea is as follows:

Pick any point u in M, and define $u_0 = u$. Apply successively the mapping T to obtain the orbit of u_0:

$u_1 = Tu_0$, $u_2 = Tu_1 = T^2u_0$,

$u_3 = Tu_2 = T^3u_0, \cdots, u_n = T^nu_0, \cdots$. In view of the hypothesis relating T and φ in (4), it can be shown that $\{u_n\}$ is a Cauchy sequence in M and $\{\varphi(u_n)\}$ is a non-increasing sequence of (positive) real numbers.

If after a finite number of iterations the sequence of entropies $\varphi(u_n)$ stabilizes (i.e., $\varphi(u_{\bar{k}}) = \varphi(u_{\bar{k}+1}) = \cdots$ for some $\bar{k}$), then $u_{\bar{k}}$ is a fixed point of T. Otherwise, letting $\ell_0 = \lim_{n\to\infty} u_n$, we obtain $\varphi(u_0) > \varphi(u_1) > \cdots > \varphi(u_n) > \cdots > \varphi(\ell_0)$.

Next, we repeat the iteration process with starting point ℓ_0, that is we consider the orbit ℓ_0, $T^2\ell_0, \cdots,$ $T^n\ell_0, \cdots$. And so on. "Eventually", the entropy stabilizes at a minimum value and we obtain a corresponding fixed point v of the mapping T. Several illustrative examples will be given in Chapter 7.

CHAPTER 2

A VARIATIONAL PRINCIPLE

In this chapter, a variational principle due to I.Ekeland will be considered which deals with minimization of functions on a complete metric space. This principle is also known as Ekeland's Variational Principal (EVP).

Its main idea is to consider a lower semi-continuous function $\varphi\colon M \to \mathbf{R}$ which is bounded from below, say by 0. When an arbitrary $\varepsilon > 0$ is given and u in M is such that $\varphi(u) \le \varepsilon$, we can always find a point v in the neighborhood of u such that $\varphi(v) \le \varepsilon$ and the "Newton quotient" $\frac{\varphi(v) - \varphi(w)}{d(v, w)}$ is also small for every $w \ne v$ with $\varphi(w) \le \varphi(v)$. In particular, if M is a Banach space and $\varphi\colon M \to \mathbf{R}$ is differentiable, we can find $v \in M$ such that $\varphi(v) \le \varepsilon$ and $|\varphi'(v)| \le \sqrt{\varepsilon}$.

We will prove Ekeland's Variational Principle (EVP) which will be used to prove other results. Moreover, EVP will play an important role in showing completeness of a metric space.

First of all, we need some tools to prove the Variational Principle, so let us consider the following lemmas below.

Lemma 1. Let (M,d) be a metric space and $\Phi: M \to \mathbf{R} \cup \{+\infty\}$, $\Phi \neq +\infty$ be a given function. For any elements $v, w \in M$ define $w \leq v$ if and only if $\Phi(v), \Phi(w) \in \mathbf{R}$ and $\Phi(w) + d(v, w) \leq \Phi(v)$. Then $\leq$ is a partial order on M.

Proof: First, it is clear that the reflexive property $v \leq v$ holds true. Next, we check the antisymmetric property, *i.e.*, if $w \leq v$ and $v \leq w$, then $w = v$. Indeed, assume that

$$\Phi(w) + d(v, w) \leq \Phi(v) \quad \text{and} \quad \Phi(v) + d(w, v) \leq \Phi(w) .$$

By adding these two inequalities, we obtain

$$\Phi(w) + 2d(w, v) + \Phi(v) \leq \Phi(v) + \Phi(w) .$$

Since each value of Φ above is a finite number, cancellation gives $2d(w, v) \leq 0$, *i.e.*, $d(w, v) \leq 0$. It follows that $d(w, v) = 0$. Therefore, $v = w$.

Finally, let us check the <u>transitive property</u>. Suppose $u \le v$ and $v \le w$. Then, $\Phi(u) + d(v, u) \le \Phi(v)$ and $\Phi(v) + d(w, v) \le \Phi(w)$. By adding,

$$\Phi(u) + d(v, u) + d(w, v) + \Phi(v) \le \Phi(v) + \Phi(w) .$$

By cancellation, $\Phi(u) + d(v, u) + d(w, v) \le \Phi(w)$. And the triangular inequality gives

$$\Phi(u) + d(w, u) \le \Phi(u) + d(v, u) + d(w, v) \le \Phi(w) .$$

Thus, $u \le w$. The proof of Lemma 1 is complete.

Lemma 2. Let (M,d) be a metric space and $\Phi: M \to \mathbf{R} \cup \{+\infty\}$, $\Phi \ne +\infty$ be given. Let $\le$ be the partial order in Lemma 1. Given $u \in M$ with $\Phi(u) \in \mathbf{R}$, define $S = \{w \in M \mid w \le u\}$. If Φ is lower semi-continuous (l.s.c.), then S is closed.

<u>Proof</u>: Let $\{w_n\}$ be a sequence in S such that w_n converges to w in M. We need to show that $w \in S$. Since $w_n \in S$, we have that $w_n \le u$ for all $n \in \mathbf{N}$, and Lemma 1 gives

$$\Phi(w_n) + d(u, w_n) \le \Phi(u) \quad \forall n \in \mathbf{N} . \tag{2.1}$$

By convergence of $\{w_n\}$, for any given $\varepsilon \ge 0$, there exists $N = N(\varepsilon)$ such that whenever n is greater than N, we have $d(w_n, w) \le \varepsilon$. On the other hand, by lower semi-continuity of the function Φ and, increasing $N(\varepsilon)$ if necessary, it follows that

$$(2.2) \qquad -\varepsilon \le \Phi(w_n) - \Phi(w)$$

for all $n \ge N(\varepsilon)$. Inequalities (2.1) and (2.2) give us

$$-\varepsilon + d(u, w_n) \le \Phi(u) - \Phi(w) ,$$

hence,

$$\Phi(w) + d(u, w_n) \le \Phi(u) + \varepsilon .$$

Finally, the triangle inequality gives

$$\Phi(w) + d(u, w) \le \Phi(w) + d(u, w_n) + d(w_n, w) \le \Phi(u) + \varepsilon + d(w_n, w) \le \Phi(u) + 2\varepsilon$$

hence

$$\Phi(w) + d(u, v) \le \Phi(u) + 2\varepsilon .$$

Since $\varepsilon > 0$ was given arbitrarily, we conclude that $\Phi(w) + d(u, w) \le \Phi(u)$. By Lemma 1, $w \le u$, that is , $w \in S$. Therefore, S is closed.

The next lemma is used to prove EVP in this chapter. Furthermore, it will lead us to a more surprising result shown in Chapter 3.

Lemma 3: Let (M,d) be a **complete** metric space. If $\{S_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of nonempty closed subsets of M, i.e., $S_1 \supset S_2 \supset S_3 \supset \cdots$, such that

$$\lim_{n\to\infty}(\operatorname{diam} S_n) = 0, \text{ then } \bigcap_{n=1}^{\infty} S_n = \{x\} \text{ for some } x \in M$$

(see e,g. Klambauer [1]).

Note: Details of this proof will be given in Theorem 3.

Now, we have obtained some basic facts in order to prove Ekeland's Variational Principle (I. Ekeland [1], [2]). As we have seen in the abstract, EVP always gives a better approximation than the value at given point. The following is an exact statement of EVP.

Theorem 1: Let (M,d) be a complete metric space and let $\Phi: M \to \mathbf{R} \cup \{+\infty\}$, $\Phi \neq +\infty$, be a l.s.c. function which is bounded from below. Let $\varepsilon > 0$ be given and $u \in M$ be such that $\Phi(u) \leq \inf_M \Phi + \varepsilon$. Then, there exists $v \in M$ such that

(i) $\Phi(v) \leq \Phi(u)$

(ii) $d(u, v) \leq 1$

(iii) $\Phi(w) > \Phi(v) - \varepsilon d(v, w)$

for each $w \neq v$ in M.

Proof: We first remark that, by Lemma 1, with d replaced by εd, $\leq$ is a partial ordering on M. Now, let $\{S_n\}_{n=0}^{\infty}$ be the non-increasing sequence of non-empty closed subsets of M, defined as follows:

Using the given $u \in M$, we shall construct inductively a sequence $\{u_n\}$ starting with $u_0 = u$. Suppose $u_0, \cdots, u_n$ are known and the sets $S_0, \ldots, S_n$ are given by

$$S_0 = \{w \in M \mid w \leq u_0\}$$
$$S_1 = \{w \in M \mid w \leq u_1\}$$
$$\vdots$$
$$S_n = \{w \in M \mid w \leq u_n\}.$$

Then, we pick $u_{n+1} \in S_n$ such that $\Phi(u_{n+1}) \leq \inf_{S_n} \Phi + \frac{1}{n+1}$ and define $S_{n+1} = \{w \in M \mid w \leq u_{n+1}\}$.

Claim 1: $\{S_n\}$ is a non-decreasing sequence that is

$$S_{n+1} \subset S_n \text{ for } n = 1, 2, 3, \cdots.$$

Proof: By definition, $u_{n+1} \in S_n$, i.e., $u_{n+1} \leq u_n$. Therefore, if $x \in S_{n+1}$, i.e., $x \leq u_{n+1}$, we obtain $x \leq u_n$. It follows that $x \in S_n$. Thus, $S_{n+1} \subset S_n$.

Claim 2: $\lim_{n\to\infty} (\operatorname{diam} S_n) = 0$.

Proof: Recall that $u_{n+1} \in S_n$ was picked such that

$\Phi(u_{n+1}) \leq \inf_{S_n} \Phi + \frac{1}{n+1}$. If $w \in S_{n+1}$, then $w \leq u_{n+1} \leq u_n$.

By Lemma 1, $\Phi(w) + d(u_{n+1}, w) \leq \Phi(u_{n+1})$. Hence,

$$d(u_{n+1}, w) \leq \Phi(u_{n+1}) - \Phi(w)$$

$$\leq \inf_{S_n} \Phi + \frac{1}{n+1} - \Phi(w)$$

$$\leq \inf_{S_n} \Phi + \frac{1}{n+1} - \inf_{S_n} \Phi$$

$$= \frac{1}{n+1} .$$

Thus, $d(u_{n+1}, w) \leq \frac{1}{n+1}$ for an arbitrary element $w \in S_{n+1}$.

The triangle inequality $d(w, z) \leq d(w, u_{n+1}) + d(z, u_{n+1})$ applied to arbitrary elements $w, z \in S_{n+1}$ shows that the diameter of S_{n+1} is at most $\frac{2}{n+1}$, i.e., $\operatorname{diam} S_n \leq \frac{2}{n}$. Hence, $\lim_{n\to\infty} (\operatorname{diam} S_n) = 0$.

Now, by Lemma 2, each S_n is closed in view of the lower semi-continuity of the function Φ . In addition, by Claim 2, we have that $\lim_{n\to\infty} (\operatorname{diam} S_n) = 0$. Since $S_{n+1} \subset S_n$ for $n = 1, 2, \cdots$ and M is complete, we conclude by Lemma 3 that $\bigcap_{n=0}^{\infty} S_n = \{v\}$ for some $v \in M$.

Next, let us show that $\Phi(v) \leq \Phi(u)$ and $d(u, v) \leq 1$. We know that $\bigcap_{n=0}^{\infty} S_n = \{v\}$. In particular, $v \in S_0$, i.e., $v \leq u_0 = u$. By Lemma 1 (with d replaced by εd), $\Phi(v) + \varepsilon d(u, v) \leq \Phi(u)$. Therefore,

(2.3) $$\Phi(v) \leq \Phi(u) - \varepsilon d(u, v) \leq \Phi(u) ,$$

which shows (i).

Now, let us rewrite (3) as

$$\varepsilon d(u, v) \le \Phi(u) - \Phi(v) \quad . \tag{2.4}$$

By our assumption, we have

$$\Phi(u) \le \inf_M \Phi + \varepsilon \quad . \tag{2.5}$$

Clearly, $\Phi(v) \ge \inf_M \Phi$. Hence,

$$-\Phi(v) \le -\inf_M \Phi \, . \tag{2.6}$$

By substituting (2.5) and (2.6) into (2.4), we obtain

$$\begin{aligned} \varepsilon d(u, v) &\le \Phi(u) - \Phi(v) \\ &\le [\inf_M \Phi + \varepsilon] - \inf_M \Phi \\ &= \varepsilon \end{aligned}$$

Consequently, $\varepsilon d(u, v) \le \varepsilon$, i.e., $d(u, v) \le 1$ which concludes the proof of part (ii).

Finally, we must show that (iii) is true, i.e., $\Phi(w) > \Phi(v) - \varepsilon d(v, w)$ for every element $w \ne v$ in M. Recalling the definition of the partial order $\le$, this is equivalent to showing the implication $w \le v \Rightarrow w = v$.

So, assume $w \le v$. Since $v \in S_n$ for all $n \in \mathbf{N}$, we get $w \le v \le u_n$ for all $n \in \mathbf{N}$. Hence, $w \le u_n$ for all $n \in \mathbf{N}$, i.e., $w \in \bigcap_{n=1}^{\infty} S_n = \{v\}$, i.e., $w = v$.

The proof of Theorem 1 is now complete.

Remark 1: Replacing the distance d by λd with $\lambda > 0$, we obtain $d(u, v) \leq 1/\lambda$ for (ii) and $\Phi(w) > \Phi(v) - \varepsilon\lambda d(v, w)$ for (iii). By choosing $\lambda = \varepsilon^{-1/2}$, we get the following interesting result (Theorem 2) which guarantees an ε-approximation of the infinimum of a differentiable function on a Banach space together with an $\varepsilon^{1/2}$-bound for the derivative.

Theorem 2: Let X be a Banach space, $\varphi: X \to \mathbf{R}$ be a function bounded from below, and differentiable on X. Then, for each $\varepsilon > 0$ and for each $u \in X$ such that

$$\varphi(u) \leq \inf_X \varphi + \varepsilon \ ,$$

there exists $v \in X$ such that

(i) $\varphi(v) \leq \varphi(u)$

(ii) $\|u - v\| \leq \varepsilon^{1/2}$

(iii) $\|\varphi'(v)\| \leq \varepsilon^{1/2}$.

Proof of (i) and (ii): They immediately follow from Theorem 1 by taking $M = X$, $\Phi = \varphi$ and, for $\varepsilon > 0$ given, by choosing $\lambda = \varepsilon^{-1/2}$ like in Remark 1. If u satisfies $\varphi(u) \leq \inf_X \varphi + \varepsilon$, then there exists $v \in X$ such that (i) and (ii) hold. Moreover, for all $w \neq v$ in X, $\varphi(w) > \varphi(v) - \varepsilon^{1/2}\|v - w\|$.

Before proving part (iii), let us recall the definition of differentiability for a function $\varphi: X \to \mathbf{R}$.

Definition: φ is differentiable at v if there exists a linear functional $\ell_v : X \to \mathbf{R}$ [necessarily unique and denoted by $\varphi'(v)$] such that

$$\varphi(v + w) = \varphi(v) + \langle \varphi'(v), w \rangle + o(\|w\|) .$$

By using this definition, if $w = th$ with $t > 0$ and $\|h\| = 1$, then $\varphi(v + th) = \varphi(v) + \langle \varphi'(v), th \rangle + o(\|th\|)$. Since $\|h\| = 1$ and $\varphi'(v)$ is linear, we get $\varphi(v + th) - \varphi(v) = t\langle \varphi'(v), h \rangle + o(|t|)$, or, dividing by $t \neq 0$, $\dfrac{\varphi(v + th) - \varphi(v)}{t} = \langle \varphi'(v), h \rangle + \dfrac{o(|t|)}{t}$.

As t goes to zero, $\dfrac{o(|t|)}{t}$ also goes to zero. Consequently, we obtain

$$\lim_{t \to 0} \frac{\varphi(v + th) - \varphi(v)}{t} = \langle \varphi'(v), h \rangle .$$

<u>Proof of (iii)</u>: Let us take $w = v + th$, where $t > 0$ and $h \in X$ is such that $\|h\| = 1$. By the proof of (ii), we have

$\varphi(w) > \varphi(v) - \varepsilon^{1/2}\|v - w\|$. Hence,

$\varphi(v + th) > \varphi(v) - \varepsilon^{1/2}\|v - (v + th)\|$. So,

$\varphi(v + th) - \varphi(v) > -\varepsilon^{1/2}\|th\|$. But, $\|h\| = 1$ and $t > 0$ yield

$\varphi(v + th) - \varphi(v) > -\varepsilon^{1/2}t$. Dividing by $t > 0$, it follows that

$\dfrac{\varphi(v + th) - \varphi(v)}{t} > -\varepsilon^{1/2}$.

Now, the function φ is differentiable at v so that $\varphi'(v)$ exists, i.e., we obtain

$$\langle\varphi'(v), h\rangle = \lim_{t\to 0}\frac{\varphi(v + th) - \varphi(v)}{t} \geq -\varepsilon^{1/2} .$$

Thus, $\langle\varphi'(v), h\rangle \geq -\varepsilon^{1/2}$.

Notice that this is true for any h with $\|h\| = 1$. Thus, we can replace h by -h to get $\langle\varphi'(v), -h\rangle \geq -\varepsilon^{1/2}$. It follows that $-\langle\varphi'(v), h\rangle \geq -\varepsilon^{1/2}$, i.e., $\langle\varphi'(v), h\rangle \leq \varepsilon^{1/2}$. Hence, $-\varepsilon^{1/2} \leq \langle\varphi'(v), h\rangle \leq \varepsilon^{1/2}$.

Therefore, $\|\varphi'(v)\|_{X^*} = \sup_{|h|=1}|\langle\varphi'(v), h\rangle| \leq \sqrt{\varepsilon}$. Consequently, we conclude

$$\|\varphi'(v)\| \leq \sqrt{\varepsilon} .$$

Now, that we have proved Ekeland's Variational Principle and one of its important consequences, we will conclude chapter 2.

CHAPTER 3

CHARACTERIZATION OF A COMPLETE METRIC SPACE

In this chapter, the notion of completeness for a metric space will be characterized. As already seen in the abstract, there are four different statements. Our main goal is to conclude that all the statements are equivalent.

In the mean time, a theorem due to James Caristi will be introduced, and Ekeland's Variational Principle will be discussed again. Eventually, one will discover that both theorems (Caristi's and EVP) are equivalent.

Finally, we will summarize this chapter in Theorem 6.

The main reason for Ekeland's Variational Principle (EVP) to hold true is the fact that M is complete. In this chapter, we will characterize completeness through various statements.

Corollary 1: Let (M,d) be a complete metric space. Given a function $F : M \to \mathbf{R} \cup \{+\infty\}$, $F \neq +\infty$, which is continuous and bounded from below, and given $\varepsilon > 0$, there exists $v \in M$ such that

(a) $F(v) \leq \inf_{M} F + \varepsilon$

(b) $F(w) + \varepsilon d(v, w) > F(v)$ for all $w \neq v$ in M.

Proof: The proof is an immediate consequence of Theorem 1.

Next, we will take a look at an expanded version of Lemma 3 [see Chapter 2]. We notice that the following theorem is one characterization of a complete metric space.

Theorem 3: Let (M,d) be a metric space. Then M is **complete** if and only for any non-increasing sequence $\{S_n\}_{n=1}^{\infty}$ of nonempty closed subsets of M, i.e., $S_1 \supset S_2 \supset S_3 \supset \cdots$, such that $\lim_{n\to\infty} (\operatorname{diam} S_n) = 0$, one has $\bigcap_{n=1}^{\infty} S_n = \{x\}$.

Proof: As usual, let us prove the forward statement first, then prove the converse statement. Assume M is complete and let $\{S_n\}_{n=1}^{\infty}$ be a non-increasing sequence of nonempty closed subsets of M such that $\lim_{n\to\infty}(\operatorname{diam} S_n) = 0$. Given $n \in \mathbf{N}$, let $x_n \in S_n$. If $m \ge n$, then $d(x_m, x_n) \le \operatorname{diam} S_n$. Since $\operatorname{diam} S_n \to 0$ as $n \to \infty$, we get $d(x_m, x_n) \to 0$ as $m, n \to \infty$, hence $\{x_n\}$ is a Cauchy sequence. We are assuming that M is a complete metric space, so that every Cauchy sequence converges to some point in M. So, let us say $\lim_{n\to\infty} x_n = x$ for some x in M.

Now, for any arbitrary $k \in \mathbf{N}$, we obtain $x_j \in S_k$ whenever $j \ge k$. Therefore, as each S_k is closed, it follows that $x \in S_k$. Since we took an arbitrary $k \in \mathbf{N}$, we get $x \in \bigcap_{n=1}^{\infty} S_n$.

Finally, let us show uniqueness. If we assume that $y \in \bigcap_{n=1}^{\infty} S_n$, then $d(x, y) \le \operatorname{diam} S_n$ for all $n \in \mathbf{N}$. However, the diameter of S_n tends to zero as n tends to infinity, hence $d(x,y)=0$. Consequently, we obtain $\bigcap_{n=1}^{\infty} S_n = \{x\}$.

Next, let us prove the converse. Suppose M has the non-decreasing closed set property. Let $\{x_n\}$ be a Cauchy sequence in M. For each $n \in \mathbf{N}$, let

$$S_n = \overline{\{x_m \mid m \geq n\}} = \mathrm{cl}\{x_m \mid m \geq n\}.$$

Then, $\{S_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of closed set. And, the fact that $\{x_n\}$ is a Cauchy sequence implies that

$$(3.1) \qquad \lim_{n\to\infty} (\mathrm{diam}\, S_n) = 0$$

by our definition of S_n .

Let $\bigcap_{n=1}^{\infty} S_n = \{x\}$. If $\varepsilon > 0$ is given, then there exists $n_0 \in \mathbf{N}$ such that $\mathrm{diam}\, S_{n_0} \leq \varepsilon$ by (3.1) above. On the other hand, $x \in S_{n_0} = \overline{\{x_m \mid m \geq n_0\}}$. Thus, if $n \geq n_0$, then $d(x_n, x) \leq \varepsilon$, which shows that $\{x_n\}$ converges to x. We conclude that M is complete.

Q.E.D.

Now, before we prove Theorem 4, let us consider the following lemma. It is used in the proof of that theorem.

Lemma 4: Let (M,d) be a metric space. Given a Cauchy sequence $\{y_n\}$ in M, the limit $\lim_{n\to\infty} d(y_n, x)$ exists for any $x \in M$, so that the mapping $F: M \to \mathbf{R}$,

$$F(x) = \lim_{n\to\infty} d(y_n, x) \quad , \text{ is well-defined.}$$

Proof: Given $x \in M$, we notice that $\{d(y_n, x)\}$ is a real-valued Cauchy sequence in view of the triangle inequality:

$$|d(y_m, x) - d(y_n, x)| \le d(y_m, y_n)$$

and the fact that $\{y_n\}$ is a Cauchy sequence. Since **R** is complete, it follows that the sequence $\{d(y_n, x)\}$ converges. Q.E.D.

Theorem 4 characterizes a complete metric space M in terms of a weaker version of EVP (cf. Sullivan [1]).

Theorem 4: Let (M,d) be a metric space. Then M is **complete** if and only if for every function $F: M \to \mathbf{R} \cup \{+\infty\}$, $F \ne +\infty$, which is continuous and bounded from below, and for every $\varepsilon > 0$, there exists $v \in M$ such that

(a) $F(v) \le \inf_M F + \varepsilon$

(b) $F(w) + \varepsilon d(v, w) > F(v)$ for all $w \ne v$ in M.

Proof: Notice that the direct statement of showing part (a) and (b) is exactly the contents of Corollary 1. Hence, it suffices to prove the converse statement.

Let (M,d) be an arbitrary metric space for which (a) and (b) above hold true. Let $\{y_n\}$ be a Cauchy sequence in M. Recalling Lemma 4, consider the function $F\colon M \to \mathbf{R}$ given by

$$F(x) = \lim_n d(y_n, x) .$$

Claim: F is (uniformly) continuous and $\inf_M F = 0$.

<u>Proof</u>: The continuity (in fact, uniform continuity) of F follows from the triangle inequality:

$$\left|d(y_n, x) - d(y_n, y)\right| \le d(x, y),$$

for which (passing the limit as $n \to \infty$) we obtain

$$\left|F(x) - F(y)\right| \le d(x, y) .$$

On the other hand, $F(y_j) = \lim_{n \to \infty} d(y_n, y_j) \le \varepsilon$ for all $j \ge N_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\inf_M F = 0$.

Now, to show the completeness of M, we will find a $v \in M$ such that $d(y_n, v) \to 0$, i.e., $F(v) = 0$. Let $0 < \varepsilon < 1$. Then, there exists a $v \in M$ such that

(3.2) $\qquad F(v) \le \inf_M F + \varepsilon \qquad$ and

(3.3) $\qquad F(w) + \varepsilon d(v, w) > F(v)$ for all $w \ne v$ in M .

Since $\inf_M F = 0$, (3.2) can be written as $F(v) \le \varepsilon$. Pick $\hat{w} = y_p$ where p is large enough so that $F(y_p) \le \eta$ ($\eta > 0$

given). As before, we can do it based on the definition of F and the fact that $\{y_n\}$ is a Cauchy sequence.

Consequently, for any $\eta > 0$, we obtain

$$\begin{aligned} d(\hat{w}, v) &\le \lim_n d(\hat{w}, y_n) + \lim_n d(v, y_n) \\ &= F(\hat{w}) + F(v) \\ &\le \eta + \varepsilon \end{aligned}$$

Hence, $d(\hat{w}, v) \le \eta + \varepsilon$.

Let us recall that for all $w \ne v$ in M, $F(w) + \varepsilon d(v, w) > F(v)$.

Also recall that, $F(\hat{w}) \le \eta$. Therefore, we have

$$\begin{aligned} F(v) &\le F(\hat{w}) + \varepsilon d(v, \hat{w}) \\ &\le F(\hat{w}) + \varepsilon(\eta + \varepsilon) \\ &= F(\hat{w}) + \varepsilon\eta + \varepsilon^2 \\ &\le \eta + \varepsilon\eta + \varepsilon^2 . \end{aligned}$$

Since $\eta > 0$ is arbitrarily, we obtain

$$F(v) \le \varepsilon^2.$$

By repeating this argument, we get

$$F(v) \le \varepsilon^n \text{ for all } n \ge 1.$$

Since $0 < \varepsilon < 1$, we conclude that $F(v) = 0$ by letting $n \to \infty$. Thus, $\inf_M F = 0$ is attained at $v \in M$. We have proved the completeness of M.

Q.E.D.

According to **Banach Fixed Point Theorem** (BFPT), when one has a contraction mapping T on M, then T has a unique fixed point. However, one can obtain a fixed point for T without having a contraction mapping, if T satisfies certain conditions. In the proof of the forward part of the following theorem, we use EVP to show existence of a fixed point under suitable conditions on T (cf. Mawhin-Willem [1]). The original (different) proof is due to James Caristi [1]. A variant of the proof of the converse statement can be found in de Figueiredo [1].

Theorem 5: Let (M,d) be a metric space. Then M is **complete** if and only if for every function $\varphi: M \to \mathbf{R}$, $\varphi \geq 0$ which is l.s.c. and for every mapping $T: M \to M$ satisfying $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all $u \in M$, then T has a fixed point.

<u>Proof</u>: First, let us show that T has a fixed point. Clearly, φ is bounded from below, since $\varphi \geq 0$. WLOG $\inf_M \varphi = 0$. Hence, we have conditions of applying Corollary 1, i.e., if $\varepsilon > 0$ is given, then there exists $v \in M$ such that

(3.4) $\varphi(v) \leq \inf_M \varphi + \varepsilon = \varepsilon$

(3.5) $\varphi(w) + \varepsilon d(v, w) \geq \varphi(v)$, for all w in M.

Now, we show that T has v as a fixed point, i.e.,

$Tv = v$. By our assumption, the mapping T has the property that $d(v, Tv) \le \varphi(v) - \varphi(Tv)$.

Hence,

$$(3.6) \qquad \varphi(v) \ge d(v, Tv) + \varphi(Tv) \ .$$

On the other hand, letting $w = Tv$ in (3.4), we obtain

$$(3.7) \qquad \varphi(Tv) + \varepsilon d(v, Tv) \ge \varphi(v) \ .$$

Inequalities (3.6) and (3.7) give us

$$\varphi(Tv) + \varepsilon d(v, Tv) \ge d(v, Tv) + \varphi(Tv) \quad , \text{i.e.,}$$

$$(3.8) \qquad \varepsilon d(v, Tv) \ge d(v, Tv) \ .$$

By taking $\varepsilon = \frac{1}{2}$ in (3.8), we get $\frac{1}{2} d(v, Tv) \ge d(v, Tv)$.

Therefore, $d(v,Tv) = 0$, so that $Tv = v$, i.e., T has a fixed point.

Now, let us prove the converse statement. Suppose that, for any $T: M \to M$ and $\varphi: M \to \mathbf{R}$ satisfying the conditions stated in Theorem 5, T has a fixed point in M, i.e., $Tv = v$ for some v in M. We want to show that M is complete. Suppose it is not so. Then, by Theorem 4, we can find $\varepsilon > 0$ such that, for any given $u \in M$, there exists $w \ne u$ such that $\varphi(u) \ge \varphi(w) + \varepsilon d(u, w)$.

Define $F(u) = \{w \in M \mid w \ne u \ , \varphi(u) \ge \varphi(w) + \varepsilon d(u, w)\}$.

Applying **The Axiom of Choice**, if we have a map $F: M \to 2^M$, i.e., F(u) is a subset of M for each u, then there exists a map $f: M \to M$, $u \mapsto f(u)$ such that $f(u) \in F(u)$ for each u. Thus, we can find a map $T: M \to M$ (namely, $T = f$) such that

$$\text{(3.9)} \qquad Tu \neq u \text{ for all } u \text{ in } M.$$

On the other hand, by construction we have $\varphi(u) \geq \varphi(Tu) + \varepsilon d(u, Tu)$, so that T has a fixed point in M. Hence, $Tv = v$ for some v in M, so that we have just reached a contradiction with (3.9). Therefore, M is complete. Q.E.D.

Since we have been discussing Caristi's Theorem, let us take a look at another related result. In particular, the next proposition will be brought up, when we discuss an iteration process in Chapter 6.

Proposition 1: Let (M,d) be a complete metric space. Suppose $T: M \to M$ is a contraction mapping, i.e., $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in M$ and some $0 \leq \alpha < 1$. Let $\varphi: M \to \mathbf{R}$ be the continuous function defined by $\varphi(x) = \frac{1}{1-\alpha} d(x, Tx)$. Then, φ satisfies the assumption in Caristi's Theorem.

<u>Proof</u>: Since T is a contraction mapping, we have $d(Tx, T^2x) \leq \alpha d(x, Tx)$, hence

(3.10) $$-d(Tx, T^2x) \geq -\alpha d(x, Tx) \ .$$

On the other hand, by definition of φ, we have

(3.11) $$\varphi(x) = \frac{1}{1-\alpha} d(x, Tx)$$

(3.12) $$\varphi(Tx) = \frac{1}{1-\alpha} d(Tx, T^2x) \ .$$

(3.11)-(3.12) imply $\varphi(x) - \varphi(Tx) = \dfrac{1}{1-\alpha}\left[d(x, Tx) - d(Tx, T^2x)\right]$.

Using (3.10), we obtain $\varphi(x) - \varphi(Tx) \geq \dfrac{1}{1-\alpha}\left[d(x, Tx) - \alpha d(x, Tx)\right]$

$$\geq \frac{1-\alpha}{1-\alpha} d(x, Tx) \ .$$

Therefore, $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all x in M, as we wanted to show. Consequently, by Caristi's Theorem, there exists some $v \in M$ such that $Tv = v$. The fact that T is a contraction implies that the fixed point v is unique. Indeed, if $Tw = w$, then $d(w, v) = d(Tw, Tv) \leq \alpha d(w, v)$ and, since $0 \leq \alpha < 1$, we obtain $d(w, v) = 0$, i.e., $w = v$.
Q.E.D.

We shall now give an independent proof of the converse statement of Theorem 5 without using EVP directly.

Proposition 2: Let (M,d) be a metric space. Suppose that, for every function $\varphi: M \to \mathbf{R}$, $\varphi \geq 0$ which is l.s.c. and for every mapping $T: M \to M$ satisfying $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all $u \in M$, it follows that T has a fixed point. Then M is a **complete** metric space.

Proof: Let us point out that is suffices to show that $(M, \delta d)$ is complete for some $\delta > 0$, since d and δd are equivalent distances on M. Therefore, we may substitute d by δd in above, and in fact, we will choose $0 < \delta < 1$.

Let $\{y_n\}$ be a Cauchy sequence in M. Define $\varphi: M \to \mathbf{R}$ by $\varphi(x) = \lim_{n\to\infty} d(x, y_n)$.

If there exists $x_0 \in M$ such that $\varphi(x_0) = 0$, then the proof is finished, since $\varphi(x_0) = \lim_{n\to\infty} d(x_0, y_n) = 0$.

Assume by contradiction that $\varphi(x) > 0$ for all $x \in M$. Define a set A_x by $A_x = \{y \in M \mid \varphi(y) \leq \varphi(x) - \delta d(x, y)\}$.

Case 1: Suppose there exists $x_0 \in M$ such that $A_{x_0} = \{x_0\}$, i.e., if y is different from x_0, then $\varphi(y) > \varphi(x_0) - \delta d(x_0, y)$.

In view of Theorem 4, φ is (uniformly) continuous, and since $\{y_n\}$ is a Cauchy sequence, it follows that $\inf_M \varphi = 0$. Moreover, for any $\eta > 0$, there exists $N = N(\eta)$ such that whenever $n \geq N$, we have $0 < \varphi(y_n) \leq \eta$.

By hypothesis, $\varphi(y_n) > \varphi(x_0) - \delta d(x_0, y_n)$. Hence, $\varphi(y_n) + \delta d(x_0, y_n) > \varphi(x_0)$, so that $\eta + \delta d(x_0 y_n) > \varphi(x_0)$ for all $n \geq N$. Passing to the limit as $n \to \infty$, we obtain $\eta + \delta\varphi(x_0) \geq \varphi(x_0)$. Since $\eta > 0$ is arbitrary, we get $\delta\varphi(x_0) \geq \varphi(x_0)$, hence $\delta \geq 1$ (Since we are assuming $\varphi(x_0) > 0$). But this contradicts our choice of $0 < \delta < 1$. Therefore, Case 1 does not occur.

Case 2: For every $x \in M$, $A_x \supsetneq \{x\}$, i.e., given $x \in M$,

there exists $y \in A_x$ such that $y \neq x$.

Consider the mapping $\Im: M \to 2^M$ defined by $\Im(x) = A_x \setminus \{x\}$. By **The Axiom of Choice**, there exists a map $T: M \to M$ such that $Tx \in \Im(x) = A_x \setminus \{x\}$ for all x in M. So,

$$(3.13) \qquad Tx \neq x \text{ for all } x \in M.$$

Now, the fact that $Tx \in A_x$ for every x in M means that $\delta d(Tx, x) \leq \varphi(x) - \varphi(Tx)$ for every x in M. Therefore, by our hypothesis, there **must** exist $v \in M$ such that $Tv = v$. However, this is in contradiction with (3.13).

Therefore, our assumption that $\varphi(x) > 0$ for all x in M can not hold and we conclude that $\varphi(x_0) = 0$ for some $x_0 \in M$. In other words, given an arbitrary Cauchy sequence $\{y_n\}$ in M, there exists $x_0 \in M$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. The metric space (M,d) is necessarily complete.

Q.E.D.

Our next result gives an independent proof of the direct statement of theorem 5, without using EVP directly, in the case that M is compact.

Proposition 3: Suppose M is a compact metric space. Let $\varphi: M \to \mathbf{R}$, $\varphi \geq 0$, be l.s.c. and $T: M \to M$ be such that $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all $u \in M$. Then, T has a fixed point.

Proof: Since M is compact and φ is l.s.c., we have that there exists $v \in M$ such that $\varphi(v) = \inf_M \varphi = a$ (Royden [1]). By assumption, $\varphi(Tv) \leq \varphi(v)$ since $d(v, Tv) \geq 0$. But then, $\varphi(Tv) = \varphi(v)$ since $\varphi(v)$ is the minimum of φ. Therefore,

$$d(v, Tv) \leq \varphi(v) - \varphi(Tv) = 0 \quad ,$$

i.e.,

$$d(v, Tv) = 0 \quad ,$$

so that v is a fixed point of T.

Note that the above argument shows that each w in $\varphi^{-1}(a) = \{ u \in M \mid \varphi(u) = a \}$ is a fixed point of T.

Q.E.D.

Before we conclude this chapter, let us consider the following theorem. It gives us a summary of Chapter 3.

Theorem 6: Let (M,d) be a metric space. Then, the following statements are equivalent:

(a) M is complete.

(b) For any non-increasing sequence $\{S_n\}_{n=1}^{\infty}$ of nonempty closed subsets of M, i.e., $S_1 \supset S_2 \supset S_3 \supset \cdots$, such that $\lim_{n\to\infty}(\text{diam } S_n) = 0$, one has $\bigcap_{n=1}^{\infty} S_n = \{x\}$ for some x in M.

(c) For every function $F: M \to \mathbf{R} \cup \{+\infty\}$, which is continuous and bounded from below, and for every $\varepsilon > 0$, there exists $v \in M$ such that

(i) $F(v) \le \inf_M F + \varepsilon$

(ii) $F(w) + \varepsilon d(v, w) > F(v)$ for all $w \ne v$ in M.

(d) For every function $\varphi: M \to \mathbf{R}$, $\varphi \ge 0$ which is l.s.c. and for every mapping $T: M \to M$

satisfying $d(u, Tu) \le \varphi(u) - \varphi(Tu)$ for all

$u \in M$, then T has a fixed point.

Proof: By Theorem 3, (a) is true if and only if (b) is true. Moreover, by Theorem 5, (a) is true if and only if (d) is true.

We should notice that (c) and (d) are equivalent. They indicate that a weak form of EVP and Caristi's Theorem are equivalent. In fact, if we assume (c), then Corollary 1 gives (d). Conversely, the proof that (d) implies (c) can be shown by assuming (d) and not(c). It gives a contradiction, as we have seen the proof in Theorem 5.

CHAPTER 4

FINDING A FIXED POINT BY TRANSFINITE INDUCTION

In this chapter, an alternative method of finding a fixed point will be discussed. As we know, whenever one has a contraction mapping T on a complete metric space, then T has a fixed point.

On the other hand, when an arbitrary T is given, one needs some additional condition to find a fixed point. In fact, under the conditions in Theorem 6 (d), transfinite induction (Caristi [1]) will be used to find a fixed point. Furthermore, we should keep in mind that a fixed point obtained under those conditions may not be unique.

In a paper titled "Fixed Point Theorem For Mappings Satisfying Inwardness Conditions" written by James Caristi, there is a theorem which gives the idea of how to prove Theorem 8 in this chapter. So let us quote the theorem in that paper.

Theorem 7: Let (M,d) be a complete metric space, K a closed subset of M. Suppose $T: K \to K$ is an arbitrary function and $F: K \to M$ is continuous. If there exists a real number $r < 0$ such that $d(Tu, F \circ Tu) \leq d(u, Fu) + rd(u, Tu)$ for all $u \in K$, then T has a fixed point.

We shall prove Theorem 8 below by letting $r = -1$, replacing $d(u, Fu)$ with $\varphi(u)$ and making minor modifications in the proof of the above result.

Theorem 8: Let (M,d) be a complete metric space. Given a mapping $T: M \to M$. If there exists a lower semi-continuous function $\varphi: M \to \mathbf{R}$ where $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all $u \in M$, then T has a fixed point.

Proof: Let Γ be the collection of ordinals less than the first uncountable ordinal Ω, and let r_0 be the first element of Γ. Let $x_0 \in M$ and set $x_{r_0} = x_0$.

For fixed $\alpha \in \Gamma$, suppose that for all $r \in \Gamma$ with $r < \alpha$, we have defined $x_r \in M$ in such a way that

(i) If $r = r^1 + 1$, then $x_r = T(x_{r^1})$, and

(ii) If $r_n \to r$, then $x_{r_n} \to x_r$.

To complete the induction, we must define x_α. Suppose that $\alpha = \alpha^1 + 1$, i.e., α is the successor of α^1. Since α can not be successor of any other ordinal besides α^1, we define $x_\alpha = T(x_{\alpha^1})$.

Suppose $\alpha_n \to \alpha$. To define x_α, we must show that if $\beta_n \to \alpha$, then there is $x \in M$ such that $x_{\alpha_n} \to x$ and $x_{\beta_n} \to x$. By Well-Ordering of Γ, we can define a sequence $\{r_n\}$ consisting the elements of $\{\alpha_n\} \cup \{\beta_n\}$ and which is non-decreasing.

Claim 1: If $\xi_0 < \xi < \alpha$, then $\varphi(x_\xi) \le \varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \xi} d(x_r, x_{r+1})$.

Proof: We will use transfinite induction on ξ.

Suppose $\xi = \xi_0 + 1$. Then, by (i) above, we have

(4.1) $$x_\xi = T(x_{\xi_0}),$$

hence,
$$\begin{aligned}\varphi(x_\xi) &= \varphi(T(x_{\xi_0}))\\ &\le \varphi(x_{\xi_0}) - d(x_{\xi_0}, T(x_{\xi_0})) \qquad \text{[By hypothesis]}\\ &= \varphi(x_{\xi_0}) - d(x_{\xi_0}, x_\xi). \qquad \text{[By (4.1)]}\end{aligned}$$

Now, let us assume that the claim is true for $\xi = \eta$ and the summation term is finite, i.e.,

(a) $$\sum_{\xi_0 \le r < \eta} d(x_r, x_{r+1}) < +\infty$$

(b) $$\varphi(x_\eta) \le \varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \eta} d(x_r, x_{r+1}).$$

Then, if $\xi = \eta + 1$, (i) gives

(4.2) $$x_\xi = T(x_\eta),$$

hence,
$$\begin{aligned}\varphi(x_\xi) &= \varphi(T(x_\eta))\\ &\le \varphi(x_\eta) - d(x_\eta, T(x_\eta)) \qquad \text{[By hypothesis]}\\ &= \varphi(x_\eta) - d(x_\eta, x_\xi) \qquad \text{[By (4.2)]}\\ &= \left[\varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \eta} d(x_r, x_{r+1})\right] - d(x_\eta, x_\xi) \qquad \text{[by (b)]}\\ &= \varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \xi} d(x_r, x_{r+1}).\end{aligned}$$

Finally, let $\xi_n \to \xi$ and assume the claim is true for $\xi = \xi_n$ and we know that the summation term corresponding to each ξ_n is finite, i.e.,

(c) $$\sum_{\xi_0 \le r < \xi_n} d(x_r, x_{r+1}) < +\infty \quad \text{and}$$

(d) $$\varphi(x_{\xi_n}) \le \varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \xi_n} d(x_r, x_{r+1}) \text{ for all } n \in \mathbf{N}.$$

Now, let $S_n = \sum_{\xi_0 \le r < \xi_n} d(x_r, x_{r+1})$ and $S = \sum_{\xi_0 \le r < \xi} d(x_r, x_{r+1})$. By the induction hypothesis, we have

$$0 \le \varphi(x_{\xi_n}) \le \varphi(x_{\xi_0}) - \sum_{\xi_0 \le r < \xi_n} d(x_r, x_{r+1}) = \varphi(x_{\xi_0}) - S_n \text{ , so that}$$

$0 \le S_n \le \varphi(x_{\xi_0})$ for all $n \in \mathbf{N}$. Hence, $\{S_n\}$ is bounded.

Clearly,

(e) $\{S_n\}$ is non-decreasing and

(f) $$S_n = \sum_{\xi_0 \le r < \xi_n} d(x_r, x_{r+1}) \le \sum_{\xi_0 \le r < \xi} d(x_r, x_{r+1}) = S \text{ , i.e.,}$$

$S_n \le S$ for all $n \in \mathbf{N}$.

Therefore, we obtain

(4.3) $$\lim_{n \to \infty} S_n \le S .$$

Let $\{\delta_i\}$ be an enumeration of $\{ r \mid \xi_0 \le r < \xi \}$. Then,

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} d(x_{\delta_i}, x_{\delta_{i+1}}).$$

Now, given any $n \in \mathbf{N}$, pick large m so that $\xi_m \geq \max\{\delta_1, \delta_2, \cdots, \delta_n\}$. Since the partial sum S_n is non-decreasing, we get $S_m \geq \sum_{i=1}^{n} d\left(x_{\delta_i}, x_{\delta_{i+1}}\right)$. Hence, $S \leq \sup\{S_m\} = \lim_{m\to\infty} S_m \leq S$. Consequently, we obtain

$$(4.4) \qquad \lim_{n\to\infty} S_n = S ,$$

where S is a finite value.

Recall, recalling (b) and using the fact that φ is l.s.c., we obtain

$$\varphi\left(x_\xi\right) \leq \varphi\left(x_{\xi_0}\right) - \sum_{\xi_0 \leq r < \xi} d\left(x_r, x_{r+1}\right)$$

after passing to the limit, which concludes the proof of claim 1.

<u>Claim 2</u>: Let $\xi_n < \alpha$ for $n = 0, 1, 2, \cdots$, and $\{\xi_n\}$ be non-decreasing. Define $\varepsilon_n = \varphi\left(x_{\xi_{n-1}}\right) - \varphi\left(x_{\xi_n}\right)$. Then $\sum \varepsilon_n < +\infty$.

<u>Proof</u>: By Claim 1, we have

$$(4.5) \qquad \varphi\left(x_{\xi_n}\right) \leq \varphi\left(x_{\xi_{n-1}}\right) - \sum_{\xi_{n-1} \leq r < \xi_n} d\left(x_r, x_{r+1}\right) ,$$

hence

$$-\varphi\left(x_{\xi_n}\right) \geq -\left[\varphi\left(x_{\xi_{n-1}}\right) - \sum_{\xi_{n-1} \leq r < \xi_n} d\left(x_r, x_{r+1}\right)\right] ,$$

so that $\varepsilon_n = \varphi(x_{\xi_{n-1}}) - \varphi(x_{\xi_n}) \geq \sum_{\xi_{n-1} \leq r < \xi_n} d(x_r, x_{r+1}) \geq 0$.

Hence, $\varphi(x_{\xi_{n-1}}) - \varphi(x_{\xi_n}) \geq 0$, i.e., $\varphi(x_{\xi_{n-1}}) \geq \varphi(x_{\xi_n}) \geq 0$.

Thus, the sequence $\{\varphi(x_{\xi_n})\}$ is non-increasing. Therefore,

$$(4.6) \qquad \lim_{n\infty \to \infty} \varphi(x_{\xi_n}) = L \quad \text{for some } L \geq 0 \ .$$

Now, let us add the ε_i's :

$$\sum_{i=1}^{n} \varepsilon_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$$

$$= \left[\varphi(x_{\xi_0}) - \varphi(x_{\xi_1})\right] + \left[\varphi(x_{\xi_1}) - \varphi(x_{\xi_2})\right] + \cdots + \left[\varphi(x_{\xi_{n-1}}) - \varphi(x_{\xi_n})\right]$$

$$= \varphi(x_{\xi_0}) - \varphi(x_{\xi_n}) \quad \text{[By telescoping cancellation]} .$$

Hence, $$\sum \varepsilon_n = \lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon_i$$

$$= \varphi(x_{\xi_0}) - \lim_{n\to\infty} \varphi(x_{\xi_n}) < +\infty \qquad \text{[By (4.6)]} .$$

Claim 2 is finally proved.

<u>Claim 3</u>: $d(x_{r_n}, x_{r_{n+1}}) \leq \sum_{r_n \leq r < r_{n+1}} d(x_r, x_{r+1})$.

Again, we will use transfinite induction on r_{n+1} to prove the claim.

For $r_{n+1} = r_n + 1$, we get $d(x_{r_n}, x_{r_{n+1}}) = d(x_{r_n}, x_{r_n+1})$.

Assume the claim is true for $r_{n+1} = \eta$, so that

(4.7) $$d\left(x_{r_n}, x_\eta\right) \le \sum_{r_n \le r < \eta} d\left(x_r, x_{r+1}\right) .$$

Then, for $r_{n+1} = \eta + 1$, we have

$$\begin{aligned} d\left(x_{r_n}, x_{r_{n+1}}\right) &\le d\left(x_{r_n}, x_\eta\right) + d\left(x_\eta, x_{r_{n+1}}\right) \\ &\le \sum_{r_n \le r < \eta} d\left(x_r, x_{r+1}\right) + d\left(x_\eta, x_{r_{n+1}}\right) \qquad \text{[By (4.7)]} \\ &= \sum_{r_n \le r < r_{n+1}} d\left(x_r, x_{r+1}\right) . \end{aligned}$$

Now, assume that

(4.8) $$\lim_{i \to \infty} \xi_i = r_{n+1} ,$$

and that the claim is true for $r_{n+1} = \xi_i$, $i = 1, 2, \cdots$, i.e.,

(4.9) $$d\left(x_{r_n}, x_{\xi_i}\right) \le \sum_{r_n \le r < \xi_i} d\left(x_r, x_{r+1}\right) .$$

In view of (4.4), given any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ so that if $i \ge N$, then

(4.10) $$d\left(x_{\xi_i}, x_{r_{n+1}}\right) \le \varepsilon .$$

Hence,
$$\begin{aligned} d\left(x_{r_n}, x_{r_{n+1}}\right) &\le d\left(x_{r_n}, x_{\xi_N}\right) + d\left(x_{\xi_N}, x_{r_{n+1}}\right) \\ &\le \sum_{r_n \le r < \xi_N} d\left(x_r, x_{r+1}\right) + \varepsilon \end{aligned}$$

[By (4.9) and (4.10)].

Since $\varepsilon > 0$ is arbitrary, we conclude the proof of Claim 3.

Now, by Claim 3, we obtain the following:

$$(4.11) \qquad d\left(x_{r_n}, x_{r_{n+1}}\right) \le \sum_{r_n \le r < r_{n+1}} d(x_r, x_{r+1}) \ .$$

By Claim 1, $\varphi\left(x_{r_{n+1}}\right) \le \varphi\left(x_{r_n}\right) - \sum_{r_n \le r < r_{n+1}} d(x_r, x_{r+1})$, so that

$$\sum_{r_n \le r < r_{n+1}} d(x_r, x_{r+1}) \le \varphi\left(x_{r_n}\right) - \varphi\left(x_{r_{n+1}}\right) \ .$$

By letting $\varepsilon_n = \varphi\left(x_{r_n}\right) - \varphi\left(x_{r_{n+1}}\right)$, we obtain

$$(4.12) \qquad \sum_{\xi_n \le r < r_{n+1}} d(x_r, x_{r+1}) \le \varepsilon_n \ .$$

On the other hand, (4.11) and (4.12) yield

$$d\left(x_{r_n}, x_{r_{n+1}}\right) \le \sum_{r_n \le r < r_{n+1}} d(x_r, x_{r+1}) \le \varepsilon_n \ .$$

In view of (4.5) in Claim 2, $\left\{x_{r_n}\right\}$ is a Cauchy sequence. Since M is complete, there exists $x \in M$ such that $x_{r_n} \to x$, hence $x_{\alpha_n} \to x$ and $x_{\beta_n} \to x$.

By transfinite induction, we have defined a subset $K_1 \subset M$ as $K_1 = \{ x_r \mid r \in \Gamma \}$.

Let $m = \inf\{ \varphi(x) \mid x \in K_1\}$. Choose a sequence $\left\{ x_{r_j} \right\} \subset K_1$ such that $\{r_i\}$ is increasing and $\varphi\left(x_{r_i}\right) \to m$.

<u>Notice</u>: (g) $\{r_j\}$ is countable.

(h) Ω is the first uncountable ordinal.

By (g) and (h), $\{r_i\}$ can not converge to Ω, otherwise Ω would be a countable ordinal. Since $\{r_j\}$ is increasing, it must converge to its upper bound, say $r < \Omega$. Hence,

$$\lim_{j\to\infty} \varphi\left(x_{r_j}\right) = \varphi(x_r)$$

$$= m$$

$$= \inf\{\ \varphi(x) \mid x \in K_1\ \}\ .$$

If $r + 1 \in \Gamma$, then

$$\varphi(x_{r+1}) = \varphi(T(x_r)) \qquad \text{[By definition]}$$

$$\leq \varphi(x_r) - d(x_r, T(x_r))\ .$$

If $d(x_r, T(x_r)) > 0$, then $\varphi(x_{r+1}) < \varphi(x_r)$. Since $\varphi(x_r) = m = \inf\{\ \varphi(x) \mid x \in K_1\ \}$, this can not happen by minimality of $\varphi(x_r)$. Therefore,

$$d(x_r, T(x_r)) = 0,$$

i.e.,

$$T(x_r) = x_r\ .$$

We have finally obtained a fixed point of the mapping T.

Q.E.D.

CHAPTER 5

APPLICATION TO A MINIMUM/MAXIMUM PRINCIPLE AT INFINITY

In this chapter, an application to real valued functions will be discussed.

We will be focusing on a special case of the **Minimum Principle at Infinity** (Chen and Xin [1]).

Let us consider a twice differentiable function $f: \mathbf{R} \to \mathbf{R}$, bounded from below. Without loss of generality, assume $f(x) \geq 0$ for all $x \in \mathbf{R}$ and $\inf_{\mathbf{R}} f = 0$. We will show that, given an arbitrary $\varepsilon > 0$, we can find a point v in $\mathbf{R}$ satisfying

(a) $f(v) \leq \varepsilon$

(b) $|f'(v)| \leq \varepsilon$

(c) $f''(v) \geq -2\sqrt{\varepsilon}$.

Unfortunately, one can not obtain a two-sided bound for f'' as in the case of f' in (b).

We should observe that if $\inf_{\mathbf{R}} f = 0$ is attained at some $v \in \mathbf{R}$, then necessarily

(i) $f(v) = 0$

(ii) $f'(v) = 0$

(iii) $f''(v) \geq 0$.

On the other hand, even in the case that $\inf_{\mathbf{R}} f$ is not attained, one can say that there exists a "minimum at infinity." More precisely, we will show that (i), (ii), (iii) hold for arbitrary $\varepsilon > 0$, from which one has the following theorem.

Theorem 9 (Minimum Principle at Infinity) (Chen and Xin [1]) If $f \in C^2(\mathbf{R})$, $\inf_{\mathbf{R}} f = 0$, and $f(x) > 0$ for all $x \in \mathbf{R}$, then there is a sequence $\{v_n\} \subset \mathbf{R}$ such that

(5.1) $$\lim_{n\to\infty} f(v_n) = 0$$

(5.2) $$\lim_{n\to\infty} f'(v_n) = 0$$

(5.3) $$\liminf_{n\to\infty} f''(v_n) \geq 0$$

(5.4) $$|v_n| \to +\infty \quad .$$

The proof of Theorem 9 follows from the following result.

Theorem 10: Let $f \in C^2(\mathbf{R})$, $\inf_{\mathbf{R}} f = 0$, and $f(x) > 0$ for all $x \in \mathbf{R}$. Given $0 < \varepsilon < 1$, there exists $v \in \mathbf{R}$ such that

(a) $\quad 0 < f(v) \le 2\varepsilon$

(b) $\quad |f'(v)| \le \varepsilon$

(c) $\quad f''(v) \ge -2\sqrt{\varepsilon}$.

<u>Proof</u>: First of all, let us find $\hat{v} \in \mathbf{R}$ such that both $f(\hat{v})$ and $|f'(\hat{v})|$ are small. Indeed, by our assumption and applying EVP, we can obtain $\hat{v}$ such that

$$0 < f(\hat{v}) \le \frac{\varepsilon^2}{4} \tag{5.5}$$

$$|f'(\hat{v})| \le \frac{\varepsilon}{2} \ . \tag{5.6}$$

Therefore, the function $g: \mathbf{R} \to \mathbf{R}$ defined by $g(u) = f(u) + |f'(u)|$ is continuous and satisfies

$$\begin{aligned} g(\hat{v}) &= f(\hat{v}) + |f'(\hat{v})| \\ &\le \frac{\varepsilon^2}{4} + \frac{\varepsilon}{2} \ . \\ &\le \varepsilon \, . \end{aligned}$$

In particular, $\inf_{\mathbf{R}} g = 0$ as $\varepsilon > 0$ was arbitrary. Moreover, $f(u) > 0$ for all $u \in \mathbf{R}$ implies $g(u) > 0$ for all $u \in \mathbf{R}$.

Now, let us apply EVP to g. Given $\varepsilon > 0$ and $\hat{v} \in \mathbf{R}$ as above, there exists $v \in \mathbf{R}$ such that

(d) $\quad g(v) \le g(\hat{v}) \le \varepsilon$

(e) $\quad |v - \hat{v}| \le \sqrt{\varepsilon}$

(f) $\quad g(w) \ge g(v) - \sqrt{\varepsilon}|w - v|$ for all $w \in \mathbf{R}$.

Let us analyze the following possible cases.

Case 1: $f'(v) > 0$.

By continuity of f', we have $f'(u)>0$ in some neighborhood $W = (\hat{v} - \delta, \hat{v} + \delta)$ of $\hat{v}$ and

(g) $$g(w) = f(w) + |f'(w)|$$
$$= f(w) + f'(w) \quad \text{for all } w \in W.$$

Writing $w = v + t$, $0 < |t| < \delta$, and using (f) and (g), we get

$$f(v+t) + f'(v+t) \ge f(v) + f'(v) - |t|\sqrt{\varepsilon},$$

i.e., $$f(v+t) - f(v) + f'(v+t) - f'(v) \ge -|t|\sqrt{\varepsilon}.$$

If $t > 0$, we obtain $\dfrac{f(v+t) - f(v)}{t} + \dfrac{f'(v+t) - f'(v)}{t} \ge -\sqrt{\varepsilon}$.

If $t < 0$, we obtain $\dfrac{f(v+t) - f(v)}{t} + \dfrac{f'(v+t) - f'(v)}{t} \le \sqrt{\varepsilon}$.

Letting $t \to 0$ in these two inequalities gives us

$$-\sqrt{\varepsilon} \le f'(v) + f''(v) \le \sqrt{\varepsilon}.$$

Since $f'(v) \le \varepsilon$ by (d), we obtain

$$-2\sqrt{\varepsilon} \le -\sqrt{\varepsilon} - \varepsilon$$

$$\leq -\sqrt{\varepsilon} - f'(v)$$

$$\leq f''(v)$$

$$\leq \sqrt{\varepsilon} - f'(v)$$

$$\leq \sqrt{\varepsilon} \leq 2\sqrt{\varepsilon} .$$

Thus,

$$(5.7) \qquad |f''(v)| \leq 2\sqrt{\varepsilon} .$$

Case 2: $f'(v) < 0$.

Similarly to Case 1, we obtain

$$(5.8) \qquad |f''(v)| \leq 2\sqrt{\varepsilon} .$$

Case 3: $f'(v) = 0$.

In this case, we will have that either

(5.9a) $f'(y)$ does not change sign in some neighborhood $(v - r, v + r)$ of v

or

(5.9b) $f'(y)$ changes sign in every neighborhood $(v - r, v + r)$ of v.

Subcase (5.9a):

Since we have either $f'(y) \geq 0$ [or $f'(y) \leq 0$] for all $y \in (y - r, y + r)$, and $f'(v) = 0$ by assumption, we immediately obtain $f''(v) = 0$.

Subcase (5.9b):

By continuity of f and f′, there exists $\delta > 0$ such that whenever $|u - v| \le \delta$, we get

(5.10) $$|f(u) - f(v)| \le \varepsilon$$

and (5.11) $$|f'(u) - f'(v)| \le \varepsilon .$$

(5.10) gives us

$$0 < f(u)$$
$$\le \varepsilon + f(v)$$
$$\le \varepsilon + \varepsilon = 2\varepsilon .$$

On the other hand, (5.11) and the fact that $f'(v) = 0$ yield

$$|f'(u)| \le \varepsilon .$$

Let $r > 0$ be such that $0 < r < \delta$. Since f′ changes sign in every neighborhood $(v - r, v + r)$ of v, there exist $v_1, v_2 \in (v - r, v + r)$ so that $f'(v_1) f'(v_2) < 0$.

WLOG, assume $v_1 < v_2$. Note that v_1 and v_2 depend on $r > 0$.

If $f'(v_2) > 0 > f'(v_1)$, then we are finished in view of the **Mean Value Theorem (MVT)** applied to f′ on the interval $[v_1, v_2]$. Indeed by MVT, there exists $w \in (v_1, v_2)$ such that $f''(w) = \dfrac{f'(v_2) - f'(v_1)}{v_2 - v_1}$. Since $f'(v_2) > f'(v_1)$ and $v_2 > v_1$, we obtain $f''(w) > 0$.

Moreover, in case $v_1 < v_2 < v$, we choose v_i with $f'(v_i) < 0$ and we follow the same argument above for the two points v_i and v. And if $v < v_1 < v_2$, then we pick v_j with $f'(v_j) > 0$ and again use the same argument for the two points v and v_j. In either case, it follows that

$f''(w) >$ for some $w \in (v_i, v)$ or $w \in (v, v_j)$ respectively.

And since w is δ-close to v, we also obtain that $f(w)$ and $|f'(w)|$ are small, namely $0 < f(w) \le 2\varepsilon$ and $|f'(w)| \le \varepsilon$ (see (5.10), (5.11)).

The only remaining possibility is that $v_1 < v < v_2$ with $f'(v_2) < 0 < f'(v_1)$.

Now, if there exists $f'(c) \ge 0$ for some $c \in (v, v + r]$, then we are finished in view of the previous MVT argument, recalling that $f'(v) = 0$.

So, assume $f'(y) < 0$ for all $y \in (v, v + r]$. Let A_z be the set defined by $A_z = \{ f'(x) \mid v + r \le x \le z \}$. If z is close to

$v + r$, then $A_z \subset (-\infty, 0)$ by continuity of f'. Clearly, if $z_1 < z_2$, then $A_{z_1} \subset A_{z_2}$.

Let $S_z = \sup A_z = \sup\{ a \in \mathbf{R} \mid a \in A_z \}$. Hence, if $y_1 < y_2$, we have $S_{y_1} \subset S_{y_2}$.

If $S_z = S_{v+r}$ for all $z > v + r$, then $f'(z) = f'(z + r) = -\alpha < 0$ for all $z > v + r$. Therefore, we have $f(z) = f(v + r) - \alpha[z - (v + r)]$ for $z \geq v + r$ by the **Fundamental Theorem of Calculus**, so that $f(z) \to -\infty$ as $z \to \infty$. This can not happen since we are assuming $f(x) > 0$ for all $x \in \mathbf{R}$.

Therefore, we must have $0 > S_{z_0} > S_{v+r}$ for some $z_0 > v + r$, where $S_{v+r} = f'(v + r)$. However, this tells us that there is some $s \in (v + r, z_0)$ such that $f'(s) = S_{z_0} > S_{v+r} = f'(v + r)$. By the same MVT argument used earlier, we conclude that there exists $w \in (v + r, s)$ such that $f''(w) > 0$.

Moreover, $0 > f'(w) > f'(v + r) \geq -\varepsilon$. Since $f'(y) < 0$ for all $y \in (v + r, s)$, we have that f is decreasing on $(v + r, s)$, so that $0 < f(w) \leq 2\varepsilon$.

We have finished the proof of Theorem 8. Q.E.D.

<u>Proof of Theorem 9</u>: For each $n = 2, 3, \cdots$, take $\varepsilon = \dfrac{1}{n}$ and $v = v_n$ satisfying (a), (b), (c) in Theorem 10. We obtain a sequence $\{v_n\}$ satisfying

$$0 < f(v_n) \leq \frac{2}{n}$$

$$|f'(v_n)| \leq \frac{1}{n}$$

$$f''(v_n) \geq -\frac{2}{\sqrt{n}}$$

There must exist a subsequence of $\{v_n\}$ (still denoted $\{v_n\}$) such that $|v_n| \to +\infty$. Otherwise we have boundedness of $\{v_n\}$, and therefore there exists a subsequence converging to some $\tilde{v} \in \mathbf{R}$. Necessarily, $f(\tilde{v}) = 0$, contrary to our assumption that $f(x) > 0$ for all $x \in \mathbf{R}$. Q.E.D.

Remark: We can have a **Maximum Principle at Infinity** by replacing f by -f.

CHAPTER 6

FINDING A FIXED POINT BY ITERATION

In this chapter, we will again be discussing the question of finding fixed points. We will use an informal iteration process instead of the formal proof by transfinite induction.

Simply speaking, we pick an arbitrary point $u \in M$. Then, we construct a sequence $\{u_n\}$ in the following manner:

$u_0 = u$, $u_1 = Tu_0$, $u_2 = Tu_1 = T^2 u_0$, $\cdots$,

$u_n = Tu_{n-1} = T^n u_0$, $\cdots$, where T is a given self-map on M and $\varphi: M \to \mathbf{R}$ is a l.s.c. with $\varphi \geq 0$.

Assuming that φ and T are related as in Theorem 8, so that the entropy function φ decreases along orbits $\{T^n u\}$ of T, we will conclude that when φ "stabilizes" (i.e., reaches a minimum at a certain $v = v_n$), then v is a fixed point of T.

Moreover, it will turn out that we can possibly get multiple fixed points for T, in contrast to the case of BFPT.

The transfinite induction proof of Theorem 8 (Caristi [1]) is a formalization of possibly infinitely many infinite iterations of the mapping T, as a natural way of getting fixed points.

So, let us pick an arbitrary element $u \in M$ and consider the sequence

$$u_0^{(0)} = u$$

$$u_1^{(0)} = Tu_0^{(0)}$$

$$u_2^{(0)} = Tu_1^{(0)} = T^2u_0^{(0)}$$

$$\vdots$$

$$u_n^{(0)} = Tu_{n-1}^{(0)} = T^nu_0^{(0)}$$

$$\vdots \quad .$$

One can show that if T satisfies $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all u in M, then the sequence constructed above is a Cauchy sequence. Indeed, we have the following lemma.

Lemma 5: Let (M,d) be a complete metric space. Consider a mapping $T: M \to M$ satisfying $d(u, Tu) \leq \varphi(u) - \varphi(Tu)$ for all $u \in M$, where $\varphi: M \to \mathbf{R}$ is a l.s.c. and $\varphi \geq 0$. Let $v \in M$ be an arbitrary point. Then, the sequence $\{T^nv\}$ is a Cauchy sequence, hence convergent.

Proof: Set $v_0 = v$ and $v_n = T^n v$, $n = 1, 2, \cdots$. By assumption, we have $d(v_0, Tv_0) \le \varphi(v_0) - \varphi(Tv_0)$, hence $\varphi(Tv_0) \le \varphi(v_0)$, since $d(Tv, v) \ge 0$, i.e., $0 \le \varphi(v_1) \le \varphi(v_0)$. In general, we have $0 \le \varphi(v_{n+1}) = \varphi(Tv_n) \le \varphi(v_n)$ for all $n \in \mathbf{N}$.

Therefore, $\{\varphi(v_n)\}$ is a bounded monotone sequence. Let $\lim_{n\to\infty} \varphi(v_n) = L \ge 0$. Then, $\{\varphi(v_n)\}$ is a Cauchy sequence, so that $\varphi(v_n) - \varphi(v_{n+p}) \to 0$ as $n \to \infty$ (uniformly for $p \ge 1$). Now, adding the inequalities

$$d(v_n, v_{n+1}) \le \varphi(v_n) - \varphi(v_{n+1})$$

$$d(v_{n+1}, v_{n+2}) \le \varphi(v_{n+1}) - \varphi(v_{n+2})$$

$$\vdots$$

$$d(v_{n+p-1}, v_{n+p}) \le \varphi(v_{n+p-1}) - \varphi(v_{n+p}),$$

we obtain

$$d(v_n, v_{n+1}) + \cdots + d(v_{n+p-1}, v_{n+p}) \le \varphi(v_n) - \varphi(v_{n+p}).$$

so that by the triangle inequality, we conclude that $d(v_n, v_{n+p}) \le \varphi(v_n) - \varphi(v_{n+p}) \to 0$ as $n \to \infty$ (uniformly for $p \ge 1$). Therefore, $\{v_n\}$ is a Cauchy sequence.

Finally, since M is complete, the sequence $\{v_n\} = \{T^n v\}$ converges to some point $\bar{v}$ in M. Q.E.D.

If one starts with a different initial point $w \in M$, one obtains a perhaps different fixed point $\hat{w}$ [see examples later on]. Therefore, each $u \in M$ will give rise to a fixed point $\hat{u} \in M$, so that multiple fixed points may be expected in general.

We recall that, when T is a contraction mapping, one can take as "**entropy**" the function

$$\varphi(u) = \frac{1}{1-\alpha} d(u, Tu) \quad \text{[cf. Proposition 1].}$$

In this case, it turns out that φ **stabilizes** after the **first (infinite) iteration** $u_0^{(0)} = u$, $u_1^{(0)} = Tu_0^{(0)}$, $u_2^{(0)} = T^2 u_0^{(0)}, \cdots, u_n^{(0)} = T^n u_0^{(0)}, \cdots$

$$\rightarrow \ell^{(0)} = \lim_{n\to\infty} u_n^{(0)} .$$

In other words, $T\ell^{(0)} = \ell^{(0)}$.

Furthermore, the limit $\ell^{(0)}$ at the end of first infinite iteration does not depend on the initial point u (We have uniqueness by BFPT).

Now, let $u \in M$ be given. We will consider sequences generated by various iterations. Namely, let

$$u_0^{(0)} = u, \; u_1^{(0)} = Tu_0^{(0)}, \cdots, u_n^{(0)} = T^n u_0^{(0)}, \cdots$$

By Lemma 5, $\{u_n^{(0)}\} \rightarrow \ell^{(0)}$.

Let $u_0^{(1)} = \ell^{(0)}$ and define

$$u_1^{(1)} = Tu_0^{(1)}, \cdots, u_n^{(1)} = T^n u_0^{(1)}, \cdots$$

Again, by Lemma 5, $\{u_n^{(1)}\} \to \ell^{(1)}$, and so on.

We obtain convergent sequences $\{u_n^{(0)}\}$, $\{u_n^{(1)}\}$, $\cdots$, $\{u_n^{(k)}\}$, $\cdots$, where $u_0^{(k)} = \lim_{n\to\infty} u_n^{(k-1)}$, and which are such that φ decreases along each one of them in order. In other words,

$$\varphi\left(u_0^{(0)}\right) \geq \varphi\left(u_1^{(0)}\right) \geq \cdots \geq \varphi\left(u_n^{(0)}\right) \geq \cdots \geq \varphi\left(\ell^{(0)}\right),$$

$$\varphi\left(\ell^{(0)}\right) = \varphi\left(u_0^{(1)}\right) \geq \varphi\left(u_1^{(1)}\right) \geq \cdots \geq \varphi\left(u_n^{(1)}\right) \geq \cdots \geq \varphi\left(\ell^{(1)}\right),$$

$$\varphi\left(\ell^{(1)}\right) = \varphi\left(u_0^{(2)}\right) \geq \varphi\left(u_1^{(2)}\right) \geq \cdots \geq \varphi\left(u_n^{(2)}\right) \geq \cdots \geq \varphi\left(\ell^{(2)}\right),$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

In fact, the iteration process above only stops at a certain $\hat{u} \in M$ for which the entropy φ reaches a minimum value. In general, $\hat{u}$ depends on the choice of initial point $u_0^{(0)} = u$.

Suppose that, in fact, φ stabilizes at a certain $\hat{u} \in M$. We claim that $\hat{u}$ is a <u>fixed point</u> for T, i.e., $T\hat{u} = \hat{u}$. Indeed, it is clear that $0 \leq d(\hat{u}, T\hat{u}) \leq \varphi(\hat{u}) - \varphi(T\hat{u}) = 0$ gives $T\hat{u} = \hat{u}$.

CHAPTER 7

SOME ILLUSTRATIVE MODEL EXAMPLES

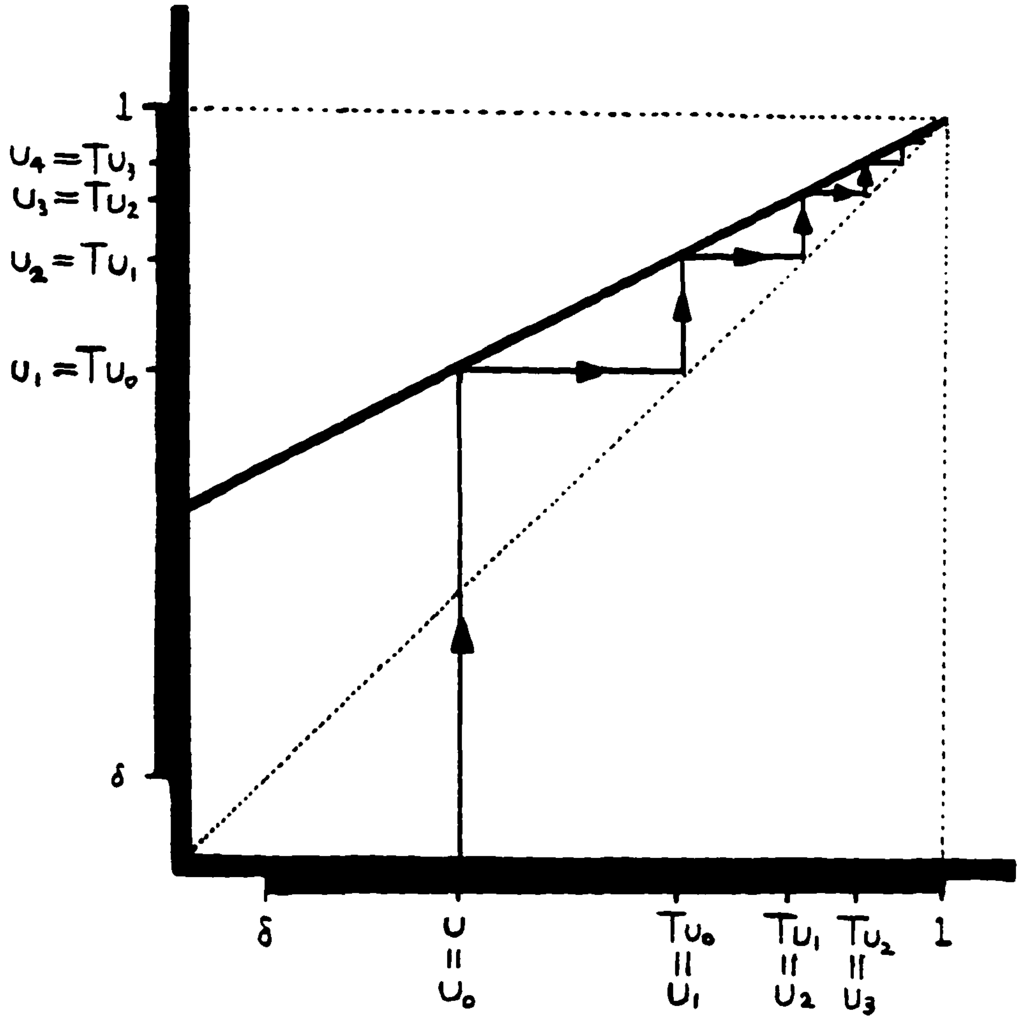
Figures 1 to 3 illustrate three model examples of how the iteration process works.

Consider the complete metric space $M = [\delta, 1]$, where $0 < \delta < 1$. Define the map $\varphi\colon M \to \mathbf{R}$ by $\varphi(u) = \frac{1}{u}$. We will consider self-maps $T\colon M \to M$ such that $d(u, Tu) \le \varphi(u) - \varphi(Tu)$ for all $u \in M$, i.e.,

$$|u - Tu| \le \frac{1}{u} - \frac{1}{Tu}.$$

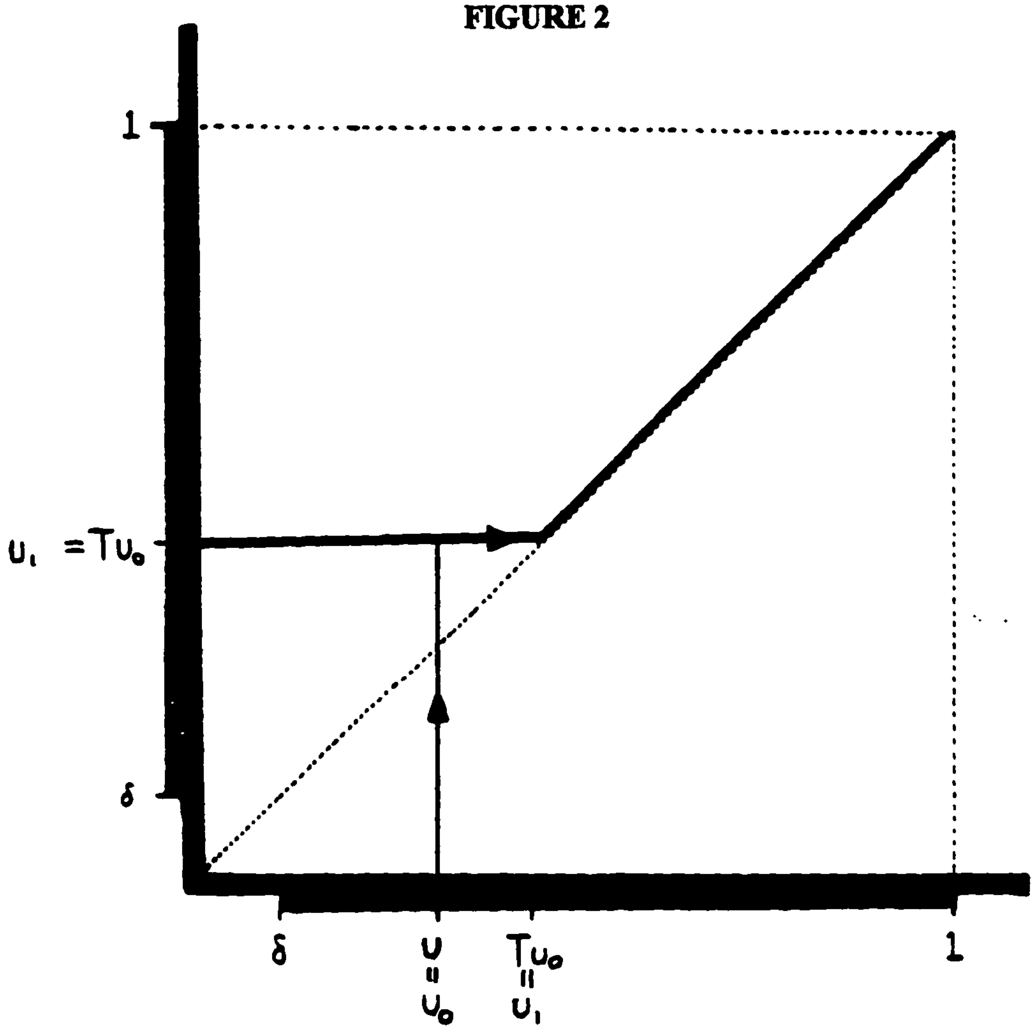
A simple calculation shows that the graph of T must be in the upper left triangular region of $[\delta, 1] \times [\delta, 1]$ (see figures).

FIGURE 1



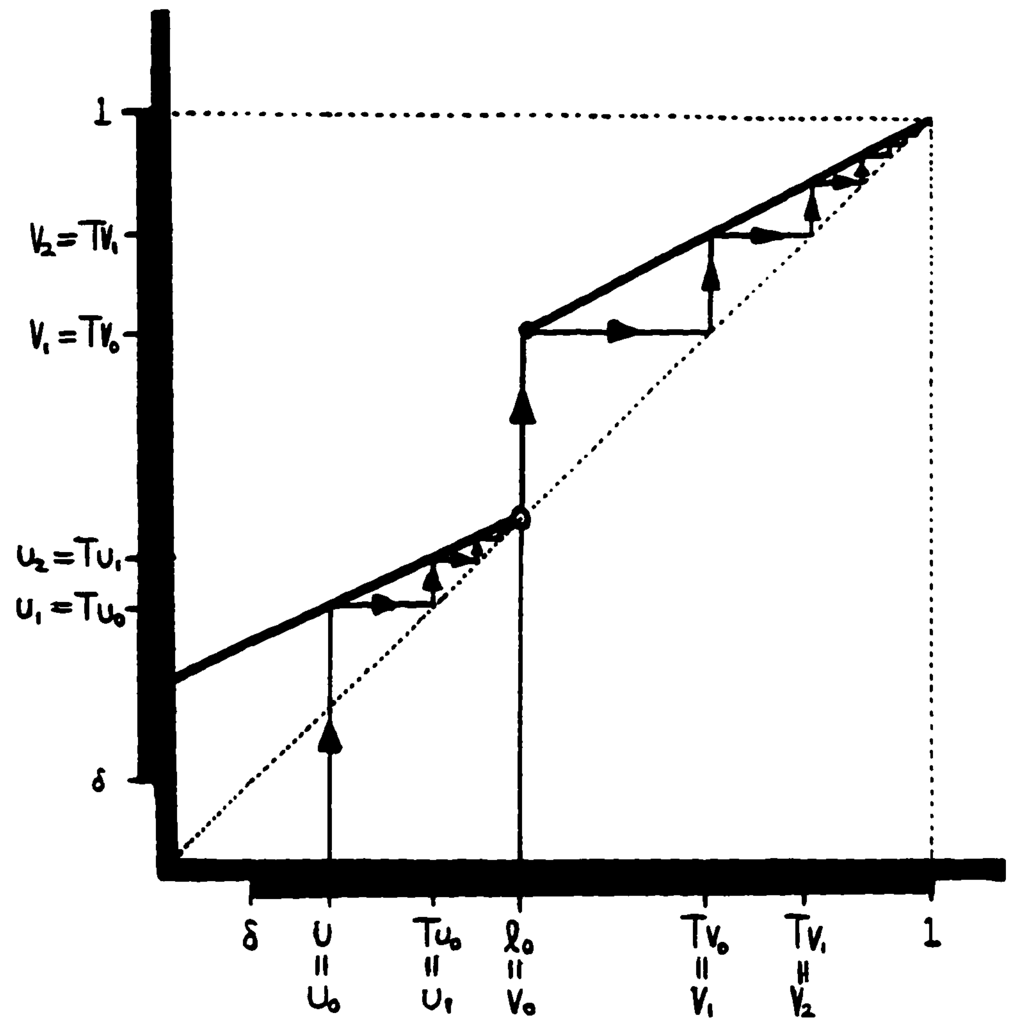
Infinitely many iterations are needed. T has a unique fixed point u = 1 (Note that T is a linear contraction).

FIGURE 2



A fixed point can be obtained by finitely many iterations (one or two in this case). There are infinitely many fixed points.

FIGURE 3



A fixed point can be obtained by finitely many applications (two, in this case) of infinitely many iterations.

SUMMARY

Let us take a look at the main points once again.

In Chapter 2, we considered Ekeland's Variational Principle (EVP). In a sense, it gave us a better approximation than the point which we had picked as our initial approximation. Moreover, we saw that EVP played an important role in showing completeness of a metric space throughout this thesis.

In Chapter 3, we studied the notion of completeness for a metric space by means of different approaches. In addition, we showed that EVP and Caristi's theorem are equivalent, while characterizing the notion of completeness.

In Chapter 4, the fixed point theorem considered in Chapter 3, due to Caristi, was discussed and a formal direct proof was presented. Probably, the most well-known fixed point theorem is **Banach Fixed Point Theorem** (BFPT). However, it requires that the mapping T on the complete metric space M be a contraction. We discussed a different approach without the requirement of contraction, but instead considering a real valued mapping φ closely related to T. In that chapter, we reproduced Caristi's proof with minor modifications. The formal proof used transfinite induction arguments and, for the sake of clarity, we decided to reserve Chapter 6 for a more informal presentation of that result and its main idea.

In Chapter 5, we presented an application by using some of the results obtained so far, such as EVP. The topic considered was a version of the **Minimum/ Maximum Principle at Infinity**. Essentially, the (say) minimum principle at infinity says the following. Given a twice differentiable function $f: \mathbf{R} \to \mathbf{R}$ which is bounded from below, if its infinimum α is not attained, then there must exist an <u>unbounded</u> sequence $\{v_n\}$ such that $f(v_n)$ approaches α, $f'(v_n)$ goes to zero, while $f''(v_n)$ goes to a non-negative value.

In Chapter 6, in order to better understand the main idea behind Theorem 8 and its proof, we presented that result through a series of iteration processes. We assumed that φ and T are related as in Theorem 8, so that the entropy function φ decreases along orbits $\{T^n u\}$ of T.

Then, we concluded that, whenever φ reaches a minimum value for some $v = v_n$, then v is a fixed point of T. We should remark that we can possibly obtain multiple fixed points depending on the choice of the initial point.

INDEX OF SYMBOLS

N	set of natural numbers
R	set of real numbers
$x \in A$	x is an element of A
$A \subset B$	A is a subset of B
$\{ x \in X \mid P(x) \}$	set of x in X with P(x)
$\{x\}$	singleton
$[a, b]$	closed interval
(a, b)	open interval
$\{x_i\}_{i=1}^{n}$	finite sequence
$\{x_i\}_{i=1}^{\infty}$	infinite sequence
2^A	set of subsets of A
$A \cap B$	intersection
$A \cup B$	union
$C(\mathbf{R})$	set of continuous functions on **R**
$C^2(\mathbf{R})$	space of twice differentiable functions
$\overline{A}$	closure of A
$T^n = T \circ T \circ \cdots \circ T$	composition n times

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