On numerical inversion of the moment generating function

Andy M Tsang
University of Nevada, Las Vegas

Follow this and additional works at: https://digitalscholarship.unlv.edu/rtds

Repository Citation
https://digitalscholarship.unlv.edu/rtds/3306

This Thesis is brought to you for free and open access by Digital Scholarship@UNLV. It has been accepted for inclusion in UNLV Retrospective Theses & Dissertations by an authorized administrator of Digital Scholarship@UNLV. For more information, please contact digitalscholarship@unlv.edu.
INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.
ON NUMERICAL INVERSION OF
THE MOMENT GENERATING
FUNCTION

by

Andy M. Tsang

A thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Science
in
Mathematics

Department of Mathematics
University of Nevada, Las Vegas
May 1997
The Thesis of Andy M. Tsang for the degree of Master of Science in Mathematics is approved.

Chairperson, Ashok K. Singh, Ph. D.

Examinining Committee Member, Malwane Ananda, Ph. D.

Examinining Committee Member, Rohan Dalpatadu, Ph. D.

Graduate Faculty Representative, Sidkazem Taghva, Ph. D.

Dean of the Graduate College, Ronald W. Smith, Ph. D.

University of Nevada, Las Vegas
May 1997
ABSTRACT

The main purpose of this thesis is to apply an algorithm for the numerical inversion of the Laplace transform that recovers the probability density function (PDF) of a sum of nonnegative continuous random variables. The Laplace transform is used in many disciplines. For example, in actuarial sciences, a common application is to study the distribution of the sum of nonnegative independent random variables. Because it is a popular method, numerical techniques have been developed to invert the Laplace transform. In the discrete case, by using a moment generating function (MGF) of a sum of independent discrete variables, the distribution can be analytically determined. In the continuous case, if the MGF fails to determine the distribution of a sum of nonnegative continuous independent variables analytically, then the PDF of the sum will be recovered by numerically inverting the Laplace transform.
# TABLE OF CONTENTS

ABSTRACT ..................................................................................................................... iii

LIST OF FIGURES ......................................................................................................... vi

ACKNOWLEDGMENTS ............................................................................................. vii

CHAPTER 1 MATHEMATICAL EXPECTATION: THE LAPLACE TRANSFORM ............................................................................................................ 1
   Laplace Transform in Mathematical Sciences ................................................... 1
   Laplace Transform in Probability ................................................................... 2
   Moment Generating Function, MGF ................................................................. 3
   Numerical Inversion of the Laplace Transform ................................................. 4

CHAPTER 2 ANALYTICAL APPROACH .................................................................... 6
   Mathematical Formulation of the MGF and the Laplace Transform ............ 6
   Recovering the PDF of S in the Discrete Case by the MGF, $\psi_S(t)$ ........... 7
   Recovering the PDF of S in the Continuous Case by $\psi_S(t)$ ..................... 13

CHAPTER 3 NUMERICAL INVERSION OF THE LAPLACE TRANSFORM .. 17
   Methods for Numerically Inverting the Laplace Transform ......................... 17
   Recovering the PDF by the Stehfest Method .................................................. 19
   Evaluation of the Stehfest Method ................................................................ 20
   Mean Square Error, MSE .............................................................................. 20
   Monte Carlo Simulation of S ........................................................................ 23

CHAPTER 4 CONCLUSIONS ................................................................................... 28

APPENDIX I LIST OF TABLES ................................................................................ 29
   Table 2-1: $L_X(z)$ and $\psi_X(t)$ for Common Discrete and Continuous
   Distributions ....................................................................................................... 30
   Table 2-2: The Distribution of the Sum, S, of $Bin(n=3, p=0.5)$ and
   $Poi(\lambda=1)$ .................................................................................................. 30
   Table 3-1: Laplace Transform-Inverse Laplace Transform ......................... 31
   Table 3-2: Analytical (f) and Numerical (ILT) Inverses $L(z)$
   for F1 and F2 ................................................................................................. 32
Table 3-3: Analytical (f) and Numerical (ILT) Inverses $L(z)$
for F3 and F4 .................................................................33

Table 3-4: The Analytical (True PDF) and the Numerical (ILT) Inverse $L_{g}(z)$
for the PDF of $Gam(\alpha=3, \theta=2)$ .........................................................34

Table 3-5: Numerically Recovering Six PDFs of S by the
Stehfest Method .................................................................35

Table 3-6: Comparison between Monte Carlo Simulation and Numerical
Inverses $L_{g}(z)$ (ILT) Based on the Mean and Variance .........................36

APPENDIX II COMPUTER LANGUAGE CODES ........................................37
BASIC Program for the Stehfest Method ...........................................38
Pascal Program for the Stehfest Method .............................................40
C++ Program for the Stehfest Method ..............................................42
SAS Program for the Stehfest Method ..............................................44

APPENDIX III SAS MONTE CARLO SIMULATION OF S ........................45
SAS Program for Monte Carlo Simulation of S .................................46
SAS Macro for Dynamic Data Exchange (DDE) to MicroSoft Excel ......48

REFERENCES ........................................................................50
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Histogram of $X_1 \sim Bin(n_1=4, p_1=0.1)$</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>Histogram of $X_2 \sim Bin(n_2=3, p_2=0.9)$</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>Histogram of $S=X_1+X_2$</td>
<td>10</td>
</tr>
<tr>
<td>2.4</td>
<td>Histogram of $X_1 \sim Bin(n_1=3, p_1=0.5)$</td>
<td>12</td>
</tr>
<tr>
<td>2.5</td>
<td>Histogram of $X_2 \sim Pois(\lambda=1)$</td>
<td>12</td>
</tr>
<tr>
<td>2.6</td>
<td>Histogram of $S=X_1+X_2$</td>
<td>12</td>
</tr>
<tr>
<td>3.1</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 1</td>
<td>25</td>
</tr>
<tr>
<td>3.2</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 2</td>
<td>25</td>
</tr>
<tr>
<td>3.3</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 3</td>
<td>25</td>
</tr>
<tr>
<td>3.4</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 4</td>
<td>26</td>
</tr>
<tr>
<td>3.5</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 5</td>
<td>26</td>
</tr>
<tr>
<td>3.6</td>
<td>Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(z)$ for Case 6</td>
<td>26</td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

I extend my appreciation to Dr. Ashok K. Singh of the Department of Mathematics at the University of Nevada, Las Vegas for his advice, suggestions, and guidance throughout my studies. I would also like to thank Dr. R. Dalpatadu and Dr. M. M. A. Ananda for their help.
CHAPTER 1

MATHEMATICAL EXPECTATION: THE LAPLACE TRANSFORM

Laplace Transform in Mathematical Sciences

The Laplace transform, named after the great French mathematician P. S. Laplace in 1782, is routinely used by engineers and scientists to solve problems. It has various applications. By applying the Laplace transform, mathematical problems can be solved more easily; however, obtaining the inverse of the Laplace transform can be difficult.

Boyce and DiPrima [4] mention that the Laplace transform is a powerful integral transform method for solving differential equations. They also discuss the use of the Laplace transform for linear differential equations with constant coefficients given the initial conditions solved by the Laplace transform.

Yet the Laplace transform has wider applicability than merely serving as a tool for solving linear differential equations. In 1966, Bellman et al. [2] illustrated the application of the technique in Biology, Economics, Engineering, and Physics. Many physical scientific processes could be formulated by using differential equations. Many of these problems can be simplified by the Laplace transform: the ordinary and
the blood described by time-dependent differential equations in chemotherapy. Moreover, the Laplace transform has been used in many diverse disciplines: actuarial science, hydrology, petroleum engineering, mathematical biology, mathematical economics, and mathematical physics.

Besides discussing the Laplace transform in various mathematical sciences areas, Bellman et al. point out that reducing the transcendence level of an equation is perhaps the most important use of applying the Laplace transform. They also point out the difficulties in developing a general method of inversion of the Laplace transform.

**Laplace Transform in Probability**

The Moment Generating Function (MGF) is a very common tool in mathematical probability and statistics. Karr [13] and Panjer and Willmot [14] indicate that the MGF is a special case of the Laplace transform.

In addition to discussing the relationship between the Laplace transform and MGF, Haight [11], and Panjer and Willmot [14] point out that the Laplace inverse of the MGF is the probability density function (PDF). They also note that the distribution of a sum (S) of independent nonnegative random variables can be determined by the Laplace transform, although sometimes this approach can fail. Subsequently, Haight shows that a way to solve the problem of recovering the PDF of
S is through the numerical inversion of the Laplace transform, except when the Laplace transform \( L_S(t) \) can be inverted analytically.

The Laplace transform method is especially useful in determination of the PDF of a sum (S) of independent nonnegative random variables. This is due to the fact that the MGF of a sum of random variables equals the product of the individual MGFs (Dwass [10]). In case the product of the MGFs cannot be inverted analytically, the technique of numerical inversion using the Laplace transform can be used. In the review written by Davies and Martin [8], they state that there are different methods of the numerical inversion of the Laplace transform. Each method is based on various fundamental concepts and conditions, and, in general, good solutions exist only for specific types of Laplace transform functions (e.g., exponential, trigonometric, and reciprocal functions).

**Moment Generating Function, MGF**

The Moment Generating Function (MGF) is a mathematical expectation in mathematical probability. Hogg and Craig [12] note several common uses of the MGF. Since the MGF is a unique function of the random variable and completely determines the distribution of the random variable, the distribution of a random variable or the sum of independent random variables can be recognized by the form of the MGF. Another application of the MGF is the ready computation of the common probability distribution parameters: the mean (\( \mu \)) and the variance (\( \sigma^2 \)). By taking
the first and second derivative of the MGF, $\mu$ and $\sigma^2$ can be obtained that describe the distribution of a random variable. Furthermore, as explored in this thesis, the PDF, which indicates the likelihood of each possible event, can often be recovered from the MGF.

One of the most important uses of the MGF is in analytically determining the distributions of the sum, $S$, of independent random variables, which come from the well-known probability distributions. By using the MGF, the PDF of the sum of discrete random variables can be determined analytically. Occasionally, the distribution of the sum of independent continuous random variables cannot be analytically determined.

**Numerical Inversion of the Laplace Transform**

In practice, inversion of the Laplace transform for many problems is based on a numerical approach. So far, no one best method has been derived to cover all function types. Bellman et al. [2] note that it is impossible to find a perfect numerical inversion formula to satisfy all possible problems, and that numerical inversion of the Laplace transform is an ill-posed problem. As such, we want to develop a set of techniques to solve different subclasses of problems.

Davies and Martin [8] reviewed 14 numerical techniques using a set of 16 test transforms. The review includes methods based on the real numbers and the complex numbers. For our purpose, only the methods based on the real numbers will be
considered, because, in the case of a MGF, all the solutions fall on the real line.

According to Davies and Martin [8], the problems associated with selecting a suitable method for inverting the Laplace transform are: (1.) applicability to a variety of common types of inversion problems; (2.) numerical accuracy; and (3.) relative computation time, programming, and implementation. In addition, the earliest technique, called the Widder technique, was published in 1964 by Cost [6]; and the latest technique was implemented in 1976 by Crump[7]. Another numerical inversion of the Laplace transform, which was developed by Cope [5] in 1990, is called the Convergence of Piessens' method.

Following the discussion in the Davies and Martin review, we have selected the method of Gaver-Stehfest (see Stehfest [20]). Because of the simplicity of the fundamental concept of the Gaver-Stehfest method, which has good accuracy over a fairly wide range of functions as indicated in the review, the Gaver-Stehfest method will be used to invert the Laplace transform in determining the PDF of the sum of nonnegative continuous random variables.
CHAPTER 2

ANALYTICAL APPROACH

Mathematical Formulation of the MGF and the Laplace Transform

Suppose that there is a positive number $h$ such that for $-h < t < h$ the mathematical expectation $E(e^{tX})$ exists. The moment generating function is denoted by $\varphi_X(t) = E(e^{tX})$. If $X$ is a continuous type of random variable, then

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx.$$  

If $X$ is a discrete type of random variable, then

$$E(e^{tX}) = \sum_x e^{tx} f(x).$$  

This expectation is called the moment-generating function of $X$ and is denoted by $\varphi_X(t)$, where the subscript represents the random variable. That is,

$$\varphi_X(t) = E(e^{tX}).$$  

If $X$ is a nonnegative random variable, then $E(e^{tX}) = \int_0^{\infty} e^{tx} f(x) \, dx$ or $E(e^{tX}) = \sum_x e^{tx} f(x)$.

The dummy variable $t$ can be replaced by $-t$. According to Panjer and Willmot [14] the relationship between the MGF and the Laplace transform $L_X(t)$ is

$$\psi_X(-t) = E(e^{-tX}) = L_X(t).$$  

The mean and variance of $X$ can be defined by $\psi_X(t)$ and $L_X(z)$ as follows:
If \( S = X_1 + X_2 + X_3 + \ldots + X_m \), where all \( X_i \)'s are stochastically independent random variables, then

\[
\mu = \psi'(0) = -L'(0) \quad \text{and} \quad \sigma^2 = \psi''(0) - [\psi'(0)]^2 = L''(0) - [L'(0)]^2.
\]

To obtain \( L_S(z) \) analytically, \( S \) must be a nonnegative continuous random variable. This will be satisfied whenever \( S \) is the sum of nonnegative continuous independent random variables.

In brief, Table 2-1 contains \( L_X(z) \) and \( \psi_X(t) \) for some common discrete and continuous distributions that can be used to solve analytically for the PDF of a sum \( S \) of independent random variables. If there is no suitable \( \psi_S(t) \) for \( S \), the \( \mu_S \) and \( \sigma_S^2 \) can be still computed by eq. 2.1 and eq. 2.2. In order to recover the PDF of \( S \), either \( \psi_S(t) \) or \( L_S(t) \) can be analytically determined by eq. 2.3. The details of the analytical method will be discussed in the following sections.

**Recovering the PDF of S in the Discrete Case by the MGF, \( \psi_S(t) \)**

In order to understand how to use \( \psi_S(t) \) to recover the PDF of \( S \), three illustrative examples will be presented. The first case is the determination of the PDF of \( S \) based on \( \psi_S(t) \) of a well-known random variable by simplification of \( \psi_S(t) \).
According to eq.2.3, $\psi_S(t)$ is the product of a summation series. The second case is based on the expansion of $\psi_S(t)$ with a finite summation series. In general, if

$$S = \sum_{i=1}^{n} X_i$$

and all the $X_i$'s are independent random variables, then

$$E_S(e^{-st}) = \sum_{s \in S} P(S = s) \cdot e^{-st}$$

(eq. 2.4)

$$= \sum_{s \in S} \left( \sum_{(x_1, x_2, \ldots, x_s) \in \{x_1 + x_2 + \ldots + x_s = s\}} \left(\prod_{i=1}^{n} P(X_i)\right) \cdot e^{-st} \right).$$

In the simple case ($n=2$), where $S = X_1 + X_2$:

$$E_S(e^{-st}) = \sum_{s \in S} P(S = s) \cdot e^{-st}$$

(eq. 2.5)

$$= \sum_{s \in S} \left( \sum_{(x_1, x_2) \in \{x_1 + x_2 = s\}} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \right) \cdot e^{-st}.$$ 

The third case is based on the expansion of $\psi_S(t)$ with an infinite summation series.

In the first case, the distribution of $S$ is the sum of two binomial distributions with a common $p$:

$$\psi_S(t) = \psi_{X_1}(t) \cdot \psi_{X_2}(t)$$

$$= (1 - p + pe^t)^{n_1} \cdot (1 - p + pe^t)^{n_2}$$

$$= (1 - p + pe^t)^{n_1+n_2}.$$ 

Relying on the assertion of the uniqueness of the MGF, $S$ is a binomial distribution with $n=n_1+n_2$ and $p=p$. Thus, the PDF of $S$ is:

$$f(x) = \binom{n_1 + n_2}{x} p^x (1 - p)^{n_1+n_2-x}, \quad x = 0, 1, 2, 3, \ldots, n_1 + n_2,$$

$$= 0 \quad elsewhere.$$
Suppose the distribution of \( S \) cannot be identified by the form of \( \psi_s(t) \). In the second case of a finite summation series, the PDF of \( S \) can be solved analytically.

The distribution of \( X_1 \) is positively skewed (see Figure 2.1), while the distribution of \( X_2 \) is negatively skewed (see Figure 2.2). Let \( S \) be the sum of these two independent variables \( X_1 \) and \( X_2 \), which belong to non-identical binomial distributions.

The following example illustrates the method of expansion of \( \psi_s(t) \) with a finite summation series in order to find the PDF of \( S \). Let \( X_1 \) and \( X_2 \) be stochastically independent with binomial distributions \( \text{Bin}(n_1=4, p_1=0.1) \) and \( \text{Bin}(n_2=3, p_2=0.9) \), respectively. We define the random variable \( S \) by \( S = X_1 + X_2 \). The MGF of \( S \) is

\[
\psi_s(t) = \psi_{x_1}(t) \cdot \psi_{x_2}(t)
\]

\[
= \sum_{s \in S} \left( \sum_{(x_1, x_2) \mid x_1 + x_2 = s} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \right) e^{-st}
\]

\[
= \sum_{s=0}^{7} \left( \sum_{(x_1, x_2) \mid x_1 + x_2 = s} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \right) e^{-st}
\]

\[
= \sum_{x_1=x_2=0}^{x_1+x_2=7} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \cdot e^{-st} + \ldots
\]

\[
\psi_s(t) = \frac{6561}{10000000} + \frac{180063}{10000000} e^{-t} + \frac{1673540}{10000000} e^{-2t} + \frac{1100943}{2000000} e^{-3t} + \frac{448967}{2000000} e^{-4t}
\]

\[
+ \frac{363069}{10000000} e^{-5t} + \frac{26487}{10000000} e^{-6t} + \frac{729}{10000000} e^{-7t}
\]
Clearly, the discrete random variable $S$ does not have a binomial distribution (see Figure 2.3) and has no similarity with the distributions of $X_1$ and $X_2$. While the possible outcomes of $X_1$ are 0, 1, 2, 3, 4 and of $X_2$ are 0, 1, 2, 3, the possible outcomes of $S$ are 0, 1, 2, 3, 4, 5, 6, 7. According to the definition of $\psi_S(i)$, the coefficient of each term becomes the probability of $S=s$. For instance, the probability that $S=0$ is 0.0006561, the probability that $S=1$ is 0.0180063, and so on.

**Figure 2.1** Histogram of $X_1 \sim Bin(n_1=4, p_1=0.1)$

**Figure 2.2** Histogram of $X_2 \sim Bin(n_2=3, p_2=0.9)$

**Figure 2.3** Histogram of $S=X_1+X_2$
The third example of this section illustrates the method of expansion of $\psi_s(t)$ with an infinite summation series. Let $X_1$ and $X_2$ be stochastically independent with a binomial distribution $Bin(n=3, p=0.5)$ and a Poisson distribution $Poi(\lambda=1)$, respectively. Notice that $X_1$ is a finite set, while $X_2$ is an infinite set. We define the random variable $S$ by $S=X_1+X_2$, where $S$ is an infinite set. Let $\text{min}(s, n)$ be the lower integer between $s$ and $n$. The MGF of $S$ is

$$\psi_s(t) = \psi_{X_1}(t) \cdot \psi_{X_2}(t)$$

$$= \sum_{s=0}^{\infty} \left( \sum_{x_1+x_2=s} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \right) \cdot e^{-ts}$$

$$= \sum_{x_1+x_2=0}^{\infty} \left( \sum_{x_1, x_2} P(X_1 = x_1) \cdot P(X_2 = s - x_1) \right) \cdot e^{-ts}$$

$$= \left[ P(X_1 = 0) \cdot P(X_2 = 0) \right] + \left[ P(X_1 = 0) \cdot P(X_2 = 1 - 0) \cdot P(X_1 = 1) \cdot P(X_2 = 1 - 1) \right] \cdot e^{-ts}$$

$$+ \left[ P(X_1 = 0) \cdot P(X_2 = 2) + P(X_1 = 1) \cdot P(X_2 = 1) \cdot P(X_2 = 0) \right] \cdot e^{-2ts} + \cdots$$

$$= \sum_{s=0}^{\infty} P(S = s) \cdot e^{ts}$$

where $S=0, 1, 2, 3, \ldots$

After expanding the infinite summation series in $\psi_s(t)$, the PDF of $S$ can be recovered in the coefficient of each term. Table 2.2 depicts the probability distribution over $S$ from 0 to 9, including either relative frequency or cumulative relative frequency. In addition, the histogram of $S$ is provided on page 12 (see Figure.
2.6). For example, the probability that \( S=0 \) is 0.045985, the probability that \( S=1 \) is 0.183940, and so on.

![Figure 2.4](image)

**Figure 2.4** Histogram of \( X_1 \sim Bin(n_1=3, p_1=0.5) \)

![Figure 2.5](image)

**Figure 2.5** Histogram of \( X_2 \sim Poi(\lambda=1) \)

![Figure 2.6](image)

**Figure 2.6** Histogram of \( S=X_1+X_2 \)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Recovering the PDF of S in the Continuous Case by $\psi_S(t)$

According to Bellman et al. [2], the important of the Laplace transform is to reduce the transcendence level of an equation. Many problems are related to differential equations. Sometimes, complicated mathematical equations can be simplified by the Laplace transform. In mathematical probability, the PDF of the sum of nonnegative independent random variables can be easily recovered from the Laplace transform. Without applying the Laplace transform, determination the PDF of the sum of independent random variables can be complicated.

Suppose that $X_1$ and $X_2$ are two stochastically independent random variables. We would like (1.) to show that $S=X_1+X_2$ has a well-known probability distribution and (2.) to recover the PDF. It is possible to obtain an explicit formula for the integrating PDFs of their sum. This method is based on the change of variables in multiple integrals called Jacobian of the transformation. However, some experience in integrating probability density functions is required. Sometimes, the identification of the boundary of the domain and the integration of the joint probability density function can be difficult. In the following two examples, the PDF of $S$ will be obtained by two different approaches. The first approach demonstrates the degree of difficulty in obtaining the PDF of $S$ by the Jacobian of the transformation. The second approach demonstrates how to solve the PDF of $S$ by the Laplace transform method.
In the first approach -- the Jacobian of the transformation -- the recovery of the PDF of S is based on the joint probability density function of \( X_1 \) and \( X_2 \) and the Jacobian determinant \( J \). After the Jacobian of the transformation has been obtained, the PDF of S can be recovered by computing the suitable marginal probability density function.

Let \( X_1 \) and \( X_2 \) be two stochastically independent random variables with \( X_1 \sim \text{exponential distribution} \), and \( X_2 \sim \text{gamma distribution} \). The joint probability density function of \( X_1 \) and \( X_2 \) is

\[
f(x_1, x_2) = \frac{1}{\theta} e^{\frac{-x_1}{\theta}} \cdot \frac{1}{\Gamma(\alpha)\theta^a} x_2^{a-1} e^{\frac{-x_2}{\theta}}, \quad 0 \leq x_1 < \infty, \ 0 \leq x_2 < \infty
\]

\[
= 0, \quad \text{elsewhere},
\]

where \( \theta > 0 \) and \( \alpha > 0 \). The transformation is from \((x_1, x_2)\) to \((s, w)\). Let \( S = X_1 + X_2 \) and \( W = \frac{x_1}{X_1 + X_2} \). The explicit form of the inverse transformation is

\[
x_1 = sw, \quad x_2 = s - sw.
\]

Hence, the Jacobian determinant is

\[
J = \begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial w} \end{vmatrix} = \begin{vmatrix} w & s \\ 1 - w & -s \end{vmatrix} = -sw + sw - s = -s,
\]

and \(|J| = s\).

The joint probability density function of \( S \) and \( W \) is

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
\[ g(s, w) = f(x_1 = sw, x_2 = (s - sw)|J| \]

\[ = \frac{1}{\theta} e^{-sw/\theta} \frac{1}{\Gamma(\alpha) \theta^\alpha} (s - sw)^{\alpha-1} e^{-(s-sw)/\theta} s \]

\[ = \frac{1}{\Gamma(\alpha)} \frac{1}{\theta^{\alpha+1}} s^{\alpha-1} e^{-\theta(1-w)^{\alpha-1}} \]

\[ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} (1 - w)^{\alpha-1} \frac{1}{\Gamma(\alpha + 1)} \frac{1}{\theta^{\alpha+1}} s^{\alpha-1} e^{-\theta s}, \quad 0 < s < \infty, \quad 0 \leq w < 1. \]

After integrating with respect to \( w \) over the joint probability density function of \( S \) and \( W \), we obtain the marginal probability density function of \( S \):

\[ m(s) = \int_0^1 g(s, w) \, dw \]

\[ = \int_0^1 e^{-sw/\theta} \frac{1}{\Gamma(\alpha) \theta^\alpha} (s - sw)^{\alpha-1} e^{-(s-sw)/\theta} s \, dw \]

\[ = \frac{1}{\Gamma(\alpha)} \frac{1}{\theta^{\alpha+1}} s^{\alpha-1} e^{-\theta(1-w)^{\alpha-1}} \, dw \]

\[ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} (1 - w)^{\alpha-1} \, dw \frac{1}{\Gamma(\alpha + 1)} \frac{1}{\theta^{\alpha+1}} s^{\alpha-1} e^{-\theta s} \]

\[ = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} \frac{1}{\theta^{\alpha+1}} s^{\alpha-1} e^{-\theta s}, \quad 0 < s < \infty, \quad 0 \leq w < 1. \]

The result becomes the PDF of the nonnegative random variable \( S \), where \( S \) has a gamma distribution with parameters \( \alpha^* = \alpha + 1 \) and \( \theta^* = \theta \). For readers familiar with mathematical probability, it is obvious that the continuous random variable \( W \) has a beta distribution with parameters \( \alpha^* = 1 \) and \( \beta^* = \alpha \).

The second approach is based on the Laplace transform. As mentioned in the introduction, the MGF and the Laplace transform of a nonnegative random variable
are equivalent, where \( \psi_X(-z) = L_X(z) \). The previous problem can be solved easily with a MGF table of the common random variables; such a table is available in many mathematical probability text books. By multiplying the Laplace transform of the exponential distribution \( X_1 \) with parameter \( \theta^* = \theta \) and the gamma distribution \( X_2 \), the Laplace transform of \( S = X_1 + X_2 \) becomes

\[
L_S(z) = L_{X_1}(z)L_{X_2}(z) \\
= \psi_{X_1}(t = -z)\psi_{X_2}(t = -z) \\
= \frac{1}{1 - \theta(-z)} \cdot \frac{1}{(1 - \theta(-z))^\alpha} \\
= \frac{1}{(1 + \theta z)^{\alpha+1}}
\]

Thus, the continuous random variable \( S \) has a gamma distribution with parameters \( \alpha^* = \alpha + 1 \) and \( \theta^* = \theta \) (see Table 2-1).

Of the two methods to recover the PDF of \( S \) in this example, it is clear that the Laplace transform is a more effective method. However, in some cases, when the expression of the Laplace transform is not available in a table, the numerical inversion of the Laplace transform can be used to recover the PDF of the sum of nonnegative continuous independent random variables.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
CHAPTER 3

NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

Methods for Numerically Inverting the Laplace Transform

For the last few decades, a large number of different numerical methods have been proposed for the inversion of the Laplace transform. Because of the increasing demand for scientific applications to conduct the numerical inversion of the Laplace transform, various theoretical methods have been developed to solve different problems. According to Davies and Martin [7], methods for numerically inverting the Laplace transform can be theoretically classified into the following six categories:

1. methods which compute a sample;
2. methods which expand \( f(t) \) in exponential functions;
3. methods based on Gaussian quadrature of the inversion integral;
4. methods which use a bilinear transformation of \( p \);
5. representation by a Fourier series; and
6. the Padé approximation.

Detailed discussions of these methods are available (see references below). However, instead of explaining the theoretical concept of methods for numerically inverting the Laplace transform, we will provide some resource information on these methods taken from the Davies and Martin’s review [7].

17

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
In the first category (number (1) above), two techniques were proposed: “Widder” (Cost [6], 1964) and “Gaver-Stehfest” (Stehfest [20], 1970). In the second category, four techniques were proposed: “Legendre polynomials” (Papoulis [16], 1956), “Bellman et al.” (Bellman et al. [2], 1966), “Trigonometric” (Papoulis [16], 1956), and “Schapery” (Cost [6], 1964). In the third category, two techniques were proposed: “Gaussian quadrature” (Piessens [17], 1971) and “Schmittroth” (Schmittroth [20], 1960). In the fourth category, three techniques were proposed: “Laguerre-Weeks” (Weeks [25], 1966), “Laguerre-Piessens-Branders” (Piessens and Branders [19], 1971), and “Chebyshev” (Piessens [18], 1972). In the fifth category, three methods were proposed: “Dubner-Abate” (Dubner and Abate [9], 1968), “Silverberg-Durbin” (Silverberg [21], 1970), and “Crump” (Crump [7], 1976). In the last category, only the technique of “Padé Approximation” was proposed. Because a number of papers have been published on the use of Padé approximation to the Laplace transform, no techniques were evaluated in the review. Interested readers are referred to the reference of Longman [14] and Longman [15].
Recovering the PDF by the Stehfest Method

According to Stehfest [15], the algorithm for the extrapolation formula is

\[ f(T) = \frac{\ln 2}{T} \sum_{i=1}^{N} V_i \mu \left( \frac{\ln 2}{T} \cdot i \right), \]

where \( V_i = (-1)^{\frac{N}{2} + i} \cdot \left( \sum_{k=\left\lfloor \frac{i}{2} \right\rfloor}^{N} \frac{N}{2 - k} \cdot \frac{(2k)!}{k! \cdot (k-1)! \cdot (i-k)! \cdot (2k-i)!} \right), \) and \( N \) must be even.

The algorithm is given in Stehfest's paper [15]. It is composed of a double finite summation series. This numerical inversion of the Laplace transform has good accuracy over a fairly wide range of functions as Davies and Martin note [3]. With an optimal value of \( N \), the PDF will be recovered by the algorithm.

To perform the inversion of the Laplace transform, the algorithm of the Stehfest method has been coded into four different computer programming languages (Appendix II). They are: BASIC [21], Pascal [22], C++ [3], and SAS [1] languages. The output of BASIC, Pascal, and C++ will be in the file called "invlt.out." The output of the SAS statistical programming language will be on the output panel while the software is active. These outputs are the results of the inversion of the Laplace transforms.

According to Davies and Martin [3], the Stehfest method is a powerful and easy-to-understand extrapolation formula. The algorithm has been successfully ported to many popular computer languages, due, in parts to its simplicity and ease of
implementation. By using simple conditional computer language statements and several logical loops, the results of the inverse of the Laplace transforms are calculated. Borland C++ was chosen for use on a PC-486 machine to complete the numerical inversion for the Laplace transform, since the operating system allows for double-precision and 32-bit computing architecture. Finally, the Stehfest method was selected as the technique for inverting the Laplace transform. The following discussion of the numerical inversion of the probabilistic L(z) is based on the Stehfest method using the C++ programming language. The N of the Stehfest method was selected to be 40 in this thesis.

**Evaluation of the Stehfest Method**

**Mean Square Error, MSE**

In the evaluation of the Stehfest method, four algebraic functions and six probability density functions which are the sum of two nonnegative continuous independent random variables are selected. The four selected algebraic functions have analytical answers available for proper evaluation of the numerical inversion of the Laplace transform (ILT). In other words, they are available in tables of Laplace transforms and inverse Laplace transforms. These four algebraic functions -- F1, F2, F3, and F4 (on Table 3-1) -- illustrate the application of the numerical method. The accompanying L(z)'s are available in tables of Laplace transforms and inverse
Laplace transforms (see Table 3-1), which are the functions discussed in Stehfest’s paper. In order to evaluate the result of the numerical approximation, an indicator is defined by the Mean Square Error (MSE):

\[
MSE = \frac{\sum_{i=1}^{40} (ILT_x f(t) - true_x f(t))^2}{40} = \frac{\sum_{i=1}^{40} Diff_i^2}{40}
\]

(3.1)

where the \( ILT_x f(t) \) is the numerical solution for the inverse of \( L_x(z) \), and the \( true_x f(t) \) is the algebraic solution for the inverse of \( L_x(z) \). Since the MSEs of \( f_1 \), \( f_2 \), and \( f_3 \), respectively, have lower values with \( 1.05 \cdot 10^{-11}, 8.76 \cdot 10^{-11}, \) and \( 7.24 \cdot 10^{-13} \) at \( N=40 \) (see Table 3-2 and Table 3-3), the following numerical inversion Laplace transforms will use the Stehfest method with \( N=40 \). We expect a fair approximation.

Secondly, three common parent probability distributions are selected in our study to demonstrate how to recover the PDF of the sum of two nonnegative continuous independent random variables by using the Stehfest method. The PDF, \( L(z) \), \( \mu \), and \( \sigma^2 \) are summarized as follows:

<table>
<thead>
<tr>
<th>Probability Distribution</th>
<th>PDF</th>
<th>( L_x(z) = \psi_x(t=-z) )</th>
<th>( \mu )</th>
<th>( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential ( Exp(\theta) )</td>
<td>( f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} )</td>
<td>( L(z) = \frac{1}{1-\theta(-z)} )</td>
<td>( \theta )</td>
<td>( \theta^2 )</td>
</tr>
<tr>
<td>Gamma ( Gam(\alpha, \theta) )</td>
<td>( f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} )</td>
<td>( L(z) = \frac{1}{(1-\theta(-z))^\alpha} )</td>
<td>( \alpha\theta )</td>
<td>( \alpha\theta^2 )</td>
</tr>
<tr>
<td>Chi-Square ( \chi^2(r) )</td>
<td>( f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} )</td>
<td>( L(z) = \frac{1}{(1-2(-z))^{\frac{r}{2}}} )</td>
<td>( r )</td>
<td>( 2r )</td>
</tr>
</tbody>
</table>
The numerical inversion of the Laplace transform will be applied in order to
compute the PDF of \( S = X_1 + X_2 \) for the three probability distributions according to the
following six cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>Laplace Transform Function for ( S, L_S(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \text{Exp}(\theta = 1) )</td>
<td>( \text{Exp}(\theta = 3) )</td>
<td>( \frac{1}{1 + z} \cdot \frac{1}{1 + 3z} )</td>
</tr>
<tr>
<td>2</td>
<td>( \chi^2(r=1) )</td>
<td>( \text{Exp}(\theta = 1) )</td>
<td>( \frac{1}{\sqrt{1 + 2z}} \cdot \frac{1}{z} )</td>
</tr>
<tr>
<td>3</td>
<td>( \text{Gam}(\alpha = 1, \theta = 2) )</td>
<td>( \text{Gam}(\alpha = 2, \theta = 1) )</td>
<td>( \frac{1}{(1 + 2z)} \cdot \frac{1}{(1 + z)^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \text{Exp}(\theta = 0.5) )</td>
<td>( \text{Gam}(\alpha = 1, \theta = 2.5) )</td>
<td>( \frac{1}{1 + 0.5z} \cdot \frac{1}{(1 + 2.5z)} )</td>
</tr>
<tr>
<td>5</td>
<td>( \text{Exp}(\theta = 1) )</td>
<td>( \text{Gam}(\alpha = 1.5, \theta = 0.5) )</td>
<td>( \frac{1}{1 + z} \cdot \frac{1}{(1 + 0.5z)^3} )</td>
</tr>
<tr>
<td>6</td>
<td>( \text{Exp}(\theta = 2) )</td>
<td>( \text{Gam}(\alpha = 2, \theta = 2) )</td>
<td>( \frac{1}{(1 + 2z)} \cdot \frac{1}{(1 + 2z)^2} )</td>
</tr>
</tbody>
</table>

In order to evaluate the accuracy of the numerical approximation, we begin
with case 6. Notice that by the property of eq. 2.3 on page 7, \( L_S(z) \) is simplified into
\( \psi_S(t) \):

\[
L_S(z) = \frac{1}{(1 + 2z)^3}
\]
\[
= \frac{1}{(1 - 2t)^3}
\]
\[
= \psi_S(t = -z)
\]

According to the mathematical expression, we can identify the corresponding
probability distribution of \( \psi_S(t) \). It is clear that \( S \) has a gamma distribution with
parameters $\alpha=3$ and $\theta=2$. By using the PDF of $\text{Gam}(\alpha=3, \theta=2)$, the numerical approximation of case 6 and the analytical PDF are obtained (Table 3-4). The MSE is $7.78 \times 10^{-11}$ for case 6. This result indicates that it can successfully recover the PDF for case 6. In Table 3-5, the results of inversion of the Laplace transform in the Stehfest method contain all six PDFs of the selected sum $S$ of the nonnegative continuous independent random variables. Theoretically, the negative values in the table should be replaced by zeros, since all values of the PDF of any random variable must be nonnegative. In order to evaluate the results of the estimate of PDF of case 1 to case 5, an alternative method needs to be applied. It is clear that there is no way to obtain the analytical PDF of case 1 to case 5. Further discussion needs to be conducted based on Monte Carlo simulation.

Monte Carlo Simulation of $S$

SAS statistical software will be used to perform the simulation with 100,000 iterations for six cases in the previous discussion. The simulation program is in Appendix I. Let $S$ be $X+Y$, where $X$ and $Y$ are two stochastically independent random variables. In SAS, we use the random generator to produce many different kinds of random numbers such as exponential random numbers, chi-square random numbers, and gamma random numbers. By choosing a random number for $X$ and a random number for $Y$ at each iteration, we will ultimately obtain 100,000 random numbers for $S$. After choosing the class interval with a length of 1 unit, the frequency
table of S can be easily created in SAS. Then, we repeatedly apply the "Proc Freq" statistical procedure for all six cases. By using a macro in SAS (see Appendix III), all six frequency tables can be collected together and dynamically sent to a MicroSoft Excel electronic spread sheet. Hence, the comparison between numerical inversion of the Laplace transform (ILT) on Table 3-5 and the Monte Carlo Simulation (Sim) of all six cases can be summarized into six histograms generated in MicroSoft Excel as follows:
Figure 3.1 Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(t)$ for case 1

Figure 3.2 Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(t)$ for case 2

Figure 3.3 Histogram of the Sum: Simulation vs. Numerical Inverse $L_S(t)$ for case 3

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Figure 3.4 Histogram of the Sum: Simulation vs. Numerical Inverse $L_5(t)$ for case 4

Figure 3.5 Histogram of the Sum: Simulation vs. Numerical Inverse $L_5(t)$ for case 5

Figure 3.6 Histogram of the Sum: Simulation vs. Numerical Inverse $L_5(t)$ for case 6
From the six figures, we see that case 6 has the best fit, but case 5 is not fitted very well. Based on the figures, we can observe the detailed probability of each sample outcome for $S$. In addition, the sufficient statistics of the mean and variance are also provided in Table 3-6. If $S=X+Y$, where $X$ and $Y$ are independent variables, then (see Panjer and Willmot [16]):

(eq. 3.2) \[ E(S) = E(X) + E(Y) \]

(eq. 3.3) \[ Var(S) = Var(X) + Var(Y). \]

With these two properties, the alternative summary reveals the performance of the Monte Carlo Simulation and the numerical inversion of the Laplace transform (ILT): The Stehfest method accurately recovers the probability density function for all six cases.
CONCLUSIONS

A simple method for computing the PDF of sums of discrete independent random variables which may not be identically distributed via the MGF is illustrated. Our method has applications in actuarial science. The Stehfest method of numerical inversion of the Laplace transform is used to compute the PDF of sums of continuous independent random variables. The accuracy of the result is verified by using Monte Carlo simulation.
APPENDIX I

LIST OF TABLES
Table 2-1: $L_X(z)$ and $\psi_X(t)$ for Common Discrete and Continuous Distributions

<table>
<thead>
<tr>
<th></th>
<th>$\psi_X(t)$</th>
<th>$L_X(z)$</th>
<th>$\mu$</th>
<th>$\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli $Ber(p)$</td>
<td>$1 - p + pe^t$</td>
<td>$p$</td>
<td>$p(1-p)$</td>
<td></td>
</tr>
<tr>
<td>Binomial $Bin(n,p)$</td>
<td>$(1 - p + pe^t)^n$</td>
<td>$np$</td>
<td>$np(1-p)$</td>
<td></td>
</tr>
<tr>
<td>Geometric $Geo(\mu)$</td>
<td>$\frac{pe^t}{1 - (1 - p)e^t}$</td>
<td>$\frac{1}{p}$</td>
<td>$\frac{1 - p}{p^2}$</td>
<td></td>
</tr>
<tr>
<td>Poisson $Poi(\lambda)$</td>
<td>$e^{\lambda(e^t - 1)}$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
<td></td>
</tr>
<tr>
<td>Exponential $Exp(\theta)$</td>
<td>$\frac{1}{1 - \theta t}$</td>
<td>$\frac{1}{1 + \theta z}$</td>
<td>$\theta$</td>
<td>$\theta^2$</td>
</tr>
<tr>
<td>Chi-Square $\chi^2(r)$</td>
<td>$\frac{1}{(1 - 2t)^{\frac{r}{2}}}$</td>
<td>$\frac{1}{(1 + 2z)^{\frac{r}{2}}}$</td>
<td>$r$</td>
<td>$2r$</td>
</tr>
<tr>
<td>Gamma $Gamma(\alpha, \theta)$</td>
<td>$\frac{1}{(1 - \theta t)^{\alpha}}$</td>
<td>$\frac{1}{(1 + \theta z)^{\alpha}}$</td>
<td>$\alpha\theta$</td>
<td>$\alpha\theta^2$</td>
</tr>
</tbody>
</table>

Table 2-2: The Distribution of the Sum, S, of $Bin(n=3, p=0.5)$ and $Poi(\lambda=1)$.

<table>
<thead>
<tr>
<th>s</th>
<th>Prob(S=s)</th>
<th>Prob(S≤s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.045985</td>
<td>0.045985</td>
</tr>
<tr>
<td>1</td>
<td>0.183940</td>
<td>0.229925</td>
</tr>
<tr>
<td>2</td>
<td>0.298902</td>
<td>0.528827</td>
</tr>
<tr>
<td>3</td>
<td>0.260581</td>
<td>0.789408</td>
</tr>
<tr>
<td>4</td>
<td>0.139871</td>
<td>0.929279</td>
</tr>
<tr>
<td>5</td>
<td>0.052116</td>
<td>0.981395</td>
</tr>
<tr>
<td>6</td>
<td>0.014626</td>
<td>0.996021</td>
</tr>
<tr>
<td>7</td>
<td>0.003266</td>
<td>0.999287</td>
</tr>
<tr>
<td>8</td>
<td>0.000603</td>
<td>0.999890</td>
</tr>
<tr>
<td>9</td>
<td>0.000095</td>
<td>0.999985</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 3-1: Laplace Transform-Inverse Laplace Transform

<table>
<thead>
<tr>
<th>Case</th>
<th>$L(z)$</th>
<th>$f(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: F1</td>
<td>$\frac{1}{\sqrt{z}}$</td>
<td>$\frac{1}{\sqrt{\pi t}}$</td>
</tr>
<tr>
<td>2: F2</td>
<td>$\frac{\ln(z)}{z}$</td>
<td>$-0.57722 - \ln(t)$</td>
</tr>
<tr>
<td>3: F3</td>
<td>$\frac{1}{z + 1}$</td>
<td>$e^{-t}$</td>
</tr>
<tr>
<td>4: F4</td>
<td>$\frac{\pi}{\sqrt{2z^2}} \cdot e^{\frac{1}{2t}}$</td>
<td>$\sin(\sqrt{2t})$</td>
</tr>
</tbody>
</table>
Table 3-2: Analytical (f) and Numerical (ILT) Inverses $L(z)$ for F1 and F2

<table>
<thead>
<tr>
<th>t</th>
<th>f1</th>
<th>ILT</th>
<th>Diff^2</th>
<th>f2</th>
<th>ILT</th>
<th>Diff^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.56419</td>
<td>0.56419</td>
<td>9.13E-12</td>
<td>-0.57722</td>
<td>-0.57722</td>
<td>4.79E-12</td>
</tr>
<tr>
<td>2</td>
<td>0.39894</td>
<td>0.42984</td>
<td>4.42E-12</td>
<td>-1.27037</td>
<td>-1.27037</td>
<td>4.67E-12</td>
</tr>
<tr>
<td>3</td>
<td>0.32574</td>
<td>0.32574</td>
<td>6.26E-11</td>
<td>-1.67583</td>
<td>-1.67584</td>
<td>2.55E-11</td>
</tr>
<tr>
<td>4</td>
<td>0.28209</td>
<td>0.28210</td>
<td>2.28E-12</td>
<td>-1.96351</td>
<td>-1.96351</td>
<td>4.74E-12</td>
</tr>
<tr>
<td>5</td>
<td>0.25231</td>
<td>0.25231</td>
<td>2.27E-11</td>
<td>-2.18666</td>
<td>-2.18665</td>
<td>1.22E-10</td>
</tr>
<tr>
<td>6</td>
<td>0.23033</td>
<td>0.23034</td>
<td>3.11E-11</td>
<td>-2.36898</td>
<td>-2.36898</td>
<td>2.53E-11</td>
</tr>
<tr>
<td>7</td>
<td>0.19947</td>
<td>0.19947</td>
<td>1.11E-12</td>
<td>-2.65666</td>
<td>-2.65666</td>
<td>4.75E-12</td>
</tr>
<tr>
<td>8</td>
<td>0.18806</td>
<td>0.18806</td>
<td>1.32E-12</td>
<td>-2.77444</td>
<td>-2.77444</td>
<td>4.57E-11</td>
</tr>
<tr>
<td>9</td>
<td>0.17841</td>
<td>0.17841</td>
<td>1.14E-11</td>
<td>-2.87981</td>
<td>-2.87979</td>
<td>1.22E-10</td>
</tr>
<tr>
<td>10</td>
<td>0.17011</td>
<td>0.17010</td>
<td>5.66E-11</td>
<td>-2.97512</td>
<td>-2.97509</td>
<td>4.21E-10</td>
</tr>
<tr>
<td>11</td>
<td>0.16287</td>
<td>0.16287</td>
<td>1.56E-11</td>
<td>-3.06213</td>
<td>-3.06213</td>
<td>2.53E-11</td>
</tr>
<tr>
<td>12</td>
<td>0.15648</td>
<td>0.15648</td>
<td>2.65E-12</td>
<td>-3.14217</td>
<td>-3.14217</td>
<td>7.15E-13</td>
</tr>
<tr>
<td>13</td>
<td>0.15079</td>
<td>0.15078</td>
<td>8.4E-12</td>
<td>-3.21628</td>
<td>-3.21627</td>
<td>1.17E-10</td>
</tr>
<tr>
<td>14</td>
<td>0.14567</td>
<td>0.14568</td>
<td>1.96E-11</td>
<td>-3.28527</td>
<td>-3.28528</td>
<td>4.57E-11</td>
</tr>
<tr>
<td>15</td>
<td>0.14105</td>
<td>0.14105</td>
<td>5.7E-13</td>
<td>-3.34981</td>
<td>-3.34981</td>
<td>4.74E-12</td>
</tr>
<tr>
<td>16</td>
<td>0.13684</td>
<td>0.13683</td>
<td>2.22E-12</td>
<td>-3.41043</td>
<td>-3.41042</td>
<td>7.24E-11</td>
</tr>
<tr>
<td>17</td>
<td>0.13298</td>
<td>0.13298</td>
<td>6.59E-13</td>
<td>-3.46759</td>
<td>-3.46758</td>
<td>4.6E-11</td>
</tr>
<tr>
<td>18</td>
<td>0.12943</td>
<td>0.12943</td>
<td>5.47E-13</td>
<td>-3.52166</td>
<td>-3.52165</td>
<td>5.01E-11</td>
</tr>
<tr>
<td>19</td>
<td>0.12616</td>
<td>0.12615</td>
<td>5.68E-13</td>
<td>-3.57295</td>
<td>-3.57294</td>
<td>1.22E-10</td>
</tr>
<tr>
<td>20</td>
<td>0.12312</td>
<td>0.12311</td>
<td>2.29E-12</td>
<td>-3.62174</td>
<td>-3.62173</td>
<td>6.99E-11</td>
</tr>
<tr>
<td>21</td>
<td>0.12029</td>
<td>0.12028</td>
<td>2.82E-11</td>
<td>-3.66826</td>
<td>-3.66824</td>
<td>4.21E-10</td>
</tr>
<tr>
<td>22</td>
<td>0.11764</td>
<td>0.11764</td>
<td>3.72E-12</td>
<td>-3.71271</td>
<td>-3.71272</td>
<td>1.76E-12</td>
</tr>
<tr>
<td>23</td>
<td>0.11516</td>
<td>0.11517</td>
<td>7.78E-12</td>
<td>-3.75527</td>
<td>-3.75528</td>
<td>2.54E-11</td>
</tr>
<tr>
<td>24</td>
<td>0.11284</td>
<td>0.11284</td>
<td>8.95E-12</td>
<td>-3.79610</td>
<td>-3.79610</td>
<td>2.86E-11</td>
</tr>
<tr>
<td>25</td>
<td>0.11065</td>
<td>0.11065</td>
<td>1.31E-12</td>
<td>-3.83532</td>
<td>-3.83532</td>
<td>7.23E-13</td>
</tr>
<tr>
<td>26</td>
<td>0.10858</td>
<td>0.10857</td>
<td>2.33E-11</td>
<td>-3.87306</td>
<td>-3.87304</td>
<td>3.93E-10</td>
</tr>
<tr>
<td>27</td>
<td>0.10662</td>
<td>0.10662</td>
<td>4.23E-12</td>
<td>-3.90942</td>
<td>-3.90941</td>
<td>1.17E-10</td>
</tr>
<tr>
<td>28</td>
<td>0.10477</td>
<td>0.10477</td>
<td>2.03E-12</td>
<td>-3.94452</td>
<td>-3.94452</td>
<td>1.17E-13</td>
</tr>
<tr>
<td>29</td>
<td>0.10301</td>
<td>0.10301</td>
<td>9.84E-12</td>
<td>-3.97842</td>
<td>-3.97842</td>
<td>4.52E-11</td>
</tr>
<tr>
<td>30</td>
<td>0.10133</td>
<td>0.10133</td>
<td>2.05E-12</td>
<td>-4.01121</td>
<td>-4.01121</td>
<td>1.57E-12</td>
</tr>
<tr>
<td>31</td>
<td>0.09974</td>
<td>0.09974</td>
<td>2.76E-13</td>
<td>-4.04296</td>
<td>-4.04295</td>
<td>4.67E-12</td>
</tr>
<tr>
<td>32</td>
<td>0.09821</td>
<td>0.09821</td>
<td>8.36E-18</td>
<td>-4.07373</td>
<td>-4.07372</td>
<td>1.88E-11</td>
</tr>
<tr>
<td>33</td>
<td>0.09676</td>
<td>0.09676</td>
<td>1.08E-12</td>
<td>-4.10358</td>
<td>-4.10357</td>
<td>7.24E-11</td>
</tr>
<tr>
<td>34</td>
<td>0.09537</td>
<td>0.09537</td>
<td>2.43E-11</td>
<td>-4.13257</td>
<td>-4.13258</td>
<td>1.88E-10</td>
</tr>
<tr>
<td>35</td>
<td>0.09403</td>
<td>0.09403</td>
<td>3.3E-13</td>
<td>-4.16074</td>
<td>-4.16073</td>
<td>4.6E-11</td>
</tr>
<tr>
<td>36</td>
<td>0.09275</td>
<td>0.09275</td>
<td>2.22E-11</td>
<td>-4.18814</td>
<td>-4.18812</td>
<td>5.11E-10</td>
</tr>
<tr>
<td>37</td>
<td>0.09152</td>
<td>0.09152</td>
<td>2.67E-13</td>
<td>-4.21481</td>
<td>-4.21480</td>
<td>5.02E-11</td>
</tr>
<tr>
<td>38</td>
<td>0.09034</td>
<td>0.09034</td>
<td>1.23E-12</td>
<td>-4.24078</td>
<td>-4.24078</td>
<td>1.9E-13</td>
</tr>
<tr>
<td>39</td>
<td>0.08921</td>
<td>0.08920</td>
<td>2.85E-12</td>
<td>-4.26610</td>
<td>-4.26609</td>
<td>1.23E-10</td>
</tr>
</tbody>
</table>

MSE 1.05E-11 8.76E-11

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
Table 3-3: Analytical (f) and Numerical (ILT) Inverses \( L(x) \) for F3 and F4

<table>
<thead>
<tr>
<th>t</th>
<th>f3</th>
<th>ILT</th>
<th>Diff(^2)</th>
<th>f4</th>
<th>ILT</th>
<th>Diff(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.36788</td>
<td>0.36788</td>
<td>1.43E-16</td>
<td>0.98777</td>
<td>0.98777</td>
<td>7.51E-13</td>
</tr>
<tr>
<td>2</td>
<td>0.13534</td>
<td>0.13534</td>
<td>1.1E-15</td>
<td>0.90930</td>
<td>0.90930</td>
<td>5.4E-12</td>
</tr>
<tr>
<td>3</td>
<td>0.04979</td>
<td>0.04979</td>
<td>8.78E-16</td>
<td>0.63816</td>
<td>0.63816</td>
<td>1.14E-12</td>
</tr>
<tr>
<td>4</td>
<td>0.01832</td>
<td>0.01832</td>
<td>1.61E-13</td>
<td>0.30807</td>
<td>0.30807</td>
<td>2.82E-11</td>
</tr>
<tr>
<td>5</td>
<td>0.00674</td>
<td>0.00674</td>
<td>2.34E-13</td>
<td>-0.02068</td>
<td>-0.02068</td>
<td>4.96E-12</td>
</tr>
<tr>
<td>6</td>
<td>0.00248</td>
<td>0.00248</td>
<td>8.45E-14</td>
<td>-0.31695</td>
<td>-0.31694</td>
<td>2.57E-11</td>
</tr>
<tr>
<td>7</td>
<td>0.00091</td>
<td>0.00091</td>
<td>1.52E-12</td>
<td>-0.56470</td>
<td>-0.56471</td>
<td>8.32E-11</td>
</tr>
<tr>
<td>8</td>
<td>0.00034</td>
<td>0.00034</td>
<td>2.45E-12</td>
<td>-0.75680</td>
<td>-0.75679</td>
<td>1.89E-10</td>
</tr>
<tr>
<td>9</td>
<td>0.00012</td>
<td>0.00012</td>
<td>1.23E-12</td>
<td>-0.89168</td>
<td>-0.89169</td>
<td>1.5E-10</td>
</tr>
<tr>
<td>10</td>
<td>0.00005</td>
<td>0.00005</td>
<td>1.95E-14</td>
<td>-0.97128</td>
<td>-0.97128</td>
<td>5.15E-11</td>
</tr>
<tr>
<td>11</td>
<td>0.00002</td>
<td>0.00002</td>
<td>7.37E-13</td>
<td>-0.99976</td>
<td>-0.99975</td>
<td>2.96E-11</td>
</tr>
<tr>
<td>12</td>
<td>0.00001</td>
<td>0.00000</td>
<td>2.54E-12</td>
<td>-0.96264</td>
<td>-0.96264</td>
<td>2.67E-13</td>
</tr>
<tr>
<td>13</td>
<td>0.00000</td>
<td>0.00000</td>
<td>3.75E-12</td>
<td>-0.92618</td>
<td>-0.92619</td>
<td>5.6E-12</td>
</tr>
<tr>
<td>14</td>
<td>0.00000</td>
<td>0.00000</td>
<td>3.58E-12</td>
<td>-0.83695</td>
<td>-0.83695</td>
<td>3.16E-11</td>
</tr>
<tr>
<td>15</td>
<td>0.00000</td>
<td>0.00000</td>
<td>2.44E-12</td>
<td>-0.72150</td>
<td>-0.72151</td>
<td>2.79E-10</td>
</tr>
<tr>
<td>16</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.18E-12</td>
<td>-0.58618</td>
<td>-0.58617</td>
<td>7.26E-12</td>
</tr>
<tr>
<td>17</td>
<td>0.00000</td>
<td>0.00000</td>
<td>3.06E-13</td>
<td>-0.43698</td>
<td>-0.43698</td>
<td>1.81E-12</td>
</tr>
<tr>
<td>18</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.68E-15</td>
<td>-0.27942</td>
<td>-0.27942</td>
<td>1.82E-11</td>
</tr>
<tr>
<td>19</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.47E-13</td>
<td>-0.11849</td>
<td>-0.11849</td>
<td>4.46E-13</td>
</tr>
<tr>
<td>20</td>
<td>0.00000</td>
<td>0.00000</td>
<td>5.18E-13</td>
<td>0.04136</td>
<td>0.04133</td>
<td>1.03E-09</td>
</tr>
<tr>
<td>21</td>
<td>0.00000</td>
<td>0.00000</td>
<td>8.94E-13</td>
<td>0.19627</td>
<td>0.19624</td>
<td>8.04E-10</td>
</tr>
<tr>
<td>22</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.17E-12</td>
<td>0.34296</td>
<td>0.34294</td>
<td>4.57E-10</td>
</tr>
<tr>
<td>23</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.28E-12</td>
<td>0.47867</td>
<td>0.47863</td>
<td>2.27E-09</td>
</tr>
<tr>
<td>24</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.25E-12</td>
<td>0.60121</td>
<td>0.60117</td>
<td>2.23E-09</td>
</tr>
<tr>
<td>25</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.11E-12</td>
<td>0.70886</td>
<td>0.70880</td>
<td>3.53E-09</td>
</tr>
<tr>
<td>26</td>
<td>0.00000</td>
<td>0.00000</td>
<td>9.23E-13</td>
<td>0.80037</td>
<td>0.80034</td>
<td>1.31E-09</td>
</tr>
<tr>
<td>27</td>
<td>0.00000</td>
<td>0.00000</td>
<td>7.11E-13</td>
<td>0.87493</td>
<td>0.87491</td>
<td>2.32E-10</td>
</tr>
<tr>
<td>28</td>
<td>0.00000</td>
<td>0.00000</td>
<td>5.09E-13</td>
<td>0.93209</td>
<td>0.93210</td>
<td>2.28E-10</td>
</tr>
<tr>
<td>29</td>
<td>0.00000</td>
<td>0.00000</td>
<td>3.38E-13</td>
<td>0.97176</td>
<td>0.97185</td>
<td>6.92E-09</td>
</tr>
<tr>
<td>30</td>
<td>0.00000</td>
<td>0.00000</td>
<td>2.01E-13</td>
<td>0.99417</td>
<td>0.99432</td>
<td>2.09E-08</td>
</tr>
<tr>
<td>31</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.1E-13</td>
<td>0.99980</td>
<td>1.00004</td>
<td>5.8E-08</td>
</tr>
<tr>
<td>32</td>
<td>0.00000</td>
<td>0.00000</td>
<td>4.76E-14</td>
<td>0.98936</td>
<td>0.98969</td>
<td>1.11E-07</td>
</tr>
<tr>
<td>33</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.23E-14</td>
<td>0.96376</td>
<td>0.96421</td>
<td>2.05E-07</td>
</tr>
<tr>
<td>34</td>
<td>0.00000</td>
<td>0.00000</td>
<td>8.75E-16</td>
<td>0.92406</td>
<td>0.92461</td>
<td>3.03E-07</td>
</tr>
<tr>
<td>35</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.74E-15</td>
<td>0.87146</td>
<td>0.87213</td>
<td>4.44E-07</td>
</tr>
<tr>
<td>36</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.23E-14</td>
<td>0.80726</td>
<td>0.80803</td>
<td>5.87E-07</td>
</tr>
<tr>
<td>37</td>
<td>0.00000</td>
<td>0.00000</td>
<td>2.54E-14</td>
<td>0.73282</td>
<td>0.73367</td>
<td>7.28E-07</td>
</tr>
<tr>
<td>38</td>
<td>0.00000</td>
<td>0.00000</td>
<td>4.11E-14</td>
<td>0.64954</td>
<td>0.65045</td>
<td>8.31E-07</td>
</tr>
<tr>
<td>39</td>
<td>0.00000</td>
<td>0.00000</td>
<td>5.66E-14</td>
<td>0.55867</td>
<td>0.55980</td>
<td>8.81E-07</td>
</tr>
<tr>
<td>40</td>
<td>0.00000</td>
<td>0.00000</td>
<td>6.85E-14</td>
<td>0.46223</td>
<td>0.46310</td>
<td>7.67E-07</td>
</tr>
</tbody>
</table>

MSE 7.42E-13 1.24E-07
Table 3-4: The Analytical (True PDF) and the Numerical (ILT) Inverse $L_g(z)$ for the PDF of $\text{Gam}(\alpha=3, \theta=2)$

<table>
<thead>
<tr>
<th>t</th>
<th>True PDF</th>
<th>ILT</th>
<th>Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.037908</td>
<td>0.037908</td>
<td>2.76E-17</td>
</tr>
<tr>
<td>2</td>
<td>0.091970</td>
<td>0.091970</td>
<td>2.65E-16</td>
</tr>
<tr>
<td>3</td>
<td>0.125511</td>
<td>0.125511</td>
<td>8.31E-14</td>
</tr>
<tr>
<td>4</td>
<td>0.135335</td>
<td>0.135335</td>
<td>2.91E-17</td>
</tr>
<tr>
<td>5</td>
<td>0.128258</td>
<td>0.128257</td>
<td>8.81E-13</td>
</tr>
<tr>
<td>6</td>
<td>0.112021</td>
<td>0.112021</td>
<td>2.17E-14</td>
</tr>
<tr>
<td>7</td>
<td>0.092479</td>
<td>0.092478</td>
<td>1E-12</td>
</tr>
<tr>
<td>8</td>
<td>0.073263</td>
<td>0.073264</td>
<td>1.73E-12</td>
</tr>
<tr>
<td>9</td>
<td>0.056239</td>
<td>0.056244</td>
<td>1.96E-11</td>
</tr>
<tr>
<td>10</td>
<td>0.042112</td>
<td>0.042117</td>
<td>2.73E-11</td>
</tr>
<tr>
<td>11</td>
<td>0.030906</td>
<td>0.030913</td>
<td>4.2E-11</td>
</tr>
<tr>
<td>12</td>
<td>0.022309</td>
<td>0.022311</td>
<td>6.06E-12</td>
</tr>
<tr>
<td>13</td>
<td>0.015880</td>
<td>0.015877</td>
<td>8.64E-12</td>
</tr>
<tr>
<td>14</td>
<td>0.011171</td>
<td>0.011161</td>
<td>9.8E-11</td>
</tr>
<tr>
<td>15</td>
<td>0.007778</td>
<td>0.007766</td>
<td>1.48E-10</td>
</tr>
<tr>
<td>16</td>
<td>0.005367</td>
<td>0.005349</td>
<td>3.28E-10</td>
</tr>
<tr>
<td>17</td>
<td>0.003675</td>
<td>0.003663</td>
<td>1.51E-10</td>
</tr>
<tr>
<td>18</td>
<td>0.002499</td>
<td>0.002490</td>
<td>8.73E-11</td>
</tr>
<tr>
<td>19</td>
<td>0.001689</td>
<td>0.001684</td>
<td>2.53E-11</td>
</tr>
<tr>
<td>20</td>
<td>0.001135</td>
<td>0.001134</td>
<td>3.24E-13</td>
</tr>
<tr>
<td>21</td>
<td>0.000759</td>
<td>0.000761</td>
<td>4.12E-12</td>
</tr>
<tr>
<td>22</td>
<td>0.000505</td>
<td>0.000516</td>
<td>1.23E-10</td>
</tr>
<tr>
<td>23</td>
<td>0.000335</td>
<td>0.000347</td>
<td>1.53E-10</td>
</tr>
<tr>
<td>24</td>
<td>0.000221</td>
<td>0.000237</td>
<td>2.42E-10</td>
</tr>
<tr>
<td>25</td>
<td>0.000146</td>
<td>0.000164</td>
<td>3.41E-10</td>
</tr>
<tr>
<td>26</td>
<td>0.000095</td>
<td>0.000112</td>
<td>2.74E-10</td>
</tr>
<tr>
<td>27</td>
<td>0.000062</td>
<td>0.000078</td>
<td>2.43E-10</td>
</tr>
<tr>
<td>28</td>
<td>0.000041</td>
<td>0.000053</td>
<td>1.48E-10</td>
</tr>
<tr>
<td>29</td>
<td>0.000027</td>
<td>0.000038</td>
<td>1.32E-10</td>
</tr>
<tr>
<td>30</td>
<td>0.000017</td>
<td>0.000026</td>
<td>8.11E-11</td>
</tr>
<tr>
<td>31</td>
<td>0.000011</td>
<td>0.000016</td>
<td>2.76E-11</td>
</tr>
<tr>
<td>32</td>
<td>0.000007</td>
<td>0.000006</td>
<td>9.68E-13</td>
</tr>
<tr>
<td>33</td>
<td>0.000005</td>
<td>0.000001</td>
<td>1.45E-11</td>
</tr>
<tr>
<td>34</td>
<td>0.000003</td>
<td>0.000001</td>
<td>4.72E-12</td>
</tr>
<tr>
<td>35</td>
<td>0.000002</td>
<td>-0.000003</td>
<td>2.9E-11</td>
</tr>
<tr>
<td>36</td>
<td>0.000001</td>
<td>-0.000003</td>
<td>2.17E-11</td>
</tr>
<tr>
<td>37</td>
<td>0.000001</td>
<td>-0.000007</td>
<td>6.27E-11</td>
</tr>
<tr>
<td>38</td>
<td>0.000001</td>
<td>-0.000008</td>
<td>7.63E-11</td>
</tr>
<tr>
<td>39</td>
<td>0.000000</td>
<td>-0.000009</td>
<td>8.17E-11</td>
</tr>
<tr>
<td>40</td>
<td>0.000000</td>
<td>-0.000010</td>
<td>1.08E-10</td>
</tr>
</tbody>
</table>

MSE 7.78E-11
Table 3-5 : Numerically Recovering Six PDFs of S by the Stehfest Method
- (the values in the shaded area should be replace by zeros)

<table>
<thead>
<tr>
<th>t</th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.174326</td>
<td>0.35075</td>
<td>0.109423</td>
<td>0.267492</td>
<td>0.444919</td>
<td>0.037908</td>
</tr>
<tr>
<td>2</td>
<td>0.189041</td>
<td>0.223361</td>
<td>0.194418</td>
<td>0.215507</td>
<td>0.282702</td>
<td>0.09197</td>
</tr>
<tr>
<td>3</td>
<td>0.159046</td>
<td>0.126329</td>
<td>0.170776</td>
<td>0.10078</td>
<td>0.049403</td>
<td>0.135335</td>
</tr>
<tr>
<td>4</td>
<td>0.091069</td>
<td>0.037532</td>
<td>0.117007</td>
<td>0.067645</td>
<td>0.018696</td>
<td>0.128257</td>
</tr>
<tr>
<td>5</td>
<td>0.066428</td>
<td>0.020456</td>
<td>0.07975</td>
<td>0.045357</td>
<td>0.006964</td>
<td>0.112021</td>
</tr>
<tr>
<td>6</td>
<td>0.048029</td>
<td>0.011233</td>
<td>0.052189</td>
<td>0.030406</td>
<td>0.002582</td>
<td>0.092478</td>
</tr>
<tr>
<td>7</td>
<td>0.034573</td>
<td>0.006228</td>
<td>0.033268</td>
<td>0.020382</td>
<td>0.000954</td>
<td>0.073264</td>
</tr>
<tr>
<td>8</td>
<td>0.024831</td>
<td>0.003486</td>
<td>0.020851</td>
<td>0.013662</td>
<td>0.00036</td>
<td>0.056244</td>
</tr>
<tr>
<td>9</td>
<td>0.017814</td>
<td>0.001968</td>
<td>0.01292</td>
<td>0.009157</td>
<td>0.000129</td>
<td>0.042117</td>
</tr>
<tr>
<td>10</td>
<td>0.012773</td>
<td>0.001122</td>
<td>0.00795</td>
<td>0.006138</td>
<td>3.97E-05</td>
<td>0.030913</td>
</tr>
<tr>
<td>11</td>
<td>0.009155</td>
<td>0.000643</td>
<td>0.004869</td>
<td>0.004114</td>
<td>7.86E-06</td>
<td>0.022311</td>
</tr>
<tr>
<td>12</td>
<td>0.006561</td>
<td>0.00037</td>
<td>0.002977</td>
<td>0.002758</td>
<td>4.35E-06</td>
<td>0.015877</td>
</tr>
<tr>
<td>13</td>
<td>0.004702</td>
<td>0.000212</td>
<td>0.001818</td>
<td>0.001849</td>
<td>-1.63E-05</td>
<td>0.011161</td>
</tr>
<tr>
<td>14</td>
<td>0.003369</td>
<td>0.000123</td>
<td>0.001114</td>
<td>0.001239</td>
<td>-3.55E-06</td>
<td>0.007766</td>
</tr>
<tr>
<td>15</td>
<td>0.002414</td>
<td>6.76E-05</td>
<td>0.000679</td>
<td>0.000831</td>
<td>-9.40E-07</td>
<td>0.005349</td>
</tr>
<tr>
<td>16</td>
<td>0.00173</td>
<td>4.11E-05</td>
<td>0.000419</td>
<td>0.000557</td>
<td>-2.99E-06</td>
<td>0.003663</td>
</tr>
<tr>
<td>17</td>
<td>0.00124</td>
<td>2.34E-05</td>
<td>0.000259</td>
<td>0.000374</td>
<td>2.02E-06</td>
<td>0.00249</td>
</tr>
<tr>
<td>18</td>
<td>0.000888</td>
<td>1.34E-05</td>
<td>0.000158</td>
<td>0.000251</td>
<td>2.22E-06</td>
<td>0.001684</td>
</tr>
<tr>
<td>19</td>
<td>0.000636</td>
<td>7.29E-06</td>
<td>9.52E-05</td>
<td>0.000169</td>
<td>2.61E-06</td>
<td>0.001134</td>
</tr>
<tr>
<td>20</td>
<td>0.000456</td>
<td>2.54E-06</td>
<td>5.85E-05</td>
<td>0.000113</td>
<td>8.98E-06</td>
<td>0.000761</td>
</tr>
<tr>
<td>21</td>
<td>0.000327</td>
<td>3.27E-06</td>
<td>3.05E-05</td>
<td>7.61E-05</td>
<td>7.48E-06</td>
<td>0.000516</td>
</tr>
<tr>
<td>22</td>
<td>0.000234</td>
<td>1.05E-06</td>
<td>1.56E-05</td>
<td>5.11E-05</td>
<td>5.24E-06</td>
<td>0.000347</td>
</tr>
<tr>
<td>23</td>
<td>0.000168</td>
<td>8.41E-07</td>
<td>5.81E-06</td>
<td>3.43E-05</td>
<td>7.13E-07</td>
<td>0.000237</td>
</tr>
<tr>
<td>24</td>
<td>0.00012</td>
<td>1.38E-06</td>
<td>-2.39E-06</td>
<td>2.29E-05</td>
<td>1.14E-05</td>
<td>0.000164</td>
</tr>
<tr>
<td>25</td>
<td>8.63E-05</td>
<td>3.61E-07</td>
<td>8.55E-07</td>
<td>1.52E-05</td>
<td>-2.45E-07</td>
<td>0.000112</td>
</tr>
<tr>
<td>26</td>
<td>6.18E-05</td>
<td>4.89E-07</td>
<td>-6.53E-06</td>
<td>9.96E-06</td>
<td>7.11E-06</td>
<td>7.80E-05</td>
</tr>
<tr>
<td>27</td>
<td>4.43E-05</td>
<td>-2.92E-07</td>
<td>-1.22E-05</td>
<td>6.40E-06</td>
<td>6.79E-06</td>
<td>5.29E-05</td>
</tr>
<tr>
<td>28</td>
<td>3.16E-05</td>
<td>5.26E-07</td>
<td>-1.00E-05</td>
<td>3.98E-06</td>
<td>1.01E-06</td>
<td>3.80E-05</td>
</tr>
<tr>
<td>29</td>
<td>2.25E-05</td>
<td>7.73E-07</td>
<td>-5.02E-06</td>
<td>2.33E-06</td>
<td>3.18E-06</td>
<td>2.62E-05</td>
</tr>
<tr>
<td>30</td>
<td>1.60E-05</td>
<td>4.33E-07</td>
<td>-5.74E-06</td>
<td>1.22E-06</td>
<td>-1.82E-06</td>
<td>1.64E-05</td>
</tr>
<tr>
<td>31</td>
<td>1.13E-05</td>
<td>9.65E-07</td>
<td>-3.20E-06</td>
<td>4.68E-07</td>
<td>2.16E-06</td>
<td>6.22E-06</td>
</tr>
<tr>
<td>32</td>
<td>7.86E-06</td>
<td>9.21E-07</td>
<td>-2.26E-06</td>
<td>-2.78E-08</td>
<td>1.76E-06</td>
<td>8.44E-07</td>
</tr>
<tr>
<td>33</td>
<td>5.39E-06</td>
<td>8.22E-07</td>
<td>-3.87E-06</td>
<td>-3.33E-07</td>
<td>4.91E-07</td>
<td>8.19E-07</td>
</tr>
<tr>
<td>34</td>
<td>3.61E-06</td>
<td>4.18E-07</td>
<td>-3.86E-06</td>
<td>-5.15E-07</td>
<td>-5.63E-06</td>
<td>-3.46E-06</td>
</tr>
<tr>
<td>35</td>
<td>2.33E-06</td>
<td>1.36E-06</td>
<td>-3.43E-07</td>
<td>-6.07E-07</td>
<td>-1.89E-06</td>
<td>-3.42E-06</td>
</tr>
<tr>
<td>36</td>
<td>1.40E-06</td>
<td>7.42E-07</td>
<td>-1.06E-06</td>
<td>-6.34E-07</td>
<td>1.79E-06</td>
<td>-7.13E-06</td>
</tr>
<tr>
<td>37</td>
<td>7.43E-07</td>
<td>8.79E-07</td>
<td>3.23E-07</td>
<td>-6.17E-07</td>
<td>-1.10E-06</td>
<td>-8.23E-06</td>
</tr>
<tr>
<td>38</td>
<td>2.81E-07</td>
<td>1.08E-06</td>
<td>4.75E-07</td>
<td>-5.63E-07</td>
<td>6.18E-06</td>
<td>-8.72E-06</td>
</tr>
<tr>
<td>39</td>
<td>-3.80E-08</td>
<td>6.89E-07</td>
<td>1.11E-06</td>
<td>-4.94E-07</td>
<td>-7.37E-06</td>
<td>-1.02E-05</td>
</tr>
<tr>
<td>40</td>
<td>-3.80E-08</td>
<td>6.89E-07</td>
<td>1.11E-06</td>
<td>-4.94E-07</td>
<td>-7.37E-06</td>
<td>-1.02E-05</td>
</tr>
</tbody>
</table>
Table 3-6: Comparison between Monte Carlo Simulation and Numerical Inverses $L_S(z)$ (ILT) Based on the Mean and Variance

<table>
<thead>
<tr>
<th>Case</th>
<th>true</th>
<th>Monte Carlo Simulation</th>
<th>ILT</th>
<th>Mean</th>
<th>true</th>
<th>Monte Carlo Simulation</th>
<th>ILT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4.00997</td>
<td>3.99797</td>
<td></td>
<td>10</td>
<td>10.10665</td>
<td>10.00756</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.99648</td>
<td>1.97532</td>
<td></td>
<td>3</td>
<td>3.01346</td>
<td>3.09792</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>3.99698</td>
<td>4.00187</td>
<td></td>
<td>6</td>
<td>5.95578</td>
<td>5.98297</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3.02109</td>
<td>2.99304</td>
<td></td>
<td>6.5</td>
<td>6.58634</td>
<td>6.54150</td>
</tr>
<tr>
<td>5</td>
<td>1.75</td>
<td>1.75697</td>
<td>1.74934</td>
<td></td>
<td>1.375</td>
<td>1.39686</td>
<td>1.38633</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6.01036</td>
<td>6.00249</td>
<td></td>
<td>12</td>
<td>12.01471</td>
<td>12.02719</td>
</tr>
</tbody>
</table>
APPENDIX II

COMPUTER LANGUAGE CODES
BASIC Program for the Stehfest Method

```basic
LET N=20
LET Nh=N/2
LET FilenameS="invlap.out"
Open #1: Name FilenameS, Create Newold
Erase #1
Printsl: " Inversion Laplace Transformation at N: ", N
DIM V(l0), G(l0), H(l;Nh)
MAT reDimV(0:N), G(0:N), H(l:Nh)
Def Min(a,b)
if a>b then let Min=b else let Min=a
end Def
Def Sign(x)
if Mod(x,2)=0 then let Sign=0 else let Sign=1
end Def
LET G(0)=1
FOR i=l to N
LET G(i)=G(i-1)*i
NEXT i
LET H(l)=2/G(Nh-l)
FOR i=2 to Nh
LET H(i) = (i"Nh)* (G(2*i))/(G(Nh-i)*G(i)*G(i-l)
next i
let sn=2*Sign(Nh)-1
for i=1 to M
let V(i)=0
for k=int((i+1)/2) to Min(i,Nh)
let V(i)=V(i)+H(k)/ (G(i-k)*G(2*k-i) )
next k
let V(i)=sn*V(i)
let sn=-sn
next i
for w=l to 6
if w=l then Printsl: " let P(l)=1/sqr(s)"
if w=2 then Printsl: " let P(2)=log(s)/ (s)"
if w=3 then Printsl: " let P(3)=1/(s^4)"
if w=4 then Printsl: " let P(4)=1/(s+1)"
if w=5 then Printsl: "let P(5)=sqr(3.1415927/(2*s"^3))*exp(-(1/(2*s)))"
if w=6 then Printsl: "let P(6)=(s-1)^3)/(s^4)"
for t=1 to 20
let Fa=0
for i=1 to N
let s=a*i
if w=1 then let P(1)=1/sqr(s)
if w=2 then let P(2)=log(s)/ (s)"
if w=3 then let P(3)=1/(s^4)"
if w=4 then let P(4)=1/(s+1)"
if w=5 then let P(5)=sqr(3.1415927/(2*s"^3))*exp(-(1/(2*s)))"
if w=6 then let P(6)=(s-1)^3)/(s^4)"
let Fa=Fa+V(i)*P(w)
next i
```

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
let Fa = a*Fa
print#1: w, t, fa, V(t)
next t
next w
End
Pascal Program for the Stehfest Method

```pascal
program invlap;
uses
  WinCrt;  { Allows Writeln, Readln, cursor movement, etc. }
const N=20;
  Nh=10; (N/2)
type arrone=array [0..N] of extended;
arrtwo=array [1..Nh] of extended;
arrthree=array [1..6] of extended;
var t,i, ih, k, sn,w:integer;
s,Fa,a: extended;
sign, V,G:arrone;
H:arrtwo;
Piarrthree;
log:text;
function Min(a,b:integer): integer;
begin
  if a>b then Min:=b
  else Min:=a;
  end; ( Min)
begin
  assign(log, 'invlap.out' )  ;
  rewrite(log);
  G[0]:=1;
  for i:=1 to N do G[i]:=G[i-1]*i;
  H[1]:=2/G[Nh-1] ;
  for i:=2 to Nh do
    H[i]:=exp(Nh*ln(i))* (G[2*i])/ (G[Nh-i]*G[i]*G[i-1] )  ;
  if (Nh mod 2)=0 then sn:=-l
  else sn:=1; (endif)
  for i:=1 to N do
    begin
      V[i]:=0 ;
      for k:=trunc((i+1)/2) to Min(i,Nh) do V[i] :=V[i]+H[k]/(G[i-k]*G[2*k-i]);
      {  V[i]:=sn*V[i];
      sign[i]:=sn;
      sn:=-sn;
      end;
    writeln;
    writeln(log,'for N : ',N:3);
    for w :=1 to 6 do begin
      if w=1 then writeln(log,' P(1) : LTF of Exp(theta=5)+Exp(10) ')
      else if w=2 then writeln(log,' P(2) : LTF of Chisq(r=1)+Exp(theta=1) ')
      else if w=3 then writeln(log,' P(3) : LTF of Gamma(a=2,theta=2) ')
      else if w=4 then writeln(log,' P(4) : LTF of Exp(theta=2)+Gamma(a=2,theta=4) ')
      else if w=5 then
        begin
          writeln(log,' P(5) : LTF of Exp(theta=2)+Gamma(a=2,theta=2) ')
          else writeln(log,' P(6) : LTF of Normal(u=6,s2=1)+ Exp(theta=1) ')
          end;
  end;
end;
```
for t := 1 to 20 do begin
   Fa:=0;
   a:=ln(2)/t;
   for i:=1 to N do begin
      s:=a*i;
      if w=1 then P[1]:=1/((1+(5*s))*(1+(10*s)))
      else if w=2 then P[2]:=1/((sqrt(1+(2*s)))*(1+s))
      else if w=3 then P[3]:=1/((1+(2*s))*(1+s))
      else if w=4 then P[4]:=1/((1+2*s)*(1+(4*s))*(1+(2*s))
      else if w=5 then P[5]:=1/((1+(2*s))*(1+(2*s))*(1+(2*s))
      else if w=6 then P[6]:=exp((-6*s)+(s*s/2))/(1+s);
      Fa:=Fa+(V[i]*P[w])*sign[i];
   end;  {endfor i}
   Fa:=a*Fa;
   writeln(log,t:5,Fa:28:10);
end;  {endfor t}
end;  {end for w}
close(log);
end.
C++ Program for the Stehfest Method

```cpp
/* The following program is the algorithm for the Stehfest method. */
/* It computes the inverses of the following Laplace transforms */
/* at N=20: */
/* P1(s)=1/sqrt(s) */
/* P2(s)=log(s)/s */
/* P3(s)=1/(s^4) */
/* P4(s)=1/(s+1) */
/* P5(s)=sqrt(3.1415927/(2*s^3))*exp(-1/(2*s)) */
/* P6(s)=(s-1)^3/(s^4) */
/* The algorithm was derived by Stehfest [15], */
/* coded by Tsang in Borland C++ language. */

#include <iostream.h>
#include <math.h>
#include <fstream.h>

int min(int nl, int n2);
maint()
{
    int N, Nh, t, i, ih, sn, w, k, n1, n2, sign[21];
    long double s, Fa, a;
    long double V[21], G[21], P[6], H[11]; // double -long double
    ofstream fp; fp.open("invlap.out", ios::app);
    fp.precision(40);
    N=20;
    Nh=N/2;
    G[0]=1;
    for (i=1; i<=N ; i++) { G[i]=G[i-1]*i; } /* endfor */
    for (i=2; i<=Nh ; i++) { H[i]=pow(i,Nh)*G[2-i]*G[i]*G[i-1]; } /* endfor */
    if (fmod(Nh,2)==0) { sn=-1; } else { sn=1; }
    for (i=1; i<=N ; i++) { V[i]=0;
        for (k=floor((i+1)/2); k<=min(i,Nh); k++) { V[i]=V[i]+(H[k]/G[i-k]*G[2*k-i]); } /* endfor */
        sign[i]=sn;
        sn=-sn; } /* endfor */
    fp << "Inversion of Laplace Transformation 368 N: " " N: " " 
    for(i=1; i<=N; i++) { fp << "i: " " V[i] " " 
    switch(w) {
        case (1): { fp << " ** P1: 1/sqrt(s) ** " " 
        break;
        case (2): { fp << " ** P2: log(s)/s ** " " 
        break;
        case (3): { fp << " ** P3: 1/(s^4) ** " " 
        break;
        case (4): { fp << " ** P4: 1/(s+1) ** " " 
        break;
        case (5): { fp << " ** P5: sqrt(3.1415927/(2*s^3))*exp(-1/(2*s)) ** " " 
        break;
        case (6): { fp << " ** P6: pow((s-1)^3/(pow(s,4) ** " " 
        break;
        } */
```

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
t=1;
while(t<=20){
    Fa=0;
    a=log(2)/t;
    i=1;
    while(i<=N) {
        s=a*i;
        switch(w) {
            case (1): [ P[1]= 1/sqrt(s) ;
                        break;]
            case (2): [ P[2]=log(s)/s ;
                        break;]
            case (3): [ P[3]=1/(s*s*s*s) ;
                        break;]
            case (4): [ P[4]=1/(s+1);
                        break;]
            case (5): [ P[5]=sqrt(3.1415926535897932385/(2*s*s*s)) *exp(-1/(2*s)) ;
                        break;]
            case (6): [ P[6]= pow((s-1),3)/pow(s,4);
                        break;]
            Fa+=(V[i]*P[w])*sign[i];
            i+=1;
        } /* endfor i */
    Fa*=a;
    fp << t << "\t" << Fa << "\n";
    t+=1;
} /* endfor t */
w+=1;
} /* endwhile */
fp.close();
return 0;
}

int min(int n1, int n2) {
    int minimum; // Local value.
    minimum = (n1 < n2) ? (n1) : (n2);
    return(minimum);
}
SAS Program for the Stehfest Method

/********************************************
/* The following program is the algorithm for the Stehfest method.
/* It computes the inverses of the following Laplace transforms
/* at N=20:
/* P1(s)=1/sqrt(s)
/* P2(s)=log(s)/(s)
/* P3(s)=1/(s^4)
/* P4(s)=1/(s+1)
/* P5(s)=sqrt(3.1415927/(2*(s^3)))*exp(-1/(2*s))
/* P6(s)=(s-1)^3/(s^4)
/* The algorithm was derived by Stehfest [15],
/* coded by Tsang in SAS statistical language.
/********************************************/
data test;
N=20;
Nh=N/2;
array P(6); array V(21); array G(21); array H(10); G(1)=1;
do i=1 to N;
G(i+1)=G(i)*i;
end;
H(1)=2/G(Nh);
do i=2 to Nh;
H(i)=(i*Nh)*G(2*i+1)/(G(Nh-i+1)*G(i+1)*G(i));
end;
if mod(Nh,2)=0 then sn=-1;
else sn=1;
do i=1 to N;
V(i+1)=0;
do k=floor((i+1)/2) to Min(i,Nh);
V(i+1)=(H(k)/(G(i-k+1)*G(2*k-i+1)));;
end;
V(i+1)=V(i+1)*sn;
sn=sn*-1;
end;
put 'N:' N;
do w=1 to 6;
if w=1 then put ' P(1)=1/sqrt(s)' ;
else if w=2 then put ' P(2)=log(s)/s' ;
else if w=3 then put ' P(3)=1/(s**4)' ;
else if w=4 then put ' P(4)=1/(s+1)' ;
else if w=5 then put ' P(5)=sqrt(3.1415927/(2*(s**3)))*exp(-1/(2*s))' ;
else if w=6 then put ' P(6)=(s-1)**3/(s**4)' ;
do t=1 to 20;
T=-log(2)/t;
Fa=0;
do i=1 to N;
s=i*T;
if w=1 then P(1)=1/sqrt(s);
else if w=2 then P(2)=log(s)/s;
else if w=3 then P(3)=1/(s**4);
else if w=4 then P(4)=1/(s+1);
else if w=5 then P(5)=sqrt(3.1415927/(2*(s**3)))*exp(-1/(2*s)) ;
else if w=6 then P(6)=(s-1)**3/(s**4);
Fa+=(V(i+1)*P(w));
end;
Fa=a*Fa;
put 't:' t 'Fa:' Fa 'V:' V(t+1);
end;
run;

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
APPENDIX III

SAS MONTE CARLO SIMULATION OF S
SAS Program for Monte Carlo Simulation of Six Distributions of $S$

/* Simulated six random variables, the sums, are:

Case 1: $S = \text{Exp}(\theta=1) + \text{Exp}(\theta=3)$
Case 2: $S = \chi^2(\nu=1) + \text{Exp}(\theta=1)$
Case 3: $S = \text{Gam}(\alpha=1, \theta=2) + \text{Gam}(\alpha = 2, \theta=1)$
Case 4: $S = \text{Exp}(\theta=0.5) + \text{Gam}(\alpha = 1.5, \theta=0.5)$
Case 5: $S = \text{Exp}(\theta=1) + \text{Gam}(\alpha = 1.5, \theta=2)$
Case 6: $S = \text{Exp}(\theta=2) + \text{Gam}(\alpha = 2, \theta=2)$

*/

/* written by Tsang in SAS statistical language.*/

/*----------------------- Case 1 -----------------------*/
data casecur; /*exp(theta=1)+exp(theta=3)*/;
do i=1 to 100000;
x=ranexp(198765)*1; /*exp(theta=1)*/;
y=ranexp(2097456)*3; /*exp(theta=3)*/;
s=x+y;
sc=floor(s)+1;
output;
end;
run;
proc means;
run;
proc freq data=casecur;
table sc/noprint out=freqout1;run;

/*----------------------- Case 2 -----------------------*/
data casecur; /*chisq(r=1)+exp(theta=1)*/;
do i=1 to 100000;
\$x=2*rangam(198765,0.5); /*Chisq with r=1*/;
y=ranexp(2097456)*1; /*exp(theta=1)*/;
s=x+y;
sc=(floor(s*10/5)+1)*5/10;
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout2;run;

/*----------------------- Case 3 -----------------------*/
data casecur; /*gamma(a=1,theta=2)+gamma(a=2,theta=1)*/;
do i=1 to 100000;
x=2*rangam(198765,1); /*Chisq with r=1*/;
y=rangam(2092345,2); /*gamma(a=1,theta=2.5)*/;
s=x+y;
sc=(floor(s*10/5)+1)*5/10;
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout3;run;

/*----------------------- Case 4 -----------------------*/
data casecur; /*exp(theta=0.5)+gamma(a=1,theta=2.5)*/;
do i=1 to 100000;
x=ranexp(198765)*0.5; /*exp(theta=0.5)*/;
y=2.5*rangam(2092345,1); /*gamma(a=1,theta=2.5)*/;
s=x+y;
sc=(floor(s*10/5)+1)*5/10;
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout4;run;

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout4;run;
/*.............................. Case 5 ...................................... */;
data casecur; 
/*exp(theta=1)+gamma(a=1.5,theta=0.5)*/;
do i=1 to 100000;
x=ranexp(198765)*1; /*exp(theta=1)*/;
y=0.5*rangam(2092345,1.5); /*gamma(a=1.5,theta=0.5)*/;
s=x+y;
*sc=(floor(s*10/5)+1)*5/10;
sc=(floor(s))+1;
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout5;run;
/*.............................. Case 6 ...................................... */;
data casecur; 
/*exp(theta=2)+gamma(a=2,theta=2)*/;
do i=1 to 100000;
x=ranexp(198765)*2; /*exp(theta=2)*/;
y=2*rangam(2092345,2); /*gamma(a=2,theta=2)*/;
s=x+y;
sc=(floor(s))+1;
output;
end;
run;
proc means;
run;
proc freq;
table sc/noprint out=freqout6;run;
data all; set freqout1(in=case1) freqout2(in=case2) freqout3(in=case3) freqout4(in=case4) freqout5(in=case5) freqout6(in=case6);
if case1 then case=1;
else if case2 then case=2;
else if case3 then case=3;
else if case4 then case=4;
else if case5 then case=5;
else if case6 then case=6;
run;
SAS Macro for Dynamic Data Exchange (DDE) to Microsoft Excel

/** Obtain all the information of the given data set **/
/** such as variables' names, the number of observations... **/
%macro excel (datin);
proc contents data=&datin out=work.headinfo short noprint;
run;
data _null_; 
set work.headinfo end=last;
call symput ('var'!!trim(left(put(_n_,8.))),name);
if last then do;
   call symput ('no_vars',trim(left(put(_n_,8.))));
   call symput ('no_rows',trim(left(put(nobs+1,8.))));
end; 
run;
data varname;set work.headinfo;
   keep name;
run;
proc sort data=varname;by name;
proc transpose data=varname out=res1;
   var name;
run;
data res1;set res1;
   keep col1-c&no_vars;
run;
filename exceltmp dde "excel | Sheet1 ! r1c1:r&no_rows.c&no_vars" notab;
data _null_; 
   file exceltmp;
set Matin;
   put 
      %do loop=1 %to &no_vars;
      &var&loop '09'x 
   %end ;
run;
filename exceltmp clear;
%mend excel;

/** Obtain all the given data exchange the data without variables' names **/
%macro excel1 (datin);
proc contents data=&datin out=work.headinfo short noprint;
run;
data _null_; 
   set work.headinfo end=last;
call symput ('var'!!trim(left(put(_n_,8.))),name);
if last then do;
   call symput ('no_vars',trim(left(put(_n_,8.))));
   call symput ('no_rows',trim(left(put(nobs+1,8.))));
end; 
run;
filename exceltmp dde "excel | Sheet1 ! r1c1:r&no_rows.c&no_vars" notab;
data _null_; 
   file exceltmp;
set &datin;
   put 
      %do loop=1 %to &no_vars;
      &var&loop '09'x 
   %end ;
run;
filename exceltmp clear;
%mend excel1;
; /* Assign the data for data exchange i.e. Data=work.all */
Xexcel (all);
/* update the variables' name to Microsoft Excel */
/* always use res1 that represent the information of variables' names */
Xexcel(res1);
run;
REFERENCES


Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.


