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Notes on Linear Divisible Sequences and Their Construction: A Computational Approach

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NOTES ON LINEAR DIVISIBLE SEQUENCES AND THEIR CONSTRUCTION: A COMPUTATIONAL APPROACH

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ABSTRACT

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In this Masters thesis, we examine linear divisible sequences. A linear divisible sequence is any sequence $\{a_n\}_{n\geq 0}$ that can be expressed by a linear homogeneous recursion relation that is also a divisible sequence. A sequence $\{a_n\}_{n\geq 0}$ is called a divisible sequence if it has the property that if $n|m$, then $a_n|a_m$. A sequence of numbers $\{a_n\}_{n\geq 0}$ is called a linear homogeneous recurrence sequence of order m if it can be written in the form

$$a_{n+m} = p_1 a_{n+m-1} + p_2 a_{n+m-2} + \cdots + p_{m-1} a_{n+1} + p_m a_n, \quad n \geq 0,$$

for some constants $p_1, p_2, ..., p_m$ with $p_m \neq 0$ and initial conditions $a_0, a_1, ..., a_{m-1}$. We focus on taking products, powers, and products of powers of second order linear divisible sequences in order to construct higher order linear divisible sequences. We hope to find a pattern in these constructions so that we can easily form higher order linear divisible sequence.

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CHAPTER 1

INTRODUCTION

In this thesis we examine the construction of higher order linear divisible sequences. A linear divisible sequence is any sequence of numbers $\{a_n\}_{n\geq 0}$ that can be expressed as a linear homogeneous recurrence relation that is also a divisible sequence. We also look at polynomial linear divisible sequences. A polynomial linear divisible sequence is any sequence of polynomials $\{a_n(x)\}_{n\geq 0}$ that can be expressed as a linear homogeneous recurrence relation that is also a divisible sequence. For the rest of this thesis, we will define $\{a_n\}$ to mean $\{a_n\}_{n\geq 0}$ and $\{a_n(x)\}$ to mean $\{a_n(x)\}_{n\geq 0}$.

A sequence of numbers $\{a_n\}$ is called a divisibility sequence if it has the property that whenever $n|m$, then $a_n|a_m$. Our definition of divides in the integral domain states that if R is an integral domain and $a, b \in R$, then we say $a|b$ if there exists $k \in R$ such that $ak = b$. Thus, if $\{a_n\}$ is a sequence of elements of the ring of integers $\mathbb{Z}$, then $a_n|a_m$ means there is a $k \in \mathbb{Z}$ such that $a_n k = a_m$. A sequence of polynomials $\{a_n(x)\}$ is a divisibility sequence if it has the property that whenever $n|m$, then $a_n(x)|a_m(x)$. This would mean there exists a polynomial $k(x)$ such that $a_n(x)k(x) = a_m(x)$.

In [2] we get a good history on divisible sequences. The concept of divisibility sequences were first discussed by Lucas [12] in 1878. However the term divisibility sequence first appeared in the 1930s in works by Hall [7], Lehmer [11], and Ward [15]. More recent works on divisibility sequence can be seen in works by Bézivin, Pethö, and Van Der Poorten [1]; Silverman [14]; as well as He and Shiue [9]. Also in the bibliography in [5], one can find an extensive list of works on recurrence sequences, including divisibility sequences. In fact, Lehmer [11] did a lot of work with non-integer sequences such as $u_{n+2} = \sqrt{\ell}u_{n+1} + bu_n$ for $u_0 = 0$, $u_1 = 1$ where $\ell, b \in \mathbb{Z}$ and $\gcd(\ell, b) = 1$.

A sequence of numbers $\{a_n\}$ is called a linear homogeneous recurrence sequence of order m if

$$a_{n+m} = p_1 a_{n+m-1} + p_2 a_{n+m-2} + \cdots + p_{m-1} a_{n+1} + p_m a_n, \tag{1.1}$$

for any $n \geq 0$, constants $p_1, p_2, ..., p_m$ with $p_m \neq 0$, and initial conditions $a_0, a_1, ..., a_{m-1}$. Since equation (1.1) is linear, we know that if the sequences $\{a_n\}$ and $\{b_n\}$ are recurrence sequences that satisfy equation (1.1) and c is a non-zero constant, then the sequence $\{ca_n + b_n\}$ also satisfies equation (1.1).

Suppose we have a solution to (1.1) that is the geometric series $\{a_n\}$ where $a_n = \alpha^n$ for some α. Then we have

$$\alpha^{n+m} = a_{n+m} = p_1 \alpha^{n+m-1} + p_2 \alpha^{n+m-2} + \cdots + p_{m-1} \alpha^{n+1} + p_m \alpha^n, \quad n \geq 0.$$

Moving everything to one side and dividing by α^n, we get

$$P_m(\alpha) = \alpha^m - p_1 \alpha^{m-1} - p_2 \alpha^{m-2} - \cdots - p_{m-1} \alpha - p_m = 0. \tag{1.2}$$

Thus, the sequence $\{a_n\}$ where $a_n = \alpha^n$ satisfies equation (1.1) if and only if α is a solution to equation (1.2). Equation (1.2) is called the characteristic equation and its roots are called characteristic roots.

Suppose the characteristic equation (1.2) has m distinct roots, $\{\alpha_k\}_{k=1}^m$, then α_k^n is a solution to the recurrence relation for all k. Therefore, the sequence $\{a_n\}$ satisfies the recurrence relation if and only if

$$a_{n+m} = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_{m-1} \alpha_{m-1}^n + A_m \alpha_m^n, \tag{1.3}$$

for all n. The constants $\{A_k\}$ depend on the $\{p_k\}$ and the initial conditions.

Suppose the characteristic equation (1.2) has $i \leq m$ distinct roots, $\{\alpha_k\}_{k=1}^i$ with each α_k having multiplicity j_k, $k = 1, 2, \ldots, i$. Then, for each α_k, we know α_k^n, $n\alpha_k^n$, $n^2\alpha_k^n$, $\ldots$, $n^{j_k-1}\alpha_k^n$ are all solutions to the recurrence relation. Therefore, the sequence $\{a_n\}$ satisfies the recurrence relation if and only if

$$\begin{aligned} a_n = {} & (A_{1,0} + A_{1,1} n + A_{1,2} n^2 + \cdots + A_{1,j_1-1} n^{j_1-1}) \alpha_1^n \\ & + (A_{2,0} + A_{2,1} n + A_{2,2} n^2 + \cdots + A_{2,j_2-1} n^{j_2-1}) \alpha_2^n \\ & \vdots \\ & + (A_{i,0} + A_{i,1} n + A_{i,2} n^2 + \cdots + A_{i,j_i-1} n^{j_i-1}) \alpha_i^n, \end{aligned} \tag{1.4}$$

for all n. The constants $\{A_{k,j}\}$ is depend on the $\{p_k\}$ and the initial conditions.

Both equations (1.3) and (1.4) are called the general solution of a recurrence relation, where equation (1.3) is a special case of equation (1.4). They can be seen in many combinatorics books, including in Chen and Koh [3] on page 235, and are proven in Roberts and Tesmam [13] on pages 362-363. Thus, if we know the roots of our characteristic equation, then we can rewrite it as

$$P_m(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_{m-1})(x - \alpha_m) = 0 \tag{1.5}$$

if the roots are all distinct, and as

$$P_m(x) = (x - \alpha_1)^{j_1}(x - \alpha_2)^{j_2}\cdots(x - \alpha_i)^{j_i} = 0 \tag{1.6}$$

if we only have $i \leq m$ distinct roots.

A sequence of polynomials $\{a_n(x)\}$ is called a linear homogeneous recurrence relation of order m if it can be written in the form

$$a_{n+m}(x) = p_1(x)a_{n+m-1}(x) + p_2(x)a_{n+m-2}(x) + \cdots + p_{m-1}(x)a_{n+1}(x) + p_m(x)a_n(x), n \geq 0, \tag{1.7}$$

for some polynomials $p_1(x), p_2(x), ..., p_m(x)$ with $p_m(x) \neq 0$ and initial conditions $a_0(x), a_1(x), ..., a_{m-1}(x)$. We can find the characteristic equation and general forms of the linear homogeneous recurrence relation of a polynomial sequence in the same manner as we did for sequences of numbers.

We start off our study of linear divisible sequences by examining second order linear divisible sequences in Chapter 2. In Chapters 3 through 5, we construct higher order linear divisible sequences by taking various products and powers of second order linear divisible sequences. In Chapter 6, we take various products and powers of second order polynomial linear divisible sequences to construct higher order linear divisible sequences.

CHAPTER 2

SECOND ORDER LINEAR DIVISIBLE SEQUENCES

A sequence of numbers $\{a_n\}$ is called a second order linear homogeneous recurrence relation if it satisfies the equation

$$a_{n+2} = pa_{n-1} + qa_n, \quad n \geq 0, \tag{2.1}$$

for constant p, non-zero constant q, and initial conditions a_0 and a_1. If we let α and β be roots of the polynomial $x^2 - px - q = 0$, where α and β satisfy $\alpha + \beta = p$ and $\alpha\beta = -q$, then the general solution of $\{a_n\}$ is

$$a_n = \begin{cases} \left(\frac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta; \\ na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & \text{if } \alpha = \beta. \end{cases} \tag{2.2}$$

This formula can be seen in many papers including He and Shiue [8].

A sequence of polynomial $\{a_n(x)\}$ is called a second order linear homogeneous recurrence relation if it satisfies the equation

$$a_{n+2}(x) = p(x)a_{n-1}(x) + q(x)a_n(x), \quad n \geq 0, \tag{2.3}$$

for polynomials $p(x)$, non-zero polynomial $q(x)$, and initial conditions $a_0(x)$ and $a_1(x)$. If we let $\alpha(x)$ and $\beta(x)$ be roots of the polynomial $t^2 - p(x)t - q(x) = 0$, where $\alpha(x)$ and $\beta(x)$ satisfy $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$, then the general solution of $\{a_n(x)\}$ is

$$a_n = \begin{cases} \left(\frac{a_1(x) - \beta(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\alpha^n(x) - \left(\frac{a_1(x) - \alpha(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\beta^n(x), & \text{if } \alpha(x) \neq \beta(x); \\ na_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x), & \text{if } \alpha(x) = \beta(x). \end{cases} \tag{2.4}$$

Again this formula can be seen in many papers including He and Shiue [8].

Next, we examine under what conditions the sequence generated by a second order linear homogeneous recurrence relation is a linear divisible sequence.

Theorem 2.1. *Let $\{a_n\}$ be sequence of elements in an integral domain R, defined by a second order linear homogeneous recurrence relation of the form (2.1), such that $p, q \in R$ and an arbitrary $a_1 \in R$. Then $\{a_n\}$ is a divisible sequence if $a_0 = 0$.*

Proof. Let $\{a_n\}$ be sequence of numbers in an integral domain R, defined by a second order linear homogeneous recurrence relation of the form (2.1), such that $p, q \in R$ and an arbitrary $a_1 \in R$. Then, $\{a_n\}$ has characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then, R, the integral domain our sequence is in, is dependent on α, β, a_1, and a_0.

Let $a_0 = 0$ and $n|m$, meaning there exists an integer j such that $nj = m$. By substituting 0 in for a_0 in equation (2.2), it becomes

$$a_n = \begin{cases} \left(\frac{a_1}{\alpha-\beta}\right)(\alpha^n - \beta^n), & \text{if } \alpha \neq \beta; \\ na_1\alpha^{n-1}, & \text{if } \alpha = \beta. \end{cases} \tag{2.5}$$

Case 1: Let $\alpha \neq \beta$. Then from equation (2.5) we have

$$\begin{aligned} \frac{a_m}{a_n} &= \frac{\left(\frac{a_1}{\alpha-\beta}\right)(\alpha^m - \beta^m)}{\left(\frac{a_1}{\alpha-\beta}\right)(\alpha^n - \beta^n)} \\ &= \frac{\alpha^m - \beta^m}{\alpha^n - \beta^n} \\ &= \frac{(\alpha^n)^j - (\beta^n)^j}{\alpha^n - \beta^n}. \end{aligned}$$

Our next step is to show $\frac{(\alpha^n)^j - (\beta^n)^j}{\alpha^n - \beta^n}$ is in our integral domain R. To do this we will use the following Girard-Waring identities that can be found in many works, including the work by He and Shiue[10], and proven in works like Comtet [4] and Gould [6]:

$$x^n + y^n = \sum_{0 \leq k \leq [n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (x+y)^{n-2k} (xy)^k \tag{2.6}$$

and

$$\frac{x^{n+1} - y^{n+1}}{x - y} = \sum_{0 \leq k \leq [n/2]} (-1)^k \binom{n-k}{k} (x+y)^{n-2k} (xy)^k. \tag{2.7}$$

It is important to note that $\frac{n}{n-k}\binom{n-k}{k}$ from equation (2.6) is an integer when n and k are integers because

$$\begin{aligned} \frac{n}{n-k}\binom{n-k}{k} &= \frac{n(n-k)!}{(n-k)k!(n-2k)!} \\ &= \frac{n(n-k-1)!(n-k)}{(n-k)k!(n-2k)!} \\ &= \frac{n(n-k-1)!}{k!(n-2k)!} \end{aligned}$$

$$
\begin{aligned}
&= \frac{((n-k)+k)(n-k-1)!}{k!(n-2k)!} \\
&= \frac{(n-k)! + (k(n-k-1)!)}{k!(n-2k)!} \\
&= \frac{(n-k)!}{k!(n-2k)!} + \frac{k(n-k-1)!}{k!(n-2k)!} \\
&= \frac{(n-k)!}{k!(n-2k)!} + \frac{k(n-k-1)!}{k(k-1)!(n-2k)!} \\
&= \frac{(n-k)!}{k!(n-2k)!} + \frac{(n-k-1)!}{(k-1)!(n-2k)!} \\
&= \binom{n-k}{k} + \binom{n-k-1}{k-1}.
\end{aligned}
$$

Thus by equation (2.7) we have

$$
\frac{(\alpha^n)^j - (\beta^n)^j}{\alpha^n - \beta^n} = \sum_{0 \le k \le [(j-1)/2]} (-1)^k \binom{j-k-1}{k} (\alpha^n + \beta^n)^{j-2k-1} (\alpha^n \beta^n)^k \tag{2.8}
$$

and by equation (2.6) we have

$$
\alpha^n + \beta^n = \sum_{0 \le k \le [n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (\alpha+\beta)^{n-2k} (\alpha\beta)^k. \tag{2.9}
$$

Since, $\alpha + \beta = p$ and $\alpha\beta = -q$, we know $(\alpha+\beta)^{n-2k} \in R$ and $(\alpha\beta)^k \in R$ because integral domains are closed. Thus, by equation (2.9), we know $\alpha^n + \beta^n \in R$. Then since, $\alpha^n \beta^n = (-q)^n$, we know $(\alpha^n \beta^n)^k \in R$, and since, $\alpha^n + \beta^n \in R$, we know $(\alpha^n + \beta^n)^{j-2k-1} \in R$. Thus, by equation (2.8), we know $\frac{(\alpha^n)^j - (\beta^n)^j}{\alpha^n - \beta^n} \in R$. Thus, $\frac{a_m}{a_n} \in R$, meaning $\{a_n\}$ is a divisible sequence when $\alpha \neq \beta$.

Case 2: Let $\alpha = \beta$. Note that $\alpha = \beta$ only happens when $x^2 - px - q = 0$ is a perfect square trinomial, which happens when $p^2 + 4q = 0$. Thus we have $2\alpha = p$ and $\alpha^2 = -q$. Then from equation (2.5), we have

$$
\begin{aligned}
\frac{a_m}{a_n} &= \frac{m a_1 \alpha^{m-1}}{n a_1 \alpha^{n-1}} \\
&= \frac{nj a_1 \alpha^{nj-1}}{n a_1 \alpha^{n-1}} \\
&= j \alpha^{nj-n}.
\end{aligned}
$$

Since our characteristic equation is monic, and its discriminate is zero, we know $\alpha \in R$. Since, $\alpha \in R$, we know $j\alpha^{nj-n} \in R$. Thus, $\frac{a_m}{a_n} \in R$, meaning $\{a_n\}$ is a divisible sequence when $\alpha = \beta$.

Therefore, if $a_0 = 0$, then $\{a_n\}$ is a divisible sequence. □

Note that, if R is an intergral domain, then $R(x)$ an integral domain. Thus, by Theorem 2.1, any sequence of polynomials that can be defined by (2.3) with coefficients in an integral domain R and an arbitrary $a_1(x) \in R(x)$ is a polynomial linear divisible sequence if $a_0(x) = 0$.

By substituting 0 in for $a_0(x)$ in equation (2.4), it becomes

$$a_n(x) = \begin{cases} \left(\frac{a_1(x)}{\alpha(x)-\beta(x)}\right)(\alpha^n(x) - \beta^n(x)), & \text{if } \alpha(x) \neq \beta(x); \\ n a_1(x)\alpha^{n-1}(x), & \text{if } \alpha(x) = \beta(x). \end{cases} \tag{2.10}$$

Based on equation (2.5), we can define many second order linear divisible sequences by one of the following sequences

$$\left\{ W_n(a_1, \alpha, \beta) = a_1 \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\} \tag{2.11}$$

where a_1, α, and β are non-zero constants with $\alpha \neq \beta$, or

$$\left\{ W_n(a_1, \alpha, \beta) = n a_1 \alpha^{n-1} \right\} \tag{2.12}$$

where a_1, α, and β are non-zero constants with $\alpha = \beta$. These sequence can be represented by the second order linear homogeneous recurrence relation, $W_{n+2} = (\alpha+\beta)W_{n+1} - \alpha\beta W_n$ with initial conditions $W_1 = a_1$ and $W_0 = 0$.

Based on equation (2.10), we can also define many second order polynomial linear divisible sequences by one of the following sequences

$$\left\{ W_n\left(a_1(x), \alpha(x), \beta(x)\right) = a_1(x) \frac{(\alpha(x))^n - (\beta(x))^n}{\alpha(x) - \beta(x)} \right\} \tag{2.13}$$

where $a_1(x)$, $\alpha(x)$, and $\beta(x)$ are non-zero polynomials with $\alpha(x) \neq \beta(x)$, or

$$\left\{ W_n(a_1(x), \alpha(x), \beta(x)) = n a_1(x) \left(\alpha(x)\right)^{n-1} \right\} \tag{2.14}$$

where $a_1(x)$, $\alpha(x)$, and $\beta(x)$ are non-zero constants with $\alpha(x) = \beta(x)$. These sequence can be represented by the second order linear homogeneous recurrence relation, $W_{n+2}(x) = (\alpha(x)+\beta(x))W_{n+1}(x) - \alpha(x)\beta(x)W_n(x)$ with initial conditions $W_1(x) = a_1(x)$ and $W_0(x) = 0$.

We now come up with some second order linear divisible sequences and second order polynomial linear divisible sequences in the form $\{W_n(a_1, \alpha, \beta)\}$ and $\{W_n(a_1(x), \alpha(x), \beta(x))\}$ respectively. We will be using some of these sequence in our examples throughout this thesis.

Example 2.1. First, we define the sequence $\left\{W_n\left(1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)\right\}$. Then we see $\alpha+\beta = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1$ and $\alpha\beta = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = -1$. Thus, $\left\{W_n\left(1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)\right\}$ is the second order linear divisible sequence defined by $W_{n+2} = W_{n+1} + W_n$ with $W_0 = 0$ and $W_1 = 1$. This is the Fibonacci sequence, $\{F_n\}$.

Example 2.2. Next, we define the sequence $\{W_n\left(1, 1+\sqrt{2}, 1-\sqrt{2}\right)\}$. Then we see $\alpha + \beta = \left(1+\sqrt{2}\right) + \left(1-\sqrt{2}\right) = 2$ and $\alpha\beta = \left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right) = -1$. Thus, $\{W_n\left(1, 1+\sqrt{2}, 1-\sqrt{2}\right)\}$ is the second order linear divisible sequence defined by $W_{n+2} = 2W_{n+1} + W_n$ with $W_0 = 0$ and $W_1 = 1$. This is the Pell number sequence, $\{P_n\}$.

Example 2.3. Next, we define the sequence $\{W_n(1, 2, 1)\}$. Then we see $\alpha + \beta = 3$ and $\alpha\beta = 2$. Thus, $\{W_n(1, 2, 1)\}$ is the second order linear divisible sequence defined by $W_{n+2} = 3W_{n+1} - 2W_n$ with $W_0 = 0$ and $W_1 = 1$. This is the Mersenne number sequence, $\{M_n\}$.

Example 2.4. Next, we define the sequence $\{W_n(1, 1, 1)\}$. Then we see $\alpha + \beta = 2$ and $\alpha\beta = 1$. Thus, $\{W_n(1, 1, 1)\}$ is the second order linear divisible sequence defined by$W_{n+2} = 2W_{n+1} - 1W_n$ with $W_0 = 0$ and $W_1 = 1$. This is the sequence of natural numbers including zero which we will denote as $\{N_n\}$.

Example 2.5. Next, we define the sequence $\{W_n(1, \sqrt{2}, \sqrt{3})\}$. Then we see $\alpha+\beta = \sqrt{2}+\sqrt{3}$ and $\alpha\beta = \sqrt{6}$. Thus, $\{W_n(1, \sqrt{2}, \sqrt{3})\}$ is the second order linear divisible sequence defined by$W_{n+2} = \left(\sqrt{2}+\sqrt{3}\right)W_{n+1} - \sqrt{6}W_n$ with $W_0 = 0$ and $W_1 = 1$. Note that this is a linear divisible sequence in the integral domain $\mathbb{Z}\left(\sqrt{2}, \sqrt{3}\right)$.

Example 2.6. [10] Next, we consider $\{a_n\}$ to be a geometric sequence. Then $\{S_n\}$, the sequence of partial sums of $\{a_n\}$, is a linear divisible sequence. If a is the first term of the sequence and r is the ratio of the terms, then $S_n = a\frac{1-r^n}{1-r}$, which is in the form of $\{W_n(a, 1, r)\}$, is a linear divisible sequence. Thus $\{S_n\}$, can be written as the second order linear divisible sequence defined by $S_{n+2} = (1+r)S_{n+1} - rS_n$ for $S_1 = a$ and $S_0 = 0$. Note that $\{S_n\}$ is a sequence of integers when a and r are integers.

Example 2.7. Next, we define the sequence $\left\{W_n\left(1, \frac{x+\sqrt{x^2+4}}{2}, \frac{x-\sqrt{x^2+4}}{2}\right)\right\}$. Then $\alpha(x)+\beta(x) = \frac{x+\sqrt{x^2+4}}{2} + \frac{x-\sqrt{x^2+4}}{2} = x$ and $\alpha(x)\beta(x) = \left(\frac{x+\sqrt{x^2+4}}{2}\right)\left(\frac{x-\sqrt{x^2+4}}{2}\right) = -1$. Thus, $\left\{W_n\left(1, \frac{x+\sqrt{x^2+4}}{2}, \frac{x-\sqrt{x^2+4}}{2}\right)\right\}$ is the second order polynomial linear divisible sequence defined by $W_{n+2} = xW_{n+1} + W_n$ with $W_0 = 0$ and $W_1 = 1$. This is a sequence known as the Fibonacci polynomials, $\{F_n(x)\}$.

Example 2.8. Next, we define the sequence $\left\{W_n\left(1, x+\sqrt{x^2+4}, x-\sqrt{x^2+4}\right)\right\}$. Then $\alpha(x)+\beta(x) = x+\sqrt{x^2+4}+x-\sqrt{x^2+4} = 2x$ and $\alpha(x)\beta(x) = \left(x+\sqrt{x^2+4}\right)\left(x-\sqrt{x^2+4}\right) = -1$. Thus, $\left\{W_n\left(1, x+\sqrt{x^2+4}, x-\sqrt{x^2+4}\right)\right\}$ is the second order polynomial linear divisible sequence defined by $W_{n+2} = 2xW_{n+1} + W_n$ with $W_0 = 0$ and $W_1 = 1$. This is the sequence of Chebyshev polynomials of the second kind that are denoted $\{U_n(x)\}$.

Example 2.9. Next, we define the sequence $\{W_n(1, x, 1)\}$. Then $\alpha(x)+\beta(x) = x+1$ and $\alpha(x)\beta(x) = x$. Thus, $\{W_n(1, x, 1)\}$ is the second order polynomial linear divisible sequence defined by $W_{n+2} = (x+1)W_{n+1} - xW_n$ with $W_0 = 0$ and $W_1 = 1$ which is the sequence known as repunits base x. This is also the sequence $\{0, 1, 1+x, 1+x+x^2, 1+x+x^2+x^3, \ldots\}$.

Example 2.10. Next, we define the sequence $\{W_n(1, x, x)\}$. Then $\alpha(x)+\beta(x) = 2x$ and $\alpha(x)\beta(x) = x^2$. Thus, $\{W_n(1, x, x)\}$ is the second order polynomial linear divisible sequence defined by $W_{n+2} = 2xW_{n+1} - x^2W_n$ with $W_0 = 0$ and $W_1 = 1$.

CHAPTER 3

PRODUCTS OF SECOND ORDER LINEAR DIVISIBLE SEQUENCES

Here we start our construction of higher order linear divisible sequence. We construct these higher order linear divisible sequences by taking various products and powers of second order linear divisible sequences. These products and powers are defined term by term. This type of construction was started by He and Shiue in [9]. Throughout the rest of this thesis we will use $\{w_n\}$ to represent the sequence constructed by taking these product and powers of second order linear divisible sequences.

In this chapter, we discuss taking products of multiple distinct second order linear divisible sequences. We start with the results of He and Shiue in [9] where they examined multiplying two distinct second order linear divisible sequences. We then move on to the product of three distinct second order linear divisible sequences and the product of four distinct second order linear divisible sequences. We define this product term by term; thus, $\{w_n\}$ is the sequence $\{a_{0_1}a_{0_2}\cdots a_{0_i}, a_{1_1}a_{1_2}\cdots a_{1_i}, a_{2_1}a_{2_2}\cdots a_{2_i}, \ldots\}$. It is important to note that the product of divisible sequences is a divisible sequence.

Since we are multiplying linear homogeneous recurrence relations, it is important to show what this multiplication produces. When we multiply two linear homogeneous recurrence relations term by term, we construct a new linear homogeneous recurrence relation. We show this by multiplying the general forms of the two linear homogeneous recurrence relations. Then, we show that the product is in the general form of a new linear homogeneous recurrence relation.

Theorem 3.1. *If $\{a_n\}$ and $\{b_n\}$ are linear homogeneous recurrence sequences, then the sequence of term by term products $\{w_n = a_n b_n\}$ is a linear homogeneous recurrence sequence.*

Proof. Let $\{a_n\}$ be a linear homogeneous recurrence sequence of order m_1 with $s \le m_1$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_s$ with multiplicities $j_1, j_2, \ldots, j_s$. Then, by equation (1.4), we know each element of $\{a_n\}$ can be expressed as

$$
\begin{aligned}
a_n = &\left(A_{1,0} + A_{1,1}n + \cdots + A_{1,j_1-1}n^{j_1-1}\right)\alpha_1^n \\
&+ \left(A_{2,0} + A_{2,1}n + \cdots + A_{2,j_2-1}n^{j_2-1}\right)\alpha_2^n \\
&\ \vdots \\
&+ \left(A_{s,0} + A_{s,1}n + \cdots + A_{s,j_s-1}n^{j_s-1}\right)\alpha_s^n.
\end{aligned}
$$

Let $\{b_n\}$ be a linear homogeneous recurrence sequence of order m_2 with $t \le m_2$ distinct roots $\beta_1, \beta_2, \ldots, \beta_t$ with multiplicities $k_1, k_2, \ldots, k_t$. Then, by equation (1.4), we know each element of $\{b_n\}$ can be expressed as

$$
\begin{aligned}
b_n = &\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right)\beta_1^n \\
&+ \left(B_{2,0} + B_{2,1}n + \cdots + B_{2,k_2-1}n^{k_2-1}\right)\beta_2^n \\
&\ \vdots \\
&+ \left(B_{t,0} + B_{t,1}n + \cdots + B_{t,k_t-1}n^{k_t-1}\right)\beta_t^n.
\end{aligned}
$$

Since we are multiplying term by term we know that each element of $\{w_n\}$ can be expressed as

$$
\begin{aligned}
w_n = &\left(A_{1,0} + A_{1,1}n + \cdots + A_{1,j_1-1}n^{j_1-1}\right)\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right)(\alpha_1\beta_1)^n \\
&+ \left(A_{2,0} + A_{2,1}n + \cdots + A_{2,j_2-1}n^{j_2-1}\right)\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right)(\alpha_2\beta_1)^n \\
&\ \vdots \\
&+ \left(A_{s,0} + A_{s,1}n + \cdots + A_{s,j_s-1}n^{j_s-1}\right)\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right)(\alpha_s\beta_1)^n \\
&+ \left(A_{1,0} + A_{1,1}n + \cdots + A_{1,j_1-1}n^{j_1-1}\right)\left(B_{2,0} + B_{2,1}n + \cdots + B_{2,k_2-1}n^{k_2-1}\right)(\alpha_1\beta_2)^n \\
&\ \vdots \\
&+ \left(A_{s,0} + A_{s,1}n + \cdots + A_{s,j_s-1}n^{j_s-1}\right)\left(B_{t,0} + B_{t,1}n + \cdots + B_{t,k_t-1}n^{k_t-1}\right)(\alpha_s\beta_t)^n.
\end{aligned}
$$

Distributing the above we get

$$
\begin{aligned}
w_n = &\left(A_{1,0}\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right) + A_{1,1}n\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right) + \right. \\
&\left. \cdots + A_{1,j_1-1}n^{j_1-1}\left(B_{1,0} + B_{1,1}n + \cdots + B_{1,k_1-1}n^{k_1-1}\right)\right)(\alpha_1\beta_1)^n
\end{aligned}
$$

$$+\left(A_{2,0}\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)+A_{2,1}n\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)+\right.$$
$$\left.\cdots+A_{2,j_2-1}n^{j_2-1}\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)\right)(\alpha_2\beta_1)^n$$
$$\vdots$$
$$+\left(A_{s,0}\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)+A_{s,1}n\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)+\right.$$
$$\left.\cdots+A_{s,j_s-1}n^{j_s-1}\left(B_{1,0}+B_{1,1}n+\cdots+B_{1,k_1-1}n^{k_1-1}\right)\right)(\alpha_s\beta_1)^n$$
$$+\left(A_{1,0}\left(B_{2,0}+B_{2,1}n+\cdots+B_{2,k_2-1}n^{k_2-1}\right)+A_{1,1}n\left(B_{2,0}+B_{2,1}n+\cdots+B_{2,k_2-1}n^{k_2-1}\right)+\right.$$
$$\left.\cdots+A_{1,j_1-1}n^{j_1-1}\left(B_{2,0}+B_{2,1}n+\cdots+B_{2,k_2-1}n^{k_2-1}\right)\right)(\alpha_1\beta_2)^n$$
$$\vdots$$
$$+\left(A_{s,0}\left(B_{t,0}+B_{t,1}n+\cdots+B_{t,k_t-1}n^{k_t-1}\right)+A_{s,1}n\left(B_{t,0}+B_{t,1}n+\cdots+B_{t,k_t-1}n^{k_t-1}\right)+\right.$$
$$\left.\cdots+A_{s,j_s-1}n^{j_s-1}\left(B_{t,0}+B_{t,1}n+\cdots+B_{t,k_t-1}n^{k_t-1}\right)\right)(\alpha_s\beta_t)^n.$$

Distributing again we get

$$w_n=\left(A_{1,0}B_{1,0}+A_{1,0}B_{1,1}n+\cdots+A_{1,0}B_{1,k_1-1}n^{k_1-1}+A_{1,1}B_{1,0}n+A_{1,1}B_{1,1}n^2+\cdots+A_{1,1}B_{1,k_1-1}n^{k_1}+\right.$$
$$\left.\cdots+A_{1,j_1-1}B_{1,0}n^{j_1-1}+A_{1,j_1-1}B_{1,1}n^{j_1}+\cdots+A_{1,j_1-1}B_{1,k_1-1}n^{j_1+k_1-2}\right)(\alpha_1\beta_1)^n$$
$$+\left(A_{2,0}B_{1,0}+A_{2,0}B_{1,1}n+\cdots+A_{2,0}B_{1,k_1-1}n^{k_1-1}+A_{2,1}B_{1,0}n+A_{2,1}B_{1,1}n^2+\cdots+A_{2,1}B_{1,k_1-1}n^{k_1}+\right.$$
$$\left.\cdots+A_{2,j_2-1}B_{1,0}n^{j_2-1}+A_{2,j_2-1}B_{1,1}n^{j_2}+\cdots+A_{2,j_2-1}B_{1,k_1-1}n^{j_2+k_1-2}\right)(\alpha_2\beta_1)^n$$
$$\vdots$$
$$+\left(A_{s,0}B_{1,0}+A_{s,0}B_{1,1}n+\cdots+A_{s,0}B_{1,k_1-1}n^{k_1-1}+A_{s,1}B_{1,0}n+A_{s,1}B_{1,1}n^2+\cdots+A_{s,1}B_{1,k_1-1}n^{k_1}+\right.$$
$$\left.\cdots+A_{s,j_1-1}B_{1,0}n^{j_1-1}+A_{s,j_1-1}B_{1,1}n^{j_1}+\cdots+A_{s,j_s-1}B_{1,k_1-1}n^{j_s+k_1-2}\right)(\alpha_s\beta_1)^n$$
$$+\left(A_{1,0}B_{2,0}+A_{1,0}B_{2,1}n+\cdots+A_{1,0}B_{2,k_2-1}n^{k_2-1}+A_{1,1}B_{2,0}n+A_{1,1}B_{2,1}n^2+\cdots+A_{1,1}B_{2,k_2-1}n^{k_2}+\right.$$
$$\left.\cdots+A_{1,j_1-1}B_{2,0}n^{j_1-1}+A_{1,j_1-1}B_{2,1}n^{j_1}+\cdots+A_{1,j_1-1}B_{2,k_2-1}n^{j_1+k_2-2}\right)(\alpha_1\beta_2)^n$$
$$\vdots$$
$$+\left(A_{s,0}B_{t,0}+A_{s,0}B_{t,1}n+\cdots+A_{s,0}B_{t,k_t-1}n^{k_1-1}+A_{s,1}B_{t,0}n+A_{s,1}B_{t,1}n^2+\cdots+A_{s,1}B_{t,k_t-1}n^{k_1}+\right.$$
$$\left.\cdots+A_{s,j_s-1}B_{t,0}n^{j_1-1}+A_{s,j_s-1}B_{t,1}n^{j_1}+\cdots+A_{s,j_s-1}B_{1,k_t-1}n^{j_s+k_t-2}\right)(\alpha_s\beta_t)^n.$$

Now by combining like terms in each parentheses based of powers of n, we get

$$
\begin{aligned}
w_n = \big(&A_{1,0}B_{1,0} + (A_{1,0}B_{1,1} + A_{1,1}B_{1,0})\, n + (A_{1,0}B_{1,2} + A_{1,1}B_{1,1} + A_{1,2}B_{1,0})\, n^2 + \\
&\cdots + A_{1,j_1-1}B_{1,k_1-1}n^{j_1+k_1-2}\big)\, (\alpha_1\beta_1)^n \\
+ \big(&A_{2,0}B_{1,0} + (A_{2,0}B_{1,1} + A_{2,1}B_{1,0})\, n + (A_{2,0}B_{1,2} + A_{2,1}B_{1,1} + A_{2,2}B_{1,0})\, n^2 + \\
&\cdots + A_{2,j_2-1}B_{1,k_1-1}n^{j_2+k_1-2}\big)\, (\alpha_2\beta_1)^n \\
&\vdots \\
+ \big(&A_{s,0}B_{1,0} + (A_{s,0}B_{1,1} + A_{s,1}B_{1,0})\, n + (A_{s,0}B_{1,2} + A_{s,1}B_{1,1} + A_{s,2}B_{1,0})\, n^2 + \\
&\cdots + A_{s,j_s-1}B_{1,k_1-1}n^{j_s+k_1-2}\big)\, (\alpha_s\beta_1)^n \\
+ \big(&A_{1,0}B_{2,0} + (A_{1,0}B_{2,1} + A_{1,1}B_{2,0})\, n + (A_{1,0}B_{2,2} + A_{1,1}B_{2,1} + A_{1,2}B_{2,0})\, n^2 + \\
&\cdots + A_{1,j_1-1}B_{2,k_2-1}n^{j_1+k_2-2}\big)\, (\alpha_1\beta_2)^n \\
&\vdots \\
+ \big(&A_{s,0}B_{t,0} + (A_{s,0}B_{t,1} + A_{s,1}B_{t,0})\, n + (A_{s,0}B_{t,2} + A_{s,1}B_{t,1} + A_{s,2}B_{t,0})\, n^2 + \\
&\cdots + A_{s,j_s-1}B_{t,k_t-1}n^{j_s+k_t-2}\big)\, (\alpha_s\beta_t)^n .
\end{aligned}
$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic function has roots $\alpha_1\beta_1, \ldots, \alpha_s\beta_1, \alpha_2\beta_1, \ldots, \alpha_s\beta_t$ with multiplicities at least $j_1+k_1-1, \ldots, j_s+k_1-1, j_1+k_2-1, \ldots, j_s+k_t-1$. Therefore, the sequence of term by term products of two linear homogeneous recurrence relations can be expressed as a linear homogeneous recurrence relation. □

Next, we look at the equations created by multiplying a finite number of second order linear divisible sequences. Let $\{a_{n_1}\}$, $\{a_{n_2}\}$, …, $\{a_{n_i}\}$ be second order linear divisible sequences that satisfy equation (2.1) with $a_{0_i} = 0$ for all i. Then $\{a_{n_i}\}$ has a characteristic equation $x^2 - p_ix - q_i = 0$ with roots α_i and β_i such that $\alpha_i + \beta_i = p_i$ and $\alpha_i\beta_i = -q_i$. Since each $\{a_{n_i}\}$ has $a_{0_i} = 0$, they can be expressed using equation (2.5). Since the order of multiplication does not matter, for simplicity, we will say all sequences with double roots will be written first. This means that if there is one sequence in our product with a double root, we will call that sequence $\{a_{n_1}\}$. If there are two sequences with double roots in our product we will call them sequences $\{a_{n_1}\}$ and $\{a_{n_2}\}$. Then the sequence $\{w_n = a_{n_1}a_{n_2}\cdots a_{n_i}\}$ has one of the following expressions depending on how many of the characteristic equations have distinct roots.

$$w_n = \begin{cases} \prod_{k=1}^{i}\left(\frac{a_{1_k}}{\alpha_k-\beta_k}\right)(\alpha_k^n-\beta_k^n), & \text{if } \alpha_k \neq \beta_k \text{ for all } k \leq i; \\ \left(\prod_{k=2}^{i}\left(\frac{a_{1_k}}{\alpha_k-\beta_k}\right)(\alpha_k^n-\beta_k^n)\right)\left(na_{1_1}\alpha_1^{n-1}\right), & \text{if } \alpha_1 = \beta_1 \text{ and } \alpha_k \neq \beta_k \\ & \text{for } 2 \leq k \leq i; \\ \left(\prod_{k=3}^{i}\left(\frac{a_{1_k}}{\alpha_k-\beta_k}\right)(\alpha_k^n-\beta_k^n)\right)\left(\prod_{m=1}^{2} na_{1_m}\alpha_m^{n-1}\right), & \text{if } \alpha_m = \beta_m \text{ for } m = 1,2 \text{ and} \\ & \alpha_k \neq \beta_k \text{ for } 3 \leq k \leq i\ ; \\ \vdots & \\ \left(\prod_{k=\ell+1}^{i}\left(\frac{a_{1_k}}{\alpha_k-\beta_k}\right)(\alpha_k^n-\beta_k^n)\right)\left(\prod_{m=1}^{\ell} na_{1_m}\alpha_m^{n-1}\right), & \text{if } \alpha_m = \beta_m \text{ for } 1 \leq m \leq \ell \text{ and} \\ & \alpha_k \neq \beta_k \text{ for } \ell+1 \leq k \leq i; \\ \vdots & \\ \left(\prod_{k=i-1}^{i}\left(\frac{a_{1_k}}{\alpha_k-\beta_k}\right)(\alpha_k^n-\beta_k^n)\right)\left(\prod_{m=1}^{i-2} na_{1_m}\alpha_m^{n-1}\right), & \text{if } \alpha_m = \beta_m \text{ for } 1 \leq m \leq i-2 \\ & \alpha_k \neq \beta_k \text{ for } k = i-1, i; \\ \left(\left(\frac{a_{1_i}}{\alpha_i-\beta_i}\right)(\alpha_i^n-\beta_i^n)\right)\left(\prod_{m=1}^{i-1} na_{1_m}\alpha_m^{n-1}\right), & \text{if } \alpha_m = \beta_m \text{ for } 1 \leq m \leq i-1, \\ & \text{and } \alpha_i \neq \beta_i; \\ \prod_{m=1}^{i} na_{1_m}\alpha_m^{n-1}, & \text{if } \alpha_m = \beta_m, \text{ for all } m \leq i. \end{cases} \tag{3.1}$$

Next we will prove some common equalities that will be used throughout this type of construction.

Lemma 3.2. *If $x^2 - px - q = 0$ is a quadratic equation with roots α and β such that $\alpha + \beta = p$ and $\alpha\beta = -q$ then*

(a) $\alpha^2 + \beta^2 = p^2 + 2q$.

(b) $\alpha^4 + \beta^4 = (p^2 + 2q)^2 - 2q^2$.

(c) $\alpha^2 + \alpha\beta + \beta^2 = p^2 + q$.

(d) $\alpha^2 - \alpha\beta + \beta^2 = p^2 + 3q$.

(e) $\alpha^4 - \alpha^2\beta^2 + \beta^4 = (p^2 + 2q)^2 - 3q^2$.

(f) $\alpha^8 + \beta^8 = ((p^2 + 2q)^2 - 2q^2)^2 - 2q^4$.

Proof. Let $x^2 + px + q = 0$ be a quadratic equation with roots α and β such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Thus, we have

(a) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + 2q$.

(b) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = (p^2 + 2q)^2 - 2q^2$.

(c) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta = (\alpha + \beta)^2 - \alpha\beta = p^2 + q$.

(d) $\alpha^2 - \alpha\beta + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta - \alpha\beta = (\alpha + \beta)^2 - 3\alpha\beta = p^2 + 3q$.

(e) $\alpha^4 - \alpha^2\beta^2 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 - \alpha^2\beta^2 = (\alpha^2 + \beta^2)^2 - 3\alpha^2\beta^2 = (p^2 + 2q)^2 - 3q^2$.

(f) $\alpha^8 + \beta^8 = (\alpha^4 + \beta^4)^2 - 2\alpha^4\beta^4 = ((p^2 + 2q)^2 - 2q^2)^2 - 2q^4$.

□

3.1

Product of Two Distinct Second Order Linear Divisible Sequences

In this section we will multiply two distinct second order linear divisible sequences in order to come up with a single higher order linear divisible sequence. This multiplication constructs a fourth order linear divisible sequence.

Theorem 3.3. [9] *Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - p_1x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, and the sequence $\{b_n\}$ has a characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$. Then $\{w_n = a_nb_n\}$ is a linear divisible sequence that satisfies the fourth order linear homogeneous recurrence relation*

$$w_{n+4} = p_1p_2w_{n+3} + \left(p_1^2q_2 + p_2^2q_1 + 2q_1q_2\right) w_{n+2} + p_1p_2q_1q_2w_{n+1} - q_1^2q_2^2w_n \tag{3.2}$$

for $n \geq 0$ with initial conditions $w_3 = a_3b_3$, $w_2 = a_2b_2$, $w_1 = a_1b_1$, and $w_0 = a_0b_0 = 0$.

Proof. Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - p_1x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, and the sequence $\{b_n\}$ have the characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$.

Case 1: Let both characteristic functions have distinct roots, meaning $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. Then from equation (3.1), we have

$$\begin{aligned} w_n &= a_n b_n \\ &= \left(\frac{a_1}{\alpha_1 - \beta_1}\right)(\alpha_1^n - \beta_1^n)\left(\frac{b_1}{\alpha_2 - \beta_2}\right)(\alpha_2^n - \beta_2^n) \\ &= \left(\frac{a_1 b_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)}\right)\left((\alpha_1\alpha_2)^n - (\alpha_1\beta_2)^n - (\alpha_2\beta_1)^n + (\beta_1\beta_2)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2$, $\alpha_1\beta_2$, $\alpha_2\beta_1$, and $\beta_1\beta_2$ each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have four roots, which is how many characteristic roots we need for a fourth order linear divisible sequence. Thus, the characteristic equation is

$$\begin{aligned} (x-\alpha_1\alpha_2)(x-\alpha_1\beta_2)(x-\beta_1\alpha_2)(x-\beta_1\beta_2) &= x^4 - (\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1+\beta_1\beta_2)\,x^3 \\ &\quad + \left(\alpha_1^2\alpha_2\beta_2+\alpha_1\beta_1\alpha_2^2+2\alpha_1\alpha_2\beta_1\beta_2+\alpha_1\beta_1\beta_2^2+\alpha_2\beta_1^2\beta_2\right)x^2 \\ &\quad - \left(\alpha_1^2\alpha_2^2\beta_1\beta_2+\alpha_1^2\alpha_2\beta_1\beta_2^2+\alpha_1\alpha_2^2\beta_1^2\beta_2+\alpha_1\alpha_2\beta_1^2\beta_2^2\right)x+\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2. \end{aligned}$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (3.2), we have

$$\begin{aligned} \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2 &= \alpha_1(\alpha_2 + \beta_2) + \beta_1(\alpha_2 + \beta_2) \\ &= (\alpha_2 + \beta_2)(\alpha_1 + \beta_1) \\ &= p_1 p_2. \end{aligned}$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (3.2), we have

$$\begin{aligned} \alpha_1^2\alpha_2\beta_2+\alpha_1\beta_1\alpha_2^2+2\alpha_1\alpha_2\beta_1\beta_2+\alpha_1\beta_1\beta_2^2+\alpha_2\beta_1^2\beta_2 &= \alpha_1\beta_1\left(\alpha_2^2+\beta_2^2\right)+\alpha_2\beta_2\left(\alpha_1^2+\beta_1^2\right)+2\alpha_1\alpha_2\beta_1\beta_2 \\ &= -q_1\left(p_2^2+2q_2\right)-q_2\left(p_1^2+2q_1\right)+2q_1q_2 \end{aligned}$$

$$=-p_2^2q_1-2q_1q_2-p_1^2q_2-2q_1q_2+2q_1q_2$$

$$=-\left(p_2^2q_1+p_1^2q_2+2q_1q_2\right).$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (3.2), we have

$$\begin{aligned}\alpha_1^2\alpha_2^2\beta_1\beta_2+\alpha_1^2\alpha_2\beta_1\beta_2^2+\alpha_1\alpha_2^2\beta_1^2\beta_2+\alpha_1\alpha_2\beta_1^2\beta_2^2 &= \alpha_1\alpha_2\beta_1\beta_2\left(\alpha_1\alpha_2+\alpha_1\beta_2+\beta_1\alpha_2+\beta_1\beta_2\right)\\ &= \alpha_1\alpha_2\beta_1\beta_2\left(\alpha_2+\beta_2\right)\left(\alpha_1+\beta_1\right)\\ &= p_1p_2q_1q_2.\end{aligned}$$

Looking at the constant, which becomes the coefficient of w_n in equation (3.2), we have

$$\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2=q_1^2q_2^2.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (3.2).

Case 2: Let one characteristic function have duplicate roots and the other have distinct roots. WLOG we can say the characteristic function of $\{a_n\}$ has the duplicate root, meaning $\alpha_1=\beta_1$ and $\alpha_2\neq\beta_2$. Then from equation (3.1), we have

$$\begin{aligned}w_n &= a_nb_n\\ &= \left(\frac{na_1b_1}{\alpha_2-\beta_2}\right)(\alpha_2^n-\beta_2^n)\alpha_1^{n-1}\\ &= \left(\frac{na_1b_1}{\alpha_1(\alpha_2-\beta_2)}\right)((\alpha_1\alpha_2)^n-(\alpha_1\beta_2)^n)\\ &= \left(\frac{na_1b_1}{\alpha_1(\alpha_2-\beta_2)}\right)(\alpha_1\alpha_2)^n-\left(\frac{na_1b_1}{\alpha_1(\alpha_2-\beta_2)}\right)(\alpha_1\beta_2)^n.\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2$ and $\alpha_1\beta_2$ each with a multiplicity of at least two. We will let each of them have multiplicity two since that means we will have four roots, which is how many characteristic roots we need for a fourth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n=a_nb_n\}$ are $\alpha_1\alpha_2$, $\alpha_1\alpha_2$, $\alpha_1\beta_2$, and $\alpha_1\beta_2$, then the characteristic equation is

$$(x-\alpha_1\alpha_2)\,(x-\alpha_1\beta_2)\,(x-\alpha_1\alpha_2)\,(x-\alpha_1\beta_2)\,.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$ and $\alpha_1\alpha_1 = -q_1$.

Case 3: Let both characteristic functions have duplicate roots, meaning $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Then from equation (3.1), we have

$$w_n = a_n b_n = n^2 a_1 b_1 \alpha_1^{n-1} \alpha_2^{n-1} = \frac{n^2 a_1 b_1}{\alpha_1 \alpha_2} (\alpha_1 \alpha_2)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root $\alpha_1\alpha_2$ with a multiplicity of at least three. We will let it have multiplicity four since that means we will have four roots, which is how many characteristic roots we need for a fourth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n b_n\}$ are $\alpha_1\alpha_2$, $\alpha_1\alpha_2$, $\alpha_1\alpha_2$, and $\alpha_1\alpha_2$, then the characteristic equation is

$$(x-\alpha_1\alpha_2)\,(x-\alpha_1\alpha_2)\,(x-\alpha_1\alpha_2)\,(x-\alpha_1\alpha_2)\,.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 and β_2 with α_2 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, and $\alpha_2\alpha_2 = -q_2$.

Therefore, when we multiply two distinct second order linear divisible sequences, we can construct a fourth order linear divisible sequence defined by recurrence relation (3.2). It is easy to see from our definition of $\{w_n = a_n b_n\}$ that $w_3 = a_3 b_3$, $w_2 = a_2 b_2$, $w_1 = a_1 b_1$, and $w_0 = a_0 b_0 = 0$. □

Note that in He and Shiue [9] they only proved case 1 from Theorem 3.3. We prove the other cases here so that we can see that the recurrence relation (3.2) still works when the roots of one or more characteristic equations are the same.

Also note that in case one we chose the multiplicity of the roots to be one as that was the simplest multiplicity to work with. It may be that if we let one or more of the roots have a higher multiplicity, we could have constructed a different linear homogeneous recurrence relation that works for the same sequence. For example if we had let all the roots have multiplicity two then our characteristic equation would have been $\prod_{i=1}^{4}(x - r_i)^2$. This would have constructed a different linear homogeneous recurrence relation that is of order eight.

In later cases we chose multiplicities in such a way to show the linear homogeneous recurrence relation we constructed in case one works when one or more of the sequences have duplicate roots. Again, we may be able to come up with different linear homogeneous recurrence relations by choosing multiplicities that are higher or lower that would work in these cases.

We will be choosing the multiplicities of roots in the same manner in future constructions in this thesis. In those cases, we may also create different linear homogeneous recurrence relations by making a different choice for the multiplicities of roots.

Next, we have examples that take the product of two second order linear divisible sequences to construct fourth order linear divisible sequences.

Example 3.1. Using the Fibonacci sequence and the sequence of natural numbers including zero, we define the sequence $\{w_n = F_n N_n\}$. Then, by Theorem 3.3, we get a fourth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 2w_{n+3} + w_{n+2} - 2w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n N_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	6	6	48	9	306	12	1728	15	9150	18	46512
1	1	4	12	7	91	10	550	13	3029	16	15792	19	79439
2	2	5	25	8	168	11	976	14	5278	17	27149	20	135300

Table 3.1: Terms of the sequence $\{w_n = F_n N_n\}$

Example 3.2. Using the Pell number sequence and the sequence of natural numbers including zero, we define the sequence $\{w_n = P_n N_n\}$. Then, by Theorem 3.3, we get a fourth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 4w_{n+3} - 2w_{n+2} - 4w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n N_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	15	6	420	9	8865	12	166320	15	2925375	18	49395780
1	1	4	48	7	1183	10	23780	13	434993	16	7533312	19	125877071
2	4	5	145	8	3264	11	63151	14	1130948	17	19323713	20	319888560

Table 3.2: Terms of the sequence $\{w_n = P_n N_n\}$

Example 3.3. Using the Mersenne number sequence and the sequence of natural numbers including zero, we define the sequence $\{w_n = M_n N_n\}$. Then, by Theorem 3.3, we get a fourth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 6w_{n+3} - 13w_{n+2} + 12w_{n+1} - 4w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n N_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	21	6	378	9	4599	12	49140	15	491505	18	4718574
1	1	4	60	7	889	10	10230	13	106483	16	1048560	19	9961453
2	6	5	155	8	2040	11	22517	14	229362	17	2228207	20	20971500

Table 3.3: Terms of the sequence $\{w_n = M_n N_n\}$

3.2

Product of Three Distinct Second Order Linear Divisible Sequences

In this section we will multiply three distinct second order linear divisible sequences in order to come up with a single higher order linear divisible sequence. This multiplication constructs an eighth order linear divisible sequences.

Theorem 3.4. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = c_0 = 0$ and a_1, b_1, c_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - p_1 x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, the sequence $\{b_n\}$ has a characteristic equation $x^2 - p_2 x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$, and the sequence $\{c_n\}$ has a characteristic equation $x^2 - p_3 x - q_3 = 0$ with roots α_3 and β_3, such that $\alpha_3 + \beta_3 = p_3$ and $\alpha_3\beta_3 = -q_3$. Then $\{w_n = a_n b_n c_n\}$ is a linear divisible sequence that satisfies* *as the eighth order linear homogeneous recurrence relation*

$$\begin{aligned}
w_{n+8} =& p_1p_2p_3w_{n+7} + \left(p_2^2p_3^2q_1 + p_1^2p_3^2q_2 + p_1^2p_2^2q_3 + 2p_3^2q_1q_2 + 2p_2^2q_1q_3 + 2p_1^2q_2q_3 + 4q_1q_2q_3\right) w_{n+6} \\
&+ \left(p_1p_2p_3^3q_1q_2 + p_1p_2^3p_3q_1q_3 + p_1^3p_2p_3q_2q_3 + 5p_1p_2p_3q_1q_2q_3\right) w_{n+5} \\
&- \left(p_1^4q_2^2q_3^2 + p_2^4q_1^2q_3^2 + p_3^4q_1^2q_2^2 - p_1^2p_2^2p_3^2q_1q_2q_3 + 4p_1^2q_1q_2^2q_3^2 + 4p_2^2q_1^2q_2q_3^2 + 4p_3^2q_1^2q_2^2q_3 + 6q_1^2q_2^2q_3^2\right) w_{n+4} \\
&+ q_1q_2q_3\left(p_1p_2p_3^3q_1q_2 + p_1p_2^3p_3q_1q_3 + p_1^3p_2p_3q_2q_3 + 5p_1p_2p_3q_1q_2q_3\right) w_{n+3} \\
&+ q_1^2q_2^2q_3^2\left(p_2^2p_3^2q_1 + p_1^2p_3^2q_2 + p_1^2p_2^2q_3 + 2p_3^2q_1q_2 + 2p_2^2q_1q_3 + 2p_1^2q_2q_3 + 4q_1q_2q_3\right) w_{n+2} \\
&- p_1p_2p_3q_1^3q_2^3q_3^3w_{n+1} - q_1^4q_2^4q_3^4w_n
\end{aligned} \tag{3.3}$$

for $n \geq 0$ *with initial conditions* $w_i = a_ib_ic_i$ *for* $0 \leq i \leq 7$.

Proof. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = c_0 = 0$ and a_1, b_1, c_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - p_1x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, the sequence $\{b_n\}$ have the characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$, and the sequence $\{c_n\}$ have the characteristic equation $x^2 - p_3x - q_3 = 0$ with roots α_3 and β_3, such that $\alpha_3 + \beta_3 = p_3$ and $\alpha_3\beta_3 = -q_3$.

Case 1: Let each characteristic function have distinct roots, meaning $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$, and $\alpha_3 \neq \beta_3$. Then from equation (3.1) we have

$$\begin{aligned}
w_n =& a_nb_nc_n \\
=& \left(\frac{a_1b_1c_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)}\right)(\alpha_1^n - \beta_1^n)(\alpha_2^n - \beta_2^n)(\alpha_3^n - \beta_3^n) \\
=& \left(\frac{a_1b_1c_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)}\right)\left((\alpha_1\alpha_2)^n - (\alpha_1\beta_2)^n - (\alpha_2\beta_1)^n + (\beta_1\beta_2)^n\right)(\alpha_3^n - \beta_3^n) \\
=& \left(\frac{a_1b_1c_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)}\right)\left((\alpha_1\alpha_2\alpha_3)^n - (\alpha_1\alpha_2\beta_3)^n - (\alpha_1\beta_2\alpha_3)^n + (\alpha_1\beta_2\beta_3)^n\right. \\
&\left. -(\beta_1\alpha_2\alpha_3)^n + (\beta_1\alpha_2\beta_3)^n + (\beta_1\beta_2\alpha_3)^n - (\beta_1\beta_2\beta_3)^n\right).
\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha_1\alpha_2\alpha_3$, $r_2 = \alpha_1\alpha_2\beta_3$, $r_3 = \alpha_1\beta_2\alpha_3$, $r_4 = \alpha_1\beta_2\beta_3$, $r_5 = \beta_1\alpha_2\alpha_3$, $r_6 = \beta_1\alpha_2\beta_3$, $r_7 = \beta_1\beta_2\alpha_3$, and $r_8 = \beta_1\beta_2\beta_3$ each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have eight roots, which is how many characteristic roots we need for an eighth order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{8}(x-r_i)=x^8-\left(\sum_{1\le i\le 8} r_i\right)x^7+\ldots+(-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 8} r_{i_1}\cdots r_{i_k}\right)x^{8-k},\ \text{for } k\le 8.$$

Looking at the coefficient of x^7, which becomes the coefficient of w_{n+7} in equation (3.3), we have

$$\begin{aligned}\sum_{1\le i\le 8} r_i =&\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\beta_3+\alpha_1\beta_2\alpha_3+\alpha_1\beta_2\beta_3+\beta_1\alpha_2\alpha_3+\beta_1\alpha_2\beta_3+\beta_1\beta_2\alpha_3+\beta_1\beta_2\beta_3\\ =&\alpha_1\left(\alpha_2\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2+\beta_2\beta_3\right)+\beta_1\left(\alpha_2\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2+\beta_2\beta_3\right)\\ =&\left(\alpha_1+\beta_1\right)\left(\alpha_2\alpha_3+\alpha_2\beta_3+\alpha_3\beta_2+\beta_2\beta_3\right)\\ =&\left(\alpha_1+\beta_1\right)\left(\alpha_2\left(\alpha_3+\beta_3\right)+\beta_2\left(\alpha_3+\beta_3\right)\right)\\ =&\left(\alpha_1+\beta_1\right)\left(\alpha_2+\beta_2\right)\left(\alpha_3+\beta_3\right)\\ =&p_1p_2p_3.\end{aligned}$$

Looking at the coefficient of x^6, which becomes the coefficient of w_{n+6} in equation (3.3), we have

$$\begin{aligned}\sum_{1\le i<j\le 8} r_ir_j =&\alpha_1\alpha_2^2\alpha_3^2\beta_1+\alpha_1^2\alpha_2\alpha_3^2\beta_2+\alpha_2\alpha_3^2\beta_1^2\beta_2+\alpha_1\alpha_3^2\beta_1\beta_2^2+\alpha_1^2\alpha_2^2\alpha_3\beta_3+\alpha_2^2\alpha_3\beta_1^2\beta_3+\alpha_1^2\alpha_3\beta_2^2\beta_3\\ &+\alpha_3\beta_1^2\beta_2^2\beta_3+\alpha_1\alpha_2^2\beta_1\beta_3^2+\alpha_1^2\alpha_2\beta_2\beta_3^2+\alpha_2\beta_1^2\beta_2\beta_3^2+\alpha_1\beta_1\beta_2^2\beta_3^2+2\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2\\ &+2\alpha_1\alpha_2^2\alpha_3\beta_1\beta_3+2\alpha_1^2\alpha_2\alpha_3\beta_2\beta_3+2\alpha_2\alpha_3\beta_1^2\beta_2\beta_3+2\alpha_1\alpha_3\beta_1\beta_2^2\beta_3+2\alpha_1\alpha_2\beta_1\beta_2\beta_3^2\\ &+4\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\\ =&\alpha_1\beta_1\left(\alpha_2^2\alpha_3^2+\alpha_2^2\beta_3^2+\alpha_3^2\beta_2^2+\beta_2^2\beta_3^2\right)+\alpha_2\beta_2\left(\alpha_1^2\alpha_3^2+\alpha_1^2\beta_3^2+\alpha_3^2\beta_1^2+\beta_1^2\beta_3^2\right)\\ &+\alpha_3\beta_3\left(\alpha_1^2\alpha_2^2+\alpha_1^2\beta_2^2+\alpha_2^2\beta_1^2+\beta_1^2\beta_2^2\right)+2\alpha_1\alpha_2\beta_1\beta_2\left(\alpha_3^2+\beta_3^2\right)\\ &+2\alpha_1\alpha_3\beta_1\beta_3\left(\alpha_2^2+\beta_2^2\right)+2\alpha_2\alpha_3\beta_2\beta_3\left(\alpha_1^2+\beta_1^2\right)+4\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\\ =&\alpha_1\beta_1\left(\alpha_2^2+\beta_2^2\right)\left(\alpha_3^2+\beta_3^2\right)+\alpha_2\beta_2\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_3^2+\beta_3^2\right)+\alpha_3\beta_3\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_2^2+\beta_2^2\right)\\ &+2\alpha_1\alpha_2\beta_1\beta_2\left(\alpha_3^2+\beta_3^2\right)+2\alpha_1\alpha_3\beta_1\beta_3\left(\alpha_2^2+\beta_2^2\right)+2\alpha_2\alpha_3\beta_2\beta_3\left(\alpha_1^2+\beta_1^2\right)+4\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\\ =&-q_1\left(p_2^2+2q_2\right)\left(p_3^2+2q_3\right)-q_2\left(p_1^2+2q_1\right)\left(p_3^2+2q_3\right)-q_3\left(p_1^2+2q_1\right)\left(p_2^2+2q_2\right)\\ &+2q_1q_2\left(p_3^2+2q_3\right)+2q_1q_3\left(p_2^2+2q_2\right)+2q_2q_3\left(p_1^2+2q_1\right)-4q_1q_2q_3\\ =&-p_2^2p_3^2q_1-p_1^2p_3^2q_2-p_1^2p_2^2q_3-2p_3^2q_1q_2-2p_2^2q_1q_3-2p_1^2q_2q_3-4q_1q_2q_3.\end{aligned}$$

Looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (3.3), we have

$$
\begin{aligned}
\sum_{1\le i<j<k\le 8} r_i r_j r_k =& \alpha_1^2\alpha_2^2\alpha_3^3\beta_1\beta_2 + \alpha_1\alpha_2^2\alpha_3^3\beta_1^2\beta_2 + \alpha_1^2\alpha_2\alpha_3^3\beta_1\beta_2^2 + \alpha_1\alpha_2\alpha_3^3\beta_1^2\beta_2^2 + \alpha_1^2\alpha_2^3\alpha_3^2\beta_1\beta_3 \\
&+ \alpha_1\alpha_2^3\alpha_3^2\beta_1^2\beta_3 + \alpha_1^3\alpha_2^2\alpha_3^2\beta_2\beta_3 + \alpha_2^2\alpha_3^2\beta_1^3\beta_2\beta_3 + \alpha_1^3\alpha_2\alpha_3^2\beta_2^2\beta_3 + \alpha_2\alpha_3^2\beta_1^3\beta_2^2\beta_3 \\
&+ \alpha_1^2\alpha_3^2\beta_1\beta_2^3\beta_3 + \alpha_1\alpha_3^2\beta_1^2\beta_2^3\beta_3 + \alpha_1^2\alpha_2^3\alpha_3\beta_1\beta_3^2 + \alpha_1\alpha_2^3\alpha_3\beta_1^2\beta_3^2 + \alpha_1^3\alpha_2^2\alpha_3\beta_2\beta_3^2 \\
&+ \alpha_2^2\alpha_3\beta_1^3\beta_2\beta_3^2 + \alpha_1^3\alpha_2\alpha_3\beta_2^2\beta_3^2 + \alpha_2\alpha_3\beta_1^3\beta_2^2\beta_3^2 + \alpha_1^2\alpha_3\beta_1\beta_2^3\beta_3^2 + \alpha_1\alpha_3\beta_1^2\beta_2^3\beta_3^2 \\
&+ \alpha_1^2\alpha_2^2\beta_1\beta_2\beta_3^3 + \alpha_1\alpha_2^2\beta_1^2\beta_2\beta_3^3 + \alpha_1^2\alpha_2\beta_1\beta_2^2\beta_3^3 + \alpha_1\alpha_2\beta_1^2\beta_2^2\beta_3^3 \\
&+ 4\alpha_1^2\alpha_2^2\alpha_3^2\beta_1\beta_2\beta_3 + 4\alpha_1\alpha_2^2\alpha_3^2\beta_1^2\beta_2\beta_3 + 4\alpha_1^2\alpha_2\alpha_3^2\beta_1\beta_2^2\beta_3 + 4\alpha_1\alpha_2\alpha_3^2\beta_1^2\beta_2^2\beta_3 \\
&+ 4\alpha_1^2\alpha_2^2\alpha_3\beta_1\beta_2\beta_3^2 + 4\alpha_1\alpha_2^2\alpha_3\beta_1^2\beta_2\beta_3^2 + 4\alpha_1^2\alpha_2\alpha_3\beta_1\beta_2^2\beta_3^2 + 4\alpha_1\alpha_2\alpha_3\beta_1^2\beta_2^2\beta_3^2 \\
=& \left(\alpha_1+\beta_1\right)\left(\alpha_2+\beta_2\right)\left(\alpha_3+\beta_3\right)\big(\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2 + \alpha_1\alpha_2^2\alpha_3\beta_1\beta_3 + \alpha_1^2\alpha_2\alpha_3\beta_2\beta_3 \\
&+\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 + \alpha_2\alpha_3\beta_1^2\beta_2\beta_3 + \alpha_1\alpha_3\beta_1\beta_2^2\beta_3 + \alpha_1\alpha_2\beta_1\beta_2\beta_3^2\big) \\
=& \left(\alpha_1+\beta_1\right)\left(\alpha_2+\beta_2\right)\left(\alpha_3+\beta_3\right)\big(\alpha_1\alpha_2\beta_1\beta_2\left(\alpha_3^2+\beta_3^2\right) + \alpha_1\alpha_3\beta_1\beta_3\left(\alpha_2^2+\beta_2^2\right) \\
&+\alpha_2\alpha_3\beta_2\beta_3\left(\alpha_1^2+\beta_1^2\right) + \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3\big) \\
=& p_1p_2p_3\left(q_1q_2\left(p_3^2+2q_3\right) + q_1q_3\left(p_2^2+2q_2\right) + q_2q_3\left(p_1^2+2q_1\right) - q_1q_2q_3\right) \\
=& p_1p_2p_3^3q_1q_2 + p_1p_2^3p_3q_1q_3 + p_1^3p_2p_3q_2q_3 + 5p_1p_2p_3q_1q_2q_3.
\end{aligned}
$$

Looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (3.3), we have

$$
\begin{aligned}
\sum_{1\le i_1<\cdots<i_4\le 8} r_{i_1}\cdots r_{i_4} =& \alpha_1^2\alpha_2^2\alpha_3^4\beta_1^2\beta_2^2 + \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2\beta_3^4 + \alpha_1^2\alpha_2^4\alpha_3^2\beta_1^2\beta_3^2 + \alpha_1^2\alpha_3^2\beta_1^2\beta_2^4\beta_3^2 + \alpha_1^4\alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 \\
&+ \alpha_2^2\alpha_3^2\beta_1^4\beta_2^2\beta_3^2 + \alpha_1\alpha_2\alpha_3\beta_1^3\beta_2^3\beta_3^3 + \alpha_1\alpha_2\alpha_3^3\beta_1^3\beta_2^3\beta_3 + \alpha_1\alpha_2^3\alpha_3\beta_1^3\beta_2\beta_3^3 \\
&+ \alpha_1^3\alpha_2\alpha_3\beta_1\beta_2^3\beta_3^3 + \alpha_1^3\alpha_2^3\alpha_3^3\beta_1\beta_2\beta_3 + \alpha_1\alpha_2^3\alpha_3^3\beta_1^3\beta_2\beta_3 + \alpha_1^3\alpha_2\alpha_3^3\beta_1\beta_2^3\beta_3 \\
&+ \alpha_1^3\alpha_2^3\alpha_3\beta_1\beta_2\beta_3^3 + 2\alpha_1^2\alpha_2^3\alpha_3^3\beta_1^2\beta_2\beta_3 + 2\alpha_1^3\alpha_2^2\alpha_3^3\beta_1\beta_2^2\beta_3 + 2\alpha_1\alpha_2^2\alpha_3^3\beta_1^3\beta_2^2\beta_3 \\
&+ 2\alpha_1^2\alpha_2\alpha_3^3\beta_1^2\beta_2^3\beta_3 + 2\alpha_1^3\alpha_2^3\alpha_3^2\beta_1\beta_2\beta_3^2 + 2\alpha_1\alpha_2^3\alpha_3^2\beta_1^3\beta_2\beta_3^2 + 2\alpha_1^3\alpha_2\alpha_3^2\beta_1\beta_2^3\beta_3^2 \\
&+ 2\alpha_1\alpha_2\alpha_3^2\beta_1^3\beta_2^3\beta_3^2 + 2\alpha_1^2\alpha_2^3\alpha_3\beta_1^2\beta_2\beta_3^3 + 2\alpha_1^3\alpha_2^2\alpha_3\beta_1\beta_2^2\beta_3^3 + 2\alpha_1\alpha_2^2\alpha_3\beta_1^3\beta_2^2\beta_3^3 \\
&+ 2\alpha_1^2\alpha_2\alpha_3\beta_1^2\beta_2^3\beta_3^3 + 4\alpha_1^2\alpha_2^2\alpha_3^3\beta_1^2\beta_2^2\beta_3 + 4\alpha_1^2\alpha_2^3\alpha_3^2\beta_1^2\beta_2\beta_3^2 + 4\alpha_1^3\alpha_2^2\alpha_3^2\beta_1\beta_2^2\beta_3^2 \\
&+ 4\alpha_1\alpha_2^2\alpha_3^2\beta_1^3\beta_2^2\beta_3^2 + 4\alpha_1^2\alpha_2\alpha_3^2\beta_1^2\beta_2^3\beta_3^2 + 4\alpha_1^2\alpha_2^2\alpha_3\beta_1^2\beta_2^2\beta_3^3 + 8\alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2 \\
=& \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2\left(\alpha_3^4+\beta_3^4\right) + \alpha_1^2\alpha_3^2\beta_1^2\beta_3^2\left(\alpha_2^4+\beta_2^4\right) + \alpha_2^2\alpha_3^2\beta_2^2\beta_3^2\left(\alpha_1^4+\beta_1^4\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \left(\alpha_1^2\alpha_2^2\alpha_3^2 + \alpha_2^2\alpha_3^2\beta_1^2 + \alpha_1^2\alpha_3^2\beta_2^2 + \alpha_3^2\beta_1^2\beta_2^2 + \alpha_1^2\alpha_2^2\beta_3^2 + \alpha_2^2\beta_1^2\beta_3^2\right.\\
&\left.+\alpha_1^2\beta_2^2\beta_3^2 + \beta_1^2\beta_2^2\beta_3^2\right) + 2\alpha_1^2\alpha_2\alpha_3\beta_1^2\beta_2\beta_3 \left(\alpha_2^2\alpha_3^2 + \alpha_2^2\beta_3^2 + \alpha_3^2\beta_2^2 + \beta_2^2\beta_3^2\right)\\
&+ 2\alpha_1\alpha_2^2\alpha_3\beta_1\beta_2^2\beta_3 \left(\alpha_1^2\alpha_3^2 + \alpha_1^2\beta_3^2 + \alpha_3^2\beta_1^2 + \beta_1^2\beta_3^2\right)\\
&+ 2\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2\beta_3^2 \left(\alpha_1^2\alpha_2^2 + \alpha_1^2\beta_2^2 + \alpha_2^2\beta_1^2 + \beta_1^2\beta_2^2\right) + 4\alpha_1^2\alpha_2^2\alpha_3\beta_1^2\beta_2^2\beta_3 \left(\alpha_3^2 + \beta_3^2\right)\\
&+ 4\alpha_1^2\alpha_2\alpha_3^2\beta_1^2\beta_2\beta_3^2 \left(\alpha_2^2 + \beta_2^2\right) + 4\alpha_1\alpha_2^2\alpha_3^2\beta_1\beta_2^2\beta_3^2 \left(\alpha_1^2 + \beta_1^2\right) + 8\alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2\\
=&\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2 \left(\alpha_3^4 + \beta_3^4\right) + \alpha_1^2\alpha_3^2\beta_1^2\beta_3^2 \left(\alpha_2^4 + \beta_2^4\right) + \alpha_2^2\alpha_3^2\beta_2^2\beta_3^2 \left(\alpha_1^4 + \beta_1^4\right)\\
&+ \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_2^2 + \beta_2^2\right)\left(\alpha_3^2 + \beta_3^2\right)\\
&+ 2\alpha_1^2\alpha_2\alpha_3\beta_1^2\beta_2\beta_3 \left(\alpha_2^2 + \beta_2^2\right)\left(\alpha_3^2 + \beta_3^2\right) + 2\alpha_1\alpha_2^2\alpha_3\beta_1\beta_2^2\beta_3 \left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_3^2 + \beta_3^2\right)\\
&+ 2\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2\beta_3^2 \left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_2^2 + \beta_2^2\right) + 4\alpha_1^2\alpha_2^2\alpha_3\beta_1^2\beta_2^2\beta_3 \left(\alpha_3^2 + \beta_3^2\right)\\
&+ 4\alpha_1^2\alpha_2\alpha_3^2\beta_1^2\beta_2\beta_3^2 \left(\alpha_2^2 + \beta_2^2\right) + 4\alpha_1\alpha_2^2\alpha_3^2\beta_1\beta_2^2\beta_3^2 \left(\alpha_1^2 + \beta_1^2\right) + 8\alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2\\
=&q_1^2q_2^2 \left(\left(p_3^2 + 2q_3\right)^2 - 2q_3^2\right) + q_1^2q_3^2 \left(\left(p_2^2 + 2q_2\right)^2 - 2q_2^2\right) + q_2^2q_3^2 \left(\left(p_1^2 + 2q_1\right)^2 - 2q_1^2\right)\\
&- q_1q_2q_3 \left(p_1^2 + 2q_1\right)\left(p_2^2 + 2q_2\right)\left(p_3^2 + 2q_3\right) + 2q_1^2q_2q_3 \left(p_2^2 + 2q_2\right)\left(p_3^2 + 2q_3\right)\\
&+ 2q_1q_2^2q_3 \left(p_1^2 + 2q_1\right)\left(p_3^2 + 2q_3\right) + 2q_1q_2q_3^2 \left(p_1^2 + 2q_1\right)\left(p_2^2 + 2q_2\right)\\
&- 4q_1^2q_2^2q_3 \left(p_3^2 + 2q_3\right) - 4q_1^2q_2q_3^2 \left(p_2^2 + 2q_2\right) - 4q_1q_2^2q_3^2 \left(p_1^2 + 2q_1\right) + 8q_1^2q_2^2q_3^2\\
=&p_1^4q_2^2q_3^2 + p_2^4q_1^2q_3^2 + p_3^4q_1^2q_2^2 - p_1^2p_2^2p_3^2q_1q_2q_3 + 4p_1^2q_1q_2^2q_3^2 + 4p_2^2q_1^2q_2q_3^2 + 4p_3^2q_1^2q_2^2q_3\\
&+ 6q_1^2q_2^2q_3^2.
\end{aligned}
$$

When $1 \leq i_1 < \cdots < i_5 \leq 8$, we can show that $r_{i_1}\cdots r_{i_5} = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3(r_ir_jr_k)$ where $r_i, r_j, r_k \in \{r_{i_1}, \ldots, r_{i_5}\}$. For each $r_{i_1}\cdots r_{i_5}$, there exists $r_s, r_t \in \{r_{i_1}, \ldots, r_{i_5}\}$, such that $r_sr_t = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$. This means $r_{i_1}\cdots r_{i_5} = r_sr_t(r_ir_jr_k) = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3(r_ir_jr_k)$. For example, if we take $r_1\cdots r_5$, then we can see that $r_4r_5 = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3$, which means $r_1\cdots r_5 = \alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3(r_1r_2r_3)$.

Thus, looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (3.3), we have

$$
\begin{aligned}
\sum_{1\leq i_1<\cdots<i_5\leq 8} r_{i_1}\cdots r_{i_5} =&\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3 \left(\sum_{1\leq i<j<k\leq 8} r_ir_jr_k\right)\\
=&- q_1q_2q_3 \left(p_1p_2p_3^3q_1q_2 + p_1p_2^3p_3q_1q_3 + p_1^3p_2p_3q_2q_3 + 5p_1p_2p_3q_1q_2q_3\right).
\end{aligned}
$$

Since we calculated $\sum_{1\leq i<j<k\leq 8} r_ir_jr_k$ as the coefficient of x^5, above we can just replace it here.

When $1 \leq i_1 < \cdots < i_6 \leq 8$, we can show that $r_{i_1} \cdots r_{i_6} = \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2(r_i r_j)$ where $r_i, r_j \in \{r_{i_1}, \ldots, r_{i_6}\}$. For each $r_{i_1} \cdots r_{i_6}$, there exists $r_{s_1}, \ldots, r_{s_4} \in \{r_{i_1}, \ldots, r_{i_6}\}$, such that $r_{s_1} \cdots r_{s_4} = \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2$. This means $r_{i_1} \cdots r_{i_6} = r_{s_1} \cdots r_{s_4}(r_i r_j) = \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2(r_i r_j)$. For example if we take $r_1 \cdots r_6$ we can see that $r_3r_4r_5r_6 = \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2$, which means $r_1 \cdots r_6 = \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2(r_1r_2)$.

Thus, looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (3.3), we have

$$\begin{aligned}\sum_{1\leq i_1<\cdots<i_6\leq 8} r_{i_1} \cdots r_{i_6} =& \alpha_1^2\alpha_2^2\alpha_3^2\beta_1^2\beta_2^2\beta_3^2 \left(\sum_{1\leq i<j\leq 8} r_i r_j\right)\\ =& q_1^2q_2^2q_3^2\left(-p_2^2p_3^2q_1 - p_1^2p_3^2q_2 - p_1^2p_2^2q_3 - 2\,p_3^2q_1q_2 - 2\,p_2^2q_1q_3 - 2\,p_1^2q_2q_3 - 4\,q_1q_2q_3\right).\end{aligned}$$

Since we calculated $\sum_{1\leq i<j\leq 8} r_i r_j$ as the coefficient of x^6 above, we can just replace it here.

When $1 \leq i_1 < \cdots < i_7 \leq 8$, we can show that $r_{i_1} \cdots r_{i_7} = \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3(r_i)$ where $r_i \in \{r_{i_1}, \ldots, r_{i_7}\}$. For each $r_{i_1} \cdots r_{i_7}$, there exists $r_{s_1}, \ldots, r_{s_6} \in \{r_{i_1}, \ldots, r_{i_7}\}$, such that $r_{s_1} \cdots r_{s_6} = \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3$. This means $r_{i_1} \cdots r_{i_7} = r_{s_1} \cdots r_{s_6}(r_i) = \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3(r_i)$. For example, if we take $r_1 \cdots r_7$, we can see that $r_2 \cdots r_7 = \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3$, which means $r_1 \cdots r_7 = \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3(r_1)$.

Thus, looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (3.3), we have

$$\begin{aligned}\sum_{1\leq i_1<\cdots<i_7\leq 8} r_{i_1} \cdots r_{i_7} =& \alpha_1^3\alpha_2^3\alpha_3^3\beta_1^3\beta_2^3\beta_3^3 \left(\sum_{1\leq i\leq 8} r_i\right)\\ =& -p_1p_2p_3q_1^3q_2^3q_3^3.\end{aligned}$$

Since we calculated $\sum_{1\leq i\leq 8} r_i r_j$ as the coefficient of x^7 above, we can just replace it here.

Looking at the constant, which becomes the coefficient of w_n in equation (3.3), we have

$$\sum_{1\leq i_1<\cdots<i_8\leq 8} r_{i_1} \cdots r_{i_8} = \alpha_1^4\alpha_2^4\alpha_3^4\beta_1^4\beta_2^4\beta_3^4 = q_1^4q_2^4q_3^4.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (3.3).

Case 2: Let one characteristic function have duplicate roots and the other two have distinct roots. WLOG we can say the characteristic function of $\{a_n\}$ has the duplicate root, meaning $\alpha_1 = \beta_1$, $\alpha_2 \neq \beta_2$, and $\alpha_3 \neq \beta_3$. Then, from equation (3.1), we have

$$w_n = a_n b_n c_n$$

$$
\begin{aligned}
&= \left(\frac{na_1b_1c_1}{(\alpha_2-\beta_2)(\alpha_3-\beta_3)}\right)(\alpha_2^n-\beta_2^n)(\alpha_3^n-\beta_3^n)\,\alpha_1^{n-1}\\
&= \left(\frac{na_1b_1c_1}{(\alpha_2-\beta_2)(\alpha_3-\beta_3)}\right)\left((\alpha_2\alpha_3)^n-(\alpha_2\beta_3)^n-(\alpha_3\beta_2)^n+(\beta_2\beta_3)^n\right)\alpha_1^{n-1}\\
&= \left(\frac{na_1b_1c_1}{\alpha_1(\alpha_2-\beta_2)(\alpha_3-\beta_3)}\right)\left((\alpha_1\alpha_2\alpha_3)^n-(\alpha_1\alpha_2\beta_3)^n-(\alpha_1\alpha_3\beta_2)^n+(\alpha_1\beta_2\beta_3)^n\right).
\end{aligned}
$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2\alpha_3$, $\alpha_1\alpha_2\beta_3$, $\alpha_1\beta_2\alpha_3$, and $\alpha_1\beta_2\beta_3$ each with a multiplicity of at least two. We will let each of them have multiplicity two since that means we will have eight roots, which is how many characteristic roots we need for an eighth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_nb_nc_n\}$ are $r_1=\alpha_1\alpha_2\alpha_3$, $r_2=\alpha_1\alpha_2\beta_3$, $r_3=\alpha_1\beta_2\alpha_3$, $r_4=\alpha_1\beta_2\beta_3$, $r_5=\alpha_1\alpha_2\alpha_3$, $r_6=\alpha_1\alpha_2\beta_3$, $r_7=\alpha_1\beta_2\alpha_3$, and $r_8=\alpha_1\beta_2\beta_3$, then the characteristic equation is

$$
\prod_{i=1}^{8}(x-r_i) = x^8 - \left(\sum_{1\le i\le 8} r_i\right)x^7 + \ldots + (-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 8} r_{i_1}\cdots r_{i_k}\right)x^{8-k}, \text{ for } k\le 8.
$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 throughout. This works because, in this case, $\alpha_1+\alpha_1=p_1$ and $\alpha_1\alpha_1=-q_1$.

Case 3: Let two characteristic functions have duplicate roots and the other one have distinct roots. WLOG we can say the characteristic functions of $\{a_n\}$ and $\{b_n\}$ have the duplicate root, meaning $\alpha_1=\beta_1$, $\alpha_2=\beta_2$, and $\alpha_3\neq\beta_3$. Then, from equation (3.1), we have

$$
\begin{aligned}
w_n =& a_nb_nc_n\\
&= \left(\frac{n^2a_1b_1c_1}{(\alpha_3-\beta_3)}\right)(\alpha_3^n-\beta_3^n)\,\alpha_1^{n-1}\alpha_2^{n-1}\\
&= \left(\frac{n^2a_1b_1c_1}{\alpha_1\alpha_2(\alpha_3-\beta_3)}\right)\left((\alpha_1\alpha_2\alpha_3)^n-(\alpha_1\alpha_2\beta_3)^n\right)\\
&= \left(\frac{n^2a_1b_1c_1}{\alpha_1\alpha_2(\alpha_3-\beta_3)}\right)(\alpha_1\alpha_2\alpha_3)^n-\left(\frac{n^2a_1b_1c_1}{\alpha_1\alpha_2(\alpha_3-\beta_3)}\right)(\alpha_1\alpha_2\beta_3)^n.
\end{aligned}
$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2\alpha_3$ and $\alpha_1\alpha_2\beta_3$ each with a multiplicity of at least three. We will let each of them have multiplicity four since that means we will have eight roots, which is how many characteristic roots we need for an eighth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n b_n c_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3$, $r_2 = \alpha_1\alpha_2\beta_3$, $r_3 = \alpha_1\alpha_2\alpha_3$, $r_4 = \alpha_1\alpha_2\beta_3$, $r_5 = \alpha_1\alpha_2\alpha_3$, $r_6 = \alpha_1\alpha_2\beta_3$, $r_7 = \alpha_1\alpha_2\alpha_3$, and $r_8 = \alpha_1\alpha_2\beta_3$, then the characteristic equation is

$$\prod_{i=1}^{8}(x - r_i) = x^8 - \left(\sum_{1\le i\le 8} r_i\right) x^7 + \ldots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 8} r_{i_1}\cdots r_{i_k}\right) x^{8-k}, \text{ for } k \le 8.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 and β_2 with α_2 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, and $\alpha_2\alpha_2 = -q_2$.

Case 4: Let each characteristic functions have duplicate roots, meaning $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, and $\alpha_3 = \beta_3$. Then, from equation (3.1), we have

$$w_n = a_n b_n c_n = n^3 a_1 b_1 c_1 \alpha_1^{n-1}\alpha_2^{n-1}\alpha_3^{n-1} = \frac{n^3 a_1 b_1 c_1}{\alpha_1\alpha_2\alpha_3}(\alpha_1\alpha_2\alpha_3)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root $\alpha_1\alpha_2\alpha_3$ with a multiplicity of at least four. We will let it have multiplicity eight since that means we will have eight roots, which is how many characteristic roots we need for an eighth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n b_n c_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3$, $r_2 = \alpha_1\alpha_2\alpha_3$, $r_3 = \alpha_1\alpha_2\alpha_3$, $r_4 = \alpha_1\alpha_2\alpha_3$, $r_5 = \alpha_1\alpha_2\alpha_3$, $r_6 = \alpha_1\alpha_2\alpha_3$, $r_7 = \alpha_1\alpha_2\alpha_3$, and $r_8 = \alpha_1\alpha_2\alpha_3$, then the characteristic equation is

$$\prod_{i=1}^{8}(x - r_i) = x^8 - \left(\sum_{1\le i\le 8} r_i\right) x^7 + \ldots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 8} r_{i_1}\cdots r_{i_k}\right) x^{8-k}, \text{ for } k \le 8.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1, β_2 with α_2, and β_3 with α_3 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, $\alpha_2\alpha_2 = -q_2$, $\alpha_3 + \alpha_3 = p_3$, and $\alpha_3\alpha_3 = -q_3$.

Therefore, when we multiply three distinct second order linear divisible sequences, we can construct a eighth order linear divisible sequence defined by recurrence relation (3.3). It is easy to see from our definition of $\{w_n = a_n b_n c_n\}$ that $w_i = a_i b_i c_i$ for $0 \le i \le 7$ □

Next, we have an example that takes the product of three second order linear divisible sequences in order to construct an eighth order linear divisible sequence.

Example 3.4. Using the Fibonacci sequence, Pell number sequence and Mersenne number sequences we define a sequence $\{w_n = F_n P_n M_n\}$. Then, by Theorem 3.4, we get an eighth order linear divisible sequence that satisfies the linear homogeneous recurrence relation

$$w_{n+8} = 6w_{n+7} + 27w_{n+6} - 66w_{n+5} - 253w_{n+4} - 132w_{n+3} + 108w_{n+2} + 48w_{n+1} - 16w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n P_n M_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	4495	10	133798170	15	3898134346750	20	113458232405776500
1	1	6	35280	11	1045912603	16	30454847443440	21	886399585423924390
2	6	7	279019	12	8172964800	17	237932181378643	22	6925050871102681014
3	70	8	2184840	13	63860418883	18	1858866142205520	23	54102376390964996119
4	540	9	17113390	14	498941217762	19	14522530081665223	24	422678043468647366400

Table 3.4: Terms of the sequence $\{w_n = F_n P_n M_n\}$

3.3

Product of Four Distinct Second Order Linear Divisible Sequences

In this section, we will multiply four distinct second order linear divisible sequences in order to come up with a single higher order linear divisible sequence. This multiplication constructs a sixteenth order linear divisible sequence.

Theorem 3.5. *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = c_0 = d_0 = 0$ and a_1, b_1, c_1, d_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - p_1x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, the sequence $\{b_n\}$ has a characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$, the sequence $\{c_n\}$ has a characteristic equation $x^2 - p_3x - q_3 = 0$ with roots α_3 and β_3, such that $\alpha_3 + \beta_3 = p_3$ and $\alpha_3\beta_3 = -q_3$, and the sequence $\{d_n\}$ has a characteristic equation $x^2 - p_4x - q_4 = 0$ with roots α_4 and β_4, such that $\alpha_4 + \beta_4 = p_4$ and $\alpha_4\beta_4 = -q_4$. Then, $\{w_n = a_nb_nc_nd_n\}$ is a linear divisible sequence that satisfies the sixteenth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+16} =& p_1p_2p_3p_4w_{n+15} + \big(p_2^2p_3^2p_4^2q_1 + p_1^2p_3^2p_4^2q_2 + p_1^2p_2^2p_4^2q_3 + p_1^2p_2^2p_3^2q_4 + 2p_3^2p_4^2q_1q_2 + 2p_2^2p_4^2q_1q_3 \\ & + 2p_1^2p_4^2q_2q_3 + 2p_2^2p_3^2q_1q_4 + 2p_1^2p_3^2q_2q_4 + 2p_1^2p_2^2q_3q_4 + 4p_4^2q_1q_2q_3 + 4p_3^2q_1q_2q_4 + 4p_2^2q_1q_3q_4 \end{aligned}$$

$$+4p_1^2q_2q_3q_4+8q_1q_2q_3q_4\big)\,w_{n+14}+\big(p_1p_2p_3^3p_4^3q_1q_2+p_1p_2^3p_3p_4^3q_1q_3+p_1^3p_2p_3p_4^3q_2q_3+p_1p_2^3p_3^3p_4q_1q_4$$

$$+p_1^3p_2p_3^3p_4q_2q_4+p_1^3p_2^3p_3p_4q_3q_4+5p_1p_2p_3p_4^3q_1q_2q_3+5p_1p_2p_3^3p_4q_1q_2q_4+5p_1p_2^3p_3p_4q_1q_3q_4$$

$$+5p_1^3p_2p_3p_4q_2q_3q_4+19p_1p_2p_3p_4q_1q_2q_3q_4\big)\,w_{n+13}-\big(p_3^4p_4^4q_1^2q_2^2+p_2^4p_4^4q_1^2q_3^2+p_1^4p_4^4q_2^2q_3^2+p_2^4p_3^4q_1^2q_4^2$$

$$+p_1^4p_3^4q_2^2q_4^2+p_1^4p_2^4q_3^2q_4^2-p_1^2p_2^2p_3^2p_4^4q_1q_2q_3-p_1^2p_2^2p_3^4p_4^2q_1q_2q_4-p_1^2p_2^4p_3^2p_4^2q_1q_3q_4-p_1^4p_2^2p_3^2p_4^2q_2q_3q_4$$

$$+4p_3^2p_4^4q_1^2q_2^2q_3+4p_2^2p_4^4q_1^2q_2q_3^2+4p_1^2p_4^4q_1q_2^2q_3^2+4p_3^4p_4^2q_1^2q_2^2q_4+4p_2^4p_4^2q_1^2q_3^2q_4+4p_1^4p_4^2q_2^2q_3^2q_4$$

$$+4p_2^2p_3^4q_1^2q_2q_4^2+4p_1^2p_3^4q_1q_2^2q_4^2+4p_2^4p_3^2q_1^2q_3q_4^2+4p_1^4p_3^2q_2^2q_3q_4^2+4p_1^2p_2^4q_1q_3^2q_4^2+4p_1^4p_2^2q_2q_3^2q_4^2$$

$$+6p_4^4q_1^2q_2^2q_3^2+6p_3^4q_1^2q_2^2q_4^2+6p_2^4q_1^2q_3^2q_4^2+6p_1^4q_2^2q_3^2q_4^2-9p_1^2p_2^2p_3^2p_4^2q_1q_2q_3q_4+16p_3^2p_4^2q_1^2q_2^2q_3q_4$$

$$+16p_2^2p_4^2q_1^2q_2q_3^2q_4+16p_1^2p_4^2q_1q_2^2q_3^2q_4+16p_2^2p_3^2q_1^2q_2q_3q_4^2+16p_1^2p_3^2q_1q_2^2q_3q_4^2+16p_1^2p_2^2q_1q_2q_3^2q_4^2$$

$$+24p_4^2q_1^2q_2^2q_3^2q_4+24p_3^2q_1^2q_2^2q_3q_4^2+24p_2^2q_1^2q_2q_3^2q_4^2+24p_1^2q_1q_2^2q_3^2q_4^2+28q_1^2q_2^2q_3^2q_4^2\big)\,w_{n+12}$$

$$+\big(p_1^3p_2^3p_3^3p_4^3q_1q_2q_3q_4-p_1p_2p_3^3p_4^5q_1^2q_2^2q_3-p_1p_2^3p_3p_4^5q_1^2q_2q_3^2-p_1^3p_2p_3p_4^5q_1q_2^2q_3^2-p_1p_2p_3^5p_4^3q_1^2q_2^2q_4$$

$$-p_1p_2^5p_3p_4^3q_1^2q_3^2q_4-p_1^5p_2p_3p_4^3q_2^2q_3^2q_4-p_1p_2^3p_3^5p_4q_1^2q_2q_4^2-p_1^3p_2p_3^5p_4q_1q_2^2q_4^2-p_1p_2^5p_3^3p_4q_1^2q_3q_4^2$$

$$-p_1^5p_2p_3^3p_4q_2^2q_3q_4^2-p_1^3p_2^5p_3p_4q_1q_3^2q_4^2-p_1^5p_2^3p_3p_4q_2q_3^2q_4^2-5p_1p_2p_3p_4^5q_1^2q_2^2q_3^2-5p_1p_2p_3^5p_4q_1^2q_2^2q_4^2$$

$$-5p_1p_2^5p_3p_4q_1^2q_3^2q_4^2-5p_1^5p_2p_3p_4q_2^2q_3^2q_4^2-9p_1p_2p_3^3p_4^3q_1^2q_2^2q_3q_4-9p_1p_2^3p_3p_4^3q_1^2q_2q_3^2q_4$$

$$-9p_1^3p_2p_3p_4^3q_1q_2^2q_3^2q_4-9p_1p_2^3p_3^3p_4q_1^2q_2q_3q_4^2-9p_1^3p_2p_3^3p_4q_1q_2^2q_3q_4^2-9p_1^3p_2^3p_3p_4q_1q_2q_3^2q_4^2$$

$$-31p_1p_2p_3p_4^3q_1^2q_2^2q_3^2q_4-31p_1p_2p_3^3p_4q_1^2q_2^2q_3q_4^2-31p_1p_2^3p_3p_4q_1^2q_2q_3^2q_4^2-31p_1^3p_2p_3p_4q_1q_2^2q_3^2q_4^2$$

$$-63p_1p_2p_3p_4q_1^2q_2^2q_3^2q_4^2\big)\,w_{n+11}-\big(p_1^2p_2^2p_3^4p_4^4q_1^2q_2^2q_3q_4+p_1^2p_2^4p_3^2p_4^4q_1^2q_2q_3^2q_4+p_1^4p_2^2p_3^2p_4^4q_1q_2^2q_3^2q_4$$

$$+p_1^2p_2^4p_3^4p_4^2q_1^2q_2q_3q_4^2+p_1^4p_2^2p_3^4p_4^2q_1q_2^2q_3q_4^2+p_1^4p_2^4p_3^2p_4^2q_1q_2q_3^2q_4^2-p_2^2p_3^2p_4^6q_1^3q_2^2q_3^2-p_1^2p_3^2p_4^6q_1^2q_2^3q_3^2$$

$$-p_1^2p_2^2p_4^6q_1^2q_2^2q_3^3-p_2^2p_3^6p_4^2q_1^3q_2^2q_4^2-p_1^2p_3^6p_4^2q_1^2q_2^3q_4^2-p_2^6p_3^2p_4^2q_1^3q_3^2q_4^2-p_1^6p_3^2p_4^2q_2^3q_3^2q_4^2-p_1^2p_2^6p_4^2q_1^2q_3^3q_4^2$$

$$-p_1^6p_2^2p_4^2q_2^2q_3^3q_4^2-p_1^2p_2^2p_3^6q_1^2q_2^2q_4^3-p_1^2p_2^6p_3^2q_1^2q_3^2q_4^3-p_1^6p_2^2p_3^2q_2^2q_3^2q_4^3-2p_3^2p_4^6q_1^3q_2^3q_3^2-2p_2^2p_4^6q_1^3q_2^2q_3^3$$

$$-2p_1^2p_4^6q_1^2q_2^3q_3^3-2p_3^6p_4^2q_1^3q_2^3q_4^2-2p_2^6p_4^2q_1^3q_3^3q_4^2-2p_1^6p_4^2q_2^3q_3^3q_4^2-2p_2^2p_3^6q_1^3q_2^2q_4^3-2p_1^2p_3^6q_1^2q_2^3q_4^3$$

$$-2p_2^6p_3^2q_1^3q_3^2q_4^3-2p_1^6p_3^2q_2^3q_3^2q_4^3-2p_1^2p_2^6q_1^2q_3^3q_4^3-2p_1^6p_2^2q_2^2q_3^3q_4^3-4p_4^6q_1^3q_2^3q_3^3-4p_3^6q_1^3q_2^3q_4^3-4p_2^6q_1^3q_3^3q_4^3$$

$$-4p_1^6q_2^3q_3^3q_4^3+5p_1^2p_2^2p_3^2p_4^4q_1^2q_2^2q_3^2q_4+5p_1^2p_2^2p_3^4p_4^2q_1^2q_2^2q_3q_4^2+5p_1^2p_2^4p_3^2p_4^2q_1^2q_2q_3^2q_4^2$$

$$+5p_1^4p_2^2p_3^2p_4^2q_1q_2^2q_3^2q_4^2-6p_2^2p_3^2p_4^4q_1^3q_2^2q_3^2q_4-6p_1^2p_3^2p_4^4q_1^2q_2^3q_3^2q_4-6p_1^2p_2^2p_4^4q_1^2q_2^2q_3^3q_4$$

$$-6p_2^2p_3^4p_4^2q_1^3q_2^2q_3q_4^2-6p_1^2p_3^4p_4^2q_1^2q_2^3q_3q_4^2-6p_2^4p_3^2p_4^2q_1^3q_2q_3^2q_4^2-6p_1^4p_3^2p_4^2q_1q_2^3q_3^2q_4^2-6p_1^2p_2^4p_4^2q_1^2q_2q_3^3q_4^2$$

$$- 6p_1^4p_2^2p_4^2q_1q_2^2q_3^3q_4^2 - 6p_1^2p_2^2p_3^4q_1^2q_2^2q_3q_4^3 - 6p_1^2p_2^4p_3^2q_1^2q_2q_3^2q_4^3 - 6p_1^4p_2^2p_3^2q_1q_2^2q_3^2q_4^3 - 12p_3^2p_4^4q_1^3q_2^3q_3^2q_4$$

$$- 12p_2^2p_4^4q_1^3q_2^2q_3^3q_4 - 12p_1^2p_4^4q_1^2q_2^3q_3^3q_4 - 12p_3^4p_4^2q_1^3q_2^3q_3q_4^2 - 12p_2^4p_4^2q_1^3q_2q_3^3q_4^2 - 12p_1^4p_4^2q_1q_2^3q_3^3q_4^2$$

$$- 12p_2^2p_3^4q_1^3q_2^2q_3q_4^3 - 12p_1^2p_3^4q_1^2q_2^3q_3q_4^3 - 12p_2^4p_3^2q_1^3q_2q_3^2q_4^3 - 12p_1^4p_3^2q_1q_2^3q_3^2q_4^3 - 12p_1^2p_2^4q_1^2q_2q_3^3q_4^3$$

$$- 12p_1^4p_2^2q_1q_2^2q_3^3q_4^3 + 12p_1^2p_2^2p_3^2p_4^2q_1^2q_2^2q_3^2q_4^2 - 24p_4^4q_1^3q_2^3q_3^3q_4 - 24p_3^4q_1^3q_2^3q_3q_4^3 - 24p_2^4q_1^3q_2q_3^3q_4^3$$

$$- 24p_1^4q_1q_2^3q_3^3q_4^3 - 31p_2^2p_3^2p_4^2q_1^3q_2^2q_3^2q_4^2 - 31p_1^2p_3^2p_4^2q_1^2q_2^3q_3^2q_4^2 - 31p_1^2p_2^2p_4^2q_1^2q_2^2q_3^3q_4^2$$

$$- 31p_1^2p_2^2p_3^2q_1^2q_2^2q_3^2q_4^3 - 46p_3^2p_4^2q_1^3q_2^3q_3^2q_4^2 - 46p_2^2p_4^2q_1^3q_2^2q_3^3q_4^2 - 46p_1^2p_4^2q_1^2q_2^3q_3^3q_4^2 - 46p_2^2p_3^2q_1^3q_2^2q_3^2q_4^3$$

$$- 46p_1^2p_3^2q_1^2q_2^3q_3^2q_4^3 - 46p_1^2p_2^2q_1^2q_2^2q_3^3q_4^3 - 60p_4^2q_1^3q_2^3q_3^3q_4^2 - 60p_3^2q_1^3q_2^3q_3^2q_4^3 - 60p_2^2q_1^3q_2^2q_3^3q_4^3$$

$$-60p_1^2q_1^2q_2^3q_3^3q_4^3 - 56q_1^3q_2^3q_3^3q_4^3\big)\, x_{n+10} + \big(p_1p_2^3p_3^3p_4^5q_1^3q_2^2q_3^2q_4 + p_1^3p_2p_3^3p_4^5q_1^2q_2^3q_3^2q_4$$

$$+ p_1^3p_2^3p_3p_4^5q_1^2q_2^2q_3^3q_4 + p_1p_2^3p_3^5p_4^3q_1^3q_2^2q_3q_4^2 + p_1^3p_2p_3^5p_4^3q_1^2q_2^3q_3q_4^2 + p_1p_2^5p_3^3p_4^3q_1^3q_2q_3^2q_4^2$$

$$+ p_1^5p_2p_3^3p_4^3q_1q_2^3q_3^2q_4^2 + p_1^3p_2^5p_3p_4^3q_1^2q_2q_3^3q_4^2 + p_1^5p_2^3p_3p_4^3q_1q_2^2q_3^3q_4^2 + p_1^3p_2^3p_3^5p_4q_1^2q_2^2q_3q_4^3$$

$$+ p_1^3p_2^5p_3^3p_4q_1^2q_2q_3^2q_4^3 + p_1^5p_2^3p_3^3p_4q_1q_2^2q_3^2q_4^3 - p_1p_2p_3p_4^7q_1^3q_2^3q_3^3 - p_1p_2p_3^7p_4q_1^3q_2^3q_4^3 - p_1p_2^7p_3p_4q_1^3q_3^3q_4^3$$

$$- p_1^7p_2p_3p_4q_2^3q_3^3q_4^3 + 2p_1p_2p_3^3p_4^5q_1^3q_2^3q_3^2q_4 + 2p_1p_2^3p_3p_4^5q_1^3q_2^2q_3^3q_4 + 2p_1^3p_2p_3p_4^5q_1^2q_2^3q_3^3q_4$$

$$+ 2p_1p_2p_3^5p_4^3q_1^3q_2^3q_3q_4^2 + 2p_1p_2^5p_3p_4^3q_1^3q_2q_3^3q_4^2 + 2p_1^5p_2p_3p_4^3q_1q_2^3q_3^3q_4^2 + 2p_1p_2^3p_3^5p_4q_1^3q_2^2q_3q_4^3$$

$$+ 2p_1^3p_2p_3^5p_4q_1^2q_2^3q_3q_4^3 + 2p_1p_2^5p_3^3p_4q_1^3q_2q_3^2q_4^3 + 2p_1^5p_2p_3^3p_4q_1q_2^3q_3^2q_4^3 + 2p_1^3p_2^5p_3p_4q_1^2q_2q_3^3q_4^3$$

$$+ 2p_1^5p_2^3p_3p_4q_1q_2^2q_3^3q_4^3 - 3p_1p_2p_3p_4^5q_1^3q_2^3q_3^3q_4 - 3p_1p_2p_3^5p_4q_1^3q_2^3q_3q_4^3 - 3p_1p_2^5p_3p_4q_1^3q_2q_3^3q_4^3$$

$$- 3p_1^5p_2p_3p_4q_1q_2^3q_3^3q_4^3 + 3p_1^3p_2^3p_3^3p_4^3q_1^2q_2^2q_3^2q_4^2 + 14p_1p_2^3p_3^3p_4^3q_1^3q_2^2q_3^2q_4^2 + 14p_1^3p_2p_3^3p_4^3q_1^2q_2^3q_3^2q_4^2$$

$$+ 14p_1^3p_2^3p_3p_4^3q_1^2q_2^2q_3^3q_4^2 + 14p_1^3p_2^3p_3^3p_4q_1^2q_2^2q_3^2q_4^3 + 24p_1p_2p_3^3p_4^3q_1^3q_2^3q_3^2q_4^2 + 24p_1p_2^3p_3p_4^3q_1^3q_2^2q_3^3q_4^2$$

$$+ 24p_1^3p_2p_3p_4^3q_1^2q_2^3q_3^3q_4^2 + 24p_1p_2^3p_3^3p_4q_1^3q_2^2q_3^2q_4^3 + 24p_1^3p_2p_3^3p_4q_1^2q_2^3q_3^2q_4^3 + 24p_1^3p_2^3p_3p_4q_1^2q_2^2q_3^3q_4^3$$

$$+ 26p_1p_2p_3p_4^3q_1^3q_2^3q_3^3q_4^2 + 26p_1p_2p_3^3p_4q_1^3q_2^3q_3^2q_4^3 + 26p_1p_2^3p_3p_4q_1^3q_2^2q_3^3q_4^3 + 26p_1^3p_2p_3p_4q_1^2q_2^3q_3^3q_4^3$$

$$+43p_1p_2p_3p_4q_1^3q_2^3q_3^3q_4^3\big)\, w_{n+9} - \big(p_4^8q_1^4q_2^4q_3^4 + p_3^8q_1^4q_2^4q_4^4 + p_2^8q_1^4q_3^4q_4^4 + p_1^8q_2^4q_3^4q_4^4 + p_2^4p_3^4p_4^4q_1^4q_2^2q_3^2q_4^2$$

$$+ p_1^4p_3^4p_4^4q_1^2q_2^4q_3^2q_4^2 + p_1^4p_2^4p_4^4q_1^2q_2^2q_3^4q_4^2 + p_1^4p_2^4p_3^4q_1^2q_2^2q_3^2q_4^4 + p_1^2p_2^2p_3^2p_4^6q_1^3q_2^3q_3^3q_4 + p_1^2p_2^2p_3^6p_4^2q_1^3q_2^3q_3q_4^3$$

$$+ p_1^2p_2^6p_3^2p_4^2q_1^3q_2q_3^3q_4^3 + p_1^6p_2^2p_3^2p_4^2q_1q_2^3q_3^3q_4^3 + 2p_1^2p_2^2p_3^4p_4^4q_1^3q_2^3q_3^2q_4^2 + 2p_1^2p_2^4p_3^2p_4^4q_1^3q_2^2q_3^3q_4^2$$

$$+ 2p_1^4p_2^2p_3^2p_4^4q_1^2q_2^3q_3^3q_4^2 + 2p_1^2p_2^4p_3^4p_4^2q_1^3q_2^2q_3^2q_4^3 + 2p_1^4p_2^2p_3^4p_4^2q_1^2q_2^3q_3^2q_4^3 + 2p_1^4p_2^4p_3^2p_4^2q_1^2q_2^2q_3^3q_4^3$$

$$+ 4p_2^2p_3^4p_4^4q_1^4q_2^3q_3^2q_4^2 + 4p_1^2p_3^4p_4^4q_1^3q_2^4q_3^2q_4^2 + 4p_2^4p_3^2p_4^4q_1^4q_2^2q_3^3q_4^2 + 4p_1^4p_3^2p_4^4q_1^2q_2^4q_3^3q_4^2$$

$$+ 4p_1^2p_2^4p_4^4q_1^3q_2^2q_3^4q_4^2 + 4p_1^4p_2^2p_4^4q_1^2q_2^3q_3^4q_4^2 + 4p_2^4p_3^4p_4^2q_1^4q_2^2q_3^2q_4^3 + 4p_1^4p_3^4p_4^2q_1^2q_2^4q_3^2q_4^3$$

$$+ 4p_1^4p_2^4p_4^2q_1^2q_2^2q_3^4q_4^3 + 4p_1^2p_2^4p_3^4q_1^3q_2^2q_3^2q_4^4 + 4p_1^4p_2^2p_3^4q_1^2q_2^3q_3^2q_4^4 + 4p_1^4p_2^4p_3^2q_1^2q_2^2q_3^3q_4^4 + 4p_3^4p_4^4q_1^4q_2^4q_3^2q_4^2$$

$$+ 4p_2^4p_4^4q_1^4q_2^2q_3^4q_4^2 + 4p_1^4p_4^4q_1^2q_2^4q_3^4q_4^2 + 4p_2^4p_3^4q_1^4q_2^2q_3^2q_4^4 + 4p_1^4p_3^4q_1^2q_2^4q_3^2q_4^4 + 4p_1^4p_2^4q_1^2q_2^2q_3^4q_4^4$$

$$+ 8p_4^6q_1^4q_2^4q_3^4q_4 + 8p_3^6q_1^4q_2^4q_3q_4^4 + 8p_2^6q_1^4q_2q_3^4q_4^4 + 8p_1^6q_1q_2^4q_3^4q_4^4 + 16p_2^2p_3^2p_4^4q_1^4q_2^3q_3^3q_4^2$$

$$+ 16p_1^2p_3^2p_4^4q_1^3q_2^4q_3^3q_4^2 + 16p_1^2p_2^2p_4^4q_1^3q_2^3q_3^4q_4^2 + 16p_2^2p_3^4p_4^2q_1^4q_2^3q_3^2q_4^3 + 16p_1^2p_3^4p_4^2q_1^3q_2^4q_3^2q_4^3$$

$$+ 16p_2^4p_3^2p_4^2q_1^4q_2^2q_3^3q_4^3 + 16p_1^4p_3^2p_4^2q_1^2q_2^4q_3^3q_4^3 + 16p_1^2p_2^4p_4^2q_1^3q_2^2q_3^4q_4^3 + 16p_1^4p_2^2p_4^2q_1^2q_2^3q_3^4q_4^3$$

$$+ 16p_1^2p_2^2p_3^4q_1^3q_2^3q_3^2q_4^4 + 16p_1^2p_2^4p_3^2q_1^3q_2^2q_3^3q_4^4 + 16p_1^4p_2^2p_3^2q_1^2q_2^3q_3^3q_4^4 + 16p_3^2p_4^4q_1^4q_2^4q_3^3q_4^2$$

$$+ 16p_2^2p_4^4q_1^4q_2^3q_3^4q_4^2 + 16p_1^2p_4^4q_1^3q_2^4q_3^4q_4^2 + 16p_3^4p_4^2q_1^4q_2^4q_3^2q_4^3 + 16p_2^4p_4^2q_1^4q_2^2q_3^4q_4^3 + 16p_1^4p_4^2q_1^2q_2^4q_3^4q_4^3$$

$$+ 16p_2^2p_3^4q_1^4q_2^3q_3^2q_4^4 + 16p_1^2p_3^4q_1^3q_2^4q_3^2q_4^4 + 16p_2^4p_3^2q_1^4q_2^2q_3^3q_4^4 + 16p_1^4p_3^2q_1^2q_2^4q_3^3q_4^4 + 16p_1^2p_2^4q_1^3q_2^2q_3^4q_4^4$$

$$+ 16p_1^4p_2^2q_1^2q_2^3q_3^4q_4^4 + 18p_1^2p_2^2p_3^2p_4^4q_1^3q_2^3q_3^3q_4^2 + 18p_1^2p_2^2p_3^4p_4^2q_1^3q_2^3q_3^2q_4^3 + 18p_1^2p_2^4p_3^2p_4^2q_1^3q_2^2q_3^3q_4^3$$

$$+ 18p_1^4p_2^2p_3^2p_4^2q_1^2q_2^3q_3^3q_4^3 + 82p_1^2p_2^2p_3^2p_4^2q_1^3q_2^3q_3^3q_4^3 + 36p_4^4q_1^4q_2^4q_3^4q_4^2 + 36p_3^4q_1^4q_2^4q_3^2q_4^4 + 36p_2^4q_1^4q_2^2q_3^4q_4^4$$

$$+ 36p_1^4q_1^2q_2^4q_3^4q_4^4 + 64p_2^2p_3^2p_4^2q_1^4q_2^3q_3^3q_4^3 + 64p_1^2p_3^2p_4^2q_1^3q_2^4q_3^3q_4^3 + 64p_1^2p_2^2p_4^2q_1^3q_2^3q_3^4q_4^3$$

$$+ 64p_1^2p_2^2p_3^2q_1^3q_2^3q_3^3q_4^4 + 64p_3^2p_4^2q_1^4q_2^4q_3^3q_4^3 + 64p_2^2p_4^2q_1^4q_2^3q_3^4q_4^3 + 64p_1^2p_4^2q_1^3q_2^4q_3^4q_4^3 + 64p_2^2p_3^2q_1^4q_2^3q_3^3q_4^4$$

$$+ 64p_1^2p_3^2q_1^3q_2^4q_3^3q_4^4 + 64p_1^2p_2^2q_1^3q_2^3q_3^4q_4^4 + 80p_4^2q_1^4q_2^4q_3^4q_4^3 + 80p_3^2q_1^4q_2^4q_3^3q_4^4 + 80p_2^2q_1^4q_2^3q_3^4q_4^4$$

$$+80p_1^2q_1^3q_2^4q_3^4q_4^4 + 70q_1^4q_2^4q_3^4q_4^4\big) w_{n+8} + q_1q_2q_3q_4 \big(p_1p_2^3p_3^3p_4^5q_1^3q_2^2q_3^2q_4 + p_1^3p_2p_3^3p_4^5q_1^2q_2^3q_3^2q_4$$

$$+ p_1^3p_2^3p_3p_4^5q_1^2q_2^2q_3^3q_4 + p_1p_2^3p_3^5p_4^3q_1^3q_2^2q_3q_4^2 + p_1^3p_2p_3^5p_4^3q_1^2q_2^3q_3q_4^2 + p_1p_2^5p_3^3p_4^3q_1^3q_2q_3^2q_4^2$$

$$+ p_1^5p_2p_3^3p_4^3q_1q_2^3q_3^2q_4^2 + p_1^3p_2^5p_3p_4^3q_1^2q_2q_3^3q_4^2 + p_1^5p_2^3p_3p_4^3q_1q_2^2q_3^3q_4^2 + p_1^3p_2^3p_3^5p_4q_1^2q_2^2q_3q_4^3$$

$$+ p_1^3p_2^5p_3^3p_4q_1^2q_2q_3^2q_4^3 + p_1^5p_2^3p_3^3p_4q_1q_2^2q_3^2q_4^3 - p_1p_2p_3p_4^7q_1^3q_2^3q_3^3 - p_1p_2p_3^7p_4q_1^3q_2^3q_4^3 - p_1p_2^7p_3p_4q_1^3q_3^3q_4^3$$

$$- p_1^7p_2p_3p_4q_2^3q_3^3q_4^3 + 2p_1p_2p_3^3p_4^5q_1^3q_2^3q_3^2q_4 + 2p_1p_2^3p_3p_4^5q_1^3q_2^2q_3^3q_4 + 2p_1^3p_2p_3p_4^5q_1^2q_2^3q_3^3q_4$$

$$+ 2p_1p_2p_3^5p_4^3q_1^3q_2^3q_3q_4^2 + 2p_1p_2^5p_3p_4^3q_1^3q_2q_3^3q_4^2 + 2p_1^5p_2p_3p_4^3q_1q_2^3q_3^3q_4^2 + 2p_1p_2^3p_3^5p_4q_1^3q_2^2q_3q_4^3$$

$$+ 2p_1^3p_2p_3^5p_4q_1^2q_2^3q_3q_4^3 + 2p_1p_2^5p_3^3p_4q_1^3q_2q_3^2q_4^3 + 2p_1^5p_2p_3^3p_4q_1q_2^3q_3^2q_4^3 + 2p_1^3p_2^5p_3p_4q_1^2q_2q_3^3q_4^3$$

$$+ 2p_1^5p_2^3p_3p_4q_1q_2^2q_3^3q_4^3 - 3p_1p_2p_3p_4^5q_1^3q_2^3q_3^3q_4 - 3p_1p_2p_3^5p_4q_1^3q_2^3q_3q_4^3 - 3p_1p_2^5p_3p_4q_1^3q_2q_3^3q_4^3$$

$$- 3p_1^5p_2p_3p_4q_1q_2^3q_3^3q_4^3 + 3p_1^3p_2^3p_3^3p_4^3q_1^2q_2^2q_3^2q_4^2 + 14p_1p_2^3p_3^3p_4^3q_1^3q_2^2q_3^2q_4^2 + 14p_1^3p_2p_3^3p_4^3q_1^2q_2^3q_3^2q_4^2$$

$$+ 14p_1^3p_2^3p_3p_4^3q_1^2q_2^2q_3^3q_4^2 + 14p_1^3p_2^3p_3^3p_4q_1^2q_2^2q_3^2q_4^3 + 24p_1p_2p_3^3p_4^3q_1^3q_2^3q_3^2q_4^2 + 24p_1p_2^3p_3p_4^3q_1^3q_2^2q_3^3q_4^2$$

$$+ 24p_1^3p_2p_3p_4^3q_1^2q_2^3q_3^3q_4^2 + 24p_1p_2^3p_3^3p_4q_1^3q_2^2q_3^2q_4^3 + 24p_1^3p_2p_3^3p_4q_1^2q_2^3q_3^2q_4^3 + 24p_1^3p_2^3p_3p_4q_1^2q_2^2q_3^3q_4^3$$

$$+ 26p_1p_2p_3p_4^3q_1^3q_2^3q_3^3q_4^2 + 26p_1p_2p_3^3p_4q_1^3q_2^3q_3^2q_4^3 + 26p_1p_2^3p_3p_4q_1^3q_2^2q_3^3q_4^3 + 26p_1^3p_2p_3p_4q_1^2q_2^3q_3^3q_4^3$$

$$+43p_1p_2p_3p_4q_1^3q_2^3q_3^3q_4^3\right) w_{n+7} - q_1^2q_2^2q_3^2q_4^2 \left(p_1^2p_2^2p_3^4p_4^4q_1^2q_2^2q_3q_4 + p_1^2p_2^4p_3^2p_4^4q_1^2q_2q_3^2q_4$$

$$+ p_1^4p_2^2p_3^2p_4^4q_1q_2^2q_3^2q_4 + p_1^2p_2^4p_3^4p_4^2q_1^2q_2q_3q_4^2 + p_1^4p_2^2p_3^4p_4^2q_1q_2^2q_3q_4^2 + p_1^4p_2^4p_3^2p_4^2q_1q_2q_3^2q_4^2$$

$$- p_2^2p_3^2p_4^6q_1^3q_2^2q_3^2 - p_1^2p_3^2p_4^6q_1^2q_2^3q_3^2 - p_1^2p_2^2p_4^6q_1^2q_2^2q_3^3 - p_2^2p_3^6p_4^2q_1^3q_2^2q_4^2 - p_1^2p_3^6p_4^2q_1^2q_2^3q_4^2 - p_2^6p_3^2p_4^2q_1^3q_3^2q_4^2$$

$$- p_1^6p_3^2p_4^2q_2^3q_3^2q_4^2 - p_1^2p_2^6p_4^2q_1^2q_3^3q_4^2 - p_1^6p_2^2p_4^2q_2^2q_3^3q_4^2 - p_1^2p_2^2p_3^6q_1^2q_2^2q_4^3 - p_1^2p_2^6p_3^2q_1^2q_3^2q_4^3 - p_1^6p_2^2p_3^2q_2^2q_3^2q_4^3$$

$$- 2p_3^2p_4^6q_1^3q_2^3q_3^2 - 2p_2^2p_4^6q_1^3q_2^2q_3^3 - 2p_1^2p_4^6q_1^2q_2^3q_3^3 - 2p_3^6p_4^2q_1^3q_2^3q_4^2 - 2p_2^6p_4^2q_1^3q_3^3q_4^2 - 2p_1^6p_4^2q_2^3q_3^3q_4^2$$

$$- 2p_2^2p_3^6q_1^3q_2^2q_4^3 - 2p_1^2p_3^6q_1^2q_2^3q_4^3 - 2p_2^6p_3^2q_1^3q_3^2q_4^3 - 2p_1^6p_3^2q_2^3q_3^2q_4^3 - 2p_1^2p_2^6q_1^2q_3^3q_4^3 - 2p_1^6p_2^2q_2^2q_3^3q_4^3$$

$$- 4p_4^6q_1^3q_2^3q_3^3 - 4p_3^6q_1^3q_2^3q_4^3 - 4p_2^6q_1^3q_3^3q_4^3 - 4p_1^6q_2^3q_3^3q_4^3 + 5p_1^2p_2^2p_3^2p_4^4q_1^2q_2^2q_3^2q_4 + 5p_1^2p_2^2p_3^4p_4^2q_1^2q_2^2q_3q_4^2$$

$$+ 5p_1^2p_2^4p_3^2p_4^2q_1^2q_2q_3^2q_4^2 + 5p_1^4p_2^2p_3^2p_4^2q_1q_2^2q_3^2q_4^2 - 6p_2^2p_3^2p_4^4q_1^3q_2^2q_3^2q_4 - 6p_1^2p_3^2p_4^4q_1^2q_2^3q_3^2q_4$$

$$- 6p_1^2p_2^2p_4^4q_1^2q_2^2q_3^3q_4 - 6p_2^2p_3^4p_4^2q_1^3q_2^2q_3q_4^2 - 6p_1^2p_3^4p_4^2q_1^2q_2^3q_3q_4^2 - 6p_2^4p_3^2p_4^2q_1^3q_2q_3^2q_4^2 - 6p_1^4p_3^2p_4^2q_1q_2^3q_3^2q_4^2$$

$$- 6p_1^2p_2^4p_4^2q_1^2q_2q_3^3q_4^2 - 6p_1^4p_2^2p_4^2q_1q_2^2q_3^3q_4^2 - 6p_1^2p_2^2p_3^4q_1^2q_2^2q_3q_4^3 - 6p_1^2p_2^4p_3^2q_1^2q_2q_3^2q_4^3 - 6p_1^4p_2^2p_3^2q_1q_2^2q_3^2q_4^3$$

$$- 12p_3^2p_4^4q_1^3q_2^3q_3^2q_4 - 12p_2^2p_4^4q_1^3q_2^2q_3^3q_4 - 12p_1^2p_4^4q_1^2q_2^3q_3^3q_4 - 12p_3^4p_4^2q_1^3q_2^3q_3q_4^2 - 12p_2^4p_4^2q_1^3q_2q_3^3q_4^2$$

$$- 12p_1^4p_4^2q_1q_2^3q_3^3q_4^2 - 12p_2^2p_3^4q_1^3q_2^2q_3q_4^3 - 12p_1^2p_3^4q_1^2q_2^3q_3q_4^3 - 12p_2^4p_3^2q_1^3q_2q_3^2q_4^3 - 12p_1^4p_3^2q_1q_2^3q_3^2q_4^3$$

$$- 12p_1^2p_2^4q_1^2q_2q_3^3q_4^3 - 12p_1^4p_2^2q_1q_2^2q_3^3q_4^3 + 12p_1^2p_2^2p_3^2p_4^2q_1^2q_2^2q_3^2q_4^2 - 24p_4^4q_1^3q_2^3q_3^3q_4 - 24p_3^4q_1^3q_2^3q_3q_4^3$$

$$- 24p_2^4q_1^3q_2q_3^3q_4^3 - 24p_1^4q_1q_2^3q_3^3q_4^3 - 31p_2^2p_3^2p_4^2q_1^3q_2^2q_3^2q_4^2 - 31p_1^2p_3^2p_4^2q_1^2q_2^3q_3^2q_4^2 - 31p_1^2p_2^2p_4^2q_1^2q_2^2q_3^3q_4^2$$

$$- 31p_1^2p_2^2p_3^2q_1^2q_2^2q_3^2q_4^3 - 46p_3^2p_4^2q_1^3q_2^3q_3^2q_4^2 - 46p_2^2p_4^2q_1^3q_2^2q_3^3q_4^2 - 46p_1^2p_4^2q_1^2q_2^3q_3^3q_4^2 - 46p_2^2p_3^2q_1^3q_2^2q_3^2q_4^3$$

$$- 46p_1^2p_3^2q_1^2q_2^3q_3^2q_4^3 - 46p_1^2p_2^2q_1^2q_2^2q_3^3q_4^3 - 60p_4^2q_1^3q_2^3q_3^3q_4^2 - 60p_3^2q_1^3q_2^3q_3^2q_4^3 - 60p_2^2q_1^3q_2^2q_3^3q_4^3$$

$$-60p_1^2q_1^2q_2^3q_3^3q_4^3 - 56q_1^3q_2^3q_3^3q_4^3\right) w_{n+6} + q_1^3q_2^3q_3^3q_4^3 \left(p_1^3p_2^3p_3^3p_4^3q_1q_2q_3q_4 - p_1p_2p_3^3p_4^5q_1^2q_2^2q_3$$

$$- p_1p_2^3p_3p_4^5q_1^2q_2q_3^2 - p_1^3p_2p_3p_4^5q_1q_2^2q_3^2 - p_1p_2p_3^5p_4^3q_1^2q_2^2q_4 - p_1p_2^5p_3p_4^3q_1^2q_3^2q_4 - p_1^5p_2p_3p_4^3q_2^2q_3^2q_4$$

$$- p_1p_2^3p_3^5p_4q_1^2q_2q_4^2 - p_1^3p_2p_3^5p_4q_1q_2^2q_4^2 - p_1p_2^5p_3^3p_4q_1^2q_3q_4^2 - p_1^5p_2p_3^3p_4q_2^2q_3q_4^2 - p_1^3p_2^5p_3p_4q_1q_3^2q_4^2$$

$$- p_1^5p_2^3p_3p_4q_2q_3^2q_4^2 - 5p_1p_2p_3p_4^5q_1^2q_2^2q_3^2 - 5p_1p_2p_3^5p_4q_1^2q_2^2q_4^2 - 5p_1p_2^5p_3p_4q_1^2q_3^2q_4^2 - 5p_1^5p_2p_3p_4q_2^2q_3^2q_4^2$$

$$- 9p_1p_2p_3^3p_4^3q_1^2q_2^2q_3q_4 - 9p_1p_2^3p_3p_4^3q_1^2q_2q_3^2q_4 - 9p_1^3p_2p_3p_4^3q_1q_2^2q_3^2q_4 - 9p_1p_2^3p_3^3p_4q_1^2q_2q_3q_4^2$$

$$- 9p_1^3p_2p_3^3p_4q_1q_2^2q_3q_4^2 - 9p_1^3p_2^3p_3p_4q_1q_2q_3^2q_4^2 - 31p_1p_2p_3p_4^3q_1^2q_2^2q_3^2q_4 - 31p_1p_2p_3^3p_4q_1^2q_2^2q_3q_4^2$$

$$
\begin{aligned}
&-31p_1p_2^3p_3p_4q_1^2q_2q_3^2q_4^2 - 31p_1^3p_2p_3p_4q_1q_2^2q_3^2q_4^2 - 63p_1p_2p_3p_4q_1^2q_2^2q_3^2q_4^2\big)\, w_{n+5} \\
&- q_1^4q_2^4q_3^4q_4^4\,\big(p_3^4p_4^4q_1^2q_2^2 + p_2^4p_4^4q_1^2q_3^2 + p_1^4p_4^4q_2^2q_3^2 + p_2^4p_3^4q_1^2q_4^2 + p_1^4p_3^4q_2^2q_4^2 + p_1^4p_2^4q_3^2q_4^2 - p_1^2p_2^2p_3^2p_4^4q_1q_2q_3 \\
&- p_1^2p_2^2p_3^4p_4^2q_1q_2q_4 - p_1^2p_2^4p_3^2p_4^2q_1q_3q_4 - p_1^4p_2^2p_3^2p_4^2q_2q_3q_4 + 4p_3^2p_4^4q_1^2q_2^2q_3 + 4p_2^2p_4^4q_1^2q_2q_3^2 \\
&+ 4p_1^2p_4^4q_1q_2^2q_3^2 + 4p_3^4p_4^2q_1^2q_2^2q_4 + 4p_2^4p_4^2q_1^2q_3^2q_4 + 4p_1^4p_4^2q_2^2q_3^2q_4 + 4p_2^2p_3^4q_1^2q_2q_4^2 + 4p_1^2p_3^4q_1q_2^2q_4^2 \\
&+ 4p_2^4p_3^2q_1^2q_3q_4^2 + 4p_1^4p_3^2q_2^2q_3q_4^2 + 4p_1^2p_2^4q_1q_3^2q_4^2 + 4p_1^4p_2^2q_2q_3^2q_4^2 + 6p_4^4q_1^2q_2^2q_3^2 + 6p_3^4q_1^2q_2^2q_4^2 + 6p_2^4q_1^2q_3^2q_4^2 \\
&+ 6p_1^4q_2^2q_3^2q_4^2 - 9p_1^2p_2^2p_3^2p_4^2q_1q_2q_3q_4 + 16p_3^2p_4^2q_1^2q_2^2q_3q_4 + 16p_2^2p_4^2q_1^2q_2q_3^2q_4 + 16p_1^2p_4^2q_1q_2^2q_3^2q_4 \\
&+ 16p_2^2p_3^2q_1^2q_2q_3q_4^2 + 16p_1^2p_3^2q_1q_2^2q_3q_4^2 + 16p_1^2p_2^2q_1q_2q_3^2q_4^2 + 24p_4^2q_1^2q_2^2q_3^2q_4 + 24p_3^2q_1^2q_2^2q_3q_4^2 \\
&+24p_2^2q_1^2q_2q_3^2q_4^2 + 24p_1^2q_1q_2^2q_3^2q_4^2 + 28q_1^2q_2^2q_3^2q_4^2\big)\, w_{n+4} + q_1^5q_2^5q_3^5q_4^5\,\big(p_1p_2p_3^3p_4^3q_1q_2 + p_1p_2^3p_3p_4^3q_1q_3 \\
&+ p_1^3p_2p_3p_4^3q_2q_3 + p_1p_2^3p_3^3p_4q_1q_4 + p_1^3p_2p_3^3p_4q_2q_4 + p_1^3p_2^3p_3p_4q_3q_4 + 5p_1p_2p_3p_4^3q_1q_2q_3 \\
&+5p_1p_2p_3^3p_4q_1q_2q_4 + 5p_1p_2^3p_3p_4q_1q_3q_4 + 5p_1^3p_2p_3p_4q_2q_3q_4 + 19p_1p_2p_3p_4q_1q_2q_3q_4\big)\, w_{n+3} \\
&+ q_1^6q_2^6q_3^6q_4^6\,\big(p_2^2p_3^2p_4^2q_1 + p_1^2p_3^2p_4^2q_2 + p_1^2p_2^2p_4^2q_3 + p_1^2p_2^2p_3^2q_4 + 2p_3^2p_4^2q_1q_2 + 2p_2^2p_4^2q_1q_3 + 2p_1^2p_4^2q_2q_3 \\
&+ 2p_2^2p_3^2q_1q_4 + 2p_1^2p_3^2q_2q_4 + 2p_1^2p_2^2q_3q_4 + 4p_4^2q_1q_2q_3 + 4p_3^2q_1q_2q_4 + 4p_2^2q_1q_3q_4 + 4p_1^2q_2q_3q_4 \\
&+8q_1q_2q_3q_4\big)\, w_{n+2} + p_1p_2p_3p_4q_1^7q_2^7q_3^7q_4^7w_{n+1} - q_1^8q_2^8q_3^8q_4^8w_n \qquad (3.4)
\end{aligned}
$$

for $n \geq 0$ *with initial conditions* $w_i = a_ib_ic_id_i$ *for* $0 \leq i \leq 15$.

Proof. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = c_0 = d_0 = 0$ and a_1, b_1, c_1, d_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - p_1x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, the sequence $\{b_n\}$ have the characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$, the sequence $\{c_n\}$ have the characteristic equation $x^2 - p_3x - q_3 = 0$ with roots α_3 and β_3, such that $\alpha_3 + \beta_3 = p_3$ and $\alpha_3\beta_3 = -q_3$, and the sequence $\{d_n\}$ have the characteristic equation $x^2 - p_4x - q_4 = 0$ with roots α_4 and β_4, such that $\alpha_4 + \beta_4 = p_4$ and $\alpha_4\beta_4 = -q_4$.

Case 1: Let each characteristic function have distinct roots, meaning $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$, $\alpha_3 \neq \beta_3$, and $\alpha_4 \neq \beta_4$. Then, by equation (3.1), we have

$$
\begin{aligned}
w_n =& a_nb_nc_nd_n \\
=& \left(\frac{a_1b_1c_1d_1}{(\alpha_1-\beta_1)(\alpha_2-\beta_2)(\alpha_3-\beta_3)(\alpha_4-\beta_4)}\right)(\alpha_1^n - \beta_1^n)(\alpha_2^n - \beta_2^n)(\alpha_3^n - \beta_3^n)(\alpha_4^n - \beta_4^n)
\end{aligned}
$$

$$= \left(\frac{a_1 b_1 c_1 d_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)(\alpha_4 - \beta_4)} \right) ((\alpha_1\alpha_2)^n - (\alpha_1\beta_2)^n - (\alpha_2\beta_1)^n + (\beta_1\beta_2)^n) (\alpha_3^n - \beta_3^n)(\alpha_4^n - \beta_4^n)$$

$$= \left(\frac{a_1 b_1 c_1 d_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)(\alpha_4 - \beta_4)} \right) ((\alpha_1\alpha_2\alpha_3)^n - (\alpha_1\alpha_2\beta_3)^n - (\alpha_1\beta_2\alpha_3)^n + (\alpha_1\beta_2\beta_3)^n$$

$$-(\beta_1\alpha_2\alpha_3)^n + (\beta_1\alpha_2\beta_3)^n + (\beta_1\beta_2\alpha_3)^n - (\beta_1\beta_2\beta_3)^n) (\alpha_4^n - \beta_4^n)$$

$$= \left(\frac{a_1 b_1 c_1 d_1}{(\alpha_1 - \beta_1)(\alpha_2 - \beta_2)(\alpha_3 - \beta_3)(\alpha_4 - \beta_4)} \right) ((\alpha_1\alpha_2\alpha_3\alpha_4)^n - (\alpha_1\alpha_2\alpha_3\beta_4)^n - (\alpha_1\alpha_2\beta_3\alpha_4)^n + (\alpha_1\alpha_2\beta_3\beta_4)^n$$

$$- (\alpha_1\beta_2\alpha_3\alpha_4)^n + (\alpha_1\beta_2\alpha_3\beta_4)^n + (\alpha_1\beta_2\beta_3\alpha_4)^n - (\alpha_1\beta_2\beta_3\beta_4)^n - (\beta_1\alpha_2\alpha_3\alpha_4)^n + (\beta_1\alpha_2\alpha_3\beta_4)^n$$

$$+(\beta_1\alpha_2\beta_3\alpha_4)^n - (\beta_1\alpha_2\beta_3\beta_4)^n + (\beta_1\beta_2\alpha_3\alpha_4)^n - (\beta_1\beta_2\alpha_3\beta_4)^n - (\beta_1\beta_2\beta_3\alpha_4)^n + (\beta_1\beta_2\beta_3\beta_4)^n).$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_2 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_3 = \alpha_1\alpha_2\beta_3\alpha_4$, $r_4 = \alpha_1\alpha_2\beta_3\beta_4$, $r_5 = \alpha_1\beta_2\alpha_3\alpha_4$, $r_6 = \alpha_1\beta_2\alpha_3\beta_4$, $r_7 = \alpha_1\beta_2\beta_3\alpha_4$, $r_8 = \alpha_1\beta_2\beta_3\beta_4$, $r_9 = \beta_1\alpha_2\alpha_3\alpha_4$, $r_{10} = \beta_1\alpha_2\alpha_3\beta_4$, $r_{11} = \beta_1\alpha_2\beta_3\alpha_4$, $r_{12} = \beta_1\alpha_2\beta_3\beta_4$, $r_{13} = \beta_1\beta_2\alpha_3\alpha_4$, $r_{14} = \beta_1\beta_2\alpha_3\beta_4$, $r_{15} = \beta_1\beta_2\beta_3\alpha_4$, and $r_{16} = \beta_1\beta_2\beta_3\beta_4$ each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have sixteen roots, which is how many characteristic roots we need for an sixteenth order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{16}(x - r_i) = x^{16} - \left(\sum_{1 \le i \le 16} r_i \right) x^{15} + \cdots + (-1)^k \left(\sum_{1 \le i_1 < \cdots < i_k \le 16} r_{i_1} \cdots r_{i_k} \right) x^{16-k}, \text{ for} k \le 16.$$

Looking at the coefficient of x^{15}, which becomes the coefficient of w_{n+15} in equation (3.4), we have

$$\begin{aligned}
\sum_{1 \le i \le 16} r_i =& \alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_3\beta_4 + \alpha_1\alpha_2\beta_3\alpha_4 + \alpha_1\alpha_2\beta_3\beta_4 + \alpha_1\beta_2\alpha_3\alpha_4 + \alpha_1\beta_2\alpha_3\beta_4 + \alpha_1\beta_2\beta_3\alpha_4 \\
&+ \alpha_1\beta_2\beta_3\beta_4 + \beta_1\alpha_2\alpha_3\alpha_4 + \beta_1\alpha_2\alpha_3\beta_4 + \beta_1\alpha_2\beta_3\alpha_4 + \beta_1\alpha_2\beta_3\beta_4 + \beta_1\beta_2\alpha_3\alpha_4 + \beta_1\beta_2\alpha_3\beta_4 \\
&+ \beta_1\beta_2\beta_3\alpha_4 + \beta_1\beta_2\beta_3\beta_4 \\
=& \alpha_1 (\alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\beta_2 + \alpha_2\alpha_4\beta_3 + \alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_4 + \alpha_3\beta_2\beta_4 + \alpha_2\beta_3\beta_4 + \beta_2\beta_3\beta_4) \\
&+ \beta_1 (\alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\beta_2 + \alpha_2\alpha_4\beta_3 + \alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_4 + \alpha_3\beta_2\beta_4 + \alpha_2\beta_3\beta_4 + \beta_2\beta_3\beta_4) \\
=& (\alpha_1 + \beta_1) (\alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\beta_2 + \alpha_2\alpha_4\beta_3 + \alpha_4\beta_2\beta_3 + \alpha_2\alpha_3\beta_4 + \alpha_3\beta_2\beta_4 + \alpha_2\beta_3\beta_4 + \beta_2\beta_3\beta_4) \\
=& (\alpha_1 + \beta_1) (\alpha_2 (\alpha_3\alpha_4 + \alpha_4\beta_3 + \alpha_3\beta_4 + \beta_3\beta_4) + \beta_2 (\alpha_3\alpha_4 + \alpha_4\beta_3 + \alpha_3\beta_4 + \beta_3\beta_4)) \\
=& (\alpha_1 + \beta_1) (\alpha_2 + \beta_2) (\alpha_3\alpha_4 + \alpha_4\beta_3 + \alpha_3\beta_4 + \beta_3\beta_4) \\
=& (\alpha_1 + \beta_1) (\alpha_2 + \beta_2) (\alpha_3 (\alpha_4 + \beta_4) + \beta_3 (\alpha_4 + \beta_4))
\end{aligned}$$

$$= (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_3)(\alpha_4 + \beta_4)$$

$$= p_1 p_2 p_3 p_4.$$

For the coefficient of x^{14} through x^8, we will only be showing the final form of the coefficient. All the multiplication of the roots, grouping of the terms, factoring of the groups, substitution and simplifying of the coefficient was done with Sage, a computer algebra program. The outcome from Sage can be found in the appendix. Note that because of how Sage works, we denote α_1 as $a1$, β_1 as $b1$, p_1 as $p1$, and q_1 as $q1$ inside Sage. Other subscripts are denoted in the same manner.

Looking at the coefficient of x^{14}, which becomes the coefficient of w_{n+14} in equation (3.4), we have

$$\begin{aligned}\sum_{1\le i<j\le 16} r_i r_j = & -\left(p_2^2p_3^2p_4^2q_1 + p_1^2p_3^2p_4^2q_2 + p_1^2p_2^2p_4^2q_3 + p_1^2p_2^2p_3^2q_4 + 2p_3^2p_4^2q_1q_2 + 2p_2^2p_4^2q_1q_3\right.\\ & + 2p_1^2p_4^2q_2q_3 + 2p_2^2p_3^2q_1q_4 + 2p_1^2p_3^2q_2q_4 + 2p_1^2p_2^2q_3q_4 + 4p_4^2q_1q_2q_3 + 4p_3^2q_1q_2q_4\\ & \left. + 4p_2^2q_1q_3q_4 + 4p_1^2q_2q_3q_4 + 8q_1q_2q_3q_4\right).\end{aligned}$$

Looking at the coefficient of x^{13}, which becomes the coefficient of w_{n+13} in equation (3.4), we have

$$\begin{aligned}\sum_{1\le i<j<k\le 16} r_i r_j r_k = & \, p_1p_2p_3^3p_4^3q_1q_2 + p_1p_2^3p_3p_4^3q_1q_3 + p_1^3p_2p_3p_4^3q_2q_3 + p_1p_2^3p_3^3p_4q_1q_4\\ & + p_1^3p_2p_3^3p_4q_2q_4 + p_1^3p_2^3p_3p_4q_3q_4 + 5p_1p_2p_3p_4^3q_1q_2q_3 + 5p_1p_2p_3^3p_4q_1q_2q_4\\ & + 5p_1p_2^3p_3p_4q_1q_3q_4 + 5p_1^3p_2p_3p_4q_2q_3q_4 + 19p_1p_2p_3p_4q_1q_2q_3q_4.\end{aligned}$$

Looking at the coefficient of x^{12}, which becomes the coefficient of w_{n+12} in equation (3.4), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_4\le 16} r_{i_1}\cdots r_{i_4} = & \, p_3^4p_4^4q_1^2q_2^2 + p_2^4p_4^4q_1^2q_3^2 + p_1^4p_4^4q_2^2q_3^2 + p_2^4p_3^4q_1^2q_4^2 + p_1^4p_3^4q_2^2q_4^2 + p_1^4p_2^4q_3^2q_4^2\\ & - p_1^2p_2^2p_3^2p_4^4q_1q_2q_3 - p_1^2p_2^2p_3^4p_4^2q_1q_2q_4 - p_1^2p_2^4p_3^2p_4^2q_1q_3q_4 - p_1^4p_2^2p_3^2p_4^2q_2q_3q_4\\ & + 4p_3^2p_4^4q_1^2q_2^2q_3 + 4p_2^2p_4^4q_1^2q_2q_3^2 + 4p_1^2p_4^4q_1q_2^2q_3^2 + 4p_3^4p_4^2q_1^2q_2^2q_4 + 4p_2^4p_4^2q_1^2q_3^2q_4\\ & + 4p_1^4p_4^2q_2^2q_3^2q_4 + 4p_2^2p_3^4q_1^2q_2q_4^2 + 4p_1^2p_3^4q_1q_2^2q_4^2 + 4p_2^4p_3^2q_1^2q_3q_4^2 + 4p_1^4p_3^2q_2^2q_3q_4^2\\ & + 4p_1^2p_2^4q_1q_3^2q_4^2 + 4p_1^4p_2^2q_2q_3^2q_4^2 + 6p_4^4q_1^2q_2^2q_3^2 + 6p_3^4q_1^2q_2^2q_4^2 + 6p_2^4q_1^2q_3^2q_4^2 + 6p_1^4q_2^2q_3^2q_4^2\\ & - 9p_1^2p_2^2p_3^2p_4^2q_1q_2q_3q_4 + 16p_3^2p_4^2q_1^2q_2^2q_3q_4 + 16p_2^2p_4^2q_1^2q_2q_3^2q_4 + 16p_1^2p_4^2q_1q_2^2q_3^2q_4\\ & + 16p_2^2p_3^2q_1^2q_2q_3q_4^2 + 16p_1^2p_3^2q_1q_2^2q_3q_4^2 + 16p_1^2p_2^2q_1q_2q_3^2q_4^2 + 24p_4^2q_1^2q_2^2q_3^2q_4\\ & + 24p_3^2q_1^2q_2^2q_3q_4^2 + 24p_2^2q_1^2q_2q_3^2q_4^2 + 24p_1^2q_1q_2^2q_3^2q_4^2 + 28q_1^2q_2^2q_3^2q_4^2.\end{aligned}$$

Looking at the coefficient of x^{11}, which becomes the coefficient of w_{n+11} in equation (3.4), we have

$$
\begin{aligned}
\sum_{1\le i_1<\cdots<i_5\le 16} r_{i_1}\cdots r_{i_5} =& p_1^3p_2^3p_3^3p_4^3q_1q_2q_3q_4 - p_1p_2p_3^3p_4^5q_1^2q_2^2q_3 - p_1p_2^3p_3p_4^5q_1^2q_2q_3^2 - p_1^3p_2p_3p_4^5q_1q_2^2q_3^2 \\
& - p_1p_2p_3^5p_4^3q_1^2q_2^2q_4 - p_1p_2^5p_3p_4^3q_1^2q_3^2q_4 - p_1^5p_2p_3p_4^3q_2^2q_3^2q_4 - p_1p_2^3p_3^5p_4q_1^2q_2q_4^2 \\
& - p_1^3p_2p_3^5p_4q_1q_2^2q_4^2 - p_1p_2^5p_3^3p_4q_1^2q_3q_4^2 - p_1^5p_2p_3^3p_4q_2^2q_3q_4^2 - p_1^3p_2^5p_3p_4q_1q_3^2q_4^2 \\
& - p_1^5p_2^3p_3p_4q_2q_3^2q_4^2 - 5p_1p_2p_3p_4^5q_1^2q_2^2q_3^2 - 5p_1p_2p_3^5p_4q_1^2q_2^2q_4^2 - 5p_1p_2^5p_3p_4q_1^2q_3^2q_4^2 \\
& - 5p_1^5p_2p_3p_4q_2^2q_3^2q_4^2 - 9p_1p_2p_3^3p_4^3q_1^2q_2^2q_3q_4 - 9p_1p_2^3p_3p_4^3q_1^2q_2q_3^2q_4 \\
& - 9p_1^3p_2p_3p_4^3q_1q_2^2q_3^2q_4 - 9p_1p_2^3p_3^3p_4q_1^2q_2q_3q_4^2 - 9p_1^3p_2p_3^3p_4q_1q_2^2q_3q_4^2 \\
& - 9p_1^3p_2^3p_3p_4q_1q_2q_3^2q_4^2 - 31p_1p_2p_3p_4^3q_1^2q_2^2q_3^2q_4 - 31p_1p_2p_3^3p_4q_1^2q_2^2q_3q_4^2 \\
& - 31p_1p_2^3p_3p_4q_1^2q_2q_3^2q_4^2 - 31p_1^3p_2p_3p_4q_1q_2^2q_3^2q_4^2 - 63p_1p_2p_3p_4q_1^2q_2^2q_3^2q_4^2.
\end{aligned}
$$

Looking at the coefficient of x^{10}, which becomes the coefficient of w_{n+10} in equation (3.4), we have

$$
\begin{aligned}
\sum_{1\le i_1<\cdots<i_6\le 16} r_{i_1}\cdots r_{i_6} =& p_1^2p_2^2p_3^4p_4^4q_1^2q_2^2q_3q_4 + p_1^2p_2^4p_3^2p_4^4q_1^2q_2q_3^2q_4 + p_1^4p_2^2p_3^2p_4^4q_1q_2^2q_3^2q_4 + p_1^2p_2^4p_3^4p_4^2q_1^2q_2q_3q_4^2 \\
& + p_1^4p_2^2p_3^4p_4^2q_1q_2^2q_3q_4^2 + p_1^4p_2^4p_3^2p_4^2q_1q_2q_3^2q_4^2 - p_2^2p_3^2p_4^6q_1^3q_2^2q_3^2 - p_1^2p_3^2p_4^6q_1^2q_2^3q_3^2 \\
& - p_1^2p_2^2p_4^6q_1^2q_2^2q_3^3 - p_2^2p_3^6p_4^2q_1^3q_2^2q_4^2 - p_1^2p_3^6p_4^2q_1^2q_2^3q_4^2 - p_2^6p_3^2p_4^2q_1^3q_3^2q_4^2 - p_1^6p_3^2p_4^2q_2^3q_3^2q_4^2 \\
& - p_1^2p_2^6p_4^2q_1^2q_3^3q_4^2 - p_1^6p_2^2p_4^2q_2^2q_3^3q_4^2 - p_1^2p_2^2p_3^6q_1^2q_2^2q_4^3 - p_1^2p_2^6p_3^2q_1^2q_3^2q_4^3 - p_1^6p_2^2p_3^2q_2^2q_3^2q_4^3 \\
& - 2p_3^2p_4^6q_1^3q_2^3q_3^2 - 2p_2^2p_4^6q_1^3q_2^2q_3^3 - 2p_1^2p_4^6q_1^2q_2^3q_3^3 - 2p_3^6p_4^2q_1^3q_2^3q_4^2 - 2p_2^6p_4^2q_1^3q_3^3q_4^2 \\
& - 2p_1^6p_4^2q_2^3q_3^3q_4^2 - 2p_2^2p_3^6q_1^3q_2^2q_4^3 - 2p_1^2p_3^6q_1^2q_2^3q_4^3 - 2p_2^6p_3^2q_1^3q_3^2q_4^3 - 2p_1^6p_3^2q_2^3q_3^2q_4^3 \\
& - 2p_1^2p_2^6q_1^2q_3^3q_4^3 - 2p_1^6p_2^2q_2^2q_3^3q_4^3 - 4p_4^6q_1^3q_2^3q_3^3 - 4p_3^6q_1^3q_2^3q_4^3 - 4p_2^6q_1^3q_3^3q_4^3 - 4p_1^6q_2^3q_3^3q_4^3 \\
& + 5p_1^2p_2^2p_3^2p_4^4q_1^2q_2^2q_3^2q_4 + 5p_1^2p_2^2p_3^4p_4^2q_1^2q_2^2q_3q_4^2 + 5p_1^2p_2^4p_3^2p_4^2q_1^2q_2q_3^2q_4^2 \\
& + 5p_1^4p_2^2p_3^2p_4^2q_1q_2^2q_3^2q_4^2 - 6p_2^2p_3^2p_4^4q_1^3q_2^2q_3^2q_4 - 6p_1^2p_3^2p_4^4q_1^2q_2^3q_3^2q_4 - 6p_1^2p_2^2p_4^4q_1^2q_2^2q_3^3q_4 \\
& - 6p_2^2p_3^4p_4^2q_1^3q_2^2q_3q_4^2 - 6p_1^2p_3^4p_4^2q_1^2q_2^3q_3q_4^2 - 6p_2^4p_3^2p_4^2q_1^3q_2q_3^2q_4^2 - 6p_1^4p_3^2p_4^2q_1q_2^3q_3^2q_4^2 \\
& - 6p_1^2p_2^4p_4^2q_1^2q_2q_3^3q_4^2 - 6p_1^4p_2^2p_4^2q_1q_2^2q_3^3q_4^2 - 6p_1^2p_2^2p_3^4q_1^2q_2^2q_3q_4^3 - 6p_1^2p_2^4p_3^2q_1^2q_2q_3^2q_4^3 \\
& - 6p_1^4p_2^2p_3^2q_1q_2^2q_3^2q_4^3 - 12p_3^2p_4^4q_1^3q_2^3q_3^2q_4 - 12p_2^2p_4^4q_1^3q_2^2q_3^3q_4 - 12p_1^2p_4^4q_1^2q_2^3q_3^3q_4 \\
& - 12p_3^4p_4^2q_1^3q_2^3q_3q_4^2 - 12p_2^4p_4^2q_1^3q_2q_3^3q_4^2 - 12p_1^4p_4^2q_1q_2^3q_3^3q_4^2 - 12p_2^2p_3^4q_1^3q_2^2q_3q_4^3 \\
& - 12p_1^2p_3^4q_1^2q_2^3q_3q_4^3 - 12p_2^4p_3^2q_1^3q_2q_3^2q_4^3 - 12p_1^4p_3^2q_1q_2^3q_3^2q_4^3 - 12p_1^2p_2^4q_1^2q_2q_3^3q_4^3
\end{aligned}
$$

$$
\begin{aligned}
&- 12p_1^4p_2^2q_1q_2^2q_3^3q_4^3 + 12p_1^2p_2^2p_3^2p_4^2q_1^2q_2^2q_3^2q_4^2 - 24p_4^4q_1^3q_2^3q_3^3q_4 - 24p_3^4q_1^3q_2^3q_3q_4^3 \\
&- 24p_2^4q_1^3q_2q_3^3q_4^3 - 24p_1^4q_1q_2^3q_3^3q_4^3 - 31p_2^2p_3^2p_4^2q_1^3q_2^2q_3^2q_4^2 - 31p_1^2p_3^2p_4^2q_1^2q_2^3q_3^2q_4^2 \\
&- 31p_1^2p_2^2p_4^2q_1^2q_2^2q_3^3q_4^2 - 31p_1^2p_2^2p_3^2q_1^2q_2^2q_3^2q_4^3 - 46p_3^2p_4^2q_1^3q_2^3q_3^2q_4^2 - 46p_2^2p_4^2q_1^3q_2^2q_3^3q_4^2 \\
&- 46p_1^2p_4^2q_1^2q_2^3q_3^3q_4^2 - 46p_2^2p_3^2q_1^3q_2^2q_3^2q_4^3 - 46p_1^2p_3^2q_1^2q_2^3q_3^2q_4^3 - 46p_1^2p_2^2q_1^2q_2^2q_3^3q_4^3 \\
&- 60p_4^2q_1^3q_2^3q_3^3q_4^2 - 60p_3^2q_1^3q_2^3q_3^2q_4^3 - 60p_2^2q_1^3q_2^2q_3^3q_4^3 - 60p_1^2q_1^2q_2^3q_3^3q_4^3 - 56q_1^3q_2^3q_3^3q_4^3.
\end{aligned}
$$

Looking at the coefficient of x^9, which becomes the coefficient of w_{n+9} in equation (3.4), we have

$$
\begin{aligned}
\sum_{1\leq i_1<\cdots<i_7\leq 16} r_{i_1}\cdots r_{i_7} =& p_1p_2^3p_3^3p_4^5q_1^3q_2^2q_3^2q_4 + p_1^3p_2p_3^3p_4^5q_1^2q_2^3q_3^2q_4 + p_1^3p_2^3p_3p_4^5q_1^2q_2^2q_3^3q_4 + p_1p_2^3p_3^5p_4^3q_1^3q_2^2q_3q_4^2 \\
&+ p_1^3p_2p_3^5p_4^3q_1^2q_2^3q_3q_4^2 + p_1p_2^5p_3^3p_4^3q_1^3q_2q_3^2q_4^2 + p_1^5p_2p_3^3p_4^3q_1q_2^3q_3^2q_4^2 + p_1^3p_2^5p_3p_4^3q_1^2q_2q_3^3q_4^2 \\
&+ p_1^5p_2^3p_3p_4^3q_1q_2^2q_3^3q_4^2 + p_1^3p_2^3p_3^5p_4q_1^2q_2^2q_3q_4^3 + p_1^3p_2^5p_3^3p_4q_1^2q_2q_3^2q_4^3 + p_1^5p_2^3p_3^3p_4q_1q_2^2q_3^2q_4^3 \\
&- p_1p_2p_3p_4^7q_1^3q_2^3q_3^3 - p_1p_2p_3^7p_4q_1^3q_2^3q_4^3 - p_1p_2^7p_3p_4q_1^3q_3^3q_4^3 - p_1^7p_2p_3p_4q_2^3q_3^3q_4^3 \\
&+ 2p_1p_2p_3^3p_4^5q_1^3q_2^3q_3^2q_4 + 2p_1p_2^3p_3p_4^5q_1^3q_2^2q_3^3q_4 + 2p_1^3p_2p_3p_4^5q_1^2q_2^3q_3^3q_4 \\
&+ 2p_1p_2p_3^5p_4^3q_1^3q_2^3q_3q_4^2 + 2p_1p_2^5p_3p_4^3q_1^3q_2q_3^3q_4^2 + 2p_1^5p_2p_3p_4^3q_1q_2^3q_3^3q_4^2 \\
&+ 2p_1p_2^3p_3^5p_4q_1^3q_2^2q_3q_4^3 + 2p_1^3p_2p_3^5p_4q_1^2q_2^3q_3q_4^3 + 2p_1p_2^5p_3^3p_4q_1^3q_2q_3^2q_4^3 \\
&+ 2p_1^5p_2p_3^3p_4q_1q_2^3q_3^2q_4^3 + 2p_1^3p_2^5p_3p_4q_1^2q_2q_3^3q_4^3 + 2p_1^5p_2^3p_3p_4q_1q_2^2q_3^3q_4^3 \\
&- 3p_1p_2p_3p_4^5q_1^3q_2^3q_3^3q_4 - 3p_1p_2p_3^5p_4q_1^3q_2^3q_3q_4^3 - 3p_1p_2^5p_3p_4q_1^3q_2q_3^3q_4^3 \\
&- 3p_1^5p_2p_3p_4q_1q_2^3q_3^3q_4^3 + 3p_1^3p_2^3p_3^3p_4^3q_1^2q_2^2q_3^2q_4^2 + 14p_1p_2^3p_3^3p_4^3q_1^3q_2^2q_3^2q_4^2 \\
&+ 14p_1^3p_2p_3^3p_4^3q_1^2q_2^3q_3^2q_4^2 + 14p_1^3p_2^3p_3p_4^3q_1^2q_2^2q_3^3q_4^2 + 14p_1^3p_2^3p_3^3p_4q_1^2q_2^2q_3^2q_4^3 \\
&+ 24p_1p_2p_3^3p_4^3q_1^3q_2^3q_3^2q_4^2 + 24p_1p_2^3p_3p_4^3q_1^3q_2^2q_3^3q_4^2 + 24p_1^3p_2p_3p_4^3q_1^2q_2^3q_3^3q_4^2 \\
&+ 24p_1p_2^3p_3^3p_4q_1^3q_2^2q_3^2q_4^3 + 24p_1^3p_2p_3^3p_4q_1^2q_2^3q_3^2q_4^3 + 24p_1^3p_2^3p_3p_4q_1^2q_2^2q_3^3q_4^3 \\
&+ 26p_1p_2p_3p_4^3q_1^3q_2^3q_3^3q_4^2 + 26p_1p_2p_3^3p_4q_1^3q_2^3q_3^2q_4^3 + 26p_1p_2^3p_3p_4q_1^3q_2^2q_3^3q_4^3 \\
&+ 26p_1^3p_2p_3p_4q_1^2q_2^3q_3^3q_4^3 + 43p_1p_2p_3p_4q_1^3q_2^3q_3^3q_4^3.
\end{aligned}
$$

Looking at the coefficient of x^8, which becomes the coefficient of w_{n+8} in equation (3.4), we have

$$
\begin{aligned}
\sum_{1\leq i_1<\cdots<i_8\leq 16} r_{i_1}\cdots r_{i_8} =& p_4^8q_1^4q_2^4q_3^4 + p_3^8q_1^4q_2^4q_4^4 + p_2^8q_1^4q_3^4q_4^4 + p_1^8q_2^4q_3^4q_4^4 + p_2^4p_3^4p_4^4q_1^4q_2^2q_3^2q_4^2 \\
&+ p_1^4p_3^4p_4^4q_1^2q_2^4q_3^2q_4^2 + p_1^4p_2^4p_4^4q_1^2q_2^2q_3^4q_4^2 + p_1^4p_2^4p_3^4q_1^2q_2^2q_3^2q_4^4 + p_1^2p_2^2p_3^2p_4^6q_1^3q_2^3q_3^3q_4
\end{aligned}
$$

$$+ p_1^2p_2^2p_3^6p_4^2q_1^3q_2^3q_3q_4^3 + p_1^2p_2^6p_3^2p_4^2q_1^3q_2q_3^3q_4^3 + p_1^6p_2^2p_3^2p_4^2q_1q_2^3q_3^3q_4^3$$

$$+ 2p_1^2p_2^2p_3^4p_4^4q_1^3q_2^3q_3^2q_4^2 + 2p_1^2p_2^4p_3^2p_4^4q_1^3q_2^2q_3^3q_4^2 + 2p_1^4p_2^2p_3^2p_4^4q_1^2q_2^3q_3^3q_4^2$$

$$+ 2p_1^2p_2^4p_3^4p_4^2q_1^3q_2^2q_3^2q_4^3 + 2p_1^4p_2^2p_3^4p_4^2q_1^2q_2^3q_3^2q_4^3 + 2p_1^4p_2^4p_3^2p_4^2q_1^2q_2^2q_3^3q_4^3$$

$$+ 4p_2^2p_3^4p_4^4q_1^4q_2^3q_3^2q_4^2 + 4p_1^2p_3^4p_4^4q_1^3q_2^4q_3^2q_4^2 + 4p_2^4p_3^2p_4^4q_1^4q_2^2q_3^3q_4^2 + 4p_1^4p_3^2p_4^4q_1^2q_2^4q_3^3q_4^2$$

$$+ 4p_1^2p_2^4p_4^4q_1^3q_2^2q_3^4q_4^2 + 4p_1^4p_2^2p_4^4q_1^2q_2^3q_3^4q_4^2 + 4p_2^4p_3^4p_4^2q_1^4q_2^2q_3^2q_4^3 + 4p_1^4p_3^4p_4^2q_1^2q_2^4q_3^2q_4^3$$

$$+ 4p_1^4p_2^4p_4^2q_1^2q_2^2q_3^4q_4^3 + 4p_1^2p_2^4p_3^4q_1^3q_2^2q_3^2q_4^4 + 4p_1^4p_2^2p_3^4q_1^2q_2^3q_3^2q_4^4 + 4p_1^4p_2^4p_3^2q_1^2q_2^2q_3^3q_4^4$$

$$+ 4p_3^4p_4^4q_1^4q_2^4q_3^2q_4^2 + 4p_2^4p_4^4q_1^4q_2^2q_3^4q_4^2 + 4p_1^4p_4^4q_1^2q_2^4q_3^4q_4^2 + 4p_2^4p_3^4q_1^4q_2^2q_3^2q_4^4$$

$$+ 4p_1^4p_3^4q_1^2q_2^4q_3^2q_4^4 + 4p_1^4p_2^4q_1^2q_2^2q_3^4q_4^4 + 8p_4^6q_1^4q_2^4q_3^4q_4 + 8p_3^6q_1^4q_2^4q_3q_4^4 + 8p_2^6q_1^4q_2q_3^4q_4^4$$

$$+ 8p_1^6q_1q_2^4q_3^4q_4^4 + 16p_2^2p_3^2p_4^4q_1^4q_2^3q_3^3q_4^2 + 16p_1^2p_3^2p_4^4q_1^3q_2^4q_3^3q_4^2 + 16p_1^2p_2^2p_4^4q_1^3q_2^3q_3^4q_4^2$$

$$+ 16p_2^2p_3^4p_4^2q_1^4q_2^3q_3^2q_4^3 + 16p_1^2p_3^4p_4^2q_1^3q_2^4q_3^2q_4^3 + 16p_2^4p_3^2p_4^2q_1^4q_2^2q_3^3q_4^3$$

$$+ 16p_1^4p_3^2p_4^2q_1^2q_2^4q_3^3q_4^3 + 16p_1^2p_2^4p_4^2q_1^3q_2^2q_3^4q_4^3 + 16p_1^4p_2^2p_4^2q_1^2q_2^3q_3^4q_4^3$$

$$+ 16p_1^2p_2^2p_3^4q_1^3q_2^3q_3^2q_4^4 + 16p_1^2p_2^4p_3^2q_1^3q_2^2q_3^3q_4^4 + 16p_1^4p_2^2p_3^2q_1^2q_2^3q_3^3q_4^4$$

$$+ 16p_3^2p_4^4q_1^4q_2^4q_3^3q_4^2 + 16p_2^2p_4^4q_1^4q_2^3q_3^4q_4^2 + 16p_1^2p_4^4q_1^3q_2^4q_3^4q_4^2 + 16p_3^4p_4^2q_1^4q_2^4q_3^2q_4^3$$

$$+ 16p_2^4p_4^2q_1^4q_2^2q_3^4q_4^3 + 16p_1^4p_4^2q_1^2q_2^4q_3^4q_4^3 + 16p_2^2p_3^4q_1^4q_2^3q_3^2q_4^4 + 16p_1^2p_3^4q_1^3q_2^4q_3^2q_4^4$$

$$+ 16p_2^4p_3^2q_1^4q_2^2q_3^3q_4^4 + 16p_1^4p_3^2q_1^2q_2^4q_3^3q_4^4 + 16p_1^2p_2^4q_1^3q_2^2q_3^4q_4^4 + 16p_1^4p_2^2q_1^2q_2^3q_3^4q_4^4$$

$$+ 18p_1^2p_2^2p_3^2p_4^4q_1^3q_2^3q_3^3q_4^2 + 18p_1^2p_2^2p_3^4p_4^2q_1^3q_2^3q_3^2q_4^3 + 18p_1^2p_2^4p_3^2p_4^2q_1^3q_2^2q_3^3q_4^3$$

$$+ 18p_1^4p_2^2p_3^2p_4^2q_1^2q_2^3q_3^3q_4^3 + 82p_1^2p_2^2p_3^2p_4^2q_1^3q_2^3q_3^3q_4^3 + 36p_4^4q_1^4q_2^4q_3^4q_4^2 + 36p_3^4q_1^4q_2^4q_3^2q_4^4$$

$$+ 36p_2^4q_1^4q_2^2q_3^4q_4^4 + 36p_1^4q_1^2q_2^4q_3^4q_4^4 + 64p_2^2p_3^2p_4^2q_1^4q_2^3q_3^3q_4^3 + 64p_1^2p_3^2p_4^2q_1^3q_2^4q_3^3q_4^3$$

$$+ 64p_1^2p_2^2p_4^2q_1^3q_2^3q_3^4q_4^3 + 64p_1^2p_2^2p_3^2q_1^3q_2^3q_3^3q_4^4 + 64p_3^2p_4^2q_1^4q_2^4q_3^3q_4^3 + 64p_2^2p_4^2q_1^4q_2^3q_3^4q_4^3$$

$$+ 64p_1^2p_4^2q_1^3q_2^4q_3^4q_4^3 + 64p_2^2p_3^2q_1^4q_2^3q_3^3q_4^4 + 64p_1^2p_3^2q_1^3q_2^4q_3^3q_4^4 + 64p_1^2p_2^2q_1^3q_2^3q_3^4q_4^4$$

$$+ 80p_4^2q_1^4q_2^4q_3^4q_4^3 + 80p_3^2q_1^4q_2^4q_3^3q_4^4 + 80p_2^2q_1^4q_2^3q_3^4q_4^4 + 80p_1^2q_1^3q_2^4q_3^4q_4^4 + 70q_1^4q_2^4q_3^4q_4^4.$$

When $1 \le i_1 < \cdots < i_9 \le 16$, we can show that $r_{i_1} \cdots r_{i_9} = \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4(r_{j_1} \cdots r_{j_7})$ where $r_{j_1}, \ldots, r_{j_7} \in \{r_{i_1}, \ldots, r_{i_9}\}$. For each $r_{i_1} \cdots r_{i_9}$, there exists $r_s, r_t \in \{r_{i_1}, \ldots, r_{i_9}\}$, such that $r_s r_t = \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4$. This means $r_{i_1} \cdots r_{i_9} = r_s r_t (r_{j_1} \cdots r_{j_7}) = \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4(r_{j_1} \cdots r_{j_7})$. For example, if we take $r_1 \cdots r_9$, we can see that $r_8 r_9 = \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4$, which means $r_1 \cdots r_9 = \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4(r_1 \cdots r_7)$.

Thus, looking at the coefficient of x^7, which becomes the coefficient of w_{n+7} in equation (3.4), we have

$$\begin{aligned}
\sum_{1\le i_1<\cdots<i_9\le 16} r_{i_1}\cdots r_{i_9} =& \alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4\left(\sum_{1\le j_1<\cdots<j_7\le 16} r_{j_1}\cdots r_{j_7}\right)\\
=& q_1q_2q_3q_4\left(p_1p_2^3p_3^3p_4^5q_1^3q_2^2q_3^2q_4 + p_1^3p_2p_3^3p_4^5q_1^2q_2^3q_3^2q_4 + p_1^3p_2^3p_3p_4^5q_1^2q_2^2q_3^3q_4\right.\\
&+ p_1p_2^3p_3^5p_4^3q_1^3q_2^2q_3q_4^2 + p_1^3p_2p_3^5p_4^3q_1^2q_2^3q_3q_4^2 + p_1p_2^5p_3^3p_4^3q_1^3q_2q_3^2q_4^2\\
&+ p_1^5p_2p_3^3p_4^3q_1q_2^3q_3^2q_4^2 + p_1^3p_2^5p_3p_4^3q_1^2q_2q_3^3q_4^2 + p_1^5p_2^3p_3p_4^3q_1q_2^2q_3^3q_4^2\\
&+ p_1^3p_2^3p_3^5p_4q_1^2q_2^2q_3q_4^3 + p_1^3p_2^5p_3^3p_4q_1^2q_2q_3^2q_4^3 + p_1^5p_2^3p_3^3p_4q_1q_2^2q_3^2q_4^3\\
&- p_1p_2p_3p_4^7q_1^3q_2^3q_3^3 - p_1p_2p_3^7p_4q_1^3q_2^3q_4^3 - p_1p_2^7p_3p_4q_1^3q_3^3q_4^3 - p_1^7p_2p_3p_4q_2^3q_3^3q_4^3\\
&+ 2p_1p_2p_3^3p_4^5q_1^3q_2^3q_3^2q_4 + 2p_1p_2^3p_3p_4^5q_1^3q_2^2q_3^3q_4 + 2p_1^3p_2p_3p_4^5q_1^2q_2^3q_3^3q_4\\
&+ 2p_1p_2p_3^5p_4^3q_1^3q_2^3q_3q_4^2 + 2p_1p_2^5p_3p_4^3q_1^3q_2q_3^3q_4^2 + 2p_1^5p_2p_3p_4^3q_1q_2^3q_3^3q_4^2\\
&+ 2p_1p_2^3p_3^5p_4q_1^3q_2^2q_3q_4^3 + 2p_1^3p_2p_3^5p_4q_1^2q_2^3q_3q_4^3 + 2p_1p_2^5p_3^3p_4q_1^3q_2q_3^2q_4^3\\
&+ 2p_1^5p_2p_3^3p_4q_1q_2^3q_3^2q_4^3 + 2p_1^3p_2^5p_3p_4q_1^2q_2q_3^3q_4^3 + 2p_1^5p_2^3p_3p_4q_1q_2^2q_3^3q_4^3\\
&- 3p_1p_2p_3p_4^5q_1^3q_2^3q_3^3q_4 - 3p_1p_2p_3^5p_4q_1^3q_2^3q_3q_4^3 - 3p_1p_2^5p_3p_4q_1^3q_2q_3^3q_4^3\\
&- 3p_1^5p_2p_3p_4q_1q_2^3q_3^3q_4^3 + 3p_1^3p_2^3p_3^3p_4^3q_1^2q_2^2q_3^2q_4^2 + 14p_1p_2^3p_3^3p_4^3q_1^3q_2^2q_3^2q_4^2\\
&+ 14p_1^3p_2p_3^3p_4^3q_1^2q_2^3q_3^2q_4^2 + 14p_1^3p_2^3p_3p_4^3q_1^2q_2^2q_3^3q_4^2 + 14p_1^3p_2^3p_3^3p_4q_1^2q_2^2q_3^2q_4^3\\
&+ 24p_1p_2p_3^3p_4^3q_1^3q_2^3q_3^2q_4^2 + 24p_1p_2^3p_3p_4^3q_1^3q_2^2q_3^3q_4^2 + 24p_1^3p_2p_3p_4^3q_1^2q_2^3q_3^3q_4^2\\
&+ 24p_1p_2^3p_3^3p_4q_1^3q_2^2q_3^2q_4^3 + 24p_1^3p_2p_3^3p_4q_1^2q_2^3q_3^2q_4^3 + 24p_1^3p_2^3p_3p_4q_1^2q_2^2q_3^3q_4^3\\
&+ 26p_1p_2p_3p_4^3q_1^3q_2^3q_3^3q_4^2 + 26p_1p_2p_3^3p_4q_1^3q_2^3q_3^2q_4^3 + 26p_1p_2^3p_3p_4q_1^3q_2^2q_3^3q_4^3\\
&\left.+26p_1^3p_2p_3p_4q_1^2q_2^3q_3^3q_4^3 + 43p_1p_2p_3p_4q_1^3q_2^3q_3^3q_4^3\right).
\end{aligned}$$

Since we calculated $\sum_{1\le j_1<\cdots<j_7\le 16} r_{j_1}\cdots r_{j_7}$ as the coefficient of x^9 above, we can just replace it here.

When $1 \le i_1 < \cdots < i_{10} \le 16$, we can show that $r_{i_1}\cdots r_{i_{10}} = \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2(r_{j_1}\cdots r_{j_6})$ where $r_{j_1},\dots,r_{j_6} \in \{r_{i_1},\dots,r_{i_{10}}\}$. For each $r_{i_1}\cdots r_{i_{10}}$, there exists $r_{s_1},\dots,r_{s_4} \in \{r_{i_1},\dots,r_{i_{10}}\}$, such that $r_{s_1}\cdots r_{s_4}$ $= \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2$. This means $r_{i_1}\cdots r_{i_{10}} = r_{s_1}\cdots r_{s_4}(r_{j_1}\cdots r_{j_6}) = \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2(r_{j_1}\cdots r_{j_6})$. For example, if we take $r_1\cdots r_{10}$, then we can see that $r_7r_8r_9r_{10} = \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2$, which means

$r_1 \cdots r_{10} = \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2(r_1 \cdots r_6)$.

Thus, looking at the coefficient of x^6, which becomes the coefficient of w_{n+6} in equation (3.4), we have

$$
\begin{aligned}
\sum_{1\leq i_1<\cdots<i_{10}\leq 16} r_{i_1}\cdots r_{i_{10}} =& \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\beta_1^2\beta_2^2\beta_3^2\beta_4^2\left(\sum_{1\leq j_1<\cdots<j_6\leq 16} r_{j_1}\cdots r_{j_6}\right)\\
=& q_1^2q_2^2q_3^2q_4^2\,(p_1^2p_2^2p_3^4p_4^4q_1^2q_2^2q_3q_4 + p_1^2p_2^4p_3^2p_4^4q_1^2q_2q_3^2q_4 + p_1^4p_2^2p_3^2p_4^4q_1q_2^2q_3^2q_4\\
&+ p_1^2p_2^4p_3^4p_4^2q_1^2q_2q_3q_4^2 + p_1^4p_2^2p_3^4p_4^2q_1q_2^2q_3q_4^2 + p_1^4p_2^4p_3^2p_4^2q_1q_2q_3^2q_4^2 - p_2^2p_3^2p_4^6q_1^3q_2^2q_3^2\\
&- p_1^2p_3^2p_4^6q_1^2q_2^3q_3^2 - p_1^2p_2^2p_4^6q_1^2q_2^2q_3^3 - p_2^2p_3^6p_4^2q_1^3q_2^2q_4^2 - p_1^2p_3^6p_4^2q_1^2q_2^3q_4^2\\
&- p_2^6p_3^2p_4^2q_1^3q_3^2q_4^2 - p_1^6p_3^2p_4^2q_2^3q_3^2q_4^2 - p_1^2p_2^6p_4^2q_1^2q_3^3q_4^2 - p_1^6p_2^2p_4^2q_2^2q_3^3q_4^2\\
&- p_1^2p_2^2p_3^6q_1^2q_2^2q_4^3 - p_1^2p_2^6p_3^2q_1^2q_3^2q_4^3 - p_1^6p_2^2p_3^2q_2^2q_3^2q_4^3 - 2p_3^2p_4^6q_1^3q_2^3q_3^2 - 2p_2^2p_4^6q_1^3q_2^2q_3^3\\
&- 2p_1^2p_4^6q_1^2q_2^3q_3^3 - 2p_3^6p_4^2q_1^3q_2^3q_4^2 - 2p_2^6p_4^2q_1^3q_3^3q_4^2 - 2p_1^6p_4^2q_2^3q_3^3q_4^2 - 2p_2^2p_3^6q_1^3q_2^2q_4^3\\
&- 2p_1^2p_3^6q_1^2q_2^3q_4^3 - 2p_2^6p_3^2q_1^3q_3^2q_4^3 - 2p_1^6p_3^2q_2^3q_3^2q_4^3 - 2p_1^2p_2^6q_1^2q_3^3q_4^3 - 2p_1^6p_2^2q_2^2q_3^3q_4^3\\
&- 4p_4^6q_1^3q_2^3q_3^3 - 4p_3^6q_1^3q_2^3q_4^3 - 4p_2^6q_1^3q_3^3q_4^3 - 4p_1^6q_2^3q_3^3q_4^3 + 5p_1^2p_2^2p_3^2p_4^4q_1^2q_2^2q_3^2q_4\\
&+ 5p_1^2p_2^2p_3^4p_4^2q_1^2q_2^2q_3q_4^2 + 5p_1^2p_2^4p_3^2p_4^2q_1^2q_2q_3^2q_4^2 + 5p_1^4p_2^2p_3^2p_4^2q_1q_2^2q_3^2q_4^2\\
&- 6p_2^2p_3^2p_4^4q_1^3q_2^2q_3^2q_4 - 6p_1^2p_3^2p_4^4q_1^2q_2^3q_3^2q_4 - 6p_1^2p_2^2p_4^4q_1^2q_2^2q_3^3q_4 - 6p_2^2p_3^4p_4^2q_1^3q_2^2q_3q_4^2\\
&- 6p_1^2p_3^4p_4^2q_1^2q_2^3q_3q_4^2 - 6p_2^4p_3^2p_4^2q_1^3q_2q_3^2q_4^2 - 6p_1^4p_3^2p_4^2q_1q_2^3q_3^2q_4^2 - 6p_1^2p_2^4p_4^2q_1^2q_2q_3^3q_4^2\\
&- 6p_1^4p_2^2p_4^2q_1q_2^2q_3^3q_4^2 - 6p_1^2p_2^2p_3^4q_1^2q_2^2q_3q_4^3 - 6p_1^2p_2^4p_3^2q_1^2q_2q_3^2q_4^3 - 6p_1^4p_2^2p_3^2q_1q_2^2q_3^2q_4^3\\
&- 12p_3^2p_4^4q_1^3q_2^3q_3^2q_4 - 12p_2^2p_4^4q_1^3q_2^2q_3^3q_4 - 12p_1^2p_4^4q_1^2q_2^3q_3^3q_4 - 12p_3^4p_4^2q_1^3q_2^3q_3q_4^2\\
&- 12p_2^4p_4^2q_1^3q_2q_3^3q_4^2 - 12p_1^4p_4^2q_1q_2^3q_3^3q_4^2 - 12p_2^2p_3^4q_1^3q_2^2q_3q_4^3 - 12p_1^2p_3^4q_1^2q_2^3q_3q_4^3\\
&- 12p_2^4p_3^2q_1^3q_2q_3^2q_4^3 - 12p_1^4p_3^2q_1q_2^3q_3^2q_4^3 - 12p_1^2p_2^4q_1^2q_2q_3^3q_4^3 - 12p_1^4p_2^2q_1q_2^2q_3^3q_4^3\\
&+ 12p_1^2p_2^2p_3^2p_4^2q_1^2q_2^2q_3^2q_4^2 - 24p_4^4q_1^3q_2^3q_3^3q_4 - 24p_3^4q_1^3q_2^3q_3q_4^3 - 24p_2^4q_1^3q_2q_3^3q_4^3\\
&- 24p_1^4q_1q_2^3q_3^3q_4^3 - 31p_2^2p_3^2p_4^2q_1^3q_2^2q_3^2q_4^2 - 31p_1^2p_3^2p_4^2q_1^2q_2^3q_3^2q_4^2 - 31p_1^2p_2^2p_4^2q_1^2q_2^2q_3^3q_4^2\\
&- 31p_1^2p_2^2p_3^2q_1^2q_2^2q_3^2q_4^3 - 46p_3^2p_4^2q_1^3q_2^3q_3^2q_4^2 - 46p_2^2p_4^2q_1^3q_2^2q_3^3q_4^2 - 46p_1^2p_4^2q_1^2q_2^3q_3^3q_4^2\\
&- 46p_2^2p_3^2q_1^3q_2^2q_3^2q_4^3 - 46p_1^2p_3^2q_1^2q_2^3q_3^2q_4^3 - 46p_1^2p_2^2q_1^2q_2^2q_3^3q_4^3 - 60p_4^2q_1^3q_2^3q_3^3q_4^2\\
&-60p_3^2q_1^3q_2^3q_3^2q_4^3 - 60p_2^2q_1^3q_2^2q_3^3q_4^3 - 60p_1^2q_1^2q_2^3q_3^3q_4^3 - 56q_1^3q_2^3q_3^3q_4^3\,).
\end{aligned}
$$

Since we calculated $\sum_{1\leq j_1<\cdots<j_6\leq 16} r_{j_1}\cdots r_{j_6}$ as the coefficient of x^{10} above, we can just replace it here.

When $1 \leq i_1 < \cdots < i_{11} \leq 16$, we can show that $r_{i_1} \cdots r_{i_{11}} = \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3(r_{j_1} \cdots r_{j_5})$ where $r_{j_1}, \ldots, r_{j_5} \in \{r_{i_1}, \ldots, r_{i_{11}}\}$. For each $r_{i_1} \cdots r_{i_{11}}$, there exists $r_{s_1}, \ldots, r_{s_6} \in \{r_{i_1}, \ldots, r_{i_{11}}\}$, such that $r_{s_1} \cdots r_{s_6} = \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3$. This means $r_{i_1} \cdots r_{i_{11}} = r_{s_1} \cdots r_{s_6}(r_{j_1} \cdots r_{j_5}) = \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3(r_{j_1} \cdots r_{j_5})$. For example, if we take $r_1 \cdots r_{11}$, then we can see that $r_6r_7r_8r_9r_{10}r_{11} = \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3$, which means $r_1 \cdots r_{11} = \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3(r_1 \cdots r_5)$.

Thus, looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (3.4), we have

$$\begin{aligned}
\sum_{1\leq i_1<\cdots<i_{11}\leq 16} r_{i_1}\cdots r_{i_{11}} =& \alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^3\beta_1^3\beta_2^3\beta_3^3\beta_4^3 \left(\sum_{1\leq j_1<\cdots<j_5\leq 16} r_{j_1}\cdots r_{j_5} \right) \\
=& q_1^3q_2^3q_3^3q_4^3 \left(p_1^3p_2^3p_3^3p_4^3q_1q_2q_3q_4 - p_1p_2p_3^3p_4^5q_1^2q_2^2q_3 - p_1p_2^3p_3p_4^5q_1^2q_2q_3^2 \right. \\
& - p_1^3p_2p_3p_4^5q_1q_2^2q_3^2 - p_1p_2p_3^5p_4^3q_1^2q_2^2q_4 - p_1p_2^5p_3p_4^3q_1^2q_3^2q_4 - p_1^5p_2p_3p_4^3q_2^2q_3^2q_4 \\
& - p_1p_2^3p_3^5p_4q_1^2q_2q_4^2 - p_1^3p_2p_3^5p_4q_1q_2^2q_4^2 - p_1p_2^5p_3^3p_4q_1^2q_3q_4^2 - p_1^5p_2p_3^3p_4q_2^2q_3q_4^2 \\
& - p_1^3p_2^5p_3p_4q_1q_3^2q_4^2 - p_1^5p_2^3p_3p_4q_2q_3^2q_4^2 - 5p_1p_2p_3p_4^5q_1^2q_2^2q_3^2 - 5p_1p_2p_3^5p_4q_1^2q_2^2q_4^2 \\
& - 5p_1p_2^5p_3p_4q_1^2q_3^2q_4^2 - 5p_1^5p_2p_3p_4q_2^2q_3^2q_4^2 - 9p_1p_2p_3^3p_4^3q_1^2q_2^2q_3q_4 - 9p_1p_2^3p_3p_4^3q_1^2q_2q_3^2q_4 \\
& - 9p_1^3p_2p_3p_4^3q_1q_2^2q_3^2q_4 - 9p_1p_2^3p_3^3p_4q_1^2q_2q_3q_4^2 - 9p_1^3p_2p_3^3p_4q_1q_2^2q_3q_4^2 \\
& - 9p_1^3p_2^3p_3p_4q_1q_2q_3^2q_4^2 - 31p_1p_2p_3p_4^3q_1^2q_2^2q_3^2q_4 - 31p_1p_2p_3^3p_4q_1^2q_2^2q_3q_4^2 \\
& \left. -31p_1p_2^3p_3p_4q_1^2q_2q_3^2q_4^2 - 31p_1^3p_2p_3p_4q_1q_2^2q_3^2q_4^2 - 63p_1p_2p_3p_4q_1^2q_2^2q_3^2q_4^2 \right).
\end{aligned}$$

Since we calculated $\sum_{1\leq j_1<\cdots<j_5\leq 16} r_{j_1}\cdots r_{j_5}$ as the coefficient of x^{11} above, we can just replace it here.

When $1 \leq i_1 < \cdots < i_{12} \leq 16$, we can show that $r_{i_1} \cdots r_{i_{12}} = \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4(r_{j_1} \cdots r_{j_4})$ where $r_{j_1}, \ldots, r_{j_4} \in \{r_{i_1}, \ldots, r_{i_{12}}\}$. For each $r_{i_1} \cdots r_{i_{12}}$, there exists $r_{s_1}, \ldots, r_{s_8} \in \{r_{i_1}, \ldots, r_{i_{12}}\}$, such that $r_{s_1} \cdots r_{s_8} = \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4$. This means $r_{i_1} \cdots r_{i_{12}} = r_{s_1} \cdots r_{s_8}(r_{j_1} \cdots r_{j_4}) = \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4(r_{j_1} \cdots r_{j_4})$. For example, if we take $r_1 \cdots r_{12}$, then we can see that $r_5 \cdots r_{12} = \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4$, this means $r_1 \cdots r_{12} = \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4(r_1 \cdots r_4)$.

Thus, looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (3.4), we have

$$\begin{aligned}
\sum_{1\leq i_1<\cdots<i_{12}\leq 16} r_{i_1}\cdots r_{i_{12}} =& \alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4\beta_1^4\beta_2^4\beta_3^4\beta_4^4 \left(\sum_{1\leq j_1<\cdots<j_4\leq 16} r_{j_1}\cdots r_{j_4} \right) \\
=& q_1^4q_2^4q_3^4q_4^4 \left(p_3^4p_4^4q_1^2q_2^2 + p_2^4p_4^4q_1^2q_3^2 + p_1^4p_4^4q_2^2q_3^2 + p_2^4p_3^4q_1^2q_4^2 + p_1^4p_3^4q_2^2q_4^2 + p_1^4p_2^4q_3^2q_4^2 \right. \\
& - p_1^2p_2^2p_3^2p_4^4q_1q_2q_3 - p_1^2p_2^2p_3^4p_4^2q_1q_2q_4 - p_1^2p_2^4p_3^2p_4^2q_1q_3q_4 - p_1^4p_2^2p_3^2p_4^2q_2q_3q_4
\end{aligned}$$

$$+ 4p_3^2p_4^4q_1^2q_2^2q_3 + 4p_2^2p_4^4q_1^2q_2q_3^2 + 4p_1^2p_4^4q_1q_2^2q_3^2 + 4p_3^4p_4^2q_1^2q_2^2q_4 + 4p_2^4p_4^2q_1^2q_3^2q_4$$
$$+ 4p_1^4p_4^2q_2^2q_3^2q_4 + 4p_2^2p_3^4q_1^2q_2q_4^2 + 4p_1^2p_3^4q_1q_2^2q_4^2 + 4p_2^4p_3^2q_1^2q_3q_4^2 + 4p_1^4p_3^2q_2^2q_3q_4^2$$
$$+ 4p_1^2p_2^4q_1q_3^2q_4^2 + 4p_1^4p_2^2q_2q_3^2q_4^2 + 6p_4^4q_1^2q_2^2q_3^2 + 6p_3^4q_1^2q_2^2q_4^2 + 6p_2^4q_1^2q_3^2q_4^2 + 6p_1^4q_2^2q_3^2q_4^2$$
$$- 9p_1^2p_2^2p_3^2p_4^2q_1q_2q_3q_4 + 16p_3^2p_4^2q_1^2q_2^2q_3q_4 + 16p_2^2p_4^2q_1^2q_2q_3^2q_4 + 16p_1^2p_4^2q_1q_2^2q_3^2q_4$$
$$+ 16p_2^2p_3^2q_1^2q_2q_3q_4^2 + 16p_1^2p_3^2q_1q_2^2q_3q_4^2 + 16p_1^2p_2^2q_1q_2q_3^2q_4^2 + 24p_4^2q_1^2q_2^2q_3^2q_4$$
$$+24p_3^2q_1^2q_2^2q_3q_4^2 + 24p_2^2q_1^2q_2q_3^2q_4^2 + 24p_1^2q_1q_2^2q_3^2q_4^2 + 28q_1^2q_2^2q_3^2q_4^2\big).$$

Since we calculated $\sum_{1\le j_1<\cdots<j_4\le 16} r_{j_1}\cdots r_{j_4}$ as the coefficient of x^{12} above, we can just replace it here.

When $1 \le i_1 < \cdots < i_{13} \le 16$, we can show that $r_{i_1}\cdots r_{i_{13}} = \alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5(r_ir_jr_k)$ where $r_i, r_j, r_k \in \{r_{i_1},\ldots,r_{i_{13}}\}$. For each $r_{i_1}\cdots r_{i_{13}}$, there exists $r_{s_1},\ldots,r_{s_{10}} \in \{r_{i_1},\ldots,r_{i_{13}}\}$, such that $r_{s_1}\cdots r_{s_{10}} = \alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5$. This means $r_{i_1}\cdots r_{i_{13}} = r_{s_1}\cdots r_{s_{10}}(r_ir_jr_k) = \alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5(r_ir_jr_k)$. For example, if we take $r_1\cdots r_{13}$, then we can see that $r_4\cdots r_{13} = \alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5$, which means $r_1\cdots r_{13} = \alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5(r_1r_2r_3)$.

Thus, looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (3.4), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_{13}\le 16} r_{i_1}\cdots r_{i_{13}} =&\alpha_1^5\alpha_2^5\alpha_3^5\alpha_4^5\beta_1^5\beta_2^5\beta_3^5\beta_4^5\left(\sum_{1\le i<j<k\le 16} r_ir_jr_k\right)\\ =&q_1^5q_2^5q_3^5q_4^5\big(p_1p_2p_3^3p_4^3q_1q_2 + p_1p_2^3p_3p_4^3q_1q_3 + p_1^3p_2p_3p_4^3q_2q_3 + p_1p_2^3p_3^3p_4q_1q_4\\ &+ p_1^3p_2p_3^3p_4q_2q_4 + p_1^3p_2^3p_3p_4q_3q_4 + 5p_1p_2p_3p_4^3q_1q_2q_3 + 5p_1p_2p_3^3p_4q_1q_2q_4\\ &+5p_1p_2^3p_3p_4q_1q_3q_4 + 5p_1^3p_2p_3p_4q_2q_3q_4 + 19p_1p_2p_3p_4q_1q_2q_3q_4\big).\end{aligned}$$

Since we calculated $\sum_{1\le i<j<k\le 16} r_ir_jr_k$ as the coefficient of x^{13} above, we can just replace it here.

When $1 \le i_1 < \cdots < i_{14} \le 16$, we can show that $r_{i_1}\cdots r_{i_{14}} = \alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6(r_ir_j)$ where $r_i, r_j \in \{r_{i_1},\ldots,r_{i_{14}}\}$. For each $r_{i_1}\cdots r_{i_{14}}$, there exists $r_{s_1},\ldots,r_{s_{12}} \in \{r_{i_1},\ldots,r_{i_{14}}\}$, such that $r_{s_1}\cdots r_{s_{12}} = \alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6$. This means $r_{i_1}\cdots r_{i_{14}} = r_{s_1}\cdots r_{s_{12}}(r_ir_j) = \alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6(r_ir_j)$. For example, if we take $r_1\cdots r_{14}$, then we can see that $r_3\cdots r_{14} = \alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6$, which means $r_1\cdots r_{14} = \alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6(r_1r_2)$.

Thus, looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (3.4), we have

$$\sum_{1\le i_1<\cdots<i_{14}\le 16} r_{i_1}\cdots r_{i_{14}} =\alpha_1^6\alpha_2^6\alpha_3^6\alpha_4^6\beta_1^6\beta_2^6\beta_3^6\beta_4^6\left(\sum_{1\le i<j\le 16} r_ir_j\right)$$

$$= -\, q_1^6 q_2^6 q_3^6 q_4^6 \left(p_2^2p_3^2p_4^2q_1 + p_1^2p_3^2p_4^2q_2 + p_1^2p_2^2p_4^2q_3 + p_1^2p_2^2p_3^2q_4 + 2p_3^2p_4^2q_1q_2 \right.$$
$$+ 2p_2^2p_4^2q_1q_3 + 2p_1^2p_4^2q_2q_3 + 2p_2^2p_3^2q_1q_4 + 2p_1^2p_3^2q_2q_4 + 2p_1^2p_2^2q_3q_4 + 4p_4^2q_1q_2q_3$$
$$\left. +4p_3^2q_1q_2q_4 + 4p_2^2q_1q_3q_4 + 4p_1^2q_2q_3q_4 + 8q_1q_2q_3q_4\right).$$

Since we calculated $\sum_{1\le i<j\le 16} r_ir_j$ as the coefficient of x^{14} above, we can just replace it here.

When $1 \le i_1 < \cdots < i_{15} \le 16$, we can show that $r_{i_1}\cdots r_{i_{15}} = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7(r_i)$ where $r_i \in \{r_{i_1},\ldots,r_{i_{15}}\}$. For each $r_{i_1}\cdots r_{i_{15}}$, there exists an $r_{s_1},\ldots,r_{s_{14}} \in \{r_{i_1},\ldots,r_{i_{15}}\}$, such that $r_{s_1}\cdots r_{s_{14}} = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7$. This means $r_{i_1}\cdots r_{i_{15}} = r_{s_1}\cdots r_{s_{14}}(r_i) = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7(r_i)$. For example, if we take $r_1\cdots r_{15}$, then we can see that $r_2\cdots r_{15} = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7$, which means $r_1\cdots r_{15} = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7(r_1)$.

Thus, looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (3.4), we have

$$\sum_{1\le i_1<\cdots<i_{15}\le 16} r_{i_1}\cdots r_{i_{15}} = \alpha_1^7\alpha_2^7\alpha_3^7\alpha_4^7\beta_1^7\beta_2^7\beta_3^7\beta_4^7\left(\sum_{1\le i\le 16} r_i\right) = p_1p_2p_3p_4q_1^7q_2^7q_3^7q_4^7$$

Since we calculated $\sum_{1\le i\le 16} r_i$ as the coefficient of x^{15} above, we can just replace it here.

Looking at the constant, which becomes the coefficient of w_n in equation (3.4), we have

$$\sum_{1\le i_1<\cdots<i_{16}\le 16} r_{i_1}\cdots r_{i_16} = \alpha_1^8\alpha_2^8\alpha_3^8\alpha_4^8\beta_1^8\beta_2^8\beta_3^8\beta_4^8 = q_1^8q_2^8q_3^8q_4^8.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (3.4).

Case 2: Let one characteristic function have duplicate roots and the other three have distinct roots. WLOG we can say the characteristic function of $\{a_n\}$ has the duplicate root, meaning $\alpha_1 = \beta_1$, $\alpha_2 \ne \beta_2$, $\alpha_3 \ne \beta_3$, and $\alpha_4 \ne \beta_4$. Then, from equation (3.1), we have

$$\begin{aligned}
w_n =& a_nb_nc_nd_n \\
=& \left(\frac{na_1b_1c_1d_1}{(\alpha_2-\beta_2)(\alpha_3-\beta_3)(\alpha_4-\beta_4)}\right)(\alpha_2^n-\beta_2^n)(\alpha_3^n-\beta_3^n)(\alpha_4^n-\beta_4^n)\alpha_1^{n-1} \\
=& \left(\frac{na_1b_1c_1d_1}{\alpha_1(\alpha_2-\beta_2)(\alpha_3-\beta_3)(\alpha_4-\beta_4)}\right)\left((\alpha_1\alpha_2\alpha_3\alpha_4)^n - (\alpha_1\alpha_2\alpha_3\beta_4)^n - (\alpha_1\alpha_2\beta_3\alpha_4)^n + (\alpha_1\alpha_2\beta_3\beta_4)^n\right. \\
& \left. -(\alpha_1\beta_2\alpha_3\alpha_4)^n + (\alpha_1\beta_2\alpha_3\beta_4)^n + (\alpha_1\beta_2\beta_3\alpha_4)^n - (\alpha_1\beta_2\beta_3\beta_4)^n\right).
\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2\alpha_3\alpha_4$, $\alpha_1\alpha_2\alpha_3\beta_4$,

$\alpha_1\alpha_2\beta_3\alpha_4$, $\alpha_1\alpha_2\beta_3\beta_4$, $\alpha_1\beta_2\alpha_3\alpha_4$, $\alpha_1\beta_2\alpha_3\beta_4$, $\alpha_1\beta_2\beta_3\alpha_4$, and $\alpha_1\beta_2\beta_3\beta_4$ each with a multiplicity of at least two. We will let each of them have multiplicity two since that means we will have sixteen roots, which is how many characteristic roots we need for a sixteenth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_nb_nc_nd_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_2 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_3 = \alpha_1\alpha_2\beta_3\alpha_4$, $r_4 = \alpha_1\alpha_2\beta_3\beta_4$, $r_5 = \alpha_1\beta_2\alpha_3\alpha_4$, $r_6 = \alpha_1\beta_2\alpha_3\beta_4$, $r_7 = \alpha_1\beta_2\beta_3\alpha_4$, $r_8 = \alpha_1\beta_2\beta_3\beta_4$, $r_9 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{10} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{11} = \alpha_1\alpha_2\beta_3\alpha_4$, $r_{12} = \alpha_1\alpha_2\beta_3\beta_4$, $r_{13} = \alpha_1\beta_2\alpha_3\alpha_4$, $r_{14} = \alpha_1\beta_2\alpha_3\beta_4$, $r_{15} = \alpha_1\beta_2\beta_3\alpha_4$, and $r_{16} = \alpha_1\beta_2\beta_3\beta_4$, then the characteristic equation is

$$\prod_{i=1}^{16}(x - r_i) = x^{16} - \left(\sum_{1\le i\le 16} r_i\right) x^{15} + \cdots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 16} r_{i_1}\cdots r_{i_k}\right) x^{16-k}, \text{ for} k \le 16.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$ and $\alpha_1\alpha_1 = -q_1$.

Case 3: Let two characteristic functions have duplicate roots and the other two have distinct roots. WLOG we can say the characteristic functions of $\{a_n\}$ and $\{b_n\}$ have the duplicate roots, meaning $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 \ne \beta_3$, and $\alpha_4 \ne \beta_4$. Then, from equation (3.1), we have

$$\begin{aligned} w_n &= a_nb_nc_nd_n \\ &= \left(\frac{n^2a_1b_1c_1d_1}{(\alpha_3-\beta_3)(\alpha_4-\beta_4)}\right)(\alpha_3^n - \beta_3^n)(\alpha_4^n - \beta_4^n)\alpha_1^{n-1}\alpha_2^{n-1} \\ &= \left(\frac{n^2a_1b_1c_1d_1}{\alpha_1\alpha_2(\alpha_3-\beta_3)(\alpha_4-\beta_4)}\right)\left((\alpha_1\alpha_2\alpha_3\alpha_4)^n - (\alpha_1\alpha_2\alpha_3\beta_4)^n - (\alpha_1\alpha_2\beta_3\alpha_4)^n + (\alpha_1\alpha_2\beta_3\beta_4)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2\alpha_3\alpha_4$, $\alpha_1\alpha_2\alpha_3\beta_4$, $\alpha_1\alpha_2\beta_3\alpha_4$, and $\alpha_1\alpha_2\beta_3\beta_4$ each with a multiplicity of at least three. We will let each of them have multiplicity four since that means we will have sixteen roots, which is how many characteristic roots we need for a sixteenth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_nb_nc_nd_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_2 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_3 = \alpha_1\alpha_2\beta_3\alpha_4$, $r_4 = \alpha_1\alpha_2\beta_3\beta_4$, $r_5 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_6 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_7 = \alpha_1\alpha_2\beta_3\alpha_4$, $r_8 = \alpha_1\alpha_2\beta_3\beta_4$, $r_9 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{10} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{11} = \alpha_1\alpha_2\beta_3\alpha_4$, $r_{12} = \alpha_1\alpha_2\beta_3\beta_4$, $r_{13} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{14} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{15} = \alpha_1\alpha_2\beta_3\alpha_4$, and $r_{16} = \alpha_1\alpha_2\beta_3\beta_4$, then the characteristic equation is

$$\prod_{i=1}^{16}(x - r_i) = x^{16} - \left(\sum_{1\le i\le 16} r_i\right) x^{15} + \cdots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 16} r_{i_1}\cdots r_{i_k}\right) x^{16-k}, \text{ for} k \le 16.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 and β_2 with α_2 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, and $\alpha_2\alpha_2 = -q_2$.

Case 4: Let three characteristic functions have duplicate roots and the other have distinct roots. WLOG we can say the characteristic functions of $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ have the duplicate roots, meaning $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, and $\alpha_4 \neq \beta_4$. Then, from equation (3.1), we have

$$\begin{aligned} w_n =& a_n b_n c_n d_n \\ =& \left(\frac{n^3 a_1 b_1 c_1 d_1}{(\alpha_4 - \beta_4)}\right)(\alpha_4^n - \beta_4^n)\alpha_1^{n-1}\alpha_2^{n-1}\alpha_3^{n-1} \\ =& \left(\frac{n^3 a_1 b_1 c_1 d_1}{\alpha_1\alpha_2\alpha_3(\alpha_4 - \beta_4)}\right)((\alpha_1\alpha_2\alpha_3\alpha_4)^n - (\alpha_1\alpha_2\alpha_3\beta_4)^n). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1\alpha_2\alpha_3\alpha_4$ and $\alpha_1\alpha_2\alpha_3\beta_4$ each with a multiplicity of at least four. We will let each of them have multiplicity eight since that means we will have sixteen roots, which is how many characteristic roots we need for a sixteenth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n b_n c_n d_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_2 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_3 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_4 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_5 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_6 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_7 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_8 = \alpha_1\alpha_2\alpha_3\beta_4$, $r_9 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{10} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{11} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{12} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{13} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{14} = \alpha_1\alpha_2\alpha_3\beta_4$, $r_{15} = \alpha_1\alpha_2\alpha_3\alpha_4$, and $r_{16} = \alpha_1\alpha_2\alpha_3\beta_4$, then the characteristic equation is

$$\prod_{i=1}^{16}(x - r_i) = x^{16} - \left(\sum_{1\le i\le 16} r_i\right)x^{15} + \cdots + (-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 16} r_{i_1}\cdots r_{i_k}\right)x^{16-k}, \text{ for} k \le 16.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1, β_2 with α_2, and β_3 with α_3 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, $\alpha_2\alpha_2 = -q_2$, $\alpha_3 + \alpha_3 = p_3$, and $\alpha_3\alpha_3 = -q_3$.

Case 5: Let each characteristic functions have duplicate roots, meaning $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, and $\alpha_4 = \beta_4$. Then, from equation (3.1), we have

$$w_n = a_n b_n c_n d_n = n^4 a_1 b_1 c_1 d_1 \alpha_1^{n-1}\alpha_2^{n-1}\alpha_3^{n-1}\alpha_4^{n-1} = \frac{n^4 a_1 b_1 c_1 d_1}{\alpha_1\alpha_2\alpha_3\alpha_4}(\alpha_1\alpha_2\alpha_3\alpha_4)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root $\alpha_1\alpha_2\alpha_3\alpha_4$ with a multiplicity of at least five. We will let it have multiplicity sixteen since that means we will have sixteen roots, which is how many characteristic roots we need for a sixteenth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n b_n c_n d_n\}$ are $r_1 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_2 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_3 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_4 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_5 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_6 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_7 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_8 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_9 = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{10} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{11} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{12} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{13} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{14} = \alpha_1\alpha_2\alpha_3\alpha_4$, $r_{15} = \alpha_1\alpha_2\alpha_3\alpha_4$, and $r_{16} = \alpha_1\alpha_2\alpha_3\alpha_4$, then the characteristic equation is

$$\prod_{i=1}^{16}(x - r_i) = x^{16} - \left(\sum_{1\le i\le 16} r_i\right) x^{15} + \cdots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 16} r_{i_1}\cdots r_{i_k}\right) x^{16-k}, \text{ for} k \le 16.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1, β_2 with α_2, and β_3 with α_3 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, $\alpha_2\alpha_2 = -q_2$, $\alpha_3 + \alpha_3 = p_3$, $\alpha_3\alpha_3 = -q_3$, $\alpha_4 + \alpha_4 = p_4$, and $\alpha_4\alpha_4 = -q_4$.

Therefore, when we multiply four distinct second order linear divisible sequences we can construct a sixteenth order linear divisible sequence defined by recurrence relation (3.4). It is easy to see from our definition of $\{w_n = a_n b_n c_n d_n\}$ that $w_i = a_i b_i c_i d_i$ for $0 \le i \le 15$ □

Next, we have an example that takes the product of four second order linear divisible sequences to construct a sixteenth order linear divisible sequence.

Example 3.5. Using the Fibonacci sequence, Pell number sequence, Mersenne number sequences, and the sequence of natural numbers including zero we define a sequence $\{w_n = F_n P_n M_n N_n\}$. Then, by Theorem 3.5, we get a sixteenth order linear divisible sequence that satisfies the recurrence relation

$$\begin{aligned} w_{n+16} =& 12w_{n+15} + 18w_{n+14} - 456w_{n+13} - 443w_{n+12} + 6336w_{n+11} + 11106w_{n+10} - 27468w_{n+9} \\ & - 87873w_{n+8} - 54936w_{n+7} + 44424w_{n+6} + 50688w_{n+5} - 7088w_{n+4} - 14592w_{n+3} \\ & + 1152w_{n+2} + 1536w_{n+1} - 256w_n, \end{aligned}$$

for $n \ge 0$. The table below shows some terms of the sequence $\{w_n = F_n P_n M_n N_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	7	1953133	14	6985177048668	21	18614391293902412190
1	1	8	17478720	15	58472015201250	22	152351119164258982308
2	12	9	154020510	16	487277559095040	23	1244354656992194910737
3	210	10	1337981700	17	4044847083436931	24	10144273043247536793600
4	2160	11	11505038633	18	33459590559699360	25	82554933399852260719375
5	22475	12	98075577600	19	275928071551639237	26	670763926581706461658908
6	211680	13	830185445479	20	2269164648115530000	27	5441936114229817195931490

Table 3.5: Terms of the sequence $\{w_n = F_n P_n M_n N_n\}$

CHAPTER 4

POWERS OF SECOND ORDER LINEAR DIVISIBLE SEQUENCES

In this chapter, we will look at taking powers of a single second order linear divisible sequence. We start with the work done by He and Shiue in [9] where they squared a single second order linear divisible sequence and cubed a single second order linear divisible sequence. We then move on to the forth, fifth, and sixth powers of a single second order linear divisible sequence. We take these powers term by term; thus, $\{w_n\}$ is the sequence $\left\{a_0^j, a_1^j, a_2^j, \ldots\right\}$.

We start with looking at what the powers of the general forms of second order linear divisible sequences will look like. Let $\{a_n\}$ be a second order linear divisible sequences that satisfies equation (2.1) with $a_0 = 0$. Then $\{a_n\}$ has a characteristic function $x^2 - px - q = 0$ with roots α and β such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Since $\{a_n\}$ is a second order divisible sequences it can be expressed by equation (2.5). Then the sequence $\{w_n = a_n^j\}$ has one of the following expressions depending on weather the roots of the characteristic equation of $\{a_n\}$ are distinct or not.

$$w_n = \begin{cases} \left(\frac{a_1}{\alpha-\beta}\right)^j (\alpha^n - \beta^n)^j, & \text{if } \alpha \neq \beta; \\ n^j a_1^j \left(\alpha^{n-1}\right)^j, & \text{if } \alpha = \beta. \end{cases} \tag{4.1}$$

4.1

Square of a Second Order Linear Divisible Sequences

In this section, we will square a second order linear divisible sequence in order to come up with a single higher order linear divisible sequence. This squaring constructs a third order linear divisible sequences.

Theorem 4.1. [9] *Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose that the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then $\left\{w_n = a_n^2\right\}$ is a linear divisible* *sequence that satisfies the third order linear homogeneous recurrence relation*

$$w_{n+3} = \left(p^2 + q\right) w_{n+2} + q \left(p^2 + q\right) w_{n+1} - q^3 w_n \tag{4.2}$$

for $n \geq 0$ with initial conditions $w_2 = a_2^2$, $w_1 = a_1^2$, and $w_0 = a_0^2 = 0$.

Proof. Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$.

Case 1: Let the characteristic function have distinct roots, meaning $\alpha \neq \beta$. Then, by equation (4.1), we have

$$\begin{aligned} w_n &= a_n^2 \\ &= \left(\frac{a_1}{\alpha - \beta}\right)^2 (\alpha^n - \beta^n)^2 \\ &= \left(\frac{a_1^2}{(\alpha - \beta)^2}\right) \left(\left(\alpha^2\right)^n - 2\left(\alpha\beta\right)^n + \left(\beta^2\right)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots α^2, $\alpha\beta$, and β^2 each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have three roots, which is how many characteristic roots we need for a third order linear divisible sequence. Thus, the characteristic equation is

$$\left(x - \alpha^2\right)\left(x - \alpha\beta\right)\left(x - \beta^2\right) = x^3 - \left(\alpha^2 + \alpha\beta + \beta^2\right) x^2 + \left(\alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3\right) x - \alpha^3\beta^3.$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (4.2), we have

$$\begin{aligned} \alpha^2 + \alpha\beta + \beta^2 &= \alpha^2 + 2\alpha\beta + \beta^2 - \alpha\beta \\ &= \left(\alpha + \beta\right)^2 - \alpha\beta \\ &= p^2 + q. \end{aligned}$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (4.2), we have

$$\begin{aligned} \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 &= \alpha\beta\left(\alpha^2 + \alpha\beta + \beta^2\right) \\ &= \alpha\beta\left(\alpha^2 + 2\alpha\beta + \beta^2 - \alpha\beta\right) \end{aligned}$$

$$= \alpha\beta\left((\alpha+\beta)^2 - \alpha\beta\right)$$
$$= q\left(p^2 + q\right).$$

Looking at the constant, which becomes the coefficient of w_n in equation (4.2), we have

$$\alpha^3\beta^3 = (\alpha\beta)^3 = (-q)^3 = -q^3.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (4.2).

Case 2: Let the characteristic function have a duplicate root, meaning $\alpha = \beta$. Then, by equation (4.1), we have

$$w_n = a_n^2 = n^2 a_1^2 \left(\alpha^2\right)^{n-1} = \frac{n^2 a_1^2}{\alpha^2}\left(\alpha^2\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^2 with a multiplicity of at least three. We will let it have multiplicity three since that means we will have three roots, which is how many characteristic roots we need for a third order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n = a_n^2\right\}$ are α^2, α^2, and α^2, then the characteristic equation is

$$\left(x - \alpha^2\right)\left(x - \alpha^2\right)\left(x - \alpha^2\right).$$

At this point, this case becomes the same as case 1 by simply replacing β with α throughout. This works because, in this case, $\alpha + \alpha = p$ and $\alpha\alpha = -q$.

Therefore, when we take the square of a second order linear divisible sequence, we can construct a third order linear divisible sequence defined by recurrence relation (4.2). It is easy to see by how we define $\{w_n = a_n^2\}$ that $w_2 = a_2^2$, $w_1 = a_1^2$, and $w_0 = a_0^2 = 0$. □

Note that in He and Shiue [9] they only proved case 1 from Theorem 4.1. The second case is proven here so that we can see that the recurrence relation (4.2) still works when the roots of the characteristic equation are the same.

Next, we have examples that square second order linear divisible sequences to construct third order linear divisible sequences.

Example 4.1. [9]Using the Fibonacci sequence, we define the sequence $\{w_n = F_n^2\}$. Then, by Theorem 4.1, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+3} = 2w_{n+2} + 2w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	4	6	64	9	1156	12	20736	15	372100	18	6677056
1	1	4	9	7	169	10	3025	13	54289	16	974169	19	17480761
2	1	5	25	8	441	11	7921	14	142129	17	2550409	20	45765225

Table 4.1: Terms of the sequence $\{w_n = F_n^2\}$

Example 4.2. [9] Using the Pell number sequence, we define the sequence $\{w_n = P_n^2\}$. Then, by Theorem 4.1, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+3} = 5w_{n+2} + 5w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	25	6	4900	9	970225	12	192099600	15	38034750625	18	7530688524100
1	1	4	144	7	28561	10	5654884	13	1119638521	16	221682772224	19	43892069261881
2	4	5	841	8	166464	11	32959081	14	6525731524	17	1292061882721	20	255821727047184

Table 4.2: Terms of the sequence $\{w_n = P_n^2\}$

Example 4.3. [9] Using the Mersenne number sequence, we define the sequence $\{w_n = M_n^2\}$. Then, by Theorem 4.1, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+3} = 7w_{n+2} - 14w_{n+1} + 8w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = M_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	49	6	3969	9	261121	12	16769025	15	1073676289	18	68718952449
1	1	4	225	7	16129	10	1046529	13	67092481	16	4294836225	19	274876858369
2	9	5	961	8	65025	11	4190209	14	268402689	17	17179607041	20	1099509530625

Table 4.3: Terms of the sequence $\{w_n = M_n^2\}$

Example 4.4. Using the sequence of natural numbers including zero, we define the sequence $\{w_n = N_n^2\}$.

Then, by Theorem 4.1, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+3} = 3w_{n+2} - 3w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = N_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	9	6	36	9	81	12	144	15	225	18	324
1	1	4	16	7	49	10	100	13	169	16	256	19	361
2	4	5	25	8	64	11	121	14	196	17	289	20	400

Table 4.4: Terms of the sequence $\{w_n = N_n^2\}$

4.2

Cube of a Second Order Linear Divisible Sequences

In this section we will cube a second order linear divisible sequence in order to come up with a single higher order linear divisible sequence. This cubing constructs a fourth order linear divisible sequences.

Theorem 4.2. [9] *Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then $\{w_n = a_n^3\}$ is a linear divisible sequence that satisfies the fourth order linear homogeneous recurrence relation*

$$w_{n+4} = p\left(p^2 + 2q\right) w_{n+3} + q\left(p^2 + q\right)\left(p^2 + 2q\right) w_{n+2} - pq^3\left(p^2 + 2q\right) w_{n+1} - q^6 w_n \tag{4.3}$$

for $n \geq 0$ with initial conditions $w_3 = a_3^3$, $w_2 = a_2^3$, $w_1 = a_1^3$, and $w_0 = a_0^3 = 0$.

Proof. Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$.

Case 1: Let the characteristic function have distinct roots, meaning $\alpha \neq \beta$. Then, by equation (4.1), we have

$$\begin{aligned}
w_n &= a_n^3 \\
&= \left(\frac{a_1}{\alpha - \beta}\right)^3 (\alpha^n - \beta^n)^3 \\
&= \left(\frac{a_1^3}{(\alpha - \beta)^3}\right)\left(\left(\alpha^3\right)^n - 3\left(\alpha^2\beta\right)^n + 3\left(\alpha\beta^2\right)^n - \left(\beta^3\right)^n\right).
\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots α^3, $\alpha^2\beta$, $\alpha\beta^2$, and β^3 each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have four roots, which is how many characteristic roots we need for a fourth order linear divisible sequence. Thus, the characteristic equation is

$$
\begin{aligned}
&\left(x-\alpha^3\right)\left(x-\alpha^2\beta\right)\left(x-\alpha\beta^2\right)\left(x-\beta^3\right) \\
&= x^4-\left(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3\right)x^3+\left(\alpha^5\beta+\alpha^4\beta^2+2\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5\right)x^2 \\
&\quad -\left(\alpha^6\beta^3+\alpha^5\beta^4+\alpha^4\beta^5+\alpha^3\beta^6\right)x+\alpha^6\beta^6.
\end{aligned}
$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (4.3), we have

$$
\begin{aligned}
\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3 &= (\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2+\alpha^2\beta+\alpha\beta^2 \\
&= (\alpha+\beta)^3-2\alpha^2\beta-2\alpha\beta^2 \\
&= (\alpha+\beta)^3-2\alpha\beta\left(\alpha+\beta\right) \\
&= p^3+2pq \\
&= p\left(p^2+2q\right).
\end{aligned}
$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (4.3), we have

$$
\begin{aligned}
\alpha^5\beta+\alpha^4\beta^2+2\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5 &= \alpha\beta\left(\alpha^4+\alpha^3\beta+2\alpha^2\beta^2+\alpha\beta^3+\beta^4\right) \\
&= \alpha\beta\left(\left(\alpha^2+\beta^2\right)^2-2\alpha^2\beta^2+\alpha^3\beta+2\alpha^2\beta^2+\alpha\beta^3\right) \\
&= \alpha\beta\left(\left(\alpha^2+\beta^2\right)^2+\alpha\beta\left(\alpha^2+\beta^2\right)\right) \\
&= \alpha\beta\left(\left((\alpha+\beta)^2-2\alpha\beta\right)^2+\alpha\beta\left((\alpha+\beta)^2-2\alpha\beta\right)\right) \\
&= -q\left(\left(p^2+2q\right)^2-q\left(p^2+2q\right)\right) \\
&= -q\left(p^4+4p^2q+4q^2-p^2q-2q^2\right) \\
&= -q\left(p^4+3p^2q+2q^2\right) \\
&= -q\left(p^2+2q\right)\left(p^2+q\right).
\end{aligned}
$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (4.3), we have

$$\begin{aligned}\alpha^6\beta^3+\alpha^5\beta^4+\alpha^4\beta^5+\alpha^3\beta^6 &= \alpha^3\beta^3\left(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3\right)\\ &= \alpha^3\beta^3\left((\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2+\alpha^2\beta+\alpha\beta^2\right)\\ &= \alpha^3\beta^3\left((\alpha+\beta)^3-2\alpha^2\beta-2\alpha\beta^2\right)\\ &= \alpha^3\beta^3\left((\alpha+\beta)^3-2\alpha\beta\left(\alpha+\beta\right)\right)\\ &= -q^3\left(p^3+2pq\right)\\ &= -pq^3\left(p^2+2q\right).\end{aligned}$$

Looking at the constant, which becomes the coefficient of w_n in equation (4.3), we have

$$\alpha^6\beta^6=(\alpha\beta)^6=(-q)^6=q^6.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (4.3).

Case 2: Let the characteristic function have a duplicate root, meaning $\alpha=\beta$. Then, by equation (4.1,) we have

$$w_n=a_n^3=n^3a_1^3\left(\alpha^3\right)^{n-1}=\frac{n^3a_1^3}{\alpha^3}\left(\alpha^3\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^3 with a multiplicity of at least four. We will let it have multiplicity four since that means we will have four roots, which is how many characteristic roots we need for a fourth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n=a_n^3\right\}$ are α^3, α^3, α^3, and α^3, then the characteristic equation is

$$\left(x-\alpha^3\right)\left(x-\alpha^3\right)\left(x-\alpha^3\right)\left(x-\alpha^3\right).$$

At this point, this case becomes the same as case 1 by simply replacing β with α throughout the proof of that case. This works because, in this case, $\alpha+\alpha=p$ and $\alpha\alpha=-q$.

Therefore, when we take the cube of a second order linear divisible sequence, we can construct a fourth order linear divisible sequence defined by recurrence relation (4.3). It is easy to see by how we define $\{w_n=a_n^3\}$ that $w_3=a_3^3$, $w_2=a_2^3$, $w_1=a_1^3$, and $w_0=a_0^3=0$. □

Note that in He and Shiue [9] they only proved case 1 from Theorem 4.2. The second case is proven here so that we can see that the recurrence relation (4.3) still works when the roots of the characteristic equation are the same.

Next, we have examples that cube second order linear divisible sequences to construct forth order linear divisible sequences.

Example 4.5. [9] Using the Fibonacci sequence, we define the sequence $\{w_n = F_n^3\}$. Then, by Theorem 4.2, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 3w_{n+3} + 6w_{n+2} - 3w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^3\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	8	6	512	9	39304	12	2985984	15	226981000	18	17253512704
1	1	4	27	7	2197	10	166375	13	12649337	16	961504803	19	73087061741
2	1	5	125	8	9261	11	704969	14	53582633	17	4073003173	20	309601747125

Table 4.5: Terms of the sequence $\{w_n = F_n^3\}$

Example 4.6. [9] Using the Pell number sequence, we define the sequence $\{w_n = P_n^3\}$. Then, by Theorem 4.2, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 12w_{n+3} + 30w_{n+2} - 12w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n^3\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	343000	12	2662500456000	18	206657907547204610000
1	1	7	4826809	13	37464224551181	19	2907897430955111700029
2	8	8	67917312	14	527161643971768	20	40917221940918370907520
3	125	9	955671625	15	7417727240640625	21	575749004603813264071250
4	1728	10	13447314152	16	104375343011770368	22	8101403286394301751067120
5	24389	11	189218084021	17	1468672529408250769	23	11399539501412404337235241

Table 4.6: Terms of the sequence $\{w_n = P_n^3\}$

Example 4.7. [9] Using of the Mersenne sequence, we define the sequence $\{w_n = M_n^3\}$. Then, by Theorem 4.2, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 15w_{n+3} - 70w_{n+2} + 120w_{n+1} - 64w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = M_n^3\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	250047	12	68669157375	18	18014192351838207
1	1	7	2048383	13	549554511871	19	144114363443707903
2	27	8	16581375	14	4397241253887	20	1152918206075109375
3	343	9	133432831	15	35181150961663	21	9223358842721533951
4	3375	10	1070599167	16	281462092005375	22	73786923518292656127
5	29791	11	8577357823	17	2251748274470911	23	590295599252498284543

Table 4.7: Terms of the sequence $\left\{w_n = M_n^3\right\}$

Example 4.8. Using the sequence of natural numbers including zero, we define the sequence $\left\{w_n = N_n^3\right\}$. Then, by Theorem 4.2, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+4} = 4w_{n+3} - 6w_{n+2} + 4w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = N_n^3\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	27	6	216	9	729	12	1728	15	3375	18	5832
1	1	4	64	7	343	10	1000	13	2197	16	4096	19	6859
2	8	5	125	8	512	11	1331	14	2744	17	4913	20	8000

Table 4.8: Terms of the sequence $\left\{w_n = N_n^3\right\}$

4.3

Fourth Power of a Second Order Linear Divisible Sequences

In this section, we will find the fourth power a second order linear divisible sequence in order to come up with a single higher order linear divisible sequence. Raising a second order linear divisible sequences to the fourth power constructs a fifth order linear divisible sequence.

Theorem 4.3. *Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then $\left\{w_n = a_n^4\right\}$ is a linear divisible sequence that satisfies the fifth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+5} = &\left(p^4 + 3\,p^2q + q^2\right) w_{n+4} + \left(p^6q + 5\,p^4q^2 + 7\,p^2q^3 + 2\,q^4\right) w_{n+3} \\ &- \left(p^6q^3 + 5\,p^4q^4 + 7\,p^2q^5 + 2\,q^6\right) w_{n+2} - \left(p^4q^6 + 3\,p^2q^7 + q^8\right) w_{n+1} + q^{10}w_n \end{aligned} \tag{4.4}$$

for $n \geq 0$ with initial conditions $w_4 = a_4^4$, $w_3 = a_3^4$, $w_2 = a_2^4$, $w_1 = a_1^4$, and $w_0 = a_0^4 = 0$.

Proof. Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition

$a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$.

Case 1: Let the characteristic function have distinct roots, meaning $\alpha \neq \beta$. Then, by equation (4.1), we have

$$\begin{aligned} w_n &= a_n^4 \\ &= \left(\frac{a_1}{\alpha - \beta}\right)^4 (\alpha^n - \beta^n)^4 \\ &= \left(\frac{a_1^4}{(\alpha - \beta)^4}\right)\left(\left(\alpha^4\right)^n - 4\left(\alpha^3\beta\right)^n + 6\left(\alpha^2\beta^2\right)^n - 4\left(\alpha\beta^3\right)^n + \left(\beta^4\right)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots α^4, $\alpha^3\beta$, $\alpha^2\beta^2$, $\alpha\beta^3$, and β^4 each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have five roots, which is how many characteristic roots we need for a fifth order linear divisible sequence. Thus, the characteristic equation is

$$\begin{aligned} &\left(x - \alpha^4\right)\left(x - \alpha^3\beta\right)\left(x - \alpha^2\beta^2\right)\left(x - \alpha\beta^3\right)\left(x - \beta^4\right) \\ &= x^5 - \left(\alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right)x^4 + \left(\alpha^7\beta + \alpha^6\beta^2 + 2\,\alpha^5\beta^3 + 2\,\alpha^4\beta^4 + 2\,\alpha^3\beta^5 + \alpha^2\beta^6 + \alpha\beta^7\right)x^3 \\ &\quad - \left(\alpha^9\beta^3 + \alpha^8\beta^4 + 2\,\alpha^7\beta^5 + 2\,\alpha^6\beta^6 + 2\,\alpha^5\beta^7 + \alpha^4\beta^8 + \alpha^3\beta^9\right)x^2 \\ &\quad + \left(\alpha^{10}\beta^6 + \alpha^9\beta^7 + \alpha^8\beta^8 + \alpha^7\beta^9 + \alpha^6\beta^{10}\right)x - \alpha^{10}\beta^{10} \end{aligned}$$

Looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (4.4), we have

$$\begin{aligned} \alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4 &= \left(\left(\alpha^2 + \beta^2\right)^2 + \alpha^3\beta - \alpha^2\beta^2 + \alpha\beta^3\right) \\ &= \left(\left(\alpha^2 + \beta^2\right)^2 + \alpha\beta\left(\alpha^2 - \alpha\beta + \beta^2\right)\right) \\ &= \left(\left((\alpha + \beta)^2 - 2\,\alpha\beta\right)^2 + \alpha\beta\left((\alpha + \beta)^2 - 3\,\alpha\beta\right)\right) \\ &= \left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right) \\ &= p^4 + 3p^2q + q^2. \end{aligned}$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (4.4), we have

$$\alpha^7\beta + \alpha^6\beta^2 + 2\,\alpha^5\beta^3 + 2\,\alpha^4\beta^4 + 2\,\alpha^3\beta^5 + \alpha^2\beta^6 + \alpha\beta^7 = \left(\alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right)\left(\alpha^2 + \beta^2\right)\alpha\beta$$

$$= -\left(\left(p^2+2\,q\right)^2 - q\left(p^2+3\,q\right)\right)\left(p^2+2\,q\right)q$$
$$= -\left(p^6q+5p^4q^2+7p^2q^3+2q^4\right).$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (4.4), we have

$$\alpha^9\beta^3+\alpha^8\beta^4+2\,\alpha^7\beta^5+2\,\alpha^6\beta^6+2\,\alpha^5\beta^7+\alpha^4\beta^8+\alpha^3\beta^9 = \left(\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4\right)\left(\alpha^2+\beta^2\right)\alpha^3\beta^3$$
$$= -\left(\left(p^2+2\,q\right)^2 - q\left(p^2+3\,q\right)\right)\left(p^2+2\,q\right)q^3$$
$$= -\left(p^6q^3+5p^4q^4+7p^2q^5+2q^6\right).$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (4.4), we have

$$\alpha^{10}\beta^6+\alpha^9\beta^7+\alpha^8\beta^8+\alpha^7\beta^9+\alpha^6\beta^{10} = \left(\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4\right)\alpha^6\beta^6$$
$$= \left(\left(p^2+2\,q\right)^2 - q\left(p^2+3\,q\right)\right)q^6$$
$$= p^4q^6+3p^2q^7+q^8.$$

Looking at the constant, which becomes the coefficient of w_n in equation (4.4), we have

$$\alpha^{10}\beta^{10} = q^{10}.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (4.4).

Case 2: Let the characteristic function have a duplicate root, meaning $\alpha = \beta$. Then, by equation (4.1), we have

$$w_n = a_n^4 = n^4a_1^4\left(\alpha^4\right)^{n-1} = \frac{n^4a_1^4}{\alpha^4}\left(\alpha^4\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^4 with a multiplicity of at least five. We will let it have multiplicity five since that means we will have five roots, which is how many characteristic roots we need for a fifth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n = a_n^4\right\}$ are α^4, α^4, α^4, α^4, and α^4, then the characteristic equation is

$$\left(x-\alpha^4\right)\left(x-\alpha^4\right)\left(x-\alpha^4\right)\left(x-\alpha^4\right)\left(x-\alpha^4\right).$$

At this point, this case becomes the same as case 1 by simply replacing β with α throughout. This works because, in this case, $\alpha + \alpha = p$ and $\alpha\alpha = -q$.

Therefore, when we take the fourth power of a second order linear divisible sequence, we can construct a fifth order linear divisible sequence defined by recurrence relation (4.4). It is easy to see by how we define $\{w_n = a_n^4\}$ that $w_4 = a_4^4$, $w_3 = a_3^4$, $w_2 = a_2^4$, $w_1 = a_1^4$, and $w_0 = a_0^4 = 0$. $\square$

Next, we have examples that take the fourth pour given second order linear divisible sequences to construct fifth order linear divisible sequences.

Example 4.9. Using the Fibonacci sequence, we define the sequence $\{w_n = F_n^4\}$. Then, by Theorem 4.3, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+5} = 5w_{n+4} + 15w_{n+3} - 15w_{n+2} - 5w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^4\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	625	10	9150625	15	138458410000	20	2094455819300625
1	1	6	4096	11	62742241	16	949005240561	21	14355614096087056
2	1	7	28561	12	429981696	17	6504586067281	22	98394841894789441
3	16	8	194481	13	2947295521	18	44583076827136	23	674408281676875201
4	81	9	1336336	14	20200652641	19	305577005139121	24	4622463123273547776

Table 4.9: Terms of the sequence $\{w_n = F_n^4\}$

Example 4.10. Using the Pell number sequence, we define the sequence $\{w_n = P_n^4\}$. Then, by Theorem 4.3, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+5} = 29w_{n+4} + 174w_{n+3} - 174w_{n+2} - 29w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n^4\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	707281	10	31977713053456	15	1446642255105937890625
1	1	6	24010000	11	1086301020364561	16	49143251500917865906176
2	16	7	815730721	12	36902256320160000	17	1669423908780535158363841
3	625	8	27710263296	13	1253590417707067441	18	56711269647011436280810000
4	20736	9	941336550625	14	42585171923327362576	19	1926513744089758912159658161

Table 4.10: Terms of the sequence $\{w_n = P_n^4\}$

Example 4.11. Using the Mersenne number sequence, we define the sequence $\left\{w_n = M_n^4\right\}$. Then, by Theorem 4.3, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+5} = 31w_{n+4} - 310w_{n+3} + 1240w_{n+2} - 1984w_{n+1} + 1024w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = M_n^4\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	923521	10	1095222947841	15	1152780773560811521
1	1	6	15752961	11	17557851463681	16	18445618199572250625
2	81	7	260144641	12	281200199450625	17	295138898083176775681
3	2401	8	4228250625	13	4501401006735361	18	4722294425687923097601
4	50625	9	68184176641	14	72040003462430721	19	75557287266811285340161

Table 4.11: Terms of the sequence $\left\{w_n = M_n^4\right\}$

Example 4.12. Using the sequence of natural numbers including zero, we define the sequence $\left\{w_n = N_n^4\right\}$. Then, by Theorem 4.3, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+5} = 5w_{n+4} - 10w_{n+3} + 10w_{n+2} - 5w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = N_n^4\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	81	6	1296	9	6561	12	20736	15	50625	18	104976
1	1	4	256	7	2401	10	10000	13	28561	16	65536	19	130321
2	16	5	625	8	4096	11	14641	14	38416	17	83521	20	160000

Table 4.12: Terms of the sequence $\left\{w_n = N_n^4\right\}$

4.4

Fifth Power of a Second Order Linear Divisible Sequences

In this section, we will find the fifth power of a second order linear divisible sequence in order to come up with a single higher order linear divisible sequence. Raising a second order linear divisible sequences to the fifth power constructs a sixth order linear divisible sequence.

Theorem 4.4. *Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then $\left\{w_n = a_n^5\right\}$ is a linear divisible sequence that* *satisfies the sixth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+6} = &\left(p^5+4p^3q+3pq^2\right)w_{n+5} + \left(p^8q+7p^6q^2+16p^4q^3+13p^2q^4+3q^5\right)w_{n+4} \\ &- \left(p^9q^3+8p^7q^4+22p^5q^5+23p^3q^6+6pq^7\right)w_{n+3} \\ &- \left(p^8q^6+7p^6q^7+16p^4q^8+13p^2q^9+3q^{10}\right)w_{n+2} \\ &+ \left(p^5q^{10}+4p^3q^{11}+3pq^{12}\right)w_{n+1} + q^{15}w_n \end{aligned} \tag{4.5}$$

for $n \geq 0$ with initial conditions $w_i = a_i^5$ for $0 \leq i \leq 5$.

Proof. Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$.

Case 1: Let the characteristic function have distinct roots, meaning $\alpha \neq \beta$. Then, by equation (4.1), we have

$$\begin{aligned} w_n &= a_n^5 \\ &= \left(\frac{a_1}{\alpha-\beta}\right)^5 (\alpha^n - \beta^n)^5 \\ &= \left(\frac{a_1^5}{(\alpha-\beta)^5}\right)\left(\left(\alpha^5\right)^n - 5\left(\alpha^4\beta\right)^n + 10\left(\alpha^3\beta^2\right)^n - 10\left(\alpha^2\beta^3\right)^n + 5\left(\alpha\beta^4\right)^n - \left(\beta^5\right)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha^5$, $r_2 = \alpha^4\beta$, $r_3 = \alpha^3\beta^2$, $r_4 = \alpha^2\beta^3$, $r_5 = \alpha\beta^4$, and $r_6 = \beta^5$ each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{6}(x - r_i) = x^6 - \left(\sum_{1\leq i\leq 6} r_i\right)x^5 + \cdots + (-1)^k\left(\sum_{1\leq i_1<\cdots<i_k\leq 6} r_{i_1}\cdots r_{i_k}\right)x^{6-k}, \text{ for } k \leq 6.$$

Looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (4.5), we have

$$\begin{aligned} \sum_{1\leq i\leq 6} r_i &= \alpha^5 + \alpha^4\beta + \alpha^3\beta^2 + \alpha^2\beta^3 + \alpha\beta^4 + \beta^5 \\ &= \left(\alpha^2+\alpha\beta+\beta^2\right)\left(\alpha^2-\alpha\beta+\beta^2\right)(\alpha+\beta) \\ &= \left(p^2+q\right)\left(p^2+3q\right)p \\ &= p^5+4p^3q+3pq^2. \end{aligned}$$

Looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (4.5), we have

$$\begin{aligned}\sum_{1\le i<j\le 6} r_i r_j &= \alpha^9\beta+\alpha^8\beta^2+2\alpha^7\beta^3+2\alpha^6\beta^4+3\alpha^5\beta^5+2\alpha^4\beta^6+2\alpha^3\beta^7+\alpha^2\beta^8+\alpha\beta^9\\ &= \left(\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4\right)\left(\alpha^2+\alpha\beta+\beta^2\right)\left(\alpha^2-\alpha\beta+\beta^2\right)\alpha\beta\\ &= -\left(\left(p^2+2q\right)^2-q\left(p^2+3q\right)\right)\left(p^2+q\right)\left(p^2+3q\right)q\\ &= -\left(p^8q+7p^6q^2+16p^4q^3+13p^2q^4+3q^5\right).\end{aligned}$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (4.5), we have

$$\begin{aligned}\sum_{1\le i<j<k\le 6} r_i r_j r_k &= \alpha^{12}\beta^3+\alpha^{11}\beta^4+2\alpha^{10}\beta^5+3\alpha^9\beta^6+3\alpha^8\beta^7+3\alpha^7\beta^8+3\alpha^6\beta^9+2\alpha^5\beta^{10}+\alpha^4\beta^{11}+\alpha^3\beta^{12}\\ &= \left(\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4\right)\left(\alpha^2-\alpha\beta+\beta^2\right)\left(\alpha^2+\beta^2\right)\left(\alpha+\beta\right)\alpha^3\beta^3\\ &= -\left(\left(p^2+2q\right)^2-q\left(p^2+3q\right)\right)\left(p^2+3q\right)\left(p^2+2q\right)pq^3\\ &= -\left(p^9q^3+8p^7q^4+22p^5q^5+23p^3q^6+6pq^7\right).\end{aligned}$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (4.5), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_4\le 6} r_{i_1}\cdots r_{i_4} &= \alpha^{14}\beta^6+\alpha^{13}\beta^7+2\alpha^{12}\beta^8+2\alpha^{11}\beta^9+3\alpha^{10}\beta^{10}+2\alpha^9\beta^{11}+2\alpha^8\beta^{12}+\alpha^7\beta^{13}+\alpha^6\beta^{14}\\ &= \left(\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4\right)\left(\alpha^2+\alpha\beta+\beta^2\right)\left(\alpha^2-\alpha\beta+\beta^2\right)\alpha^6\beta^6\\ &= \left(\left(p^2+2q\right)^2-q\left(p^2+3q\right)\right)\left(p^2+q\right)\left(p^2+3q\right)q^6\\ &= p^8q^6+7p^6q^7+16p^4q^8+13p^2q^9+3q^{10}.\end{aligned}$$

Note here for x^4, x^3, and x^2, we are using the result for $\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4$ that was shown in Theorem 4.3. Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (4.5), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_5\le 6} r_{i_1}\cdots r_{i_5} &= \alpha^{15}\beta^{10}+\alpha^{14}\beta^{11}+\alpha^{13}\beta^{12}+\alpha^{12}\beta^{13}+\alpha^{11}\beta^{14}+\alpha^{10}\beta^{15}\\ &= \left(\alpha^2+\alpha\beta+\beta^2\right)\left(\alpha^2-\alpha\beta+\beta^2\right)\left(\alpha+\beta\right)\alpha^{10}\beta^{10}\\ &= \left(p^2+q\right)\left(p^2+3q\right)pq^{10}\\ &= p^5q^{10}+4p^3q^{11}+3pq^{12}.\end{aligned}$$

Looking at the constant, which becomes the coefficient of w_n in equation (4.5), we have

$$\sum_{1\leq i_1<\cdots<i_6\leq 6} r_{i_1}\cdots r_{i_6} = \alpha^{15}\beta^{15} = -q^{15}.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (4.5).

Case 2: Let the characteristic function have a duplicate root, meaning $\alpha = \beta$. Then, by equation (4.1), we have

$$w_n = a_n^5 = n^5 a_1^5 \left(\alpha^5\right)^{n-1} = \frac{n^5 a_1^5}{\alpha^5}\left(\alpha^5\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^5 with a multiplicity of at least six. We will let it have multiplicity six since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n = a_n^5\right\}$ are α^5, α^5, α^5, α^5, α^5, and α^5, then the characteristic equation is

$$\left(x-\alpha^5\right)\left(x-\alpha^5\right)\left(x-\alpha^5\right)\left(x-\alpha^5\right)\left(x-\alpha^5\right)\left(x-\alpha^5\right).$$

At this point, this case becomes the same as case 1 by simply replacing β with α throughout. This works because, in this case, $\alpha+\alpha = p$ and $\alpha\alpha = -q$.

Therefore, when we take the fifth power of a second order linear divisible sequence, we can construct a sixth order linear divisible sequence defined by recurrence relation (4.5). It is easy to see by how we define $\{w_n = a_n^5\}$ that $w_i = a_i^5$ for $0 \leq i \leq 5$ □

Next, we have examples that take the fifth power of second order linear divisible sequences to construct sixth order linear divisible sequences.

Example 4.13. Using the Fibonacci sequence, we define the sequence $\left\{w_n = F_n^5\right\}$. Then, by Theorem 4.4, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 8w_{n+5} + 40w_{n+4} - 60w_{n+3} - 40w_{n+2} + 8w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = F_n^5\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	32768	12	61917364224	18	115202670521319424
1	1	7	371293	13	686719856393	19	1277617458486664901
2	1	8	4084101	14	7615646045657	20	14168993617568728125
3	32	9	45435424	15	84459630100000	21	157136551895768914976
4	243	10	503284375	16	936668172433707	22	1742671044798615789551
5	3125	11	5584059449	17	10387823949447757	23	19326518128014212635057

Table 4.13: Terms of the sequence $\{w_n = F_n^5\}$

Example 4.14. Using the Pell number sequence, we define the sequence $\{w_n = P_n^5\}$. Then, by Theorem 4.4, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 70w_{n+5} + 1015w_{n+4} - 2436w_{n+3} - 1015w_{n+2} + 70w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n^5\}$.

n	w_n	n	w_n	n	w_n
0	0	7	137858491849	14	34401153583102310036144 32
1	1	8	11305787424768	15	2821314058020355371191406 25
2	32	9	927216502365625	16	231382153906801606403366584 32
3	3125	10	76043001641118368	17	18976157934478377286254360624 49
4	248832	11	6236454157912944701	18	1556276332780252535561616101000 00
5	20511149	12	511465272597417600000	19	127633635445927585767791607193645 49
6	1680700000	13	41946388966896183643301	20	10467514382898667811648616099940423 68

Table 4.14: Terms of the sequence $\{w_n = P_n^5\}$

Example 4.15. Using the Mersenne number sequence, we define the sequence $\{w_n = M_n^5\}$. Then, by Theorem 4.4, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 63w_{n+5} - 1302w_{n+4} + 11160w_{n+3} - 41664w_{n+2} + 645126w_{n+1} + 32768w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = M_n^5\}$.

n	w_n	n	w_n	n	w_n
0	0	7	33038369407	14	1180231376725002502143
1	1	8	1078203909375	15	37773167607267111108607
2	243	9	34842114263551	16	1208833588708967444709375
3	16807	10	1120413075641343	17	38684150510660063165284351
4	759375	11	35940921946155007	18	1237916427633109224574418943
5	28629151	12	1151514816750309375	19	39613703469254688357136990207
6	992436543	13	36870975646169341951	20	1267644555610660532401787109375

Table 4.15: Terms of the sequence $\{w_n = M_n^5\}$

Example 4.16. Using the sequence of natural numbers including zero, we define the sequence $\{w_n = N_n^5\}$.

Then, by Theorem 4.4, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 6w_{n+5} - 15w_{n+4} + 20w_{n+3} - 15w_{n+2} + 6w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = N_n^5\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	243	6	7776	9	59049	12	248832	15	759375	18	1889568
1	1	4	1024	7	16807	10	100000	13	371293	16	1048576	19	2476099
2	32	5	3125	8	32768	11	161051	14	537824	17	1419857	20	3200000

Table 4.16: Terms of the sequence $\{w_n = N_n^5\}$

4.5

Sixth Power of a Second Order Linear Divisible Sequences

In this section we will find find the sixth power a second order divisible sequence in order to come up with a single higher order linear divisible sequence. Raising a second order linear divisible sequences to the sixth power constructs a seventh order linear divisible sequence.

Theorem 4.5. *Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$. Then $\{w_n = a_n^6\}$ is a linear divisible sequence that satisfies the seventh order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+7} = &\left(p^6 + 5p^4q + 6p^2q^2 + q^3\right) w_{n+6} + \left(p^{10}q + 9p^8q^2 + 29p^6q^3 + 40p^4q^4 + 22p^2q^5 + 3q^6\right) w_{n+5} \\ &- \left(p^{12}q^3 + 11p^{10}q^4 + 46p^8q^5 + 90p^6q^6 + 81p^4q^7 + 28p^2q^8 + 3q^9\right) w_{n+4} \\ &- \left(p^{12}q^6 + 11p^{10}q^7 + 46p^8q^8 + 90p^6q^9 + 81p^4q^{10} + 28p^2q^{11} + 3q^{12}\right) w_{n+3} \\ &+ \left(p^{10}q^{10} + 9p^8q^{11} + 29p^6q^{12} + 40p^4q^{13} + 22p^2q^{14} + 3q^{15}\right) w_{n+2} \\ &+ \left(p^6q^{15} + 5p^4q^{16} + 6p^2q^{17} + q^{18}\right) w_{n+1} - q^{21}w_n \end{aligned} \tag{4.6}$$

for $n \geq 0$ with initial conditions $w_i = a_i^6$ for $0 \leq i \leq 6$.

Proof. Let $\{a_n\}$ be a second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - px - q = 0$ with roots α and β, such that $\alpha + \beta = p$ and $\alpha\beta = -q$.

Case 1: Let the characteristic function have distinct roots, meaning $\alpha \neq \beta$. Then, by equation (4.1), we have

$$\begin{aligned}
w_n &= a_n^5 \\
&= \left(\frac{a_1}{\alpha-\beta}\right)^6 (\alpha^n - \beta^n)^6 \\
&= \left(\frac{a_1^5}{(\alpha-\beta)^5}\right)\left(\left(\alpha^6\right)^n - 6\left(\alpha^5\beta\right)^n + 15\left(\alpha^4\beta^2\right)^n - 20\left(\alpha^3\beta^3\right)^n + 15\left(\alpha^2\beta^4\right)^n - 6\left(\alpha\beta^5\right)^n + \left(\beta^6\right)^n\right).
\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha^6$, $r_2 = \alpha^5\beta$, $r_3 = \alpha^4\beta^2$, $r_4 = \alpha^3\beta^3$, $r_5 = \alpha^2\beta^4$, $r_6 = \alpha\beta^5$, and $r_7 = \beta^6$ each with a multiplicity of at least one. We will let each of them have multiplicity one since that means we will have seven roots, which is how many characteristic roots we need for a seventh order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{7}(x - r_i) = x^7 - \left(\sum_{1\le i\le 7} r_i\right)x^6 + \cdots + (-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 7} r_{i_1}\cdots r_{i_k}\right)x^{7-k}, \text{ for } k \le 7.$$

Looking at the coefficient of x^6, which becomes the coefficient of w_{n+6} in equation (4.6), we have

$$\begin{aligned}
\sum_{1\le i\le 7} r_i &= \alpha^6 + \alpha^5\beta + \alpha^4\beta^2 + \alpha^3\beta^3 + \alpha^2\beta^4 + \alpha\beta^5 + \beta^6 \\
&= \left(\alpha^2+\beta^2\right)\left(\alpha^4 - \alpha^2\beta^2 + \beta^4\right) + \alpha\beta\left(\alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right) \\
&= \left(p^2+2q\right)\left(\left(p^2+2q\right)^2 - 3q^2\right) - q\left(\left(p^2+2\,q\right)^2 - q\left(p^2+3\,q\right)\right) \\
&= p^6 + 5p^4q + 6p^2q^2 + q^3.
\end{aligned}$$

Looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (4.6), we have

$$\begin{aligned}
\sum_{1\le i<j\le 7} r_ir_j &= \alpha^{11}\beta + \alpha^{10}\beta^2 + 2\alpha^9\beta^3 + 2\alpha^8\beta^4 + 3\alpha^7\beta^5 + 3\alpha^6\beta^6 + 3\alpha^5\beta^7 + 2\alpha^4\beta^8 + 2\alpha^3\beta^9 + \alpha^2\beta^{10} + \alpha\beta^{11} \\
&= \left(\alpha^6 + \alpha^5\beta + \alpha^4\beta^2 + \alpha^3\beta^3 + \alpha^2\beta^4 + \alpha\beta^5 + \beta^6\right)\left(\alpha^2 + \alpha\beta + \beta^2\right)\left(\alpha^2 - \alpha\beta + \beta^2\right)\alpha\beta \\
&= -\left(\left(p^2+2q\right)\left(\left(p^2+2q\right)^2 - 3q^2\right) - q\left(\left(p^2+2\,q\right)^2 - q\left(p^2+3\,q\right)\right)\right)\left(p^2+q\right)\left(p^2+3q\right)q \\
&= -\left(p^{10}q + 9p^8q^2 + 29p^6q^3 + 40p^4q^4 + 22p^2q^5 + 3q^6\right).
\end{aligned}$$

Looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (4.6), we have

$$\sum_{1\le i<j<k\le 7} r_ir_jr_k = \alpha^{15}\beta^3 + \alpha^{14}\beta^4 + 2\alpha^{13}\beta^5 + 3\alpha^{12}\beta^6 + 4\alpha^{11}\beta^7 + 4\alpha^{10}\beta^8 + 5\alpha^9\beta^9 + 4\alpha^8\beta^{10} + 4\alpha^7\beta^{11}$$

$$+ 3\alpha^6\beta^{12} + 2\alpha^5\beta^{13} + \alpha^4\beta^{14} + \alpha^3\beta^{15}$$

$$= \left(\alpha^6 + \alpha^5\beta + \alpha^4\beta^2 + \alpha^3\beta^3 + \alpha^2\beta^4 + \alpha\beta^5 + \beta^6\right)\left(\alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right)$$

$$\times \left(\alpha^2 - \alpha\beta + \beta^2\right)\alpha^3\beta^3$$

$$= -\left(\left(p^2 + 2q\right)\left(\left(p^2 + 2q\right)^2 - 3q^2\right) - q\left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right)\right)$$

$$\times \left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right)\left(p^2 + 3q\right)q^3$$

$$= -\left(p^{12}q^3 + 11p^{10}q^4 + 46p^8q^5 + 90p^6q^6 + 81p^4q^7 + 28p^2q^8 + 3q^9\right).$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (4.6), we have

$$\sum_{1\le i_1<\cdots<i_4\le 7} r_{i_1}\cdots r_{i_4} = \alpha^{18}\beta^6 + \alpha^{17}\beta^7 + 2\alpha^{16}\beta^8 + 3\alpha^{15}\beta^9 + 4\alpha^{14}\beta^{10} + 4\alpha^{13}\beta^{11} + 5\alpha^{12}\beta^{12}$$

$$+ 4\alpha^{11}\beta^{13} + 4\alpha^{10}\beta^{14} + 3\alpha^9\beta^{15} + 2\alpha^8\beta^{16} + \alpha^7\beta^{17} + \alpha^6\beta^{18}$$

$$= \left(\alpha^6 + \alpha^5\beta + \alpha^4\beta^2 + \alpha^3\beta^3 + \alpha^2\beta^4 + \alpha\beta^5 + \beta^6\right)\left(\alpha^4 + \alpha^3\beta + \alpha^2\beta^2 + \alpha\beta^3 + \beta^4\right)$$

$$\times \left(\alpha^2 - \alpha\beta + \beta^2\right)\alpha^6\beta^6$$

$$= \left(\left(p^2 + 2q\right)\left(\left(p^2 + 2q\right)^2 - 3q^2\right) - q\left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right)\right)$$

$$\times \left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right)\left(p^2 + 3q\right)q^6$$

$$= p^{12}q^6 + 11p^{10}q^7 + 46p^8q^8 + 90p^6q^9 + 81p^4q^{10} + 28p^2q^{11} + 3q^{12}.$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (4.6), we have

$$\sum_{1\le i_1<\cdots<i_5\le 7} r_{i_1}\cdots r_{i_5} = \alpha^{20}\beta^{10} + \alpha^{19}\beta^{11} + 2\alpha^{18}\beta^{12} + 2\alpha^{17}\beta^{13} + 3\alpha^{16}\beta^{14} + 3\alpha^{15}\beta^{15} + 3\alpha^{14}\beta^{16}$$

$$+ 2\alpha^{13}\beta^{17} + 2\alpha^{12}\beta^{18} + \alpha^{11}\beta^{19} + \alpha^{10}\beta^{20}$$

$$= \left(\alpha^6 + \alpha^5\beta + \alpha^4\beta^2 + \alpha^3\beta^3 + \alpha^2\beta^4 + \alpha\beta^5 + \beta^6\right)\left(\alpha^2 + \alpha\beta + \beta^2\right)$$

$$\times \left(\alpha^2 - \alpha\beta + \beta^2\right)\alpha^{10}\beta^{10}$$

$$= \left(\left(p^2 + 2q\right)\left(\left(p^2 + 2q\right)^2 - 3q^2\right) - q\left(\left(p^2 + 2\,q\right)^2 - q\left(p^2 + 3\,q\right)\right)\right)\left(p^2 + q\right)$$

$$\times \left(p^2 + 3q\right)q^{10}$$

$$= p^{10}q^{10} + 9p^8q^{11} + 29p^6q^{12} + 40p^4q^{13} + 22p^2q^{14} + 3q^{15}.$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (4.6), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_6\le 7} r_{i_1}\cdots r_{i_6} &= \alpha^{21}\beta^{15}+\alpha^{20}\beta^{16}+\alpha^{19}\beta^{17}+\alpha^{18}\beta^{18}+\alpha^{17}\beta^{19}+\alpha^{16}\beta^{20}+\alpha^{15}\beta^{21}\\ &= \left(\alpha^6+\alpha^5\beta+\alpha^4\beta^2+\alpha^3\beta^3+\alpha^2\beta^4+\alpha\beta^5+\beta^6\right)\alpha^{15}\beta^{15}\\ &= -\left(\left(p^2+2q\right)\left(\left(p^2+2q\right)^2-3q^2\right)-q\left(\left(p^2+2\,q\right)^2-q\left(p^2+3\,q\right)\right)\right)q^{15}\\ &= -\left(p^6q^{15}+5p^4q^{16}+6p^2q^{17}+q^{18}\right)\end{aligned}$$

Looking at the constant, which becomes the coefficient of w_n in equation (4.6), we have

$$\sum_{1\le i_1<\cdots<i_6\le 7} r_{i_1}\cdots r_{i_7} = \alpha^{21}\beta^{15} = -q^{21}.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (4.6).

Case 2: Let the characteristic function have a duplicate root, meaning $\alpha=\beta$. Then, by equation (4.1), we have

$$w_n = a_n^6 = n^6a_1^6\left(\alpha^6\right)^{n-1} = \frac{n^6a_1^6}{\alpha^6}\left(\alpha^6\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^6 with a multiplicity of at least seven. We will let it have multiplicity seven since that means we will have seven roots, which is how many characteristic roots we need for a seventh order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n=a_n^6\right\}$ are α^6, α^6, α^6, α^6, α^6, α^6, and α^6, then the characteristic equation is

$$\left(x-\alpha^6\right)\left(x-\alpha^6\right)\left(x-\alpha^6\right)\left(x-\alpha^6\right)\left(x-\alpha^6\right)\left(x-\alpha^6\right)\left(x-\alpha^6\right).$$

At this point, this case becomes the same as case 1 by simply replacing β with α throughout. This works because, in this case, $\alpha+\alpha=p$ and $\alpha\alpha=-q$.

Therefore, when we take the sixth power of a second order linear divisible sequence, we can construct a seventh order linear divisible sequence defined by recurrence relation (4.6). It is easy to see by how we define $\{w_n=a_n^6\}$ that $w_i=a_i^6$ for $0\le i\le 6$. □

Next, we have examples that take the sixth power of second order linear divisible sequences to construct seventh order linear divisible sequences.

Example 4.17. Using the Fibonacci sequence, we define the sequence $\left\{w_n = F_n^6\right\}$. Then, by Theorem 4.5, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+7} = 13w_{n+6} + 104w_{n+5} - 260w_{n+4} - 260w_{n+3} + 104w_{n+2} + 13w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = F_n^6\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	262144	12	8916100448256	18	297683700627089391616
1	1	7	4826809	13	160005726539569	19	5341718593932745951081
2	1	8	85766121	14	2871098559212689	20	95853241822852445765625
3	64	9	1544804416	15	51520374361000000	21	1720016697051086543327296
4	729	10	27680640625	16	924491486192068809	22	30864446874428284248737761
5	15625	11	496981290961	17	16589354847268067929	23	553840029994503291482828449

Table 4.17: Terms of the sequence $\left\{w_n = F_n^6\right\}$

Example 4.18. Using the Pell number sequence, we define the sequence $\left\{w_n = P_n^6\right\}$. Then, by Theorem 4.5, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+7} = 169w_{n+6} + 5915w_{n+5} - 34307w_{n+4} - 34307w_{n+3} + 5915w_{n+2} + 169w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = P_n^6\right\}$.

n	w_n	n	w_n	n	w_n
0	0	6	117649000000	12	7088908678200207936000000
1	1	7	23298085122481	13	1403568121221313200888494761
2	64	8	4612761269305344	14	277899398875017080933981045824
3	15625	9	913308254830140625	15	55022677416541980626660400390625
4	2985984	10	180830257902579479104	16	10894212228824721394610989562855424
5	594823321	11	35803483320578215528441	17	2156998998638429219913518292389091361

Table 4.18: Terms of the sequence $\left\{w_n = P_n^6\right\}$

Example 4.19. Using the Mersenne number sequence, we define the sequence $\left\{w_n = M_n^6\right\}$. Then, by Theorem 4.5, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+7} = 127w_{n+6} - 5334w_{n+5} + 94488w_{n+4} - 755904w_{n+3} + 2731008w_{n+2} - 4161536w_{n+1} + 2097152w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = M_n^6\right\}$.

n	w_n	n	w_n	n	w_n
0	0	6	62523502209	12	4715453174592516890625
1	1	7	4195872914689	13	302010161517773079920641
2	729	8	274941996890625	14	19335730644885715992608769
3	117649	9	17804320388674561	15	1237713382987321429695725569
4	11390625	10	1146182576381093889	16	79220909236042181489028890625
5	887503681	11	73571067223779299329	17	5070370291582725139136985169921

Table 4.19: Terms of the sequence $\left\{w_n = M_n^6\right\}$

Example 4.20. Using the sequence of natural numbers including zero, we define the sequence $\left\{w_n = N_n^6\right\}$. Then, by Theorem 4.5, we get a third order linear divisible sequence that satisfies the recurrence relation

$$w_{n+7} = 7w_{n+6} - 21w_{n+5} + 35w_{n+4} - 35w_{n+3} + 21w_{n+2} - 7w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = N_n^6\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	729	6	46656	9	531441	12	2985984	15	11390625	18	34012224
1	1	4	4096	7	117649	10	1000000	13	4826809	16	16777216	19	47045881
2	64	5	15625	8	262144	11	1771561	14	7529536	17	24137569	20	64000000

Table 4.20: Terms of the sequence $\left\{w_n = N_n^6\right\}$

CHAPTER 5

PRODUCTS OF POWERS

In this chapter, we will be multiplying second order linear divisible sequence sequence that have been raised to powers. First, we will look at taking the product of the square of a second order linear divisible sequence sequence times a different second order linear divisible sequence sequence not raised to any power. Second, we will look at the product of the squares of two distinct second order linear divisible sequence sequence. This product is defined term by term; thus, the sequence $\{w_n\}$ is the sequence $\left\{a_{0_1}^{k_1} a_{0_2}^{k_2} \cdots a_{0_i}^{k_i}, a_{1_1}^{k_1} a_{1_2}^{k_2} \cdots a_{1_i}^{k_i}, a_{2_1}^{k_1} a_{2_2}^{k_2} \cdots a_{2_i}^{k_i}, \ldots\right\}$.

5.1

Product of the Square of a Second Order Times a Second Order

In this section, we look at multiplying the square of one second order linear divisible sequence by a different second order linear divisible sequence in order to come up with a single higher order linear divisible sequence. This multiplication constructs a sixth order linear divisible sequences.

Theorem 5.1. *Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - p_1 x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, and the sequence $\{b_n\}$ has a characteristic equation $x^2 - p_2 x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$. Then $\left\{w_n = a_n^2 b_n\right\}$ is a linear divisible that satisfies the sixth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+6} = {} & \left(p_1^2 p_2 + p_2 q_1\right) w_{n+5} + \left(p_1^4 q_2 + p_1^2 p_2^2 q_1 + 4p_1^2 q_1 q_2 + p_2^2 q_1^2 + 3q_1^2 q_2\right) w_{n+4} \\ & - \left(p_1^4 p_2 q_1 q_2 + 2p_1^2 p_2 q_1^2 q_2 - 2p_2 q_1^3 q_2 - p_2^2 p_2 q_1^3\right) w_{n+3} - \left(p_1^4 q_1^2 q_2^2 + p_1^2 p_2^2 q_1^3 q_2\right. \\ & \left. + 4p_1^2 q_1^3 q_2^2 + p_2^2 q_1^4 q_2 + 3q_1^4 q_2^2\right) w_{n+2} + \left(p_1^2 p_2 q_1^4 q_2^2 + p_2 q_1^5 q_2^2\right) w_{n+1} + q_1^6 q_2^3 w_n. \end{aligned} \tag{5.1}$$

for $n \geq 0$ and initial conditions $w_i = a_i^2 b_i$ for $0 \leq i \leq 5$.

Proof. Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - p_1 x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, and the sequence $\{b_n\}$ have the characteristic equation $x^2 - p_2 x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$.

Case 1: Let both characteristic functions have distinct roots, meaning $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$\begin{aligned}
w_n &= a_n^2 b_n \\
&= \left(\frac{a_1}{\alpha_1 - \beta_1}\right)^2 (\alpha_1^n - \beta_1^n)^2 \left(\frac{b_1}{\alpha_2 - \beta_2}\right) (\alpha_2^n - \beta_2^n) \\
&= \left(\frac{a_1^2 b_1}{(\alpha_1 - \beta_1)^2 (\alpha_2 - \beta_2)}\right) \left(\left(\alpha_1^2\right)^n - 2\left(\alpha_1\beta_1\right)^n + \left(\beta_1^2\right)^n\right) (\alpha_2^n - \beta_2^n) \\
&= \left(\frac{a_1^2 b_1}{(\alpha_1 - \beta_1)^2 (\alpha_2 - \beta_2)}\right) \left(\left(\alpha_1^2\alpha_2\right)^n - 2\left(\alpha_1\alpha_2\beta_1\right)^n + \left(\alpha_2\beta_1^2\right)^n - \left(\alpha_1^2\beta_2\right)^n + 2\left(\alpha_1\beta_1\beta_2\right)^n - \left(\beta_1^2\beta_2\right)^n\right).
\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha_1^2\alpha_2$, $r_2 = \alpha_1\alpha_2\beta_1$, $r_3 = \alpha_2\beta_1^2$, $r_4 = \alpha_1^2\beta_2$, $r_5 = \alpha_1\beta_1\beta_2$, and $r_6 = \beta_1^2\beta_2$ each with a multiplicity of at least one. We will let them have multiplicity one since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{6}(x - r_i) = x^6 - \left(\sum_{1\leq i\leq 6} r_i\right) x^5 + \cdots + (-1)^k \left(\sum_{1\leq i_1<\cdots<i_k\leq 6} r_{i_1}\cdots r_{i_k}\right) x^{6-k}, \text{ for } k \leq 6.$$

Looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (5.1), we have

$$\begin{aligned}
\sum_{1\leq i\leq 6} r_i &= \alpha_1^2\alpha_2 + \alpha_1\alpha_2\beta_1 + \alpha_2\beta_1^2 + \alpha_1^2\beta_2 + \alpha_1\beta_1\beta_2 + \beta_1^2\beta_2 \\
&= \left(\alpha_1^2 + \beta_1^2 + \alpha_1\beta_1\right)(\alpha_2 + \beta_2) \\
&= \left((\alpha_1 + \beta_1)^2 - \alpha_1\beta_1\right)(\alpha_2 + \beta_2) \\
&= \left(p_1^2 + q_1\right) p_2 \\
&= p_1^2 p_2 + p_2 q_1.
\end{aligned}$$

Looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (5.1), we have

$$\begin{aligned}
\sum_{1\le i<j\le 6} r_i r_j &= \alpha_1^3\alpha_2^2\beta_1 + \alpha_1^2\alpha_2^2\beta_1^2 + \alpha_1\alpha_2^2\beta_1^3 + \alpha_1^4\alpha_2\beta_2 + 2\alpha_1^3\alpha_2\beta_1\beta_2 + 3\alpha_1^2\alpha_2\beta_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1^3\beta_2 \\
&\quad + \alpha_2\beta_1^4\beta_2 + \alpha_1^3\beta_1\beta_2^2 + \alpha_1^2\beta_1^2\beta_2^2 + \alpha_1\beta_1^3\beta_2^2 \\
&= \left(\alpha_1^2\alpha_2\beta_2 + \alpha_2\beta_1^2\beta_2 + \alpha_1\alpha_2^2\beta_1 + \alpha_1\beta_1\beta_2^2 + \alpha_1\alpha_2\beta_1\beta_2\right)\left(\alpha_1^2 + \alpha_1\beta_1 + \beta_1^2\right) \\
&= \left(\alpha_2\beta_2\left(\alpha_1^2+\beta_1^2\right) + \alpha_1\beta_1\left(\alpha_2^2+\beta_2^2\right) + \alpha_1\alpha_2\beta_1\beta_2\right)\left(\alpha_1^2+\beta_1^2+\alpha_1\beta_1\right) \\
&= \left(-q_2\left(p_1^2+2q_1\right) - q_1\left(p_2^2+2q_2\right) + q_1q_2\right)\left(p_1^2+q_1\right) \\
&= -\left(p_1^4q_2 + p_1^2p_2^2q_1 + 4p_1^2q_1q_2 + p_2^2q_1^2 + 3q_1^2q_2\right)
\end{aligned}$$

Looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (5.1), we have

$$\begin{aligned}
\sum_{1\le i<j<k\le 6} r_i r_j r_k &= \alpha_1^3\alpha_2^3\beta_1^3 + \alpha_1^5\alpha_2^2\beta_1\beta_2 + 2\alpha_1^4\alpha_2^2\beta_1^2\beta_2 + 3\alpha_1^3\alpha_2^2\beta_1^3\beta_2 + 2\alpha_1^2\alpha_2^2\beta_1^4\beta_2 + \alpha_1\alpha_2^2\beta_1^5\beta_2 \\
&\quad + \alpha_1^5\alpha_2\beta_1\beta_2^2 + 2\alpha_1^4\alpha_2\beta_1^2\beta_2^2 + 3\alpha_1^3\alpha_2\beta_1^3\beta_2^2 + 2\alpha_1^2\alpha_2\beta_1^4\beta_2^2 + \alpha_1\alpha_2\beta_1^5\beta_2^2 + \alpha_1^3\beta_1^3\beta_2^3 \\
&= \left(\alpha_1^4\alpha_2\beta_2 + \alpha_2\beta_1^4\beta_2 + \alpha_1^2\alpha_2^2\beta_1^2 + \alpha_1^2\beta_1^2\beta_2^2 + 2\alpha_1^3\alpha_2\beta_1\beta_2 + 2\alpha_1^2\alpha_2\beta_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1^3\beta_2\right) \\
&\quad \times\left(\alpha_2+\beta_2\right)\alpha_1\beta_1 \\
&= \left(\alpha_2\beta_2\left(\alpha_1^4+\beta_1^4\right) + \alpha_1^2\beta_1^2\left(\alpha_2^2+\beta_2^2\right) + 2\alpha_1\alpha_2\beta_1\beta_2\left(\alpha_1^2+\beta_1^2+\alpha_1\beta_1\right)\right)\left(\alpha_2+\beta_2\right)\alpha_1\beta_1 \\
&= -\left(-q_2\left(\left(p_1^2+2q_1\right)^2 - 2q_1^2\right) + q_1^2\left(p_2^2+2q_2\right) + 2q_1q_2\left(p_1^2+q_1\right)\right)p_2q_1 \\
&= p_1^4p_2q_1q_2 + 2p_1^2p_2q_1^2q_2 - 2p_2q_1^3q_2 - p_2^2p_2q_1^3.
\end{aligned}$$

Looking at the coefficient of x^2, which becomes the coefficient of w_{n+2} in equation (5.1), we have

$$\begin{aligned}
\sum_{1\le i_1<\cdots<i_4\le 6} r_{i_1}\cdots r_{i_4} &= \alpha_1^5\alpha_2^3\beta_1^3\beta_2 + \alpha_1^4\alpha_2^3\beta_1^4\beta_2 + \alpha_1^3\alpha_2^3\beta_1^5\beta_2 + \alpha_1^6\alpha_2^2\beta_1^2\beta_2^2 + 2\alpha_1^5\alpha_2^2\beta_1^3\beta_2^2 + 3\alpha_1^4\alpha_2^2\beta_1^4\beta_2^2 \\
&\quad + 2\alpha_1^3\alpha_2^2\beta_1^5\beta_2^2 + \alpha_1^2\alpha_2^2\beta_1^6\beta_2^2 + \alpha_1^5\alpha_2\beta_1^3\beta_2^3 + \alpha_1^4\alpha_2\beta_1^4\beta_2^3 + \alpha_1^3\alpha_2\beta_1^5\beta_2^3 \\
&= \left(\alpha_1^2\alpha_2\beta_2 + \alpha_2\beta_1^2\beta_2 + \alpha_1\alpha_2^2\beta_1 + \alpha_1\beta_1\beta_2^2 + \alpha_1\alpha_2\beta_1\beta_2\right)\left(\alpha_1^2+\alpha_1\beta_1+\beta_1^2\right)\alpha_1^2\alpha_2\beta_1^2\beta_2 \\
&= \left(\alpha_2\beta_2\left(\alpha_1^2+\beta_1^2\right) + \alpha_1\beta_1\left(\alpha_2^2+\beta_2^2\right) + \alpha_1\alpha_2\beta_1\beta_2\right)\left(\alpha_1^2+\beta_1^2+\alpha_1\beta_1\right)\alpha_1^2\beta_1^2\alpha_2\beta_2 \\
&= -\left(-q_2\left(p_1^2+2q_1\right) - q_1\left(p_2^2+2q_2\right) + q_1q_2\right)\left(p_1^2+q_1\right)q_1^2q_2 \\
&= p_1^4q_1^2q_2^2 + p_1^2p_2^2q_1^3q_2 + 4p_1^2q_1^3q_2^2 + p_2^2q_1^4q_2 + 3q_1^4q_2^2.
\end{aligned}$$

Looking at the coefficient of x, which becomes the coefficient of w_{n+1} in equation (5.1), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_5\le 6} r_{i_1}\cdots r_{i_5} &= \alpha_1^6\alpha_2^3\beta_1^4\beta_2^2+\alpha_1^5\alpha_2^3\beta_1^5\beta_2^2+\alpha_1^4\alpha_2^3\beta_1^6\beta_2^2+\alpha_1^6\alpha_2^2\beta_1^4\beta_2^3+\alpha_1^5\alpha_2^2\beta_1^5\beta_2^3+\alpha_1^4\alpha_2^2\beta_1^6\beta_2^3\\ &= \left(\alpha_1^2+\beta_1^2+\alpha_1\beta_1\right)\left(\alpha_2+\beta_2\right)\alpha_1^4\beta_1^4\alpha_2^2\beta_2^2\\ &= \left(p_1^2+q_1\right)p_2q_1^4q_2^2\\ &= p_1^2p_2q_1^4q_2^2+p_2q_1^5q_2^2.\end{aligned}$$

Looking at the constant, which becomes the coefficient of w_n in equation (5.1), we have

$$\sum_{1\le i_1<\cdots<i_5\le 6} r_{i_1}\cdots r_{i_5} = \alpha_1^6\alpha_2^3\beta_1^6\beta_2^3 = -q_1^6q_2^3.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (5.1).

Case 2: Let the characteristic function of $\{a_n\}$ have duplicate roots and the characteristic function of $\{b_n\}$ have distinct roots, meaning $\alpha_1=\beta_1$ and $\alpha_2\neq\beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$\begin{aligned}w_n &= a_n^2b_n\\ &= \left(\frac{n^2a_1^2b_1}{\alpha_2-\beta_2}\right)\left(\alpha_2^n-\beta_2^n\right)\left(\alpha_1^2\right)^{n-1}\\ &= \left(\frac{n^2a_1^2b_1}{\alpha_1^2(\alpha_2-\beta_2)}\right)\left(\left(\alpha_1^2\alpha_2\right)^n-\left(\alpha_1^2\beta_2\right)^n\right)\\ &= \left(\frac{n^2a_1^2b_1}{\alpha_1^2(\alpha_2-\beta_2)}\right)\left(\alpha_1^2\alpha_2\right)^n-\left(\frac{n^2a_1^2b_1}{\alpha_1^2(\alpha_2-\beta_2)}\right)\left(\alpha_1^2\beta_2\right)^n.\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1^2\alpha_2$ and $\alpha_1^2\beta_2$ each with a multiplicity of at least three. We will let them have multiplicity three since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n=a_n^2b_n\right\}$ are $r_1=\alpha_1^2\alpha_2$, $r_2=\alpha_1^2\alpha_2$, $r_3=\alpha_1^2\alpha_2$, $r_4=\alpha_1^2\beta_2$, $r_5=\alpha_1^2\beta_2$, and $r_6=\alpha_1^2\beta_2$, then the characteristic equation is

$$\prod_{i=1}^{6}(x-r_i)=x^6-\left(\sum_{1\le i\le 6} r_i\right)x^5+\cdots+(-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 6} r_{i_1}\cdots r_{i_k}\right)x^{6-k},\ \text{for } k\le 6.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$ and $\alpha_1\alpha_1 = -q_1$.

Case 3: Let the characteristic function of $\{a_n\}$ have distinct roots and the characteristic function of $\{b_n\}$ have duplicate roots, meaning $\alpha_1 \neq \beta_1$ and $\alpha_2 = \beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$\begin{aligned} w_n &= a_n^2 b_n \\ &= \left(\frac{na_1^2 b_1}{(\alpha_1 - \beta_1)^2}\right) (\alpha_1^n - \beta_1^n)^2 (\alpha_2)^{n-1} \\ &= \left(\frac{na_1^2 b_1}{\alpha_2(\alpha_1 - \beta_1)^2}\right) \left(\left(\alpha_1^2\alpha_2\right)^n - 2\left(\alpha_1\alpha_2\beta_1^2\right)^n + \left(\alpha_2\beta_1^2\right)^n\right). \end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1^2\alpha_2$, $\alpha_1\alpha_2\beta_1$, and $\alpha_2\beta_1^2$ each with a multiplicity of at least two. We will let them have multiplicity two since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n = a_n^2 b_n\right\}$ are $r_1 = \alpha_1^2\alpha_2$, $r_2 = \alpha_1\alpha_2\beta_1$, $r_3 = \alpha_2\beta_1^2$, $r_4 = \alpha_1^2\alpha_2$, $r_5 = \alpha_1\alpha_2\beta_1$, and $r_6 = \alpha_2\beta_1^2$, then the characteristic equation is

$$\prod_{i=1}^{6}(x - r_i) = x^6 - \left(\sum_{1 \leq i \leq 6} r_i\right) x^5 + \cdots + (-1)^k \left(\sum_{1 \leq i_1 < \cdots < i_k \leq 6} r_{i_1} \cdots r_{i_k}\right) x^{6-k}, \text{ for } k \leq 6.$$

At this point, this case becomes the same as case 1 by simply replacing β_2 with α_2 throughout. This works because, in this case, $\alpha_2 + \alpha_2 = p_2$ and $\alpha_2\alpha_2 = -q_2$.

Case 4: Let both characteristic functions have duplicate roots, meaning $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$w_n = a_n^2 b_n = n^3 a_1^2 b_1 \left(\alpha_1^2\right)^{n-1} \alpha_2^{n-1} = \frac{n^3 a_1^2 b_1}{\alpha_1^2 \alpha_2} \left(\alpha_1^2 \alpha_2\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root $\alpha_1^2\alpha_2$ with a multiplicity of at least six. We will let it have multiplicity six since that means we will have six roots, which is how many characteristic roots we need for a sixth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\{w_n = a_n^2 b_n\}$ are $r_1 = \alpha_1^2\alpha_2$, $r_2 = \alpha_1^2\alpha_2$, $r_3 = \alpha_1^2\alpha_2$, $r_4 = \alpha_1^2\alpha_2$, $r_5 = \alpha_1^2\alpha_2$, and $r_6 = \alpha_1^2\alpha_2$, then the characteristic equation is

$$\prod_{i=1}^{6}(x - r_i) = x^6 - \left(\sum_{1\le i\le 6} r_i\right) x^5 + \cdots + (-1)^k \left(\sum_{1\le i_1<\cdots<i_k\le 6} r_{i_1}\cdots r_{i_k}\right) x^{6-k}, \text{ for } k \le 6.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 and β_2 with α_2 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, and $\alpha_2\alpha_2 = -q_2$.

Therefore, when we multiply the square one second order linear divisible sequence by a different second order linear divisible sequence, we can construct a sixth order linear divisible sequence defined by recurrence relation (5.1). It is easy to see by how we define $\{w_n = a_n^2 b_n\}$ that $w_i = a_i^2 b_i$ for $0 \le i \le 5$. □

Next, we have examples that take the square of a second order linear divisible sequences and multiplies it by a different second order linear divisible sequence to construct sixth order linear divisible sequences.

Example 5.1. Using the Fibonacci sequence and the Pell number sequence, we define the sequence $\{w_n = F_n^2 P_n\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 4w_{n+5} + 16w_{n+4} - 6w_{n+3} + 16w_{n+2} + 4w_{n+1} + w_n,$$

for $n \ge 0$. The table below shows some terms of the sequence $\{w_n = F_n^2 P_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	20	6	4480	9	1138660	12	287400960	15	72568802500	18	18323243845760
1	1	4	108	7	28561	10	7193450	13	1816564229	16	458669938608	19	115811947027949
2	2	5	725	8	179928	11	45474461	14	11481464878	17	2899021855801	20	731988596166300

Table 5.1: Terms of the sequence $\{w_n = F_n^2 P_n\}$

Example 5.2. Using the Pell number sequence and the Fibonacci sequence, we define the sequence $\{w_n = P_n^2 F_n\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 5w_{n+5} + 40w_{n+4} + 21w_{n+3} - 40w_{n+2} + 5w_{n+1} + w_n,$$

for $n \ge 0$. The table below shows some terms of the sequence $\{w_n = P_n^2 F_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	4205	10	311018620	15	23201197881250	20	1730633983474199760
1	1	6	39200	11	2933358209	16	218800896185088	21	16320905155410328850
2	4	7	371293	12	27662342400	17	2063422826705437	22	153915816638460784604
3	50	8	3495744	13	260875775393	18	19459299146274400	23	1451517453316876370977
4	432	9	32987650	14	2460200784548	19	183512741583924461	24	13688670604054528051200

Table 5.2: Terms of the sequence $\left\{w_n = P_n^2 F_n\right\}$

Example 5.3. Using the Fibonacci sequence and the Mersenne number sequence, we define the sequence $\left\{w_n = F_n^2 M_n\right\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 6w_{n+5} + 2w_{n+4} - 33w_{n+3} + 4w_{n+2} + 24w_{n+1} - 8w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = F_n^2 M_n\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	3	28	6	4032	9	590716	12	84913920	15	12192600700	18	1750343491008
1	1	4	135	7	21463	10	3094575	13	444681199	16	63842165415	19	9164935742407
2	3	5	775	8	112455	11	16214287	14	2328499407	17	334284658039	20	47988270804375

Table 5.3: Terms of the sequence $\left\{w_n = F_n^2 M_n\right\}$

Example 5.4. Using the Mersenne number sequence and the Fibonacci sequence, we define the sequence $\left\{w_n = M_n^2 F_n\right\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 7w_{n+5} + 7w_{n+4} - 66w_{n+3} - 28w_{n+2} + 112w_{n+1} + 64w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = M_n^2 F_n\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n	n	w_n
0	0	5	4805	10	57559095	15	654942536290	20	7438181974678125
1	1	6	31752	11	372928601	16	4239003354075	21	48140971199703746
2	9	7	209677	12	2414739600	17	27435832444477	22	311575058462033199
3	98	8	1365525	13	15632548073	18	177569773128216	23	2016556621114666993
4	675	9	8878114	14	101187813753	19	1149260144840789	24	13051430164267840800

Table 5.4: Terms of the sequence $\left\{w_n = M_n^2 F_n\right\}$

Example 5.5. Using the Pell number sequence and the Mersenne number sequence, we define the sequence $\left\{w_n = P_n^2 M_n\right\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 15w_{n+5} - 25w_{n+4} - 159w_{n+3} - 50w_{n+2} + 60w_{n+1} - 8w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = P_n^2 M_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	308700	12	786647862000	18	1974117281773146300
1	1	7	3627247	13	9170959125511	19	23012041317103803847
2	12	8	42448320	14	106911059557692	20	268248267438500962800
3	175	9	495784975	15	1246284673729375	21	3126932447247755029975
4	2160	10	5784946332	16	14527980477699840	22	36450204475983625105692
5	26071	11	67467238807	17	169351843030124191	23	424894771592145805342927

Table 5.5: Terms of the sequence $\{w_n = P_n^2 M_n\}$

Example 5.6. Using the Mersenne number sequence and the Pell number sequence, we define the sequence $\{w_n = M_n^2 P_n\}$. Then, by Theorem 5.1, we get a sixth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+6} = 14w_{n+5} - 35w_{n+4} - 84w_{n+3} + 140w_{n+2} + 224w_{n+1} + 64w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = M_n^2 P_n\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	277830	12	232418686500	18	188579236500070290
1	1	7	2725801	13	2244981506741	19	1821089148272187221
2	18	8	26530200	14	21682106022798	20	17586026022895357500
3	245	9	257204185	15	209393718262225	21	169825852089472725965
4	2700	10	2488645962	16	2022146329489200	22	1639984283429427377622
5	27869	11	24055989869	17	19527870347827249	23	15837092972393610747769

Table 5.6: Terms of the sequence $\{w_n = M_n^2 P_n\}$

5.2

Product of the Squares of Two Second Order

In this section, we look at multiplying the squares of two distinct second order linear divisible sequences in order to come up with a single higher order linear divisible sequence. This multiplication constructs a ninth order linear divisible sequences.

Theorem 5.2. *Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic* *equation* $x^2-p_1x-q_1=0$ *with roots* α_1 *and* β_1*, such that* $\alpha_1+\beta_1=p_1$ *and* $\alpha_1\beta_1=-q_1$*, and the sequence* $\{b_n\}$ *has a characteristic equation* $x^2-p_2x-q_2=0$ *with roots* α_2 *and* β_2*, such that* $\alpha_2+\beta_2=p_2$ *and* $\alpha_2\beta_2=-q_2$*. Then* $\left\{w_n=a_n^2b_n^2\right\}$ *is a linear divisible sequence that satisfies the ninth order linear homogeneous recurrence relation*

$$\begin{aligned}
w_{n+9}=&\left(p_1^2p_2^2+p_1^2q_2+p_2^2q_1+q_1q_2\right)w_{n+8}+\left(p_1^2p_2^4q_1+p_1^4p_2^2q_2+p_2^4q_1^2+p_1^4q_2^2+6p_1^2p_2^2q_1q_2\right.\\
&\left.+5p_2^2q_1^2q_2+5p_1^2q_1q_2^2+4q_1^2q_2^2\right)w_{n+7}+\left(p_1^4p_2^4q_1q_2-p_2^6q_1^3-p_1^6q_2^3+2p_1^2p_2^4q_1^2q_2+2p_1^4p_2^2q_1q_2^2\right.\\
&\left.+4p_1^2p_2^2q_1^2q_2^2-5p_2^4q_1^3q_2-5p_1^4q_1q_2^3-7p_2^2q_1^3q_2^2-7p_1^2q_1^2q_2^3-4q_1^3q_2^3\right)w_{n+6}-\left(p_1^6q_1q_2^4+p_2^6q_1^4q_2\right.\\
&+p_1^6p_2^2q_1q_2^3+p_1^2p_2^6q_1^3q_2+p_1^4p_2^4q_1^2q_2^2+7p_1^2p_2^4q_1^3q_2^2+7p_1^4p_2^2q_1^2q_2^3+6p_2^4q_1^4q_2^2+6p_1^4q_1^2q_2^4\\
&\left.+17p_1^2p_2^2q_1^3q_2^3+11p_2^2q_1^4q_2^3+11p_1^2q_1^3q_2^4+6q_1^4q_2^4\right)w_{n+5}+q_1q_2\left(p_1^6q_1q_2^4+p_2^6q_1^4q_2+p_1^6p_2^2q_1q_2^3\right.\\
&+p_1^2p_2^6q_1^3q_2+p_1^4p_2^4q_1^2q_2^2+7p_1^2p_2^4q_1^3q_2^2+7p_1^4p_2^2q_1^2q_2^3+6p_2^4q_1^4q_2^2+6p_1^4q_1^2q_2^4+17p_1^2p_2^2q_1^3q_2^3\\
&\left.+11p_2^2q_1^4q_2^3+11p_1^2q_1^3q_2^4+6q_1^4q_2^4\right)w_{n+4}-q_1^3q_2^3\left(p_1^4p_2^4q_1q_2-p_2^6q_1^3-p_1^6q_2^3+2p_1^2p_2^4q_1^2q_2\right.\\
&\left.+2p_1^4p_2^2q_1q_2^2+4p_1^2p_2^2q_1^2q_2^2-5p_2^4q_1^3q_2-5p_1^4q_1q_2^3-7p_2^2q_1^3q_2^2-7p_1^2q_1^2q_2^3-4q_1^3q_2^3\right)w_{n+3}\\
&-q_1^5q_2^5\left(p_1^2p_2^4q_1+p_1^4p_2^2q_2+p_2^4q_1^2+p_1^4q_2^2+6p_1^2p_2^2q_1q_2+5p_2^2q_1^2q_2+5p_1^2q_1q_2^2+4q_1^2q_2^2\right)w_{n+2}\\
&-q_1^7q_2^7\left(p_1^2p_2^2+p_1^2q_2+p_2^2q_1+q_1q_2\right)w_{n+1}-q_1^9q_2^9w_n
\end{aligned}\tag{5.2}$$

for $n\geq 0$ *and initial conditions* $w_i=a_i^2b_i^2$ *for* $0\leq i\leq 8$*.*

Proof. Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0=b_0=0$ and a_1, b_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2-p_1x-q_1=0$ with roots α_1 and β_1, such that $\alpha_1+\beta_1=p_1$, and $\alpha_1\beta_1=-q_1$, and the sequence $\{b_n\}$ have the characteristic equation $x^2-p_2x-q_2=0$ with roots α_2 and β_2, such that $\alpha_2+\beta_2=p_2$ and $\alpha_2\beta_2=-q_2$.

Case 1: Let both characteristic functions have distinct roots, meaning $\alpha_1\neq\beta_1$ and $\alpha_2\neq\beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$\begin{aligned}
w_n=&a_n^2b_n^2\\
=&\left(\frac{a_1}{\alpha_1-\beta_1}\right)^2(\alpha_1^n-\beta_1^n)^2\left(\frac{b_1}{\alpha_2-\beta_2}\right)^2(\alpha_2^n-\beta_2^n)^2\\
=&\left(\frac{a_1^2b_1^2}{(\alpha_1-\beta_1)^2(\alpha_2-\beta_2)^2}\right)\left(\left(\alpha_1^2\right)^n-2\left(\alpha_1\beta_1\right)^n+\left(\beta_1^2\right)^n\right)\left(\left(\alpha_2^2\right)^n-2\left(\alpha_2\beta_2\right)^n+\left(\beta_2^2\right)^n\right)
\end{aligned}$$

$$= \left(\frac{a_1^2 b_1}{(\alpha_1-\beta_1)^2(\alpha_2-\beta_2)}\right)\left(\left(\alpha_1^2\alpha_2^2\right)^n - 2\left(\alpha_1^2\alpha_2\beta_2\right)^n + \left(\alpha_1^2\beta_2^2\right)^n - 2\left(\alpha_1\alpha_2^2\beta_1\right)^n + 4\left(\alpha_1\alpha_2\beta_1\beta_2\right)^n\right.$$
$$\left. -2\left(\alpha_1\beta_1\beta_2\right)^2 + \left(\alpha_2^2\beta_1^2\right)^n - 2\left(\alpha_2\beta_1^2\beta_2\right)^n + \left(\beta_1^2\beta_2^2\right)^n\right).$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $r_1 = \alpha_1^2\alpha_2^2$, $r_2 = \alpha_1^2\alpha_2\beta_2$, $r_3 = \alpha_1^2\beta_2^2$, $r_4 = \alpha_1\alpha_2^2\beta_1$, $r_5 = \alpha_1\alpha_2\beta_1\beta_2$, $r_6 = \alpha_1\beta_1\beta_2^2$, $r_7 = \alpha_2^2\beta_1^2$, $r_8 = \alpha_2\beta_1^2\beta_2$, and $r_9 = \beta_1^2\beta_2^2$. We will let each of them have multiplicity one since that means we will have nine roots, which is how many characteristic roots we need for a ninth order linear divisible sequence. Thus, the characteristic equation is

$$\prod_{i=1}^{9}(x-r_i) = x^9 - \left(\sum_{1\le i\le 9} r_i\right)x^8 + \cdots + (-1)^k\left(\sum_{1\le i_1<\cdots<i_k\le 9} r_{i_1}\cdots r_{i_k}\right)x^{9-k}, \text{ for } k \le 9.$$

Looking at the coefficient of x^8, which becomes the coefficient of w_{n+8} in equation (5.2), we have

$$\begin{aligned}\sum_{1\le i\le 9} r_i =& \alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_2\beta_2 + \alpha_1^2\beta_2^2 + \alpha_1\alpha_2^2\beta_1 + \alpha_1\alpha_2\beta_1\beta_2 + \alpha_1\beta_1\beta_2^2 + \alpha_2^2\beta_1^2 + \alpha_2\beta_1^2\beta_2 + \beta_1^2\beta_2^2\\ =& \left(\alpha_1^2 + \alpha_1\beta_1 + \beta_1^2\right)\left(\alpha_2^2 + \alpha_2\beta_2 + \beta_2^2\right)\\ =& (p_1^2+q_1)(p_2^2+q_2)\\ =& p_1^2p_2^2 + p_1^2q_2 + p_2^2q_1 + q_1q_2.\end{aligned}$$

Looking at the coefficient of x^7, which becomes the coefficient of w_{n+7} in equation (5.2), we have

$$\begin{aligned}\sum_{1\le i<j\le 9} r_ir_j =& \alpha_1^3\alpha_2^4\beta_1 + \alpha_1^2\alpha_2^4\beta_1^2 + \alpha_1\alpha_2^4\beta_1^3 + \alpha_1^4\alpha_2^3\beta_2 + 2\alpha_1^3\alpha_2^3\beta_1\beta_2 + 3\alpha_1^2\alpha_2^3\beta_1^2\beta_2 + 2\alpha_1\alpha_2^3\beta_1^3\beta_2 + \alpha_2^3\beta_1^4\beta_2\\ &+ \alpha_1^4\alpha_2^2\beta_2^2 + 3\alpha_1^3\alpha_2^2\beta_1\beta_2^2 + 4\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2 + 3\alpha_1\alpha_2^2\beta_1^3\beta_2^2 + \alpha_2^2\beta_1^4\beta_2^2 + \alpha_1^4\alpha_2\beta_2^3 + 2\alpha_1^3\alpha_2\beta_1\beta_2^3\\ &+ 3\alpha_1^2\alpha_2\beta_1^2\beta_2^3 + 2\alpha_1\alpha_2\beta_1^3\beta_2^3 + \alpha_2\beta_1^4\beta_2^3 + \alpha_1^3\beta_1\beta_2^4 + \alpha_1^2\beta_1^2\beta_2^4 + \alpha_1\beta_1^3\beta_2^4\\ =& \left(\alpha_1^2+\alpha_1\beta_1+\beta_1^2\right)\left(\alpha_2^2+\alpha_2\beta_2+\beta_2^2\right)\left(\alpha_1\alpha_2^2\beta_1 + \alpha_1^2\alpha_2\beta_2 + \alpha_2\beta_1^2\beta_2 + \alpha_1\beta_1\beta_2^2\right)\\ =& \left(\alpha_1^2+\alpha_1\beta_1+\beta_1^2\right)\left(\alpha_2^2+\alpha_2\beta_2+\beta_2^2\right)\left(\alpha_1\beta_1\left(\alpha_2^2+\beta^2\right) + \alpha_2\beta_2\left(\alpha_1^2+\beta_2^2\right)\right)\\ =& (p_1^2+q_1)(p_2^2+q_2)(-q_1(p_2^2+2q_2) - q_2(p_1^2+2q_1))\\ =& -(p_1^2p_2^4q_1 + p_1^4p_2^2q_2 + p_2^4q_1^2 + p_1^4q_2^2 + 6p_1^2p_2^2q_1q_2 + 5p_2^2q_1^2q_2 + 5p_1^2q_1q_2^2 + 4q_1^2q_2^2).\end{aligned}$$

Looking at the coefficient of x^6, which becomes the coefficient of w_{n+6} in equation (5.2), we have

$$\sum_{1\le i<j<k\le 9} r_ir_jr_k = \alpha_1^6\alpha_2^3\beta_2^3 + \alpha_2^3\beta_1^6\beta_2^3 + \alpha_1^3\alpha_2^6\beta_1^3 + \alpha_1^3\beta_1^3\beta_2^6 + \alpha_1^5\alpha_2^5\beta_1\beta_2 + \alpha_1\alpha_2^5\beta_1^5\beta_2 + \alpha_1^5\alpha_2\beta_1\beta_2^5$$

$$
\begin{aligned}
&+ \alpha_1\alpha_2\beta_1^5\beta_2^5 + 2\alpha_1^5\alpha_2^4\beta_1\beta_2^2 + 2\alpha_1\alpha_2^4\beta_1^5\beta_2^2 + 2\alpha_1^5\alpha_2^2\beta_1\beta_2^4 + 2\alpha_1\alpha_2^2\beta_1^5\beta_2^4 + 2\alpha_1^4\alpha_2^5\beta_1^2\beta_2 \\
&+ 2\alpha_1^2\alpha_2^5\beta_1^4\beta_2 + 2\alpha_1^4\alpha_2\beta_1^2\beta_2^5 + 2\alpha_1^2\alpha_2\beta_1^4\beta_2^5 + 3\alpha_1^5\alpha_2^3\beta_1\beta_2^3 + 3\alpha_1\alpha_2^3\beta_1^5\beta_2^3 + 3\alpha_1^3\alpha_2^5\beta_1^3\beta_2 \\
&+ 3\alpha_1^3\alpha_2\beta_1^3\beta_2^5 + 4\alpha_1^4\alpha_2^4\beta_1^2\beta_2^2 + 4\alpha_1^2\alpha_2^4\beta_1^4\beta_2^2 + 4\alpha_1^4\alpha_2^2\beta_1^2\beta_2^4 + 4\alpha_1^2\alpha_2^2\beta_1^4\beta_2^4 + 6\alpha_1^4\alpha_2^3\beta_1^2\beta_2^3 \\
&+ 6\alpha_1^2\alpha_2^3\beta_1^4\beta_2^3 + 6\alpha_1^3\alpha_2^4\beta_1^3\beta_2^2 + 6\alpha_1^3\alpha_2^2\beta_1^3\beta_2^4 + 8\alpha_1^3\alpha_2^3\beta_1^3\beta_2^3 \\
=&\alpha_2^3\beta_2^3\left(\alpha_1^4 - \alpha_1^2\beta_1^2 + \beta_1^4\right)\left(\alpha_1^2 + \beta_1^2\right) + \alpha_1^3\beta_1^3\left(\alpha_2^4 - \alpha_2^2\beta_2^2 + \beta_2^4\right)\left(\alpha_2^2 + \beta_2^2\right) \\
&+ \alpha_1\alpha_2\beta_1\beta_2\left(\alpha_1^4 + \beta_1^4\right)\left(\alpha_2^4 + \beta_2^4\right) + 2\alpha_1\alpha_2^2\beta_1\beta_2^2\left(\alpha_1^4 + \beta_1^4\right)\left(\alpha_2^2 + \beta_2^2\right) \\
&+ 2\alpha_1^2\alpha_2\beta_1^2\beta_2\left(\alpha_2^4 + \beta_2^4\right)\left(\alpha_1^2 + \beta_1^2\right) + 3\alpha_1\alpha_2^3\beta_1\beta_2^3\left(\alpha_1^4 + \beta_1^4\right) + 3\alpha_1^3\alpha_2\beta_1^3\beta_2\left(\alpha_2^4 + \beta_2^4\right) \\
&+ 4\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2\left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_2^2 + \beta_2^2\right) + 6\alpha_1^2\alpha_2^3\beta_1^2\beta_2^3\left(\alpha_1^2 + \beta_1^2\right) + 6\alpha_1^3\alpha_2^2\beta_1^3\beta_2^2\left(\alpha_2^2 + \beta_2^2\right) \\
&+ 8\alpha_1^3\alpha_2^3\beta_1^3\beta_2^3 \\
=&-q_2^3\left(\left(p_1^2 + 2q_1\right)^2 - 3q_1^2\right)\left(p_1^2 + 2q_1\right) - q_1^3\left(\left(p_2^2 + 2q_2\right)^2 - 3q_2\right)\left(p_2^2 + 2q_2\right) \\
&+ q_1q_2\left(\left(p_1^2 + 2q_1\right)^2 - 2q_1^2\right)\left(\left(p_2^2 + 2q_2\right)^2 - 2q_2^2\right) - 2q_1q_2^2\left(\left(p_1^2 + 2q_1\right)^2 - 2q_1^2\right)\left(p_2^2 + 2q_2\right) \\
&- 2q_1^2q_2\left(\left(p_2^2 + 2q_2\right)^2 - 2q_2^2\right)\left(p_1^2 + 2q_1\right) + 3q_1q_2^3\left(\left(p_1^2 + 2q_1\right)^2 - 2q_1^2\right) \\
&+ 3q_1^3q_2\left(\left(p_2^2 + 2q_2\right)^2 - 2q_2^2\right) + 4q_1^2q_2^2\left(p_1^2 + 2q_1\right)\left(p_2^2 + 2q_2\right) - 6q_1^2q_2^3\left(p_1^2 + 2q_1\right) \\
&- 6q_1^3q_2^2\left(p_2^2 + 2q_2\right) + 8q_1^3q_2^3 \\
=&p_1^4p_2^4q_1q_2 - p_2^6q_1^3 - p_1^6q_2^3 + 2p_1^2p_2^4q_1^2q_2 + 2p_1^4p_2^2q_1q_2^2 + 4p_1^2p_2^2q_1^2q_2^2 - 5p_2^4q_1^3q_2 - 5p_1^4q_1q_2^3 \\
&- 7p_2^2q_1^3q_2^2 - 7p_1^2q_1^2q_2^3 - 4q_1^3q_2^3.
\end{aligned}
$$

Looking at the coefficient of x^5, which becomes the coefficient of w_{n+5} in equation (5.2), we have

$$
\begin{aligned}
\sum_{1\le i_1<\cdots<i_4\le 9} r_{i_1}\cdots r_{i_4} =&\alpha_1^7\alpha_2^5\beta_1\beta_2^3 + \alpha_1\alpha_2^5\beta_1^7\beta_2^3 + \alpha_1^7\alpha_2^4\beta_1\beta_2^4 + \alpha_1\alpha_2^4\beta_1^7\beta_2^4 + \alpha_1^7\alpha_2^3\beta_1\beta_2^5 + \alpha_1\alpha_2^3\beta_1^7\beta_2^5 \\
&+ \alpha_1^5\alpha_2^7\beta_1^3\beta_2 + \alpha_1^4\alpha_2^7\beta_1^4\beta_2 + \alpha_1^3\alpha_2^7\beta_1^5\beta_2 + \alpha_1^5\alpha_2\beta_1^3\beta_2^7 + \alpha_1^4\alpha_2\beta_1^4\beta_2^7 + \alpha_1^3\alpha_2\beta_1^5\beta_2^7 \\
&+ \alpha_1^6\alpha_2^6\beta_1^2\beta_2^2 + \alpha_1^2\alpha_2^6\beta_1^6\beta_2^2 + \alpha_1^6\alpha_2^2\beta_1^2\beta_2^6 + \alpha_1^2\alpha_2^2\beta_1^6\beta_2^6 + 3\alpha_1^6\alpha_2^5\beta_1^2\beta_2^3 + 3\alpha_1^2\alpha_2^5\beta_1^6\beta_2^3 \\
&+ 3\alpha_1^6\alpha_2^3\beta_1^2\beta_2^5 + 3\alpha_1^2\alpha_2^3\beta_1^6\beta_2^5 + 3\alpha_1^5\alpha_2^6\beta_1^3\beta_2^2 + 3\alpha_1^3\alpha_2^6\beta_1^5\beta_2^2 + 3\alpha_1^5\alpha_2^2\beta_1^3\beta_2^6 \\
&+ 3\alpha_1^3\alpha_2^2\beta_1^5\beta_2^6 + 4\alpha_1^6\alpha_2^4\beta_1^2\beta_2^4 + 4\alpha_1^2\alpha_2^4\beta_1^6\beta_2^4 + 4\alpha_1^4\alpha_2^6\beta_1^4\beta_2^2 + 4\alpha_1^4\alpha_2^2\beta_1^4\beta_2^6 \\
&+ 7\alpha_1^5\alpha_2^5\beta_1^3\beta_2^3 + 7\alpha_1^3\alpha_2^5\beta_1^5\beta_2^3 + 7\alpha_1^5\alpha_2^3\beta_1^3\beta_2^5 + 7\alpha_1^3\alpha_2^3\beta_1^5\beta_2^5 + 8\alpha_1^5\alpha_2^4\beta_1^3\beta_2^4 \\
&+ 8\alpha_1^3\alpha_2^4\beta_1^5\beta_2^4 + 8\alpha_1^4\alpha_2^5\beta_1^4\beta_2^3 + 8\alpha_1^4\alpha_2^3\beta_1^4\beta_2^5 + 10\alpha_1^4\alpha_2^4\beta_1^4\beta_2^4
\end{aligned}
$$

$$
\begin{aligned}
=&\alpha_1\alpha_2^3\beta_1\beta_2^3\left(\alpha_1^4-\alpha_1^2\beta_1^2+\beta_1^4\right)\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_2^2+\alpha_2\beta_2+\beta_2^2\right)\\
&+\alpha_1^3\alpha_2\beta_1^3\beta_2\left(\alpha_2^4-\alpha_2^2\beta_2^2+\beta_2^4\right)\left(\alpha_2^2+\beta_2^2\right)\left(\alpha_1^2+\alpha_1\beta_1+\beta_1^2\right)\\
&+\alpha_1^2\alpha_2^2\beta_1^2\beta_2^2\left(\alpha_1^4+\beta_1^4\right)\left(\alpha_2^4+\beta_2^4\right)+3\alpha_1^2\alpha_2^3\beta_1^2\beta_2^3\left(\alpha_1^4+\beta_1^4\right)\left(\alpha_2^2+\beta_2^2\right)\\
&+3\alpha_1^3\alpha_2^2\beta_1^3\beta_2^2\left(\alpha_2^4+\beta_2^4\right)\left(\alpha_1^2+\beta_1^2\right)+4\alpha_1^2\alpha_2^4\beta_1^2\beta_2^4\left(\alpha_1^4+\beta_1^4\right)\\
&+4\alpha_1^4\alpha_2^2\beta_1^4\beta_2^2\left(\alpha_2^4+\beta_2^4\right)+7\alpha_1^3\alpha_2^3\beta_1^3\beta_2^3\left(\alpha_1^2+\beta_1^2\right)\left(\alpha_2^2+\beta_2^2\right)\\
&+8\alpha_1^3\alpha_2^4\beta_1^3\beta_2^4\left(\alpha_1^2+\beta_1^2\right)+8\alpha_1^4\alpha_2^3\beta_1^4\beta_2^3\left(\alpha_2^2+\beta_2^2\right)+10\alpha_1^4\alpha_2^4\beta_1^4\beta_2^4\\
=&q_1q_2^3\left(\left(p_1^2+2q_1\right)^2-3q_1^2\right)\left(p_1^2+2q_1\right)\left(p_2^2+q_2\right)\\
&+q_1^3q_2\left(\left(p_2^2+2q_2\right)^2-3q_2^2\right)\left(p_2^2+2q_2\right)\left(p_1^2+q_1\right)\\
&+q_1^2q_2^2\left(\left(p_1^2+2q_1\right)^2-2q_1^2\right)\left(\left(p_2^2+2q_2\right)^2-2q_2^2\right)\\
&-3q_1^2q_2^3\left(\left(p_1^2+2q_1\right)^2-2q_1^2\right)\left(p_2^2+2q_2\right)-3q_1^3q_2^2\left(\left(p_2^2+2q_2\right)^3-2q_2^2\right)\left(p_1^2+2q_1\right)\\
&+4q_1^2q_2^4\left(\left(p_1^2+2q_1\right)^2-2q_1^2\right)+4q_1^4q_2^2\left(\left(p_2^2+2q_2\right)^2-2q_2^2\right)\\
&+7q_1^3q_2^3\left(p_1^2+2q_1\right)\left(p_2^2+2q_2\right)-8q_1^3q_2^4\left(p_1^2+2q_1\right)-8q_1^4q_2^3\left(p_2^2+2q_2\right)+10q_1^4q_2^4\\
=&p_1^6q_1q_2^4+p_2^6q_1^4q_2+p_1^6p_2^2q_1q_2^3+p_1^2p_2^6q_1^3q_2+p_1^4p_2^4q_1^2q_2^2+7p_1^2p_2^4q_1^3q_2^2+7p_1^4p_2^2q_1^2q_2^3\\
&+6p_2^4q_1^4q_2^2+6p_1^4q_1^2q_2^4+17p_1^2p_2^2q_1^3q_2^3+11p_2^2q_1^4q_2^3+11p_1^2q_1^3q_2^4+6q_1^4q_2^4.
\end{aligned}
$$

When $1 \le i_1 < \cdots < i_5 \le 9$, we can show that $r_{i_1}\cdots r_{i_5} = \alpha_1\alpha_2\beta_1\beta_2(r_{j_1}\cdots r_{j_4})$ where $r_{j_1},\ldots,r_{j_4} \in \{r_{i_1},\ldots,r_{i_5}\}$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is one of the roots in $r_{i_1}\cdots r_{i_5}$, then we have $r_{i_1}\cdots r_{i_5} = \alpha_1\alpha_2\beta_1\beta_2(r_{j_1}\cdots r_{j_4})$ where $r_{j_1},\ldots,r_{j_4} \in \{r_{i_1},\ldots,r_{i_5}\}$ and $r_{j_1},\ldots,r_{j_4} \ne r_5$. For example, $r_1r_2r_3r_4r_5 = \alpha_1\alpha_2\beta_1\beta_2(r_1r_2r_3r_4)$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is not one of the roots in $r_{i_1}\cdots r_{i_5}$, then there exists $r_s, r_t \in \{r_{i_1},\ldots,r_{i_5}\}$, such that $r_sr_t = \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2 = \alpha_1\alpha_2\beta_1\beta_2 r_5$. This means $r_{i_1}\cdots r_{i_5} = \alpha_1\alpha_2\beta_1\beta_2(r_ir_jr_kr_5)$ where $r_i,r_j,r_k \in \{r_{i_1},\ldots,r_{i_5}\}$ and $r_i,r_j,r_k \ne r_5$. For example, in $r_1r_2r_3r_4r_6$ we can see $r_4r_6 = \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2 = \alpha_1\alpha_2\beta_1\beta_2r_5$, which means $r_1r_2r_3r_4r_6 = \alpha_1\alpha_2\beta_1\beta_2(r_1r_2r_3r_5)$.

Thus, looking at the coefficient of x^4, which becomes the coefficient of w_{n+4} in equation (5.2), we have

$$
\begin{aligned}
\sum_{1\le i_1<\cdots<i_5\le 9} r_{i_1}\cdots r_{i_5} =&\alpha_1\alpha_2\beta_1\beta_2\left(\sum_{1\le j_1<\cdots<j_4\le 9} r_{j_1}\cdots r_{j_4}\right)\\
=&q_1q_2\left(p_1^6q_1q_2^4+p_2^6q_1^4q_2+p_1^6p_2^2q_1q_2^3+p_1^2p_2^6q_1^3q_2+p_1^4p_2^4q_1^2q_2^2+7p_1^2p_2^4q_1^3q_2^2\right.\\
&\left.+7p_1^4p_2^2q_1^2q_2^3+6p_2^4q_1^4q_2^2+6p_1^4q_1^2q_2^4+17p_1^2p_2^2q_1^3q_2^3+11p_2^2q_1^4q_2^3+11p_1^2q_1^3q_2^4+6q_1^4q_2^4\right).
\end{aligned}
$$

Since we calculated $\sum_{1\le j_1<\cdots<j_4\le 9} r_{j_1}\cdots r_{j_4}$ as the coefficient of x^5 above, we can just replace it here.

When $1 \le i_1 < \cdots < i_6 \le 9$, we can show that $r_{i_1}\cdots r_{i_6} = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3(r_ir_jr_k)$ where $r_i, r_j, r_k \in \{r_{i_1}, \ldots, r_{i_6}\}$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is one of the roots, then there exists $r_s, r_t \in \{r_{i_1}, \ldots, r_{i_6}\}$ with $r_s, r_t \ne r_5$, such that $r_sr_t = \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2$. This means $r_{i_1}\cdots r_{i_6} = r_sr_tr_5(r_ir_jr_k) = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3(r_ir_jr_k)$ where $r_i, r_j, r_k \in \{r_{i_1}, \ldots, r_{i_6}\}$ and $r_i, r_j, r_k \ne r_5$. For example, in $r_1\cdots r_6$ we can see $r_4r_6 = \alpha_1^2\alpha_2^2\beta_1^2\beta_2^2$, which means $r_1\cdots r_6 = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3(r_1r_2r_3)$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is not one of the roots in $r_{i_1}\cdots r_{i_6}$, then there exists $r_{s_1}, \ldots, r_{s_4} \in \{r_{i_1}, \ldots, r_{i_6}\}$, such that $r_{s_1}\cdots r_{s_4} = \alpha_1^4\alpha_2^4\beta_1^4\beta_2^4 = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3r_5$. This means $r_{i_1}\cdots r_{i_6} = r_{s_1}\cdots r_{s_4}(r_ir_j) = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3(r_ir_jr_5)$ where $r_i, r_j \in \{r_{i_1}, \ldots, r_{i_6}\}$ and $r_i, r_j \ne r_5$. For example, in $r_1r_2r_3r_4r_6r_7$ we can see $r_3r_4r_6r_7 = \alpha_1^4\alpha_2^4\beta_1^4\beta_2^4 = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3r_5$, which means $r_1r_2r_3r_4r_6r_7 = \alpha_1^3\alpha_2^3\beta_1^3\beta_2^3(r_1r_2r_5)$.

Thus looking at the coefficient of x^3, which becomes the coefficient of w_{n+3} in equation (5.2), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_6\le 9} r_{i_1}\cdots r_{i_6} =&\alpha_1^3\alpha_2^3\beta_1^3\beta_2^3\left(\sum_{1\le i<j<k\le 9} r_ir_jr_k\right)\\ =&q_1^3q_2^3\left(p_1^4p_2^4q_1q_2 - p_2^6q_1^3 - p_1^6q_2^3 + 2p_1^2p_2^4q_1^2q_2 + 2p_1^4p_2^2q_1q_2^2 + 4p_1^2p_2^2q_1^2q_2^2 - 5p_2^4q_1^3q_2\right.\\ &\left.-5p_1^4q_1q_2^3 - 7p_2^2q_1^3q_2^2 - 7p_1^2q_1^2q_2^3 - 4q_1^3q_2^3\right).\end{aligned}$$

Since we calculated $\sum_{1\le i<j<k\le 9} r_ir_jr_k$ as the coefficient of x^6 above, we can just replace it here.

When $1 \le i_1 < \cdots < i_7 \le 9$, we can show that $r_{i_1}\cdots r_{i_7} = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5(r_ir_j)$ where $r_i, r_j \in \{r_{i_1}, \ldots, r_{i_7}\}$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is one of the roots, then there exists $r_{s_1}, \ldots, r_{s_4} \in \{r_{i_1}, \ldots, r_{i_7}\}$ with $r_{s_1}, \ldots, r_{s_4} \ne r_5$, such that $r_{s_1}\cdots r_{s_4} = \alpha_1^4\alpha_2^4\beta_1^4\beta_2^4$. This means $r_{i_1}\cdots r_{i_7} = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5(r_ir_j)$ where $r_i, r_j \in \{r_{i_1}, \ldots, r_{i_7}\}$ and $r_i, r_j \ne r_5$. For example, in $r_1\cdots r_7$ we can see $r_3r_4r_6r_7 = \alpha_1^4\alpha_2^4\beta_1^4\beta_2^4$, which means $r_1\cdots r_7 = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5(r_1r_2)$. If $r_5 = \alpha_1\alpha_2\beta_1\beta_2$ is not one of the roots in $r_{i_1}\cdots r_{i_7}$, then there exists $r_{s_1}, \ldots, r_{s_6} \in \{r_{i_1}, \ldots, r_{i_7}\}$, such that $r_{s_1}\cdots r_{s_6} = \alpha_1^6\alpha_2^6\beta_1^6\beta_2^6 = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5r_5$. This means $r_{i_1}\cdots r_{i_7} = r_{s_1}\cdots r_{s_6}(r_i) = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5(r_ir_5)$ where $r_i \in \{r_{i_1}, \ldots, r_{i_7}\}$ and $r_i \ne r_5$. For example, in $r_1r_2r_3r_4r_6r_7r_8$ we can see $r_2r_3r_4r_6r_7r_8 = \alpha_1^6\alpha_2^6\beta_1^6\beta_2^6 = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5r_5$, which means $r_1r_2r_3r_4r_6r_7r_8 = \alpha_1^5\alpha_2^5\beta_1^5\beta_2^5(r_1r_5)$.

Thus looking at the coefficient of x^2,which becomes the coefficient of w_{n+2} in equation (5.2), we have

$$\begin{aligned}\sum_{1\le i_1<\cdots<i_7\le 9} r_{i_1}\cdots r_{i_7} =&\alpha_1^5\alpha_2^5\beta_1^5\beta_2^5\left(\sum_{1\le i<j\le 9} r_ir_j\right)\\ =&-q_1^5q_2^5\left(p_1^2p_2^4q_1 + p_1^4p_2^2q_2 + p_2^4q_1^2 + p_1^4q_2^2 + 6p_1^2p_2^2q_1q_2 + 5p_2^2q_1^2q_2 + 5p_1^2q_1q_2^2 + 4q_1^2q_2^2\right).\end{aligned}$$

Since we calculated $\sum_{1\le i<j\le 9} r_ir_j$ as the coefficient of x^7 above we can just replace it here.

When $1 \leq i_1 < \cdots < i_8 \leq 9$ we can show that $r_{i_1} \cdots r_{i_8} = \alpha_1^7 \alpha_2^7 \beta_1^7 \beta_2^7 (r_i)$ where $r_i \in \{r_{i_1}, \ldots, r_{i_8}\}$. If $r_5 = \alpha_1 \alpha_2 \beta_1 \beta_2$ is one of the roots, then there exists $r_{s_1}, \ldots, r_{s_6} \in \{r_{i_1}, \ldots, r_{i_8}\}$, such that $r_{s_1} \cdots r_{s_6} = \alpha_1^6 \alpha_2^6 \beta_1^6 \beta_2^6$. This means $r_{i_1} \cdots r_{i_8} = \alpha_1^7 \alpha_2^7 \beta_1^7 \beta_2^7 (r_i)$ where $r_i \in \{r_{i_1}, \ldots, r_{i_8}\}$ and $r_i \neq r_5$. For example in $r_1 \cdots r_8$ we can see $r_2 r_3 r_4 r_6 r_7 r_8 = \alpha_1^6 \alpha_2^6 \beta_1^6 \beta_2^6$, which means $r_1 \cdots r_8 = \alpha_1^7 \alpha_2^7 \beta_1^7 \beta_2^7 (r_1)$. If $r_5 = \alpha_1 \alpha_2 \beta_1 \beta_2$ is not one of the roots, then we have $r_1 r_2 r_3 r_4 r_6 r_7 r_8 r_9 = \alpha_1^8 \alpha_2^8 \beta_1^8 \beta_2^8 = \alpha_1^7 \alpha_2^7 \beta_1^7 \beta_2^7 r_5$.

Thus looking at the coefficient of x which becomes the coefficient of w_{n+1} in equation (5.2), we have

$$\begin{aligned}\sum_{1 \leq i_1 < \cdots < i_8 \leq 9} r_{i_1} \cdots r_{i_8} =& \alpha_1^7 \alpha_2^7 \beta_1^7 \beta_2^7 \left(\sum_{1 \leq i \leq 9} r_i \right) \\ =& q_1^7 q_2^7 \left(p_1^2 p_2^2 + p_1^2 q_2 + p_2^2 q_1 + q_1 q_2 \cdot \right).\end{aligned}$$

Since we calculated $\sum_{1 \leq i \leq 9} r_i$ as the coefficient of x^8 above we can just replace it here.

Looking at the constant, which becomes the coefficient of w_n in equation (5.2), we have

$$\sum_{1 \leq i_1 < \cdots < i_8 \leq 9} r_{i_1} \cdots r_{i_8} = \alpha_1^9 \alpha_2^9 \beta_1^9 \beta_2^9 = q_1^9 q_2^9.$$

Thus, we see that all coefficients of the characteristic equation match their corresponding coefficients in the linear homogeneous recurrence relation (5.2).

Case 2: Let one characteristic function have duplicate roots and the other have distinct roots. WLOG we can say the characteristic function of $\{a_n\}$ has the duplicate root, meaning $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$\begin{aligned}w_n =& a_n^2 b_n^2 \\ =& \left(\frac{n a_1 b_1}{\alpha_2 - \beta_2} \right)^2 (\alpha_2^n - \beta_2^n)^2 \left(\alpha_1^2 \right)^{n-1} \\ =& \left(\frac{n^2 a_1^2 b_1^2}{\alpha_1^2 (\alpha_2 - \beta_2)^2} \right) \left(\left(\alpha_1^2 \alpha_2^2 \right)^n - 2 \left(\alpha_1^2 \alpha_2 \beta_2 \right)^n + \left(\alpha_1^2 \beta_2^2 \right)^n \right).\end{aligned}$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has roots $\alpha_1^2 \alpha_2^2$, $\alpha_1^2 \alpha_2 \beta_2$, and $\alpha_1^2 \beta_2^2$ each with a multiplicity of at least three. We will let each of them have multiplicity three since that means we will have nine roots, which is how many characteristic roots we need for a ninth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{ w_n = a_n^2 b_n^2 \right\}$ are $r_1 = \alpha_1^2 \alpha_2^2$, $r_2 = \alpha_1^2 \alpha_2 \beta_2$, $r_3 = \alpha_1^2 \beta_2^2$,

$r_4 = \alpha_1^2\alpha_2^2$, $r_5 = \alpha_1^2\alpha_2\beta_2$, $r_6 = \alpha_1^2\beta_2^2$, $r_7 = \alpha_1^2\alpha_2^2$, $r_8 = \alpha_1^2\alpha_2\beta_2$, and $r_9 = \alpha_1^2\beta_2^2$, then the characteristic equation is

$$\prod_{i=1}^{9}(x - r_i) = x^9 - \left(\sum_{1\leq i\leq 9} r_i\right) x^8 + \cdots + (-1)^k \left(\sum_{1\leq i_1<\cdots<i_k\leq 9} r_{i_1}\cdots r_{i_k}\right) x^{9-k}, \text{ for } k \leq 9.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 throughout. This works because, in this case, $\alpha_1 + \alpha_1 = p_1$ and $\alpha_1\alpha_1 = -q_1$.

Case 3: Let both characteristic functions have duplicate roots, meaning $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Then, by using a combination of equations (3.1) and (4.1), we have

$$w_n = a_n^2 b_n^2 = n^4 a_1^2 b_1^2 \left(\alpha_1^2\right)^{n-1} \left(\alpha_2^2\right)^{n-1} = \frac{n^4 a_1^2 b_1^2}{\alpha_1^2\alpha_2^2} \left(\alpha_1^2\alpha_2^2\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root $\alpha_1^2\alpha_2^2$ each with a multiplicity of at least nine. We will let it have multiplicity nine since that means we will have nine roots, which is how many characteristic roots we need for a ninth order linear divisible sequence. Thus, if the roots of the characteristic equation of $\left\{w_n = a_n^2 b_n^2\right\}$ are $r_1 = \alpha_1^2\alpha_2^2$,$r_2 = \alpha_1^2\alpha_2^2$, $r_3 = \alpha_1^2\alpha_2^2$, $r_4 = \alpha_1^2\alpha_2^2$, $r_5 = \alpha_1^2\alpha_2^2$, $r_6 = \alpha_1^2\alpha_2^2$, $r_7 = \alpha_1^2\alpha_2^2$, $r_8 = \alpha_1^2\alpha_2^2$, and $r_9 = \alpha_1^2\alpha_2^2$,then the characteristic equation is

$$\prod_{i=1}^{9}(x - r_i) = x^9 - \left(\sum_{1\leq i\leq 9} r_i\right) x^8 + \cdots + (-1)^k \left(\sum_{1\leq i_1<\cdots<i_k\leq 9} r_{i_1}\cdots r_{i_k}\right) x^{9-k}, \text{ for } k \leq 9.$$

At this point, this case becomes the same as case 1 by simply replacing β_1 with α_1 and β_2 with α_2 throughout. This works because, in this case since, $\alpha_1 + \alpha_1 = p_1$, $\alpha_1\alpha_1 = -q_1$, $\alpha_2 + \alpha_2 = p_2$, and $\alpha_2\alpha_2 = -q_2$.

Therefore, when we multiply the square two second order linear divisible sequence, we can construct a ninth order linear divisible sequence defined by recurrence relation (5.2). It is easy to see by how we define $\{w_n = a_n^2 b_n^2\}$ that $w_i = a_i^2 b_i^2$ for $0 \leq i \leq 8$. □

Next, we have examples that take the square of second order linear divisible sequences and multiplies it by the square of a different second order linear divisible sequence to construct ninth order linear divisible sequences.

Example 5.7. Using the Fibonacci sequence and the Pell number sequence, we define the sequence $\{w_n = F_n^2 P_n^2\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+9} = 10w_{n+8} + 90w_{n+7} - 117w_{n+6} - 520w_{n+5} + 520w_{n+4} + 117w_{n+3} - 90w_{n+2} - 10w_{n+1} + w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^2 P_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	313600	12	3983377305600	18	50282828993973049600
1	1	7	4826809	13	60784055666569	19	767266772562388171441
2	4	8	73410624	14	927495695774596	20	11707738898202961376400
3	100	9	1121580100	15	14152730707562500	21	178648627831121459592100
4	1296	10	17106024100	16	215956484534681856	22	2726003028483778956121444
5	21025	11	261068880601	17	3295286254248582889	23	41596135659701726163087889

Table 5.7: Terms of the sequence $\{w_n = F_n^2 P_n^2\}$

Example 5.8. Using the Fibonacci sequence and the Mersenne number sequence, we define the sequence $\{w_n = F_n^2 M_n^2\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+9} = 14w_{n+8} - 14w_{n+7} - 305w_{n+6} + 588w_{n+5} + 1176w_{n+4} - 2440w_{n+3} - 448w_{n+2} + 1792w_{n+1} - 512w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^2 M_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	254016	12	347722502400	18	458840293763310144
1	1	7	2725801	13	3642383701009	19	4805056665579338809
2	9	8	28676025	14	38147805784881	20	50319301058697515625
3	196	9	301855876	15	399514947136900	21	526951070751957203716
4	2025	10	3165750225	16	4183896310472022	22	5518305860421069987489
5	24025	11	33190645489	17	43815024413829769	23	57788463091283012018401

Table 5.8: Terms of the sequence $\{w_n = F_n^2 M_n^2\}$

Example 5.9. Using the Fibonacci sequence and the sequence of natural numbers including zero, we define the sequence $\{w_n = F_n^2 N_n^2\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+9} = 6w_{n+8} - 6w_{n+7} - 19w_{n+6} + 24w_{n+5} + 24w_{n+4} - 19w_{n+3} - 6w_{n+2} + 6w_{n+1} - w_n,$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = F_n^2 N_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	2304	12	2985984	18	2163366144
1	1	7	8281	13	9174841	19	6310554721
2	4	8	28224	14	27857284	20	18306090000
3	36	9	93636	15	83722500	21	52838377956
4	144	10	302500	16	249387264	22	151820888164
5	625	11	958441	17	737068201	23	434427310321

Table 5.9: Terms of the sequence $\left\{w_n = F_n^2 N_n^2\right\}$

Example 5.10. Using the Pell number sequence and the Mersenne number sequence, we define the sequence $\left\{w_n = P_n^2 M_n^2\right\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$\begin{aligned} w_{n+9} =& 35w_{n+8} - 245w_{n+7} - 923w_{n+6} + 6090w_{n+5} + 12180w_{n+4} - 7384w_{n+3} - 7840w_{n+2} \\ &+ 4480w_{n+1} - 512w_n, \end{aligned}$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = P_n^2 M_n^2\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	19448100	12	3221322994890000	18	517501026595857890520900
1	1	7	460660369	13	75119326197060601	19	12064914106020402007532089
2	36	8	10824321600	14	1751523888733668036	20	281278427029326147068010000
3	1225	9	253346122225	15	40837009904090430625	21	6557649508678076708867101225
4	32400	10	5918000097636	16	952091200606059014400	22	152883201984231546731679272676
5	808201	11	138105437837929	17	22197115417801407838561	23	3564275255241275447720314832689

Table 5.10: Terms of the sequence $\left\{w_n = P_n^2 M_n^2\right\}$

Example 5.11. Using the Pell number sequence and the sequence of natural numbers including zero, we define the sequence $\left\{w_n = P_n^2 N_n^2\right\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+9} = 15w_{n+8} - 60w_{n+7} - 28w_{n+6} + 330w_{n+5} + 330w_{n+4} - 28w_{n+3} - 60w_{n+2} + 15w_{n+1} - w_n$$

for $n \geq 0$. The table below shows some terms of the sequence $\left\{w_n = P_n^2 N_n^2\right\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	176400	12	27662342400	18	2439943081808400
1	1	7	1399489	13	189218910049	19	15845037003539041
2	16	8	10653696	14	1279043378704	20	102328690818873600
3	225	9	78588225	15	8557818890625	21	657547887222360225
4	2304	10	565488400	16	56750789689344	22	4206157487042799376
5	21025	11	3988048801	17	373405884106369	23	26794595833640213569

Table 5.11: Terms of the sequence $\left\{w_n = P_n^2 N_n^2\right\}$

Example 5.12. Using the Mersenne number sequence and the sequence of natural numbers including zero, we define the sequence $\{w_n = M_n^2 N_n^2\}$. Then, by Theorem 5.2, we get a ninth order linear divisible sequence that satisfies the recurrence relation

$$w_{n+9} = 21w_{n+8} - 189w_{n+7} + 955w_{n+6} - 2982w_{n+5} + 5964w_{n+4} - 7640w_{n+3} + 6048w_{n+2} - 2688w_{n+1} + 512w_n$$

for $n \geq 0$. The table below shows some terms of the sequence $\{w_n = M_n^2 N_n^2\}$.

n	w_n	n	w_n	n	w_n	n	w_n
0	0	6	142884	12	2414739600	18	22264940593476
1	1	7	790321	13	11338629289	19	99230545871209
2	36	8	4161600	14	52606927044	20	439803812250000
3	441	9	21150801	15	241577165025	21	1939536661709241
4	3600	10	104652900	16	1099478073600	22	8514613985411556
5	24025	11	507015289	17	4964906434849	23	37225056794837521

Table 5.12: Terms of the sequence $\{w_n = M_n^2 N_n^2\}$

CHAPTER 6

POLYNOMIAL LINEAR DIVISIBLE SEQUENCES

In this chapter, we construct higher order polynomial linear divisible sequences. We construct these by taking products, powers, and products of powers of polynomial linear divisible sequence in the same manner we did for constructing higher order linear divisible sequences.

6.1

Products of Polynomial Linear Divisible Sequences

Like we did for second order linear divisible sequences, we can talk about the products of second order polynomial linear divisible sequences. Again we define this product term by term; thus, $\{w_n(x)\}$ is the sequence $\{a_{0_1}(x)a_{0_2}(x)\cdots a_{0_i}(x), a_{1_1}(x)a_{1_2}(x)\cdots a_{1_i}(x), a_{2_1}(x)a_{2_2}(x)\cdots a_{2_i}(x), \ldots\}$.

If we multiply two distinct second order polynomial linear divisible sequences, then we construct a forth order polynomial linear divisible sequence.

Theorem 6.1. [9] *Let $\{a_n(x)\}$ and $\{b_n(x)\}$ be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions $a_0(x) = b_0(x) = 0$ and $a_1(x)$, $b_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p_1(x)t - q_1(x) = 0$ with roots $\alpha_1(x)$ and $\beta_1(x)$, such that $\alpha_1(x) + \beta_1(x) = p_1(x)$ and $\alpha_1(x)\beta_1(x) = -q_1(x)$, and the sequence $\{b_n(x)\}$ has a characteristic equation $t^2 - p_2(x)t - q_2(x) = 0$ with roots $\alpha_2(x)$ and $\beta_2(x)$, such that $\alpha_2(x) + \beta_2(x) = p_2(x)$ and $\alpha_2(x)\beta_2(x) = -q_2(x)$. Then $\{w_n(x) = a_n(x)b_n(x)\}$ is a polynomial linear divisible sequence that satisfies the fourth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+4}(x) =& p_1(x)p_2(x)w_{n+3}(x) + \left(p_1^2(x)q_2(x) + p_2^2(x)q_1(x) + 2q_1(x)q_2(x)\right) w_{n+2}(x) \\ &+ p_1(x)p_2(x)q_1(x)q_2(x)w_{n+1}(x) - q_1^2(x)q_2^2(x)w_n(x) \end{aligned} \tag{6.1}$$

for $n \geq 0$ with initial conditions $w_3(x) = a_3(x)b_3(x)$, $w_2(x) = a_2(x)b_2(x)$, $w_1(x) = a_1(x)b_1(x)$, and $w_0(x) = a_0(x)b_0(x)$.

If we multiply three distinct second order polynomial linear divisible sequences, then we construct a eighth order polynomial linear divisible sequence.

Theorem 6.2. *Let $\{a_n(x)\}$, $\{b_n(x)\}$, and $\{c_n(x)\}$ be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions $a_0(x) = b_0(x) = c_0(x) = 0$ and $a_1(x)$, $b_1(x)$, $c_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p_1(x)t - q_1(x) = 0$ with roots $\alpha_1(x)$ and $\beta_1(x)$, such that $\alpha_1(x) + \beta_1(x) = p_1(x)$ and $\alpha_1(x)\beta_1(x) = -q_1(x)$, the sequence $\{b_n(x)\}$ has a characteristic equation $t^2 - p_2(x)t - q_2(x) = 0$ with roots $\alpha_2(x)$ and $\beta_2(x)$, such that $\alpha_2(x) + \beta_2(x) = p_2(x)$ and $\alpha_2(x)\beta_2(x) = -q_2(x)$, and the sequence $\{c_n(x)\}$ has a characteristic equation $t^2 - p_3(x)t - q_3(x) = 0$ with roots $\alpha_3(x)$ and $\beta_3(x)$, such that $\alpha_3(x) + \beta_3(x) = p_3(x)$ and $\alpha_3(x)\beta_3(x) = -q_3(x)$. Then $\{w_n(x) = a_n(x)b_n(x)c_n(x)\}$ is a polynomial linear divisible sequence that satisfies the eighth order linear homogeneous recurrence relation*

$$\begin{aligned}
w_{n+8}(x) = {} & p_1(x)p_2(x)p_3(x)w_{n+7}(x) + \big(p_2^2(x)p_3^2(x)q_1(x) + p_1^2(x)p_3^2(x)q_2(x) + p_1^2(x)p_2^2(x)q_3(x) \\
& +2p_3^2(x)q_1(x)q_2(x) + 2p_2^2(x)q_1(x)q_3(x) + 2p_1^2(x)q_2(x)q_3(x) + 4q_1(x)q_2(x)q_3(x)\big)\, w_{n+6}(x) \\
& + \big(p_1(x)p_2(x)p_3^3(x)q_1(x)q_2(x) + p_1(x)p_2^3(x)p_3(x)q_1(x)q_3(x) + p_1^3(x)p_2(x)p_3(x)q_2(x)q_3(x) \\
& +5p_1(x)p_2(x)p_3(x)q_1(x)q_2(x)q_3(x)\big)\, w_{n+5}(x) - \big(p_1^4(x)q_2^2(x)q_3^2(x) + p_2^4(x)q_1^2(x)q_3^2(x) \\
& + p_3^4(x)q_1^2(x)q_2^2(x) - p_1^2(x)p_2^2(x)p_3^2(x)q_1(x)q_2(x)q_3(x) + 4p_1^2(x)q_1(x)q_2^2(x)q_3^2(x) \\
& +4p_2^2(x)q_1^2(x)q_2(x)q_3^2(x) + 4p_3^2(x)q_1^2(x)q_2^2(x)q_3(x) + 6q_1^2(x)q_2^2(x)q_3^2(x)\big)\, w_{n+4}(x) \\
& + q_1(x)q_2(x)q_3(x)\,\big(p_1(x)p_2(x)p_3^3(x)q_1(x)q_2(x) + p_1(x)p_2^3(x)p_3(x)q_1(x)q_3(x) \\
& +p_1^3(x)p_2(x)p_3(x)q_2(x)q_3(x) + 5p_1(x)p_2(x)p_3(x)q_1(x)q_2(x)q_3(x)\big)\, w_{n+3}(x) \\
& + q_1^2(x)q_2^2(x)q_3^2(x)\,\big(p_2^2(x)p_3^2(x)q_1(x) + p_1^2(x)p_3^2(x)q_2(x) + p_1^2(x)p_2^2(x)q_3(x) \\
& +2p_3^2(x)q_1(x)q_2(x) + 2p_2^2(x)q_1(x)q_3(x) + 2p_1^2(x)q_2(x)q_3(x) + 4q_1(x)q_2(x)q_3(x)\big)\, w_{n+2}(x) \\
& - p_1(x)p_2(x)p_3(x)q_1^3(x)q_2^3(x)q_3^3(x)w_{n+1}(x) - q_1^4(x)q_2^4(x)q_3^4(x)w_n(x) \qquad (6.2)
\end{aligned}$$

for $n \geq 0$ with initial conditions $w_i(x) = a_i(x)b_i(x)c_i(x)$ for $0 \leq i \leq 7$.

If we multiply three distinct second order polynomial linear divisible sequences, then we construct a sixteenth order polynomial linear divisible sequence.

Theorem 6.3. *Let $\{a_n(x)\}$, $\{b_n(x)\}$, $\{c_n(x)\}$, and $\{d_n(x)\}$ be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions $a_0(x) = b_0(x) = c_0(x) = d_0(x) = 0$ and $a_1(x)$, $b_1(x)$, $c_1(x)$, $d_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p_1(x)t - q_1(x) = 0$ with roots $\alpha_1(x)$ and $\beta_1(x)$, such that $\alpha_1(x) + \beta_1(x) = p_1(x)$ and $\alpha_1(x)\beta_1(x) = -q_1(x)$, the sequence $\{b_n(x)\}$ has a characteristic equation $t^2 - p_2(x)t - q_2(x) = 0$ with roots $\alpha_2(x)$ and $\beta_2(x)$, such that $\alpha_2(x) + \beta_2(x) = p_2(x)$ and $\alpha_2(x)\beta_2(x) = -q_2(x)$, the sequence $\{c_n(x)\}$ has a characteristic equation $t^2 - p_3(x)t - q_3(x) = 0$ with roots $\alpha_3(x)$ and $\beta_3(x)$, such that $\alpha_3(x) + \beta_3(x) = p_3(x)$ and $\alpha_3(x)\beta_3(x) = -q_3(x)$, and the sequence $\{d_n(x)\}$ has a characteristic equation $t^2 - p_4(x)t - q_4 = 0$ with roots $\alpha_4(x)$ and $\beta_4(x)$, such that $\alpha_4(x) + \beta_4(x) = p_4(x)$ and $\alpha_4(x)\beta_4(x) = -q_4(x)$. Then $\{w_n(x) = a_n(x)b_n(x)c_n(x)d_n(x)\}$ is a sixteenth order polynomial linear divisible sequence with initial conditions $w_i(x) = a_i(x)b_i(x)c_i(x)d_i(x)$ for $0 \leq i \leq 15$.*

Note that the linear homogeneous recurrence relation constructed here is similar to recurrence relation (3.4) by replacing p_i^k with $p_i^k(x)$, q_i^k with $q_i^k(x)$, and w_{n+j} with $w_{n+j}(x)$ for $1 \leq i \leq 4$, $1 \leq k \leq 8$, and $0 \leq j \leq 16$. For this reason the recurrence relation is not reproduced here due to length.

The proofs of Theorems 6.1, 6.2, and 6.3 are similar to the proofs of Theorems 3.3, 3.4, and 3.5 respectively.

6.2

Powers of Polynomial Linear Divisible Sequences

Like we did for second order linear divisible sequences, we can talk about the powers of second order polynomial linear divisible sequences. Again we define these powers term by term; thus, $\{w_n(x)\}$ is the sequence $\left\{a_0^k(x), a_1^k(x), a_2^k(x), \ldots\right\}$.

If we square a second order polynomial linear divisible sequences, then we construct a third order polynomial linear divisible sequence.

Theorem 6.4. [9] *Let $\{a_n(x)\}$ be a second order polynomial linear divisible sequence that can be defined by (2.3) with initial condition $a_0(x) = 0$ and $a_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic* *equation* $t^2 - p(x)t - q(x) = 0$ *with roots* $\alpha(x)$ *and* $\beta(x)$*, such that* $\alpha(x) + \beta(x) = p(x)$ *and* $\alpha(x)\beta(x) = -q(x)$*. Then* $\left\{w_n(x) = a_n^2(x)\right\}$ *is a polynomial linear divisible sequence that satisfies the third order linear homogeneous recurrence relation*

$$w_{n+3}(x) = \left(p^2(x) + q(x)\right) w_{n+2}(x) + q(x)\left(p^2(x) + q(x)\right) w_{n+1}(x) - q^3(x) w_n(x) \tag{6.3}$$

for $n \geq 0$ *with initial conditions* $w_2(x) = a_2^2(x)$*,* $w_1(x) = a_1^2(x)$*, and* $w_0(x) = a_0^2(x)$*.*

If we cube a second order polynomial linear divisible sequences, then we construct a forth order polynomial linear divisible sequence.

Theorem 6.5. *Let* $\{a_n(x)\}$ *be a second order polynomial linear divisible sequence that can be defined by (2.3) with initial condition* $a_0(x) = 0$ *and* $a_1(x)$ *arbitrary. Suppose the sequence* $\{a_n(x)\}$ *has a characteristic equation* $t^2 - p(x)t - q(x) = 0$ *with roots* $\alpha(x)$ *and* $\beta(x)$*, such that* $\alpha(x) + \beta(x) = p(x)$ *and* $\alpha(x)\beta(x) = -q(x)$*. Then* $\left\{w_n(x) = a_n^3(x)\right\}$ *is a polynomial linear divisible sequence that satisfies the fourth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+4}(x) =& p(x)\left(p^2(x) + 2q(x)\right) w_{n+3}(x) + q(x)\left(p^2(x) + q(x)\right)\left(p^2(x) + 2q(x)\right) w_{n+2}(x) \\ & - p(x)q^3(x)\left(p^2(x) + 2q(x)\right) w_{n+1}(x) - q^6(x) w_n(x) \end{aligned} \tag{6.4}$$

for $n \geq 0$ *with initial conditions* $w_3(x) = a_3^3(x)$*,* $w_2(x) = a_2^3(x)$*,* $w_1(x) = a_1^3(x)$*, and* $w_0(x) = a_0^3(x)$*.*

If we take the forth power of a second order polynomial linear divisible sequences, then we construct a fifth order polynomial linear divisible sequence.

Theorem 6.6. *Let* $\{a_n(x)\}$ *be a second order polynomial linear divisible sequence that can be defined by (2.3) with initial condition* $a_0(x) = 0$ *and* $a_1(x)$ *arbitrary. Suppose the sequence* $\{a_n(x)\}$ *has a characteristic equation* $t^2 - p(x)t - q(x) = 0$ *with roots* $\alpha(x)$ *and* $\beta(x)$*, such that* $\alpha(x) + \beta(x) = p(x)$ *and* $\alpha(x)\beta(x) = -q(x)$*. Then* $\left\{w_n(x) = a_n^4(x)\right\}$ *is a polynomial linear divisible sequence that satisfies the fifth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+5}(x) = & \left(p^4(x) + 3p^2(x)q(x) + q^2(x)\right) w_{n+4}(x) + \left(p^6(x)q(x) + 5p^4(x)q^2(x) + 7p^2(x)q^3(x)\right. \\ & \left. + 2q^4(x)\right) w_{n+3}(x) - \left(p^6(x)q^3(x) + 5p^4(x)q^4(x) + 7p^2(x)q^5(x) + 2q^6(x)\right) w_{n+2}(x) \\ & - \left(p^4(x)q^6(x) + 3p^2(x)q^7(x) + q^8(x)\right) w_{n+1}(x) + q^{10}(x) w_n(x) \end{aligned} \tag{6.5}$$

for $n \geq 0$ *with initial conditions* $w_4(x) = a_4^4(x)$*,* $w_3(x) = a_3^4(x)$*,* $w_2(x) = a_2^4(x)$*,* $w_1(x) = a_1^4(x)$*, and* $w_0(x) = a_0^4(x)$*.*

If we take the fifth power of a second order polynomial linear divisible sequences, then we construct a sixth order polynomial linear divisible sequence.

Theorem 6.7. *Let $\{a_n(x)\}$ be a second order polynomial linear divisible sequence that can be defined by (2.3) with initial condition $a_0(x) = 0$ and $a_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p(x)t - q(x) = 0$ with roots $\alpha(x)$ and $\beta(x)$, such that $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Then $\left\{w_n(x) = a_n^5(x)\right\}$ is a polynomial linear divisible sequence that satisfies the sixth order linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+6}(x) = {} & \left(p^5(x) + 4p^3(x)q(x) + 3p(x)q^2(x)\right) w_{n+5}(x) + \left(p^8(x)q(x) + 7p^6(x)q^2(x) + 16p^4(x)q^3(x)\right. \\ & \left. +13p^2(x)q^4(x) + 3q^5(x)\right) w_{n+4}(x) - \left(p^9(x)q^3(x) + 8p^7(x)q^4(x) + 22p^5(x)q^5(x)\right. \\ & \left. +23p^3(x)q^6(x) + 6p(x)q^7(x)\right) w_{n+3}(x) - \left(p^8(x)q^6(x) + 7p^6(x)q^7(x) + 16p^4(x)q^8(x)\right. \\ & \left. +13p^2(x)q^9(x) + 3q^{10}(x)\right) w_{n+2}(x) + \left(p^5(x)q^{10}(x) + 4p^3(x)q^{11}(x) + 3p(x)q^{12}(x)\right) w_{n+1}(x) \\ & + q^{15}(x)w_n(x) \end{aligned} \tag{6.6}$$

for $n \geq 0$ with initial conditions $w_i(x) = a_i^5(x)$ for $0 \leq i \leq 5$.

If we take the sixth power of a second order polynomial linear divisible sequences, then we construct a seventh order polynomial linear divisible sequence.

Theorem 6.8. *Let $\{a_n(x)\}$ be a second order polynomial linear divisible sequence that can be defined by (2.3) with initial condition $a_0(x) = 0$ and $a_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p(x)t - q(x) = 0$ with roots $\alpha(x)$ and $\beta(x)$, such that $\alpha(x) + \beta(x) = p(x)$ and $\alpha(x)\beta(x) = -q(x)$. Then $\left\{w_n(x) = a_n^6(x)\right\}$ is a polynomial linear divisible sequence that satisfies the seventh order* *linear homogeneous recurrence relation*

$$\begin{aligned} w_{n+7}(x) = {} & \big(p^6(x)+5p^4(x)q+6p^2(x)q^2(x)+q^3(x)\big)\, w_{n+6}(x) + \big(p^{10}(x)q+9p^8(x)q^2(x) \\ & +29p^6(x)q^3(x)+40p^4(x)q^4(x)+22p^2(x)q^5(x)+3q^6(x)\big)\, w_{n+5}(x) - \big(p^{12}(x)q^3(x) \\ & +11p^{10}(x)q^4(x)+46p^8(x)q^5(x)+90p^6(x)q^6(x)+81p^4(x)q^7(x)+28p^2(x)q^8(x) \\ & +3q^9(x)\big)\, w_{n+4}(x) - \big(p^{12}(x)q^6(x)+11p^{10}(x)q^7(x)+46p^8(x)q^8(x)+90p^6(x)q^9(x) \\ & +81p^4(x)q^{10}(x)+28p^2(x)q^{11}(x)+3q^{12}(x)\big)\, w_{n+3}(x) + \big(p^{10}(x)q^{10}(x)+9p^8(x)q^{11}(x) \\ & +29p^6(x)q^{12}(x)+40p^4(x)q^{13}(x)+22p^2(x)q^{14}(x)+3q^{15}(x)\big)\, w_{n+2}(x) + \big(p^6(x)q^{15}(x) \\ & +5p^4(x)q^{16}(x)+6p^2(x)q^{17}(x)+q^{18}(x)\big)\, w_{n+1}(x) - q^{21}(x)w_n(x) \end{aligned} \tag{6.7}$$

for $n \geq 0$ *with initial conditions* $w_i(x) = a_i^6(x)$ *for* $0 \leq i \leq 6$.

The proofs for Theorems 6.4, 6.5, 6.6, 6.7, and 6.8 are similar to the proofs of Theorems 4.1, 4.2, 4.3, 4.4, and 4.5 respectively.

6.3

Products of Powers of Polynomial Linear Divisible Sequences

Like we did for second order linear divisible sequences, we can talk about the products of powers of second order polynomial linear divisible sequences. Again we define these products of powers term by term: thus, $\{w_n(x)\}$ is the sequence $\Big\{a_{0_1}^{k_1}(x)a_{0_2}^{k_2}(x)\cdots a_{0_i}^{k_i}(x), a_{1_1}^{k_1}(x)a_{1_2}^{k_2}(x)\cdots a_{1_i}^{k_i}(x), a_{2_1}^{k_1}(x)a_{2_2}^{k_2}(x)\cdots a_{2_i}^{k_i}(x), \ldots\Big\}$.

If we square a second order polynomial linear divisible sequences and multiply it by a different second order polynomial linear divisible sequences, then we construct a sixth order polynomial linear divisible sequence.

Theorem 6.9. *Let* $\{a_n(x)\}$ *and* $\{b_n(x)\}$ *be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions* $a_0(x) = b_0(x) = 0$ *and* $a_1(x)$, $b_1(x)$ *arbitrary. Suppose the sequence* $\{a_n(x)\}$ *has a characteristic equation* $t^2 - p_1(x)t - q_1(x) = 0$ *with roots* $\alpha_1(x)$ *and* $\beta_1(x)$, *such that* $\alpha_1(x) + \beta_1(x) = p_1(x)$ *and* $\alpha_1(x)\beta_1(x) = -q_1(x)$, *and the sequence* $\{b_n(x)\}$ *has a characteristic equation* $t^2 - p_2(x)t - q_2(x) = 0$ *with roots* $\alpha_2(x)$ *and* $\beta_2(x)$, *such that* $\alpha_2(x)+\beta_2(x) = p_2(x)$ *and* $\alpha_2(x)\beta_2(x) = -q_2(x)$. *Then* $\{w_n(x) = a_n^2(x)b_n(x)\}$ *is a polynomial linear divisible sequence that satisfies the sixth order linear* *homogeneous recurrence relation*

$$w_{n+6}(x) = \left(p_1^2(x)p_2(x) + p_2(x)q_1(x)\right) w_{n+5}(x) + \left(p_1^4(x)q_2(x) + p_1^2(x)p_2^2(x)q_1(x) + 4p_1^2(x)q_1(x)q_2(x)\right.$$
$$+p_2^2(x)q_1^2(x) + 3q_1^2(x)q_2(x)\big) w_{n+4}(x) - \left(p_1^4(x)p_2(x)q_1(x)q_2(x) + 2p_1^2(x)p_2(x)q_1^2(x)q_2(x)\right.$$
$$-2p_2(x)q_1^3(x)q_2(x) - p_2^2(x)p_2(x)q_1^3(x)\big) w_{n+3}(x) - \left(p_1^4(x)q_1^2(x)q_2^2(x) + p_1^2(x)p_2^2(x)q_1^3(x)q_2(x)\right.$$
$$+4p_1^2(x)q_1^3(x)q_2^2(x) + p_2^2(x)q_1^4(x)q_2(x) + 3q_1^4(x)q_2^2(x)\big) w_{n+2}(x) + \left(p_1^2(x)p_2(x)q_1^4(x)q_2^2(x)\right.$$
$$+p_2(x)q_1^5(x)q_2^2(x)\big) w_{n+1}(x) + q_1^6(x)q_2^3(x)w_n(x). \tag{6.8}$$

for $n \geq 0$ and initial conditions $w_i(x) = a_i^2(x)b_i(x)$ for $0 \leq i \leq 5$.

If we square a second order polynomial linear divisible sequences and multiply it by the square a different second order polynomial linear divisible sequences, then we construct a ninth order polynomial linear divisible sequence.

Theorem 6.10. *Let $\{a_n(x)\}$ and $\{b_n(x)\}$ be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions $a_0(x) = b_0(x) = 0$ and $a_1(x)$, $b_1(x)$ arbitrary. Suppose the sequence $\{a_n(x)\}$ has a characteristic equation $t^2 - p_1(x)t - q_1(x) = 0$ with roots $\alpha_1(x)$ and $\beta_1(x)$, such that $\alpha_1(x) + \beta_1(x) = p_1(x)$ and $\alpha_1(x)\beta_1(x) = -q_1(x)$, and the sequence $\{b_n(x)\}$ has a characteristic equation $t^2 - p_2(x)t - q_2(x) = 0$ with roots $\alpha_2(x)$ and $\beta_2(x)$, such that $\alpha_2(x) + \beta_2(x) = p_2(x)$ and $\alpha_2(x)\beta_2 = -q_2(x)$. Then $\left\{w_n(x) = a_n^2(x)b_n^2(x)\right\}$ is a polynomial linear divisible sequence that satisfies the ninth order linear homogeneous recurrence relation*

$$w_{n+9}(x) = \left(p_1^2(x)p_2^2(x)+p_1^2(x)q_2(x)+p_2^2(x)q_1(x)+q_1(x)q_2(x)\right) w_{n+8}(x)+\left(p_1^2(x)p_2^4(x)q_1(x)\right.$$
$$+p_1^4(x)p_2^2(x)q_2(x)+p_2^4(x)q_1^2(x)+p_1^4(x)q_2^2(x)+6p_1^2(x)p_2^2(x)q_1(x)q_2(x)+5p_2^2(x)q_1^2(x)q_2(x)$$
$$+5p_1^2(x)q_1(x)q_2^2(x)+4q_1^2(x)q_2^2(x)\big) w_{n+7}(x)+\left(p_1^4(x)p_2^4(x)q_1(x)q_2(x)-p_2^6(x)q_1^3(x)-p_1^6(x)q_2^3(x)\right.$$
$$+2p_1^2(x)p_2^4(x)q_1^2(x)q_2(x)+2p_1^4(x)p_2^2(x)q_1(x)q_2^2(x)+4p_1^2(x)p_2^2(x)q_1^2(x)q_2^2(x)-5p_2^4(x)q_1^3(x)q_2(x)$$
$$-5p_1^4(x)q_1(x)q_2^3(x)-7p_2^2(x)q_1^3(x)q_2^2(x)-7p_1^2(x)q_1^2(x)q_2^3(x)-4q_1^3(x)q_2^3(x)\big) w_{n+6}(x)$$
$$-\left(p_1^6(x)q_1(x)q_2^4(x)+p_2^6(x)q_1^4(x)q_2(x)+p_1^6(x)p_2^2(x)q_1(x)q_2^3(x)+p_1^2(x)p_2^6(x)q_1^3(x)q_2(x)\right.$$
$$+p_1^4(x)p_2^4(x)q_1^2(x)q_2^2(x)+7p_1^2(x)p_2^4(x)q_1^3(x)q_2^2(x)+7p_1^4(x)p_2^2(x)q_1^2(x)q_2^3(x)+6p_2^4(x)q_1^4(x)q_2^2(x)$$
$$+6p_1^4(x)q_1^2(x)q_2^4(x)+17p_1^2(x)p_2^2(x)q_1^3(x)q_2^3(x)+11p_2^2(x)q_1^4(x)q_2^3(x)+11p_1^2(x)q_1^3(x)q_2^4(x)$$

$$+6q_1^4(x)q_2^4(x)\big)\, w_{n+5}(x)+q_1(x)q_2(x)\left(p_1^6(x)q_1(x)q_2^4(x)+p_2^6(x)q_1^4(x)q_2(x)+p_1^6(x)p_2^2(x)q_1(x)q_2^3(x)\right.$$

$$+p_1^2(x)p_2^6(x)q_1^3(x)q_2(x)+p_1^4(x)p_2^4(x)q_1^2(x)q_2^2(x)+7p_1^2(x)p_2^4(x)q_1^3(x)q_2^2(x)+7p_1^4(x)p_2^2(x)q_1^2(x)q_2^3(x)$$

$$+6p_2^4(x)q_1^4(x)q_2^2(x)+6p_1^4(x)q_1^2(x)q_2^4(x)+17p_1^2(x)p_2^2(x)q_1^3(x)q_2^3(x)+11p_2^2(x)q_1^4(x)q_2^3(x)$$

$$+11p_1^2(x)q_1^3(x)q_2^4(x)+6q_1^4(x)q_2^4(x)\big)\, w_{n+4}(x)-q_1^3(x)q_2^3(x)\left(p_1^4(x)p_2^4(x)q_1(x)q_2(x)-p_2^6(x)q_1^3(x)\right.$$

$$-p_1^6(x)q_2^3(x)+2p_1^2(x)p_2^4(x)q_1^2(x)q_2(x)+2p_1^4(x)p_2^2(x)q_1(x)q_2^2(x)+4p_1^2(x)p_2^2(x)q_1^2(x)q_2^2(x)$$

$$-5p_2^4(x)q_1^3(x)q_2(x)-5p_1^4(x)q_1(x)q_2^3(x)-7p_2^2(x)q_1^3(x)q_2^2(x)-7p_1^2(x)q_1^2(x)q_2^3(x)$$

$$-4q_1^3(x)q_2^3(x)\big)\, w_{n+3}(x)-q_1^5(x)q_2^5(x)\left(p_1^2(x)p_2^4(x)q_1(x)+p_1^4(x)p_2^2(x)q_2(x)+p_2^4(x)q_1^2(x)+p_1^4(x)q_2^2(x)\right.$$

$$+6p_1^2(x)p_2^2(x)q_1(x)q_2(x)+5p_2^2(x)q_1^2(x)q_2(x)+5p_1^2(x)q_1(x)q_2^2(x)+4q_1^2(x)q_2^2(x)\big)\, w_{n+2}(x)$$

$$-q_1^7(x)q_2^7(x)\left(p_1^2(x)p_2^2(x)+p_1^2(x)q_2(x)+p_2^2(x)q_1(x)+q_1(x)q_2(x)\right) w_{n+1(x)}-q_1^9(x)q_2^9(x)w_n(x) \quad (6.9)$$

for $n \geq 0$ *and initial conditions* $w_i(x) = a_i^2(x)b_i^2(x)$ *for* $0 \leq i \leq 8$.

The proofs of Theorems 6.9 and 6.10 are similar to the proofs of Theorems 5.1 and 5.2 respectively.

CHAPTER 7

CONCLUSION

The main reason to continue the examination of constructions started by He and Shiue in [9] was to look for a pattern in terms of the ps and qs from the second order linear divisible sequences we were multiplying. The reason to look for a pattern is so that in the future we would not have to go through this entire construction process each time. Based on the constructions, I did not see any evidence of a pattern in multiplying distinct second order linear divisible sequences at this time. I also did not see any evidence when taking a power of a single second order linear divisible sequences at this time.

While there was no pattern that worked for every coefficient of either the product of multiple second order linear divisible sequences or for the powers of a single second order linear divisible sequence there are other things that we can learn from our constructions.

There was one pattern that did become clear as we worked on these constructions. That pattern tells us the order of the linear divisible sequence that is the result of the construction. It is important to note that the order of the linear divisible sequences was dependent on our choice of the multiplicities of the roots.

Theorem 7.1. *Let* $\{a_{n_1}\}, \{a_{n_2}\}, \ldots, \{a_{n_i}\}$ *be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions* $a_{0_i} = 0$ *and* a_{1_i} *arbitrary for all* i*. Suppose the sequence* $\{a_{n_i}\}$ *has characteristic* $x^2 - p_i x - q_i = 0$ *with roots* α_i *and* β_i*, such that* $\alpha_i + \beta_i = p_i$ *and* $\alpha_i \beta_i = -q_i$*. Then we can construct a linear divisible sequence* $\left\{w_n = a_{n_1}^{j_1} a_{n_2}^{j_2} \cdots a_{n_i}^{j_i}\right\}$ *that has the order* $(j_1+1)(j_2+1)\cdots(j_i+1)$.

Proof. It is sufficient to show this for the product of two second order linear divisible sequences. Let $\{a_n\}$ and $\{b_n\}$ be distinct second order linear divisible sequences that can be defined by (2.1) with initial conditions $a_0 = b_0 = 0$ and a_1, b_1 arbitrary. Let the sequence $\{a_n\}$ have the characteristic equation $x^2 - p_1 x - q_1 = 0$ with roots α_1 and β_1, such that $\alpha_1 + \beta_1 = p_1$ and $\alpha_1\beta_1 = -q_1$, and the sequence $\{b_n\}$ have the characteristic equation $x^2 - p_2x - q_2 = 0$ with roots α_2 and β_2, such that $\alpha_2 + \beta_2 = p_2$ and $\alpha_2\beta_2 = -q_2$.

Next, we show that $\left\{a_n^j\right\}$ can be expressed a linear homogeneous recursion relation of order $j+1$ and $\left\{b_n^k\right\}$ can be expressed a linear homogeneous recursion relation of order $k+1$. Let $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$. Then, by equation (4.1), we have

$$a_n^j = \left(\frac{a_1^j}{(\alpha_1 - \beta_1)^j}\right)(\alpha_1^n - \beta_1^n)^j = \left(\frac{a_1^j}{(\alpha_1 - \beta_1)^j}\right)\left(\sum_{s=0}^{j} (-1)^s \left(\alpha_1^{j-s}\beta_1^s\right)^n\right)$$

and

$$b_n^k = \left(\frac{b_1^k}{(\alpha_2 - \beta_2)^k}\right)(\alpha_2^n - \beta_2^n)^k = \left(\frac{b_1^k}{(\alpha_2 - \beta_2)^k}\right)\left(\sum_{t=0}^{k} (-1)^t \left(\alpha_2^{k-t}\beta_2^t\right)^n\right)$$

From the Binomial Theorem we know, $(\alpha_1^n - \beta_1^n)^j$ is a polynomial with $j+1$ terms and $(\alpha_2^n - \beta_2^n)^k$ is a polynomial with $k+1$ terms. Next, Looking at the product $w_n = a_nb_n$ we get

$$\begin{aligned} w_n &= \left(\frac{a_1^j b_1^k}{(\alpha_1 - \beta_1)^j (\alpha_2 - \beta_2)^k}\right)\left(\sum_{s=0}^{j} (-1)^s \left(\alpha_1^{j-s}\beta_1^s\right)^n\right)\left(\sum_{t=0}^{k} (-1)^t \left(\alpha_2^{k-t}\beta_2^t\right)^n\right) \\ &= \left(\frac{a_1^j b_1^k}{(\alpha_1 - \beta_1)^j (\alpha_2 - \beta_2)^k}\right)\left(\sum_{s=0}^{j}\sum_{t=0}^{k} (-1)^{s+t} \left(\alpha_1^{j-s}\beta_1^s\alpha_2^{k-t}\beta_2^t\right)^n\right). \end{aligned}$$

Since the above equations is in the form of equation (1.4), we know the sequence $\{w_n = a_nb_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the roots $\alpha_1^j\alpha_2^k$, $\alpha_1^{j-1}\beta_1\alpha_2^k, \ldots, \alpha_1^j\alpha_2^k, \ldots, \beta_1^j\beta_2^k$ each with a multiplicity of at least one. It is important to note when working out the double summation there will be no like terms. Thus, since we are multiplying a polynomial with $j+1$ term by a polynomial with $k+1$ terms we know our double summation becomes a polynomial with $(j+1)(k+1)$ terms. So, if we let all of the roots have multiplicity one then, we know the characteristic equation of $\{w_n\}$ has $(j+1)(k+1)$ roots and thus is of degree $(j+1)(k+1)$. Therefore, $\{w_n = a_nb_n\}$can be expressed as a linear homogeneous recurrence relation of order $(j+1)(k+1)$. □

Note there is no need to check the situation when one or more sequences have duplicate roots since we only want to show that we can construct a linear divisible sequence with a specific order.

Theorem 7.2. *Let* $\{a_{n_1}(x)\}, \{a_{n_2}(x)\}, \ldots, \{a_{n_i}(x)\}$ *be distinct second order polynomial linear divisible sequences that can be defined by (2.3) with initial conditions* $a_{0_i}(x) = 0$ *and* $a_{1_i}(x)$ *arbitrary for all* i. *Suppose the sequence* $\{a_{n_i}(x)\}$ *has characteristic* $t^2 - p_i(x)t - q_i(x) = 0$ *with roots* $\alpha_i(x)$ *and* $\beta_i(x)$, *such* *that* $\alpha_i(x)+\beta_i(x)=p_i(x)$ *and* $\alpha_i(x)\beta_i(x)=-q_i(x)$. *Then we can construct a polynomial linear divisible sequence* $\left\{w_n=a_{n_1}^{j_1}(x)a_{n_2}^{j_2}(x)\cdots a_{n_i}^{j_i}(x)\right\}$ *that has the order* $(j_1+1)(j_2+1)\cdots(j_i+1)$.

This means that if we were looking to construct a linear divisible sequence of a particular order, we would know how it would be constructed. The table below shows what products of second order linear divisible sequences we could take to construct a linear divisible sequence of a specific order for some smaller orders. A similar table could be constructed for polynomial linear divisible sequences.

order	products	order	products
3	$\{a_n^2\}$	18	$\{a_n^{17}\}$, $\{a_n^8b_n\}$, $\{a_n^5b_n^2\}$, $\{a_n^2b_n^2c_n\}$
4	$\{a_n^3\}$, $\{a_nb_n\}$	19	$\{a_n^{18}\}$
5	$\{a_n^4\}$	20	$\{a_n^{19}\}$, $\{a_n^9b_n\}$, $\{a_n^4b_n^3\}$, $\{a_n^4b_nc_n\}$
6	$\{a_n^5\}$, $\{a_n^3b_n\}$	21	$\{a_n^{20}\}$, $\{a_n^6b_n^2\}$
7	$\{a_n^6\}$	22	$\{a_n^{21}\}$, $\{a_n^{10}b_n\}$
8	$\{a_n^7\}$, $\{a_n^4b_n\}$, $\{a_nb_nc_n\}$	23	$\{a_n^{22}\}$
9	$\{a_n^8\}$, $\{a_n^2b_n^2\}$	24	$\{a_n^{23}\}$, $\{a_n^{11}b_n\}$, $\{a_n^7b_n^2\}$, $\{a_n^5b_n^3\}$, $\{a_n^5b_nc_n\}$, $\{a_n^2b_nc_nd_n\}$
10	$\{a_n^9\}$, $\{a_n^5b_n\}$	25	$\{a_n^{24}\}$, $\{a_n^4b_n^4\}$
11	$\{a_n^{10}\}$	26	$\{a_n^{25}\}$, $\{a_n^{12}b_n\}$
12	$\{a_n^{11}\}$, $\{a_n^6b_n\}$, $\{a_n^3b_n^2\}$, $\{a_n^2b_nc_n\}$	27	$\{a_n^{26}\}$, $\{a_n^8b_n^2\}$, $\{a_n^2b_n^2c_n^2\}$
13	$\{a_n^{12}\}$	28	$\{a_n^{27}\}$, $\{a_n^{13}b_n\}$, $\{a_n^6b_n^3\}$, $\{a_n^6b_nc_n\}$
14	$\{a_n^{13}\}$, $\{a_n^7b_n\}$	29	$\{a_n^{28}\}$
15	$\{a_n^{14}\}$, $\{a_n^4b_n^2\}$	30	$\{a_n^{29}\}$, $\{a_n^{14}b_n\}$, $\{a_n^9b_n^2\}$, $\{a_n^5b_n^4\}$, $\{a_n^5b_n^2c_n\}$
16	$\{a_n^{15}\}$, $\{a_n^7b_n\}$, $\{a_n^3b_n^3\}$, $\{a_n^3b_nc_n\}$, $\{a_nb_nc_nd_n\}$	31	$\{a_n^{30}\}$
17	$\{a_n^{16}\}$	32	$\{a_n^{31}\}$, $\{a_n^{15}b_n\}$, $\{a_n^7b_n^3\}$, $\{a_n^7b_nc_n\}$, $\{a_n^3b_n^3c_n\}$, $\{a_n^3b_nc_nd_n\}$, $\{a_nb_nc_nd_ne_n\}$

Table 7.1: Products of second order linear divisible sequences to make a specific order

It is important to note that the orders we calculated in this thesis was dependent on choosing a multiplicity of one in the case when all of our second order linear divisible sequences had distinct roots. By letting the multiplicity be different, we would construct linear homogeneous recurrence relation of different orders. Constructing these linear homogeneous recurrence relation and comparing them to the ones constructed in this thesis is left for future work.

Another observation is that any coefficient that is the sum of the product of more then half of the roots of the characteristic equation is the product of one of the coefficients that is the sum of the products of less then half of the roots of the characteristic equation times every q from each second order linear divisible sequence to some power. For example, in the proof Theorem 3.5 we showed that the coefficient of x^4, which becomes the coefficient of w_{n+4}, is equal to the coefficient of x^{12}, which becomes the coefficient of w_{n+12}, times all four of the q's to the fourth power. Note that in this case the coefficient of x^4 is the sum of the products of twelve of the roots, and the coefficient of x^{12} is the sum of the products of four of the roots. So we can see this pattern is a result of certain facts. The first is the fact that $\binom{n}{k} = \binom{n}{n-k}$. The second fact is that if we have an even number of roots, then we have matching pairs of roots whose product is the product of q's to some power, and if we have an odd number of roots, then there is one root that is the product of q's to some power and the rest of the roots are matching pairs whose product is the product of q's to some power. This is helpful that if we ever do further construction of this type we only have to work out half of the coefficients.

The next thing that stands out is that if you take the product of multiple distinct second order linear divisible sequence, then each coefficient appears to have its own pattern. This pattern is based off the number of the roots the characteristic equation that are being multiplied. We say that these coefficients appear to have a patter here because, we are not positive if all coefficients have a pattern. The reason for this is just lack of examples. For example, we only have one example of a coefficient that is the product of seven roots of a characteristic function, and one example is not enough to establish a pattern. One pattern that we do see right away is that the coefficient that is the sum of the roots of the characteristic equation is a product of all the p's from our second order liner divisible sequences. There is also a clear pattern in the coefficients that are the sum of the products of two of the roots of the characteristic equation. These patterns are helpful in that if we ever do further constructions of this type we can reduce the amount of coefficients we have to construct. The proof of these patterns is left for future work.

When taking powers of a single second order linear divisible sequence no patterns were evident. The main things that came out are some equalities that became helpful in future proofs. For example, in proof of Theorem 4.3, we showed that if $\alpha + \beta = p$ and $\alpha\beta = -q$, then

$$\alpha^4+\alpha^3\beta+\alpha^2\beta^2+\alpha\beta^3+\beta^4=p^4+3p^2q+q^2.$$

This equality was used in the proofs of some theorems that followed Theorem 4.3. So much that came out of these constructions was saving time in future constructions. Also we did see an easy way to construct a higher order LDS by taking any power of a second order LDS that can be defined by(2.1) where the characteristic equation has a duplicate root.

Theorem 7.3. *Let $\{a_n\}$ be a distinct second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Suppose the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with the duplicate root α, such that $\alpha + \alpha = p$ and $\alpha^2 = -q$. Then $\{w_n = a_n^k\}$ is a linear divisible sequence that satisfies the $k+1$ order linear homogeneous recurrence relation*

$$w_{n+k+1} = \sum_{j=1}^{k+1} (-1)^{j-1} \binom{k+1}{j} \left(\alpha^k\right)^j w_{n+k+1-j} \tag{7.1}$$

for $n \geq 0$ with initial conditions $w_i = a_i^k$, for $0 \leq i \leq k$.

Proof. Let $\{a_n\}$ be a distinct second order linear divisible sequence that can be defined by (2.1) with initial condition $a_0 = 0$ and a_1 arbitrary. Let the sequence $\{a_n\}$ has a characteristic equation $x^2 - px - q = 0$ with the duplicate root α, such that $\alpha + \alpha = p$ and $\alpha^2 = -q$. Then, by equation (4.1), we have

$$w_n = a_n^k = n^k a_1^k ((\alpha)^{n-1})^k = n^k a_1^k \left(\alpha^k\right)^{n-1} = \frac{n^k a_1^k}{\alpha^k} \left(\alpha^k\right)^n.$$

Since the above equation is in the form of equation (1.4), we know the sequence $\{w_n\}$ can be expressed as a linear homogeneous recurrence relation whose characteristic equation has the root α^k with a multiplicity of at least $k+1$. We will let it have multiplicity $k+1$ since that means we will have $k+1$ roots, which is how many characteristic roots we need for a $k+1$ order linear divisible sequence Thus, if we let α^k have multiplicity $k+1$, then the characteristic function become

$$\left(x - \alpha^k\right)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} \left(-\alpha^k\right)^j = x^{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} x^{k+1-j} \left(\alpha^k\right)^j.$$

Therefore, when we take the kth power of a second order linear divisible sequence, we can construct a $k+1$ order linear divisible sequence defined by recurrence relation (7.1). It is easy to see by how we define $w_n = a_n^k$ that $w_i = a_i^k$, for $0 \leq i \leq k$. □

While we did not come up with a pattern, the linear homogeneous recursion relations we constructed are still useful. In He and Shiue[9], they showed that certain well know fourth order linear divisible sequences are actually represented by the linear homogeneous recursion relation (3.2). Thus, these well know fourth order linear divisible sequences are the product of two distinct second order linear divisible sequences. We can now do the same thing with each of the linear homogeneous recursion relations that we constructed. So we could check if eighth order linear divisible sequences are the products of three distinct second order linear divisible sequences, or if ninth order linear divisible sequences are the products of the squares of two different second order linear divisible sequences. This is left for future work. One other possibility for future work is to see if the recurrence relations we constructed work for sequences that could be defined by (2.1) or (2.3) that are not divisible to also construct higher order sequences.

APPENDIX: COEFFICIENTS PRODUCT FOUR SEQUENCES

Factoring, susbsitition of varibles, and simplification of the coefficent of x^{14} from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x14/coefficient-x14.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^{13} from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x13/coefficient-x13.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^{12} from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x12/coefficient-x12.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^{11} from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x11/coefficient-x11.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^{10} from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x10/coefficient-x10.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^9 from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x9/coefficient-x9.pdf

Factoring, susbsitition of varibles, and simplification of the coefficent of x^8 from the characteristic polynomial in Theorem 3.5 can be found online at:

https://www.pdf-archive.com/2017/10/17/coefficient-x8/coefficient-x8.pdf

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