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A Monolithic Arbitrary Lagrangian-Eulerian Finite Element Method for an Unsteady Stokes/Parabolic Interface Problem

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A MONOLITHIC ARBITRARY LAGRANGIAN-EULERIAN FINITE ELEMENT METHOD FOR AN UNSTEADY STOKES/PARABOLIC INTERFACE PROBLEM

By

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ABSTRACT

A MONOLITHIC ALE FINITE ELEMENT METHOD FOR A STOKES/PARABOLIC INTERFACE PROBLEM WITH JUMP COEFFICIENTS

By

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In this thesis, a non-conservative arbitrary Lagrangian-Eulerian (ALE) method is developed and analyzed for a type of linearized Fluid-Structure Interaction (FSI) problem in a time dependent domain with a moving interface - an unsteady Stokes/parabolic interface problem with jump coefficients. The corresponding mixed finite element approximation is analyzed for both semi- and full discretizations based upon the so-called non-conservative ALE scheme. The stability and optimal convergence properties in the energy norm are obtained for both schemes.

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CHAPTER 1

Introduction

1.1 Introduction

This thesis will study a coupled system of partial differential equations (PDEs), which consists of an unsteady Stokes equation and a parabolic equation defined in a time-dependent domain with a moving interface. Such coupled systems of PDEs arise from many fluid-structure interaction (FSI) problems. Fluid-Structure Interaction problems describe the coupled dynamics of fluid mechanics and structure mechanics.They are classical multi-physics problems (Richter, 2010) and as such, have a diverse range of applications in engineering. A key factor in the simulation of such problems comes from the deformation of the domain due to the evolving flow acting on the surface. Specifically, we are looking at a two-way coupled system, that is, the fluid flow affects the structure at the same time that the motion of the structure affects the fluid flow.

A classic example of a two-way coupled system is an elastic structure submerged in a fluid with an inflow condition. As the flow deforms the structure, the deformation of the structure affects the flow. Thus creating feedback between both the flow and structure, i.e., the coupling is two-way. This is illustrated in Figure 1.1.

The thing that every FSI problem has in common is that the domain on which the coupled system is defined will move with respect to time, that is, the domain (often called Ω) is no longer fixed. We can then describe the domain as time dependent ($\Omega(t)$). The movement of

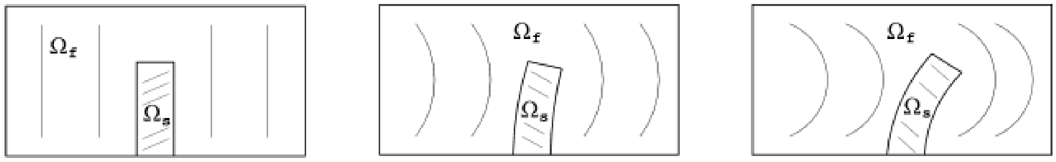

Figure 1.1. Two-way coupled FSI system (Richter, 2010)

the domain can be in the form of a rotation, translation, or deformation.

In order to take this movement into consideration, we will use the Arbitrary-Lagrangian-Eulerian (ALE) description of the model problem, and then adopt the ALE finite element method to discretize the proposed unsteady Stokes/parabolic interface problem. To that end, we first take some arbitrary invertible affine mapping from the initial domain (reference domain) to the domain at any other time in the simulation. With this mapping we can define a domain velocity ω which allows the implementation of a mesh updating algorithm that follows the moving domain. The definition of the mapping and the consequences are further discussed in Section 3.2.

The classical approach to such problems is to discretize the time dependent domain so that the Geometric Conservation Law (GCL) is preserved.

$$\left|\Omega\left(t^{n+1}\right)\right| - \left|\Omega\left(t^{n}\right)\right| = \int_{t^n}^{t^{n+1}} \frac{d}{dt}\int_{\Omega(t)} dxdt = \int_{t^n}^{t^{n+1}} \int_{\partial\Omega(t)} \mathbf{w}\cdot\vec{n}dsdt$$

The GCL is further discussed in Section 3.3. The numerical discretization that we are going to develop in this thesis, now coined the "non-conservative ALE scheme", does not actually satisfy the GCL.In what follows, the development and analysis of our numerical scheme will show that the non-conservative ALE scheme has no downsides and provides a much simpler scheme.

1.2 Outline

This thesis is divided into four sections. In Chapter 2, we provide useful preliminary results and introduce notation used in the remainder of the thesis.

Chapter 3, specifically Section 3.1 , presents the linearized FSI model problem, an unsteady Stokes/parabolic interface problem. Section 3.2 establishes the ALE mapping and some standard definitions, followed by the ALE formulation of the model problem.In Section 3.3 we make some comments on the Reynold's Tranport Theorem and its relation to the Geometric Conservation Law. We then finish this chapter with the Non-Conservative Weak form in Section 3.4.

Chapter 4 consists of the derivation of the semi-discrete scheme followed by the analysis of the stability and error estimates in Sections 4.2 and 4.3 respectively.

Chapter 5 begins with the derivation of the fully-discrete scheme. We then spend the rest of this chapter on the analysis of the error estimates in Section 5.1.

We end the thesis with a few concluding remarks in Chapter 6.

CHAPTER 2

Preliminary Notation and Results

We adopt the standard Lebesgue and Sobolev Spaces taken from Adams and Fournier (2003). Let $\Omega \subset \mathbb{R}^d$ be an open set where $m \in \mathbb{N}$, and $1 \le p \le \infty$. Let $L_p(\Omega)$ denote the linear space of measurable p^{th} power integrable functions on Ω equipped with norm $\|\cdot\|_{L^p(\Omega)}$. The Sobolev space $W^{m,p}(\Omega)$ contains functions $f \in L^p(\Omega)$ that have weak derivatives $D^\alpha f \in L^p(\Omega)$ up to m. For $1 \le p < \infty$, the norm in $W^{m,p}(\Omega)$ is denoted by

$$\|u\|_{W^{m,p}} = \left(\int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}},$$

and for $p = \infty$,

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \le m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

We also use the classical norm and seminorm notations for Sobolev Spaces. In many situations we choose to simplify this notation, we denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and omit the index $p = 2$ and Ω whenever possible, that is, $\|u\|_{W^{m,2}} = \|u\|_{H^m}$. We also denote $W^{0,p}(\Omega)$ by $L^p(\Omega)$ and omit the index $m = 0$ whenever is convenient. That is $\|u\|_{W^{0,p}} = \|u\|_{L^p}$. We also shorten this during the longer proofs to $\|u\|_{L^p} = \|u\|_0$ and $\|u\|_{H^m} = \|u\|_m$.

seriesLemma 2.1 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $1 \le p < \infty$. Then there exists a constant $M > 0$ that only depends on p and Ω such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \le M \|Du\|_{L^p(\Omega)}. \tag{2.1}$$

seriesLemma 2.2 (Cauchy-Schwarz inequality).

$$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \tag{2.2}$$

seriesLemma 2.3 (Young's inequality with ϵ). *If $a, b \in \mathbb{R}$ where $a \geq 0$ and $b \geq 0$, then we have*

$$ab \leq \epsilon\frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2},\ \forall \epsilon > 0.$$

Note that the special case where $\epsilon = 1$ is known as simply Young's inequality and will be used frequently throughout this thesis.

seriesTheorem 2.1 (Reynold's Transport Theorem (Leal, 2007), (Reynolds, 1903)). *Let $\phi(\mathbf{x}, t)$ be a smooth function defined on $\Omega_t \times (0, T)$. we have that*

$$\frac{d}{dt}\int_{\Omega_t} \phi(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_t} \left(\frac{\partial \phi}{\partial t} + \nabla\phi \cdot \omega + \phi \nabla \cdot \omega \right) d\mathbf{x} = \int_{\Omega_t} \left(\left.\frac{d\phi}{dt}\right|_{\hat{\mathbf{x}}} + \phi \nabla \cdot \omega \right) d\mathbf{x},$$

where ω is domain velocity and

$$\left.\frac{d\phi}{dt}\right|_{\hat{\mathbf{x}}} = \frac{\partial \phi}{\partial t} + \omega \cdot \nabla \phi.$$

It's worth noting that the above equality also holds on open subdomains of Ω_t.

In the following lemma, $\bar{V}_g$, Q_0^1, $\bar{W}_{h,t}$ and $M_{h,t}$ are introduced in Sections 3.4 and 4.1.

seriesLemma 2.4. *Let $(\mathbf{V_1}, \mathbf{V_2}) \in H^1 \cap L^\infty\left(0, T; \bar{V}_g\right)$, $p_1 \in L^2\left(0, T; Q_0^1\right)$ and let $\tilde{\mathbf{V}}_{\mathbf{i}}$, $\tilde{p}_1$ be the interpolation onto $\bar{W}_{h,t}$ and $M_{h,t}$ respectively. We then have*

$$\|\mathbf{V} - \tilde{\mathbf{V}}\|_r \leq Ch^{k+1-r}\|\mathbf{V}\|_{H^{k+1}},\ for\ r = 0, 1, 2\ and\ k \geq r,$$

$$\|p_1 - \tilde{p}_1\|_r \leq Ch^{k-r}\|p_1\|_{H^k},\ for\ r = 0, 1\ and\ k \geq r + 1.$$

seriesLemma 2.5. *Assume we have the same conditions as in Lemma 2.4. We then have*

$$\|\frac{d(\mathbf{V}_i - \tilde{V}_i)}{dt}\big|_{\hat{\mathbf{x}}}\|_1 \leq Ch^{r-1}\|\frac{d\mathbf{V}_i}{dt}\big|_{\hat{\mathbf{x}}}\|_{H^r}, \quad \forall r > 1.$$

This lemma can be found in Gastaldi (2001).

CHAPTER 3

The Unsteady Stokes/Parabolic Interface Problem

3.1 Model Description

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, $\mathcal{I} = (0, T]$ $(T > 0)$. Two subdomains, $\Omega_t^i := \Omega_i(t) \subset \Omega$ $(i = 1, 2)$ $(0 \leq t \leq T)$, satisfying $\overline{\Omega_t^1} \cup \overline{\Omega_t^2} = \overline{\Omega}$, $\Omega_t^1 \cap \Omega_t^2 = \emptyset$. These two subdomains are separated by an interface: $\Gamma_t := \Gamma(t) = \partial\Omega_t^1 \cap \partial\Omega_t^2$, which may move/deform along with $t \in \mathcal{I}$, which causes Ω_t^i $(i = 1, 2)$ to also change with $t \in \mathcal{I}$ and are termed as the current (Eulerian) domains with respect to $\mathbf{x}$, in contrast to their initial (reference/Lagrangian) domains, $\hat{\Omega}^i := \Omega_0^i$ $(i = 1, 2)$ with respect to $\hat{\mathbf{x}}$, where, a *flow map* is defined from $\hat{\Omega}^i$ to Ω_t^i $(i = 1, 2)$, as: $\hat{\mathbf{x}}_i \mapsto \mathbf{x}_i(\hat{\mathbf{x}}_i, t)$ such that $\mathbf{x}_i(\hat{\mathbf{x}}_i, t) = \hat{\mathbf{x}}_i + X_i(\hat{\mathbf{x}}_i, t), \forall t \in \mathcal{I}$, where X_i is the displacement field in the Lagrangian frame. The deformation gradient tensor, $\mathbf{F}_i := \nabla_{\hat{\mathbf{x}}_i}\mathbf{x}_i$, and $\mathbf{J}_i = \det(\mathbf{F}_i)$. A few examples of this type of domain are illustrated in Figure 3.1.

In what follows, we set $\hat{\psi} = \hat{\psi}(\hat{\mathbf{x}}, t)$ which equals $\psi(\mathbf{x}(\hat{\mathbf{x}}, t), t)$, and $\hat{\nabla} = \nabla_{\hat{\mathbf{x}}_i}$ $(i = 1, 2)$.

We define the Stokes equations in Ω_t^1 and the parabolic equation in Ω_t^2 with respect to $\mathbf{V}_t^i \in H^1(0, T; H^3(\Omega_t^i)), i = 1, 2$ and $p_1 \in H^1\left(0, T; H^2(\Omega_t^1)\right)$ as follows

$$
\left\{
\begin{array}{rcll}
\frac{\partial \mathbf{V}_1}{\partial t} - \nabla \cdot (\mu_1 \nabla \mathbf{V}_1) + \nabla p_1 & = & f_1, & \text{in } \Omega_t^1 \times \mathcal{I} \\
\nabla \cdot \mathbf{V}_1 & = & 0, & \text{in } \Omega_t^1 \times \mathcal{I} \\
\mathbf{V}_1 & = & g_1, & \text{on } \partial\Omega_t^1 \backslash \Gamma_t \times \mathcal{I} \\
\mathbf{V}_1(\mathbf{x}, 0) & = & \mathbf{V}_1^0, & \text{in } \hat{\Omega}^1 \\
\frac{\partial \mathbf{V}_2}{\partial t} - \nabla \cdot (\mu_2 \nabla \mathbf{V}_2) & = & f_2, & \text{in } \Omega_t^2 \times \mathcal{I} \\
\mathbf{V}_2 & = & g_2, & \text{on } \partial\Omega_t^2 \backslash \Gamma_t \times \mathcal{I} \\
\mathbf{V}_2(\mathbf{x}, 0) & = & \mathbf{V}_2^0, & \text{in } \hat{\Omega}^2 \\
\mathbf{V}_1 & = & \mathbf{V}_2, & \text{on } \Gamma_t \times \mathcal{I} \\
\left(-p_1 I + \mu_1 \nabla \mathbf{V}_1\right) n_1 + \mu_2 \nabla \mathbf{V}_2 n_2 & = & \tau, & \text{on } \Gamma_t \times \mathcal{I}
\end{array}
\right. \tag{3.1}
$$

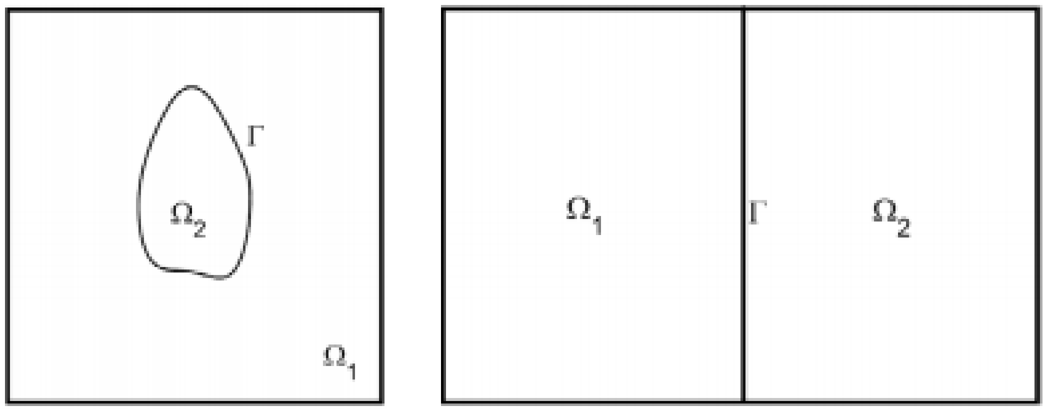

Figure 3.1. An immersed and partitioned domain (Lan et al., 2017)

3.2 ALE Mapping

With the model problem in place, we now define the affine mapping that allows us to use the ALE description of the model problem. Assume $\exists X_t^i \in H^1\left(0, T; W^{2,\infty}(\hat{\Omega}^i)^2\right)$ such that $\forall t \in \mathcal{I}$, the mapping:

$$\begin{aligned} X_t^i : \hat{\Omega}^i &\rightarrow \Omega_t^i \\ \hat{\mathbf{x}}_i &\rightarrow \mathbf{x}_i(\hat{\mathbf{x}}_i, t) \end{aligned}$$

is invertible and $(X_t^i)^{-1} \in W^{2,\infty}(\Omega_t^i)^2$. $\hat{\mathbf{x}}_i \in \hat{\Omega}^i$ is known as the reference coordinate variable. The domain velocity is then defined as

$$\omega_i : \Omega_t^i \times \mathcal{I} \rightarrow R^2, \ \ \omega_i(\mathbf{x}_i, t) = \frac{\partial X_t^i\,(\hat{\mathbf{x}}_i, t)}{\partial t} \quad for\ i = 1, 2$$

With this domain velocity, we can now define a derivative which takes this velocity into account. This is known as the ALE derivative and is defined as

$$\begin{aligned} \frac{d\mathbf{V}_i}{dt}\Big|_{\hat{\mathbf{x}}} : \Omega_t^i \times \mathcal{I} &\rightarrow R \\ (\mathbf{x}_i, t) &\rightarrow \frac{d\mathbf{V}_i}{dt}\Big|_{\hat{\mathbf{x}}}(\mathbf{x}_i, t) = \frac{\partial \mathbf{V}_i}{\partial t}(\mathbf{x}_i, t) + \left(\omega_i(\mathbf{x}_i, t) \cdot \nabla\right) \mathbf{V}_i(\mathbf{x}_i, t) \end{aligned} \tag{3.2}$$

Equipped with the domain velocity and ALE derivative, we can proceed to rewrite our problem using the ALE description. In order to do this, we note that

$$\frac{\partial \mathbf{V}_i}{\partial t}(\mathbf{x}_i, t) = \left.\frac{d\mathbf{V}_i}{dt}\right|_{\hat{\mathbf{x}}}(\mathbf{x}_i, t) - (\omega_i(\mathbf{x}_i, t) \cdot \nabla)\, \mathbf{V}_i(\mathbf{x}_i, t)$$

Substituting this into our model problem we obtain the ALE description as follows.

$$\left\{\begin{array}{rcll}
\left.\frac{d\mathbf{V}_1}{dt}\right|_{\hat{\mathbf{x}}} - (\omega_1 \cdot \nabla)\, \mathbf{V}_1 - \nabla \cdot (\mu_1 \nabla \mathbf{V}_1) + \nabla p_1 & = & f_1, & \text{in } \Omega_t^1 \times \mathcal{I} \\
\nabla \cdot \mathbf{V}_1 & = & 0, & \text{in } \Omega_t^1 \times \mathcal{I} \\
\mathbf{V}_1 & = & g_1, & \text{on } \partial\Omega_t^1 \backslash \Gamma_t \times \mathcal{I} \\
\mathbf{V}_1(\mathbf{x}, 0) & = & \mathbf{V}_1^0, & \text{in } \hat{\Omega}^1 \\
\left.\frac{d\mathbf{V}_2}{dt}\right|_{\hat{\mathbf{x}}} - (\omega_2 \cdot \nabla)\, \mathbf{V}_2 - \nabla \cdot (\mu_2 \nabla \mathbf{V}_2) & = & f_2, & \text{in } \Omega_t^2 \times \mathcal{I} \\
\mathbf{V}_2 & = & g_2, & \text{on } \partial\Omega_t^2 \backslash \Gamma_t \times \mathcal{I} \\
\mathbf{V}_2(\mathbf{x}, 0) & = & \mathbf{V}_2^0, & \text{in } \hat{\Omega}^2 \\
\omega_1 & = & \omega_2, & \text{on } \Gamma_t \times \mathcal{I} \\
\mathbf{V}_1 & = & \mathbf{V}_2, & \text{on } \Gamma_t \times \mathcal{I} \\
(-p_1 I + \mu_1 \nabla \mathbf{V}_1)\, n_1 + \mu_2 \nabla \mathbf{V}_2 n_2 & = & \tau, & \text{on } \Gamma_t \times \mathcal{I}
\end{array}\right. \tag{3.3}$$

3.3 Geometric Conservation Law

Before continuing to the discretization of our problem, it's pertinent to explore the nature of the GCL. The Geometric Conservation Law (GCL) comes as a consequence of Theorem 2.1. Letting $\phi(\mathbf{x}, t) = 1$, we see that $\frac{\partial \phi}{\partial t} = 0$. Plugging this into Theorem 2.1, we have

$$\frac{d}{dt}\Big|\Omega_t\Big| = \int_{\Omega_t} (\nabla \cdot \mathbf{w})\, d\mathbf{x} = \int_{\partial\Omega_t} \mathbf{w} \cdot \mathbf{n} ds$$

Integrating both sides from t^n to t^{n+1}, we get

seriesLemma 3.1 (Geometric Conservation Law).

$$\Big|\Omega\left(t^{n+1}\right)\Big| - \Big|\Omega\left(t^n\right)\Big| = \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_t} \mathbf{w} \cdot \mathbf{n} ds dt$$

Consider the P.D.E.

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \cdot F = 0$$

where $\mathbf{V}$ is a transported quantity and F is the flux. Choosing a test function $\tilde{V}$, integrating over the entire domain, using integration by parts and applying Theorem 2.1 to take the time derivative out of the integral we can obtain the Conservative Formulation:

$$\frac{d}{dt}\left(\mathbf{V},\tilde{V}\right)_{\Omega_t} - \left(\nabla\cdot\left(\mathbf{V}\mathbf{w}^T\right),\tilde{V}\right)_{\Omega_t} - \left(F,\nabla\tilde{V}\right)_{\Omega_t} = 0.$$

We can see that letting $\mathbf{V}=1$ and $\tilde{V}=1$ we obtain Lemma 3.1. Hence the GCL is conserved.

On the other hand, we can neglect the use of Theorem 2.1. Keeping the ALE time derivative inside of the integral gives the following non-conservative formulation.

$$\left(\left.\frac{d\mathbf{V}}{dt}\right|_{\hat{\mathbf{x}}},\tilde{V}\right)_{\Omega_t} - \left(\mathbf{w}\cdot\nabla\mathbf{V},\tilde{V}\right)_{\Omega_t} - \left(F,\nabla\tilde{V}\right)_{\Omega_t} = 0.$$

Letting $\mathbf{V}$ and $\tilde{V}$ be constants, simply yields $0=0$. Thus, this does not produce Lemma 3.1. However, the non-conservative scheme does not seem to yield any negative results. In fact, the non-conservative formulation is much simpler than the conservative forumlation to be developed and implemented on the fully discrete level, as shown in Chapter 5. This is the formulation we will be analyzing in this thesis.

3.4 Non-Conservative Weak Form

To begin, we need to introduce some Sobolev Spaces.

$$\begin{aligned}
\bar{V} &:= \{(\psi_1,\psi_2)\in H^1(\Omega_t^1)^d\times H^1(\Omega_t^2)^d \big| \psi_1=\psi_2 \text{ on } \Gamma_t\}\\
\bar{V}_g &:= \{(\psi_1,\psi_2)\in \bar{V} \big| \psi_i=g_i \text{ on } \partial\Omega_t^i\backslash\Gamma_t, i=1,2\}\\
\bar{V}_0 &:= \{(\psi_1,\psi_2)\in \bar{V} \big| \psi_i=0 \text{ on } \partial\Omega_t^i\backslash\Gamma_t, i=1,2\}\\
Q^1 &:= L^2\left(\Omega_t^1\right)\\
Q_0^1 &:= \{q\in Q^1 \big| \textstyle\int_{\Omega_t^1} q\,dx = 0\}.
\end{aligned}$$

With these spaces we can now define the monolithic weak form of model (3.3). Adding the equations of model (3.3) together, multiplying by test functions $(\psi_1,\psi_2)\in\bar{V}_0$ and applying integration by parts, we obtain the non-conservative weak form as follows. Find $(\mathbf{V}_1, \mathbf{V}_2) \in H^1 \cap L^\infty\left(0, T; \bar{V}_g\right)$ and $p_1 \in L^2\left(0, T; Q_0^1\right)$ such that

$$\sum_{i=1}^{2}\left[\left(\left.\frac{d\mathbf{V}_i}{dt}\right|_{\hat{\mathbf{x}}}, \psi_i\right)_{\Omega_t^i} + \left(\mu_i \nabla \mathbf{V}_i, \nabla \psi_i\right)_{\Omega_t^i} - \left(\left(\omega_i \cdot \nabla\right) \mathbf{V}_i, \psi_i\right)_{\Omega_t^i}\right] - \left(p_1, \nabla \cdot \psi_1\right)_{\Omega_t^1}$$
$$+ \left(\nabla \cdot \mathbf{V}_1, q_1\right)_{\Omega_t^1} = \sum_{i=1}^{2}\left(f_i, \psi_i\right)_{\Omega_t^i} + \left\langle \tau, \psi_1 \right\rangle_{\Gamma_t}, \forall \left(\psi_1, \psi_2\right) \in \bar{V}_0,\ q_1 \in Q^1 \tag{3.4}$$

CHAPTER 4

Finite Element Discretization: Semi-Discretization

Consider a quasi-uniform triangulation $\mathcal{T}^i_{h,0}$ of the continuous domain Ω^i_0. We assume that no triangle of $\mathcal{T}^i_{h,0}$ has two edges on $\partial\Omega^i_0$ and that no triangle crosses the interface Γ_t. We now define the discretization of our ALE mapping X.

4.1 Discretized ALE Mapping and the Semi-Discrete Formulation

For any $t \in \mathcal{I}$ consider the discretization of the mapping X^i_t by means of piecewise linear Lagrangian finite elements. We will denote this mapping $X^i_{h,t}$:

$$\begin{array}{l} X^i_{h,t} : \hat{\Omega}^i \to \Omega^i_t \\ \hat{X}_h^{\;i} \to X^i_{h,t}(\hat{\mathbf{x}}_i, t) \end{array}$$

where $X^i_{h,t}$ is smooth and invertible. Likewise, the discrete mesh velocity is defined as follows:

$$\omega^i_h : \Omega^i_{t,h} \times \mathcal{I} \to R^2, \quad \omega^i_h(\mathbf{x}_i, t) = \frac{dX^i_{h,t}\left(\hat{\mathbf{x}}_i, t\right)}{dt},$$

which leads to the discrete ALE time derivative:

$$\begin{array}{rcl} \dfrac{d\mathbf{V}_i}{dt}\Big|^h_{\hat{\mathbf{x}}} : \Omega^i_{t,h} \times \mathcal{I} & \to & R \\ (\mathbf{x}_i, t) & \to & \dfrac{d\mathbf{V}_i}{dt}\Big|^h_{\hat{\mathbf{x}}}(\mathbf{x}_i, t) = \dfrac{\partial \mathbf{V}_i}{\partial t}(\mathbf{x}_i, t) + \left(\omega^i_h(\mathbf{x}_i, t) \cdot \nabla\right) \mathbf{V}_i(\mathbf{x}_i, t). \end{array}$$

We will denote the image of $\mathcal{T}^i_{h,0}$ under this discrete mapping as $\mathcal{T}^i_{h,t}$. We now proceed to the definition of our discrete spaces using the classical P^2 elements for $\mathbf{V}_i$ and P^1 elements for Q^1. We can find such a mapping by look at the harmonic mapping. That is, it satisfies the following:

$$\left\{ \begin{array}{c} -\Delta X^i_{h,t} = 0, \;\text{ in }\; \hat{\Omega}^i \\ X^i_{h,t} = 0, \;\text{ on }\; \delta\hat{\Omega}^i \backslash \hat{\Gamma}_h \\ X^i_{h,t} = \Pi_h X^i_t, \;\text{ on }\; \Gamma_h \end{array} \right. \tag{4.1}$$

The discrete ALE FEM spaces are defined as follows:

$$\begin{aligned}
\bar{W}_{h,t} &= \{(\psi_{1,h}, \psi_{h,2}) \in \bar{V}_g \big| \psi_{i,h}\big|_K \in P^2(K), \forall K \in \mathcal{T}^i_{h,t}\}, \\
\bar{W}^0_{h,t} &= \{(\psi_{1,h}, \psi_{h,2}) \in \bar{V}_0 \big| \psi_{i,h}\big|_K \in P^2(K), \forall K \in \mathcal{T}^i_{h,t}\}, \\
M_{h,t} &= \{q_h \in Q^1 \big| q_h\big|_K \in P^1(K), \forall K \in \mathcal{T}^i_{h,t}\}, \\
M^0_{h,t} &= \{q_h \in Q^1_0 \big| q_h\big|_K \in P^1(K), \forall K \in \mathcal{T}^i_{h,t}\}, \\
M_h &= \{\mathbf{q}_h \in \left(L^2\left(\Omega_0\right)\right)^2 \big| \mathbf{q}_h\big|_K \in \left(P^1(K)\right)^2, \forall K \in \mathcal{T}^i_{h,0}\},
\end{aligned}$$

where $P^n(K)$ is the set of polynomials on K of degree less than or equal to n.

Now, using (3.4) and the above definitions, the corresponding finite element discretization is: Find $(\mathbf{V}_{1,h}, \mathbf{V}_{2,h}) \in \bar{W}_{h,t}$, $p_{1,h} \in M_{h,t}$ such that

$$\begin{aligned}
&\sum_{i=1}^{2} \left[\left(\frac{d\mathbf{V}_{i,h}}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \psi_{i,h} \right)_{\Omega^i_t} + \left(\mu_i \nabla \mathbf{V}_{i,h}, \nabla \psi_{i,h}\right)_{\Omega^i_t} - \left(\left(\omega_{i,h} \cdot \nabla\right) \mathbf{V}_{i,h}, \psi_{i,h}\right) \right] \qquad (4.2) \\
&- \left(p_{1,h}, \nabla \cdot \psi_{1,h}\right)_{\Omega^1_t} + \left(\nabla \cdot \mathbf{V}_{1,h}, q_{1,h}\right)_{\Omega^1_t} = \sum_{i=1}^{2} \left(f_i, \psi_{i,h}\right)_{\Omega^i_t} + \langle \tau, \psi_{1,h} \rangle_{\Gamma_t} \\
&\forall \left(\psi_{i,h}, \psi_{i,h}\right) \in \bar{W}^0_{h,t},\ q_{1,h} \in M_{h,t}.
\end{aligned}$$

The analysis of the convergence of the above scheme relies on a couple of assumptions about the discrete ALE mapping $X_{h,t}$. We assume that the following error estimate is true:

$$\|X_t - X_{h,t}\|_{L^\infty(\Omega_0)^2} + h\|\nabla\left(X_t - X_{h,t}\right)\|_{L^\infty(\Omega_0)^4} \le Ch^2|lnh|\|X_t\|_{W^{2,\infty}(\Omega_0)^2}.$$

Construction of such a mapping is discussed in Gastaldi (2001).

Assuming $\omega_h \in W^{2,\infty}(\Omega_t)^2$, then we also have the following error estimate on the domain velocity:

$$\|\omega(t) - \omega_h(t)\|_{L^\infty(\Omega_t)^2} + h\|\nabla\left(\omega(t) - \omega_h(t)\right)\|_{L^\infty(\Omega_t)^4} \le Ch^2|lnh|\|\omega(t)\|_{W^{2,\infty}(\Omega_t)^2}.$$

Finally, we assume that our triangulation $\mathcal{T}_{h,t}$ is non-degenerate with time. That is, we assume that there exists a $\rho > 0$ such that

$$diamB_K \ge \rho h\ diamK,\ \forall K \in \mathcal{T}_{h,t}$$

for all $t \in [0,T]$ and all $h \in (0,1]$, where B_k is the largest disk contained in K. We are now in a position to analyze the stability of 4.2.

4.2 Semi-Discretization Stability Analysis

seriesTheorem 4.1. *Assume the conditions for formulation (4.2) hold. Then we can obtain the following estimate for any $t \in \mathcal{I}$:*

$$\sum_{i=1}^{2}\left(\|\mathbf{V}_{i,h}\|_{L^\infty\left(0,t;L^2(\Omega_t^i)\right)} + \|\mathbf{V}_{i,h}\|_{L^2(0,t;H^1(\Omega_t^i))}\right)$$
$$\leq C\left(\sum_{i=1}^{2}\left(\|f_{i,h}\|_{L^2\left(0,t;L^2(\Omega_t^i)\right)} + \|\mathbf{V}_{i,h}(0)\|_{L^2(\Omega_0^i)}\right) + \|\tau\|_{L^2(\Gamma_t)}\right). \tag{4.3}$$

Proof. In (4.2), let $\psi_{i,h} = \mathbf{V}_{i,h},\ q_{1,h} = p_{1,h}$, then

$$\sum_{i=1}^{2}\left[\left(\frac{d\mathbf{V}_{i,h}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \mathbf{V}_{i,h}\right)_{\Omega_t^i} + (\mu_i\nabla\mathbf{V}_{i,h}, \nabla\mathbf{V}_{i,h})_{\Omega_t^i} - ((\omega_{i,h}\cdot\nabla)\,\mathbf{V}_{i,h}, \mathbf{V}_{i,h})\right] \tag{4.4}$$
$$= \sum_{i=1}^{2}(f_i, \mathbf{V}_{i,h})_{\Omega_t^i} + \langle\tau, \mathbf{V}_{1,h}\rangle_{\Gamma_t}.$$

By using the following estimates

$$\begin{aligned}
\left(\frac{d\mathbf{V}_{i,h}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \mathbf{V}_{i,h}\right)_{\Omega_t^i} &= \frac{1}{2}\left(\frac{d}{dt}\|\mathbf{V}_{i,h}\|_0^2 - (\nabla\cdot\omega_h\mathbf{V}_{i,h}, \mathbf{V}_{i,h})\right),\\
(\mu_i\nabla\mathbf{V}_{i,h}, \nabla\mathbf{V}_{i,h})_{\Omega_t^i} &= \mu_i\|\nabla\mathbf{V}_{i,h}\|_0^2 \geq C\|\mathbf{V}_{i,h}\|_1^2,\\
(p_{1,h}, \nabla\cdot\mathbf{V}_{i,h}) &= 0,
\end{aligned}$$

we then have,

$$\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_{i,h}\|_0^2 + C\|\mathbf{V}_{i,h}\|_1^2\right]$$
$$\leq \sum_{i=1}^{2}\left[(f_i, \mathbf{V}_{i,h})_{\Omega_t^i} + \frac{1}{2}(\nabla\cdot\omega_{i,h}\mathbf{V}_{i,h}, \mathbf{V}_{i,h})_{\Omega_t^i} + ((\omega_{i.h}\cdot\nabla)\,\mathbf{V}_{i,h}, \mathbf{V}_{i,h})_{\Omega_t^i}\right]$$
$$+ \langle\tau, \mathbf{V}_{1,h}\rangle_{\Gamma_t}.$$

Using the bound on $\omega_{i,h}$, Young's inequality with ϵ ,the Cauchy-Schwarz inequality and the trace theorem we have the following:

$$\left(\left(\omega_{i,h}\cdot\nabla\right)\mathbf{V}_{i,h},\mathbf{V}_{i,h}\right)_{\Omega_t^i}\leq\|\omega_{i,h}\|_\infty\|\nabla\mathbf{V}_{i,h}\|_0\|\mathbf{V}_{i,h}\|_0 \tag{4.5}$$

$$\leq\epsilon\|\mathbf{V}_{i,h}\|_1^2+C\|\mathbf{V}_{i,h}\|_0^2,$$

$$\left(\nabla\cdot\omega_{i,h}\mathbf{V}_{i,h},\mathbf{V}_{i,h}\right)_{\Omega_t^i}\leq C\|\mathbf{V}_{i,h}\|_0^2, \tag{4.6}$$

$$\left(f_{i,h},\mathbf{V}_{i,h}\right)_{\Omega_t^i}\leq\|f_{i,h}\|_0\|\mathbf{V}_{i,h}\|_0\leq C\left(\|f_{i,h}\|_0^2+\|\mathbf{V}_{i,h}\|_0^2\right), \tag{4.7}$$

$$\left\langle\tau,\mathbf{V}_{1,h}\right\rangle_{\Gamma_t}\leq\|\tau\|_{L^2(\Gamma_t)}\|\mathbf{V}_{1,h}\|_{L^2(\Gamma_t)}\leq\|\tau\|_{L^2(\Gamma_t)}\|\mathbf{V}_{1,h}\|_1 \tag{4.8}$$

$$\leq C\|\tau\|_{L^2(\Gamma_t)}^2+\epsilon\|\mathbf{V}_{1,h}\|_1^2.$$

We choose a sufficiently small ϵ, leading to

$$\sum_{i=1}^2\left[\frac{1}{2}\frac{d}{dt}\|\mathbf{V}_{i,h}\|_0^2+C\|\mathbf{V}_{i,h}\|_1^2\right]\leq\left(\sum_{i=1}^2\left(\|f_{i,h}\|_0^2+\|\mathbf{V}_{i,h}\|_0^2\right)+\|\tau\|_{L^2(\Gamma_t)}^2\right).$$

Integrating over time from 0 to t, then

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^2\left(\|\mathbf{V}_{i,h}(t)\|_0^2-\|\mathbf{V}_{i,h}^0\|_0^2\right)+\sum_{i=1}^2\int_0^t\|\mathbf{V}_{i,h}\|_1^2dt\\&\leq C\left(\sum_{i=1}^2\int_0^t\left(\|f_{i,h}\|_0^2+\|\mathbf{V}_{i,h}\|_0^2\right)dt+\int_0^t\|\tau\|_{L^2(\Gamma_t)}^2dt\right).\end{aligned} \tag{4.9}$$

Using Gronwall's inequality, we have the desired stability result:

$$\begin{aligned}&\sum_{i=1}^2\left(\|\mathbf{V}_{i,h}\|_{L^\infty\left(0,t;L^2(\Omega_t^i)\right)}+\|\mathbf{V}_{i,h}\|_{L^2(0,t;H^1(\Omega_t^i))}\right)\\&\qquad\leq C\left(\sum_{i=1}^2\left(\|f_{i,h}\|_{L^2\left(0,t;L^2(\Omega_t^i)\right)}+\|\mathbf{V}_{i,h}(0)\|_{L^2(\Omega_0^i)}\right)+\|\tau\|_{L^2(\Gamma_t)}\right).\end{aligned}$$

$\square$

4.3 Semi-Discrete Error Analysis

We begin by looking at a handful of lemmas which will help us through the error-analysis.

seriesLemma 4.1. *Assume* $\alpha, \beta, \gamma : \Omega(t) \to \mathbb{R}$ *are smooth functions. Then we have*

$$\frac{d}{dt}\left(\alpha\nabla\beta, \nabla\gamma\right)_{\Omega(t)} = \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\mathbf{x}}}^{h}\nabla\beta, \nabla\gamma\right)_{\Omega(t)} + \left(\alpha\nabla\left.\frac{d\beta}{dt}\right|_{\hat{\mathbf{x}}}^{h}, \nabla\gamma\right)_{\Omega(t)} + \left(\alpha\nabla\beta, \nabla\left.\frac{d\gamma}{dt}\right|_{\hat{\mathbf{x}}}^{h}\right)_{\Omega(t)}$$
$$-\left(\left(\nabla\omega_h + \omega_h^T\right)\alpha\nabla\beta, \nabla\gamma\right)_{\Omega(t)} + \left(\left(\nabla\cdot\omega_h\right)\alpha\nabla\beta, \nabla\gamma\right)_{\Omega(t)}.$$

Proof. Using $\nabla u = \hat{\nabla} u \cdot \hat{F}^{-1}$ where $\hat{F} = \left(\frac{d\mathbf{x}}{d\hat{\mathbf{x}}}\right)$

$$\frac{d}{dt}\left(\alpha\nabla\beta, \nabla\gamma\right)_{\Omega(t)} = \frac{d}{dt}\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}}$$
$$= \sum_{i=1}^{6} G_i,$$

where

$$G_1 = \int_{\Omega(0)} \frac{d\hat{\alpha}}{dt}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}} = \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\mathbf{x}}}^{h}\nabla\beta, \nabla\gamma\right)_{\Omega(t)},$$
$$G_2 = \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\frac{d\beta}{dt}\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}} = \left(\alpha\nabla\left.\frac{d\beta}{dt}\right|_{\hat{\mathbf{x}}}^{h}, \nabla\gamma\right)_{\Omega(t)},$$
$$G_3 = \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\frac{d\hat{F}^{-1}}{dt}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}}$$
$$= -\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\cdot\hat{\nabla}\omega_h\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}}$$
$$= -\left(\alpha\nabla\beta\left(\nabla\omega_h\right), \nabla\gamma\right)_{\Omega(t)},$$
$$G_4 = \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\frac{d\gamma}{dt}\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}} = \left(\alpha\nabla\beta, \nabla\left.\frac{d\gamma}{dt}\right|_{\hat{\mathbf{x}}}^{h}\right)_{\Omega(t)},$$
$$G_5 = \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\frac{d\hat{F}^{-1}}{dt}J\right) d\hat{\mathbf{x}}$$
$$= -\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta\cdot\hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma\cdot\hat{F}^{-1}\hat{\nabla}\omega_h\cdot\hat{F}^{-1}\right) J d\hat{\mathbf{x}}$$
$$= -\left(\alpha\nabla\beta, \nabla\gamma\left(\nabla\omega_h\right)\right)_{\Omega(t)},$$

$$G_6 = \int_{\Omega(0)} \hat{\alpha} \left(\hat{\nabla}\beta \cdot \hat{F}^{-1}\right) : \left(\hat{\nabla}\gamma \cdot \hat{F}^{-1}\right) \frac{dJ}{dt} d\hat{\mathbf{x}} = \left(\alpha \nabla\beta, \nabla\gamma \left(\nabla \cdot \omega_h\right)\right)_{\Omega(t)},$$

where G_3 and G_5 use $\frac{d\hat{F}^{-1}}{dt} = -\hat{F}^{-1}\hat{\nabla}\omega_h \hat{F}^{-1}$, which can be verified by observing that $\hat{F} \cdot \hat{F}^{-1} = I \implies \frac{d}{dt}\left(\hat{F} \cdot \hat{F}^{-1}\right) = \frac{d\hat{F}}{dt}\hat{F}^{-1} + \hat{F}\frac{d\hat{F}^{-1}}{dt} = 0$. So $\frac{d\hat{F}^{-1}}{dt} = -\hat{F}^{-1}\frac{d\hat{F}}{dt}\hat{F}^{-1}$, where $\frac{d\hat{F}}{dt} = \frac{d}{dt}\left(\hat{\nabla}\mathbf{x}\right) = \hat{\nabla}\frac{d\mathbf{x}}{dt} = \hat{\nabla}\omega_h$.

seriesLemma 4.2. *Assume $\alpha, \beta : \Omega(t) \to \mathbb{R}$ are smooth functions. Then we have*

$$\frac{d}{dt}\left(\alpha . \nabla \cdot \beta\right)_{\Omega(t)} = \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\mathbf{x}}}^{h}, \nabla \cdot \beta\right)_{\Omega(t)} + \left(\alpha, \nabla \cdot \left.\frac{d\beta}{dt}\right|_{\hat{\mathbf{x}}}^{h}\right)_{\Omega(t)} + \left(\left(\nabla \cdot \omega_h\right)\alpha, \nabla \cdot \beta\right)_{\Omega(t)}$$
$$- \left(\alpha \nabla\omega_h : \nabla\beta^T\right)_{\Omega(t)}.$$

Proof. Using $\nabla \cdot \beta = \hat{\nabla}\beta : \hat{F}^{-T}$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha . \nabla \cdot \beta\right)_{\Omega(t)} = \frac{d}{dt}\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta : \hat{F}^{-T}\right) J d\hat{\mathbf{x}} \\
&= \sum_{i=1}^{4} G_i,
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= \int_{\Omega(0)} \frac{d\hat{\alpha}}{dt}\left(\hat{\nabla}\beta : \hat{F}^{-T}\right) J d\hat{\mathbf{x}} = \left(\left.\frac{d\alpha}{dt}\right|_{\hat{\mathbf{x}}}^{h}, \nabla \cdot \beta\right)_{\Omega(t)}, \\
G_2 &= \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\frac{d\beta}{dt} : \hat{F}^{-T}\right) J d\hat{\mathbf{x}} = \left(\alpha, \nabla \cdot \left.\frac{d\beta}{dt}\right|_{\hat{\mathbf{x}}}^{h}\right)_{\Omega(t)}, \\
G_3 &= \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta : \frac{\hat{F}^{-T}}{dt}\right) J d\hat{\mathbf{x}} \\
&= -\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta : \hat{F}^{-T}\hat{\nabla}\omega_h^T \hat{F}^{-T}\right) J d\hat{\mathbf{x}} \\
&= -\int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\omega_h \hat{F}^{-T} : \hat{F}^{-T}\hat{\nabla}\beta^T\right) J d\hat{\mathbf{x}} \\
&= -\left(\alpha \nabla\omega_h : \nabla\beta^T\right)_{\Omega(t)}, \\
G_4 &= \int_{\Omega(0)} \hat{\alpha}\left(\hat{\nabla}\beta : \hat{F}^{-T}\right) \frac{dJ}{dt} d\hat{\mathbf{x}} = \left(\alpha, \nabla \cdot \beta\left(\nabla \cdot \omega_h\right)\right)_{\Omega(t)},
\end{aligned}$$

where we use $\frac{d\hat{F}^{-T}}{dt} = -\hat{F}^{-T}\frac{dF^T}{dt}\hat{F}^{-T}$ and $\frac{dF^T}{dt} = \frac{d}{dt}\left(\hat{\nabla}\mathbf{x}^T\right) = \hat{\nabla}\omega_h^T$ for G_3.

seriesLemma 4.3. *Assume $v \in \bar{W}_{h,t}$ and $q \in M_{h,t}$, then the following inf-sup condition holds*

$$\inf_{q\in M_{h,t}} \sup_{v\in\bar{W}_{h,t}} \frac{(\nabla\cdot v, q)}{\|v\|_1\|q\|_0} \geq C > 0.$$

This lemma can be found in Xu and Yang (2015).

We can now proceed to the main theorem of the section, the error estimate of the semi-discrete scheme.

seriesTheorem 4.2. *Suppose $(\mathbf{V}_1, p_1, \mathbf{V}_2)$ is the solution to (3.4) and $(\mathbf{V}_{1,h}, p_{1,h}, \mathbf{V}_{2,h})$ is the solution to (4.2), then we have the following error estimate:*

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\|\frac{d\mathbf{V}_i}{dt}\bigg|_{\hat{\mathbf{x}}} - \frac{d\mathbf{V}_{i,h}}{dt}\bigg|_{\hat{\mathbf{x}}}^h\|_{L^2\left(0,t;L^2(\Omega_t^i)\right)} + \|\mathbf{V}_i - \mathbf{V}_{i,h}\|_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)} \\
&+ \|p_1 - p_{1,h}\|_{L^2(0;t;L^2(\Omega_t^1))}\Bigg] \leq Ch^2\Bigg(\sum_{i=1}^{2}\Bigg[\|\mathbf{V}_i\|_{H^1\cap L^\infty\left(0,t;H^3(\Omega_t^i)\right)} \\
&+ \|\frac{d\mathbf{V}_i}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{L^2(0;t;H^3(\Omega_t^i))}\Bigg] + \|p_1\|_{L^\infty\left(0,t;H^2(\Omega_t^1)\right)} + \|\frac{dp_1}{dt}\bigg|_{\hat{x}}\|_{L^2(0;t;H^2(\Omega_t^1))}\Bigg).
\end{aligned} \tag{4.10}$$

Proof. Subtracting (4.2) from (3.4) and using the identity $\frac{d\mathbf{V}_i}{dt}\Big|_{\hat{\mathbf{x}}} = \frac{d\mathbf{V}_i}{dt}\Big|_{\hat{\mathbf{x}}}^h + (\omega_i - \omega_{i,h})\cdot\nabla\mathbf{V}_i$, we get the error equation:

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\left(\frac{d\mathbf{V}_i}{dt}\bigg|_{\hat{\mathbf{x}}}^h - \frac{d\mathbf{V}_{i,h}}{dt}\bigg|_{\hat{\mathbf{x}}}^h, \psi_{i,h}\right)_{\Omega_t^i} + \left(\mu_i\nabla\left(\mathbf{V}_i - \mathbf{V}_{i,h}\right), \nabla\psi_{i,h}\right)_{\Omega_t^i} - \\
&\left(\left(\omega_{i,h}\cdot\nabla\right)\left(\mathbf{V}_i - \mathbf{V}_{i,h}\right), \psi_{i,h}\right)_{\Omega_t^i}\Bigg] - \left(p_1 - p_{1,h}, \nabla\cdot\psi_{1,h}\right)_{\Omega_t^1} + \\
&\left(\nabla\cdot\left(\mathbf{V}_1 - \mathbf{V}_{i,h}\right), q_{1,h}\right)_{\Omega_t^1} = 0.
\end{aligned}$$

To proceed, we need to introduce the discrete kernel space K_h as

$$K_h := \{(\psi_{1,h}, \psi_{2,h}) \in \bar{W}_{h,t}\big|\ (\nabla\cdot\psi_{1,h}, q_{1,h})_{\Omega_t^1} = 0, \forall q_{1,h} \in M_{h,t}^0\}$$

Picking arbitrary functions $\tilde{\mathbf{V}} = \left(\tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2\right) \in K_h$ and $\tilde{p} \in M^0_{h,t}$. Let $\mathbf{V}_i - \mathbf{V}_{i,h} = \mathbf{V}_i - \tilde{\mathbf{V}}_i + \tilde{\mathbf{V}}_i - \mathbf{V}_{i,h} = \eta_i + \xi_i$, and $p_1 - p_{1,h} = p_1 - \tilde{p}_1 + \tilde{p}_1 - p_{1,h} = \alpha + \beta$, we can rewrite (4.11) as

$$\sum_{i=1}^{2}\left[\left(\frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \psi_{i,h}\right)_{\Omega^i_t} + \left(\mu_i \nabla \xi_i, \nabla \psi_{i,h}\right)_{\Omega^i_t}\right] - \left(\beta, \nabla \cdot \psi_{1,h}\right)_{\Omega^1_t}$$
$$= \sum_{i=1}^{2}\left[-\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\mathbf{x}}}, \psi_{i,h}\right)_{\Omega^i_t} - \left(\mu_i \nabla \eta_i, \nabla \psi_{i,h}\right)_{\Omega^i_t} + \left(\left(\omega_{i,h} \cdot \nabla\right)\left(\eta_i + \xi_i\right), \psi_{i,h}\right)\right]$$
$$+ \left(\alpha, \nabla \cdot \psi_{1,h}\right)_{\Omega^1_t}. \tag{4.11}$$

Choosing $\psi_{i,h} = \frac{d\xi_i}{dt}\Big|^h_{\hat{\mathbf{x}}}$, $q_{1,h} = \beta$, the error equation (4.11) becomes

$$\sum_{i=1}^{2}\left[\left(\frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} + \left(\mu_i \nabla \xi_i, \nabla \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t}\right] - \left(\beta, \nabla \cdot \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right) =$$
$$\sum_{i=1}^{2}\left[-\left(\frac{d\eta_i}{dt}\bigg|_{\hat{\mathbf{x}}}, \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} - \left(\mu_i \nabla \eta_i, \nabla \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} + \left(\left(\omega_i \cdot \nabla\right)\left(\eta_i + \xi_i\right), \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)\right]$$
$$+ \left(\alpha, \nabla \cdot \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^1_t}. \tag{4.12}$$

Using Youngs inequality with ϵ and the Cauchy-Schwarz inequality gives us the following estimates:

$$\left(\frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} = \|\frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\|^2_0, \tag{4.13}$$

$$\left(\mu_i \nabla \xi_i, \nabla \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} = \frac{1}{2}\Bigg(\frac{d}{dt}\left(\mu_i \nabla \xi_i, \nabla \xi_i\right) - \left(\frac{d\mu_i}{dt}\bigg|^h_{\hat{\mathbf{x}}} \nabla \xi_i, \nabla \xi_i\right) + \tag{4.14}$$
$$\left(\mu_i \nabla \xi_i \left(\nabla \omega_{i,h} + \nabla \omega^T_{i,h}\right), \nabla \xi_i\right) - \left(\left(\nabla \cdot \omega_{i,h}\right) \mu_i \nabla \xi_i, \nabla \xi_i\right)\Bigg),$$

$$\left(\beta, \nabla \cdot \frac{d\xi_1}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right) = \frac{d}{dt}\left(\beta, \nabla \cdot \xi_1\right) - \left(\frac{d\beta}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \nabla \cdot \xi_1\right) - \tag{4.15}$$
$$\left(\left(\nabla \cdot \omega_{1,h}\right) \beta, \nabla \cdot \xi_1\right) + \left(\beta \nabla \omega_{1,h}, \nabla \xi^T_1\right),$$

$$-\left(\frac{d\eta_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}, \frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\right)_{\Omega^i_t} \le C\|\frac{d\eta_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\|^2_0 + \epsilon\|\frac{d\xi_i}{dt}\bigg|^h_{\hat{\mathbf{x}}}\|^2_0, \tag{4.16}$$

$$-\left(\mu_i\nabla\eta_i, \nabla\frac{d\xi_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\right) = -\bigg(\frac{d}{dt}\left(\mu_i\nabla\eta_i, \nabla\xi_i\right) - \left(\frac{d\mu_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\nabla\eta_i, \nabla\xi_i\right) - \tag{4.17}$$

$$\left(\mu_i\nabla\frac{d\eta_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \nabla\xi_i\right) + \left(\mu_i\nabla\eta_i\left(\nabla\omega_{i,h} + \nabla\omega_{i,h}^T\right), \nabla\xi_i\right) - \left(\left(\nabla\cdot\omega_{i,h}\right)\mu_i\nabla\eta_i, \nabla\xi_i\right)\bigg),$$

$$\left(\left(\omega_{i,h}\cdot\nabla\right)\left(\eta_i+\xi_i\right), \frac{d\xi_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\right) \le C|\|\omega_{i,h}\|_{W^{2,\infty}}\left(\|\eta_i\|_1^2 + \|\xi_i\|_1^2\right) + \epsilon\|\frac{d\xi_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\|_0^2 \tag{4.18}$$

$$\left(\alpha, \nabla\cdot\frac{d\xi_1}{dt}\bigg|^{h}, _{\hat{\mathbf{x}}}\right) = \frac{d}{dt}\left(\alpha, \nabla\cdot\xi_1\right) - \left(\frac{d\alpha}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \nabla\cdot\xi_1\right) - \tag{4.19}$$

$$\left(\left(\nabla\cdot\omega_{1,h}\right)\alpha, \nabla\cdot\xi_1\right) + \left(\alpha\nabla\omega_{1,h}, \nabla\xi_1^T\right),$$

where (4.14), (4.17) use Lemma 4.1 and (4.15), (4.19) use Lemma 4.2.

Applying the bound on μ_i and $\omega_{i,h}$, as well as Cauchy-Schwarz and Young's inequality with ϵ, we get the following estimates

$$\begin{aligned}
&\left(\frac{d\mu_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\nabla\xi_i, \nabla\xi_i\right) \leq C\|\xi_i\|_1^2,\\
&\left(\mu_i\nabla\xi_i\left(\nabla\omega_{i,h}+\nabla\omega_{i,h}^T\right), \nabla\xi_i\right) \leq C\|\omega_{i,h}\|_{W^{2,\infty}}\|\xi_i\|_1^2,\\
&\left(\left(\nabla\cdot\omega_{i,h}\right)\mu_i\nabla\xi_i, \nabla\xi_i\right) \leq C\|\omega_{i,h}\|_{W^{2,\infty}}\|\xi_i\|_1^2,\\
&\frac{d}{dt}\left(\beta, \nabla\cdot\xi_1\right) = 0,\\
&\left(\frac{d\beta}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \nabla\cdot\xi_1\right) = 0,\\
&\left(\left(\nabla\cdot\omega_{1,h}\right)\beta, \nabla\cdot\xi_1\right) \leq \epsilon\|\beta\|_0^2 + C\|\omega_{1,h}\|_{W^{2,\infty}}\|\xi_1\|_1^2,\\
&\left(\beta\nabla\omega_{1,h}, \nabla\xi_1^T\right) \leq \epsilon\|\beta\|_0^2 + C\|\omega_{1,h}\|_{W^{2,\infty}}\|\xi_1\|_1^2,\\
&\left(\frac{d\mu_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\nabla\eta_i, \nabla\xi_i\right) \leq C\left(\|\eta_i\|_1^2 + \|\xi_i\|_1^2\right), \qquad (4.20)\\
&\left(\mu_i\nabla\frac{d\eta_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \nabla\xi_i\right) \leq C\left(\|\frac{d\eta_i}{dt}{}_{\hat{\mathbf{x}}}^{h}\|_1^2 + \|\xi_i\|_1^2\right),\\
&\left(\mu_i\nabla\eta_i\left(\nabla\omega_{i,h}+\nabla\omega_{i,h}^T\right), \nabla\xi_i\right) \leq C\|\omega_{i,h}\|_{W^{2,\infty}}\left(\|\eta_i\|_1^2 + \|\xi_i\|_1^2\right),\\
&\left(\left(\nabla\cdot\omega_{i,h}\right)\mu_i\nabla\eta_i, \nabla\xi_i\right) \leq C\|\omega_{i,h}\|_{W^{2,\infty}}\left(\|\eta_i\|_1^2 + \|\xi_i\|_1^2\right),\\
&\left(\mu_i\nabla\eta_i\left(\nabla\omega_{i,h}+\nabla\omega_{i,h}^T\right), \nabla\xi_i\right) \leq C\|\omega_{i,h}\|_{W^{2,\infty}}\left(\|\eta_i\|_1^2 + \|\xi_i\|_1^2\right),\\
&\left(\frac{d\alpha}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \nabla\cdot\xi_1\right) \leq C\left(\|\frac{d\alpha}{dt}{}_{\hat{\mathbf{x}}}^{h}\|_0^2 + \|\xi_1\|_1^2\right),\\
&\left(\left(\nabla\cdot\omega_{1,h}\right)\alpha, \nabla\cdot\xi_1\right) \leq C\left(\|\alpha\|_0^2 + \|\xi_1\|_1^2\right),\\
&\left(\alpha\nabla\omega_{1,h}, \nabla\xi_1^T\right) \leq C\left(\|\alpha\|_0^2 + \|\xi_1\|_1^2\right).
\end{aligned}$$

Applying the estimates obtained in (4.20) to (4.14), (4.15), (4.17), (4.19) and choosing ϵ

small enough, we have

$$\sum_{i=1}^{2}\left[\|\frac{d\xi_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_0^2+\frac{d}{dt}\left(\mu_i\nabla\xi_i,\nabla\xi_i\right)_{\Omega_t^i}\right]\leq$$
$$C\Bigg(\sum_{i=1}^{2}\left[\|\xi_i\|_1^2+\|\eta_i\|_1^2+\|\frac{d\eta_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_1^2+\frac{d}{dt}\left(\mu_i\nabla\eta_i,\nabla\xi_i\right)_{\Omega_t^i}\right]+ \tag{4.21}$$
$$\frac{d}{dt}\left(\alpha,\nabla\cdot\xi_1\right)_{\Omega_t^1}+\|\frac{d\alpha}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_0^2+\|\alpha\|_0^2\Bigg)+\epsilon\|\beta\|_0^2.$$

Integrating in time from 0 to t, yields

$$\int_0^t\frac{d}{dt}\left(\mu_i,\nabla\xi_i,\nabla\xi_i\right)dt=\left(\mu_i(t)\nabla\xi_i(t),\nabla\xi_i(t)\right)-\left(\mu_i(0)\nabla\xi_i(0),\nabla\xi_i(0)\right),$$
$$\int_0^t\frac{d}{dt}\left(\mu_i,\nabla\eta_i,\nabla\xi_i\right)dt=\left(\mu_i(t)\nabla\eta_i(t),\nabla\xi_i(t)\right)-\left(\mu_i(0)\nabla\eta_i(0),\nabla\xi_i(0)\right),$$
$$\int_0^t\frac{d}{dt}\left(\alpha,\nabla\xi_1\right)dt=\left(\alpha(t),\nabla\xi_1(t)\right)-\left(\alpha(0),\nabla\xi_1(0)\right),$$

where we bound the following terms using Cauchy-Schwarz and Young's inequality with ϵ,

$$\left(\mu_i(t)\nabla\xi_i(t),\nabla\xi_i(t)\right)\leq C\|\xi_i\|_1^2,$$

$$\left(\mu_i(0)\nabla\xi_i(0),\nabla\xi_i(0)\right)\leq C\|\xi_i(0)\|_1^2,$$

$$\left(\mu_i(t)\nabla\eta_i(t),\nabla\xi_i(t)\right)\leq C\|\eta_i\|_1^2+\epsilon\|\xi_i\|_1^2,$$

$$\left(\mu_i(0)\nabla\eta_i(0),\nabla\xi_i(0)\right)\leq C\left(\|\eta_i(0)\|_1^2+\|\xi_i(0)\|_1^2\right),$$

$$\left(\alpha(t),\nabla\xi_1(t)\right)\leq C\|\alpha\|_0^2+\epsilon\|\xi_1\|_1^2,$$

$$\left(\alpha(0),\nabla\xi_1(0)\right)\leq C\left(\|\alpha(0)\|_0^2+\|\xi_1(0)\|_1^2\right).$$

Applying Gronwall's Inequality, we're left with

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\frac{d\xi_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|^2_{L^2\left(0,t;L^2(\Omega_t^i)\right)}+\|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\right]\\
&\leq C\inf_{\substack{\tilde{\mathbf{V}}\in K_h\setminus\{0\},\\ \tilde{p}\in M^0_{h,t}\setminus\{0\}}}\Big(\sum_{i=1}^{2}\Big[\|\frac{d\eta_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)}+\|\eta_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}+\|\xi_i(0)\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\\
&+\|\eta_i(0)\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\Big]+\|\alpha\|^2_{L^\infty\left(0,t;L^2(\Omega_t^1)\right)}+\|\frac{d\alpha}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_{L^2\left(0,t;L^2(\Omega_t^1)\right)}\Big)^2\\
&+\epsilon\|\beta(t)\|^2_{L^2\left(0,t;L^2(\Omega_t^1)\right)}.
\end{aligned}
\tag{4.22}
$$

Using Brezzi Theory discussed in Boffi et al. (2013), we can take the infimum over the more general finite element spaces, this gives the following:

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\frac{d\xi_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|^2_{L^2\left(0,t;L^2(\Omega_t^i)\right)}+\|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\right]\\
&\leq C\inf_{\substack{\tilde{\mathbf{V}}\in \bar{W}_{h,t}\setminus\{0\},\\ \tilde{p}\in M^0_{h,t}\setminus\{0\}}}\Big(\sum_{i=1}^{2}\Big[\|\frac{d\eta_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)}+\|\eta_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\\
&+\|\xi_i(0)\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)}\Big]+\|\alpha\|^2_{L^\infty\left(0,t;L^2(\Omega_t^1)\right)}+\|\frac{d\alpha}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_{L^2\left(0,t;L^2(\Omega_t^1)\right)}\Big)^2\Big)\\
&+\epsilon\|\beta(t)\|^2_{L^2\left(0,t;L^2(\Omega_t^1)\right)}.
\end{aligned}
\tag{4.23}
$$

For the error estimate on pressure, we will use the discrete LBB condition for FSI problems discussed in Xu and Yang (2015):

$$
\begin{aligned}
\|\beta\|_L^2 &\leq \sup_{(\psi_{1,h},\psi_{2,h})\in W_{h,t}}\frac{(\nabla\cdot\psi_{1,h},\beta)}{\|(\psi_{1,h},\psi_{2,h})\|_1}\\
&=\sup_{(\psi_{1,h},\psi_{2,h})\in W_{h,t}}\frac{(\nabla\cdot\psi_{1,h},\alpha+\beta)_{\Omega_t^1}-(\nabla\cdot\psi_{1,h},\alpha)_{\Omega_t^1}}{\|(\psi_{1,h},\psi_{2,h})\|_1}\\
&\leq \sup_{(\psi_{1,h},\psi_{2,h})\in W_{h,t}}\frac{\sum_{i=1}^{2}\left[\left(\frac{d\left(\mathbf{V}_i-\mathbf{V}_{i,h}\right)}{dt}\Big|_{\hat{\mathbf{x}}}^{h},\psi_{i,h}\right)_{\Omega_t^i}+\left(\mu_i\nabla\left(\mathbf{V}_i-\mathbf{V}_{i,h}\right),\nabla\psi_{i,h}\right)_{\Omega_t^i}\right.}{\|(\psi_{1,h},\psi_{2,h})\|_1}\\
&+\sup_{(\psi_{1,h},\psi_{2,h})\in W_{h,t}}\frac{\left.-\left(\left(\omega_{i,h}\cdot\nabla\right)\left(\mathbf{V}_i-\mathbf{V}_{i,h}\right),\psi_{i,h}\right)_{\Omega_t^i}\right]-(\nabla\cdot\psi_{1,h},\alpha)_{\Omega_t^1}}{\|(\psi_{1,h},\psi_{2,h})\|_1}\\
&\leq C\sum_{i=1}^{2}\left[\|\frac{d\eta_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_0^2+\|\frac{d\xi_i}{dt}\Big|_{\hat{\mathbf{x}}}^{h}\|_0^2+\|\eta_i\|_1^2+\|\xi_i\|_1^2\right]+\|\alpha\|_0^2,
\end{aligned}
\tag{4.24}
$$

where the final inequality comes from applying the Cauchy-Schwarz inequality and separating $\mathbf{V}_i - \mathbf{V}_{i,h} = \eta_i + \xi_i$. Integrating (4.24) from 0 to t and plugging (4.23) in, we have

$$\|\beta\|^2_{L^2(0,t;L^2(\Omega_t^1))} \leq C \inf_{\substack{\tilde{\mathbf{V}} \in \bar{W}_{h,t}\setminus\{0\},\\ \tilde{p}\in M^0_{h,t}\setminus\{0\}}} \Big(\sum_{i=1}^{2} \Big[\| \frac{d\eta_i}{dt}\Big|^h_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} + \|\eta_i\|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} +$$
$$\| \frac{d\xi_i}{dt}\Big|^h_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} + \|\xi_i\|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} + \|\eta_i(0)\|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} \Big] + \|\alpha\|^2_{L^2\left(0,t;L^2(\Omega_t^1)\right)} \Big).$$

Taking ϵ small enough and applying Gronwall's inequality, (4.23) becomes

$$\begin{aligned} &\sum_{i=1}^{2} \Big[\| \frac{d\xi_i}{dt}\Big|^h_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;L^2(\Omega_t^i)\right)} + \|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)} \Big] \\ &\leq C \inf_{\substack{\tilde{\mathbf{V}} \in \bar{W}_{h,t}\setminus\{0\},\\ \tilde{p}\in M^0_{h,t}\setminus\{0\}}} \Big(\sum_{i=1}^{2} \Big[\| \frac{d\eta_i}{dt}\Big|^h_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;H^1(\Omega_t^i)\right)} + \|\eta_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)} \\ &+ \|\xi_i(0)\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)} \Big] + \|\alpha\|^2_{L^\infty\left(0,t;L^2(\Omega_t^1)\right)} + \| \frac{d\alpha}{dt}\Big|^h_{\hat{\mathbf{x}}} \|_{L^2\left(0,t;L^2(\Omega_t^1)\right)} \big)^2 \Big). \end{aligned} \tag{4.25}$$

Choosing interpolation as our arbitrary function, letting $\mathbf{V}_{i,h}(0) = \tilde{V}_{i,h}(0)$, adding (4.24) and (4.25) and using Lemmas 2.4 and 2.5, we have

$$\begin{aligned} &\sum_{i=1}^{2} \Big[\| \frac{d\xi_i}{dt}\Big|^h_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;L^2(\Omega_t^i)\right)} + \|\xi_i\|^2_{L^\infty\left(0,t;H^1(\Omega_t^i)\right)} + \|\beta\|^2_{L^2(0;t;L^2(\Omega_t^1))} \Big] \\ &\leq Ch^4 \Big(\sum_{i=1}^{2} \Big[\| \frac{d\mathbf{V}_i}{dt}\Big|_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;H^3(\Omega_t^i)\right)} + \|\mathbf{V}_i\|^2_{(H^1\cap L^\infty)\left(0,t;H^3(\Omega_t^i)\right)} \Big] + \|p_1\|^2_{L^\infty\left(0,t;H^2(\Omega_t^1)\right)} \\ &+ \| \frac{dp_1}{dt}\Big|_{\hat{\mathbf{x}}} \|^2_{L^2\left(0,t;H^2(\Omega_t^1)\right)} \Big) \end{aligned} \tag{4.26}$$

Thus, adding $\frac{d\eta_i}{dt}\big|^h_{\hat{\mathbf{x}}}$, η_i, α back in and applying the triangle inequality we have our result.

□

CHAPTER 5

Fully-Discrete Scheme

With the semi-discrete scheme taken care of, we can now move on to the fully-discrete scheme. Let $\Delta t > 0$ be the time step and $t_n = n\Delta t$ for $n = 0, ..., N$. We'll be using the backward Euler scheme for temporal discretization. We introduce the following notation:

$$\delta_t \mathbf{V}_{i,h}^{n+1} = \frac{\mathbf{V}_{i,h}^{n+1} - \mathbf{V}_{i,h}^{n} \circ X_{n+1,n}}{\Delta t},$$

where $X_{n+1,n} = X_n \circ (X_{n+1})^{-1}$. The fully discrete scheme can now be obtained.

Find $(v_{1,h}^{n+1}, v_{2,h}^{n+1}) \in \bar{W}_{n,t_{n+1}}$, $p_{1,h}^{n+1} \in M_{h,t_{n+1}}^0$ such that $v_{i,h}^0 = \tilde{v}_i(0)$ for every n and

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[\big(\delta_t \mathbf{V}_{i,h}^{n+1}, \psi_{i,h}\big)_{\Omega_{n+1}^i} + \big(\mu_i \nabla \mathbf{V}_{i,h}^{n+1}, \nabla \psi_{i,h}\big)_{\Omega_{n+1}^i} - \big(\omega_{i,h}^{n+1} \nabla \mathbf{V}_{i,h}^{n+1}, \psi_{i,h}\big)_{\Omega_{n+1}^i} \Big] \\
&- \big(p_{1,h}^{n+1}, \nabla \cdot \psi_{1,h}\big)_{\Omega_{n+1}^1} + \big(\nabla \cdot \mathbf{V}_{1,h}^{n+1}, q_{1,h}\big)_{\Omega_{n+1}^1} = \sum_{i=1}^{2} \Big[\big(f_{i,h}^{n+1}, \psi_{i,h}\big)_{\Omega_{n+1}^i} \Big] \\
&+ \langle \tau, \psi_{1,h} \rangle_{\Gamma_{t_{n+1}}}
\end{aligned} \tag{5.1}$$

holds for every $(\psi_{1,h}, \psi_{2,h}) \in \bar{W}_{h,t}^0$ and every $q_{1,h} \in M_{h,t}$. We can now move on to the error estimate for the fully discrete scheme.

5.1 Fully-Discrete Error Analysis

We'll start with a few lemmas which will allow us to perform the required analysis. For what follows, we define $X_{n,n+1} = X_{n+1} \circ (X_n)^{-1}$. We have the following lemmas.

seriesLemma 5.1. *Let $\phi^{n+1} \in \bar{W}_{h,t}$, then*

$$\|\phi^{n+1} \circ X_{n,n+1}\|_{L^2(\Omega_{t_n}^i)}^2 = \|\phi^{n+1}\|_{L^2(\Omega_{n+1}^i)}^2 - \int_{t_n}^{t_{n+1}} \left(\int_{\Omega_t} |\phi^{n+1} \circ X_{t,n+1}|^2 \nabla \cdot \omega_h dx \right) dt.$$

Proof.

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t} |\phi^{n+1}\circ X_{t,n+1}|^2 dx = \int_{\Omega_0} \frac{d}{dt}|\hat{\phi}^{n+1}|^2 J_t d\hat{\mathbf{x}} = \int_{\Omega_0} |\hat{\phi}^{n+1}|^2 \frac{dJ_t}{dt} d\hat{\mathbf{x}} \\
&= \int_{\Omega_t} |\phi^{n+1}\circ X_{t,n+1}|^2 \nabla\cdot\omega_h dx.
\end{aligned} \tag{5.2}$$

Thus

$$\begin{aligned}
&\int_{t_n}^{t^{n+1}}\int_{\Omega_t} |\phi^{n+1}\circ X_{t,n+1}|^2 \nabla\cdot\omega_h dx dt = \int_{t_n}^{t^{n+1}} \frac{d}{dt}\int_{\Omega_t} |\phi^{n+1}\circ X_{t,n+1}|^2 dx dt \\
&= \int_{\Omega_{t_{n+1}}} |\phi^{n+1}|^2 dx - \int_{\Omega_{t_n}} |\phi^{n+1}\circ X_{n,n+1}|^2 dx,
\end{aligned} \tag{5.3}$$

where rearranging gives the result. □

The following lemma considers the classical Taylor expansion technique in the context of the ALE description.

seriesLemma 5.2. *For any* $\mathbf{V}\in\bar{W}_{h,t}$*, we have*

$$\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^n,t^n)}{\Delta t} = \left.\frac{d\mathbf{V}^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \frac{\Delta t}{2}\left[\left.\frac{d^2\mathbf{V}^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}} - (\nabla\omega_h)(\omega_h)\nabla\mathbf{V}\right].$$

Proof. Expanding $\mathbf{V}(x^n,t^n)$ at x^{n+1}, we get

$$\mathbf{V}(x^n,t^n) = \mathbf{V}(x^{n+1},t^n) - \Delta x\left(\frac{\partial\mathbf{V}}{\partial x}\right)(x^{n+1},t^n) + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2\mathbf{V}}{\partial x^2}\right)(x^{n+1},t^n) + ... \tag{5.4}$$

Noting that

$$\begin{aligned}
&\left(\frac{\partial\mathbf{V}}{\partial x}\right)(x^{n+1},t^n) = \left(\frac{\partial\mathbf{V}}{\partial x}\right)(x^{n+1},t^{n+1}) - \Delta t\left(\frac{\partial^2\mathbf{V}}{\partial x\partial t}\right)(x^{n+1},t^{n+1}) + ..., \\
&\left(\frac{\partial^2\mathbf{V}}{\partial x^2}\right)(x^{n+1},t^n) = \left(\frac{\partial^2\mathbf{V}}{\partial x^2}\right)(x^{n+1},t^{n+1}) - \Delta t\left(\frac{\partial^3\mathbf{V}}{\partial x^2\partial t}\right)(x^{n+1},t^{n+1}) + ...,
\end{aligned} \tag{5.5}$$

we have,

$$\begin{aligned}
&\mathbf{V}(x^n,t^n) = \mathbf{V}(x^{n+1},t^n) - \Delta x\left(\frac{\partial\mathbf{V}}{\partial x}\right)(x^{n+1},t^{n+1}) + \Delta x\Delta t\left(\frac{\partial^2\mathbf{V}}{\partial x\partial t}\right)(x^{n+1},t^{n+1}) + \\
&\frac{(\Delta x)^2}{2}\left(\frac{\partial^2\mathbf{V}}{\partial x^2}\right)(x^{n+1},t^{n+1}) + ...
\end{aligned} \tag{5.6}$$

Thus,

$$\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^{n},t^{n})}{\Delta t}=\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^{n+1},t^{n})}{\Delta t} \\ +\frac{\mathbf{V}(x^{n+1},t^{n})-\mathbf{V}(x^{n},t^{n})}{\Delta t}. \tag{5.7}$$

Which, when expanded, gives

$$\frac{\Delta x}{\Delta t}\left(\frac{\partial \mathbf{V}}{\partial x}\right)^{n+1}-\frac{(\Delta x)^2}{2\Delta t}\left(\frac{\partial^2 \mathbf{V}}{\partial x^2}\right)^{n+1}-\Delta x\left(\frac{\partial^2 \mathbf{V}}{\partial x\partial t}\right)^{n+1} \\ +\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^{n+1},t^{n})}{\Delta t}, \tag{5.8}$$

where

$$\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^{n+1},t^{n})}{\Delta t}=\left(\frac{\partial \mathbf{V}}{\partial t}\right)^{n+1}-\frac{\Delta t}{2}\left(\frac{\partial^2 \mathbf{V}}{\partial t^2}\right)^{n+1}+\ldots \tag{5.9}$$

Since $x(\hat{\mathbf{x}},t^n)=x(\hat{\mathbf{x}},t^{n+1})-\Delta t\left(\frac{\partial x}{\partial t}\right)^{n+1}+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1}+\ldots$. We see that $\frac{\Delta x}{\Delta t}=\left(\frac{\partial x}{\partial t}\right)^{n+1}-\frac{\Delta t}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1}$.

We then have

$$\frac{\mathbf{V}(x^{n+1},t^{n+1})-\mathbf{V}(x^{n},t^{n})}{\Delta t}=\left(\frac{\partial x}{\partial t}\right)^{n+1}\left(\frac{\partial \mathbf{V}}{\partial x}\right)^{n+1}-\frac{\Delta t}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1}\left(\frac{\partial \mathbf{V}}{\partial x}\right)^{n+1} \\ -\Delta t\left(\frac{\partial x}{\partial t}\right)^{n+1}\left(\frac{\partial^2 \vec{V}}{\partial x\partial t}\right)^{n+1}-\frac{\Delta t}{2}\left[\left(\frac{\partial x}{\partial t}\right)^{n+1}\right]^2\left(\frac{\partial^2 \mathbf{V}}{\partial x^2}\right)^{n+1} \\ +\left(\frac{\partial \mathbf{V}}{\partial t}\right)^{n+1}-\frac{\Delta t}{2}\left(\frac{\partial^2 \mathbf{V}}{\partial t^2}\right)^{n+1}$$

$$=\left.\frac{d\mathbf{V}^{n+1}}{dt}\right|_{\hat{\mathbf{x}}}-\frac{\Delta t}{2}\left[\frac{\partial^2 \mathbf{V}}{\partial t^2}+\frac{\partial^2 x^{n+1}}{\partial t^2}\frac{\partial \mathbf{V}^{n+1}}{\partial x}+2\frac{\partial x^{n+1}}{\partial t}\frac{\partial^2 \mathbf{V}^{n+1}}{\partial x\partial t}+\left(\frac{\partial x^{n+1}}{\partial t}\right)^2\frac{\partial^2 \mathbf{V}^{n+1}}{\partial x^2}\right] \\ =\left.\frac{d\mathbf{V}^{n+1}}{dt}\right|_{\hat{\mathbf{x}}}-\frac{\Delta t}{2}\left[\left.\frac{d^2\mathbf{V}^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}}-\frac{dx}{dt}\frac{d\mathbf{V}}{dx}\frac{d}{dx}\left(\frac{dx}{dt}\right)\right]. \tag{5.10}$$

□

The final lemma is borrowed from Martin et al. (2009). It puts bounds on various Jacobian terms which arise.

seriesLemma 5.3. *There exists C_1 and C_2 depending on X and $h_0 > 0$ such that*

$$\|J_{X_{h,t}}\|_{L^\infty(\Omega_0)} \le C_1 \qquad \forall t \in [0,T],\ \forall h \in (0,h_0)$$

$$\|J_{X_{h,t}^{-1}}\|_{L^\infty(\Omega_0)} \le C_2 \qquad \forall t \in [0,T],\ \forall h \in (0,h_0)$$

$$\|J_t - J_n\|_\infty \le C\Delta t.$$

We can now proceed to the main theorem of the section, the fully discrete error estimate.

seriesTheorem 5.1. *Suppose $(\mathbf{V}_1, p_1, \mathbf{V}_2)$ is the solution to (3.4) and $\left(\mathbf{V}_{1,h}^{n+1}, p_{1,h}^{n+1}, \mathbf{V}_{2,h}^{n+1}\right)$ is the solution to (5.1), then we have the following error estimate:*

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\|\mathbf{V}_i^N - \mathbf{V}_{i,h}^N\|_{L^2(\Omega_N^i)} + \Delta t \sum_{j=1}^{N} \|\mathbf{V}_i^j - \mathbf{V}_{i,h}^j\|_{H^1(\Omega_j^i)}\right] \\
&\le C(h^2+\Delta t)\Bigg(\sum_{i=1}^{2}\left[\sum_{j=0}^{N}\|\mathbf{V}_i^j\|_{H^3(\Omega_j^i)} + \sum_{j=0}^{N}\|\frac{d\mathbf{V}_i^j}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{H^3(\Omega_j^i)} + \sum_{j=0}^{N}\|\frac{d^2\mathbf{V}_i^j}{dt^2}\bigg|_{\hat{\mathbf{x}}}\|_{L^2(\Omega_j^i)}\right] \\
&+ \sum_{j=1}^{N}\|p_1^j\|_{H^2(\Omega_j^1)}\Bigg)
\end{aligned} \tag{5.11}$$

Proof. We begin by adding and subtracting $\delta_t \mathbf{V}_i^{n+1}$ into (5.1).We then subtract the result from (3.4). This gives:

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\left(\frac{d\mathbf{V}_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}} - \delta_t \mathbf{V}_i^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i} + \left(\delta_t \mathbf{V}_i^{n+1} - \delta_t \mathbf{V}_{i,h}^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i} \\
&+ \left(\mu_i \nabla\left(\mathbf{V}_i^{n+1} - \mathbf{V}_{i,h}^{n+1}\right), \nabla\psi_{i,h}\right)_{\Omega_{n+1}^i} - \left(\omega_{i,h}^{n+1}\cdot\nabla\left(\mathbf{V}_i^{n+1} - \mathbf{V}_{i,h}^{n+1}\right), \psi_{i,h}\right)_{\Omega_{n+1}^i}\Bigg] \\
&- \left(p_1^{n+1} - p_{1,h}^{n+1}, \nabla\cdot\psi_{1,h}\right)_{\Omega_{n+1}^1} + \left(\nabla\cdot\left(\mathbf{V}_1^{n+1} - \mathbf{V}_{1,h}^{n+1}\right), q_{1,h}\right)_{\Omega_{n+1}^1} = 0.
\end{aligned} \tag{5.12}$$

Pick arbitrary functions $\tilde{\mathbf{V}} = \left(\tilde{\mathbf{V}}_1, \tilde{\mathbf{V}}_2\right) \in K_h$ and $\tilde{p} \in M_{h,t}^0$. Let $\mathbf{V}_i - \mathbf{V}_{i,h} = \mathbf{V}_i - \tilde{\mathbf{V}}_i +$

$\tilde{\mathbf{V}}_i - \mathbf{V}_{i,h} = \eta_i + \xi_i$, and $p_1 - p_{1,h} = p_1 - \tilde{p}_1 + \tilde{p}_1 - p_{1,h} = \alpha + \beta$, and choosing $(\psi_{1,h}, \psi_{2,h}) \in K_h$:

$$\begin{aligned}
&\sum_{i=1}^{2} \Bigg[\left(\left.\frac{d\mathbf{V}_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \delta_t \mathbf{V}_i^{n+1}, \psi_{i,h} \right)_{\Omega_{n+1}^i} + \left(\delta_t \xi_i^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i} + \\
&\left(\mu_i \nabla \xi_i^{n+1}, \nabla \psi_{i,h}\right)_{\Omega_{n+1}^i} \Bigg] - \left(\beta^{n+1}, \nabla \cdot \psi_{1,h}\right)_{\Omega_{n+1}^1} + \left(\nabla \cdot \xi_1^{n+1}, q_{1,h}\right)_{\Omega_{n+1}^1} \\
&= \sum_{i=1}^{2} \Bigg[- \left(\delta_t \eta_i^{n+1}, \psi_{i,h}\right)_{\Omega_{n+1}^i} - \left(\mu_i \nabla \eta_i^{n+1}, \nabla \psi_{i,h}\right)_{\Omega_{n+1}^i} + \\
&\left(\left(\omega_{i,h}^{n+1} \cdot \nabla\right)\left(\eta_i^{n+1} + \xi_i^{n+1}\right), \psi_{i,h}\right)_{\Omega_{n+1}^i} \Bigg] + \left(\alpha^{n+1}, \nabla \cdot \psi_{1,h}\right)_{\Omega_{n+1}^1} \\
&- \left(\nabla \cdot \eta_i^{n+1}, q_{1,h}\right)_{\Omega_{n+1}^1}.
\end{aligned} \tag{5.13}$$

Choosing $\psi_{i,h} = \xi_i^{n+1},\ q_{1,h} = \beta^{n+1}$, using Cauchy-Schwarz and Young's inequality with ϵ and noting

$$- \left(\beta^{n+1}, \nabla \cdot \xi_1^{n+1}\right)_{\Omega_{n+1}^1} + \left(\nabla \cdot \xi_1^{n+1}, \beta^{n+1}\right)_{\Omega_{n+1}^1} = \left(\nabla \cdot \eta_1^{n+1}, \beta^{n+1}\right)_{\Omega_{n+1}^1} = 0,$$

we have the following estimates:

$$\begin{aligned}
&\left(\mu_i \nabla \xi_i^{n+1}, \nabla \xi_i^{n+1}\right) \geq C \|\xi_i^{n+1}\|_1^2, \\
&\left(\mu_i \nabla \eta_i^{n+1}, \nabla \xi_i^{n+1}\right) \leq C \|\eta_i^{n+1}\|_1^2 + \epsilon \|\xi_i^{n+1}\|_1^2, \\
&\left(\left(\omega_{i,h}^{n+1} \cdot \nabla\right)\left(\eta_i^{n+1} + \xi_i^{n+1}\right), \xi_i^{n+1}\right) \leq C \left(\|\eta_i^{n+1}\|_1^2 + \|\xi_i^{n+1}\|_0^2\right) + \epsilon \|\xi_i^{n+1}\|_1^2, \\
&\left(\alpha^{n+1}, \nabla \cdot \xi_i^{n+1}\right) \leq C \|\alpha^{n+1}\|_0^2 + \epsilon \|\xi_i^{n+1}\|_1^2.
\end{aligned} \tag{5.14}$$

The term $\left(\delta_t \xi_i^{n+1}, \psi_{i,h}\right)$ is handled in the following way.

We'll change variables: $x_n = X_n \circ (X_{n+1})^{-1}(x_{n+1})$, $\psi_{i,h}^{n+1} = \hat{\psi_{i,h}} \circ X_{n+1}$. This gives:

$$\begin{aligned}
&\left(\frac{\xi_i^{n+1} - \xi_i^n \circ X_{n+1,n}}{\Delta t}, \psi_{i,h}\right)_{\Omega_{n+1}^i} = \left(\frac{\xi_i^{n+1}}{\Delta t}\right)_{\Omega_{n+1}^i} - \left(\frac{\xi_i^n \circ X_{n+1,n}}{\Delta t}, \psi_{i,h}\right)_{\Omega_{n+1}^i} \\
&= \left(\frac{\xi_i^{n+1}}{\Delta t}, \psi_{i,h}\right)_{\Omega_{n+1}^i} - \left(\frac{\xi_i^n}{\Delta t}, \psi_{i,h} \cdot \frac{J_{n+1}}{J_n}\right)_{\Omega_{n+1}^i}.
\end{aligned} \tag{5.15}$$

Choosing $\psi_{i,h} = \xi_i^{n+1}$, we have:

$$\left(\frac{\xi_i^{n+1}}{\Delta t}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} - \left(\frac{\xi_i^n}{\Delta t}, \xi_i^{n+1} \circ X_{n,n+1}\right)_{\Omega_n^i} + \left(\frac{\xi_i^n}{\Delta t}, \xi_i^{n+1} \circ X_{n,n+1}\left(\frac{J_n - J_{n+1}}{J_n}\right)\right)_{\Omega_n^i}.$$

We note $\left(\frac{\xi_i^{n+1}}{\Delta t}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} = \frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t}$, and the remaining two terms we move to the right hand side. We then have the following estimate:

$$\left(\frac{\xi_i^n}{\Delta t}, \xi_i^{n+1} \circ X_{n,n+1}\right)_{\Omega_{n+1}^i} \leq \frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1} \circ X_{n,n+1}\|_{0,n}^2}{\Delta t}. \tag{5.16}$$

Using Lemma 5.1, we have:

$$\begin{aligned}
&\left(\frac{\xi_i^n}{\Delta t}, \xi_i^{n+1} \circ X_{n,n+1}\right)_{\Omega_n^i} \leq \frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1} \circ X_{n,n+1}\|_{0,n}^2}{\Delta t} \\
&\leq \frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t} - \frac{1}{2}\int_{t_n}^{t_{n+1}}\left(\frac{1}{\Delta t}\int_{\Omega_t} |\xi_i^{n+1} \circ X_{t,n+1}|^2 \nabla\cdot\omega_{i,h} dx\right) dt \\
&\leq \frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t} + \frac{1}{2}\sup_{t\in[t_n,t_{n+1}]}\|\nabla\cdot\omega_{i,h}\|_0 \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t}\|\xi_i^{n+1} \circ X_{t,n+1}\|_{0,t}^2 dt \\
&\leq \frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t} + C\|\xi_i^{n+1}\|_{0,n+1}^2,
\end{aligned} \tag{5.17}$$

where $\|\xi_i^{n+1} \circ X_{t,n+1}\|_{0,t}^2 \leq \|J_t\|_\infty \|J_{n+1}^{-1}\|_\infty \|\xi_i^{n+1}\|_{0,n+1}^2$ is used from Lemma 5.3

Following similarly, we also have:

$$\begin{aligned}
&\left(\frac{\xi_i^n}{\Delta t}, \xi_i^{n+1} \circ X_{n,n+1}\left(\frac{J_n - J_{n+1}}{J_n}\right)\right)_{\Omega_n^i} \leq C\Delta t\left(\frac{1}{2}\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + \frac{1}{2}\frac{\|\xi_i^{n+1} \circ X_{n,n+1}\|_{0,n}^2}{\Delta t}\right) \\
&\leq C\Delta t\left(\frac{\|\xi_i^n\|_{0,n}^2}{\Delta t} + C\frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t} + \|\xi_i^{n+1}\|_{0,n+1}^2\right).
\end{aligned} \tag{5.18}$$

The terms $\left(\left.\frac{d\mathbf{V}_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \delta_t \mathbf{V}_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i}$ and $-\left(\delta_t \eta_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i}$ will be handled using Lemma 5.2:

$$\begin{aligned}
&\left(\left.\frac{d\mathbf{V}_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \delta_t \mathbf{V}_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&= \left(\frac{\Delta t}{2}\left(\left.\frac{d^2\mathbf{V}_i^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}} - \left(\frac{dx}{dt}\right)^{n+1}\left(\frac{d\mathbf{V}_i}{dx}\right)^{n+1}\frac{d}{dx}\left(\frac{dx}{dt}\right)^{n+1}\right), \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&\leq C(\Delta t)^2\left(\|\left.\frac{d^2\mathbf{V}_i^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}}\|_{0,n+1}^2 + \|\left(\nabla\omega_{i,h}\right)^{n+1}\left(\omega_{i,h}\right)^{n+1}\nabla\mathbf{V}_i^{n+1}\|_{0,n+1}^2 + \|\xi_i^{n+1}\|_{0,n+1}^2\right) \\
&\leq C(\Delta t)^2\left(\|\left.\frac{d^2\mathbf{V}_i^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}}\|_{0,n+1}^2 + \|\mathbf{V}_i^{n+1}\|_{1,n+1}^2 + \|\xi_i^{n+1}\|_{0,n+1}^2\right)
\end{aligned} \tag{5.19}$$

and

$$
\begin{aligned}
&\left(\delta_t \eta_i^{n+1}, \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&= \left(\frac{d\eta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}} - \frac{\Delta t}{2}\left(\frac{d^2\eta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}} - \left(\frac{dx}{dt}\right)^{n+1}\left(\frac{d\eta_i}{dx}\right)^{n+1}\frac{d}{dx}\left(\frac{dx}{dt}\right)^{n+1}\right), \xi_i^{n+1}\right)_{\Omega_{n+1}^i} \\
&\leq C\Bigg(\| \frac{d\eta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}} \|_{0,n+1}^2 + (\Delta t)^2 \| \frac{d^2\eta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}} \|_{0,n+1}^2 + \| \left(\nabla \omega_{i,h}\right)^{n+1} \left(\omega_{i,h}\right)^{n+1} \nabla \eta_i^{n+1} \|_{0,n+1}^2 \\
&+ \|\xi_i^{n+1}\|_{0,n+1}^2 \Bigg) \\
&\leq C\left(\| \frac{d\eta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}} \|_{0,n+1}^2 + (\Delta t)^2 \| \frac{d^2\eta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}} \|_{0,n+1}^2 + \|\eta_i^{n+1}\|_{1,n+1}^2 + \|\xi_i^{n+1}\|_{0,n+1}^2 \right),
\end{aligned}
\tag{5.20}
$$

where $\frac{\partial x}{\partial t} \in H^1(0,T;W^{1,\infty}(\Omega_t^i))$ and $\mathbf{V} \in L^2(0,T;H^2(\Omega_t^1 \cup \Omega_t^2)) \cap H^1(0,T;H^1(\Omega_t))$.

Using the estimates from 5.14 and choosing ϵ small enough, we have the following:

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\frac{1}{2}\frac{\|\xi_i^{n+1}\|_{0,n+1}^2}{\Delta t} - \frac{1}{2}\frac{\|\xi_i^{n}\|_{0,n}^2}{\Delta t} + \|\xi_i^{n+1}\|_{1,n+1}^2 \right] \\
&\leq C\Bigg(\sum_{i=1}^{2}\left[\| \frac{d\eta_i^{n+1}}{dt} \|_{0,n+1}^2 + \|\eta_i^{n+1}\|_{1,n+1}^2 + \|\xi_i^{n+1}\|_{0,n+1}^2 + \|\xi_i^{n}\|_{0,n}^2 \right] \\
&+ \|\alpha^{n+1}\|_{0,n+1}^2 + C(\Delta t)^2 \Bigg).
\end{aligned}
\tag{5.21}
$$

To achieve the global error we sum over n from 0 to N and use

$$
\sum_{j=1}^{n+1} \|\xi_i^j\|_{0,j}^2 + \sum_{j=1}^{n} \|\xi_i^j\|_{0,j}^2 \leq 2\sum_{j=1}^{n+1} \|\xi_i^j\|_{0,j}^2,
$$

as well as the Discrete Gronwall inequality to get

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\frac{1}{2}\|\xi_i^N\|_{0,N}^2 + \Delta t \sum_{j=0}^{N} \|\xi_i^j\|_{1,j}^2 \right] \\
&\leq \inf_{\substack{\tilde{\mathbf{V}} \in K_h \setminus \{0\}, \\ \tilde{p} \in M_{h,t}^0 \setminus \{0\}}} C\left(\sum_{i=1}^{2}\left[\|\xi_i^0\|_{0,0}^2 + \Delta t \sum_{j=0}^{N} \| \frac{d\eta_i^j}{dt}\bigg|_{\hat{\mathbf{x}}} \|_{0,j}^2 + \Delta t \sum_{j=0}^{N} \|\eta_i^j\|_{1,j}^2 \right]\right) \\
&+ \Delta t \sum_{j=0}^{N} \|\alpha^j\|_{0,j}^2 + C(\Delta t)^2.
\end{aligned}
\tag{5.22}
$$

By Brezzi theory discussed in Boffi et al. (2013), we extend the infimum over the more general finite element space:

$$\sum_{i=1}^{2}\left[\frac{1}{2}\|\xi_i^N\|_{0,N}^2+\Delta t\sum_{j=0}^{N}\|\xi_i^j\|_{1,j}^2\right]$$
$$\leq \inf_{\substack{\tilde{\mathbf{V}}\in W_{h,t}\setminus\{0\},\\ \tilde{p}\in M^0_{h,t}\setminus\{0\}}} C\left(\sum_{i=1}^{2}\left[\|\xi_i^0\|_{0,0}^2+\Delta t\sum_{j=0}^{N}\|\frac{d\eta_i^j}{dt}\Big|_{\hat{\mathbf{x}}}\|_{0,j}^2+\Delta t\sum_{j=0}^{N}\|\eta_i^j\|_{1,j}^2\right]\right) \tag{5.23}$$
$$+\Delta t\sum_{j=0}^{N}\|\alpha^j\|_{0,j}^2+C(\Delta t)^2.$$

Choosing interpolation for our arbitrary functions and using Lemma 2.4, we have

$$\sum_{i=1}^{2}\left[\|\xi_i^N\|_{L^2(\Omega^i_{t_N})}^2+\Delta t\sum_{j=0}^{N}\|\xi_i^j\|_{H^1(\Omega^i_j)}^2\right]$$
$$\leq C(h^4+(\Delta t)^2)\left(\sum_{i=1}^{2}\left[\sum_{j=0}^{N}\|\mathbf{V}_i^j\|_{H^3(\Omega^i_j)}^2+\sum_{j=0}^{N}\|\frac{d\mathbf{V}_i^j}{dt}\Big|_{\hat{\mathbf{x}}}\|_{H^3(\Omega^i_j)}^2+\sum_{j=0}^{N}\|\frac{d^2\mathbf{V}_i^j}{dt^2}\Big|_{\hat{\mathbf{x}}}\|_{L^2(\Omega^i_j)}^2\right]\right.$$
$$\left.+\sum_{j=0}^{N}\|p_1^j\|_{H^2(\Omega^i_j)}^2\right). \tag{5.24}$$

Adding η_i^N and η_i^j back in and using the triangle inequality, we have our result. $\square$

CHAPTER 6

Conclusion

The Stokes/Parabolic interface problem is a stepping stone to more complex fluid-structure interaction problems. The model problem is described in a moving domain Ω_t and we discuss the properties of an appropriate ALE mapping. We then write our model problem using the ALE description. We proceed to discretize the model problem in space to define its semi-discrete non-conservative ALE finite element approximation, and analyze both its stability and error estimates. We see that the semi-discrete scheme has a convergence order of $O(h^2)$. We then proceed to discretize the temperal domain using the implicit backward Euler scheme, and define the fully discrete non-conservative ALE finite element approximation. After analyzing the fully discrete scheme's error estimates, we obtain a convergence order of $O(h^2 + \Delta t)$.

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