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Numerical Study in the Conservative Arbitrary Lagrangian-Eulerian (ALE) Method for an Unsteady Stokes/Parabolic Interface Problem with Jump Coefficients and a Moving Interface

Michael Joseph Ramirez

NUMERICAL STUDY IN THE CONSERVATIVE ARBITRARY LAGRANGIAN-EULERIAN (ALE) METHOD FOR AN UNSTEADY STOKES/PARABOLIC INTERFACE PROBLEM WITH JUMP COEFFICIENTS AND A MOVING INTERFACE

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ABSTRACT

NUMERICAL STUDY IN THE CONSERVATIVE ARBITRARY LAGRANGIAN-EULERIAN (ALE) METHOD FOR AN UNSTEADY STOKES/PARABOLIC INTERFACE PROBLEM WITH JUMP COEFFICIENTS AND A MOVING INTERFACE

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Towards numerical analyses for fluid-structure interaction (FSI) problems in the future, in this thesis the arbitrary Lagrangian-Eulerian (ALE) finite element method within a conservative form is developed and analyzed for a linearized FSI problem - an unsteady Stokes/parabolic interface problem with jump coefficients and moving interface, and the corresponding mixed finite element approximation is developed and analyzed for both semi- and fully discrete schemes based upon the so-called conservative formulation. In terms of a novel H^1-projection technique, their stability and optimal convergence properties are obtained for approximating the real solution equipped with lower regularity.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

This thesis provides a numerical study of a linearized fluid-structure interaction (FSI) problem between the unsteady Stokes equations and a vector-valued parabolic equation coupled over a moving interface with jump coefficients. The study of unstable interactions, where a strain induced on a surface causes movement, has applications in both engineering and biology (Richter, 2010). The study of FSI problems are often too complex to solve analytically and are therefore done using numerical methods. In this thesis we complete a numerical study for an unsteady Stokes/parabolic interface problem using a monolithic arbitrary Lagrangian-Eulerian (ALE) approach where both Stokes variables (velocity and pressure) and a vector-valued solution to the parabolic equation are solved simultaneously.

In particular, as a foundational model for complex FSI problems, the Stokes/parabolic interface problem described in this thesis provides a type of linearized FSI problem where the fluid is modeled by Stokes equations in terms of fluid velocity and pressure; the structure is modeled by parabolic equation in terms of the structure velocity.

Body-fitted mesh methods have become the most reliable numerical approach for solving unsteady moving domain/interface problems including FSIs due to the high accuracy that is derived. The challenge is of course developing a mesh that adapts to the moving boundary/interface at all times, and which can be efficiently generated. The body-fitted mesh approach that has been adopted for this paper is the arbitrary Lagrangian-Eulerian (ALE) method due to its high practicality, where the mesh on the interface continuously accommodates to the shared interface of both the fluid and structure, and therefore satisfies the interface conditions of the FSI.

The goal of this paper is to apply a novel H^1-projection technique developed in the recently submitted paper by Lan and Sun (2019) to the conservative ALE finite element analysis for an unsteady Stokes/parabolic FSI problem, and use it to obtain optimal error estimates for both semi- and fully discrete ALE finite element schemes. In previous numerical studies (Martin et al. (2009)), a classical H^1-projection was adopted to carry out ALE-finite element analyses for single Stokes equations on a moving domain and provided limited sub-optimal convergence order due to the effect of extra approximation error from the ALE mapping. The novel H^1- projection adopted in this paper derives an optimal convergence theorem for the developed ALE finite element approximation in both semi- and full discretization since the projection includes the influence of the discrete ALE mapping inside. Moreover, the newly developed H^1-projections analysis technique for the ALE FEM utilized in this paper can be extended to a realistic FSI problem in the future.

1.2 Outline

This thesis is divided into four sections. In Chapter 2, we provide useful preliminary results and introduce notation used in the remainder of the thesis.

Chapter 3, specifically Section 3.1 , presents our model description of a linearized FSI model problem, an unsteady Stokes/parabolic interface problem. Section 3.2 establishes the ALE mapping and some standard definitions, followed by the ALE formulation of the model problem. In Section 3.3 we make some comments on the Reynold's Tranport Theorem and its relation to the Geometric Conservation Law. We then finish this chapter with the Conservative Weak form in Section 3.4.

Chapter 4, specifically Section 4.1, presents our H^1 -projection definition and error estimates derived from this projection, which are first utilized in the derivation of the semi-discrete scheme followed by the analysis of the stability and error estimates in Sections 4.2 and 4.3 respectively.

Chapter 5 begins with the derivation of the fully-discrete scheme. We then spend the rest of this chapter on the analysis of the error estimates in Section 5.1.

We end the thesis with a few concluding remarks in Chapter 6.

CHAPTER 2

PRELIMINARY NOTATION AND RESULTS

The standard functional spaces taken from Adams and Fournier (2003) are adopted for this paper. We let $\Omega \subset \mathbb{R}^d$ be an open set where $m \in \mathbb{N}$, and $1 \leq p \leq \infty$, and let $L^p(\Omega)$ denote the linear space of measurable p^{th} power integrable functions on Ω equipped with norm $\|\cdot\|_{L^p(\Omega)}$. The functional space $W^{m,p}(\Omega)$ contains functions $f \in L^p(\Omega)$ with weak derivatives $D^\alpha f \in L^p(\Omega)$ up to m. For $1 \leq p < \infty$, the norm in $W^{m,p}(\Omega)$ is denoted by

$$\|u\|_{W^{m,p}} = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha u|^p \, dx \right)^{\frac{1}{p}},$$

and for $p = \infty$,

$$\|u\|_{W^{m,\infty}} = \max_{|\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

We also use the classical norm and seminorm notations for these functional spaces. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and omit the index $p = 2$ and Ω to simplify notation when possible, that is, $\|u\|_{W^{m,2}} = \|u\|_{H^m}$. We also denote $W^{0,p}(\Omega)$ by $L^p(\Omega)$ and omit the index $m = 0$. That is $\|u\|_{W^{0,p}} = \|u\|_{L^p}$. We also use shortened notation $\|u\|_{L^p} = \|u\|_0$ and $\|u\|_{H^m} = \|u\|_m$ during longer proofs.

We introduce the following notation for inner products to be used in this paper:

$$(\Phi_i, \Psi_i)_{\Omega_i^t} = \int_{\Omega_i^t} \Phi_i \cdot \Psi_i \mathrm{dx}, \text{ where i= 1,2}$$
$$\langle \Phi_i, \Psi_i \rangle_{\Gamma^t} = \int_{\Gamma^t} \Phi_i \cdot \Psi_i \mathrm{ds}$$

Lemma 2.1 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $1 \leq p < \infty$. Then there exists a constant $M > 0$ that only depends on p and Ω such that for all $u \in W_0^{1,p}(\Omega)$*

$$\|u\|_{L^p(\Omega)} \leq M\|Du\|_{L^p(\Omega)}. \tag{2.1}$$

Lemma 2.2 (Cauchy-Schwarz inequality).

$$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}. \tag{2.2}$$

Lemma 2.3 (Young's inequality with ϵ). *If $a, b \in \mathbb{R}$ where $a \geq 0$ and $b \geq 0$, then we have*

$$ab \leq \epsilon\frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2},\ \forall\epsilon > 0.$$

Note that the special case where $\epsilon = 1$ is known as simply Young's inequality and will be used frequently throughout this thesis.

Theorem 2.1 (Reynold's Transport Theorem (Leal, 2007), (Reynolds, 1903)). *Let $\varphi(\mathbf{x}, t)$ be a smooth function defined on $\Omega^t \times (0, T)$. we have that*

$$\frac{d}{dt}\int_{\Omega^t} \varphi(\mathbf{x}, t)d\mathbf{x} = \int_{\Omega^t} \left(\frac{\partial\varphi}{\partial t} + \nabla\varphi \cdot \vec{\omega} + \varphi\nabla \cdot \vec{\omega}\right) d\mathbf{x} = \int_{\Omega^t} \left(\left.\frac{d\varphi}{dt}\right|_{\hat{\mathbf{x}}} + \varphi\nabla \cdot \vec{\omega}\right) d\mathbf{x},$$

where $\vec{\omega}$ is domain velocity and

$$\left.\frac{d\varphi}{dt}\right|_{\hat{\mathbf{x}}} = \frac{\partial\varphi}{\partial t} + \vec{\omega} \cdot \nabla\varphi.$$

It's worth noting that the above equality also holds on open subdomains of Ω^t.

CHAPTER 3

THE UNSTEADY STOKES/PARABOLIC INTERFACE PROBLEM

3.1 Model Description

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, and $T > 0$. Two subdomains, $\Omega_i^t := \Omega_i(t) \subset \Omega$ $(i = 1, 2)$ $(0 \leq t \leq T)$, satisfying $\overline{\Omega_1^t} \cup \overline{\Omega_2^t} = \overline{\Omega}$, $\Omega_1^t \cap \Omega_2^t = \emptyset$. These two subdomains are separated by an interface: $\Gamma^t := \Gamma(t) = \partial\Omega_1^t \cap \partial\Omega_1^t$, which may move/deform along with $t \in (0, T]$, which causes Ω_i^t $(i = 1, 2)$ to also change with $t \in (0, T]$ and are termed as the current (Eulerian) domains with respect to $\mathbf{x}$, in contrast to their initial (reference/Lagrangian) domains, Ω_i^0 $(i = 1, 2)$ with respect to $\hat{\mathbf{x}}$, where, a *flow map* is defined from Ω_i^0 to Ω_i^t $(i = 1, 2)$, as: $\hat{\mathbf{x}}_i \mapsto \mathbf{x}_i(\hat{\mathbf{x}}_i, t)$ such that $\mathbf{x}_i(\hat{\mathbf{x}}_i, t) = \hat{\mathbf{x}}_i + X_i(\hat{\mathbf{x}}_i, t), \forall t \in (0, T]$, where X_i is the displacement field in the Lagrangian frame. The deformation gradient tensor, $\mathbf{F}_i := \nabla_{\hat{\mathbf{x}}_i} \mathbf{x}_i$, and $\mathbf{J}_i = \det(\mathbf{F}_i)$. An example of this type of domain with an immersed case is illustrated in Figure 3.1.

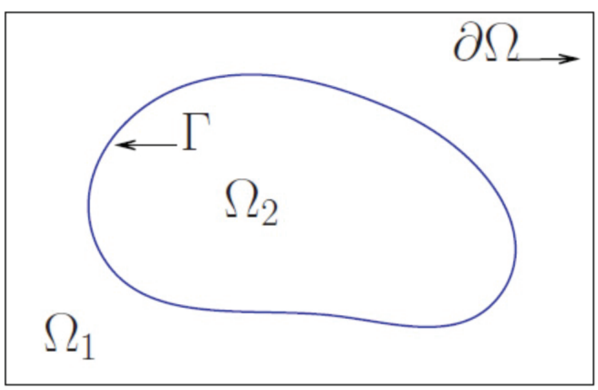

Figure 3.1. Schematic domain with the interface Γ^t between two subdomains Ω_1 and Ω_2, (Wang et al., 2018)

In what follows, we set $\hat{\psi} = \hat{\psi}(\hat{\mathbf{x}}, t)$ which equals $\psi(\mathbf{x}(\hat{\mathbf{x}}, t), t)$, and $\hat{\nabla} = \nabla_{\hat{\mathbf{x}}_i}$ $(i = 1, 2)$.

We define the Stokes equations in Ω_1^t and the parabolic equation in Ω_2^t with respect to $\vec{u}_i \in H^1(0, T; H^2(\Omega_1^t)^d \cup H^2(\Omega_2^t)^d), i = 1, 2$ and $p_1 \in L^\infty(0, T; H^1(\Omega_1^t))$ as follows

$$\left\{\begin{array}{rcll}
\frac{\partial \vec{u}_1}{\partial t} - \nabla \cdot (\mu_1 \nabla \vec{u}_1) + \nabla p_1 & = & \vec{f}_1, & \text{in } \Omega_1^t \times (0, T] \\
\nabla \cdot \vec{u}_1 & = & 0, & \text{in } \Omega_1^t \times (0, T] \\
\vec{u}_1 & = & \vec{g}_1, & \text{on } \partial\Omega_1^t \backslash \Gamma^t \times (0, T] \\
\vec{u}_1(x, 0) & = & \vec{u}_1^0, & \text{in } \Omega_1^0 \\
\frac{\partial \vec{u}_2}{\partial t} - \nabla \cdot (\mu_2 \nabla \vec{u}_2) & = & \vec{f}_2, & \text{in } \Omega_2^t \times (0, T] \\
\vec{u}_2 & = & \vec{g}_2, & \text{on } \partial\Omega_2^t \backslash \Gamma^t \times (0, T] \\
\vec{u}_2(x, 0) & = & \vec{u}_2^0, & \text{in } \Omega_2^0 \\
\vec{u}_1 & = & \vec{u}_2, & \text{on } \Gamma^t \times [0, T] \\
(-p_1 I + \mu_1 \nabla \vec{u}_1)\, \vec{n_1} + \mu_2 \nabla \vec{u_2} \vec{n_2} & = & \vec{\tau}, & \text{on } \Gamma^t \times [0, T]
\end{array}\right. \tag{3.1}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two jump coefficients, i.e., $\mu_1 \neq \mu_2$. And, $\vec{f}_i \in L^2(\Omega_i^t)(i = 1, 2)$, $\vec{\tau} \in H^{1/2}(\Gamma^t)$.

3.2 ALE Mapping

With the model problem in place, we now define the affine mapping that allows us to use the ALE description of the model problem. Assume $\exists X_i^t \in H^1\left(0, T; W^{2,\infty}(\Omega_i^0)^d\right)$ (i=1,2) such that $\forall t \in (0, T]$, the mapping:

$$\begin{aligned} X_i^t : \Omega_i^0 &\rightarrow \Omega_i^t \\ \hat{x}_i &\rightarrow x(\hat{x}, t) \end{aligned}$$

is invertible such that $(X_i^t)^{-1} \in W^{1,\infty}(\Omega_i^t)^2$. $\hat{x}_i \in \Omega_i^0$ is known as the reference coordinate variable. The domain velocity is then defined as

$$\vec{\omega_i} : \Omega_i^t \times (0, T] \rightarrow R^2, \quad \vec{\omega_i}(x, t) = \frac{\partial X_i^t\,(\hat{x}, t)}{\partial t}; for\ i = 1, 2$$

With this domain velocity, we can now define a derivative which takes this velocity into account. This is known as the ALE derivative and is defined as

$$\begin{aligned} \frac{\partial \vec{u}_i}{\partial t}\Big|_{\hat{x}} : \Omega_i^t \times (0,T] &\rightarrow R \\ (x,t) &\rightarrow \frac{\partial \vec{u}_i}{\partial t}\Big|_{\hat{x}}(x,t) = \frac{\partial \vec{u}_i}{\partial t}(x,t) + \left(\vec{\omega}_i(x,t)\cdot\nabla\right)\vec{u}_i(x,t) \end{aligned} \tag{3.2}$$

Equipped with the domain velocity and ALE derivative, we can proceed to rewrite our problem using the ALE description.

Substituting this into our model problem we obtain the ALE description as follows.

$$\begin{cases} \frac{\partial \vec{u}_1}{\partial t}\Big|_{\hat{x}} - \nabla\cdot(\mu_1\nabla\vec{u}_1) - (\vec{\omega}_2\cdot\nabla)\,\vec{u}_1 + \nabla p_1 = \vec{f}_1, & \text{in } \Omega_1^t \times (0,T] \\ \nabla\cdot\vec{u}_1 = 0, & \text{in } \Omega_1^t \times (0,T] \\ \vec{u}_1 = \vec{g}_1, & \text{on } \partial\Omega_1^t\backslash\Gamma^t\times(0,T] \\ \vec{u}_1(x,0) = \vec{u}_1^0, & \text{in } \Omega_1^0 \\ \frac{\partial \vec{u}_2}{\partial t}\Big|_{\hat{x}} - \nabla\cdot(\mu_2\nabla\vec{u}_2) - (\vec{\omega}_2\cdot\nabla)\,\vec{u}_2 = \vec{f}_2, & \text{in } \Omega_2^t \times (0,T] \\ \vec{u}_2 = \vec{g}_2, & \text{on } \partial\Omega_2^t\backslash\Gamma^t\times(0,T] \\ \vec{u}_2(x,0) = \vec{u}_2^0, & \text{in } \Omega_2^0 \\ \vec{\omega}_1 = \vec{\omega}_2, & \text{on } \Gamma^t \times [0,T] \\ \vec{u}_1 = \vec{u}_2, & \text{on } \Gamma^t \times [0,T] \\ (-p_1 I + \mu_1\nabla\vec{u}_1)\,\vec{n}_1 + \mu_2\nabla\vec{u}_2\vec{n}_2 = \vec{\tau}, & \text{on } \Gamma^t \times [0,T] \end{cases} \tag{3.3}$$

3.3 Geometric Conservation Law

The Geometric Conservation Law (GCL) comes as a consequence of Theorem 2.1. Letting $\varphi(x,t) = 1$, we see that $\frac{\partial\varphi}{\partial t} = 0$. Plugging this into Theorem 2.1, we have

$$\frac{d}{dt}\left|\Omega^t\right| = \int_{\Omega^t} (\nabla\cdot\vec{\omega})\,dx = \int_{\partial\Omega^t} \vec{\omega}\cdot\vec{n}ds$$

Integrating both sides from t^n to t^{n+1}, we get

Lemma 3.1 (Geometric Conservation Law)**.**

$$\left|\Omega\left(t^{n+1}\right)\right| - \left|\Omega\left(t^n\right)\right| = \int_{t^n}^{t^{n+1}} \int_{\partial\Omega^t} \vec{\omega}\cdot\vec{n}dsdt$$

Consider the P.D.E.

$$\frac{\partial \vec{u}}{\partial t} + \nabla \cdot F = 0$$

where $\vec{u}$ is a transported quantity and F is the flux. Choosing a test function Ψ, integrating over the entire domain, using integration by parts and applying Theorem 2.1 to take the time derivative out of the integral we can obtain the Conservative Formulation:

$$\frac{d}{dt}\left(\vec{u}, \Psi\right)_{\Omega^t} - \left(\nabla \cdot \left(\vec{u} \cdot \vec{\omega}^T\right), \Psi\right)_{\Omega^t} - \left(F, \nabla\Psi\right)_{\Omega^t} = 0.$$

We can see that letting $\vec{u} = 1$ and $\Psi = 1$ we obtain Lemma 3.1. Hence the GCL is conserved. This is the formulation we will be analyzing in this thesis.

3.4 Conservative Weak Form

To begin, we need to introduce the following functional Spaces.

$$\begin{aligned}
\bar{U} &:= \{(\psi_1, \psi_2) \in H^1(\Omega_1^t)^d \times H^1(\Omega_2^t)^d \big| \psi_1 = \psi_2 \text{on } \Gamma^t\} \\
\bar{U} &:= \{\psi_i \in H^1(\Omega_i^t)^d \big| \psi_i = \hat{\psi}_i((X_i^t)^{-1}) \forall \hat{\psi}_i \in \mathrm{H}^1(\Omega_i^0)^d,\ \mathrm{i} = 1,2\} \\
\bar{U}_g &:= \{(\psi_1, \psi_2) \in \bar{U} \big| \psi_i = g_i \text{ on } \partial\Omega_i^t \backslash \Gamma^t, i = 1,2\} \\
\bar{U}_0 &:= \{(\psi_1, \psi_2) \in \bar{U} \big| \psi_i = 0 \text{ on } \partial\Omega_i^t \backslash \Gamma^t, i = 1,2\} \\
Q^1 &:= L^2\left(\Omega_1^t\right) \\
Q_0^1 &:= \{q_1 \in Q^1 \big| \textstyle\int_{\Omega_1^t} q_1 dx = 0\}.
\end{aligned}$$

With these spaces we can now define the ALE Weak form of model (3.3). Adding the equations of model (3.3) together, multiplying by test functions $(\psi_1, \psi_2) \in \bar{U}_0$ and applying integration by parts, we obtain the conservative weak form as follows: Find $(\vec{u}_1, \vec{u}_2) \in (H^1 \cap L^\infty)\left(0, T; \bar{U}_g\right)$ and $p_1 \in L^2\left(0, T; Q_0^1\right)$ such that

$$\sum_{i=1}^{2}\left[\frac{d}{dt}\left(\vec{u}_i, \psi_i\right)_{\Omega_i^t} + \left(\mu_i \nabla \vec{u}_i, \nabla\psi_i\right)_{\Omega_i^t} - \left(\left(\vec{\omega}_i \cdot \nabla\right)\vec{u}_i, \psi_i\right)_{\Omega_i^t} - \left(\left(\nabla \cdot \vec{\omega}_i\right)\vec{u}_i, \psi_i\right)_{\Omega_i^t}\right] \tag{3.4}$$

$$- \left(p_1, \nabla \cdot \psi_1\right)_{\Omega_1^t} + \left(\nabla \cdot \vec{u}_1, q_1\right)_{\Omega_1^t} = \sum_{i=1}^{2}\left(\vec{f}_i, \psi_i\right)_{\Omega_i^t} + \langle \vec{\tau}, \psi_1 \rangle_{\Gamma^t}, \forall \left(\psi_1, \psi_2\right) \in \bar{U}_0,\ q_1 \in Q^1$$

CHAPTER 4

SEMI-DISCRETE FINITE ELEMENT APPROXIMATION

Denote the mesh size with h (0<h<1). For i=1,2, we construct the quasi-uniform triangulation $T_{h,i}^0$ in the continuous domain Ω_i^0. We assume also no triangle of $T_{h,i}^0$ has two edges on $\partial\Omega_i^0$ and that no triangle crosses the interface Γ^0. We now consider the discrete ALE mapping of X_i^t by means of piecewise linear Lagrangian finite elements denoted by $X_{h,i}^t$.

4.1 Discretized ALE Mapping and the Semi-Discrete Formulation

We define the discrete ALE mapping $X_{h,i}^t$:

$$\begin{aligned} X_{h,i}^t : \Omega_i^0 &\to \Omega_i^t \\ \hat{x} &\to x(\hat{\mathbf{x}}, t) \end{aligned}$$

where $X_{h,i}^t$ is smooth and invertible. Likewise, the discrete mesh velocity is defined as follows:

$$\vec{\omega}_{h,i} : \Omega_i^t \times (0,T] \to R^2, \quad \vec{\omega}_{h,i}(x,t) = \frac{dX_{h,i}^t(\hat{x},t)}{dt},$$

which leads to the discrete ALE time derivative:

$$\begin{aligned} \frac{\partial \vec{u}_i}{\partial t}\Big|_{\hat{x}}^h : \Omega_i^t \times (0,T] &\to R \\ (\mathbf{x},t) &\to \frac{\partial \vec{u}_i}{\partial t}\Big|_{\hat{x}}^h := \frac{\partial \vec{u}_i}{\partial t}(x,t) + (\vec{\omega}_{h,i}(x,t)\cdot\nabla)\,\vec{u}_i(x,t). \end{aligned}$$

We will denote the image of $\mathcal{T}_{h,i}^0$ under this discrete mapping as $\mathcal{T}_{h,i}^t$. We now proceed to the definition of our mixed finite element spaces using the classical P_2 elements for $\vec{u}_i$ and P_1 elements for Q_1.

The discrete ALE finite element spaces are defined as follows:

$$
\begin{array}{rcl}
\bar{W}_h^t & = & \{(\psi_{h,1},\psi_{h,2}) \in \bar{U}_g \big| \psi_{h,i}\big|_K \in P_2(K), \forall K \in T_{h,i}^t (i=1,2)\}, \\
\bar{W}_h^0 & = & \{(\psi_{h,1},\psi_{h,2}) \in \bar{U}_0 \big| \psi_{h,i}\big|_K \in P_2(K), \forall K \in T_{h,i}^t (i=1,2)\}, \\
M_h^t & = & \{q_{h,1} \in Q_1 \big| q_{h,1}\big|_K \in P_1(K), \forall K \in T_{h,1}^t\} \\
M_h^0 & = & \{q_{h,1} \in Q_1^0 \big| q_{h,1}\big|_K \in P_1(K), \forall K \in T_{h,1}^t\} \\
\tilde{W}_h^t & = & \{(\psi_{h,1},\psi_{h,2}) \in \bar{W}_h^t \big| \ (\nabla\cdot \psi_{h,1}, q_{h,1})_{\Omega_1^t} = 0, \forall q_{h,1} \in M_h^t\},
\end{array}
$$

where $P_n(K)$ is the set of polynomials on K of degree less than or equal to n.

Now, using (3.4) and the above definitions, the corresponding semi-discrete conservative ALE finite element discretization is to find $(\vec{u}_{h,1},\vec{u}_{h,2}) \in \bar{W}_h^t$, $p_{h,1} \in M_h^t$ such that

$$
\sum_{i=1}^{2} \left[\frac{d}{dt} \left(\vec{u}_{h,i}, \psi_{h,i}\right)_{\Omega_i^t} + \left(\mu_i \nabla \vec{u}_{h,i}, \nabla \psi_{h,i}\right)_{\Omega_i^t} - \left(\left(\vec{\omega}_i \cdot \nabla\right) \vec{u}_{h,i}, \psi_{h,i}\right)_{\Omega_i^t} - \left(\left(\nabla \cdot \vec{\omega}_i\right) \vec{u}_{h,i}, \psi_{h,i}\right)_{\Omega_i^t} \right]
$$

$$
- \left(p_{h,1}, \nabla \cdot \psi_{h,1}\right)_{\Omega_1^t} + \left(\nabla \cdot \vec{u}_{h,1}, q_{h,1}\right)_{\Omega_1^t} = \sum_{i=1}^{2} \left(\vec{f}_i, \psi_{h,i}\right)_{\Omega_i^t} + \langle \vec{\tau}, \psi_{h,1} \rangle_{\Gamma^t}
$$

$$
\forall \left(\psi_{h,1},\psi_{h,2}\right) \in \bar{W}_h^0, \ q_{h,1} \in M_h^t. \qquad (4.1)
$$

The analysis of the convergence of the above scheme relies on a couple of assumptions about the discrete ALE mapping $X_{h,i}^t$. We assume that the following error estimate is true:

$$
\|X_i^t - X_{h,i}^t\|_{L^\infty(\Omega_i^0)^d} + h \|\nabla \left(X_i^t - X_{h,i}^t\right)\|_{L^\infty(\Omega_i^0)^{2d}} \leq C h^2 |lnh| \|X_i^t\|_{W^{2,\infty}(\Omega_i^0)^d}.
$$

Construction of such a mapping is discussed in Gastaldi (2001).

Assuming $\vec{\omega}_{h,i} \in W^{2,\infty}(\Omega_i^t)^d$, then we also have the following error estimate on the domain velocity:

$$
\|\vec{\omega}_i(t) - \vec{\omega}_{h,i}(t)\|_{L^\infty(\Omega_i^t)^d} + h \|\nabla \left(\omega(t)_i - \vec{\omega}_{h,i}(t)\right)\|_{L^\infty(\Omega_i^t)^{2d}} \leq C h^2 |lnh| \|\vec{\omega}_i(t)\|_{W^{2,\infty}(\Omega_i^t)^d}. \quad (4.2)
$$

Finally, we assume that our triangulation $\mathcal{T}_{h,i}^t$ is non-degenerate with time. That is, we assume that there exists a $\rho > 0$ such that

$$
diam B_K \geq \rho h \ diam K, \ \forall K \in \mathcal{T}_{h,i}^t
$$

for all $t \in [0,T]$ and all $h \in (0,1]$, where B_k is the largest disk contained in K. We are now in a position to analyze the stability of 4.1.

4.2 Stability Analysis

Theorem 4.1. *The following stability result holds for the semi-discrete scheme (4.1) for any $t \in (0, T]$:*

$$\sum_{i=1}^{2}\left(\|\vec{u}_{h,i}\|_{L^\infty\left(0,t;L^2(\Omega_i^t)^d\right)} + \|\vec{u}_{h,i}\|_{L^2(0,t;H^1(\Omega_i^t)^d)}\right)$$
$$\leq C\left(\sum_{i=1}^{2}\left(\|\vec{f}_i\|_{L^2\left(0,t;L^2(\Omega_i^t)^d\right)} + \|\vec{u}_i^0\|_{L^2(\Omega_i^0)^d}\right) + \|\vec{\tau}\|_{L^2(0,T;L^2(\Gamma^t)^d)}\right). \quad (4.3)$$

Proof. In equation (4.1), let $\psi_{h,i} = \vec{u}_{h,i},\ q_{h,1} = p_{h,1}$ and use Theorem (2.1) to go back to time derivative on discrete ALE frame:

$$\sum_{i=1}^{2}\left[\left(\frac{d\vec{u}_{h,i}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \vec{u}_{h,i}\right)_{\Omega_i^t} + (\mu_i\nabla\vec{u}_{h,i}, \nabla\vec{u}_{h,i})_{\Omega_i^t} - ((\vec{\omega}_{h,i}\cdot\nabla)\,\vec{u}_{h,i}, \vec{u}_{h,i})_{\Omega_i^t}\right] \quad (4.4)$$
$$= \sum_{i=1}^{2}\left(\vec{f}_i, \vec{u}_{h,i}\right)_{\Omega_i^t} + \langle\vec{\tau}, \vec{u}_{h,1}\rangle_{\Gamma^t}.$$

By using the following estimates

$$\left(\frac{d\vec{u}_{h,i}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \vec{u}_{h,i}\right)_{\Omega_i^t} = \frac{1}{2}\left(\frac{d}{dt}\|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2 - (\vec{u}_{h,i}\nabla\cdot\vec{\omega}_{h,i}, \vec{u}_{h,i})\right),$$
$$(\mu_i\nabla\vec{u}_{h,i}, \nabla\vec{u}_{h,i})_{\Omega_i^t} = \mu_i\|\nabla\vec{u}_{h,i}\|_{0,\Omega_i^t}^2 \geq C\|\vec{u}_{h,i}\|_{1,\Omega_i^t}^2,$$

we then have,

$$\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2 + C\|\vec{u}_{h,i}\|_{1,\Omega_i^t}^2\right]$$
$$\leq \sum_{i=1}^{2}\left[\left(\vec{f}_i, \vec{u}_{h,i}\right)_{\Omega_i^t} + \frac{1}{2}(\vec{u}_{h,i}\nabla\cdot\vec{\omega}_{h,i}, \vec{u}_{h,i})_{\Omega_i^t} + ((\vec{\omega}_{h,i}\cdot\nabla)\,\vec{u}_{h,i}, \vec{u}_{h,i})_{\Omega_i^t}\right]$$
$$+ \langle\vec{\tau}, \vec{u}_{h,1}\rangle_{\Gamma^t}.$$

Using the boundedness of $\vec{\omega}_{h,i}$ due to the convergence assumption of the discrete domain velocity (4.2), Young's inequality with ϵ ,the Cauchy-Schwarz inequality and the trace theorem we have the following:

$$((\vec{\omega}_{h,i} \cdot \nabla) \vec{u}_{h,i}, \vec{u}_{h,i})_{\Omega_i^t} \leq \|\vec{\omega}_{h,i}\|_{\infty,\Omega_i^t} \|\nabla \vec{u}_{h,i}\|_{0,\Omega_i^t} \|\vec{u}_{h,i}\|_{0,\Omega_i^t} \tag{4.5}$$

$$\leq \epsilon \|\vec{u}_{h,i}\|_{1,\Omega_i^t}^2 + C\|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2,$$

$$(\vec{u}_{h,i} \nabla \cdot \vec{\omega}_{h,i}, \vec{u}_{h,i})_{\Omega_i^t} \leq C\|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2, \tag{4.6}$$

$$\left(\vec{f}_{h,i}, \vec{u}_{h,i}\right)_{\Omega_i^t} \leq \|\vec{f}_{h,i}\|_{0,\Omega_i^t} \|\vec{u}_{h,i}\|_{0,\Omega_i^t} \leq C\left(\|\vec{f}_{h,i}\|_{0,\Omega_i^t}^2 + \|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2\right), \tag{4.7}$$

$$\langle \vec{\tau}, \vec{u}_{h,1} \rangle_{\Gamma^t} \leq \|\vec{\tau}\|_{L^2(\Gamma^t)} \|\vec{u}_{h,1}\|_{L^2(\Gamma^t)} \leq C\|\vec{\tau}\|_{L^2(\Gamma^t)} \|\vec{u}_{h,1}\|_{1,\Omega_1^t} \tag{4.8}$$

$$\leq C\|\vec{\tau}\|_{L^2(\Gamma^t)}^2 + \epsilon \|\vec{u}_{h,1}\|_{1,\Omega_1^t}^2.$$

We choose a sufficiently small ϵ, leading to

$$\sum_{i=1}^{2} \left[\frac{1}{2}\frac{d}{dt}\|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2 + C\|\vec{u}_{h,i}\|_{1,\Omega_i^t}^2\right] \leq \left(\sum_{i=1}^{2}\left(\|\vec{f}_{h,i}\|_{0,\Omega_i^t}^2 + \|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2\right) + \|\vec{\tau}\|_{L^2(\Gamma^t)}^2\right).$$

Integrating over time from 0 to t, then

$$\sum_{i=1}^{2} \|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2 + \sum_{i=1}^{2}\int_0^t \|\vec{u}_{h,i}\|_{1,\Omega_i^t}^2 dt \tag{4.9}$$
$$\leq C\left(\sum_{i=1}^{2}\left(\int_0^t \left(\|\vec{f}_{h,i}\|_{0,\Omega_i^t}^2 + \|\vec{u}_{h,i}\|_{0,\Omega_i^t}^2\right) dt + \|\vec{u}_i^0\|_{L^2(\Omega_i^0)}^2\right) + \int_0^t \|\vec{\tau}\|_{L^2(\Gamma^t)}^2 dt\right).$$

Using Gronwall's inequality, we have the desired stability result in Theorem (4.1). □

4.3 Semi-Discrete Error Analysis

We begin by looking at a novel H^1 -projection, it's definition and resulting error estimates, that will help us through the error-analysis.

Definition 4.1. Assume $(\tilde{\mathbf{u}}_{\mathbf{1}}, \tilde{\mathbf{u}}_{\mathbf{2}}) \in \bar{W}_h^t$ and $\tilde{\mathbf{p}}_{\mathbf{1}} \in M_h^t$, then the following H^1 -projection for the solution to (3.1) is defined

$$\sum_{i=1}^{2}\left[\left(\mu_i\nabla\left(\vec{u}_i-\tilde{\mathbf{u}}_{\mathbf{i}}\right),\nabla\psi_{h,i}\right)_{\Omega_i^t}-\left(\left(\vec{\omega}_{h,i}\cdot\nabla\right)\left(\vec{u}_i-\tilde{\mathbf{u}}_{\mathbf{i}}\right),\psi_{h,i}\right)_{\Omega_i^t}+\kappa\left(\left(\vec{u}_i-\tilde{\mathbf{u}}_{\mathbf{i}}\right),\psi_{h,i}\right)_{\Omega_i^t}\right]$$

$$-\left(\left(p_1-\tilde{\mathbf{p}}_{\mathbf{1}}\right),\nabla\cdot\psi_{h,1}\right)_{\Omega_1^t}+\left(q_{h,1},\nabla\cdot\left(\vec{u}_i-\tilde{\mathbf{u}}_{\mathbf{i}}\right)\right)_{\Omega_1^t}=0,\forall\left(\psi_{h,1},\psi_{h,2}\right)\in\bar{W}_h^0,\ q_{h,1}\in M_h^t$$

where $\kappa=\max\left(\frac{M_1^2}{2\mu_1}+\frac{\mu_1}{2}+M_1,\frac{M_2^2}{2\mu_2}+\frac{\mu_2}{2}+M_2\right)$ and $|\vec{\omega}_{h,i}|_{0,\infty}\leq M_i(i=1,2)$.

The following lemmas for error estimates of the H^1-projection defined as above are proved in Lan and Sun (2019).

Lemma 4.1. *There exists a unique solution* $((\tilde{\mathbf{u}}_{\mathbf{1}},\tilde{\mathbf{u}}_{\mathbf{2}}),\tilde{\mathbf{p}}_{\mathbf{1}})\in\bar{W}_h^t\times M_h^t$ *such that:*

$$\sum_{i=1}^{2}\|\vec{u}_i-\tilde{\mathbf{u}}_i\|_{0,\Omega_i^t}+h\sum_{i=1}^{2}\|\vec{u}_i-\tilde{\mathbf{u}}_i\|_{1,\Omega_i^t}+h\|\ p_1-\tilde{\mathbf{p}}_{\mathbf{1}}\|_{0,\Omega_1^t}$$
$$\leq h^2\left(\sum_{i=1}^{2}\|\vec{u}_i\|_{2,\Omega_i^t}+\|p_1\|_{1,\Omega_1^t}\right).$$

Lemma 4.2. *With the same condition of Lemma 4.1, we have the following error estimate:*

$$\sum_{i=1}^{2}\|\frac{d\vec{u}_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}-\frac{d\tilde{\mathbf{u}}_{\mathbf{i}}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\|_{1,\Omega_i^t}+\|\frac{dp_1}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}-\frac{d\tilde{\mathbf{p}}_{\mathbf{1}}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\|_{0,\Omega_i^t}$$
$$\leq Ch|\ln h|\left(\|(\vec{u}_1,\vec{u}_2)\|_{2,\Omega_i^t}+\|(\frac{d\vec{u}_1}{dt}\bigg|_{\hat{\mathbf{x}}},\frac{d\vec{u}_2}{dt}\bigg|_{\hat{\mathbf{x}}})\|_{2,\Omega_i^t}+\|p_1\|_{1,\Omega_i^t}+\|\frac{dp_1}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{1,\Omega_i^t}\right),$$
$$\sum_{i=1}^{2}\|\frac{d\vec{u}_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}-\frac{d\tilde{\mathbf{u}}_{\mathbf{i}}}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}\|_{0,\Omega_i^t}$$
$$\leq Ch\left(\|\vec{u}_1\|_{2,\Omega_i^t}+\|\frac{d\vec{u}_1}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{2,\Omega_i^t}+\|\vec{u}_2\|_{2,\Omega_i^t}+\|\frac{d\vec{u}_2}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{2,\Omega_i^t}+\|p_1\|_{1,\Omega_i^t}+\|\frac{dp_1}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{1,\Omega_i^t}\right).$$

Applying the H^1-projection defined in Definition (4.1) to the ALE Weak Form (3.4), we get the following ALE weak form with projection:

$$\sum_{i=1}^{2}\left[\frac{d}{dt}\left(\vec{u}_i,\psi_{h,i}\right)_{\Omega_i^t}+\left(\mu_i\nabla\tilde{\mathbf{u}}_{\mathbf{i}},\nabla\psi_{h,i}\right)_{\Omega_i^t}-\left(\left(\vec{\omega}_{h,i}\cdot\nabla\right)\tilde{\mathbf{u}}_{\mathbf{i}},\psi_{h,i}\right)_{\Omega_1^t}-\left(\left(\nabla\cdot\vec{\omega}_{h,i}\right)\vec{u}_i,\psi_{h,i}\right)_{\Omega_1^t}\right]$$
$$-\left(\tilde{\mathbf{p}}_{\mathbf{1}},\nabla\cdot\psi_{h,1}\right)_{\Omega_1^t}+\left(q_{h,1},\nabla\cdot\tilde{\mathbf{u}}_{\mathbf{1}}\right)_{\Omega_1^t}=\sum_{i=1}^{2}\left[\left(\vec{f}_i,\psi_{h,i}\right)_{\Omega_i^t}+\kappa\left(\left(\vec{u}_i-\tilde{\mathbf{u}}_{\mathbf{i}}\right),\psi_{h,i}\right)_{\Omega_i^t}\right]+\langle\vec{\tau},\psi_{h,1}\rangle_{\Gamma^t}$$
$$\forall\left(\psi_{h,1},\psi_{h,2}\right)\in\bar{W}_h^0,\ q_{h,1}\in M_h^t. \tag{4.10}$$

We can now proceed to the main theorem of the section, the error estimate of the semi-discrete scheme.

Theorem 4.2. *Suppose* $(\vec{u}_1, p_1, \vec{u}_2)$ *is the solution to (4.10) and* $(\vec{u}_{h,1}, p_{h,1}, \vec{u}_{h,2})$ *is the solution to (4.1), then we have the following error estimate:*

$$\sum_{i=1}^{2}\left[\|\vec{u}_i - \vec{u}_{h,i}\|_{L^\infty\left(0,T;L^2(\Omega_i^t)^d\right)} + \|\vec{u}_i - \vec{u}_{h,i}\|_{L^2(0,T;H^1(\Omega_i^t)^d)}\right]$$
$$\le Ch\left(\|\vec{u}_1\|_{L^\infty(0,T,H^2(\Omega_i^t)^d)} + \|\vec{u}_2\|_{L^\infty(0,T,H^2(\Omega_i^t)^d)} + \|p_1\|_{L^\infty\left(0,t;H^1(\Omega_t^1)\right)}\right. \tag{4.11}$$
$$\left. + \|\frac{d\vec{u}_1}{dt}\bigg|_{\hat{x}}\|_{L^2(0,T;H^2(\Omega_i^t)^d)} + \|\frac{d\vec{u}_2}{dt}\bigg|_{\hat{x}}\|_{L^2(0,T;H^2(\Omega_i^t)^d)} + \|\frac{dp_1}{dt}\bigg|_{\hat{x}}\|_{L^2(0,T;H^1(\Omega_i^t))}\right).$$

Proof. Subtracting (4.1) from (4.10), we get the error equation:

$$\sum_{i=1}^{2}\left[\frac{d}{dt}\left(\vec{u_i} - \vec{u}_{h,i}, \psi_{h,i}\right)_{\Omega_i^t} + \left(\mu_i \nabla\left(\tilde{\mathbf{u}}_\mathbf{i} - \vec{u}_{h,i}\right), \nabla\psi_{h,i}\right)_{\Omega_i^t} - \left(\left(\vec{\omega}_{h,i}\cdot\nabla\right)\left(\tilde{\mathbf{u}}_\mathbf{i} - \vec{u}_{h,i}\right), \psi_{h,i}\right)_{\Omega_i^t}\right.$$
$$\left. - \left(\left(\nabla\cdot\vec{\omega}_{h,i}\right)\left(\vec{u_i} - \vec{u}_{h,i}\right), \psi_{h,i}\right)_{\Omega_i^t} - \left(\tilde{\mathbf{p}}_\mathbf{1} - p_{h,i}, \nabla\cdot\psi_{h,1}\right)_{\Omega_1^t} + \left(q_{h,1}, \nabla\cdot\left(\tilde{\mathbf{u}}_\mathbf{1} - \vec{u}_{h,i}\right)\right)_{\Omega_1^t}\right]$$
$$= \sum_{i=1}^{2}\left[\kappa\left((\vec{u_i} - \tilde{\mathbf{u}}_\mathbf{i}), \psi_{h,i}\right)_{\Omega_i^t}\right], \forall\left(\psi_{h,1}, \psi_{h,2}\right) \in \bar{W}_h^0,\ q_{h,1} \in M_h^t. \tag{4.12}$$

Picking new variables $\delta_i = \vec{u_i} - \tilde{\mathbf{u}}_\mathbf{i}$, $\sigma_i = \tilde{\mathbf{u}}_\mathbf{i} - \vec{u}_{h,i}$, $\phi = \tilde{\mathbf{p}}_\mathbf{1} - p_{h,1}$, and using Theorem (2.1) to go back to time derivative on discrete ALE frame, we can rewrite (4.12) as

$$\sum_{i=1}^{2}\left[\left(\frac{d(\delta_i+\sigma_i)}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \psi_{h,i}\right)_{\Omega_i^t} + \mu_i(\nabla\sigma_i, \nabla\psi_{h,i})_{\Omega_i^t}\right] - (\phi, \nabla\cdot\psi_{h,1})_{\Omega_1^t} + (q_{h,1}, \nabla\cdot\sigma_1)_{\Omega_1^t}$$
$$= \sum_{i=1}^{2}\left[((\vec{\omega}_{h,i}\cdot\nabla)\sigma_i, \psi_{h,i})_{\Omega_i^t} + \kappa(\delta_i, \psi_{h,i})_{\Omega_i^t}\right]. \tag{4.13}$$

Choosing $\psi_{h,i} = \sigma_i$, $q_{h,1} = \phi$, the error equation (4.13) becomes

$$\sum_{i=1}^{2}\left[\left(\frac{d\delta_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h} + \frac{d\sigma_i}{dt}\bigg|_{\hat{\mathbf{x}}}^{h}, \sigma_i\right)_{\Omega_i^t} + \mu_i\nabla\sigma_i, \nabla\sigma_i)_{\Omega_i^t}\right]$$
$$= \sum_{i=1}^{2}\left[((\vec{\omega}_{h,i}\cdot\nabla)\sigma_i, \sigma_i)_{\Omega_i^t} + \kappa(\delta_i, \sigma_i)_{\Omega_i^t}\right]. \tag{4.14}$$

Using Youngs inequality with ϵ, the Cauchy-Schwarz inequality and applying the bound of $\vec{\omega}_{h,i}$ and Lemma 4.1 for the H^1-projection error estimate, we get the following estimates on the right hand side:

$$((\vec{\omega}_{h,i}\cdot\nabla)\,\sigma_i,\sigma_i)\le\epsilon\|\nabla\sigma_i\|^2_{0,\Omega_i^t}+C\|\sigma_i\|^2_{0,\Omega_i^t} \tag{4.15}$$

$$\kappa\,(\delta_i,\sigma_i)_{\Omega_i^t}\le C\left(h^4\left(\|\vec{u}_1\|_{2,\Omega_i^t}+\|\vec{u}_2\|_{2,\Omega_i^t}+\|p_1\|_{1,\Omega_i^t}\right)^2+\|\sigma_i\|^2_{0,\Omega_i^t}\right) \tag{4.16}$$

For the left hand side terms, we note that

$$\left(\frac{d\delta_i}{dt}\Big|^h_{\hat{\mathbf{x}}}+\frac{d\sigma_i}{dt}\Big|^h_{\hat{\mathbf{x}}},\sigma_i\right)_{\Omega_i^t}=\frac{1}{2}\frac{d}{dt}\|\sigma_i\|^2_{0,\Omega_i^t}-\frac{1}{2}((\nabla\cdot\vec{\omega}_{h,i})\sigma_i,\sigma_i)_{\Omega_i^t}+\left(\frac{d\delta_i}{dt}\Big|^h_{\hat{\mathbf{x}}},\sigma_i\right)_{\Omega_i^t}, \tag{4.17}$$

$$\mu_i\,(\nabla\sigma_i,\nabla\sigma_i)_{\Omega_i^t}=\mu_i\|\nabla\sigma_i\|^2_{0,\Omega_i^t}. \tag{4.18}$$

Applying the boundedness of $\vec{\omega}_{h,i}$, Lemma 4.2 for the ALE derivative of H^1-projection error estimate as well as Cauchy-Schwarz and Young's inequality we get the following estimates:

$$\begin{aligned}\frac{1}{2}\left((\nabla\cdot\vec{\omega}_{h,i})\,\sigma_i,\sigma_i\right)_{\Omega_i^t}&\le C\|\sigma_i\|^2_{0,\Omega_i^t},\\ \left(\frac{d\delta_i}{dt}\Big|^h_{\hat{\mathbf{x}}},\sigma_i\right)_{\Omega_i^t}&\le Ch\Bigg(\Bigg(\|\vec{u}_1\|_{2,\Omega_i^t}+\|\frac{d\vec{u}_1}{dt}\Big|_{\hat{\mathbf{x}}}\|_{2,\Omega_i^t}+\|\vec{u}_2\|_{2,\Omega_i^t}+\|\frac{d\vec{u}_2}{dt}\Big|_{\hat{\mathbf{x}}}\|_{2,\Omega_i^t}\\ &\quad+\|p_1\|_{1,\Omega_i^t}+\|\frac{dp_1}{dt}\Big|_{\hat{\mathbf{x}}}\|_{1,\Omega_i^t}\Bigg)\|\sigma_i\|_{0,\Omega_i^t}\Bigg),\\ &\le C\left(h^2+\|\sigma_i\|^2_{0,\Omega_i^t}\right).\end{aligned} \tag{4.19}$$

Applying the estimates obtained above and choosing ϵ small enough, we have

$$\begin{aligned}&\sum_{i=1}^{2}\left[\frac{1}{2}\frac{d}{dt}\|\sigma_i\|^2_{0,\Omega_i^t}+\|\nabla\sigma_i\|^2_{0,\Omega_i^t}\right]\\ &\le C\sum_{i=1}^{2}\left[\|\sigma_i\|^2_{0,\Omega_i^t}+h^2+h^4\right]\\ &\le C\sum_{i=1}^{2}\left[\|\sigma_i\|^2_{0,\Omega_i^t}+h^2\right].\end{aligned} \tag{4.20}$$

Integrating in time from 0 to t, yields

$$\begin{aligned}&\sum_{i=1}^{2}\left[\frac{1}{2}\|\sigma_i\|^2_{0,\Omega_i^t}+\int_0^t\|\nabla\sigma_i\|^2_{0,\Omega_i^t}ds\right]\\ &\le\sum_{i=1}^{2}\left[\frac{1}{2}\|\sigma_i^0\|_{0,\Omega_i^0}+C\int_o^t\left(\|\sigma_i\|^2_{0,\Omega_i^t}+h^2\right)ds\right].\end{aligned} \tag{4.21}$$

Applying Gronwall's Inequality, we're left with

$$\begin{aligned}&\sum_{i=1}^{2}\left[\|\sigma_i\|^2_{L^\infty\left(0,T;L^2(\Omega_i^t)\right)}+\|\nabla\sigma_i\|^2_{L^2\left(0,T;L^2(\Omega_i^t)^4\right)}\right]\\&\leq\sum_{i=1}^{2}\left[\|\sigma_i^0\|_{L^2\left(\Omega_i^0\right)^2}+Ch^2\right].\end{aligned}\tag{4.22}$$

By adding $\|\delta_i\|^2_{L^\infty\left(0,T;L^2(\Omega_i^t)\right)}$ and $\|\nabla\delta_i\|^2_{L^2\left(0,T;L^2(\Omega_i^t)^4\right)}$ to left hand side, applying Lemmas 4.1 and 4.2 as well as the triangle inequality, and choose $\vec{u}^0_{h,i}=\tilde{\mathbf{u}}^0_i$, we have our result in (4.11). □

CHAPTER 5

FULLY-DISCRETE FINITE ELEMENT APPROXIMATION

With the semi-discrete scheme completed, we can now move on to the fully-discrete scheme. Let $\Delta t > 0$ be the time step and $t^n = n\Delta t$ for $n = 0, ..., N$ such that $t^N \leq$ T and $t^{N+1} >$ T. We'll be using the backward Euler scheme for temporal discretization. We introduce the following notation:

$$X^{n,n+1} = X^{n+1} \circ (X^n)^{-1},$$

and change variables

$$x^{n+1} = X^{n+1} \circ (X^n)^{-1}(x^n)$$

to deal with variables and test functions in different domains and on different time levels. We also define:

$$\hat{\partial}^t \varphi_i^{n+\frac{1}{2}} = \frac{(\varphi_i^{n+1}, \psi_{h,i}^{n+1})_{\Omega_i^{n+1}} - (\varphi_i^n, \psi_{h,i}^{n+1} \circ X^{n,n+1})_{\Omega_i^n}}{\Delta t},$$
$$\hat{\partial}^t \varphi_{h,i}^{n+\frac{1}{2}} = \frac{(\varphi_{h,i}^{n+1}, \psi_{h,i}^{n+1})_{\Omega_i^{n+1}} - (\varphi_{h,i}^n, \psi_{h,i}^{n+1} \circ X^{n,n+1})_{\Omega_i^n}}{\Delta t}.$$

We let J_i^t (i=1,2) denote the Jacobian matrix of the ALE mapping with its determinant given by:

$$J_i^t := det(F_i^t) = det\left(\frac{\partial X_i^t(\hat{x})}{\partial \hat{x}}\right).$$

The fully discrete scheme can now be obtained as follows:

Find $(\vec{u}_{h,1}^{n+1}, \vec{u}_{h,2}^{n+1}) \in \bar{W}_h^{n+1}$, $p_{h,1}^{n+1} \in M_h^{n+1}$ for every n=0,...,N-1 such that:

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[\hat{\partial}^t \vec{u}_{h,i}^{n+\frac{1}{2}} + \mu_i \left(\nabla \vec{u}_{h,i}^{n+1}, \nabla \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}} - \left(\left(\nabla \cdot \vec{\omega}_{h,i}^{n+1} \right) \vec{u}_{h,i}^{n+1}, \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}} \\
&- \left(\left(\vec{\omega}_{h,i}^{n+1} \cdot \nabla \right) \vec{u}_{h,i}^{n+1}, \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}} \Big] - \left(p_{h,1}^{n+1}, \nabla \cdot \psi_{h,1}^{n+1} \right)_{\Omega_1^{n+1}} + \left(\nabla \cdot \vec{u}_{h,1}^{n+1}, q_{h,1}^{n+1} \right)_{\Omega_1^{n+1}} \qquad (5.1) \\
&= \sum_{i=1}^{2} \left[\left(\vec{f}_i^{n+1}, \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}} \right] + \left\langle \vec{\tau}^{n+1}, \psi_{h,1}^{n+1} \right\rangle_{\Gamma^{n+1}}
\end{aligned}$$

holds for every $(\psi_{h,1}, \psi_{h,2}) \in \bar{W}_h^{n+1}$ and every $q_{h,1} \in M_h^{n+1}$. We can now move on to the error estimate for the fully discrete scheme.

5.1 Fully-Discrete Error Analysis

We'll start with a few lemmas which will allow us to perform the required analysis.

Lemma 5.1. *Let $\varphi_{h,i}^{n+1} \in \bar{W}_h^{n+1}$, then*

$$\|\varphi_{h,i}^{n+1} \circ X_i^{n,n+1}\|_{0,\Omega_i^n}^2 = \|\varphi_{h,i}^{n+1}\|_{0,\Omega_i^{n+1}}^2 - \int_{t^n}^{t^{n+1}} \left(\int_{\Omega_i^t} |\varphi_{h,i}^{n+1} \circ X_i^{t,n+1}|^2 \nabla \cdot \vec{\omega}_{h,i} dx \right) dt.$$

Proof.

$$\begin{aligned}
&\frac{d}{dt} \int_{\Omega_i^t} |\varphi_{h,i}^{n+1} \circ X_i^{t,n+1}|^2 dx = \int_{\Omega_i^0} \frac{d}{dt} |\hat{\varphi}_{h,i}^{n+1}|^2 J_i^t d\hat{x} = \int_{\Omega_i^0} |\hat{\varphi}_{h,i}^{n+1}|^2 \frac{dJ_i^t}{dt} d\hat{x} \\
&= \int_{\Omega_i^t} |\varphi_{h,i}^{n+1} \circ X_i^{t,n+1}|^2 \nabla \cdot \vec{\omega}_{h,i} dx.
\end{aligned} \qquad (5.2)$$

Thus

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}} \int_{\Omega_i^t} |\varphi_{h,i}^{n+1} \circ X_i^{t,n+1}|^2 \nabla \cdot \vec{\omega}_{h,i} dx dt = \int_{t^n}^{t^{n+1}} \frac{d}{dt} \int_{\Omega_i^t} |\varphi_{h,i}^{n+1} \circ X_i^{t,n+1}|^2 dx dt \\
&= \int_{\Omega_i^{n+1}} |\varphi_{h,i}^{n+1}|^2 dx - \int_{\Omega_i^n} |\varphi_{h,i}^{n+1} \circ X_i^{n,n+1}|^2 dx,
\end{aligned} \qquad (5.3)$$

where rearranging gives the result. □

The following lemma considers the classical Taylor expansion technique in the context of the ALE description.

Lemma 5.2. *For any $\varphi_i \in \bar{W}_{h,t}$, we have*

$$\frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^n,t^n)}{\Delta t} = \left.\frac{d\varphi_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \frac{\Delta t}{2}\left[\left.\frac{d^2\varphi_i^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}} - \frac{dx}{dt}\frac{d\varphi_i}{dx}\frac{d}{dx}\left(\frac{dx}{dt}\right)\right].$$

Proof. Expanding $\varphi_i(x^n,t^n)$ at x^{n+1}, we get

$$\varphi_i(x^n,t^n) = \varphi_i(x^{n+1},t^n) - \Delta x\left(\frac{\partial\varphi_i}{\partial x}\right)(x^{n+1},t^n) + \frac{(\Delta x)^2}{2}\left(\frac{\partial^2\varphi_i}{\partial x^2}\right)(x^{n+1},t^n) + ... \tag{5.4}$$

Noting that

$$\begin{aligned}\left(\frac{\partial\varphi_i}{\partial x}\right)(x^{n+1},t^n) &= \left(\frac{\partial\varphi_i}{\partial x}\right)(x^{n+1},t^{n+1}) - \Delta t\left(\frac{\partial^2\varphi_i}{\partial x\partial t}\right)(x^{n+1},t^{n+1}) + ...,\\ \left(\frac{\partial^2\varphi_i}{\partial x^2}\right)(x^{n+1},t^n) &= \left(\frac{\partial^2\varphi_i}{\partial x^2}\right)(x^{n+1},t^{n+1}) - \Delta t\left(\frac{\partial^3\varphi_i}{\partial x^2\partial t}\right)(x^{n+1},t^{n+1}) + ...,\end{aligned} \tag{5.5}$$

we have,

$$\begin{aligned}\varphi_i(x^n,t^n) &= \varphi_i(x^{n+1},t^n) - \Delta x\left(\frac{\partial\varphi_i}{\partial x}\right)(x^{n+1},t^{n+1}) + \Delta x\Delta t\left(\frac{\partial^2\varphi_i}{\partial x\partial t}\right)(x^{n+1},t^{n+1})\\ &+ \frac{(\Delta x)^2}{2}\left(\frac{\partial^2\varphi_i}{\partial x^2}\right)(x^{n+1},t^{n+1}) + ...\end{aligned} \tag{5.6}$$

Thus,

$$\frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^n,t^n)}{\Delta t} = \frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^{n+1},t^n)}{\Delta t} + \frac{\varphi_i(x^{n+1},t^n)-\varphi_i(x^n,t^n)}{\Delta t}. \tag{5.7}$$

Which, when expanded, gives

$$\frac{\Delta x}{\Delta t}\left(\frac{\partial\varphi_i}{\partial x}\right)^{n+1} - \frac{(\Delta x)^2}{2\Delta t}\left(\frac{\partial^2\varphi_i}{\partial x^2}\right)^{n+1} - \Delta x\left(\frac{\partial^2\varphi_i}{\partial x\partial t}\right)^{n+1} + \frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^{n+1},t^n)}{\Delta t}, \tag{5.8}$$

where

$$\frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^{n+1},t^n)}{\Delta t} = \left(\frac{\partial\varphi_i}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2\varphi_i}{\partial t^2}\right)^{n+1} + ... \tag{5.9}$$

Since $x(\hat{\mathbf{x}},t^n) = x(\hat{\mathbf{x}},t^{n+1}) - \Delta t\left(\frac{\partial x}{\partial t}\right)^{n+1} + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1} + ...$ We see that $\frac{\Delta x}{\Delta t} = \left(\frac{\partial x}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1}$.

We then have

$$
\begin{aligned}
&\frac{\varphi_i(x^{n+1},t^{n+1})-\varphi_i(x^n,t^n)}{\Delta t} = \left(\frac{\partial x}{\partial t}\right)^{n+1}\left(\frac{\partial \varphi_i}{\partial x}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 x}{\partial t^2}\right)^{n+1}\left(\frac{\partial \varphi_i}{\partial x}\right)^{n+1} \\
&- \Delta t\left(\frac{\partial x}{\partial t}\right)^{n+1}\left(\frac{\partial^2 \vec{V}}{\partial x \partial t}\right)^{n+1} - \frac{\Delta t}{2}\left[\left(\frac{\partial x}{\partial t}\right)^{n+1}\right]^2\left(\frac{\partial^2 \varphi_i}{\partial x^2}\right)^{n+1} \\
&+ \left(\frac{\partial \varphi_i}{\partial t}\right)^{n+1} - \frac{\Delta t}{2}\left(\frac{\partial^2 \varphi_i}{\partial t^2}\right)^{n+1} \qquad (5.10) \\
&= \left.\frac{d\varphi_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \frac{\Delta t}{2}\left[\frac{\partial^2 \varphi_i}{\partial t^2} + \frac{\partial^2 x^{n+1}}{\partial t^2}\frac{\partial \varphi_i^{n+1}}{\partial x} + 2\frac{\partial x^{n+1}}{\partial t}\frac{\partial^2 \varphi_i^{n+1}}{\partial x \partial t} + \left(\frac{\partial x^{n+1}}{\partial t}\right)^2\frac{\partial^2 \varphi_i^{n+1}}{\partial x^2}\right] \\
&= \left.\frac{d\varphi_i^{n+1}}{dt}\right|_{\hat{\mathbf{x}}} - \frac{\Delta t}{2}\left[\left.\frac{d^2 \varphi_i^{n+1}}{dt^2}\right|_{\hat{\mathbf{x}}} - \frac{dx}{dt}\frac{d\varphi_i}{dx}\frac{d}{dx}\left(\frac{dx}{dt}\right)\right].
\end{aligned}
$$

□

The next lemmas are borrowed from Martin et al. (2009). It puts bounds on various Jacobian terms which arise.

Lemma 5.3. *Due to the change of variable $X^{n,n+1}$, we have that*

$$\|\varphi_{h,i}^{n+1} \circ X_i^{n,n+1}\|_{0,\Omega_i^n}^2 \leq \|J_i^n\|_{\infty,\Omega^n}\|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}}\|\varphi_{h,i}^{n+1}\|_{0,\Omega_i^{n+1}}^2.$$

Lemma 5.4. *There exists C_1 and C_2 depending on X and $h_0 > 0$ such that for i=1,2*

$$\|J_{h,i}^t\|_{L^\infty(\Omega_0)} \leq C_1 \qquad \forall t \in [0,T],\ \forall h \in (0,h_0)$$

$$\|(J_{h,i}^t)^{-1}\|_{L^\infty(\Omega_0)} \leq C_2 \qquad \forall t \in [0,T],\ \forall h \in (0,h_0)$$

$$\|J_i^t - J_i^n\|_\infty \leq C\Delta t.$$

The final lemma is borrowed from (Lee and Xu (2016)). This provides a bound on the discrete domain velocity based on the regularity of the ALE mapping.

Lemma 5.5. *There exists an $M > 0$ such that*

$$max\{\|\vec{\omega}_i\|_{1,,\Omega_i^t}, \|\frac{\partial \vec{\omega}_i}{\partial t}\|_{1,\Omega_i^t}\} \leq M, \forall t \in [0,T](i=1,2).$$

We can now proceed to the main theorem of the section, the fully discrete error estimate.

Theorem 5.1. *Suppose $(\vec{u}_1, p_1, \vec{u}_2)$ is the solution to (4.10) and $\left(\vec{u}_{h,1}^{n+1}, p_{h,1}^{n+1}, \vec{u}_{h,2}^{n+1}\right)$ is the solution to (5.1), then we have the following error estimate:*

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\|\vec{u}_i^N - \vec{u}_{h,i}^N\|_{0,\Omega_i^N} + \Delta t\sum_{n=0}^{N-1}\|\vec{u}_i^{n+1} - \vec{u}_{h,i}^{n+1}\|_{H^1(\Omega_i^{n+1})}\right]\\
&\le C(h+\Delta t)\Bigg[\|\vec{u}_1\|_{L^\infty(0,T,H^2(\Omega_i^t)^d)} + \|\vec{u}_2\|_{L^\infty(0,T,H^2(\Omega_i^t)^d)} + \|p_1\|_{L^\infty\left(0,t;H^1(\Omega_t^1)\right)}\\
&+\|\frac{d\vec{u}_1}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{L^2(0,T;H^2(\Omega_i^t)^d)} + \|\frac{d\vec{u}_2}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{L^2(0,T;H^2(\Omega_i^t)^d)} + \|\frac{dp_1}{dt}\bigg|_{\hat{\mathbf{x}}}\|_{L^2(0,T;H^1(\Omega_i^t))}\Bigg].
\end{aligned} \tag{5.11}$$

Proof. We let equation (4.10) take values at t^{n+1} and add $\hat{\partial}^t\vec{u}_i^{n+\frac{1}{2}}$ to both sides of the equation. We then subtract equation (5.1) from this result and apply the H^1-projection to get the following error equation:

$$\begin{aligned}
&\sum_{i=1}^{2}\Bigg[\left(\hat{\partial}^t\vec{u}_i^{n+\frac{1}{2}} - \hat{\partial}^t\vec{u}_{h,i}^{n+\frac{1}{2}}\right) + \mu_i\left(\nabla\left(\tilde{\mathbf{u}}_i^{n+1} - \vec{u}_{h.i}^{n+1}\right), \nabla\psi_{h,i}^{n+1}\right)_{\Omega_i^{n+1}}\\
&-\left(\left(\vec{\omega}_{h,i}^{n+1}\cdot\nabla\right)\left(\tilde{\mathbf{u}}_i^{n+1} - \vec{u}_{h.i}^{n+1}\right), \psi_{h,i}^{n+1}\right)_{\Omega_1^{n+1}} + \left(\left(\nabla\cdot\vec{\omega}_{h,i}\right)\left(\vec{u}_{h.i}^{n+1} - \vec{u}_i^{n+1}\right), \psi_{h,i}^{n+1}\right)_{\Omega_1^{n+1}}\Bigg]\\
&-\left(\left(\tilde{\mathbf{p}}_1^{n+1} - p_{h,1}^{n+1}\right), \nabla\cdot\psi_{h,1}^{n+1}\right)_{\Omega_1^{n+1}} + \left(q_{h,1}^{n+1}, \nabla\cdot\left(\tilde{\mathbf{u}}_1^{n+1} - \vec{u}_{h,1}^{n+1}\right)\right)_{\Omega_1^{n+1}}\\
&=\sum_{i=1}^{2}\left[\kappa\left((\vec{u_i}^{n+1} - \tilde{\mathbf{u}}_i^{n+1}), \psi_{h,i}^{n+1}\right)_{\Omega_i^{n+1}}\right] - \left(\frac{d}{dt}\left(\vec{u}_i^{n+1}, \psi_{h,i}^{n+1}\right)_{\Omega_i^{n+1}} - \hat{\partial}^t\vec{u}_i^{n+\frac{1}{2}}\right),\\
&\forall\left(\psi_{h,1}^{n+1}, \psi_{h,2}^{n+1}\right)\in\bar{W}_h^{n+1},\ q_{h,1}^{n+1}\in M_h^{n+1}.
\end{aligned}$$

Picking new variables $\delta_i^{n+1} = \vec{u}_i^{n+1} - \tilde{\mathbf{u}}_i^{n+1}$, $\sigma_i^{n+1} = \tilde{\mathbf{u}}_i^{n+1} - \vec{u}_{h,i}^{n+1}$, $\phi^{n+1} = \tilde{\mathbf{p}}_1^{n+1} - p_{h,i}^{n+1}$ and reorganize terms as:

$$\sum_{i=1}^{2}\sum_{j=1}^{2}L_i^j = \sum_{i=1}^{2}\sum_{j=1}^{6}R_i^j$$

where:

$\mathrm{L}_i^1 = \hat{\partial}^t\sigma_i^{n+\frac{1}{2}},$

$L_i^2 = \mu_i\left(\nabla\sigma_i^{n+1}, \nabla\psi_{h,i}^{n+1}\right)_{\Omega_i^{n+1}},$

$R_i^1 = \left(\left(\vec{\omega}_{h,i}^{n+1}\cdot\nabla\right)\sigma_i^{n+1}, \psi_{h,i}^{n+1}\right)_{\Omega_1^{n+1}},$

$R_i^2 = \left((\nabla \cdot \vec{\omega}_{h,i}) \left(\delta_i^{n+1} + \sigma_i^{n+1} \right), \psi_{h,i}^{n+1} \right)_{\Omega_1^{n+1}},$

$R_i^3 = -\left(\phi^{n+1}, \nabla \cdot \psi_{h,1}^{n+1} \right)_{\Omega_1^{n+1}} + \left(q_{h,1}^{n+1}, \nabla \cdot \sigma_i^{n+1} \right)_{\Omega_1^{n+1}},$

$R_i^4 = \kappa \left((\delta_i^{n+1}, \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}},$

$R_i^5 = -\left(\frac{d}{dt} \left(u_i^{n+1}, \psi_{h,i}^{n+1} \right)_{\Omega_i^{n+1}} - \hat{\partial}^t \vec{u}_i^{n+\frac{1}{2}} \right),$

$R_i^6 = -\hat{\partial}^t \delta_i^{n+\frac{1}{2}},$

$\forall \left(\psi_{h,1}^{n+1}, \psi_{h,2}^{n+1} \right) \in \bar{W}_h^{n+1},\ q_{h,1}^{n+1} \in M_h^{n+1}.$

Choosing $\psi_{h,i}^{n+1} = \sigma_i^{n+1},\ q_{h,1} = \phi^{n+1}$, the error estimates for the following terms are obtained using similar methods as demonstrated in Chapter 4:

$\mathrm{L}_i^2 \geq C \|\nabla \sigma_i^{n+1}\|_{0,\Omega_i^{n+1}},$

$R_i^1 \leq \epsilon \|\nabla \sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 + C \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2,$

$R_i^2 \leq \mathrm{C} \Big(\|\delta_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 + \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 \Big)$ after using Youngs inequality,

$R_i^3 = 0,$

$R_i^4 \leq C \left(\|\delta_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 + \|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 \right).$

We start first by analyzing the L_i^1 term:

$$\hat{\partial}^t \sigma_i^{n+\frac{1}{2}} = \frac{1}{\Delta t} \Big[\left(\sigma_i^{n+1}, \sigma_i^{n+1} \right)_{\Omega_i^{n+1}} - \left(\sigma_i^n, \sigma_i^{n+1} \circ X^{n,n+1} \right)_{\Omega_i^n} \Big],$$

and apply Lemma (5.1) to get the following:

$$\frac{1}{\Delta t} \Big[\left(\sigma_i^{n+1}, \sigma_i^{n+1} \right)_{\Omega_i^{n+1}} - \left(\sigma_i^n, \sigma_i^{n+1} \circ X^{n,n+1} \right)_{\Omega_i^n} \Big]$$
$$\geq \frac{1}{2\Delta t} \Big[\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2 - \|\sigma_i^n\|_{0,\Omega_i^n}^2 + \int_{t^n}^{t^{n+1}} \left(\int_{\Omega_i^t} |\sigma_i^{n+1} \circ X^{t,n+1}|^2 \nabla \cdot \vec{\omega}_{h,i} dx \right) dt \Big]. \quad (5.12)$$

The last term satisfies the following after use of Lemmas (5.3) and (5.4) and is subsequently moved to the right hand side of our error equation:

$$\frac{1}{2\Delta t} \int_{t^n}^{t^{n+1}} \left(\int_{\Omega_i^t} |\sigma_i^{n+1} \circ X^{t,n+1}|^2 \nabla \cdot \vec{\omega}_{h,i} dx \right) dt$$

$$\leq \frac{1}{2\Delta t} \sup_{\hat{t}\in(t^n,t^{n+1})} \|J_i^{\hat{t}}(\nabla\cdot\hat{\omega}_{h,i})\|_{\infty,\Omega_i^0} \|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}} \int_{t^n}^{t^{n+1}} \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} d\hat{t}$$

$$\leq \frac{1}{2} \sup_{\hat{t}\in(t^n,t^{n+1})} \|J_i^{\hat{t}}(\nabla\cdot\hat{\omega}_{h,i})\|_{\infty,\Omega_i^0} \|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}} \|\sigma_i^{n+1}\|^2_{0,\Omega_{h,i}^{n+1}}$$

$$\leq C\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}. \tag{5.13}$$

To estimate R_i^5, let $\hat{u}_i = \vec{u}_i(\hat{t})$ and first consider the Taylor expansion of

$\frac{d}{dt}\left(\vec{u}_i, \sigma_i^{n+1}\circ X^{t,n+1}\right)_{\Omega_i^t}$ at t^{n+1}.

$$\Rightarrow \frac{d}{dt}\left(\vec{u}_i, \sigma_i^{n+1}\circ X^{t,n+1}\right)_{\Omega_i^t}\bigg|_{t^{n+1}} = \frac{1}{\Delta t}\bigg[\left(\vec{u}_i^{n+1}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} - \left(\vec{u}_i^{n}, \sigma_i^{n+1}\circ X^{n,n+1}\right)_{\Omega_i^n} + \int_{t^n}^{t^{n+1}} (\hat{t}-t^n)\frac{d^2}{d\hat{t}^2}\left(\hat{u}_i, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} d\hat{t}\bigg]. \tag{5.14}$$

Applying Theorem 2.1 on the following term from the remainder of the Taylor expansion (5.14) gives us:

$$\begin{aligned}
\frac{d^2}{d\hat{t}^2}\left(\hat{u}_i, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} &= \left(\frac{d^2\hat{u}_i}{d\hat{t}^2}\bigg|_{\hat{x}}, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} + \left(\frac{d\hat{u}_i}{d\hat{t}}\bigg|_{\hat{x}}(\nabla\cdot\vec{\omega}_i), \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} \\
&\quad + \left(\frac{d\hat{u}_i}{d\hat{t}}\bigg|_{\hat{x}}(\nabla\cdot\vec{\omega}_i) + \hat{u}_i\frac{\partial(\nabla\cdot\vec{\omega}_i)}{\partial\hat{t}}\bigg|_{\hat{x}}, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} \\
&\quad + \left(\hat{u}_i(\nabla\cdot\vec{\omega}_i)^2, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} \\
&\leq \bigg[\|\frac{d^2\hat{u}_i}{d\hat{t}^2}\bigg|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} + 2\|\nabla\cdot\vec{\omega}_i\|_{\infty,\Omega_i^{\hat{t}}}\|\frac{d\hat{u}_i}{d\hat{t}}\bigg|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} + \|\hat{u}_i\|_{0,\Omega_i^{\hat{t}}}\|\frac{\partial(\nabla\cdot\vec{\omega}_i)}{d\hat{t}}\|_{\infty,\Omega_i^{\hat{t}}} \\
&\quad + \|\nabla\cdot\vec{\omega}_i\|^2_{\infty,\Omega_i^{\hat{t}}}\|\hat{u}_i\|_{0,\Omega_i^{\hat{t}}}\bigg]\|\sigma_i^{n+1}\circ X^{\hat{t},n+1}\|_{0,\Omega_i^{\hat{t}}}.
\end{aligned} \tag{5.15}$$

We define the following notation and apply Lemma (5.5) to get:

$$\begin{aligned}
:= G(\hat{t}) = \|\frac{d^2\hat{u}_i}{d\hat{t}^2}\bigg|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} + 2\|\nabla\cdot\hat{\omega}_i\|_{\infty,\Omega_i^{\hat{t}}}\|\frac{d\hat{u}_i}{d\hat{t}}\bigg|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} \\
+\|\hat{u}_i\|_{0,\Omega_i^{\hat{t}}}\|\frac{\partial(\nabla\cdot\vec{\omega}_i)}{d\hat{t}}\|_{\infty,\Omega_i^{\hat{t}}} + \|\nabla\cdot\vec{\omega}_i\|^2_{\infty,\Omega_i^{\hat{t}}}\|\hat{u}_i\|_{0,\Omega_i^{\hat{t}}}
\end{aligned}$$

$$\leq C\left[\|\frac{d^2\hat{u}_i}{d\hat{t}^2}\Big|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} + \|\frac{d\hat{u}_i}{d\hat{t}}\Big|_{\hat{x}}\|_{0,\Omega_i^{\hat{t}}} + \|\hat{u}_i\|_{0,\Omega_i^{\hat{t}}}\right].$$

By using the results above, we obtain:

$$\begin{aligned}
R_i^5 &= -\left(\frac{d}{dt}\left(\vec{u}_i^{n+1}, \psi_{h,i}^{n+1}\right)_{\Omega_i^{n+1}} - \hat{\partial}^t \vec{u}_i^{n+\frac{1}{2}}\right) \\
&\leq \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)\frac{d^2}{d\hat{t}^2}\left(\hat{u}_i, \sigma_i^{n+1}\circ X^{\hat{t},n+1}\right)_{\Omega_i^{\hat{t}}} d\hat{t} \\
&\leq \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)G(\hat{t})\|\sigma_i^{n+1}\circ X^{\hat{t},n+1}\|_{0,\Omega_i^{\hat{t}}} d\hat{t}.
\end{aligned} \tag{5.16}$$

We continue this result using Lemma (5.3) and Youngs inequality

$$\begin{aligned}
&\leq \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} \|J_i^{\hat{t}}\|^{\frac{1}{2}}_{\infty,\Omega_i^0}\|(J_i^{n+1})^{-1}\|^{\frac{1}{2}}_{\infty,\Omega_i^{n+1}}(\hat{t}-t^n)G(\hat{t})\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}} d\hat{t} \\
&\leq \frac{C}{\Delta t}\left(\int_{t^n}^{t^{n+1}} \|J_i^{\hat{t}}\|_{\infty,\Omega_i^0}\|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}} G^2(\hat{t})d\hat{t}\right)^{\frac{1}{2}}\left(\int_{t^n}^{t^{n+1}} (\hat{t}-t^n)^2\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} d\hat{t}\right)^{\frac{1}{2}} \\
&\leq C\sqrt{\frac{\Delta t}{3}} \sup_{\hat{t}\in(t^n,t^{n+1})} \|J_i^{\hat{t}}\|_{\infty,\Omega_i^0}\|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}}\left(\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t}\right)^{\frac{1}{2}}\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}} \\
&\leq C(\Delta t) \sup_{\hat{t}\in(t^n,t^{n+1})} \|J_i^{\hat{t}}\|_{\infty,\Omega_i^0}\|(J_i^{n+1})^{-1}\|_{\infty,\Omega_i^{n+1}}\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t} + \epsilon_2\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}.
\end{aligned} \tag{5.17}$$

By Lemma 5.4 we get our error estimate for the term

$$R_i^5 \leq C\left((\Delta t)\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t} + \|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\right). \tag{5.18}$$

We estimate the R_i^6 as follows:

$$\begin{aligned}
-\hat{\partial}^t\delta_i^{n+\frac{1}{2}} &= \frac{1}{\Delta t}\left[\left(\delta_i^n, \sigma_i^{n+1}\circ X^{n,n+1}\right)_{\Omega_i^n} - \left(\delta_i^{n+1}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}\right] \\
&\leq \frac{1}{\Delta t}\left[\left(\delta_i^n\circ X^{n+1,n}\frac{J_i^n}{J_i^{n+1}}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}, -\left(\delta_i^{n+1}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}\right] \\
&\leq -\left(\frac{\delta_i^{n+1} - \delta_i^n\circ X^{n+1,n}\frac{J_i^n}{J_i^{n+1}}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} \\
&\leq -\left(\frac{\delta_i^{n+1} - \delta_i^n\circ X^{n+1,n}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}} - \left(\frac{\delta_i^n\circ X^{n+1,n} - \delta_i^n\circ X^{n+1,n}\frac{J_i^n}{J_i^{n+1}}}{\Delta t}, \sigma_i^{n+1}\right)_{\Omega_i^{n+1}}
\end{aligned}$$

The first term of R_i^6 will be handled using Lemma 5.2:

$$
\begin{aligned}
&-\left(\frac{\delta_i^{n+1}-\delta_i^n\circ X^{n+1,n}}{\Delta t},\sigma_i^{n+1}\right)_{\Omega_i^{n+1}}\\
&=-\left(\frac{d\delta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}}^h-\frac{\Delta t}{2}\left(\frac{d^2\delta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}}-\left(\frac{dx}{dt}\right)^{n+1}\left(\frac{d\delta_i^{n+1}}{dx}\right)^{n+1}\frac{d}{dx}\left(\frac{dx}{dt}\right)^{n+1}\right),\sigma_i^{n+1}\right)_{\Omega_i^{n+1}}
\end{aligned}
$$

where $\frac{\partial x}{\partial t}\in H^1(0,T;W^{1,\infty}(\Omega_i^t))$,

$$
\begin{aligned}
&\le C\bigg(\|\frac{d\delta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}}^h\|_{0,\Omega_i^{n+1}}^2+\|\frac{\Delta t}{2}\left(\frac{d^2\delta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}}-\left(\frac{dx}{dt}\right)^{n+1}\left(\frac{d\delta_i^{n+1}}{dx}\right)^{n+1}\frac{d}{dx}\left(\frac{dx}{dt}\right)^{n+1}\right)\|_{0,\Omega_i^{n+1}}^2\\
&+\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\bigg)\\
&\le C\left(\|\frac{d\delta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}}^h\|_{0,\Omega_i^{n+1}}^2+\frac{(\Delta t)^2}{4}\beta^2+\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\right)
\end{aligned}
$$

where $\beta^2=\|\left(\frac{d^2\delta_i^{n+1}}{dt^2}\bigg|_{\hat{\mathbf{x}}}-\left(\frac{dx}{dt}\right)^{n+1}\left(\frac{d\delta_i^{n+1}}{dx}\right)^{n+1}\frac{d}{dx}\left(\frac{dx}{dt}\right)^{n+1}\right)\|_{0,\Omega_i^{n+1}}\le C$ due to Lemma 4.2.

The second term of R_i^6 will be handled using Lemmas 5.1 and 5.4:

$$
-\left(\frac{\delta_i^n\circ X^{n+1,n}-\delta_i^n\circ X^{n+1,n}\frac{J_i^n}{J_i^{n+1}}}{\Delta t},\sigma_i^{n+1}\right)_{\Omega_i^{n+1}}=-\left(\frac{(\frac{J_i^{n+1}-J_i^n}{J_i^{n+1}})(\delta_i^n\circ X^{n+1,n})}{\Delta t},\sigma_i^{n+1}\right)_{\Omega_i^{n+1}}
$$

$$
\le C\|\delta_i^n\circ X^{n+1,n}\|_{0,\Omega_i^{n+1}}\|\sigma^{n+1}\|_{0,\Omega_i^{n+1}}
$$

$$
\le C((1+\Delta t)\|\delta_i^n\|_{0,\Omega_i^n}^2+\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2),
$$

where, we do an analogous estimate for δ_i^n as we do for σ_i^{n+1} in (5.12) and (5.13).

Therefore $R_i^6\le C\left(\|\delta_i^n\|_{0,\Omega_i^n}^2+\|\frac{d\delta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}}^h\|_{0,\Omega_i^{n+1}}^2+(\Delta t)^2+\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\right)$

Combining all bounds, moving all negative terms from left hand side to right hand side, and take sufficiently small ϵ we have the following:

$$
\begin{aligned}
&\sum_{i=1}^2\left[\frac{\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2}{\Delta t}-\frac{\|\sigma_i^n\|_{0,\Omega_i^n}^2}{\Delta t}+\|\nabla\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2\right]\\
&\le C\sum_{i=1}^2\bigg[\|\frac{d\delta_i^{n+1}}{dt}\bigg|_{\hat{\mathbf{x}}}^h\|_{0,\Omega_i^{n+1}}^2+\|\delta_i^{n+1}\|_{0,\Omega_i^{n+1}}^2+\|\delta_i^n\|_{0,\Omega_i^n}^2+(\Delta t)^2\\
&+\|\sigma_i^{n+1}\|_{0,\Omega_i^{n+1}}^2+(\Delta t)\int_{t^n}^{t^{n+1}}G^2(\hat{t})d\hat{t}\bigg].
\end{aligned}\tag{5.18}
$$

Combining all constants and multiplying all terms by Δt, we have the following:

$$\sum_{i=1}^{2}\Big[\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} - \|\sigma_i^{n}\|^2_{0,\Omega_i^{n}} + (\Delta t)\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\Big]$$
$$\leq C\sum_{i=1}^{2}\Big[(\Delta t)\|\frac{d\delta_i^{n+1}}{dt}\Big|^h_{\hat{\mathbf{x}}}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)\|\delta_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)\|\delta_i^{n}\|^2_{0,\Omega_i^{n}} + (\Delta t)^3 \tag{5.19}$$
$$+ (\Delta t)\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^2\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t}\Big].$$

Sum over n from 0 to N-1, applying telescoping technique:

$$\sum_{i=1}^{2}\Big[\|\sigma_i^{N}\|^2_{0,\Omega_i^{N}} - \|\sigma_i^{0}\|^2_{0,\Omega_i^{0}} + (\Delta t)\sum_{n=0}^{N-1}\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\Big]$$
$$\leq C\sum_{i=1}^{2}\Big[\sum_{n=0}^{N}(\Delta t)\|\delta_i^{n}\|^2_{0,\Omega_i^{n}} + \sum_{n=0}^{N-1}\Big((\Delta t)\|\frac{d\delta_i^{n}}{dt}\Big|^h_{\hat{\mathbf{x}}}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^3 \tag{5.20}$$
$$+ (\Delta t)\|\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^2\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t}\Big)\Big].$$

Apply Discrete Gronwall inequality to get:

$$\sum_{i=1}^{2}\Big[\|\sigma_i^{N}\|^2_{0,\Omega_i^{N}} - \|\sigma_i^{0}\|^2_{0,\Omega_i^{0}} + (\Delta t)\sum_{n=0}^{N-1}\|\nabla\sigma_i^{n+1}\|^2_{0,\Omega_i^{n+1}}\Big]$$
$$\leq C\sum_{i=1}^{2}\Big[\sum_{n=0}^{N}(\Delta t)\|\delta_i^{n}\|^2_{0,\Omega_i^{n}} + \sum_{n=0}^{N-1}\Big((\Delta t)\|\frac{d\delta_i^{n}}{dt}\Big|^h_{\hat{\mathbf{x}}}\|^2_{0,\Omega_i^{n+1}} + (\Delta t)^3 \tag{5.21}$$
$$+ (\Delta t)^2\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t}\Big)\Big].$$

For last two terms on right hand side, we note the following due to the regularity assumption of the real solution $\vec{u}_i$ (i=1,2):

$$\sum_{n=0}^{N-1}(\Delta t)^2\int_{t^n}^{t^{n+1}} G^2(\hat{t})d\hat{t} = (\Delta t)^2\int_0^T G^2(\hat{t})d\hat{t}$$
$$= (\Delta t)^2\int_0^T\left(\|\frac{\partial^2\vec{u}_i}{\partial\hat{t}^2}\Big|_{\hat{\mathbf{x}}}\|_{0,\Omega_i^{\hat{t}}} + \|\frac{\partial\vec{u}_i}{\partial\hat{t}}\Big|_{\hat{\mathbf{x}}}\|_{0,\Omega_i^{\hat{t}}} + \|\vec{u}_i\|_{0,\Omega_i^{\hat{t}}}\right)^2 d\hat{t} \leq C(\Delta t)^2$$

and

$$\sum_{n=0}^{N-1}(\Delta t)^3 \leq C(\Delta t)^3 N = C(\Delta t)^3\left(\frac{T}{\Delta t}\right) = C(\Delta t)^2$$

After applying the projection estimates and combining terms, using Poincare inequality on left hand side as well as choosing $\vec{u}^0_{h,i} = \tilde{\mathbf{u}}^0_i$, we get:

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\|\sigma_i^N\|^2_{0,\Omega_i^N} + (\Delta t)\sum_{n=0}^{N-1}\|\sigma_i^{n+1}\|^2_{1,\Omega_i^{n+1}}\right] \\
&\leq C(h+\Delta t)^2\Delta t\Bigg[\sum_{n=0}^{N}\left(\|\vec{u}_1^n\|_{2,\Omega_1^n} + \|\vec{u}_2^n\|_{2,\Omega_2^n} + \|p_1^n\|_{1,\Omega_1^n}\right)^2 \\
&+\sum_{n=0}^{N-1}\left(\|\frac{d\vec{u}_1^{n+1}}{dt}\Big|_{\hat{\mathbf{x}}}\|_{2,\Omega_1^{n+1}} + \|\frac{d\vec{u}_2^{n+1}}{dt}\Big|_{\hat{\mathbf{x}}}\|_{2,\Omega_2^{n+1}} + \|\frac{dp_1^{n+1}}{dt}\Big|_{\hat{\mathbf{x}}}\|_{1,\Omega_1^{n+1}}\right)^2\Bigg].
\end{aligned}
\tag{5.22}
$$

By taking the square root both sides of (5.22) and adding to the left hand side $\|\delta_i^N\|_{0,\Omega_i^N}$ and $(\Delta t)\|\delta_i\|_{1,\Omega_i^{n+1}}$ for i=1,2, we have our result in (5.11). □

CHAPTER 6

CONCLUSION

The Stokes/Parabolic interface problem and its ALE finite element analyses conducted in this thesis provide a foundation for more complex fluid-structure interaction problems' ALE finite element approximations and their advanced numerical analyses with optimal convergence rate according to a lower solution regularity in reality. We first provided a model description using moving domains Ω_i^t (i = 1,2) and provided the necessary properties of an appropriate Arbitrary Lagrangian-Eularian (ALE) mapping. Using the ALE description, we proceeded to discretize the spaces to define its semi-discrete conservative ALE finite element approximation, analyzing both its stability and error estimates utilizing a novel H^1 -projection technique, and demonstrated that the semi-discrete scheme has a convergence order of $O(h)$ according to a lower solution regularity. We further discretized the moving temporal domain generated by ALE mapping using the implicit backward Euler scheme, defining the fully discrete conservative ALE finite element approximation. Through additional analysis of the fully discrete scheme's error estimates with respect to the time step size Δt, and using the previously defined H^1 -projection, we obtained a convergence order of $O(h + \Delta t)$ which is consistent with the spatial convergence rate of the semi-discrete scheme, also consistent with the first order backward Euler-type time difference scheme.

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