5-1-2020

Uncertainty Quantification for Maxwell's Equations

Zhiwei Fang

UNCERTAINTY QUANTIFICATION FOR MAXWELL'S EQUATIONS

By

Zhiwei Fang

Bachelor of Science - Hospitality Administration
Tianjin University of Finance and Economics
2011

Master of Science - Mathematical Sciences
Tianjin University of Finance and Economics
2014

A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
May 2020

UNLV | GRADUATE COLLEGE

Dissertation Approval

The Graduate College
The University of Nevada, Las Vegas

February 14th, 2020

This dissertation prepared by

Zhiwei Fang

entitled

Uncertainty Quantification for Maxwell's Equations

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy - Mathematical Sciences
Department of Mathematical Sciences

Jichun Li, Ph.D.
Examination Committee Chair

Hongtao Yang, Ph.D.
Examination Committee Member

Pengtao Sun, Ph.D.
Examination Committee Member

Monika Neda, Ph.D.
Examination Committee Member

Robert Schill, Ph.D.
Graduate College Faculty Representative

Kathryn Hausbeck Korgan, Ph.D.
Graduate College Dean

ABSTRACT

UNCERTAINTY QUANTIFICATION FOR MAXWELL'S EQUATIONS

by

Zhiwei Fang

Dr. Jichun Li, Examination Committee Chair

Professor of Mathematical Sciences

University of Nevada, Las Vegas

This dissertation study three different approaches for stochastic electromagnetic fields based on the time domain Maxwell's equations and Drude's model: stochastic Galerkin method, stochastic collocation method, and Monte Carlo class methods. For each method, we study its regularity, stability, and convergence rates. Numerical experiments have been presented to verify our theoretical results. For stochastic collocation method, we also simulate the backward wave propagation in metamaterial phenomenon.

It turns out that the stochastic Galerkin method admits the best accuracy property but hugest computational workload as the resultant PDEs system is usually coupled. The Monte Carlo class methods are easy to implement and do parallel computing but the accuracy is relatively low. The stochastic collocation method inherits the advantages of both of these two methods.

ACKNOWLEDGEMENTS

I would like to thank my adviser Dr. Jichun Li for his constantly patience, selfless help, and broad knowledge. I also would like to thank him to give me this opportunity to get this degree.

I must also thank the members of my advisory committee, Dr. Pengtao Sun, Dr. Hongtao Yang, Dr. Monika Neda, and Dr. Robert A. Schill Jr. for their guidance and time.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In 1873, Maxwell came up with the famous partial differential equations (PDEs) to characterize the dynamics of electromagnetic fields, which brings the science in a new chapter. Given the material with permittivity ε and permeability μ, the electronic field $\boldsymbol{E}$, and the magnetic flux density $\mathbf{B}$ can be described by the following PDEs system

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \boldsymbol{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

supplemented with the following constitutive relations:

$$\mathbf{B} = \mu \boldsymbol{H},\ \mathbf{j} = \sigma \boldsymbol{E},\ \mathbf{D} = \varepsilon \boldsymbol{E}.$$

Here $\boldsymbol{H}$ represents the magnetic field intensity, $\mathbf{D}$ is the electric flux density, σ is the electric conductivity, and ρ is the volume charge density. By solving this system of PDEs with initial and boundary conditions, we may predict how the electromagnetic fields evolve on a domain.

Nevertheless, things are more complicated than our imagination. In practical problems, especially engineering problems, the material usually contains uncertainty, due to modeling error, measure error or other factors. These random uncertainties may change the solution of a physical model drastically. Therefore, many mathematicians have examined uncertainty quantification (UQ). Uncertainty is ubiquitous in many complex physical systems, such as electromagnetic and acoustic waves and the diffusion of thermal energy through random media, and flow and propagation driven by stochastic forces and stochastic initial conditions.

Stochastic partial differential equations (SPDEs) have played an important role in the study of uncertainty quantification in many branches of science and engineering. In the solid state field, fluctuations in the production process (such as during the lithography) of materials allow us to treat the permittivity and permeability as uncertain parameters (e.g. [1, 2]). Stochastic Maxwell equations with additive noise were investigated in [3, 4].

Due to the curse of dimensionality [5], it is very challenging to solve SPDEs efficiently. Many methods have been come up with to conquer the high dimensionality issue in SPDEs. Three major classes of numerical methods have become very popular in solving stochastic PDEs. First one is the so-called non-intrusive stochastic collocation method (cf. [6, 7, 8]), which is simple in implementation and the system of resulting equations is decoupled and hence is efficient to solve. The stochastic collocation method can achieve fast convergence when the solutions are sufficiently smooth in the random space. The second one is the intrusive stochastic Galerkin method [6, 7, 8], which shows fast convergence rates with increasing order of expansions, provided that the solution of the underlying differential equation is sufficiently smooth in the random space. However, the system of equations resulting from the stochastic Galerkin methods is in general coupled and quite expensive to solve especially for problems requiring high-dimensional random spaces. The last popular method is Monte Carlo class methods. The Monte Carlo class methods are easy to implement and friendly to parallel computing. However, the classic Monte Carlo method only has half order accuracy [9]. By using a multi-level technique, we may fix this issue [10].

In this dissertation, we will study the theory of these three popular methods for electromagnetic fields and their applications. In chapter 2, we will investigate the stochastic Galerkin method and apply it for standard Maxwell's equations. In chapter 3, we will consider the Monte Carlo class methods for Maxwell's equations. In chapter 4, stochastic collocation method has been considered for Maxwell's equations with Drude's metamaterial. Besides the theoretical analysis, we also simulate the backward wave propagation phenomenon in this chapter. We conclude this dissertation in chapter 5.

Before the next chapter, we will introduce the following notations which will be used for all chapters throughout this dissertation.

We introduce the following Hilbert spaces:

$$H(\mathrm{div};\Omega)=\{\boldsymbol{u}\in(L^2(\Omega))^3:\ \nabla\cdot\boldsymbol{u}\in(L^2(\Omega)^3\},$$
$$H(\mathrm{curl};\Omega)=\{\boldsymbol{u}\in(L^2(\Omega))^3:\ \nabla\times\boldsymbol{u}\in(L^2(\Omega))^3\},$$
$$H_0(\mathrm{curl};\Omega)=\{\boldsymbol{u}\in H(\mathrm{curl};\Omega):\ \boldsymbol{n}\times\boldsymbol{u}=0 \text{ on } \partial\Omega\},$$

where Ω is a bounded domain in $\mathbb{R}^3$, and $\boldsymbol{n}$ denotes the unit outward normal vector on $\partial\Omega$.

Suppose Ω has been partitioned by a family of regular cubic or tetrahedral mesh T^h with maximum mesh size h, and adopt the r-th ($r\geq 1$) order Raviart-Thomas-Nédélec (RTN) mixed finite element spaces $\boldsymbol{U}_h$ and $\boldsymbol{V}_h$ [15, 16]. That is, for any $r\geq 1$,

$$\boldsymbol{U}_h=\left\{\boldsymbol{u}_h\in H(\mathrm{div};\Omega):\ \boldsymbol{u}_h|_K\in(p_{r-1})^3\oplus\widetilde{p}_{r-1}\boldsymbol{x},\ \forall K\in T^h\right\},$$
$$\boldsymbol{V}_h=\left\{\boldsymbol{v}_h\in H(\mathrm{curl};\Omega):\ \boldsymbol{v}_h|_K\in(p_{r-1})^3\oplus S_r,\ \forall K\in T^h\right\},$$
$$S_r=\left\{\boldsymbol{p}\in(\widetilde{p}_r)^3:\ \boldsymbol{x}\cdot\boldsymbol{p}=0\right\},$$

or RTN cubic elements:

$$\boldsymbol{U}_h=\left\{\boldsymbol{u}_h\in H(\mathrm{div};\Omega):\ \boldsymbol{u}_h|_K\in Q_{r,r-1,r-1}\times Q_{r-1,r,r-1}\times Q_{r-1,r-1,r},\ \forall K\in T^h\right\},$$
$$\boldsymbol{V}_h=\left\{\boldsymbol{v}_h\in H(\mathrm{curl};\Omega):\ \boldsymbol{v}_h|_K\in Q_{r-1,r,r}\times Q_{r,r-1,r}\times Q_{r,r,r-1},\ \forall K\in T^h\right\}.$$

Here $\widetilde{p}_r$ denotes the space of homogeneous polynomials of degree r, and $Q_{i,j,k}$ denotes the space of polynomials whose degrees are less than or equal to $i,\ j,\ k$ in variables $x,\ y,\ z$, respectively. To impose the perfect electrical conductor (PEC) boundary condition, we denote

$$\boldsymbol{V}_h^0=\{\boldsymbol{v}\in\boldsymbol{V}_h:\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\partial\Omega\}.$$

This RTN mixed finite element is also called edge element.

In numerical simulation, we also consider the transverse electric (TE_z) mode or transverse magnetic (TM_z) mode. In this case, the lowest order (linear) edge element can be constructed by the following:

$$U_h = \{\psi_h \in L^2(\Omega) : \psi_h|_e \text{ is a constant}, \forall e \in \mathcal{T}_h\},$$
$$\boldsymbol{V}_h = \{\boldsymbol{\varphi}_h \in H(\text{curl};\Omega) : \boldsymbol{\varphi}_h|_e \in span\{\phi_i \nabla \phi_j - \phi_j \nabla \phi_i\}, i, j = 1, 2, 3, \forall e \in \mathcal{T}_h\},$$

where ϕ_i denotes the barycentric coordinates of a triangular element e. To impose the 2D perfect conducting boundary condition, we introduce the subspace

$$\boldsymbol{V}_h^0 = \{\boldsymbol{\varphi}_h \in \boldsymbol{V}_h : \ \hat{\tau} \cdot \phi_h = 0, \quad \text{on } \partial\Omega\},$$

where $\hat{\tau}$ is the unit tangential vector on $\partial\Omega$.

CHAPTER 2

STOCHASTIC GALERKIN METHODS FOR MAXWELL'S EQUATIONS WITH RANDOM INPUTS

2.1 Introduction

As mentioned in chapter 1, the system of PDEs arising from the stochastic Galerkin methods is generally coupled and quite expensive to solve especially the dimension of random variables (R.V.s) are very high. The stochastic Galerkin method is based on the polynomial chaos (PC) approximation, originally developed by Ghanem and Spanos [7] using Wiener-Hermite expansion and finite element discretization for a wide range of problems. It was later extended by Xiu and Karniadakis [11] to generalized polynomial chaos (gPC) expansion by using general orthogonal polynomials. Based on the gPC expansion and stochastic Galerkin projection, a given SPDE can be transformed into a system of deterministic PDEs (the Galerkin system) which can be solved by any existing popular numerical methods.

Compared with many current papers on numerical SPDEs, there are few works for solving stochastic Maxwell's equations, especially by gPC methods. In physical and engineering projects, uncertainties may caused by physical materials, by the source wave, and by the physical domain, etc. Therefore, the development of an efficient and high accurate algorithm for Maxwell's equations is meaningful for practical purposes and also interested for mathematicians.

The rest of this chapter is organized as follows. In section 2.2, we carry out some analysis of the gPC method for Maxwells equations. In section 2.3, we develop and analyze both the semi-discrete and fully-discrete finite element schemes for solving the system arising from the gPC method. Numerical experiments are presented in section 2.4 to support our theoretical analysis. This chapter is based on my published paper [12].

2.2 The gPC method for Maxwell's equations

Consider the following Maxwell's equation in $\mathbb{R}^3$ with random coefficients

$$\begin{aligned} \varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t \boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}) &= \nabla\times\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}), && (2.1)\\ \mu(\boldsymbol{x},\boldsymbol{y})\partial_t \boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}) &= -\nabla\times\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}), && (2.2)\end{aligned}$$

where $\boldsymbol{x}$ denotes the spatial variable in the domain $\Omega\subset\mathbb{R}^3$, and $\boldsymbol{y}=(y_1,y_2,\cdots,y_N)^\top\in\mathbb{R}^N$, $N\ge 1$, is a random vector with independent and identically distributed (i.i.d.) components. The curl operators are understood to operate on the spatial variables $\boldsymbol{x}$. What is more, we assume that the equations (2.1)-(2.2) are subject to the initial conditions

$$\boldsymbol{E}(0,\boldsymbol{x},\boldsymbol{y})=\boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}),\qquad \boldsymbol{H}(0,\boldsymbol{x},\boldsymbol{y})=\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}), \tag{2.3}$$

and the PEC boundary condition

$$\boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{0},\qquad \text{on } \partial\Omega, \tag{2.4}$$

where $\boldsymbol{n}$ denotes the unit outward normal vector on $\partial\Omega$, and $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ are given functions.

Following the standard gPC notations [13], we assume that $\{\Phi_m(\boldsymbol{y})\}_{m=1}^M$ be the N-variate orthonormal polynomials of degree up to p, where $M=\binom{N+p}{N}$. Every multivariate polynomial $\Phi_m(\boldsymbol{y})\in\{\Phi_m(\boldsymbol{y})\}_{m=1}^M$ is constructed as a product of univariate polynomials in each direction variable y_i, $i=1,\cdots,N$. That is,

$$\Phi_m(\boldsymbol{y})=\prod_{k=1}^N \phi_{m_k}(y_k),\qquad \sum_{k=1}^N m_k\le p,$$

where m_i is the degree of the univariate polynomial $\phi_{m_i}(y_i)$ for $1\le i\le N$. For the sake of accuracy and computational cost, we will adopt the weighted orthonormal polynomials. That is, we assume the univariate polynomials $\{\phi_i\}_{i=1}^N$ are orthonormal with weight $\rho_i(y_i)$

$$\mathbb{E}[\phi_j(y_i)\phi_k(y_i)]:=\int_{\Xi_i}\phi_j(y_i)\phi_k(y_i)\rho_i(y_i)dy_i=\delta_{jk},\qquad 1\le i,j,k\le N,$$

Table 2.1: Commonly used distributions (measures) and corresponding orthogonal polynomials.

Distribution	Orthogonal polynomials	Support	Alias
Gaussian	Hermite polynomials	$\mathbb{R}$	Wiener chaos (Hermite chaos)
Uniform	Legendre polynomials	$[a, b]$	Legendre chaos
Beta	Jacobi polynomials	$[a, b]$	Jacobi chaos
Gamma	Laguerre polynomials	$[0, \infty)$	Laguerre chaos
Poisson	Charlier polynomial	$\{0, 1, 2, \cdots\}$	Charlier chaos
Binomial	Krawtchouk polynomial	$\{0, 1, 2, \cdots, N\}$	Krawtchouk chaos

where $\mathbb{E}[\cdot]$ is the expectation operator, δ_{jk} is the Kronecker delta function, $\rho_i(y_i)$ is the density function for the R.V. y_i, and Ξ_i is the domain of y_i. Thus, the choice of $\phi_i(y_i)$ depends on the underlying probability density function $\rho_i(y_i)$. Table 2.1 lists some commonly used distributions and their orthogonal polynomials [14]. Let $\rho(\boldsymbol{y}) = \prod_{i=1}^{N} \rho_i(y_i)$ be the joint density function, and $\Xi = \prod_{i=1}^{N} \Xi_i$, then the N-variate basis polynomials $\{\Phi_m(\boldsymbol{y})\}_{m=1}^{M}$ are also orthonormal$\{\Phi_m(\boldsymbol{y})\}_{m=1}^{M}$

$$\mathbb{E}[\Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})] = \int_{\Xi} \Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y} = \delta_{mn}, \qquad 1 \leq m, n \leq M \tag{2.5}$$

With the weighted orthonormal basis $\{\Phi_m(\boldsymbol{y})\}_{m=1}^{M}$, we may approximate the solution of (2.1)-(2.2) as power series

$$\boldsymbol{E}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{m=1}^{\infty} \boldsymbol{E}_m(t, \boldsymbol{x})\Phi_m(\boldsymbol{y}), \qquad \boldsymbol{H}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{m=1}^{\infty} \boldsymbol{H}_m(t, \boldsymbol{x})\Phi_m(\boldsymbol{y}). \tag{2.6}$$

Remark 1. *In fact, some conditions are needed for the approximation (2.6), and the conditions are varying for different kinds for approximations. For example, the L^2 least square approximation (general Fourier series) requires the solution is bounded in L^2 norm, and Fourier coefficients converge to 0. But the proof of these conditions are not elementary. We hence simply assume these conditions are satisfied without any proof. The mathematical proof of those conditions are remained as a future work.*

Substituting (2.6) into (2.1)-(2.2), we get

$$\sum_{m=1}^{\infty} \left(\varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t \boldsymbol{E}_m(t,\boldsymbol{x}) - \nabla \times \boldsymbol{H}_m(t,\boldsymbol{x})\right)\Phi_m(\boldsymbol{y}) = 0 \tag{2.7}$$

$$\sum_{m=1}^{\infty} \left(\mu(\boldsymbol{x},\boldsymbol{y})\partial_t \boldsymbol{H}_m(t,\boldsymbol{x}) + \nabla \times \boldsymbol{E}_m(t,\boldsymbol{x})\right)\Phi_m(\boldsymbol{y}) = 0 \tag{2.8}$$

Multiplying (2.7)-(2.8) by $\Phi_k(\boldsymbol{y})\rho(\boldsymbol{y})$ for any $k \geq 1$, and then integrating the resultant, we obtain, by the orthonormality (2.5),

$$\sum_{m=1}^{\infty} A^{\varepsilon}_{k,m}\partial_t \boldsymbol{E}_m(t,\boldsymbol{x}) - \nabla \times \boldsymbol{H}_k(t,\boldsymbol{x}) = 0, \tag{2.9}$$

$$\sum_{m=1}^{\infty} A^{\mu}_{k,m}\partial_t \boldsymbol{H}_m(t,\boldsymbol{x}) + \nabla \times \boldsymbol{E}_k(t,\boldsymbol{x}) = 0, \tag{2.10}$$

where

$$A^{\varepsilon}_{k,m} = \int_{\Xi} \varepsilon(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_k(\boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y}, \qquad A^{\mu}_{k,m} = \int_{\Xi} \mu(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_k(\boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y}$$

If we truncate the approximation to p-th term, that is, consider the p-th order gPC approximations of $\boldsymbol{E}$ and $\boldsymbol{H}$, then

$$\boldsymbol{E}_M(t,\boldsymbol{x},\boldsymbol{y}) := \sum_{m=1}^{M} \widehat{\boldsymbol{E}}_m(t,\boldsymbol{x})\Phi_m(\boldsymbol{y}), \qquad \boldsymbol{H}_M(t,\boldsymbol{x},\boldsymbol{y}) := \sum_{m=1}^{M} \widehat{\boldsymbol{H}}_m(t,\boldsymbol{x})\Phi_m(\boldsymbol{y}). \tag{2.11}$$

The coefficients $\widehat{\boldsymbol{E}}_m$ and $\widehat{\boldsymbol{H}}_m$ satisfy the following PDEs system

$$\sum_{m=1}^{M} A^{\varepsilon}_{k,m}\partial_t \widehat{\boldsymbol{E}}_m(t,\boldsymbol{x}) - \nabla \times \widehat{\boldsymbol{H}}_k(t,\boldsymbol{x}) = 0, \tag{2.12}$$

$$\sum_{m=1}^{M} A^{\mu}_{k,m}\partial_t \widehat{\boldsymbol{H}}_m(t,\boldsymbol{x}) + \nabla \times \widehat{\boldsymbol{E}}_k(t,\boldsymbol{x}) = 0, \tag{2.13}$$

By using the column vector notations $\widehat{\boldsymbol{E}} = (\widehat{\boldsymbol{E}}_1, \cdots, \widehat{\boldsymbol{E}}_M)^{\top}$, $\widehat{\boldsymbol{H}} = (\widehat{\boldsymbol{H}}_1, \cdots, \widehat{\boldsymbol{H}}_M)^{\top}$, and matrices notations $A^{\varepsilon}(\boldsymbol{x}) = (A^{\varepsilon}_{k,m})_{1\leq k,m\leq M}$ and $A^{\mu} = (A^{\mu}_{k,m})_{1\leq k,m\leq M}$, the above PDEs system can be written as

$$
\begin{aligned}
A^{\varepsilon}(\boldsymbol{x})\partial_t\widehat{\boldsymbol{E}}(t,\boldsymbol{x}) - \nabla\times\widehat{\boldsymbol{H}}(t,\boldsymbol{x}) &= 0, && (2.14)\\
A^{\mu}(\boldsymbol{x})\partial_t\widehat{\boldsymbol{H}}(t,\boldsymbol{x}) - \nabla\times\widehat{\boldsymbol{E}}(t,\boldsymbol{x}) &= 0, && (2.15)
\end{aligned}
$$

which are subject to the PEC boundary condition

$$
\boldsymbol{n}\times\widehat{\boldsymbol{E}} = \boldsymbol{0}, \qquad \text{on } \partial\Omega, \tag{2.16}
$$

and the initial conditions

$$
\widehat{\boldsymbol{E}}(0,\boldsymbol{x}) = \widehat{\boldsymbol{E}}_0(\boldsymbol{x}), \qquad \widehat{\boldsymbol{H}}(0,\boldsymbol{x}) = \widehat{\boldsymbol{H}}_0(\boldsymbol{x}), \tag{2.17}
$$

where $\widehat{\boldsymbol{E}}_0(\boldsymbol{x})$ and $\widehat{\boldsymbol{H}}_0(\boldsymbol{x})$ are the gPC expansion coefficient vectors obtained by expressing the initial condition (2.3) in the form of (2.11).

For the sake of solvability of (2.1)-(2.2) and (2.14)-(2.15), we assume the following uniform boundness of the permittivity and permeability: there exists constants $\varepsilon_{\min}$, $\varepsilon_{\max}$, $\mu_{\min}$, and $\mu_{\max}$ such that

$$
0 < \varepsilon_{\min} \le \varepsilon(\boldsymbol{x},\boldsymbol{y}) \le \varepsilon_{\max}, \quad 0 < \mu_{\min} \le \mu(\boldsymbol{x},\boldsymbol{y}) \le \mu_{\max}, \quad \forall\boldsymbol{x}\in\Omega,\ \forall\boldsymbol{y}\in\mathbb{R}^N \tag{2.18}
$$

Then, we have the following theorem about the coefficient matrices $A^{\varepsilon}(\boldsymbol{x})$ and $A^{\mu}(\boldsymbol{x})$.

Theorem 1. *Under the assumption (2.18), the matrices $A^{\varepsilon}(\boldsymbol{x})$ and $A^{\mu}(\boldsymbol{x})$ are positive definite for any $\boldsymbol{x}\in\Omega$, and satisfy the following estimates*

$$
\begin{aligned}
0 < \varepsilon_{\min}\|\boldsymbol{u}\|^2_{L^2(\Omega)} \le \boldsymbol{u}^{\top}A^{\varepsilon}(\boldsymbol{x})\boldsymbol{u} \le \varepsilon_{\max}\|\boldsymbol{u}\|^2_{L^2(\Omega)}, && (2.19)\\
0 < \mu_{\min}\|\boldsymbol{u}\|^2_{L^2(\Omega)} \le \boldsymbol{u}^{\top}A^{\mu}(\boldsymbol{x})\boldsymbol{u} \le \mu_{\max}\|\boldsymbol{u}\|^2_{L^2(\Omega)}, && (2.20)
\end{aligned}
$$

for any M dimensional non-zero vector $\boldsymbol{u}$.

Proof. Let $\boldsymbol{u} = (\widehat{u}_1,\cdots,\widehat{u}_M)^{\top}$ be an arbitrary non-zero vector, and $u(\boldsymbol{y}) = \sum_{k=1}^{M}\widehat{u}_k\Phi_k(\boldsymbol{y})$.

By the definition of $A^\varepsilon(\boldsymbol{x})$, we have for any $\boldsymbol{x} \in \Omega$,

$$\begin{aligned}\boldsymbol{u}^\top A^\varepsilon(\boldsymbol{x})\boldsymbol{u} &= \sum_{k=1}^{M}\sum_{m=1}^{M} \widehat{u}_k \widehat{u}_m \int_\Xi \varepsilon(\boldsymbol{x},\boldsymbol{y})\Phi_k(\boldsymbol{y})\Phi_m(\boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y} \\ &= \int_\Xi \varepsilon(\boldsymbol{x},\boldsymbol{y})u^2(\boldsymbol{y})\rho(\boldsymbol{y}) > 0,\end{aligned} \tag{2.21}$$

which shows the positive definiteness of $A^\varepsilon(\boldsymbol{x})$. Then the boundness of (2.19) is given by (2.21) and (2.18). The conclusion for $A^\mu(\boldsymbol{x})$ follows the same argument. □ For the PDEs system (2.14)-(2.15), we have the following energy conservation property.

Theorem 2. *The solution $(\widehat{\boldsymbol{E}}(t,\boldsymbol{x}), \widehat{\boldsymbol{H}}(t,\boldsymbol{x}))$ of (2.14)-(2.15) subject to the PEC boundary condition (2.16) satisfies the following energy identity for any $t \in [0,T]$ and $k \geq 0$*

$$\left(\left\|A^{\varepsilon/2}\partial_{t^k}\widehat{\boldsymbol{E}}\right\|^2_{L^2(\Omega)} + \left\|A^{\mu/2}\partial_{t^k}\widehat{\boldsymbol{H}}\right\|^2_{L^2(\Omega)}\right)\Bigg|_{t=T} = \left(\left\|A^{\varepsilon/2}\partial_{t^k}\widehat{\boldsymbol{E}}\right\|^2_{L^2(\Omega)} + \left\|A^{\mu/2}\partial_{t^k}\widehat{\boldsymbol{H}}\right\|^2_{L^2(\Omega)}\right)\Bigg|_{t=0}. \tag{2.22}$$

where $(A^{\varepsilon/2})^2 = A^\varepsilon$ and $(A^{\mu/2})^2 = A^\mu$ are the square root of A^ε and A^μ, respectively, and ∂_{t^k} is the k-th order time derivative operator.

***Proof*.** Multiplying (2.14) and (2.15) by $\widehat{\boldsymbol{E}}$ and $\widehat{\boldsymbol{H}}$, and integrating over Ω, respectively. Then, summing up the resultants and using the PEC boundary condition (2.16), we easily see that (2.22) holds true for $k = 0$.

For high order time derivatives, we take the k-th time derivative of (2.14)-(2.15), and follow the same step as for the $k = 0$ case. □

2.3 The finite element time domain schemes

In this section, we solve the PDEs system (2.14)-(2.15) by a finite element method.

2.3.1 The semi-discrete scheme and its analysis

Let us first consider a semi-discrete scheme for (2.14)-(2.15): find $\widehat{\boldsymbol{E}}_h \in (\boldsymbol{V}_h^0)^M$, $\widehat{\boldsymbol{H}}_h \in (\boldsymbol{U}_h)^M$ such that

$$
\begin{aligned}
(A^{\varepsilon}\partial_t \widehat{\boldsymbol{E}}_h, \phi_{E,h})_{\Omega} - (\widehat{\boldsymbol{H}}_h, \nabla \times \phi_{E,h})_{\Omega} &= 0, \qquad \forall \phi_{E,h} \in (\boldsymbol{V}_h^0)^M, && (2.23)\\
(A^{\mu}\partial_t \widehat{\boldsymbol{H}}_h, \phi_{H,h})_{\Omega} + (\nabla \times \widehat{\boldsymbol{E}}_h, \phi_{H,h})_{\Omega} &= 0, \qquad \forall \phi_{H,h} \in (\boldsymbol{U}_h^0)^M, && (2.24)
\end{aligned}
$$

subject to the initial conditions

$$
\widehat{\boldsymbol{E}}_h(0,\boldsymbol{x}) = \Pi_h^c \widehat{\boldsymbol{E}}_0(\boldsymbol{x}), \qquad \widehat{\boldsymbol{H}}_h(0,\boldsymbol{x}) = \Pi_h^d \widehat{\boldsymbol{H}}_0(\boldsymbol{x}), \tag{2.25}
$$

where we denote Π_h^c the Nédélec interpolation operator and Π_h^d the L^2 projection into the space $\boldsymbol{U}_h$. It is known that Π_h^c and Π_h^d satisfy the following error estimates [15, 16]:

$$
\begin{aligned}
\|\boldsymbol{u} - \Pi_h^c \boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla \times (\boldsymbol{u} - \Pi_h^c \boldsymbol{u})\|_{L^2(\Omega)} &\le C h^r \|\boldsymbol{u}\|_{H^r(curl;\Omega)}, \quad \forall \boldsymbol{u} \in H^r(curl;\Omega), && (2.26)\\
\|\boldsymbol{v} - \Pi_h^d \boldsymbol{v}\|_{L^2(\Omega)} &\le C h^r \|\boldsymbol{v}\|_{H^r(\Omega)}, \quad \forall \boldsymbol{v} \in H^r(\Omega). && (2.27)
\end{aligned}
$$

Now, we are going to provide the error estimate for the semi-discrete scheme (2.23)-(2.24). Let $\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y})$ be the exact solution of (2.1)-(2.2) subject to the initial conditions (2.3) and the PEC boundary condition (2.4), and $\boldsymbol{E}_h(t,\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{H}_h(t,\boldsymbol{x},\boldsymbol{y})$ be the numerical solution

$$
\boldsymbol{E}_h(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{m=1}^{M} \widehat{\boldsymbol{E}}_{m,h}^t \Phi_m(\boldsymbol{y}), \qquad \boldsymbol{H}_h(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{m=1}^{M} \widehat{\boldsymbol{H}}_{m,h}^t \Phi_m(\boldsymbol{y}) \tag{2.28}
$$

where $\widehat{\boldsymbol{E}}_{m,h}$ and $\widehat{\boldsymbol{H}}_{m,h}$ are the m-th component of $\widehat{\boldsymbol{E}}_h(t,\boldsymbol{x})$ and $\widehat{\boldsymbol{H}}_h(t,\boldsymbol{x})$ of (2.23)-(2.25).

Considering the following errors

$$
\boldsymbol{E} - \boldsymbol{E}_h = (\boldsymbol{E} - \boldsymbol{E}_M) + (\boldsymbol{E}_M - \boldsymbol{E}_h), \qquad \boldsymbol{H} - \boldsymbol{H}_h = (\boldsymbol{H} - \boldsymbol{H}_M) + (\boldsymbol{H}_M - \boldsymbol{H}_h), \tag{2.29}
$$

where $\boldsymbol{E}_M$ and $\boldsymbol{H}_M$ are the gPC approximations given by (2.11). In the next theorem, we will show that the error bound is optimal and the error grows only linearly in time.

Theorem 3. *Denote the M-dimensional vectors R^E and R^H with k-th components given by*

$$R_k^E = \sum_{m=M+1}^{\infty} A_{k,m}^{\varepsilon}(\boldsymbol{x})\partial_t \boldsymbol{E}_m(t,\boldsymbol{x}), \quad R_k^H = \sum_{m=M+1}^{\infty} A_{k,m}^{\mu}(\boldsymbol{x})\partial_t \boldsymbol{H}_m(t,\boldsymbol{x}), \quad 1 \le k \le M. \tag{2.30}$$

Then, we have the optimal error estimate: for any $t \in [0,T]$,

$$\begin{aligned}
&\left(\mathbb{E}\left[\|\boldsymbol{E}-\boldsymbol{E}_h\|_{L^2(\Omega)}^2\right]\right)^{1/2} + \left(\mathbb{E}\left[\|\boldsymbol{H}-\boldsymbol{H}_h\|_{L^2(\Omega)}^2\right]\right)^{1/2} \\
\le\ & C \max_{0\le t\le T}\left(\sum_{m=M+1}^{\infty}\|\boldsymbol{E}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2 + \sum_{m=M+1}^{\infty}\|\boldsymbol{H}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2\right)^{1/2} \\
& + CTh^r \max_{0\le t\le T}\left(\|\partial_t\widehat{\boldsymbol{E}}\|_{H^r(curl;\Omega)}^2 + \|\widehat{\boldsymbol{E}}\|_{H^r(curl;\Omega)}^2 + \|\partial_t\widehat{\boldsymbol{H}}\|_{H^r(\Omega)}^2 + \|\widehat{\boldsymbol{H}}\|_{H^r(\Omega)}^2\right)^{1/2} \\
& + CT\max_{0\le t\le T}\left(\|A^{-\varepsilon/2}R^E\|_{L^2(\Omega)}^2 + \|A^{-\mu/2}R^H\|_{L^2(\Omega)}^2\right)^{1/2},
\end{aligned} \tag{2.31}$$

where the constant $C>0$ is independent of T and h, and $A^{-\varepsilon/2}$ and $A^{-\mu/2}$ are the inverses of $A^{\varepsilon/2}$ and $A^{\mu/2}$, respectively.

Proof. As shown in (2.29), we will split the errors into two parts: the gPC approximation error and the semi-discretization error. We will then prove them in order. (I) By the expansion (2.6) and (2.11), and the orthonormality condition of $\Phi_n(\boldsymbol{y})$, we easily see that the mean of the gPC approximation error

$$\begin{aligned}
\mathbb{E}\left[\|\boldsymbol{E}-\boldsymbol{E}_M\|_{L^2(\Omega)}^2\right] &:= \int_{\Xi}\|\boldsymbol{E}-\boldsymbol{E}_M\|_{L^2(\Omega)}^2\rho(\boldsymbol{y})d\boldsymbol{y} \\
&= \int_{\Xi}\int_{\Omega}\left|\sum_{m=1}^{M}(\boldsymbol{E}_m-\widehat{\boldsymbol{E}}_m)\Phi_m(\boldsymbol{y}) + \sum_{m=M+1}^{\infty}\boldsymbol{E}_m\Phi_m(\boldsymbol{y})\right|^2 d\boldsymbol{x}\rho(\boldsymbol{y})d\boldsymbol{y} \\
&= \int_{\Omega}\int_{\Xi}\left[\left|\sum_{m=1}^{M}(\boldsymbol{E}_m-\widehat{\boldsymbol{E}}_m)\Phi_m(\boldsymbol{y})\right|^2 + \left|\sum_{m=M+1}^{\infty}\boldsymbol{E}_m\Phi_m(\boldsymbol{y})\right|^2\right]\rho(\boldsymbol{y})d\boldsymbol{y}d\boldsymbol{x} \\
&= \sum_{m=1}^{M}\|\boldsymbol{E}_m(t,\boldsymbol{x})-\widehat{\boldsymbol{E}}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2 + \sum_{m=M+1}^{\infty}\|\boldsymbol{E}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2.
\end{aligned}$$

By the same argument, we have

$$\mathbb{E}\left[\|\boldsymbol{H}-\boldsymbol{H}_M\|_{L^2(\Omega)}^2\right] = \sum_{m=1}^{M}\|\boldsymbol{H}_m(t,\boldsymbol{x})-\widehat{\boldsymbol{H}}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2 + \sum_{m=M+1}^{\infty}\|\boldsymbol{H}_m(t,\boldsymbol{x})\|_{L^2(\Omega)}^2.$$

To investigate the error $\boldsymbol{E}-\boldsymbol{E}_M$ and $\boldsymbol{H}-\boldsymbol{H}_M$, let us introduce $\widetilde{\boldsymbol{E}}=(\boldsymbol{E}_1,\cdots,\boldsymbol{E}_M)^\top$ and $\widetilde{\boldsymbol{H}}=(\boldsymbol{H}_1,\cdots,\boldsymbol{H}_M)^\top$, where $\boldsymbol{E}_i$ and $\boldsymbol{H}_i$ are the coefficients in the expansion (2.6). By (2.9)-(2.10), we know that $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{H}}$ satisfy the following equations

$$A^\varepsilon(\boldsymbol{x})\partial_t\widetilde{\boldsymbol{E}}(t,\boldsymbol{x})-\nabla\times\widetilde{\boldsymbol{H}}(t,\boldsymbol{x})=R^E \tag{2.32}$$

$$A^\mu(\boldsymbol{x})\partial_t\widetilde{\boldsymbol{H}}(t,\boldsymbol{x})+\nabla\times\widetilde{\boldsymbol{E}}(t,\boldsymbol{x})=R^H \tag{2.33}$$

subject to the PEC boundary condition

$$\boldsymbol{n}\times\widetilde{\boldsymbol{E}}=\boldsymbol{0},\qquad \text{on } \partial\Omega, \tag{2.34}$$

and the initial conditions

$$\widetilde{\boldsymbol{E}}(0,\boldsymbol{x})=\widetilde{\boldsymbol{E}}_0(\boldsymbol{x}),\quad \widetilde{\boldsymbol{H}}(0,\boldsymbol{x})=\widetilde{\boldsymbol{H}}_0(\boldsymbol{x}), \tag{2.35}$$

where $\widetilde{\boldsymbol{E}}_0(\boldsymbol{x})$ and $\widetilde{\boldsymbol{H}}_0(\boldsymbol{x})$ are the gPC expansion coefficient vectors obtained by expressing the initial condition of (2.3) in the form of (2.6). Moreover, the k-th components of R^E and R^H are given by (2.30).

By (2.14)-(2.15) and (2.32)-(2.33), we obtain the following error equations in the weak form:

$$(A^\varepsilon\partial_t(\widetilde{\boldsymbol{E}}-\widehat{\boldsymbol{E}}),\boldsymbol{\varphi}_E)_\Omega-(\widetilde{\boldsymbol{H}}-\widehat{\boldsymbol{H}},\nabla\times\boldsymbol{\varphi}_E)_\Omega=(R^E,\boldsymbol{\varphi}_E)_\Omega,\forall\ \boldsymbol{\varphi}_E\in(H_0(curl;\Omega))^M, \tag{2.36}$$

$$(A^\mu\partial_t(\widetilde{\boldsymbol{H}}-\widehat{\boldsymbol{H}}),\boldsymbol{\varphi}_H)_\Omega+(\nabla\times(\widetilde{\boldsymbol{E}}-\widehat{\boldsymbol{E}}),\boldsymbol{\varphi}_H)_\Omega=(R^H,\boldsymbol{\varphi}_H)_\Omega,\forall\ \boldsymbol{\varphi}_H\in(H(div;\Omega))^M, \tag{2.37}$$

subject to the PEC boundary condition

$$\boldsymbol{n}\times(\widetilde{\boldsymbol{E}}-\widehat{\boldsymbol{E}})=\boldsymbol{0},\qquad \text{on } \partial\Omega, \tag{2.38}$$

and the initial conditions

$$(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})(0, \boldsymbol{x}) = (\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})(0, \boldsymbol{x}) = \boldsymbol{0}. \tag{2.39}$$

Choosing $\boldsymbol{\varphi}_E = 2(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})(t, \boldsymbol{x})$ and $\boldsymbol{\varphi}_H = 2(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})(t, \boldsymbol{x})$ in (2.36) and (2.37), respectively, adding the resultants together, and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right) && (2.40)\\
&= 2(R^E, \widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})_\Omega + 2(R^H, \widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})_\Omega\\
&\le \delta\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right)\\
&+ \frac{1}{\delta}\left(\|A^{-\varepsilon/2}R^E\|^2_{L^2(\Omega)} + \|A^{-\mu/2}R^H\|^2_{L^2(\Omega)}\right). && (2.41)
\end{aligned}$$

Integrating (2.41) from $t = 0$ and any $t \le T$ in time and taking the maximum of right hand side with respect to $t \in [0, T]$, we obtain

$$\begin{aligned}
&\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right)(t)\\
&\le \delta T \max_{0\le t\le T}\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right)\\
&+ \frac{T}{\delta}\max_{0\le t\le T}\left(\|A^{-\varepsilon/2}R^E\|^2_{L^2(\Omega)} + \|A^{-\mu/2}R^H\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{2.42}$$

Taking the maximum at left hand side with respect to $t \in [0, T]$, and setting $\delta = \frac{1}{2T}$, we obtain

$$\begin{aligned}
&\max_{0\le t\le T}\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right)\\
&\le 4T^2 \max_{0\le t\le T}(\|A^{-\varepsilon/2}R^E\|^2_{L^2(\Omega)} + \|A^{-\mu/2}R^H\|^2_{L^2(\Omega)}),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\max_{0\le t\le T}\left(\|A^{\varepsilon/2}(\widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widetilde{\boldsymbol{H}} - \widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)}\right)^{1/2}\\
&\le CT \max_{0\le t\le T}(\|A^{-\varepsilon/2}R^E\|^2_{L^2(\Omega)} + \|A^{-\mu/2}R^H\|^2_{L^2(\Omega)})^{1/2}.
\end{aligned} \tag{2.43}$$

(II) Multiplying (2.14)-(2.15) by $\boldsymbol{\varphi}_{E,h}$ and $\boldsymbol{\varphi}_{H,h}$ and integrating over Ω, we obtain

$$(A^{\varepsilon}\partial_t\widehat{\boldsymbol{E}}, \boldsymbol{\varphi}_{E,h})_{\Omega} - (\widehat{\boldsymbol{H}}, \nabla\times\boldsymbol{\varphi}_{E,h})_{\Omega} = 0, \quad \forall\boldsymbol{\varphi}_{E,h}\in(\boldsymbol{V}_h^0)^M, \tag{2.44}$$

$$(A^{\mu}\partial_t\widehat{\boldsymbol{H}}, \boldsymbol{\varphi}_{H,h})_{\Omega} + (\nabla\times\widehat{\boldsymbol{E}}, \boldsymbol{\varphi}_{H,h})_{\Omega} = 0, \quad \forall\boldsymbol{\varphi}_{H,h}\in(\boldsymbol{U}_h)^M. \tag{2.45}$$

Subtracting (2.23)-(2.24) from (2.44)-(2.45), we obtain the error equations:

$$(A^{\varepsilon}\partial_t(\widehat{\boldsymbol{E}}-\widehat{\boldsymbol{E}}_h), \boldsymbol{\varphi}_{E,h})_{\Omega} - (\widehat{\boldsymbol{H}}-\widehat{\boldsymbol{H}}_h, \nabla\times\boldsymbol{\varphi}_{E,h})_{\Omega} = 0, \quad \forall\boldsymbol{\varphi}_{E,h}\in(\boldsymbol{V}_h^0)^M, \tag{2.46}$$

$$(A^{\mu}\partial_t(\widehat{\boldsymbol{H}}-\widehat{\boldsymbol{H}}_h), \boldsymbol{\varphi}_{H,h})_{\Omega} + (\nabla\times(\widehat{\boldsymbol{E}}-\widehat{\boldsymbol{E}}_h), \boldsymbol{\varphi}_{H,h})_{\Omega} = 0, \quad \forall\boldsymbol{\varphi}_{H,h}\in(\boldsymbol{U}_h)^M. \tag{2.47}$$

Let us introduce the shorthand notations

$$\widehat{\boldsymbol{E}}_I := \Pi_h^c\widehat{\boldsymbol{E}}, \qquad \widehat{\boldsymbol{H}}_I := \Pi_h^d\widehat{\boldsymbol{H}}.$$

Choosing $\boldsymbol{\varphi}_{E,h} = 2(\widehat{\boldsymbol{E}}_I - \widehat{\boldsymbol{E}}_h)$ and $\boldsymbol{\varphi}_{H,h} = 2(\widehat{\boldsymbol{H}}_I - \widehat{\boldsymbol{H}}_h)$ in (2.46)-(2.47), respectively, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|A^{\varepsilon/2}(\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h)\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h)\|^2_{L^2(\Omega)}\right)\\
=\;& 2(A^{\varepsilon}\partial_t(\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}), \widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h)_{\Omega} + 2(\widehat{\boldsymbol{H}}-\widehat{\boldsymbol{H}}_I, \nabla\times(\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h))_{\Omega}\\
&+2(A^{\mu}\partial_t(\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}), \widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h)_{\Omega} - 2(\nabla\times(\widehat{\boldsymbol{E}}-\widehat{\boldsymbol{E}}_I), \widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h)_{\Omega}\\
\leq\;& A^{\varepsilon}_{\max}\left(\frac{1}{\delta}\|\partial_t(\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}})\|^2_{L^2(\Omega)} + \delta\|\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h\|^2_{L^2(\Omega)}\right)\\
&+A^{\mu}_{\max}\left(\frac{1}{\delta}\|\partial_t(\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}})\|^2_{L^2(\Omega)} + \delta\|\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h\|^2_{L^2(\Omega)}\right)\\
&+\frac{1}{\delta}\|\nabla\times(\widehat{\boldsymbol{E}}-\widehat{\boldsymbol{E}}_I)\|^2_{L^2(\Omega)} + \delta\|\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h\|^2_{L^2(\Omega)},\\
\leq\;& A^{\varepsilon}_{\max}\left(\frac{Ch^{2r}}{\delta}\|\partial_t\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \delta\|\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h\|^2_{L^2(\Omega)}\right)\\
&+A^{\mu}_{\max}\left(\frac{Ch^{2r}}{\delta}\|\partial_t\widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)} + \delta\|\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h\|^2_{L^2(\Omega)}\right)\\
&+\frac{Ch^{2r}}{\delta}\|\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \delta\|\widehat{\boldsymbol{H}}_I-\widehat{\boldsymbol{H}}_h\|^2_{L^2(\Omega)},
\end{aligned} \tag{2.48}$$

where the fact that $\nabla\times(\widehat{\boldsymbol{E}}_I-\widehat{\boldsymbol{E}}_h)\in\boldsymbol{U}_h$, the Cauchy-Schwarz inequality, and the interpolation and projection error estimates (2.26)-(2.27) has been used.

Integrating (2.48) from $t = 0$ to any $t \leq T$ in time, and taking the maximum of the right hand side with respect to t, we get

$$\begin{aligned}
&\left(\|A^{\varepsilon/2}(\widehat{\boldsymbol{E}}_I - \widehat{\boldsymbol{E}}_h)\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widehat{\boldsymbol{H}}_I - \widehat{\boldsymbol{H}}_h)\|^2_{L^2(\Omega)}\right)(t) \\
\leq \;& \left(\|A^{\varepsilon/2}(\widehat{\boldsymbol{E}}_I - \widehat{\boldsymbol{E}}_h)\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widehat{\boldsymbol{H}}_I - \widehat{\boldsymbol{H}}_h)\|^2_{L^2(\Omega)}\right)(0) \\
&+\frac{TCh^{2r}}{\delta} \max_{0\leq t\leq T} \left(\|\partial_t \widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \|\partial_t \widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)} + \|\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)}\right) \\
&+C\delta T \max_{0\leq t\leq T} \left(\|\widehat{\boldsymbol{E}}_I - \widehat{\boldsymbol{E}}_h\|^2_{L^2(\Omega)} + \|\widehat{\boldsymbol{H}}_I - \widehat{\boldsymbol{H}}_h\|^2_{L^2(\Omega)}\right). \qquad (2.49)
\end{aligned}$$

Noting that the first term on the right hand side of (2.49) is zero due to (2.25), then taking the maximum of the left hand side with respect to t, and choosing δ such that $\delta = \frac{1}{2CT}$, we have

$$\begin{aligned}
&\max_{0\leq t\leq T} \left(\|A^{\varepsilon/2}(\widehat{\boldsymbol{E}}_I - \widehat{\boldsymbol{E}}_h)\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widehat{\boldsymbol{H}}_I - \widehat{\boldsymbol{H}}_h)\|^2_{L^2(\Omega)}\right)^{1/2} \\
\leq \;& CTh^r \max_{0\leq t\leq T} \left(\|\partial_t \widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \|\partial_t \widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)} + \|\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)}\right)^{1/2}. \qquad (2.50)
\end{aligned}$$

Using the interpolation and projection error estimates (2.26)-(2.27) and the triangle inequality, from (2.50) we have

$$\begin{aligned}
&\max_{0\leq t\leq T} \left(\|A^{\varepsilon/2}(\widehat{\boldsymbol{E}} - \widehat{\boldsymbol{E}}_h)\|^2_{L^2(\Omega)} + \|A^{\mu/2}(\widehat{\boldsymbol{H}} - \widehat{\boldsymbol{H}}_h)\|^2_{L^2(\Omega)}\right)^{1/2} \\
\leq \;& CTh^r \max_{0\leq t\leq T} \left(\|\partial_t \widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \|\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)} + \|\partial_t \widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)} + \|\widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)}\right)^{1/2}. \qquad (2.51)
\end{aligned}$$

By the error definition (2.29) and the obtained error estimates (2.43) and (2.51), we conclude the proof of (2.31). □

Remark 2. *For any given small number $\varepsilon_M > 0$, under the assumption that there exists a* *sufficiently large M in (2.11) so that*

$$\max_{0\le t\le T}\left[\left(\sum_{m=M+1}^{\infty}\|\boldsymbol{E}_m(t,\boldsymbol{x})\|^2_{L^2(\Omega)}+\sum_{m=M+1}^{\infty}\|\boldsymbol{H}_m(t,\boldsymbol{x})\|^2_{L^2(\Omega)}\right)^{1/2}\right.$$
$$\left.+\left(\|A^{-\varepsilon/2}R^E\|^2_{L^2(\Omega)}+\|A^{-\mu/2}R^H\|^2_{L^2(\Omega)}\right)^{1/2}\right]\le\varepsilon_M, \qquad (2.52)$$

and the solutions $(\widehat{\boldsymbol{E}},\widehat{\boldsymbol{H}})$ of (2.14)-(2.15) are smooth enough and bounded above:

$$\max_{0\le t\le T}\left(\|\partial_t\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)}+\|\widehat{\boldsymbol{E}}\|^2_{H^r(curl;\Omega)}+\|\partial_t\widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)}+\|\widehat{\boldsymbol{H}}\|^2_{H^r(\Omega)}\right)^{1/2}\le C, \qquad (2.53)$$

then the optimal error estimate (2.31) becomes

$$\left(\mathbb{E}\left[\|\boldsymbol{E}-\boldsymbol{E}_h\|^2_{L^2(\Omega)}\right]\right)^{1/2}+\left(\mathbb{E}\left[\|\boldsymbol{H}-\boldsymbol{H}_h\|^2_{L^2(\Omega)}\right]\right)^{1/2}\le C(T+1)\varepsilon_M+CTh^r. \qquad (2.54)$$

Similar to remark 1, exact conditions for the assumption (2.52) are unclear since they involve a prior estimate of the solution and the underlying polynomial basis, which are complicate. Nevertheless, such issues have been explored for the stochastic Helmholtz equation [17] and the stochastic Darcys equation [18].

2.3.2 The fully-discrete scheme

To construct a fully discrete finite element scheme, we assume that the time interval $[0,T]$ is partitioned uniformly into $0=t_0<t_1<\cdots<t_{N_t}=T$, where $t_i=i\tau,\ i=0,\cdots,N_t$, and $\tau=\frac{T}{N_t}$ denotes the time step size. Furthermore, we introduce the following backward difference and average operators: For any discrete time solution u^n,

$$\delta_\tau u^{n+1}=\frac{u^{n+1}-u^n}{\tau},$$
$$\delta_\tau^2 u^{n+1}=\delta_\tau(\delta_\tau u^{n+1})=\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2},$$
$$\delta_\tau^{k+1}u^{n+1}=\delta_\tau(\delta_\tau^k u^{n+1}),\qquad k\ge 1$$

Leap-frog scheme

Now we can construct a leap-frog type scheme for solving (2.14)-(2.15): given proper initial approximations $\widehat{\boldsymbol{H}}_h^0 \in \boldsymbol{U}_h$ and $\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}} \in \boldsymbol{V}_h^0$, for $k \geq 0$ find $\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} \in \boldsymbol{V}_h^0$ and $\widehat{\boldsymbol{H}}_h^{k+1} \in \boldsymbol{U}_h$ such that

$$\left(A^{\varepsilon} \frac{\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} - \widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}}{\tau}, \boldsymbol{\varphi}_{E,h}\right)_{\Omega} - \left(\widehat{\boldsymbol{H}}_h^{k}, \nabla \times \boldsymbol{\varphi}_{E,h}\right)_{\Omega} = \mathbf{0}, \quad \forall \boldsymbol{\varphi}_{E,h} \in \boldsymbol{V}_h^0, \tag{2.55}$$

$$\left(A^{\mu} \frac{\widehat{\boldsymbol{H}}_h^{k+1} - \widehat{\boldsymbol{H}}_h^{k}}{\tau}, \boldsymbol{\varphi}_{H,h}\right)_{\Omega} + \left(\nabla \times \widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}, \boldsymbol{\varphi}_{H,h}\right)_{\Omega} = \mathbf{0}, \quad \forall \boldsymbol{\varphi}_{H,h} \in \boldsymbol{U}_h. \tag{2.56}$$

Notice that the above leap-frog scheme decouples the original coupled system by first solving for $\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}$ through (2.55), and then solving $\widehat{\boldsymbol{H}}_h^{k+1}$ from (2.56). We like to remark that the leapfrog scheme does not conserve the energy anymore due to time staggering, but the scheme is conditionally stable as shown below.

Theorem 4. *Let $C_{inv} > 0$ be the constant appearing in the standard inverse estimate*

$$\|\nabla \times \boldsymbol{u}_h\|_{L^2(\Omega)} \leq C_{inv} h^{-1} \|\boldsymbol{u}_h\|_{L^2(\Omega)}. \tag{2.57}$$

Then under the time step constraint

$$\tau \leq \frac{(\varepsilon_{\min}\mu_{\min})^{1/2} h}{C_{inv}}, \tag{2.58}$$

for any $k \geq 0$, the solution $\left(\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}, \widehat{\boldsymbol{H}}_h^{k+1}\right)$ of (2.55)-(2.56) satisfies the energy stability:

$$\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)}^2 + \left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)}^2 \leq 3\left[\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}}\right\|_{L^2(\Omega)}^2 + \left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{0}\right\|_{L^2(\Omega)}^2\right]. \tag{2.59}$$

Proof. In (2.55) and (2.56), we set

$$\boldsymbol{\varphi}_{E,h} = \tau\left(\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} + \widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}\right), \qquad \boldsymbol{\varphi}_{H,h} = \tau\left(\widehat{\boldsymbol{H}}_h^{k+1} + \widehat{\boldsymbol{H}}_h^{k}\right).$$

Then summing up the resultants, we have

$$
\begin{aligned}
& \left(\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)}^2 - \left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}\right\|_{L^2(\Omega)}^2\right) + \left(\left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)}^2 - \left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k}\right\|_{L^2(\Omega)}^2\right) \\
= \; & \tau\left(\widehat{\boldsymbol{H}}_h^{k}, \nabla\times\left(\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} + \widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}\right)\right)_\Omega - \tau\left(\nabla\times\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}, \widehat{\boldsymbol{H}}_h^{k+1} + \widehat{\boldsymbol{H}}_h^{k}\right)_\Omega \\
= \; & \tau\left[\left(\widehat{\boldsymbol{H}}_h^{k}, \nabla\times\widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}\right)_\Omega - \left(\widehat{\boldsymbol{H}}_h^{k+1}, \nabla\times\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right)_\Omega\right].
\end{aligned} \tag{2.60}
$$

Summing up (2.60) from $k = 0$ to any $k \leq N_t$, we get

$$
\begin{aligned}
& \left(\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)}^2 + \left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)}^2\right) - \left(\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}}\right\|_{L^2(\Omega)}^2 + \left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{0}\right\|_{L^2(\Omega)}^2\right) \\
= \; & \tau\left[\left(\widehat{\boldsymbol{H}}_h^{0}, \nabla\times\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}}\right)_\Omega - \left(\widehat{\boldsymbol{H}}_h^{k+1}, \nabla\times\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right)_\Omega\right].
\end{aligned} \tag{2.61}
$$

By using the Cauchy-Schwarz inequality, the inverse estimate (2.57), and Theorem 2, we have

$$
\begin{aligned}
& \tau\left(\widehat{\boldsymbol{H}}_h^{k+1}, \nabla\times\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right)_\Omega \leq \tau\left\|\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)} C_{inv}h^{-1}\left\|\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)} \\
\leq \; & \tau\mu_{\min}^{-1/2}\left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)} C_{inv}h^{-1}\varepsilon_{\min}^{-1/2}\left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)} \\
\leq \; & \frac{\tau h^{-1}C_{inv}\varepsilon_{\min}^{-1/2}\mu_{\min}^{-1/2}}{2}\left(\left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{k+1}\right\|_{L^2(\Omega)}^2 + \left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}\right\|_{L^2(\Omega)}^2\right).
\end{aligned} \tag{2.62}
$$

By the similar argument, we have

$$
\tau\left(\widehat{\boldsymbol{H}}_h^{0}, \nabla\times\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}}\right)_\Omega \leq \frac{\tau h^{-1}C_{inv}\varepsilon_{\min}^{-1/2}\mu_{\min}^{-1/2}}{2}\left(\left\|A^{\mu/2}\widehat{\boldsymbol{H}}_h^{0}\right\|_{L^2(\Omega)}^2 + \left\|A^{\varepsilon/2}\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}}\right\|_{L^2(\Omega)}^2\right) \tag{2.63}
$$

The proof is completed by substituting the estimates (2.62) and (2.63) into (2.61), and using the time step constraint (2.58). ◻

We like to remark that $C_v := (\varepsilon_{\min}\mu_{\min})^{-1/2}$ denotes the wave propagation speed in a medium with permittivity $\varepsilon_{\min}$ and permeability $\mu_{\min}$. With this notation, the time constraint (2.58) becomes $\tau \leq \frac{h}{C_{inv}C_v}$, which is similar to the stability constraint obtained for the leap-frog scheme developed for solving the metamaterial Drude model [16].

Modified leap-frog scheme

To further reduce the computational workload, we consider a more efficient scheme than the leap-frog scheme. Following the same idea in [13], it can be proved that the matrices $A^{\varepsilon}(\boldsymbol{x})$ and $A^{\mu}(\boldsymbol{x})$ are strictly diagonal dominant, and we can rewrite them as

$$A^{\varepsilon}(\boldsymbol{x}) = D^{\varepsilon}(\boldsymbol{x}) + S^{\varepsilon}(\boldsymbol{x}), \qquad A^{\mu}(\boldsymbol{x}) = D^{\mu}(\boldsymbol{x}) + S^{\mu}(\boldsymbol{x}), \tag{2.64}$$

where $D^{\varepsilon}(\boldsymbol{x})$, $D^{\mu}(\boldsymbol{x})$, and $S^{\varepsilon}(\boldsymbol{x})$, $S^{\mu}(\boldsymbol{x})$ are the diagonal and off-diagonal parts.

By using the Taylor expansion, we can establish the following second order backward finite difference scheme:

$$\frac{2u^{k-\frac{1}{2}} - 3u^{k-\frac{3}{2}} + u^{k-\frac{5}{2}}}{\tau} = \partial_t u^k + O(\tau^2). \tag{2.65}$$

Using (2.64)-(2.65), we propose the following modified leap-frog type scheme for solving (2.14)-(2.15): given proper initial approximations $\widehat{\boldsymbol{H}}_h^0 \in \boldsymbol{U}_h$ and $\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}} \in \boldsymbol{V}_h^0$, for $k \geq 0$, find $\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} \in \boldsymbol{V}_h^0$ and $\widehat{\boldsymbol{H}}_h^{k+1} \in \boldsymbol{U}_h$, such that

$$\begin{aligned}
&\left(D^{\varepsilon}\frac{\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} - \widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}}{\tau}, \boldsymbol{\varphi}_{E,h}\right)_{\Omega} - \left(\widehat{\boldsymbol{H}}_h^{k}, \nabla\times\boldsymbol{\varphi}_{E,h}\right)_{\Omega} \\
&\qquad + \left(S^{\varepsilon}\frac{2\widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}} - 3\widehat{\boldsymbol{E}}_h^{k-\frac{3}{2}} + \widehat{\boldsymbol{E}}_h^{k-\frac{5}{2}}}{\tau}, \boldsymbol{\varphi}_{E,h}\right)_{\Omega} = 0 \quad \forall \boldsymbol{\varphi}_{E,h} \in \boldsymbol{V}_h^0, \\
&\left(D^{\mu}\frac{\widehat{\boldsymbol{H}}_h^{k+1} - \widehat{\boldsymbol{H}}_h^{k}}{\tau}, \boldsymbol{\varphi}_{H,h}\right)_{\Omega} + \left(\nabla\times\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}, \boldsymbol{\varphi}_{H,h}\right)_{\Omega} \\
&\qquad + \left(S^{\mu}\frac{2\widehat{\boldsymbol{H}}_h^{k} - 3\widehat{\boldsymbol{H}}_h^{k-1} + \widehat{\boldsymbol{H}}_h^{k-2}}{\tau}, \boldsymbol{\varphi}_{H,h}\right)_{\Omega} = 0 \quad \forall \boldsymbol{\varphi}_{H,h} \in \boldsymbol{U}_h.
\end{aligned}$$

Though we could not strictly prove the stability of this modified leap-frog scheme at this moment, it is a high accurate scheme (second order in time) and much more efficient than the leap-frog scheme (2.55)-(2.56). We will verify this advantage in the numerical experiments given below.

2.4 Numerical experiments

In this section, we will perform some numerical tests to verify our theoretical analysis and present some applications of random Maxwells equations. All our numerical experiments are carried out on a 2017 MacBook Pro laptop with processor of 2.8GHz Intel Core i7, and memory of 16GB 2133MHz LPDDR3.

To implement the leap-frog scheme (2.55)-(2.56) for solving (2.14)-(2.15), we assume some given proper initial approximations $\widehat{\boldsymbol{H}}_h^0 \in \boldsymbol{U}_h$ and $\widehat{\boldsymbol{E}}_h^{-\frac{1}{2}} \in \boldsymbol{V}_h^0$. Then our problem reads: for $k \geq 0$, find $\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} \in \boldsymbol{V}_h^0$ and $\widehat{\boldsymbol{H}}_h^{k+1} \in \boldsymbol{U}_h$, such that

$$\left(A^{\varepsilon} \frac{\widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}} - \widehat{\boldsymbol{E}}_h^{k-\frac{1}{2}}}{\tau}, \boldsymbol{\varphi}_{E,h} \right)_{\Omega} - \left(\widehat{\boldsymbol{H}}_h^{k}, \nabla \times \boldsymbol{\varphi}_{E,h} \right)_{\Omega} = \left(\mathbf{f}, \boldsymbol{\varphi}_{E,h} \right), \quad \forall \boldsymbol{\varphi}_{E,h} \in \boldsymbol{V}_h^0,$$

$$\left(A^{\mu} \frac{\widehat{\boldsymbol{H}}_h^{k+1} - \widehat{\boldsymbol{H}}_h^{k}}{\tau}, \boldsymbol{\varphi}_{H,h} \right)_{\Omega} + \left(\nabla \times \widehat{\boldsymbol{E}}_h^{k+\frac{1}{2}}, \boldsymbol{\varphi}_{H,h} \right)_{\Omega} = \left(\mathbf{g}, \boldsymbol{\varphi}_{H,h} \right), \quad \forall \boldsymbol{\varphi}_{H,h} \in \boldsymbol{U}_h.$$

where $\mathbf{f}$ and $\mathbf{g}$ are artificial terms to test the convergence rate of the numerical scheme.

We partition the physical domain Ω into $N_e = N_t^2$ elements with N_d edges. By applying the finite element discretion, we assume the coefficients in (2.28) have the following form:

$$\widehat{\boldsymbol{E}}_{m,h}^{t} = \sum_{j=1}^{N_E} \widehat{\boldsymbol{E}}_{m,j,h}^{t} \boldsymbol{\varphi}_{E,j,h}(\boldsymbol{x}) \quad \widehat{\boldsymbol{H}}_{m,h}^{t} = \sum_{j=1}^{N_H} \widehat{\boldsymbol{H}}_{m,j,h}^{t} \boldsymbol{\varphi}_{H,j,h}(\boldsymbol{x})$$

where N_E and N_H are the number of basis functions of $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively. We hence get the algebraic equations about $\boldsymbol{E}_h$ and $\boldsymbol{H}_h$ as following:

$$\mathcal{A}_{m,n,i,j}^{\varepsilon} \otimes E_{n,j,h}^{k+1/2} = \mathcal{A}_{m,n,i,j}^{\varepsilon} \otimes E_{n,j,h}^{k-1/2} + \tau H_{n,j,h}^{k} M_{i,j} + \tau F_{n,j}^{k}$$
$$\mathcal{A}_{m,n,i,j}^{\mu} \otimes H_{n,j,h}^{k+1} = \mathcal{A}_{m,n,i,j}^{\mu} \otimes H_{n,j,h}^{k} - \tau E_{n,j,h}^{k+1/2} M_{i,j}^{\top} + \tau G_{n,j}^{k}$$

where $\mathcal{A}_{m,n,i,j}^{\varepsilon}$ and $\mathcal{A}_{m,n,i,j}^{\mu}$ are two fouth order tensors whose element by element definitions are given by

$$\mathcal{A}^{\varepsilon}_{m,n,i,j} = \int_{\Omega\times\Xi} \varepsilon(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})\boldsymbol{\varphi}_{E,i,h}(\boldsymbol{x})\cdot\boldsymbol{\varphi}_{E,j,h}(\boldsymbol{x})\rho(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$
$$\mathcal{A}^{\mu}_{m,n,i,j} = \int_{\Omega\times\Xi} \mu(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})\boldsymbol{\varphi}_{H,i,h}(\boldsymbol{x})\cdot\boldsymbol{\varphi}_{H,j,h}(\boldsymbol{x})\rho(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

and

$$F^k_{n,j} = \left[\int_{\Omega\times\Omega} \rho(\boldsymbol{y})\Phi_n(\boldsymbol{y})\mathbf{f}(t_k,\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{\varphi}_{E,j,h}(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}\right]_{M\times N_E}, \quad 1\le n\le M,\ \ 1\le j\le N_E,$$
$$G^k_{n,j} = \left[\int_{\Omega\times\Omega} \rho(\boldsymbol{y})\Phi_n(\boldsymbol{y})\mathbf{g}(t_k,\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{\varphi}_{H,j,h}(\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{y}\right]_{M\times N_H}, \quad 1\le n\le, M\ \ 1\le j\le N_H.$$

Denote $E^t_{n,j,h}$ and $H^t_{n,j,h}$ for the two second order tensors (i.e., matrices):

$$E^t_{n,j,h} = \left[\widehat{\boldsymbol{E}}^t_{n,j,h}\right]_{M\times N_E}, \quad 1\le n\le M,\ \ 1\le j\le N_E,$$
$$H^t_{n,j,t} = \left[\widehat{\boldsymbol{H}}^t_{n,j,h}\right]_{M\times N_H}, \quad 1\le n\le M,\ \ 1\le j\le N_H.$$

and $\otimes$ for a tensor product like operator:

$$\mathcal{A}^{\varepsilon}_{m,n,i,j}\otimes E^t_{n,j,h} = \sum_{m=1}^{M}\sum_{i=1}^{N_E}\widehat{\boldsymbol{E}}^t_{n,j,h}\int_{\Omega\times\Xi} \varepsilon(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})\boldsymbol{\varphi}_{E,i,h}(\boldsymbol{x})\cdot\boldsymbol{\varphi}_{E,j,h}(\boldsymbol{x})\rho(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$
$$\mathcal{A}^{\mu}_{m,n,i,j}\otimes H^t_{n,j,h} = \sum_{m=1}^{M}\sum_{i=1}^{N_H}\widehat{\boldsymbol{H}}^t_{n,j,h}\int_{\Omega\times\Xi} \mu(\boldsymbol{x},\boldsymbol{y})\Phi_m(\boldsymbol{y})\Phi_n(\boldsymbol{y})\boldsymbol{\varphi}_{H,i,h}(\boldsymbol{x})\cdot\boldsymbol{\varphi}_{H,j,h}(\boldsymbol{x})\rho(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

and $M_{i,j}$ for the stiffness matrix

$$M_{i,j} = \left[\int_{\Xi}\boldsymbol{\varphi}_{H,i,h}\cdot\nabla\times\boldsymbol{\varphi}_{E,j,h}(\boldsymbol{x})d\boldsymbol{x}\right]_{N_H\times N_E}, \quad 1\le i\le N_H,\ \ 1\le j\le N_E.$$

Therefore, once we have $E^t_{n,j,h}$ and $H^t_{n,j,h}$, the numerical solutions of $\boldsymbol{E}_h$ and $\boldsymbol{H}_h$ can be computed by the following quadratic forms:

$$\boldsymbol{E}_h(t,\boldsymbol{x},\boldsymbol{y}) = (\Phi_1(\boldsymbol{y}),\cdots,\Phi_M(\boldsymbol{y}))\cdot E^t_{n,j,h}\cdot(\boldsymbol{\varphi}_{E,1,h}(\boldsymbol{x}),\cdots,\boldsymbol{\varphi}_{E,N_E,h}(\boldsymbol{x}))^{\top}, \quad \text{at } t=t_{k+1/2},$$
$$\boldsymbol{H}_h(t,\boldsymbol{x},\boldsymbol{y}) = (\Phi_1(\boldsymbol{y}),\cdots,\Phi_M(\boldsymbol{y}))\cdot H^t_{n,j,h}\cdot(\boldsymbol{\varphi}_{H,1,h}(\boldsymbol{x}),\cdots,\boldsymbol{\varphi}_{H,N_H,h}(\boldsymbol{x}))^{\top}, \quad \text{at } t=t_k.$$

for $k = 0, 1, \cdots, N_t$.

2.4.1 Example 1: test of convergence and CPU time

For simplicity, we solve the two-dimensional TE_z mode equation:

$$\begin{aligned}
\varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t E_{x_1}(t,\boldsymbol{x},\boldsymbol{y}) &= \partial_{x_2} H(t,\boldsymbol{x},\boldsymbol{y}) + f_1(t,\boldsymbol{x},\boldsymbol{y}),\\
\varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t E_{x_2}(t,\boldsymbol{x},\boldsymbol{y}) &= -\partial_{x_1} H(t,\boldsymbol{x},\boldsymbol{y}) + f_2(t,\boldsymbol{x},\boldsymbol{y}),\\
\mu(\boldsymbol{x},\boldsymbol{y})\partial_t H(t,\boldsymbol{x},\boldsymbol{y}) &= -(\partial_{x_1} E_{x_2}(t,\boldsymbol{x},\boldsymbol{y}) - \partial_{x_2} E_{x_1}(t,\boldsymbol{x},\boldsymbol{y})) + g(t,\boldsymbol{x},\boldsymbol{y}).
\end{aligned}$$

subject to the PEC boundary condition (2.4). We solve this system on $\Omega \times \Xi \times [0, T]$, where $\Xi = \Omega = [0, 1]^2$ and $T = 10^{-5}$. The domain Ω is partitioned uniformly into $N_e = N_t^2$ rectangular elements with a total edge number N_d, where N_t is the total time steps. We solve the problem by using the lowest order edge element on Ω, hence $N_E = N_d$ and $N_H = N_e$. We choose the permittivity and permeability as follows:

$$\begin{aligned}
\varepsilon(\boldsymbol{x},\boldsymbol{y}) &= 1 + 0.1(\sin(x_1)\cos(y_1) + \cos(x_2)\sin(y_2)),\\
\mu(\boldsymbol{x},\boldsymbol{y}) &= 1 + 0.1(\cos(x_1)\sin(y_1) + \sin(x_2)\cos(y_2)),
\end{aligned}$$

for any $\boldsymbol{x} = (x_1, x_2) \in \Omega$ and $\boldsymbol{y} = (y_1, y_2) \in \Xi$. The following exact solution is used to test the accuracy of our numerical scheme:

$$\begin{aligned}
E_{x_1}(t,\boldsymbol{x},\boldsymbol{y}) &= \sin(\pi x_2)e^{-t}(1 + 0.1\sin(\pi y_1)\cos(\pi y_2)),\\
E_{x_2}(t,\boldsymbol{x},\boldsymbol{y}) &= \sin(\pi x_1)e^{-t}(1 + 0.1\cos(\pi y_1)\sin(\pi y_2)),\\
H(t,\boldsymbol{x},\boldsymbol{y}) &= \pi(\cos(\pi x_1) - \cos(\pi x_2))e^{-t}(1 + 0.1\sin(\pi y_1)\sin(\pi y_2)),
\end{aligned}$$

with appropriate source terms $\mathbf{f} = (f_1, f_2)$ and $\mathbf{g} = g$. We use orthogonal polynomials of degrees up to 5 and assume that $\boldsymbol{y}$ has a 2-dimensional uniform distribution on $[0, 1]^2$. Hence, $\rho(\boldsymbol{y}) = 1$ for $\boldsymbol{y} \in \Xi$ and we set $M = 21$. We calculate the errors of $\boldsymbol{E} := (E_x, E_y)$ and H at

Table 2.2: Errors of $\boldsymbol{E}$ and H by leap-frog scheme

Mesh	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{l^2(L^2)}$	Rate	$\|\boldsymbol{H}-\boldsymbol{H}_h\|_{l^2(L^2)}$	Rate	CPU time (s)
2×2	$8.517593e-02$	–	$3.259975e-01$	–	13.243185
4×4	$1.760587e-02$	2.2744	$8.343959e-02$	1.9925	104.981369
8×8	$4.620225e-03$	2.1022	$2.110498e-02$	2.0072	727.997221
16×16	$1.323839e-03$	1.9953	$5.162830e-03$	2.0313	5630.592848

the final time T by the following norm:

$$\|u-u_h\|^2_{l^2(L^2)} := \int_{\Xi}\rho(\boldsymbol{y})\sum_{i=1}^{N_e}|u(\boldsymbol{x}_i,\boldsymbol{y})-u_h(\boldsymbol{x}_i,\boldsymbol{y})|^2|K_i|d\boldsymbol{y},$$

where $\boldsymbol{x}_i$ is the middle point of element K_i, $|K_i|$ is the area of element K_i, and u represents $\boldsymbol{E}$ or H.

The solution errors are presented in Table 2.2, which clearly shows a second order convergence for both $\boldsymbol{E}$ and H. This is consistent with the theoretical result of leap-frog scheme shown in [16].

To test the convergence of gPC expansion on Ω, we solved the problem by using different orders of orthogonal polynomials for $\boldsymbol{E}$ and H with a fixed $N_t = 20$. Observing the error of H in Figure 2.1, we can find that the error is decreasing when the degree p of the gPC orthonormal polynomials is increasing. Note that the error stops going down further when $p \geq 4$. This is because the gPC error is so small for $p \geq 4$ that the spatial and temporal error of the scheme will dominate the total error. Therefore, in the above numerical example, we choose $p = 5$ so that the gPC error will not affect the total error.

Considering that the standard leap-frog scheme (2.55)-(2.56) involves the full matrices $A^{\varepsilon(\boldsymbol{x})}$ and $A^{\mu(\boldsymbol{x})}$, we expect that the modified leap-frog type scheme would be more efficient. By using this scheme, we just need to handle the diagonal matrices $D^{\varepsilon(\boldsymbol{x})}$ and $D^{\mu(\boldsymbol{x})}$. The CPU time and errors calculated by this modified scheme are shown in Table 2.3, which shows that the new scheme is such a CPU time saver with almost the same accuracy. Note that this example is not energy conservative, so we did not test how the energy changes with time.

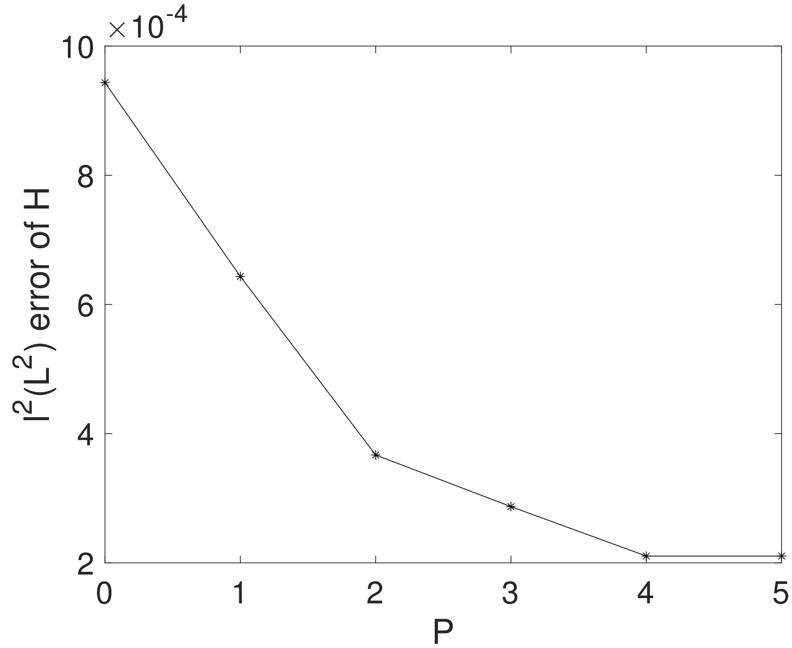


Figure 2.1: Error of gPC expansion

Table 2.3: Errors of $\boldsymbol{E}$ and H for modified leap-frog type scheme

Mesh	$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{l^2(L^2)}$	Rate	$\|\boldsymbol{H}-\boldsymbol{H}_h\|_{l^2(L^2)}$	Rate	CPU time (s)
2×2	$8.517593e-02$	–	$3.259975e-01$	–	0.893071
4×4	$1.760587e-02$	2.2744	$8.343959e-02$	1.9847	6.487596
8×8	$4.520225e-03$	2.1180	$2.110498e-02$	1.9942	44.077161
16×16	$1.284596e-03$	2.0115	$5.257084e-03$	2.0052	358.162832

2.4.2 Example 2: application with random permittivity

Here we will display one numerical experiment for wave scattering problem solved by using our method. The numerical test is done by using 1600 rectangular edge elements in the physical domain $\Omega=[-0.5,0.5]^2$. The time domain for the test is $[0,0.9]$ with $N_t=100$. We still choose $p=5$ for the orthonormal polynomial in Ξ since it is accurate enough as discussed above. A similar setup as the experiment 5.2.2 of [1] is used. Namely, we solve the scattered fields $\boldsymbol{E}^s$ and $\boldsymbol{H}^s$ governed by the following equations:

$$\begin{aligned}
\varepsilon \frac{\partial \boldsymbol{E}^s}{\partial t} &= \nabla \times \boldsymbol{H}^s + \sigma \boldsymbol{E}^s + \mathbf{S}^E, \\
\mu \frac{\partial \boldsymbol{H}^s}{\partial t} &= -\nabla \times \boldsymbol{E}^s + \mathbf{S}^H,
\end{aligned}$$

and the source term $\mathbf{S}^E$ and $\mathbf{S}^H$ are given by

$$\begin{aligned}
\mathbf{S}^E &= -(\varepsilon - \varepsilon^i)\frac{\partial \boldsymbol{E}^i}{\partial t} + (\sigma - \sigma^i)\boldsymbol{E}^i, \\
\mathbf{S}^H &= -(\mu - \mu^i)\frac{\partial \boldsymbol{H}^i}{\partial t}.
\end{aligned}$$

Here the incident field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is a solution of Maxwells equation with permittivity ε^i, permeability μ^i, and conductivity σ^i. More specifically, we set

$$\begin{aligned}
E_x^i &= \sin(\pi y)\sin(\pi t), \\
E_y^i &= \sin(\pi x)\sin(\pi t), \\
H^i &= (\cos(\pi x) - \cos(\pi y))\cos(\pi t),
\end{aligned}$$

$\sigma = \sigma^i = 0$, and

$$\varepsilon^i(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 2.25e^{0.1y} & \text{if } \boldsymbol{x} \in B(0.1), \\ 1 & \text{otherwise,} \end{cases}$$

where $B(r)$ denotes a disc centered at origin of the physical domain with radius r. In other words, $\varepsilon^i(\boldsymbol{x}, \boldsymbol{y})$ is a univariate function on $\Omega = [0, 1]$ and $\rho(y) = 1$.

A good measurement of scattering problem is the so-called radar cross section (RCS) [19]:

$$RCS(\phi) = \lim_{r \to \infty} 10 \ln \left(2\pi\rho \frac{|\boldsymbol{E}^s(\phi)|^2}{|\boldsymbol{E}^i|^2} \right) \tag{2.66}$$

where $\phi \in [-\pi, \pi]$ is the polar angle. In Figure 2.2, we plot the electronic field $\boldsymbol{E} = (E_x, E_y)$ on Ω, and in Figure 2.3 we present the mean and variance of the RCS given by (2.66).

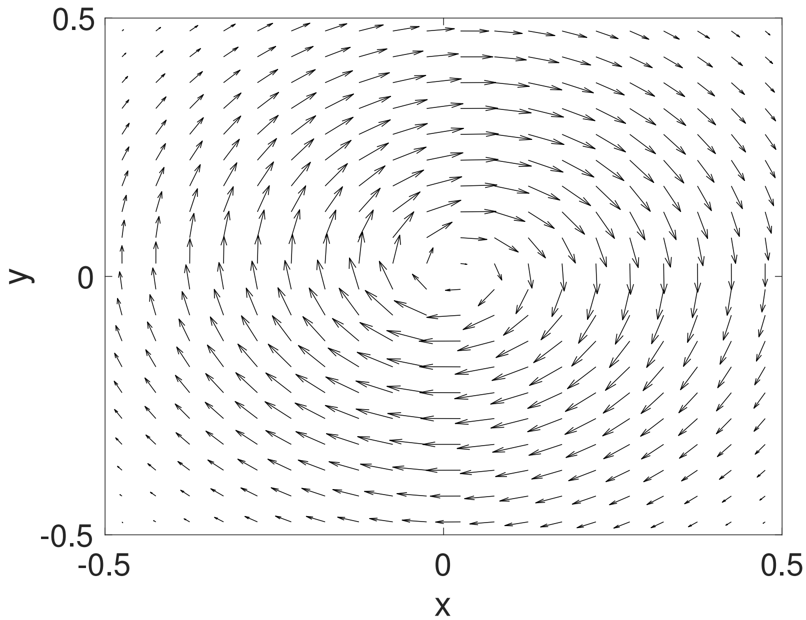


Figure 2.2: The plot of vector field $\boldsymbol{E}$.

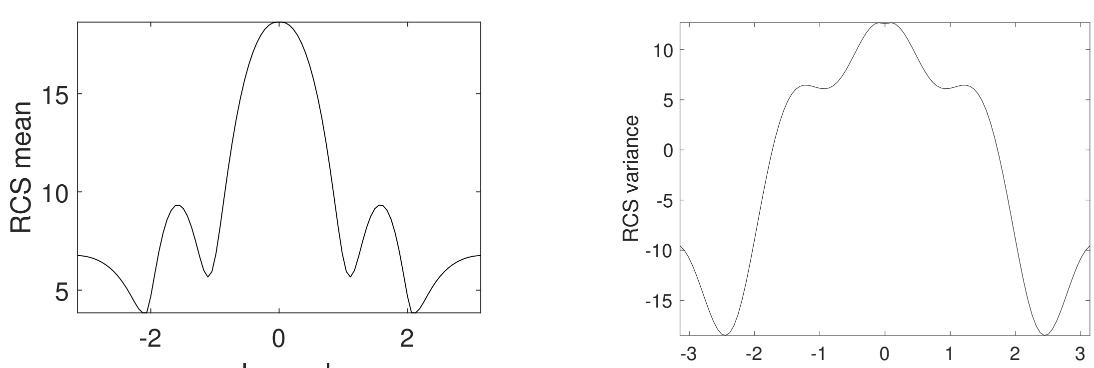


Figure 2.3: Mean and variance of RCS

CHAPTER 3

QUASI-MONTE CARLO METHODS

3.1 Introduction

In this chapter, we still consider the Maxwell's equations with random coefficients (2.1)-(2.2) in chapter 2 with the same setup. But here we assume the parameter vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_N)^\top \in [0,1]^N := \Xi$, where $\{y_i\}_{i=1}^N$ is a set of i.i.d. R.V.s on $[0,1]$. Initial condition (2.3) and boundary condition (2.4) are also be assumed to be satisfied, respectively. Boundness condition (2.18) also been assumed satisfied.

Our goal is to obtain statistical information on the solution $(\boldsymbol{E}, \boldsymbol{H})$ to (2.1)-(2.2), especially its expected value, which is defined on Ξ by

$$\mathbb{E}[u] = \int_{\Xi} u(\boldsymbol{y})\rho(\boldsymbol{y})d\boldsymbol{y}, \qquad \text{for } u = \boldsymbol{E},\ \boldsymbol{H}.$$

where $\rho(\boldsymbol{y})$ is the density function. To approximate the expected value, we can adopt the single level Monte Carlo (SLMC), multi-level Monte Carlo (MLMC) quadrature rules, and the Quasi-Monte Carlo (QMC) quadrature method. There is a huge list of the literature on the application of QMC to PDEs (especially elliptic PDEs) with random coefficients, see [20, 21, 22, 10] and references therein. To the best of our knowledge, there exist few works in the literature which study the QMC method for solving the Maxwell's equations with random inputs.

Since the solutions of (2.1)-(2.2) involve an extra parameter $\boldsymbol{y}$, we introduce the following space to measure the solutions in this chapter:

$$L^r(\Xi; V) = \{v : \Xi \mapsto V : \quad \|v\|_{L^r(\Xi;V)} < \infty\},$$

where V is a Banach space of real-valued functions on domain Ω with norm $\|\cdot\|_V$, and the space $L^r(\Xi; V)$ is equipped with the norm

$$\|v\|_{L^r(\Xi;V)} := \begin{cases} \left(\int_\Xi \|v(\cdot,\boldsymbol{y})\|_V^r \rho(\boldsymbol{y}) d\boldsymbol{y}\right)^{\frac{1}{r}} & \text{if } 0 < r < \infty, \\ \text{esssup}_{\boldsymbol{y}\in\Xi} \|v(\cdot,\boldsymbol{y})\|_V & \text{if } r = \infty. \end{cases}$$

In this chapter, we also adopt the following notations

$$\|v\|_{L^2(\Omega\times\Xi)} := \|v\|_{L^2(\Xi;L^2(\Omega))}, \quad \|v\|_{L^\infty(\Omega\times\Xi)} := \text{esssup}_{\boldsymbol{y}\in\Xi} \|v(\cdot,\boldsymbol{y})\|_{L^\infty(\Omega)}.$$

The structure of this chapter is as follows. Section 3.2 is dedicated to development and error analysis for the single level and multi-level Monte Carlo finite element methods. In section 3.3, we introduce the QMC method, establish the regularity analysis of the solution with respect to the random vector, and prove the error estimate of the QMC method. Section 3.4 presents numerical results which confirm our theoretical results. This chapter is based on my paper [23], which is under review and will be published in a journal.

3.2 The Monte Carlo finite element methods

3.2.1 Edge element for Maxwell's equations

By using the setup and notations in chapter 2, section 2.3, we have the following weak formulation for (2.1)-(2.2):

$$(\varepsilon \partial_t \boldsymbol{E}, \boldsymbol{\varphi}) = (\boldsymbol{H}, \nabla \times \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in H_0(\text{curl}; \Omega) \tag{3.1}$$

$$(\mu \partial_t \boldsymbol{H}, \boldsymbol{\psi}) = -(\nabla \times \boldsymbol{E}, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in H(\text{div}; \Omega) \tag{3.2}$$

where $(\cdot,\cdot)$ denotes the usual inner product on $L^2(\Omega)$.

To define a fully discrete scheme, we divide the time interval $[0, T]$ into M uniform subintervals by points $0 = t_0 < t_1 < \cdots < t_K = T$, where $t_k = k\tau$, and $\tau = T/K$. Moreover, we denote the k-th subinterval by $I_k = [t_{k-1}, t_k]$, and the central difference and average operators for any time series $\boldsymbol{u}^k = \boldsymbol{u}(\cdot, k\tau)$ for $0 \le k \le K$, and the central difference and average operators:

$$\delta_\tau \boldsymbol{u}^{k-\frac{1}{2}} = \frac{\boldsymbol{u}^k - \boldsymbol{u}^{k-1}}{\tau}, \qquad \overline{\boldsymbol{u}}^{k-\frac{1}{2}} = \frac{\boldsymbol{u}^k + \boldsymbol{u}^{k-1}}{2}.$$

Now we can formulate our Crank-Nicolson mixed finite element scheme: for $k = 1, 2, \cdots K$, find $\boldsymbol{E}_h^k \in \boldsymbol{V}_h^0$ and $\boldsymbol{H}_h^k \in \boldsymbol{U}_h$ such that:

$$(\varepsilon \delta_\tau \boldsymbol{E}_h^{k-\frac{1}{2}}, \boldsymbol{\varphi}_h) - (\overline{\boldsymbol{H}}_h^{k-\frac{1}{2}}, \nabla \times \boldsymbol{\varphi}_h) = 0 \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{V}_h^0, \tag{3.3}$$

$$(\mu \delta_\tau \boldsymbol{H}_h^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) + (\nabla \times \overline{\boldsymbol{E}}_h^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) = 0 \qquad \forall \boldsymbol{\psi}_h \in \boldsymbol{U}_h, \tag{3.4}$$

subject to the initial conditions

$$\boldsymbol{E}_h^0(\boldsymbol{x}, \boldsymbol{y}) = \Pi_h^c \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}) \qquad \boldsymbol{H}_h^0(\boldsymbol{x}, \boldsymbol{y}) = \Pi_h^d \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}),$$

where Π_h^d denotes the L^2 projection into space $\boldsymbol{U}_h$ and Π_h^c denotes the Nédélec interpolation on $\boldsymbol{V}_h^0$ introduced in chapter 2.

Note that the above scheme (3.3)-(3.4) can be written as follows:

$$(\varepsilon \boldsymbol{E}_h^k, \boldsymbol{\varphi}_h) - \frac{\tau}{2}(\boldsymbol{H}_h^k, \nabla \times \boldsymbol{\varphi}_h) = (\varepsilon \boldsymbol{E}_h^{k-1}, \boldsymbol{\varphi}_h) + \frac{\tau}{2}(\boldsymbol{H}_h^{k-1}, \nabla \times \boldsymbol{\varphi}_h), \tag{3.5}$$

$$(\mu \boldsymbol{H}_h^k, \boldsymbol{\psi}_h) + \frac{\tau}{2}(\nabla \times \boldsymbol{E}_h^k, \boldsymbol{\psi}_h) = (\mu \boldsymbol{H}_h^{k-1}, \boldsymbol{\psi}_h) - \frac{\tau}{2}(\nabla \times \boldsymbol{E}_h^{k-1}, \boldsymbol{\psi}_h). \tag{3.6}$$

Hence, at each time step, the coefficient matrix of (3.5)-(3.6) with the vector solution $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k)^\top$ can be written as $Q = \begin{pmatrix} A & -B \\ B^\top & D \end{pmatrix}$, which can be proved to be non-singular (cf. [16, Lemma 3.14]).

First, we have the following unconditional stability for our scheme.

Lemma 5. *For the solution* $(\boldsymbol{E}_h^k, \boldsymbol{H}_h^k)$ *of (3.3)-(3.4) and any* $k \in [1, K] \cap \mathbb{N}$*, we have*

$$\left\| \varepsilon^{\frac{1}{2}} \boldsymbol{E}_h^k \right\|_{L^2(\Omega)}^2 + \left\| \mu^{\frac{1}{2}} \boldsymbol{H}_h^k \right\|_{L^2(\Omega)}^2 = \left\| \varepsilon^{\frac{1}{2}} \boldsymbol{E}_h^0 \right\|_{L^2(\Omega)}^2 + \left\| \mu^{\frac{1}{2}} \boldsymbol{H}_h^0 \right\|_{L^2(\Omega)}^2.$$

Proof. Choosing $\boldsymbol{\varphi}_h = \tau \overline{\boldsymbol{E}}_h^{k-\frac{1}{2}}$ and $\boldsymbol{\psi}_h = \tau \overline{\boldsymbol{H}}_h^{k-\frac{1}{2}}$ in (3.3) and (3.4), respectively, and adding the results together, we have

$$\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_h^k\right\|_{L^2(\Omega)}^2-\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_h^{k-1}\right\|_{L^2(\Omega)}^2\right)+\frac{1}{2}\left(\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_h^k\right\|_{L^2(\Omega)}^2-\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_h^{k-1}\right\|_{L^2(\Omega}^2\right)=0,$$

which concludes the proof. □

Denote $C_v=\frac{1}{\sqrt{\varepsilon\mu}}$ for the wave propagation speed in a medium with permittivity ε and permeability μ. Then we can prove the following optimal error estimate for our scheme.

Theorem 6. *Suppose that the solution $(\boldsymbol{E},\boldsymbol{H})$ of (3.1)-(3.2) satisfy the following regularity:*

$$\begin{aligned}
&\varepsilon^{\frac{1}{2}}\boldsymbol{E}\in(L^\infty(0,T;H^r(curl;\Omega)))^3,\ \varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\in(L^2(0,T;H^r(curl;\Omega)))^3,\\
&\varepsilon^{\frac{1}{2}}\nabla\times\partial_{t^2}\boldsymbol{E}\in(L^2(0,T;L^2(\Omega)))^3,\\
&\mu^{\frac{1}{2}}\boldsymbol{H}\in(L^\infty(0,T;H^r(\Omega)))^3,\ \mu^{\frac{1}{2}}\nabla\times\partial_{t^2}\boldsymbol{H}\in(L^2(0,T;L^2(\Omega)))^3,
\end{aligned}$$

then for any $k\in[1,K]\cap\mathbb{N}$, we have

$$\begin{aligned}
&\left\|\varepsilon^{\frac{1}{2}}(\boldsymbol{E}_h^k-\boldsymbol{E}(t_k,\boldsymbol{x},\boldsymbol{y}))\right\|_{(L^2(\Omega))^3}+\left\|\mu^{\frac{1}{2}}(\boldsymbol{H}_h^k-\boldsymbol{H}(t_k,\boldsymbol{x},\boldsymbol{y}))\right\|_{(L^2(\Omega))^3}\\
\le\ & Ch^r\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|_{(L^2(0,T;H^r(curl;\Omega)))^3}+\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}\right\|_{(L^\infty(0,T;H^r(curl;\Omega)))^3}+\left\|\mu^{\frac{1}{2}}\boldsymbol{H}\right\|_{(L^\infty(0,T;H^r(\Omega)))^3}\right)\\
&+C\tau^2\left(\left\|\mu^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{H})\right\|_{(L^2(0,T;L^2(\Omega)))^3}+\left\|\varepsilon^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{E})\right\|_{(L^2(0,T;L^2(\Omega)))^3}\right),
\end{aligned}\tag{3.7}$$

where the constant $C>0$ is independent of h and τ, and $r\ge1$ is the degree of the finite element spaces $\boldsymbol{V}_h^0$ and $\boldsymbol{U}_h$.

Proof. Integrating (2.1) and (2.2) from t_{k-1} to t_k, multiplying the results by $\boldsymbol{\varphi}_h\in\boldsymbol{V}_h^0$ and $\boldsymbol{\psi}_h\in\boldsymbol{U}_h$ and integrating over Ω, respectively, we obtain

$$\left(\varepsilon\frac{\boldsymbol{E}^k-\boldsymbol{E}^{k-1}}{\tau},\boldsymbol{\varphi}_h\right)-\left(\frac{1}{\tau}\int_{I_k}\boldsymbol{H}ds,\nabla\times\boldsymbol{\varphi}_h\right)=0,\tag{3.8}$$

$$\left(\mu\frac{\boldsymbol{H}^k-\boldsymbol{H}^{k-1}}{\tau},\boldsymbol{\psi}_h\right)+\left(\frac{1}{\tau}\int_{I_k}\nabla\times\boldsymbol{E}ds,\boldsymbol{\psi}_h\right)=0,\tag{3.9}$$

where for simplicity we denote $u^j=u(t_j)$ for $u=\boldsymbol{E}$ or $\boldsymbol{H}$.

Let us introduce the errors

$$\widehat{\boldsymbol{E}}_h^k := \boldsymbol{E}_h^k - \boldsymbol{E}^k = (\boldsymbol{E}_h^k - \Pi_h^c \boldsymbol{E}^k) - (\boldsymbol{E}^k - \Pi_h^c \boldsymbol{E}^k) = \boldsymbol{E}_{h\xi}^k - \boldsymbol{E}_{h\eta}^k, \tag{3.10}$$

$$\widehat{\boldsymbol{H}}_h^k := \boldsymbol{H}_h^k - \boldsymbol{H}^k = (\boldsymbol{H}_h^k - \Pi_h^d \boldsymbol{H}^k) - (\boldsymbol{H}^k - \Pi_h^d \boldsymbol{H}^k) = \boldsymbol{H}_{h\xi}^k - \boldsymbol{H}_{h\eta}^k. \tag{3.11}$$

Subtracting (3.8)-(3.9) from (3.5)-(3.6), respectively, we obtain the error equations:

$$\left(\varepsilon \frac{\widehat{\boldsymbol{E}}_h^k - \widehat{\boldsymbol{E}}_h^{k-1}}{\tau}, \boldsymbol{\varphi}_h\right) - \left(\frac{\widehat{\boldsymbol{H}}_h^k + \widehat{\boldsymbol{H}}_h^{k-1}}{2}, \nabla \times \boldsymbol{\varphi}_h\right) = \left(\frac{\boldsymbol{H}^k + \boldsymbol{H}^{k-1}}{2} - \frac{1}{\tau}\int_{I_k} \boldsymbol{H} ds, \nabla \times \boldsymbol{\varphi}_h\right), \tag{3.12}$$

$$\left(\mu \frac{\widehat{\boldsymbol{H}}_h^k - \widehat{\boldsymbol{H}}_h^{k-1}}{\tau}, \boldsymbol{\psi}_h\right) + \left(\nabla \times \frac{\widehat{\boldsymbol{E}}_h^k + \widehat{\boldsymbol{E}}_h^{k-1}}{2}, \boldsymbol{\psi}_h\right) = \left(\frac{1}{\tau}\int_{I_k} \nabla \times \boldsymbol{E} ds - \nabla \times \frac{\boldsymbol{E}^k + \boldsymbol{E}^{k-1}}{2}, \boldsymbol{\psi}_h\right). \tag{3.13}$$

Using the error decomposition (3.10)-(3.11), we can rewrite the above error equations as follows:

$$\begin{aligned} & (\varepsilon \delta_\tau \boldsymbol{E}_{h\xi}^{k-\frac{1}{2}}, \boldsymbol{\varphi}_h) - (\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}, \nabla \times \boldsymbol{\varphi}_h) \\ = \; & (\varepsilon \delta_\tau \boldsymbol{E}_{h\eta}^{k-\frac{1}{2}}, \boldsymbol{\varphi}_h) - (\overline{\boldsymbol{H}}_{h\eta}^{k-\frac{1}{2}}, \nabla \times \boldsymbol{\varphi}_h) + \left(\overline{\boldsymbol{H}}^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{I_k} \boldsymbol{H} ds, \nabla \times \boldsymbol{\varphi}_h\right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} & (\mu \delta_\tau \boldsymbol{H}_{h\xi}^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) + (\nabla \times \overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) \\ = \; & (\mu \delta_\tau \boldsymbol{H}_{h\eta}^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) + (\nabla \times \overline{\boldsymbol{E}}_{h\eta}^{k-\frac{1}{2}}, \boldsymbol{\psi}_h) + \left(\frac{1}{\tau}\int_{I_k} \nabla \times \boldsymbol{E} ds - \nabla \times \overline{\boldsymbol{E}}^{k-\frac{1}{2}}, \boldsymbol{\psi}_h\right). \end{aligned} \tag{3.15}$$

Choosing $\boldsymbol{\varphi}_h = 2\tau \overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}$ in (3.14) and $\boldsymbol{\psi}_h = 2\tau \overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}$ in (3.15), then adding the resultants, we obtain

$$
\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k}\right\|^2-\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k-1}\right\|^2\right)+\left(\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{k}\right\|^2-\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{k-1}\right\|^2\right)\\
= \;& 2\tau\left(\varepsilon\delta_\tau\boldsymbol{E}_{h\eta}^{k-\frac{1}{2}},\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right)-2\tau\left(\overline{\boldsymbol{H}}_{h\eta}^{k-\frac{1}{2}},\nabla\times\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right)\\
&+2\tau\left(\overline{\boldsymbol{H}}^{k-\frac{1}{2}}-\frac{1}{\tau}\int_{I_k}\boldsymbol{H}ds,\nabla\times\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right)\\
&+2\tau\left(\mu\delta_\tau\boldsymbol{H}_{h\eta}^{k-\frac{1}{2}},\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right)+2\tau\left(\nabla\times\overline{\boldsymbol{E}}_{h\eta}^{k-\frac{1}{2}},\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right)\\
&+2\tau\left(\frac{1}{\tau}\int_{I_k}\nabla\times\boldsymbol{E}ds-\nabla\times\overline{\boldsymbol{E}}^{k-\frac{1}{2}},\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right):=\sum_{i=1}^{6}Err_i.
\end{aligned}
\tag{3.16}
$$

By the Cauchy-Schwarz inequality, the following estimate [16, Lemma 3.16]:

$$
\left\|\delta_\tau u^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}^2:=\left\|\frac{u^k-u^{k-1}}{\tau}\right\|_{L_2(\Omega)}^2\le\frac{1}{\tau}\int_{t_{k-1}}^{t_k}\|\partial_t u\|_{L_2(\Omega)}^2ds,\quad\forall\ u\in H^1((t_{k-1},t_k);L^2(\Omega)),
\tag{3.17}
$$

and the interpolation error estimates: for any $r\ge 1$,

$$
\|\boldsymbol{u}-\Pi_h^c\boldsymbol{u}\|_{L_2(\Omega)}+\|\nabla\times(\boldsymbol{u}-\Pi_h^c\boldsymbol{u})\|_{L_2(\Omega)}\le Ch^r\|\boldsymbol{u}\|_{H^r(curl;\Omega)},\forall\ \boldsymbol{u}\in H^r(curl;\Omega),
\tag{3.18}
$$

$$
\|\boldsymbol{v}-\Pi_h^d\boldsymbol{v}\|_{L_2(\Omega)}\le Ch^r\|\boldsymbol{u}\|_{H^r(\Omega)},\quad\forall\ \boldsymbol{v}\in H^r(\Omega).
\tag{3.19}
$$

we have

$$
\begin{aligned}
Err_1&\le 2\tau\left\|\varepsilon^{\frac{1}{2}}\delta_\tau\boldsymbol{E}_{h\eta}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}\left\|\varepsilon^{\frac{1}{2}}\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}\\
&\le\tau\left(\frac{1}{4\delta_*}\left\|\varepsilon^{\frac{1}{2}}\delta_\tau\boldsymbol{E}_{h\eta}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}^2+4\delta_*\left\|\varepsilon^{\frac{1}{2}}\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}^2\right)\\
&\le\frac{1}{4\delta_*}\int_{I_k}\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}_{h\eta}\right\|_{L_2(\Omega)}^2ds+2\tau\delta_*\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k}\right\|_{L_2(\Omega)}^2+\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k-1}\right\|_{L_2(\Omega)}^2\right)\\
&\le\frac{1}{\delta_*}\int_{I_k}Ch^{2r}\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|_{H^r(curl;\Omega)}^2ds+4\tau\delta_*\max_{0\le k\le K}\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k}\right\|_{L_2(\Omega)}^2.
\end{aligned}
$$

By the definition of projection Π_h^d and the property $\nabla\times\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\in\boldsymbol{U}_h$, we have $Err_2=0$, $Err_4=0$.

Using integration by parts, the following estimate [16, Lemma 3.16]:

$$\left\| \frac{u(t_{k-1})+u(t_k)}{2} - \frac{1}{\tau}\int_{I_k} u(s)ds \right\|_{L^2(\Omega)} \le \frac{\tau^3}{4}\int_{t_{k-1}}^{t_k} \|\partial_{t^2} u\|^2_{L_2(\Omega)}\, ds,$$

$$\forall\ u \in H^1((t_{k-1},t_k);L^2(\Omega)), \tag{3.20}$$

and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
Err_3 &\le 2\tau\left(\nabla\times\left(\overline{\boldsymbol{H}}^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{I_k}\boldsymbol{H}ds\right), \overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right)\\
&\le 2\tau C_v\left(\frac{1}{8\delta_*}\left\|\mu^{\frac{1}{2}}(\overline{\nabla\times\boldsymbol{H}}^{k-\frac{1}{2}} - \frac{1}{\tau}\int_{I_k}\nabla\times\boldsymbol{H}ds)\right\|^2_{L_2(\Omega)} + 2\delta_*\left\|\varepsilon^{\frac{1}{2}}\overline{\boldsymbol{E}}_{h\xi}^{k-\frac{1}{2}}\right\|^2_{L_2(\Omega)}\right)\\
&\le 2\tau C_v\left[\frac{\tau^3}{32\delta_*}\int_{I_k}\left\|\mu^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{H})\right\|^2_{L_2(\Omega)} ds + \delta_*\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k}\right\|^2_{L_2(\Omega)} + \left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k-1}\right\|^2_{L_2(\Omega)}\right)\right]\\
&\le \frac{\tau^4 C_v}{16\delta_*}\int_{I_k}\left\|\mu^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{H})\right\|^2_{L_2(\Omega)} ds + 4\tau\delta_* C_v \max_{0\le k\le K}\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{k}\right\|^2_{L_2(\Omega)}.
\end{aligned}$$

Using the Cauchy-Schwarz inequality and the interpolation error estimate (3.18), we have

$$\begin{aligned}
Err_5 &\le 2\tau\left\|\nabla\times\overline{\boldsymbol{E}}_{h\eta}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}\left\|\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right\|_{L_2(\Omega)}\\
&\le 2\tau C_v\left(\frac{1}{8\delta_*}\left\|\varepsilon^{\frac{1}{2}}\nabla\times\overline{\boldsymbol{E}}_{h\eta}^{k-\frac{1}{2}}\right\|^2_{L_2(\Omega)} + 2\delta_*\left\|\mu^{\frac{1}{2}}\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right\|^2_{L_2(\Omega)}\right)\\
&\le \frac{\tau C_v C h^{2r}}{\delta_*}\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}\right\|^2_{(L^\infty(0,T;H^r(curl;\Omega)))^3} + 4\tau C_v\delta_*\max_{0\le k\le K}\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{k}\right\|^2_{L_2(\Omega)}.
\end{aligned}$$

By the Cauchy-Schwarz inequality and (3.20), we have

$$\begin{aligned}
Err_6 &\le 2\tau C_v\left[\frac{1}{8\delta_*}\left\|\varepsilon^{\frac{1}{2}}\left(\frac{1}{\tau}\int_{I_k}\nabla\times\boldsymbol{E}ds - \overline{\nabla\times\boldsymbol{E}}^{k-\frac{1}{2}}\right)\right\|^2_{L_2(\Omega)} + 2\delta_*\left\|\mu^{\frac{1}{2}}\overline{\boldsymbol{H}}_{h\xi}^{k-\frac{1}{2}}\right\|^2_{L_2(\Omega)}\right]\\
&\le \frac{\tau^4 C_v}{16\delta_*}\int_{I_k}\left\|\varepsilon^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{E})\right\|^2_{L_2(\Omega)} ds + 4\tau C_v\delta_*\max_{0\le k\le K}\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{k}\right\|^2_{L_2(\Omega)}.
\end{aligned}$$

Substituting the above estimates of Err_i into (3.16), then summing up the resultant from

$k=1$ to $n \le K$ and using the fact that $n\tau \le T$, we have

$$
\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{n}\right\|_{L_2(\Omega)}^2 - \left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^{0}\right\|_{L_2(\Omega)}^2\right) + \left(\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{n}\right\|_{L_2(\Omega)}^2 - \left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^{0}\right\|_{L_2(\Omega)}^2\right) \\
\le\; & \frac{Ch^{2r}}{\delta_*}\int_0^{t_n}\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|_{H^r(curl;\Omega)}^2 ds + 4T\delta_*\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}\right\|_\infty^2 \\
& +\frac{\tau^4 C_v}{16\delta_*}\int_0^{t_n}\left\|\mu^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{H})\right\|_{L_2(\Omega)}^2 ds + 4T\delta_* C_v\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}\right\|_\infty^2 \\
& +\frac{TC_vCh^{2r}}{\delta_*}\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}\right\|_{(L^\infty(0,T;H^r(curl;\Omega)))^3}^2 + 4TC_v\delta_*\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}\right\|_\infty^2 \\
& +\frac{\tau^4 C_v}{16\delta_*}\int_0^{t_n}\left\|\varepsilon^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{E})\right\|_{L_2(\Omega)}^2 ds + 4TC_v\delta_*\left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}\right\|_\infty^2,
\end{aligned}
\tag{3.21}
$$

where we denote

$$
\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}\|_\infty := \max_{0\le k\le K}\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}^k\|_{L_2(\Omega)} \text{ and } \|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}\|_\infty := \max_{0\le k\le K}\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}^k\|_{L_2(\Omega)}.
$$

Taking the maximum of (3.21) with respect to n, then choosing δ_* small enough (e.g., $4T\delta_*\max(1,C_v) \le \frac{1}{3}$), we have

$$
\begin{aligned}
&\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\xi}\right\|_\infty + \left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\xi}\right\|_\infty \\
\le\; & Ch^r\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|_{(L^2(0,T;H^r(curl;\Omega)))^3} + \left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}\right\|_{(L^\infty(0,T;H^r(curl;\Omega)))^3}\right) \\
& +C\tau^2\left(\left\|\mu^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{H})\right\|_{(L^2(0,T;L^2(\Omega)))^3} + \left\|\varepsilon^{\frac{1}{2}}\nabla\times(\partial_{t^2}\boldsymbol{E})\right\|_{(L^2(0,T;L^2(\Omega)))^3}\right).
\end{aligned}
\tag{3.22}
$$

Using the interpolation error estimates (3.18)-(3.19), we obtain

$$
\begin{aligned}
&\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}_{h\eta}\right\|_{(L^2(\Omega))^3} + \left\|\mu^{\frac{1}{2}}\boldsymbol{H}_{h\eta}\right\|_{(L^2(\Omega))^3} \\
\le\; & Ch^r\left(\left\|\varepsilon^{\frac{1}{2}}\boldsymbol{E}\right\|_{(L^\infty(0,T;H^r(curl;\Omega)))^3} + \left\|\mu^{\frac{1}{2}}\boldsymbol{H}\right\|_{(L^\infty(0,T;H^r(curl;\Omega)))^3}\right).
\end{aligned}
\tag{3.23}
$$

Combining the estimates of (3.22) and (3.23), and using the triangle inequality, we complete the proof. □

3.2.2 Analysis of single level Monte Carlo method

The idea of Monte Carlo finite element method is very simple: given a sample $\boldsymbol{y}$, (3.1)-(3.2) becomes a deterministic problem in the physical domain Ω, which can be solved by any classic finite element method for Maxwell's equations [15, 16].

In practice, we are often interested in estimating the expected value (also known as expectation) of the random solutions. The expectation $\mathbb{E}[\boldsymbol{u}]$ can be estimated by a sample mean over the solution samples $\{\widehat{u}^i\}$, $i = 1, 2, \cdots M$, corresponding to M i.i.d. realizations of the random inputs:

$$\mathbb{E}[\boldsymbol{u}] \approx E_M[\boldsymbol{u}] := \frac{1}{M}\sum_{i=1}^{M} \widehat{u}^i. \tag{3.24}$$

Here $\boldsymbol{u}$ can denote either the analytic solutions $\boldsymbol{E}$ and $\boldsymbol{H}$, or the finite element solutions $\boldsymbol{E}_h$ and $\boldsymbol{H}_h$.

The following result was proved in [10] and gives a bound on the statistical error for the Monte Carlo estimator (3.24).

Lemma 7. *[10, Lemma 4.1] Let $V = L^2(\Omega)$. For any $M \in \mathbb{N}$ and $\boldsymbol{u} \in L^2(\Xi; V)$, we have*

$$\|\mathbb{E}[\boldsymbol{u}] - E_M[\boldsymbol{u}]\|_{L^2(\Omega\times\Xi)} \le \frac{1}{\sqrt{M}}\|\boldsymbol{u}\|_{L^2(\Omega\times\Xi)}.$$

Remark 3. *If we define the variance of a function $\boldsymbol{u}$ as $\sigma(\boldsymbol{u}) := \sqrt{\mathbb{E}[\|\boldsymbol{u}\|_V^2] - \|\mathbb{E}[\boldsymbol{u}]\|_V^2}$, then we have a more accurate statistical error estimate for the Monte Carlo method (cf. the proof of [10, Lemma 4.1]):*

$$\|\mathbb{E}[\boldsymbol{u}] - E_M[\boldsymbol{u}]\|_{L^2(\Omega\times\Xi)} = \frac{1}{\sqrt{M}}\sigma(\boldsymbol{u}).$$

The single level Monte Carlo method is to find out the estimator $E_M[\boldsymbol{u}]$ defined in (3.24). To this end, we pick a sequence of i.i.d. sample points $\boldsymbol{y}^i$, $i = 1, 2, \cdots M$, and compute the corresponding numerical solution $\widehat{u}^i$ of (3.5)-(3.6). The error estimate of single level Monte Carlo finite element method is given by the following theorem:

Theorem 8. *Under the same regularity assumptions given in Theorem 6, the single level Monte Carlo method (3.5)-(3.6) satisfies the following error estimate: at any time step*

$t_k = k\tau$, $k = 1, 2, \cdots, K$, we have

$$\|\mathbb{E}[\boldsymbol{E}(t_k)] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{H}(t_k)] - E_M[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)} \leq C\left(\tau^2 + h^r + M^{-\frac{1}{2}}\right)$$

Proof. For simplicity, we denote $\boldsymbol{E}^k := \boldsymbol{E}(\boldsymbol{x}, t_k)$ and $\boldsymbol{H}^k := \boldsymbol{H}(\boldsymbol{x}, t_k)$.

Using Jessen's inequality for the solution $\boldsymbol{E}$, we have

$$\begin{aligned}\|\mathbb{E}[\boldsymbol{E}(t_k)] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} &\leq \|\mathbb{E}[\boldsymbol{E}(t_k)] - \mathbb{E}[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{E}_h^k] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)}\\ &\leq \mathbb{E}\left[\|\boldsymbol{E}(t_k) - \boldsymbol{E}_h^k\|_V\right]^{\frac{1}{2}} + \|\mathbb{E}[\boldsymbol{E}_h^k] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)},\end{aligned}$$

which, along with a similar estimate for the solution $\boldsymbol{H}$, leads to

$$\|\mathbb{E}[\boldsymbol{E}(t_k)] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{H}(t_k)] - E_M[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)} \leq \sqrt{2}I + II, \tag{3.25}$$

where we denote

$$\begin{aligned}I &:= \mathbb{E}\left[\|\boldsymbol{E}(t_k) - \boldsymbol{E}_h^k\|_0^2 + \|\boldsymbol{H}(t_k) - \boldsymbol{H}_h^k\|_0^2\right]^{\frac{1}{2}},\\ II &:= \|\mathbb{E}[\boldsymbol{E}_h^k] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{H}_h^k] - E_M[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)}.\end{aligned}$$

The first term I in (3.25) measures the error of the finite element scheme and the second term II gives the statistical error. Note that the estimate of I is given by Theorem 6, with the constant coefficient C independent of vector $\boldsymbol{y}$ after taking the mean.

To bound the term II, we use Lemma 7 and Lemma 5 to obtain

$$\begin{aligned}&\varepsilon_{\min}\|\mathbb{E}[\boldsymbol{E}_h^k] - E_M[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)}^2 + \mu_{\min}\|\mathbb{E}[\boldsymbol{H}_h^k] - E_M[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)}^2\\ \leq\ &\frac{1}{M}\left(\int_\Omega\int_\Xi \varepsilon|\boldsymbol{E}_h^k|^2 + \mu|\boldsymbol{H}_h^k|^2 d\boldsymbol{y}d\boldsymbol{x}\right)\\ \leq\ &\frac{1}{M}\left(\varepsilon_{\max}\|\boldsymbol{E}_h^0\|_{L^2(\Omega\times\Xi)}^2 + \mu_{\max}\|\boldsymbol{H}_h^0\|_{L^2(\Omega\times\Xi)}^2\right) \leq \frac{C}{M}.\end{aligned}$$

This leads to $II \leq CM^{-\frac{1}{2}}$. Substituting the estimates of I and II into (3.25) concludes our proof. □

3.2.3 Analysis of multi-level Monte Carlo method

As shown in theorem 8, the single level Monte Carlo method only has half order convergent rate. This means to get a desired accuracy we have to solve the PDEs system several times, which is still a huge workload. We then enlightened by multi-level finite element methods come up with the multi-level Monte Carlo method (MLMC). The basic idea of MLMC is the solve the PDEs problems on a relatively rough mesh and then finer later. By using the linear combination, their error have been canceled partially. Finally, we use smaller workload to achieve the same accuracy compared with single level Monte Carlo method.

For the MLMC method we discretize the physical domain Ω by a sequence of nested partitions $\{\mathcal{T}_l\}_{l=1}^L$ with corresponding mesh size h_l and time step τ_l. That is, $\mathcal{T}_k \subset \mathcal{T}_{k+1}$ for $k = 1, 2, \cdots, L-1$. And then, we solve the finite element scheme (3.5)-(3.6) in the corresponding mixed finite element spaces $\boldsymbol{U}_{h_l}$ and $\boldsymbol{V}_{h_l}$. Introducing the notation $\boldsymbol{u}_0 := 0$, we can write

$$\boldsymbol{u}_L = \sum_{l=1}^{L} (\boldsymbol{u}_l - \boldsymbol{u}_{l-1}),$$

where $\boldsymbol{u}_l$ represents the solution obtained on mesh $\mathcal{T}_l$.

By the linearity of expectation, we have

$$\mathbb{E}[\boldsymbol{u}_L] = \mathbb{E}\left[\sum_{l=1}^{L} (\boldsymbol{u}_l - \boldsymbol{u}_{l-1})\right] = \sum_{l=1}^{L} \mathbb{E}[\boldsymbol{u}_l - \boldsymbol{u}_{l-1}].$$

In the MLMC method, we estimate $\mathbb{E}[\boldsymbol{u}_l - \boldsymbol{u}_{l-1}]$ by a level dependent number of samples M_l, i.e, the MLMC estimator is given by:

$$\mathbb{E}[\boldsymbol{u}_L] \approx E^{ML}[\boldsymbol{u}] := \sum_{l=1}^{L} E_{M_l}[\boldsymbol{u}_l - \boldsymbol{u}_{l-1}] \tag{3.26}$$

Theorem 9. *Under the same assumptions as Theorem 8, the MLMC finite element solution* *of (3.5)-(3.6) satisfies the following error estimate:*

$$
\begin{aligned}
&\|\mathbb{E}[\boldsymbol{E}(t_k)] - E^{ML}[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{H}(t_k)] - E^{ML}[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)} \\
\le\; & C\left(\tau_L^2 + h_L^r + \sum_{l=1}^{L}(h_l^r + \tau_l^2)M_l^{-\frac{1}{2}}\right).
\end{aligned}
$$

Proof. Similar to the proof of Theorem 8, we rewrite the error of $\boldsymbol{E}$ in two parts:

$$
\begin{aligned}
&\|\mathbb{E}[\boldsymbol{E}(t_k)] - E^{ML}[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} \\
=& \left\|\mathbb{E}[\boldsymbol{E}(t_k)] - \mathbb{E}[\boldsymbol{E}_L^k] + \mathbb{E}[\boldsymbol{E}_L^k] - \sum_{l=1}^{L} E_{M_l}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k]\right\|_{L^2(\Omega\times\Xi)} \\
\le& \|\mathbb{E}[\boldsymbol{E}(t_k)] - \mathbb{E}[\boldsymbol{E}_L^k]\|_{L^2(\Omega\times\Xi)} + \left\|\sum_{l=1}^{L}\left(\mathbb{E}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] - E_{M_l}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k]\right)\right\|_{L^2(\Omega\times\Xi)} \\
\le& \mathbb{E}\left[\|\boldsymbol{E}(t_k) - \boldsymbol{E}_L^k\|_{L^2(\Omega)}^2\right]^{\frac{1}{2}} + \left\|\sum_{l=1}^{L}\left(\mathbb{E}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] - E_{M_l}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k]\right)\right\|_{L^2(\Omega\times\Xi)}.
\end{aligned}
$$

Similar estimate holds true for $\boldsymbol{H}$.

Hence, we have

$$
\|\mathbb{E}[\boldsymbol{E}(t_k)] - E^{ML}[\boldsymbol{E}_h^k]\|_{L^2(\Omega\times\Xi)} + \|\mathbb{E}[\boldsymbol{H}(t_k)] - E^{ML}[\boldsymbol{H}_h^k]\|_{L^2(\Omega\times\Xi)} \le \sqrt{2}I + II,
$$

where we denote

$$
\begin{aligned}
I :=& \mathbb{E}\left[\|\boldsymbol{E}(t_k) - \boldsymbol{E}_L^k\|_{L^2(\Omega)}^2 + \|\boldsymbol{H}(t_k) - \boldsymbol{H}_L^k\|_{L^2(\Omega)}^2\right]^{\frac{1}{2}}, \\
II :=& \left\|\sum_{l=1}^{L}\left(\mathbb{E}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] - E_{M_l}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k]\right)\right\|_{L^2(\Omega\times\Xi)} \\
&+ \left\|\sum_{l=1}^{L}\left(\mathbb{E}[\boldsymbol{H}_l^k - \boldsymbol{H}_{l-1}^k] - E_{M_l}[\boldsymbol{H}_l^k - \boldsymbol{H}_{l-1}^k]\right)\right\|_{L^2(\Omega\times\Xi)}.
\end{aligned}
$$

The error term I is the error caused by the finite element scheme, which is given by Theorem 6.

To estimate term II, by Lemma 7, for any $1 \le l \le M$ we have:

$$\begin{aligned}
&\left\| \left(\mathbb{E}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] - E_{M_l}[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] \right) \right\|_{L^2(\Omega\times\Xi)} + \\
&\left\| \left(\mathbb{E}[\boldsymbol{H}_l^k - \boldsymbol{H}_{l-1}^k] - E_{M_l}[\boldsymbol{H}_l^k - \boldsymbol{H}_{l-1}^k] \right) \right\|_{L^2(\Omega\times\Xi)} \\
= &\left\| (\mathbb{E} - E_{M_l})[\boldsymbol{E}_l^k - \boldsymbol{E}_{l-1}^k] \right\|_{L^2(\Omega\times\Xi)} + \left\| (\mathbb{E} - E_{M_l})[\boldsymbol{H}_l^k - \boldsymbol{H}_{l-1}^k] \right\|_{L^2(\Omega\times\Xi)} \\
\le &M_l^{-\frac{1}{2}} \left(\|\boldsymbol{E}^k - \boldsymbol{E}_l^k\|_{L^2(\Omega\times\Xi)} + \|\boldsymbol{E}^k - \boldsymbol{E}_{l-1}^k\|_{L^2(\Omega\times\Xi)} \right. \\
&\left. + \|\boldsymbol{H}^k - \boldsymbol{H}_l^k\|_{L^2(\Omega\times\Xi)} + \|\boldsymbol{H}^k - \boldsymbol{H}_{l-1}^k\|_{L^2(\Omega\times\Xi)} \right) \\
\le &CM_l^{-\frac{1}{2}} (h_l^r + \tau_l^2 + h_{l-1}^r + \tau_{l-1}^2) \le CM_l^{-\frac{1}{2}} (h_l^r + \tau_l^2).
\end{aligned}$$

Hence we have the error estimate for term II: $II \le C \sum_{l=1}^{L} (h_l^r + \tau_l^2) M_l^{-\frac{1}{2}}$, which, along with the estimate of I, completes the proof. □

3.3 Quasi-Monte Carlo finite element method

Due to the slow convergence of the classical Monte Carlo method, the quasi-Monte Carlo methods have been proposed to solve stochastic elliptic equations (e.g., [24, 25]). In this section, we will analyze the usage of this method for solving the stochastic Maxwell's equations. Instead of considering the expectation of $\boldsymbol{E}$ and $\boldsymbol{H}$ directly, we will find out estimator of $\mathbb{E}[G_1(\boldsymbol{E})]$ and $\mathbb{E}[G_2(\boldsymbol{H})]$ respectively, where $G_1, G_2 : \ L^2(\Omega) \mapsto \mathbb{R}$ are some bounded linear functionals.

3.3.1 QMC integration in the finite dimensional setting

For any function F defined on $\Xi = [0,1]^n$, consider the following integral

$$I(F) := \int_{\Xi} F(\boldsymbol{y}) d\boldsymbol{y}.$$

To approximate $I(F)$, we use the N point QMC estimator given by

$$Q_N(F) := \frac{1}{N} \sum_{i=1}^{N} F(\boldsymbol{y}^{(i)}),$$

where $\{\boldsymbol{y}^{(i)}\}_{i=1}^N \subset \Xi$ is the set of points which needs to be chosen carefully. Here we just focus on the shifted rank-1 lattice rules. In these rules, the quadrature points are given by the following formula

$$\boldsymbol{y}^{(i)} = \text{frac}\left(\frac{i\mathbf{z}}{N} + \boldsymbol{\Delta}\right), \qquad i = 1, 2, \cdots N,$$

where $\mathbf{z} \in \mathbb{Z}^s$ is known as the generating vector, $\boldsymbol{\Delta} \in [0,1]^s$ is the shift, and frac$(\cdot)$ means taking the fractional part of each component in the vector. More details on the general theory and choices of quadrature points for QMC lattice rules for the s-dimensional cube can be found in [25, 26] and references therein.

To measure the error of this method, we need the following weighted and unanchored Sobolev space $\mathcal{W}_{s,\gamma}$ which is a Hilbert space containing functions defined over U, equipped with the norm

$$\begin{aligned}\|F\|^2_{\mathcal{W}_{s,\gamma}} &:= \sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}(\boldsymbol{y}_{\mathfrak{u}}; \boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}) d\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2 d\boldsymbol{y}_{\mathfrak{u}} \\ &= \sum_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{[0,1]^{|\mathfrak{u}|}} \left(\int_{[0,1]^{s-|\mathfrak{u}|}} \partial^{\mathfrak{u}} F(\boldsymbol{y}_{\mathfrak{u}}; \boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}) d\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}\right)^2 d\boldsymbol{y}_{\mathfrak{u}}\end{aligned}$$

where $\{1:s\}$ is a shorthand notation for the set of indices $\{1, 2, \cdots, s\}$, $\frac{\partial^{|\mathfrak{u}|}F}{\partial \boldsymbol{y}_{\mathfrak{u}}}$ and $\partial^{\mathfrak{u}}F$ denote the mixed first derivative with respect to the active variables y_j with $j \in \mathfrak{u}$, and $\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}}$ denotes the inactive variables y_j with $j \notin \mathfrak{u}$. And $\gamma_{\mathfrak{u}} \geq 0$ is a weight parameter associated with each group of variables $\boldsymbol{y}_{\mathfrak{u}}$, with the convention that $\gamma_{\emptyset} = 1$. If $\gamma_{\mathfrak{u}} = 0$, then we expect that the corresponding integral of the mixed first derivative is also zero, and we follow the convention "$0/0 = 0$".

The weighted spaces was first introduced by Sloan and Woźniakowski [27] and later generalized in many papers (e.g., [28, 29]). We now state the essential theorem for QMC error estimate.

Theorem 10. *[48, Theorem 4.1] Let $s, N \in \mathbb{N}$ be given, and assume $F \in \mathcal{W}_{s,\gamma}$ for a particular choice of weights γ. Then a randomly shifted lattice rule can be constructed using a component-by-component algorithm such that the root-mean-square error satisfies: for all*

$\lambda \in (\frac{1}{2}, 1]$,

$$\sqrt{\mathbb{E}_\Delta\left[|I(F) - Q_N(F)|^2\right]} \le \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^\lambda \left(\frac{2\zeta(2\lambda)}{(2\pi^2)^\lambda}\right)^{|\mathfrak{u}|}\right)^{\frac{1}{2\lambda}} \varphi(N)^{-\frac{1}{2\lambda}} \|F\|_{\mathcal{W}_{s,\gamma}} \tag{3.27}$$

where $\mathbb{E}_\Delta[\cdot]$ denotes the expectation with respect to the random shift which is uniformly distributed over Ξ, and φ is the Euler's totient function.

3.3.2 Regularity analysis with respect to the random vector

To obtain the error estimate for the QMC finite element method, we need the regularity estimate for the solution of (2.1)-(2.4) in chapter 2 with respect to the random vector.

First, we have the following energy conservation property.

Theorem 11. *For the solution $(\boldsymbol{E}, \boldsymbol{H})$ of (2.1)-(2.4) and any $m \ge 0$, we have $\forall t \in [0, T]$,*

$$(\|\varepsilon^{\frac{1}{2}} \partial_t^m \boldsymbol{E}\|^2_{L^2(\Omega\times\Xi)} + \|\mu^{\frac{1}{2}} \partial_t^m \boldsymbol{H}\|^2_{L^2(\Omega\times\Xi)})(t) = (\|\varepsilon^{\frac{1}{2}} \partial_t^m \boldsymbol{E}\|^2_{L^2(\Omega\times\Xi)} + \|\mu^{\frac{1}{2}} \partial_t^m \boldsymbol{H}\|^2_{L^2(\Omega\times\Xi)})(0),$$

here and below we denote $\partial_t^m := \partial_{t^m}$ for the m-th derivative with respect to variable t.

Proof. When $m = 0$, the proof is the same as that of [37, Lemma 2.1] even when ε and μ depend on spatial variable $\boldsymbol{x}$.

For any $m \ge 1$, taking the m-th time derivative of (2.1) and (2.2), multiplying the respective result by $\partial_t^m \boldsymbol{E}$ and $\partial_t^m \boldsymbol{H}$, then integrating over Ξ and Ω, and adding the results together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\varepsilon^{\frac{1}{2}} \partial_t^m \boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}} \partial_t^m \boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right) \\
=\ & \int_\Xi \int_\Omega \nabla \times \partial_t^m \boldsymbol{H} \cdot \partial_t^m \boldsymbol{E}\, d\boldsymbol{x} d\boldsymbol{y} - \int_\Xi \int_\Omega \nabla \times \partial_t^m \boldsymbol{E} \cdot \partial_t^m \boldsymbol{H}\, d\boldsymbol{x} d\boldsymbol{y} \\
=\ & -\int_\Xi \int_{\partial\Omega} (\boldsymbol{n} \times \partial_t^m \boldsymbol{E}) \cdot \partial_t^m \boldsymbol{H}\, ds d\boldsymbol{y} \\
=\ & -\int_\Xi \int_{\partial\Omega} \partial_t^m (\boldsymbol{n} \times \boldsymbol{E}) \cdot \partial_{t^m} \boldsymbol{H}\, ds d\boldsymbol{y} = 0,
\end{aligned} \tag{3.28}$$

where we used the PEC boundary condition (2.4) in the last step. Integrating (3.28) from

$t = 0$ to any time t concludes the proof. □

With Theorem 11, we can prove the following bound for the first derivative of the solution with respect to the random vector.

Theorem 12. *Denote the constant* $C_s^{\varepsilon\mu} := |\varepsilon^{-1}\partial_{y_i}^s \varepsilon|_{L^\infty(\Omega\times\Xi)} + |\mu^{-1}\partial_{y_i}^s \mu|_{L^\infty(\Omega\times\Xi)}$ *for any integer* $s \geq 1$*. Then for the solution* $(\boldsymbol{E}, \boldsymbol{H})$ *of (2.1)-(2.4), we have: for any* $t \in [0, T]$ *and* y_i*,*

$$\begin{aligned}
&\left(\left\| \varepsilon^{\frac{1}{2}} \partial_{y_i} \boldsymbol{E} \right\|^2_{L^2(\Omega\times\Xi)} + \left\| \mu^{\frac{1}{2}} \partial_{y_i} \boldsymbol{H} \right\|^2_{L^2(\Omega\times\Xi)} \right)(t) \\
\leq & \left[t C_1^{\varepsilon\mu} \left(\left\| \varepsilon^{\frac{1}{2}} \partial_t \boldsymbol{E} \right\|^2_{L^2(\Omega\times\Xi)} + \left\| \mu^{\frac{1}{2}} \partial_t \boldsymbol{H} \right\|^2_{L^2(\Omega\times\Xi)} \right)(0) \right. \\
& \left. + \left(\left\| \varepsilon^{\frac{1}{2}} \partial_{y_i} \boldsymbol{E} \right\|^2_{L^2(\Omega\times\Xi)} + \left\| \mu^{\frac{1}{2}} \partial_{y_i} \boldsymbol{H} \right\|^2_{L^2(\Omega\times\Xi)} \right)(0) \right] \exp(t C_1^{\varepsilon\mu}).
\end{aligned}$$

Proof. Differentiating (2.1) and (2.2) with respect to y_i, respectively, we have

$$\varepsilon \partial_t \partial_{y_i} \boldsymbol{E} + \partial_{y_i} \varepsilon \cdot \partial_t \boldsymbol{E} = \nabla \times \partial_{y_i} \boldsymbol{H}, \tag{3.29}$$

$$\mu \partial_t \partial_{y_i} \boldsymbol{H} + \partial_{y_i} \mu \cdot \partial_t \boldsymbol{H} = -\nabla \times \partial_{y_i} \boldsymbol{E}. \tag{3.30}$$

Multiplying (3.29) and (3.30) by $\partial_{y_i} \boldsymbol{E}$ and $\partial_{y_i} \boldsymbol{H}$, respectively, then integrating over Ξ and Ω, and adding the resultants together, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\left\| \varepsilon^{\frac{1}{2}} \partial_{y_i} \boldsymbol{E} \right\|^2_{L^2(\Omega\times\Xi)} + \left\| \mu^{\frac{1}{2}} \partial_{y_i} \boldsymbol{H} \right\|^2_{L^2(\Omega\times\Xi)} \right) \\
= & \int_\Xi \int_\Omega (-\partial_{y_i} \varepsilon \cdot \partial_t \boldsymbol{E} + \nabla \times \partial_{y_i} \boldsymbol{H}) \cdot \partial_{y_i} \boldsymbol{E} \; d\boldsymbol{x} d\boldsymbol{y} \\
& - \int_\Xi \int_\Omega (\partial_{y_i} \mu \cdot \partial_t \boldsymbol{H} + \nabla \times \partial_{y_i} \boldsymbol{E}) \cdot \partial_{y_i} \boldsymbol{H} \; d\boldsymbol{x} d\boldsymbol{y} \\
= & - \int_\Xi \int_\Omega \partial_{y_i} \varepsilon \cdot \partial_t \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{E} \; d\boldsymbol{x} d\boldsymbol{y} - \int_\Xi \int_\Omega \partial_{y_i} \mu \cdot \partial_t \boldsymbol{H} \cdot \partial_{y_i} \boldsymbol{H} \; d\boldsymbol{x} d\boldsymbol{y},
\end{aligned} \tag{3.31}$$

where we used integration by parts and the PEC boundary condition (2.4) in the last step, i.e.,

$$\int_\Xi \int_\Omega \nabla \times \partial_{y_i} \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{H} = \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times \partial_{y_i} \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{H} + \int_\Xi \int_\Omega \partial_{y_i} \boldsymbol{E} \cdot \nabla \times \partial_{y_i} \boldsymbol{H} = \int_\Xi \int_\Omega \partial_{y_i} \boldsymbol{E} \cdot \nabla \times \partial_{y_i} \boldsymbol{H}.$$

By the Cauchy-Schwarz inequality, from (3.31) we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{y_i}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{y_i}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\\
&\leq \left|\varepsilon^{-1}\partial_{y_i}\varepsilon\right|_{L^\infty(\Omega\times\Xi)}\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\varepsilon^{\frac{1}{2}}\partial_{y_i}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}\right)\\
&\quad+\left|\mu^{-1}\partial_{y_i}\mu\right|_{L^\infty(\Omega\times\Xi)}\frac{1}{2}\left(\left\|\mu^{\frac{1}{2}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{y_i}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\\
&\leq C_1^{\varepsilon\mu}\cdot\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\\
&+C_1^{\varepsilon\mu}\cdot\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{y_i}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{y_i}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right),
\end{aligned}
$$

where we used the notation of $C_1^{\varepsilon\mu}$ and Theorem 11 in the last step.

Using the Gronwall inequality to the last inequality concludes the proof. □

By the same technique, we can prove the following bound for the higher order derivatives of the solution with respect to the random vector.

Theorem 13. *For the solution $(\boldsymbol{E},\boldsymbol{H})$ of (2.1)-(2.4) and any $|\boldsymbol{m}|\geq 1$, we have: $\forall\, t\in[0,T]$,*

$$
\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(t)\\
&\leq \exp\left(t\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\right)\left[\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right.\\
&\left.+\int_0^t\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)dt\right],
\end{aligned}
$$

where we denote $\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}=\partial_{y_1}^{m_1}\cdots\partial_{y_n}^{m_n}$ and $\binom{\boldsymbol{m}}{\boldsymbol{s}}=\Pi_{j=1}^n\binom{m_j}{s_j}$ for any $\boldsymbol{m}=(m_1,\cdots,m_n)$ and $\boldsymbol{s}=(s_1,\cdots,s_n)$ with m_i and s_i either 0 or 1.

Proof. Taking the $|\boldsymbol{m}|$-th mixed derivative of (2.1) and (2.2) with respect to $y_1,\cdots,y_n$, respectively, we have

$$\varepsilon\partial_t\left(\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}\right) = \nabla\times\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H} - \sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{y}}^{|\boldsymbol{s}|}\varepsilon\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{E}, \tag{3.32}$$

$$\mu\partial_t\left(\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H}\right) = -\nabla\times\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E} - \sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{y}}^{|\boldsymbol{s}|}\mu\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{H}. \tag{3.33}$$

Multiplying (3.32) and (3.33) by $\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}$ and $\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H}$, respectively, then integrating over Ξ and Ω, and adding the resultants together, we easily obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\\
= & -\int_\Xi\int_\Omega\sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{y}}^{|\boldsymbol{s}|}\varepsilon\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{E}\cdot\partial_{\boldsymbol{y}}^{|\boldsymbol{m}|}\boldsymbol{E}\ d\boldsymbol{x}d\boldsymbol{y}\\
& -\int_\Xi\int_\Omega\sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{y}}^{|\boldsymbol{s}|}\mu\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{H}\cdot\partial_{\boldsymbol{\xi}}^{m}\boldsymbol{H}\ d\boldsymbol{x}d\boldsymbol{y}\\
\le & \sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\cdot\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\\
& + \sum_{1\le|\boldsymbol{s}|\le|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\cdot\frac{1}{2}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right),
\end{aligned} \tag{3.34}$$

which, along with the Gronwall inequality, completes the proof. □

Note that Theorem 13 involves the estimate

$$\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} \quad \text{and} \quad \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)},$$

which can be bounded as below.

Theorem 14. *For the solution* $(\boldsymbol{E},\boldsymbol{H})$ *of (2.1)-(2.4) and any* $|\boldsymbol{m}|, n \ge 1$*, we have:* $\forall\ t \in$

$[0,T]$,

$$
\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(t)\\
&\leq \exp\left(t\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\right)\left[\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right.\\
&\left.+\int_0^t\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}C_s^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t^{n+1}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}+\left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t^{n+1}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)dt\right].
\end{aligned}
$$

Proof. Taking the n-th derivative of (3.32) and (3.33) with respect to t, respectively, we have

$$
\varepsilon\partial_t\left(\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{E}\right) = \nabla\times\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{H}-\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{s}|}\varepsilon\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t^{n+1}\boldsymbol{E}, \tag{3.35}
$$

$$
\mu\partial_t\left(\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{H}\right) = -\nabla\times\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{E}-\sum_{1\leq|\boldsymbol{s}|\leq|\boldsymbol{m}|}\binom{\boldsymbol{m}}{\boldsymbol{s}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{s}|}\mu\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|-|\boldsymbol{s}|}\partial_t^{n+1}\boldsymbol{H}. \tag{3.36}
$$

Multiplying (3.35) and (3.36) by $\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{E}$ and $\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\partial_t^n\boldsymbol{H}$, respectively, then following the proof of Theorem 13, we easily conclude the proof. □

Using Theorem 11, and Theorem 14 recursively in Theorem 13, we can see that the higher order derivatives $(\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{E}\|^2_{L^2(\Omega\times\Xi)}+\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{H}\|^2_{L^2(\Omega\times\Xi)})(t)$ can be bounded by a linear combination of the initial values:

$$
\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}(0),\quad \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^{|\boldsymbol{m}|}\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}(0), \tag{3.37}
$$

$$
\left\|\partial_{\boldsymbol{\xi}}^{|\boldsymbol{s}|}\partial_t^l\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)}(0),\quad \left\|\partial_{\boldsymbol{\xi}}^{|\boldsymbol{s}|}\partial_t^l\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}(0),\ \forall\ 1\leq l\leq|\boldsymbol{m}|, 1\leq|\boldsymbol{s}|+l\leq|\boldsymbol{m}|. \tag{3.38}
$$

But an explicit expression for the bound is too complicated to write down due to the recursive dependence. Below we illustrate the exact bound for $m=2$.

Theorem 15. *For the solution $(\boldsymbol{E}, \boldsymbol{H})$ of (2.1)-(2.4) and any $t \in [0, T]$, we have:*

$$\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(t) \\
&\leq \exp(t(2C_1^{\varepsilon\mu} + C_2^{\varepsilon\mu}))\left\{\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right. \\
&+ 2\exp(tC_1^{\varepsilon\mu})\left[\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right. \\
&\left.+tC_1^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_t^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right] \\
&\left.+tC_2^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right\}.
\end{aligned}$$

Proof. Using Theorem 13 for $m = 2$, we have

$$\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(t) \\
\leq\ & \exp(t(2C_1^{\varepsilon\mu} + C_2^{\varepsilon\mu}))\cdot\left\{\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right. \\
&+\int_0^t\left[2C_1^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\right. \\
&\left.\left.+C_2^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)\right]dt\right\}. \qquad (3.39)
\end{aligned}$$

To bound the $\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}$ and $\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}$ terms in (3.39), we use Theorem 14 with $m = n = 1$ to obtain

$$\begin{aligned}
&\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(t) \\
\leq\ & \exp(tC_1^{\varepsilon\mu})\cdot\left[\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right. \\
&\left.+\int_0^t C_1^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_t^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)dt\right] \\
\leq\ & \exp(tC_1^{\varepsilon\mu})\cdot\left[\left(\left\|\varepsilon^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_{\boldsymbol{\xi}}\partial_t\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right. \\
&\left.+tC_1^{\varepsilon\mu}\left(\left\|\varepsilon^{\frac{1}{2}}\partial_t^2\boldsymbol{E}\right\|^2_{L^2(\Omega\times\Xi)} + \left\|\mu^{\frac{1}{2}}\partial_t^2\boldsymbol{H}\right\|^2_{L^2(\Omega\times\Xi)}\right)(0)\right],
\end{aligned}$$

where in the last step we used Theorem 11 with $m = 2$.

Substituting (3.40) into (3.39) and using the following estimates

$$\int_0^t C_1^{\varepsilon\mu} e^{tC_1^{\varepsilon\mu}} dt = e^{tC_1^{\varepsilon\mu}} - 1 \leq e^{tC_1^{\varepsilon\mu}}, \quad \int_0^t tC_1^{\varepsilon\mu} e^{tC_1^{\varepsilon\mu}} dt = te^{tC_1^{\varepsilon\mu}} - \frac{1}{C_1^{\varepsilon\mu}}(e^{tC_1^{\varepsilon\mu}} - 1) \leq te^{tC_1^{\varepsilon\mu}},$$

we conclude the proof. □

3.3.3 The error estimate

In this subsection, we are going to find out the error $G_1(\boldsymbol{E}(\cdot, \boldsymbol{y}) - \boldsymbol{E}_h(\cdot, \boldsymbol{y}))$ and $G_2(\boldsymbol{H}(\cdot, \boldsymbol{y}) - \boldsymbol{H}_h(\cdot, \boldsymbol{y}))$ where $\boldsymbol{y} \in U$ is given and $\boldsymbol{E}_h(\cdot, \boldsymbol{y})$ and $\boldsymbol{H}_h(\cdot, \boldsymbol{y})$ are the finite element solutions of (2.1)-(2.2).

Theorem 16. *Under the same conditions as Theorem 10, we have the following error estimate*

$$\begin{aligned}
&\sqrt{\mathbb{E}_\Delta \left[\left| I(G_1(\boldsymbol{E})) - Q_N(G_1(\boldsymbol{E}_h^k)) \right|^2 + \left| I(G_2(\boldsymbol{H})) - Q_N(G_2(\boldsymbol{H}_h^k)) \right|^2 \right]} \\
&\leq C \left(h^r + \tau^2 + \varphi(M)^{-\frac{1}{2\lambda}} \right),
\end{aligned}$$

where

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p} \right)$$

is the Euler's totient function. Here the product is over the distinct prime numbers dividing n.

***Proof*.** By the triangle inequality, we have

$$\mathbb{E}_\Delta \left[\left| I(G_1(\boldsymbol{E})) - Q_N(G_1(\boldsymbol{E}_h^k)) \right|^2 + \left| I(G_2(\boldsymbol{H})) - Q_N(G_2(\boldsymbol{H}_h^k)) \right|^2 \right] \leq Err_1 + Err_2,$$

where

$$\begin{aligned}
Err_1 &= \mathbb{E}_\Delta \left[|(I - Q_N)(G_1(\boldsymbol{E}))|^2 + |(I - Q_N)(G_2(\boldsymbol{H}))|^2 \right], \\
Err_2 &= \mathbb{E}_\Delta \left[Q_N(G_1(\boldsymbol{E} - \boldsymbol{E}_h^k))^2 + Q_N(G_2(\boldsymbol{H} - \boldsymbol{H}_h^k))^2 \right].
\end{aligned}$$

The first term, which is the statistical error of QMC, can be estimated by Theorem 10. In fact, by Theorem 10 and the derivative estimates given by Theorems 13-14, we have

$$\mathbb{E}_\Delta\left[|(I-Q_N)(G_1(\boldsymbol{E}))|^2\right] \leq C\varphi(M)^{-\frac{1}{\lambda}}\|\boldsymbol{E}\|^2_{\mathcal{W}_{s,\gamma}} \leq C\varphi(M)^{-\frac{1}{\lambda}}.$$

Using the similar estimates for $\boldsymbol{H}$, we have

$$Err_1 \leq C\varphi(M)^{-\frac{1}{\lambda}}.$$

For the second term, we first notice that G_1 and G_2 are bounded on V:

$$|G_1(\boldsymbol{E}-\boldsymbol{E}_h^k)| \leq \|G_1\|_{V^*}\|\boldsymbol{E}-\boldsymbol{E}_h^k\|_V, \qquad |G_2(\boldsymbol{H}-\boldsymbol{H}_h^k)| \leq \|G_2\|_{V^*}\|\boldsymbol{H}-\boldsymbol{H}_h^k\|_V.$$

Applying the property that the QMC quadrature weights $1/N$ are positive and have a sum 1, we obtain

$$\mathbb{E}_\Delta\left[Q_N(G_1(\boldsymbol{E}-\boldsymbol{E}_h^k))^2\right] \leq \mathbb{E}_\Delta\left[Q_N\left(\|G_1\|_{V^*}\|\boldsymbol{E}-\boldsymbol{E}_h^k\|_V\right)^2\right] \leq C\|\boldsymbol{E}-\boldsymbol{E}_h^k\|_V^2 \leq C(\tau^2+h^r)^2.$$

Hence, by Theorem 6 we have

$$Err_2 \leq C(\|\boldsymbol{E}-\boldsymbol{E}_h^k\|_V^2 + \|\boldsymbol{H}-\boldsymbol{H}_h^k\|_V^2) \leq C(\tau^2+h^r)^2$$

Combining the estimates Err_1 and Err_2 together, we complete the proof. □

3.4 Numerical Experiments

In this section, we present three numerical examples to verify our analysis. As shown in Sec. 3.2.1, we will apply the Crank-Nicolson scheme for the TE_z mode, which has unknowns as electric field $\boldsymbol{E} := (E_{x_1}, E_{x_2})$ and magnetic field H, with the lowest order edge element on the triangular mesh. We will compute the sample means by using the simple level Monte Carlo, multi-level Monte Carlo and QMC methods, respectively.

To test the convergence rate with an exact solution, we add additional source terms to the original governing equations. More specifically, we solve the following mixed finite element scheme: for any $k \geq 1$, find $\boldsymbol{E}_h^k := (E_{x_1,h}^k, E_{x_2,h}^k) \in \boldsymbol{V}_h^0$, $H_h^k \in U_h$ such that

$$
\begin{aligned}
(\varepsilon \boldsymbol{E}_h^k, \boldsymbol{\varphi}_h) - \frac{\tau}{2}(H_h^k, \nabla \times \boldsymbol{\varphi}_h) &= (\varepsilon \boldsymbol{E}_h^{k-1}, \boldsymbol{\varphi}_h) + \frac{\tau}{2}(H_h^{k-1}, \nabla \times \boldsymbol{\varphi}_h) + \tau \left(\boldsymbol{f}^{k-\frac{1}{2}}, \boldsymbol{\varphi}_h \right), \\
(\mu H_h^k, \psi_h) + \frac{\tau}{2}(\nabla \times \boldsymbol{E}_h^k, \psi_h) &= (\mu H_h^{k-1}, \psi_h) - \frac{\tau}{2}(\nabla \times \boldsymbol{E}_h^{k-1}, \psi_h) + \tau \left(g^{k-\frac{1}{2}}, \psi_h \right)
\end{aligned}
$$

hold true for any $\boldsymbol{\varphi}_h \in \boldsymbol{V}_h^0$ and $\psi_h \in U_h$, where $\boldsymbol{f}$ and g are the added source terms.

The finite element spaces U_h and $\boldsymbol{V}_h$ on a regular triangular mesh $\mathcal{T}_h$ of the domain $\Omega = [0,1]^2$ are chosen as 2D linear edge element.

As shown in [16], there is a superconvergence at the midpoint of triangle elements' edges for lowest order edge elements. We hence define the $\overline{P}_h u$ as the linear interpolation of u on each element $K_i \in \mathcal{T}_h$ by using the average of u on the midpoints of three edges of K_i. After this post-processing, we define the discretized l^2 error of numerical solutions as

$$
\|\mathbb{E}[u] - E_M[u_h]\|_{l^2(\Omega)}^2 = \sum_{i=1}^{N} \left|\mathbb{E}[u] - \overline{P}_h E_M[u_h]\right|^2 |K_i|
$$

where N is the partition number of Ω.

3.4.1 Single level Monte Carlo Method

For this example, we adopt the following random coefficients and exact solutions: for any $t \in (0,1]$,

$$
\begin{aligned}
\varepsilon(\boldsymbol{x}, \boldsymbol{y}) &= 1 + 0.01(y_1 x_1 + y_2 x_2 + y_3 x_1^2 + y_4 x_2^2 + y_5 x_1 x_2 + y_6 x_1^3), \\
\mu(\boldsymbol{x}, \boldsymbol{y}) &= 1 + 0.01(y_1 x_2 + y_2 x_1 + y_3 x_2^2 + y_4 x_1^2 + y_5 x_1 x_2 + y_6 x_2^3), \\
E_x(t, \boldsymbol{x}, \boldsymbol{y}) &= \sin(\pi x_1)\cos(\pi x_2) e^{-\pi t}(\varepsilon + 2\mu), \\
E_y(t, \boldsymbol{x}, \boldsymbol{y}) &= -\cos(\pi x_1)\sin(\pi x_2) e^{-\pi t}(2\varepsilon + \mu), \\
H(t, \boldsymbol{x}, \boldsymbol{y}) &= \sin(\pi x_1)\sin(\pi x_2) e^{-\pi t}(\varepsilon - 2\mu),
\end{aligned}
$$

Table 3.1: Errors of (E_{x_1}, E_{x_2}, H) obtained by the single level Monte Carlo method with the lowest edge element

Mesh	$\|\mathbb{E}[\boldsymbol{E}(T)] - E_M[\boldsymbol{E}_h^k]\|_{l^2(\Omega)}$	Rate	$\|\mathbb{E}[H(T)] - E_M[H_h^k]\|_{l^2(\Omega)}$	Rate	CPU time(s)
$N=2$	$2.025681E-01$	-	$9.855452E-01$	-	0.57096
$N=4$	$8.016480E-02$	1.9413	$2.456558E-01$	1.9868	13.29811
$N=8$	$1.846385E-02$	2.0869	$6.260263E-02$	1.9837	540.99794
$N=16$	$4.241596E-03$	2.0654	$1.580314E-02$	1.9891	43986.39742
$N=32$	$1.053961E-03$	2.0088	$3.972438E-03$	1.9921	258671.07102

where $\boldsymbol{x} = (x_1, x_2) \in \Omega$ and $\boldsymbol{y} = (y_1, \cdots, y_6) \in \Xi = [0,1]^6$, i.e., y_i are uniformly distributed random variables. The source functions $\boldsymbol{f}$ and g are obtained by plugging the exact solutions E_x, E_y, H into the governing equations.

As shown in theorem 8 To test the convergence rate, we set the number of samples for Monte Carlo test as $M = N^4$, where N is the number of partition Ω in both x and y-direction, i.e., we first partition Ω into $N \times N$ rectangles, then partition each rectangle into two triangles by connecting the diagonals. The total number of time steps is chosen as N also. All the numerical tests have been carried out by using the FEniCS package on a 2017 MacBook Pro laptop with a 2.8 GHz Intel Core i7 processor and a memory of 16 GB 2133 MHz LPDDR3. The discrete $l^2(\Omega)$ errors between the expectation of exact solution and the sample mean of numerical solution at the final time $T = 1$ is computed to check our theoretical convergence rate given in Theorem 8.

Table 3.1 shows clearly that both the errors of $\boldsymbol{E}$ and H are second order, which is due to the superconvergence phenomenon obtained for the lowest order triangular edge element [30]. Note that the finest mesh numerical test needs to solve the problem $32^4 = 1,048,576$ times, which takes about 72 hours, which shows that the simple level Monte Carlo method is impractically slow. Later, we will show that the multi-level Monte Carlo and QMC methods are much more effcient than the simple level Monte Carlo method.

To further confirm our theoretical analysis, we resolve this example by using one order higher basis functions, i.e., a second order edge element for the electric field and linear Lagrange element for the magnetic field. The numerical results are presented in Table 3.2, which clearly shows the second order error estimate for both $\boldsymbol{E}$ and H. This is consistent

Table 3.2: Errors of (E_{x_1}, E_{x_2}, H) obtained by the single level Monte Carlo method with the second order edge element

Mesh	$\|\mathbb{E}[\boldsymbol{E}(T)] - E_M[\boldsymbol{E}_h^k]\|_{l^2(\Omega)}$	Rate	$\|\mathbb{E}[H(T)] - E_M[H_h^k]\|_{l^2(\Omega)}$	Rate	CPU time(s)
$N=2$	$1.467381E-01$	-	$5.914026E-01$	-	0.53130
$N=4$	$3.563603E-02$	2.0418	$1.550685E-01$	1.9312	15.71102
$N=8$	$9.159853E-03$	2.0009	$3.790434E-02$	1.9819	733.77024
$N=16$	$2.232569E-03$	2.0075	$9.409349E-03$	1.9954	67682.74943

Table 3.3: Errors of (E_{x_1}, E_{x_2}, H) obtained by the multi-level Monte Carlo method

Mesh	$\|\mathbb{E}[\boldsymbol{E}(T)] - E_M[\boldsymbol{E}_h^k]\|_{l^2(\Omega)}$	Rate	$\|\mathbb{E}[H(T)] - E_M[H_h^k]\|_{l^2(\Omega)}$	Rate	CPU time(s)
$L=1$	$8.022609E-02$	-	$2.464436E-01$	-	0.61908
$L=2$	$1.866771E-02$	1.8555	$6.106013E-02$	2.0151	23.07685
$L=3$	$4.516066E-03$	1.7770	$1.496582E-02$	2.0165	437.75725
$L=4$	$1.758842E-03$	1.7113	$3.842888E-03$	2.0196	27501.86158
$L=5$	$4.211790E-04$	2.0621	$9.102927E-04$	2.0778	69150.66038

with Theorem 8.

3.4.2 Multi-level Monte Carlo Method

We repeat the last numerical example by using the multi-level Monte Carlo method analyzed in Sec. 3.2.3 and compute the sample mean by the telescope series of (3.26). At level l of each numerical test, we set $h_l = \tau_l = \frac{1}{2}^l$ and $M_l = 16^{L-l}l^2$, where $l = 1 \ldots L$.

As we can see from Table 3.3, the errors are still second order, which verifies Theorem 9. For the finest mesh case ($L = 5$), it requests total $\sum_{l=1}^{5} M_l = 84,510$ Monte Carlo tests. Compared to the single level method, this saves a lot in the computational time as shown in Table 3.3.

3.4.3 The QMC method

This test is used to verify Theorem 16. We used the shifted lattice rule to generate the quasi random sequence on $[0,1]^6$. Since the theoretical convergent results of QMC in this case is of $O(M^{-1+\epsilon_m})$ where $0 < \epsilon_m \ll 1$ (see [24]), we just take the total QMC test times $M = N^2$ for each test, where N is the spatial and temporal partition number. As shown in Table 3.4, both the convergent rates of $\boldsymbol{E}$ and H are about second order. Note that the QMC sample

Table 3.4: Errors of (E_{x_1}, E_{x_2}, H) obtained by the QMC method

Mesh	$\left\|\mathbb{E}[\boldsymbol{E}(T)] - E_M[\boldsymbol{E}_h^k]\right\|_{l^2(\Omega)}$	Rate	$\left\|\mathbb{E}[H(T)] - E_M[H_h^k]\right\|_{l^2(\Omega)}$	Rate	CPU time(s)
$N = 2$	$2.022678E - 01$	-	$9.852111E - 01$	-	0.13250
$N = 4$	$8.019398E - 02$	1.9454	$2.455848E - 01$	1.9872	0.71975
$N = 8$	$1.843664E - 02$	2.0936	$6.260027E - 02$	1.9844	8.207584
$N = 16$	$4.244253E - 03$	2.0758	$1.580382E - 02$	1.9903	165.89224
$N = 32$	$1.037406E - 03$	2.0325	$3.965429E - 03$	1.9947	4970.95464

for the finest temporal and spatial mesh is only $M = 32^2 = 1,024$, which is much lower than both the single level and multi-level Monte Carlo methods.

CHAPTER 4

STOCHASTIC COLLOCATION METHODS FOR METAMATERIAL MAXWELL'S EQUATIONS WITH RANDOM INPUTS

4.1 Introduction

So far, we studied the stochastic Galerkin method and Quasi Monte Carlo method for standard Maxwell's equations (2.1)-(2.2).

By taking advantage of the strength of Monte Carlo methods and the stochastic Galerkin methods, the stochastic collocation method (cf. [31, 5]) achieves fast convergence when the solutions are sufficiently smooth in the random space. More importantly, the stochastic collocation method is simple in implementation and the system of resulting equations is decoupled and hence is efficient to solve. The stochastic collocation method have been widely used to solve various problems, such as elliptic problems [32], hyperbolic equations [33]. More details can be found in recent review articles [32, 34] and monographs [35, 36, 14].

Compared to many papers published for other problems, there are not many existing works on numerical methods for solving stochastic Maxwell's equations in stochastic collocation method. In this chapter, we will shift our focus from standard Maxwell's equations to Drude's model, which is an extension of Maxwell's equations in metamaterial. We will solve stochastic Drude model by using stochastic collocation method.

The rest of this chapter is organized as follows. In section 4.2, we first present detailed regularity analysis of the metamaterial Maxwell's equations with respect to random variables. Then we establish the convergence analysis for the stochastic collocation method developed to solving this model. Numerical results are presented in section 4.3 to support our theoretical analysis. This chapter is based on my published paper [37, 38].

4.2 Maxwell's equations in metamaterial with random coefficients

By using the same setup and notations in chapter 2, we consider the following Maxwell's equations in metamaterial [16]

$$\varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}) = \nabla\times\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{J}(t,\boldsymbol{x},\boldsymbol{y}), \tag{4.1}$$

$$\mu(\boldsymbol{x},\boldsymbol{y})\partial_t\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}) = -\nabla\times\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{K}(t,\boldsymbol{x},\boldsymbol{y}), \tag{4.2}$$

$$\partial_t\boldsymbol{J}(t,\boldsymbol{x},\boldsymbol{y}) + \Gamma_e(\boldsymbol{x},\boldsymbol{y})\boldsymbol{J}(t,\boldsymbol{x},\boldsymbol{y}) = \varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}), \tag{4.3}$$

$$\partial_t\boldsymbol{K}(t,\boldsymbol{x},\boldsymbol{y}) + \Gamma_m(\boldsymbol{x},\boldsymbol{y})\boldsymbol{K}(t,\boldsymbol{x},\boldsymbol{y}) = \mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}), \tag{4.4}$$

where $\boldsymbol{J}(t,\boldsymbol{x},\boldsymbol{y})$ is the electric current density, $\boldsymbol{K}(t,\boldsymbol{x},\boldsymbol{y})$ is the magnetic current density, and $\boldsymbol{y} = (y_1, y_2, \cdots, y_d)^\top$. The system (4.1)-(4.4) are subjected to the random initial conditions

$$\boldsymbol{E}(0,\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{H}(0,\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}), \tag{4.5}$$

$$\boldsymbol{J}(0,\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{J}_0(\boldsymbol{x},\boldsymbol{y}), \quad \boldsymbol{K}(0,\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{K}_0(\boldsymbol{x},\boldsymbol{y}), \tag{4.6}$$

and the PEC boundary condition:

$$\boldsymbol{n}\times\boldsymbol{E} = \boldsymbol{0}, \quad \text{on } \partial\Omega, \tag{4.7}$$

where $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{J}_0$ and $\boldsymbol{K}_0$ are some given functions. To accommodate the uncertainty or randomness of the material, we assume that the permittivity ε, permeability μ, electric plasma frequency ω_{pe}, magnetic plasma frequency ω_{pm}, electric damping frequency Γ_{pe} and magnetic damping frequency Γ_{pm} are all random. Here and below, $\boldsymbol{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$, where $\Omega \subset \mathbb{R}^3$ is a bounded polyhedral domain with a Lipschitz boundary. For simplicity, we denote ∂_{s^j} the jth derivative with respect to variable s, e.g., $s = t$ or y_i. We like to emphasize that here and below ∇ is only for spatial variable $\boldsymbol{x}$.

To solve problem (4.1)-(4.7), we use the Lagrange interpolation approach by following

[8, 39, 40]. We first choose a set of Gauss-Lobatto collocation points $\{\boldsymbol{y}_k\}_{k=1}^{(N+1)^d} \in \Xi$, where $N+1$ denotes the number of collocation points in each random variable space. We then solve the following system of equations at each collocation point $\boldsymbol{y}_j, j = 1, \cdots, (N+1)^d$:

$$\varepsilon(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t \widehat{\boldsymbol{E}}(t, \boldsymbol{x}, \boldsymbol{y}_j) = \nabla \times \widehat{\boldsymbol{H}}(t, \boldsymbol{x}, \boldsymbol{y}_j) - \widehat{\boldsymbol{J}}(t, \boldsymbol{x}, \boldsymbol{y}_j), \tag{4.8}$$

$$\mu(\boldsymbol{x}, \boldsymbol{y}_j)\partial_t \widehat{\boldsymbol{H}}(t, \boldsymbol{x}, \boldsymbol{y}_j) = -\nabla \times \widehat{\boldsymbol{E}}(t, \boldsymbol{x}, \boldsymbol{y}_j) - \widehat{\boldsymbol{K}}(t, \boldsymbol{x}, \boldsymbol{y}_j), \tag{4.9}$$

$$\partial_t \widehat{\boldsymbol{J}}(t, \boldsymbol{x}, \boldsymbol{y}_j) + \Gamma_e(\boldsymbol{x}, \boldsymbol{y}_j)\widehat{\boldsymbol{J}}(t, \boldsymbol{x}, \boldsymbol{y}_j) = \varepsilon(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y}_j)\widehat{\boldsymbol{E}}(t, \boldsymbol{x}, \boldsymbol{y}_j), \tag{4.10}$$

$$\partial_t \widehat{\boldsymbol{K}}(t, \boldsymbol{x}, \boldsymbol{y}_j) + \Gamma_m(\boldsymbol{x}, \boldsymbol{y}_j)\widehat{\boldsymbol{K}}(t, \boldsymbol{x}, \boldsymbol{y}_j) = \mu(\boldsymbol{x}, \boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y}_j)\widehat{\boldsymbol{H}}(t, \boldsymbol{x}, \boldsymbol{y}_j), \tag{4.11}$$

subject to the initial conditions

$$\widehat{\boldsymbol{E}}(0, \boldsymbol{x}, \boldsymbol{y}_j) = \boldsymbol{E}_0(\boldsymbol{x}, \boldsymbol{y}_j), \quad \widehat{\boldsymbol{H}}(0, \boldsymbol{x}, \boldsymbol{y}_j) = \boldsymbol{H}_0(\boldsymbol{x}, \boldsymbol{y}_j), \tag{4.12}$$

$$\widehat{\boldsymbol{J}}(0, \boldsymbol{x}, \boldsymbol{y}_j) = \boldsymbol{J}_0(\boldsymbol{x}, \boldsymbol{y}_j), \quad \widehat{\boldsymbol{K}}(0, \boldsymbol{x}, \boldsymbol{y}_j) = \boldsymbol{K}_0(\boldsymbol{x}, \boldsymbol{y}_j), \tag{4.13}$$

and the PEC boundary condition:

$$\boldsymbol{n} \times \widehat{\boldsymbol{E}}(t, \boldsymbol{x}, \boldsymbol{y}_j) = \boldsymbol{0}, \quad \text{on} \quad \partial\Omega, \tag{4.14}$$

i.e., we can simply denote the approximate solution as

$$\boldsymbol{E}^N(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \widehat{\boldsymbol{E}}(t, \boldsymbol{x}, \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \ \boldsymbol{H}^N(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \widehat{\boldsymbol{H}}(t, \boldsymbol{x}, \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \tag{4.15}$$

$$\boldsymbol{J}^N(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \widehat{\boldsymbol{J}}(t, \boldsymbol{x}, \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \ \boldsymbol{K}^N(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{(N+1)^d} \widehat{\boldsymbol{K}}(t, \boldsymbol{x}, \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y}), \tag{4.16}$$

where $\mathcal{L}_k(\boldsymbol{y})$ are the tensor-product Lagrange interpolation polynomials. In Remark 4, we show that $u^N(t, \boldsymbol{x}, \boldsymbol{y})$ is just the interpolation of u, denoted as $I_N^y u = \sum_{k=1}^{(N+1)^d} u(t, \boldsymbol{x}, \boldsymbol{y}_k)\mathcal{L}_k(\boldsymbol{y})$, where $u = \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K}$.

To prove the convergence rate of this scheme, we first need to establish the regularity for the solution of our model problem (4.1)-(4.7). To simplify the notation and make the proof clear, sometimes we drop the explicit dependence of all physical parameters on $\boldsymbol{x}$ and $\boldsymbol{y}$.

Remark 4. *To justify that* $\boldsymbol{E}^N(t,\boldsymbol{x},\boldsymbol{y}) = I_N^y\boldsymbol{E}$*,* $\boldsymbol{H}^N(t,\boldsymbol{x},\boldsymbol{y}) = I_N^y\boldsymbol{H}$*,* $\boldsymbol{J}^N(t,\boldsymbol{x},\boldsymbol{y}) = I_N^y\boldsymbol{J}$ *and* $\boldsymbol{K}^N(t,\boldsymbol{x},\boldsymbol{y}) = I_N^y\boldsymbol{K}$*, we denote the errors*

$$
\begin{aligned}
\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \boldsymbol{E}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y}_j), \quad \widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) &= \boldsymbol{H}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y}_j),\\
\widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \boldsymbol{J}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \boldsymbol{J}(t,\boldsymbol{x},\boldsymbol{y}_j), \quad \widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) &= \boldsymbol{K}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \boldsymbol{K}(t,\boldsymbol{x},\boldsymbol{y}_j).
\end{aligned}
$$

Choosing $\boldsymbol{y} = \boldsymbol{y}_j$ *in (4.1)-(4.7) and subtracting the resultants from the corresponding equations of (4.8)-(4.14), we can see that* $\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$*,* $\widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$*,* $\widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$ *and* $\widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$ *satisfy the following equations:*

$$
\varepsilon(\boldsymbol{x},\boldsymbol{y}_j)\partial_t\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \nabla\times\widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j), \tag{4.17}
$$

$$
\mu(\boldsymbol{x},\boldsymbol{y}_j)\partial_t\widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = -\nabla\times\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) - \widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j), \tag{4.18}
$$

$$
\partial_t\widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) + \Gamma_e(\boldsymbol{x},\boldsymbol{y}_j)\widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \varepsilon(\boldsymbol{x},\boldsymbol{y}_j)\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y}_j)\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j), \tag{4.19}
$$

$$
\partial_t\widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) + \Gamma_m(\boldsymbol{x},\boldsymbol{y}_j)\widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \mu(\boldsymbol{x},\boldsymbol{y}_j)\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y}_j)\widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j), \tag{4.20}
$$

subject to the zero initial conditions

$$
\widehat{\boldsymbol{E}}^N(0,\boldsymbol{x},\boldsymbol{y}_j) = \widehat{\boldsymbol{H}}^N(0,\boldsymbol{x},\boldsymbol{y}_j) = 0, \tag{4.21}
$$

$$
\widehat{\boldsymbol{J}}^N(0,\boldsymbol{x},\boldsymbol{y}_j) = \widehat{\boldsymbol{K}}^N(0,\boldsymbol{x},\boldsymbol{y}_j) = 0, \tag{4.22}
$$

and the PEC boundary condition:

$$
\boldsymbol{n}\times\widehat{\boldsymbol{E}}^N(t,\boldsymbol{x},\boldsymbol{y}_j) = \boldsymbol{0}, \quad \text{on} \ \ \partial\Omega. \tag{4.23}
$$

Multiplying (4.17)-(4.20) by $\widehat{\boldsymbol{E}}^N(0,\boldsymbol{x},\boldsymbol{y}_j)$*,* $\widehat{\boldsymbol{H}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$*,* $\widehat{\boldsymbol{J}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$ *and* $\widehat{\boldsymbol{K}}^N(t,\boldsymbol{x},\boldsymbol{y}_j)$*,* *respectively, and integrating over Ω, we can easily see that (cf. proof of Lemma 3.12 in [16]):*

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\sqrt{\varepsilon}\widehat{\boldsymbol{E}}^N\right\|^2_{L^2(\Omega)} + \left\|\sqrt{\mu}\widehat{\boldsymbol{H}}^N\right\|^2_{L^2(\Omega)} + \left\|\frac{\widehat{\boldsymbol{J}}^N}{\sqrt{\varepsilon}\omega_{pe}}\right\|^2_{L^2(\Omega)} + \left\|\frac{\widehat{\boldsymbol{K}}^N}{\sqrt{\mu}\omega_{pm}}\right\|^2_{L^2(\Omega)}\right)(y_j)$$
$$+\int_\Omega \left(\frac{\Gamma_e}{\varepsilon\omega_{pe}^2}\left|\widehat{\boldsymbol{J}}^N\right|^2 + \frac{\Gamma_m}{\mu\omega_{pm}^2}\left|\widehat{\boldsymbol{K}}^N\right|^2\right) = 0. \tag{4.24}$$

Integrating (4.24) from $t = 0$ to t and using the zero initial conditions, we easily have

$$\frac{1}{2}\left(\left\|\sqrt{\varepsilon}\widehat{\boldsymbol{E}}^N\right\|^2_{L^2(\Omega)} + \left\|\sqrt{\mu}\widehat{\boldsymbol{H}}^N\right\|^2_{L^2(\Omega)} + \left\|\frac{\widehat{\boldsymbol{J}}^N}{\sqrt{\varepsilon}\omega_{pe}}\right\|^2_{L^2(\Omega)} + \left\|\frac{\widehat{\boldsymbol{K}}^N}{\sqrt{\mu}\omega_{pm}}\right\|^2_{L^2(\Omega)}\right)(t, y_j) \le 0,$$

which leads to

$$\widehat{\boldsymbol{E}}^N(0, \boldsymbol{x}, \boldsymbol{y}_j) = \widehat{\boldsymbol{H}}^N(0, \boldsymbol{x}, \boldsymbol{y}_j) = \widehat{\boldsymbol{J}}^N(0, \boldsymbol{x}, \boldsymbol{y}_j) = \widehat{\boldsymbol{K}}^N(0, \boldsymbol{x}, \boldsymbol{y}_j) = 0.$$

These justify that $\boldsymbol{E}^N(t, \boldsymbol{x}, \boldsymbol{y}) = I_N^y \boldsymbol{E}$, $\boldsymbol{H}^N(t, \boldsymbol{x}, \boldsymbol{y}) = I_N^y \boldsymbol{H}$, $\boldsymbol{J}^N(t, \boldsymbol{x}, \boldsymbol{y}) = I_N^y \boldsymbol{J}$ and $\boldsymbol{K}^N(t, \boldsymbol{x}, \boldsymbol{y}) = I_N^y \boldsymbol{K}$.

4.2.1 Regularity analysis

Lemma 17. *For problem (4.1)-(4.7) and any $t \in [0, T]$, we have*

$$\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon(\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{E}|^2 + \mu(\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{H}|^2 + \frac{1}{\varepsilon(\boldsymbol{x}, \boldsymbol{y})\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y})}|\boldsymbol{J}|^2 \right.$$
$$\left. + \frac{1}{\mu(\boldsymbol{x}, \boldsymbol{y})\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y})}|\boldsymbol{K}|^2\right)(t) d\boldsymbol{x} d\boldsymbol{y}$$
$$\le \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon(\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{E}_0|^2 + \mu(\boldsymbol{x}, \boldsymbol{y})|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon(\boldsymbol{x}, \boldsymbol{y})\omega_{pe}^2(\boldsymbol{x}, \boldsymbol{y})}|\boldsymbol{J}_0|^2 \right.$$
$$\left. + \frac{1}{\mu(\boldsymbol{x}, \boldsymbol{y})\omega_{pm}^2(\boldsymbol{x}, \boldsymbol{y})}|\boldsymbol{K}_0|^2\right) d\boldsymbol{x} d\boldsymbol{y}.$$

Proof. Multiplying (4.1)-(4.4) by $2\rho(\boldsymbol{y})\boldsymbol{E}, 2\rho(\boldsymbol{y})\boldsymbol{H}, 2\rho(\boldsymbol{y})\boldsymbol{J}$ and $2\rho(\boldsymbol{y})\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\left(\varepsilon(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{E}|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{H}|^2+\frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}|^2\right.\\
&\quad\left.+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}|^2\right)d\boldsymbol{x}d\boldsymbol{y}\\
&\quad+\int_{\Xi}\int_{\Omega}2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{J}|^2+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\boldsymbol{K}|^2\right)d\boldsymbol{x}d\boldsymbol{y}=0,
\end{aligned} \tag{4.25}$$

where in the last step we used the PEC boundary condition (4.7) in the following identity

$$\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\boldsymbol{E}\cdot\nabla\times\boldsymbol{H}-\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\boldsymbol{H}\cdot\nabla\times\boldsymbol{E}=-\int_{\Xi}\int_{\partial\Omega}\rho(\boldsymbol{y})\boldsymbol{H}\cdot\boldsymbol{n}\times\boldsymbol{E}=0.$$

Integrating (4.25) with respect to t from $t=0$ to t concludes the proof. □

Lemma 18. *Denote*

$$C_{max1}:=2\max_{\overline{\Omega}\times\overline{\Xi}}\left(\frac{1}{\varepsilon\mu},\omega_{pe}^2+\Gamma_e^2,\omega_{pm}^2+\Gamma_m^2,\frac{1}{\varepsilon^2\omega_{pe}^2},\frac{1}{\mu^2\omega_{pm}^2}\right).$$

Then for problem (4.1)-(4.7) and any $t\in[0,T]$, we have

$$\begin{aligned}
&\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\left(\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2+\frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{J}|^2\right.\\
&\left.+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}\\
\le{}& C_{max1}\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\left(\mu|\nabla\times\boldsymbol{H}_0|^2+\varepsilon|\nabla\times\boldsymbol{E}_0|^2+\varepsilon|\boldsymbol{E}_0|^2+\mu|\boldsymbol{H}_0|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.\\
&\left.+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)d\boldsymbol{x}d\boldsymbol{y}.
\end{aligned}$$

Proof. Taking the time derivative of (4.1)-(4.4), we obtain

$$\varepsilon(\boldsymbol{x},\boldsymbol{y})\partial_t(\partial_t\boldsymbol{E}) = \nabla\times(\partial_t\boldsymbol{H}) - \partial_t\boldsymbol{J}, \tag{4.26}$$

$$\mu(\boldsymbol{x},\boldsymbol{y})\partial_t(\partial_t\boldsymbol{H}) = -\nabla\times(\partial_t\boldsymbol{E}) - \partial_t\boldsymbol{K}, \tag{4.27}$$

$$\frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}\partial_t(\partial_t\boldsymbol{J}) + \frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}\partial_t\boldsymbol{J} = \partial_t\boldsymbol{E}, \tag{4.28}$$

$$\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}\partial_t(\partial_t\boldsymbol{K}) + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}\partial_t\boldsymbol{K} = \partial_t\boldsymbol{H}. \tag{4.29}$$

Multiplying (4.26)-(4.29) by $2\rho(\boldsymbol{y})\partial_t\boldsymbol{E}, 2\rho(\boldsymbol{y})\partial_t\boldsymbol{H}, 2\rho(\boldsymbol{y})\partial_t\boldsymbol{J}$ and $2\rho(\boldsymbol{y})\partial_t\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2 + \mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}|^2\right.\\
&\left.+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2\right) d\boldsymbol{x}d\boldsymbol{y}\\
&\quad+\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_t\boldsymbol{K}|^2\right) d\boldsymbol{x}d\boldsymbol{y}\\
= &\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\left[\nabla\times(\partial_t\boldsymbol{H})\cdot\partial_t\boldsymbol{E} - \nabla\times(\partial_t\boldsymbol{E})\cdot\partial_t\boldsymbol{H}\right]\\
= &-\int_\Xi\int_{\partial\Omega} 2\rho(\boldsymbol{y})\boldsymbol{n}\times(\partial_t\boldsymbol{E})\cdot\partial_t\boldsymbol{H} = 0,
\end{aligned} \tag{4.30}$$

where in the last step we used integration by parts and the PEC boundary condition (4.7).

Integrating (4.30) with respect to t from $t = 0$ to t, then using the governing equations (4.1)-(4.4) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}|^2+\frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}|^2+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}\\
\le\ &\int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{E}_0|^2+\mu(\boldsymbol{x},\boldsymbol{y})|\partial_t\boldsymbol{H}_0|^2+\frac{1}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2}|\partial_t\boldsymbol{J}_0|^2+\frac{1}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2}|\partial_t\boldsymbol{K}_0|^2\right)d\boldsymbol{x}d\boldsymbol{y}\\
\le\ &\int_\Xi\int_\Omega \rho\left[\frac{2}{\varepsilon}(|\nabla\times\boldsymbol{H}_0|^2+|\boldsymbol{J}_0|^2)+\frac{2}{\mu}(|\nabla\times\boldsymbol{E}_0|^2+|\boldsymbol{K}_0|^2)\right.\\
&\quad\left.+\frac{2}{\varepsilon\omega_{pe}^2}(\Gamma_e^2|\boldsymbol{J}_0|^2+|\boldsymbol{E}_0|^2)+\frac{2}{\mu\omega_{pm}^2}(\Gamma_m^2|\boldsymbol{K}_0|^2+|\boldsymbol{H}_0|^2)\right]d\boldsymbol{x}d\boldsymbol{y}\\
=\ &\int_\Xi\int_\Omega \rho\left[\frac{2}{\varepsilon\mu}(\mu|\nabla\times\boldsymbol{H}_0|^2+\varepsilon|\nabla\times\boldsymbol{E}_0|^2)+2(\omega_{pe}^2+\Gamma_e^2)\cdot\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.\\
&\quad\left.+2(\omega_{pm}^2+\Gamma_m^2)\cdot\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2+\frac{2}{\varepsilon^2\omega_{pe}^2}\cdot\varepsilon|\boldsymbol{E}_0|^2+\frac{2}{\mu^2\omega_{pm}^2}\cdot\mu|\boldsymbol{H}_0|^2\right]d\boldsymbol{x}d\boldsymbol{y}\\
\le\ &C_{max1}\int_\Xi\int_\Omega\rho\left(\mu|\nabla\times\boldsymbol{H}_0|^2+\varepsilon|\nabla\times\boldsymbol{E}_0|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.\\
&\left.+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2+\varepsilon|\boldsymbol{E}_0|^2+\mu|\boldsymbol{H}_0|^2\right)d\boldsymbol{x}d\boldsymbol{y}, \qquad (4.31)
\end{aligned}$$

which concludes the proof. □

In the rest of the paper, we will use the following Gronwall inequality a lot.

Lemma 19. *If Q(t) satisfies* $\frac{dQ}{dt}\le c_0Q+d_0$ *for some constant* $c_0\neq 0$ *and* d_0*, then we have*

$$Q(t)\le e^{c_0t}\left(Q(0)+\frac{d_0}{c_0}\right),\quad \forall\ t\ge 0.$$

Theorem 20. *Denote constant* C_1*:*

$$C_1=\max_{\overline{\Omega}\times\overline{\Xi}}\left(\left|\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|+|\partial_{y_i}\Gamma_e|,\left|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|+|\partial_{y_i}\Gamma_m|,\left|\frac{\partial_{y_i}\varepsilon}{\varepsilon}\right|,\left|\frac{\partial_{y_i}\mu}{\mu}\right|\right).$$

Then for any $t \in [0,T]$ and $i = 1, \cdots, d$, we have

$$\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\partial_{y_i} \boldsymbol{E}|^2 + \mu |\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\partial_{y_i} \boldsymbol{K}|^2 \right)(t) d\boldsymbol{x} d\boldsymbol{y}$$
$$\leq e^{C_1 t} \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\partial_{y_i} \boldsymbol{E}|^2 + \mu |\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\partial_{y_i} \boldsymbol{K}|^2 \right)(0) d\boldsymbol{x} d\boldsymbol{y}$$
$$+ e^{C_1 t}(1 + C_{max1}) \int_\Xi \int_\Omega \rho \left(\varepsilon |\nabla \times \boldsymbol{E}_0|^2 + \mu |\nabla \times \boldsymbol{H}_0|^2 + \varepsilon |\boldsymbol{E}_0|^2 + \mu |\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\boldsymbol{J}_0|^2 \right.$$
$$\left. + \frac{1}{\mu \omega_{pm}^2} |\boldsymbol{K}_0|^2 \right).$$

Proof. Differentiating (4.1)-(4.4) with respect to any y_i $(i = 1, \cdots, d)$, we obtain

$$\varepsilon \partial_t (\partial_{y_i} \boldsymbol{E}) - \nabla \times (\partial_{y_i} \boldsymbol{H}) + \partial_{y_i} \boldsymbol{J} = -\partial_{y_i} \varepsilon \partial_t \boldsymbol{E}, \tag{4.32}$$
$$\mu \partial_t (\partial_{y_i} \boldsymbol{H}) + \nabla \times (\partial_{y_i} \boldsymbol{E}) + \partial_{y_i} \boldsymbol{K} = -\partial_{y_i} \mu \partial_t \boldsymbol{H}, \tag{4.33}$$
$$\partial_t (\partial_{y_i} \boldsymbol{J}) + \Gamma_e \partial_{y_i} \boldsymbol{J} - \varepsilon \omega_{pe}^2 \partial_{y_i} \boldsymbol{E} = \partial_{y_i} (\varepsilon \omega_{pe}^2) \boldsymbol{E} - \partial_{y_i} \Gamma_e \boldsymbol{J}, \tag{4.34}$$
$$\partial_t (\partial_{y_i} \boldsymbol{K}) + \Gamma_m \partial_{y_i} \boldsymbol{K} - \mu \omega_{pm}^2 \partial_{y_i} \boldsymbol{H} = \partial_{y_i} (\mu \omega_{pm}^2) \boldsymbol{H} - \partial_{y_i} \Gamma_m \boldsymbol{K}. \tag{4.35}$$

Multiplying (4.32)-(4.35) by $2\rho(\boldsymbol{y})\partial_{y_i}\boldsymbol{E}, 2\rho(\boldsymbol{y})\partial_{y_i}\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\varepsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\frac{d}{dt} \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\partial_{y_i} \boldsymbol{E}|^2 + \mu |\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\partial_{y_i} \boldsymbol{K}|^2 \right) d\boldsymbol{x} d\boldsymbol{y}$$
$$+ \int_\Xi \int_\Omega 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})} |\partial_{y_i} \boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})} |\partial_{y_i} \boldsymbol{K}|^2 \right) d\boldsymbol{x} d\boldsymbol{y}$$
$$= \int_\Xi \int_\Omega 2\rho(\boldsymbol{y}) \left[-\partial_{y_i} \varepsilon \cdot \partial_t \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{E} - \partial_{y_i} \mu \cdot \partial_t \boldsymbol{H} \cdot \partial_{y_i} \boldsymbol{H} + \frac{\partial_{y_i}(\varepsilon \omega_{pe}^2)}{\varepsilon \omega_{pe}^2} \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{J} \right.$$
$$\left. - \frac{\partial_{y_i} \Gamma_e}{\varepsilon \omega_{pe}^2} \boldsymbol{J} \cdot \partial_{y_i} \boldsymbol{J} + \frac{\partial_{y_i}(\mu \omega_{pm}^2)}{\mu \omega_{pm}^2} \boldsymbol{H} \cdot \partial_{y_i} \boldsymbol{K} - \frac{\partial_{y_i} \Gamma_m}{\mu \omega_{pm}^2} \boldsymbol{K} \cdot \partial_{y_i} \boldsymbol{K} \right], \tag{4.36}$$

where in the last step we used the following identity:

$$\int_\Xi \int_\Omega \rho(\boldsymbol{y})(\nabla \times \partial_{y_i} \boldsymbol{H} \cdot \partial_{y_i} \boldsymbol{E} - \nabla \times \partial_{y_i} \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{H}) = -\int_\Xi \int_{\partial D} \rho(\boldsymbol{y})(\boldsymbol{n} \times \partial_{y_i} \boldsymbol{E} \cdot \partial_{y_i} \boldsymbol{H}) = 0.$$

By the Cauchy-Schwarz inequality, it is easy to see that

$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\partial_{y_i}\varepsilon\partial_t\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{E} = \int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\varepsilon}{\varepsilon}\cdot\sqrt{\varepsilon}\partial_t\boldsymbol{E}\cdot\sqrt{\varepsilon}\partial_{y_i}\boldsymbol{E}$$
$$\leq \max_{\overline{\Xi}\times\overline{\Omega}}\left(\left|\frac{\partial_{y_i}\varepsilon}{\varepsilon}\right|\right)\int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\partial_t\boldsymbol{E}|^2+\varepsilon|\partial_{y_i}\boldsymbol{E}|^2\right), \tag{4.37}$$
$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\partial_{y_i}\mu\partial_t\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{H} = \int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\mu}{\mu}\cdot\sqrt{\mu}\partial_t\boldsymbol{H}\cdot\sqrt{\mu}\partial_{y_i}\boldsymbol{H}$$
$$\leq \max_{\overline{\Xi}\times\overline{\Omega}}\left(\left|\frac{\partial_{y_i}\mu}{\mu}\right|\right)\int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\mu|\partial_t\boldsymbol{H}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2\right). \tag{4.38}$$

Similarly, we can obtain

$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}^2}\boldsymbol{E}\cdot\partial_{y_i}\boldsymbol{J} \leq \max_{\overline{\Xi}\times\overline{\Omega}}\left(\left|\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right)\int_\Xi\int_\Omega \rho(\boldsymbol{y})(\varepsilon|\boldsymbol{E}|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2), \tag{4.39}$$
$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\boldsymbol{H}\cdot\partial_{y_i}\boldsymbol{K} \leq \max_{\overline{\Xi}\times\overline{\Omega}}\left(\left|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right)\int_\Xi\int_\Omega \rho(\boldsymbol{y})(\mu|\boldsymbol{H}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2),$$
$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\Gamma_e}{\varepsilon\omega_{pe}^2}\boldsymbol{J}\cdot\partial_{y_i}\boldsymbol{J} \leq \max_{\overline{\Xi}\times\overline{\Omega}}(|\partial_{y_i}\Gamma_e|)\int_\Xi\int_\Omega \frac{\rho(\boldsymbol{y})}{\varepsilon\omega_{pe}^2}(|\boldsymbol{J}|^2+|\partial_{y_i}\boldsymbol{J}|^2), \tag{4.40}$$
$$\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\cdot\frac{\partial_{y_i}\Gamma_m}{\mu\omega_{pm}^2}\boldsymbol{K}\cdot\partial_{y_i}\boldsymbol{K} \leq \max_{\overline{\Xi}\times\overline{\Omega}}(|\partial_{y_i}\Gamma_m|)\int_\Xi\int_\Omega \frac{\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}(|\boldsymbol{K}|^2+|\partial_{y_i}\boldsymbol{K}|^2). \tag{4.41}$$

Denote constants C_2 and C_3 as follows:

$$C_2 = \max_{\overline{\Omega}\times\overline{\Xi}}\left(\left|\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|,\left|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|,|\partial_{y_i}\Gamma_e|,|\partial_{y_i}\Gamma_m|\right),\quad C_3 = \max_{\overline{\Omega}\times\overline{\Xi}}\left(\left|\frac{\partial_{y_i}\varepsilon}{\varepsilon}\right|,\left|\frac{\partial_{y_i}\mu}{\mu}\right|\right),$$

Let us introduce the notations

$$ENG_0(t) = \int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\boldsymbol{E}|^2+\mu|\boldsymbol{H}|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y},$$

and

$$ENG_1(t) = \int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2+\mu|\partial_{y_i}\boldsymbol{H}|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}.$$

Substituting (4.37)-(4.41) into (4.36), we have

$$\frac{d}{dt}ENG_1(t) \le C_2 ENG_0(t) + C_1 ENG_1(t) + C_3 \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_t \boldsymbol{E}|^2 + \mu|\partial_t \boldsymbol{H}|^2\right). \quad (4.42)$$

Applying the Gronwall inequality stated in Lemma 19 to (4.42) and Lemmas 17-18, we have

$$\begin{aligned}
& ENG_1(t) \\
& \le e^{C_1 t} \left\{ ENG_1(0) + \frac{1}{C_1} \left[C_2 ENG_0(0) + C_3 C_{max1} \left(ENG_0(0) \right.\right.\right. \\
& \left.\left.\left. + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2) \right) \right] \right\} \\
& \le e^{C_1 t} \left[ENG_1(0) + (1 + C_{max1})(ENG_0(0) + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2)) \right],
\end{aligned}$$

which concludes the proof. In the last step we used the fact that $C_2 \le C_1$ and $C_3 \le C_1$. □

Remark 5. *If the physical parameters $\varepsilon, \mu, \Gamma_e, \Gamma_m, \omega_{pe}, \omega_{pm}$ are independent of y_i, then $C_1 = C_2 = C_3 = 0$. Hence from (4.42) we easily see that Theorem 20 becomes*

$$\begin{aligned}
& \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_{y_i} \boldsymbol{E}|^2 + \mu|\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i} \boldsymbol{K}|^2 \right)(t) d\boldsymbol{x} d\boldsymbol{y} \\
\le & \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_{y_i} \boldsymbol{E}|^2 + \mu|\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i} \boldsymbol{K}|^2 \right)(0) d\boldsymbol{x} d\boldsymbol{y}.
\end{aligned}$$

In the more general case, Theorem 20 shows that if the following initial conditions are L^2 bounded:

$$\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_{y_i} \boldsymbol{E}|^2 + \mu|\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i} \boldsymbol{K}|^2 \right)(0) d\boldsymbol{x} d\boldsymbol{y} \le C,$$

$$\int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2 + \varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2 \right) \le C,$$

then the solution $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K})$ of (4.1)-(4.7) is also L^2 bounded:

$$\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_{y_i} \boldsymbol{E}|^2 + \mu|\partial_{y_i} \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i} \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i} \boldsymbol{K}|^2 \right)(t) d\boldsymbol{x} d\boldsymbol{y} \le C e^{C_1 t}.$$

This boundness guarantees that the mean squared error is $O(N^{-1})$ when the stochastic collo-

cation method is used to solve the model problem (4.1)-(4.7). Details see Theorem 27 proved later.

To prove higher order convergence, we need to show that higher derivatives with respect to the random variables are L^2 bounded. Below we just present the proofs of L^2 boundness for the second-order derivatives, which depend on the estimates of $\nabla \times \boldsymbol{u}, \nabla \times \partial_t \boldsymbol{u}$ and $\nabla \times \partial_{y_i} \boldsymbol{u}$ for $\boldsymbol{u} = \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K}$. These estimates will be proved in the following three lemmas.

Lemma 21. *Denote the constant*

$$C_1^* = \max_{\overline{\Omega} \times \overline{\Xi}} \left(\left| \frac{\nabla(\varepsilon \omega_{pe}^2)}{\varepsilon \omega_{pe}} \right| + |\nabla \Gamma_e|, \left| \frac{\nabla(\mu \omega_{pm}^2)}{\mu \omega_{pm}} \right| + |\nabla \Gamma_m|, \left| \frac{\nabla \varepsilon}{\varepsilon} \right|, \left| \frac{\nabla \mu}{\mu} \right| \right).$$

Then for any $t \in [0, T]$ and $i = 1, \cdots, d$, we have

$$\begin{aligned}
&\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\nabla \times \boldsymbol{E}|^2 + \mu |\nabla \times \boldsymbol{H}|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \boldsymbol{K}|^2 \right)(t) d\boldsymbol{x} d\boldsymbol{y} \\
&\leq e^{C_1^* t}(2 + C_{max1}) \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\nabla \times \boldsymbol{E}_0|^2 + \mu |\nabla \times \boldsymbol{H}_0|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\nabla \times \boldsymbol{J}_0|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \boldsymbol{K}_0|^2 \right) \\
&\quad + e^{C_1^* t}(1 + C_{max1}) \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\boldsymbol{E}_0|^2 + \mu |\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\boldsymbol{J}_0|^2 + \frac{1}{\mu \omega_{pm}^2} |\boldsymbol{K}_0|^2 \right).
\end{aligned}$$

Proof. Taking $\nabla\times$ of (4.1)-(4.4), and using the identity $\nabla \times (\phi \boldsymbol{u}) = \phi \nabla \times \boldsymbol{u} + \nabla \phi \times \boldsymbol{u}$ for any scalar function ϕ and vector function $\boldsymbol{u}$, we obtain

$$\varepsilon \partial_t (\nabla \times \boldsymbol{E}) - \nabla \times (\nabla \times \boldsymbol{H}) + \nabla \times \boldsymbol{J} = -\nabla \varepsilon \times \partial_t \boldsymbol{E}, \tag{4.43}$$

$$\mu \partial_t (\nabla \times \boldsymbol{H}) + \nabla \times (\nabla \times \boldsymbol{E}) + \nabla \times \boldsymbol{K} = -\nabla \mu \times \partial_t \boldsymbol{H}, \tag{4.44}$$

$$\partial_t (\nabla \times \boldsymbol{J}) + \Gamma_e \nabla \times \boldsymbol{J} - \varepsilon \omega_{pe}^2 \nabla \times \boldsymbol{E} = \nabla(\varepsilon \omega_{pe}^2) \times \boldsymbol{E} - \nabla \Gamma_e \times \boldsymbol{J}, \tag{4.45}$$

$$\partial_t (\nabla \times \boldsymbol{K}) + \Gamma_m \nabla \times \boldsymbol{K} - \mu \omega_{pm}^2 \nabla \times \boldsymbol{H} = \nabla(\mu \omega_{pm}^2) \times \boldsymbol{H} - \nabla \Gamma_m \times \boldsymbol{K}. \tag{4.46}$$

Denote

$$ENG_3(t) = \int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon |\nabla \times \boldsymbol{E}|^2 + \mu |\nabla \times \boldsymbol{H}|^2 + \frac{1}{\varepsilon \omega_{pe}^2} |\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu \omega_{pm}^2} |\nabla \times \boldsymbol{K}|^2 \right)(t) d\boldsymbol{x} d\boldsymbol{y}.$$

Multiplying (4.43)-(4.46) by $2\rho(\boldsymbol{y}) \nabla \times \boldsymbol{E}, 2\rho(\boldsymbol{y}) \nabla \times \boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\varepsilon \omega_{pe}^2} \nabla \times \boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu \omega_{pm}^2} \nabla \times \boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\begin{aligned}
&\frac{d}{dt}ENG_3(t) + \int_\Xi \int_\Omega 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \boldsymbol{K}|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
&= \int_\Xi \int_\Omega (\nabla \times \nabla \times \boldsymbol{H} \cdot 2\rho \nabla \times \boldsymbol{E} - \nabla \times \nabla \times \boldsymbol{E} \cdot 2\rho \nabla \times \boldsymbol{H}) \\
&\quad - \int_\Xi \int_\Omega (\nabla \varepsilon \times \partial_t \boldsymbol{E}) \cdot 2\rho \nabla \times \boldsymbol{E} - \int_\Xi \int_\Omega (\nabla \mu \times \partial_t \boldsymbol{H}) \cdot 2\rho \nabla \times \boldsymbol{H} \\
&\quad - \int_\Xi \int_\Omega (\nabla \Gamma_e \times \boldsymbol{J}) \cdot \frac{2\rho}{\varepsilon \omega_{pe}^2} \nabla \times \boldsymbol{J} + \int_\Xi \int_\Omega \nabla(\varepsilon \omega_{pe}^2) \times \boldsymbol{E} \cdot \frac{2\rho}{\varepsilon \omega_{pe}^2} \nabla \times \boldsymbol{J} \\
&\quad - \int_\Xi \int_\Omega (\nabla \Gamma_m \times \boldsymbol{K}) \cdot \frac{2\rho}{\mu \omega_{pm}^2} \nabla \times \boldsymbol{K} + \int_\Xi \int_\Omega \nabla(\mu \omega_{pm}^2) \times \boldsymbol{H} \cdot \frac{2\rho}{\mu \omega_{pm}^2} \nabla \times \boldsymbol{K} := \sum_{i=1}^7 Err_i.
\end{aligned} \tag{4.47}$$

Using integration by parts, (4.1), and boundary conditions $\boldsymbol{n} \times \boldsymbol{E} = 0$ and $\boldsymbol{n} \times \boldsymbol{J} = 0$, we obtain

$$\begin{aligned}
&\int_\Xi \int_\Omega \nabla \times \nabla \times \boldsymbol{H} \cdot 2\rho \nabla \times \boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\nabla \times \boldsymbol{H}) \cdot 2\rho \nabla \times \boldsymbol{E} + \int_\Xi \int_\Omega \nabla \times \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\varepsilon \partial_t \boldsymbol{E} + \boldsymbol{J}) \cdot 2\rho \nabla \times \boldsymbol{E} + \int_\Xi \int_\Omega \nabla \times \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \boldsymbol{E} \\
&= \int_\Xi \int_\Omega \nabla \times \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \boldsymbol{E},
\end{aligned}$$

which leads to $Err_1 = 0$.

By the Cauchy-Schwarz inequality and the identity

$$\boldsymbol{u} \times \boldsymbol{v} = |\boldsymbol{u}| \cdot |\boldsymbol{v}| \sin\theta, \quad \text{where } \theta \text{ is the angle between } \boldsymbol{u} \text{ and } \boldsymbol{v},$$

we have

$$\begin{aligned}
Err_2 &= -\int_\Xi \int_\Omega 2\frac{\nabla \varepsilon}{\varepsilon} \times \sqrt{\rho\varepsilon} \partial_t \boldsymbol{E} \cdot \sqrt{\rho\varepsilon} \nabla \times \boldsymbol{E} \\
&\le \left(\max_{\overline{\Omega} \times \overline{\Xi}} \left| \frac{\nabla \varepsilon}{\varepsilon} \right| \right) \left(\int_\Xi \int_\Omega \rho\varepsilon |\partial_t \boldsymbol{E}|^2 + \int_\Xi \int_\Omega \rho\varepsilon |\nabla \times \boldsymbol{E}|^2 \right).
\end{aligned}$$

Similarly, we can obtain

$$
\begin{aligned}
Err_3 &\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla\mu}{\mu}\right|\right)\left(\int_{\Xi}\int_{\Omega}\rho\mu|\partial_t \boldsymbol{H}|^2 + \int_{\Xi}\int_{\Omega}\rho\mu|\nabla\times\boldsymbol{H}|^2\right),\\
Err_4 &\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_e|\right)\int_{\Xi}\int_{\Omega}\frac{\rho}{\varepsilon\omega_{pe}^2}\left(|\boldsymbol{J}|^2 + |\nabla\times\boldsymbol{J}|^2\right),\\
Err_5 &\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right)\left(\int_{\Xi}\int_{\Omega}\rho\varepsilon|\boldsymbol{E}|^2 + \int_{\Xi}\int_{\Omega}\frac{\rho}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2\right),\\
Err_6 &\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_m|\right)\int_{\Xi}\int_{\Omega}\frac{\rho}{\mu\omega_{pm}^2}\left(|\boldsymbol{K}|^2 + |\nabla\times\boldsymbol{K}|^2\right),\\
Err_7 &\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right)\left(\int_{\Xi}\int_{\Omega}\rho\mu|\boldsymbol{H}|^2 + \int_{\Xi}\int_{\Omega}\frac{\rho}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2\right).
\end{aligned}
$$

Denote constants C_2^* and C_3^* as follows:

$$
C_2^* = \max_{\overline{\Omega}\times\overline{\Xi}}\left(\left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|, \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|, |\nabla\Gamma_e|, |\nabla\Gamma_m|\right), \quad C_3^* = \max_{\overline{\Omega}\times\overline{\Xi}}\left(\left|\frac{\nabla\varepsilon}{\varepsilon}\right|, \left|\frac{\nabla\mu}{\mu}\right|\right).
$$

Recall the notation

$$
ENG_0(t) = \int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\left(\varepsilon|\boldsymbol{E}|^2 + \mu|\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y},
$$

and substitute the above estimates into (4.47), we have

$$
\begin{aligned}
&\frac{d}{dt}ENG_3(t)\\
&\leq C_2^* ENG_0(t) + C_1^* ENG_3(t) + C_3^*\int_{\Xi}\int_{\Omega}\rho(\boldsymbol{y})\left(\varepsilon|\partial_t\boldsymbol{E}|^2 + \mu|\partial_t\boldsymbol{H}|^2\right)\\
&\leq C_1^* ENG_3(t) + C_2^* ENG_0(0) + C_3^* C_{max1}\left(ENG_0(0) + \int_{\Xi}\int_{\Omega}\rho(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2)\right), \qquad (4.48)
\end{aligned}
$$

where we used Lemmas 17-18 in the last step.

Applying the Gronwall inequality (cf. Lemma 19) to (4.48) and the facts that $C_2^* \leq C_1^*$

and $C_3^* \le C_1^*$, we have

$$
\begin{aligned}
& ENG_3(t) \\
& \le e^{C_1^* t}\left[ENG_3(0) + (1 + C_{max1})(ENG_0(0) + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2))\right] \\
& \le e^{C_1^* t}(1 + C_{max1}) \int_\Xi \int_\Omega \rho\left(\varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right) \\
& + e^{C_1^* t}(2 + C_{max1}) \int_\Xi \int_\Omega \rho\left(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}_0|^2\right),
\end{aligned}
$$

which concludes the proof. □

Remark 6. *Similar remark as Remark 5 holds true. More specifically, if the physical parameters $\varepsilon, \mu, \Gamma_e, \Gamma_m, \omega_{pe}, \omega_{pm}$ are independent of the spatial variable $\boldsymbol{x}$, then $C_1^* = C_2^* = C_3^* = 0$. In this case, Lemma 21 just becomes*

$$
\begin{aligned}
& \int_\Xi \int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla \times \boldsymbol{E}|^2 + \mu|\nabla \times \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y} \\
\le & \int_\Xi \int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla \times \boldsymbol{E}|^2 + \mu|\nabla \times \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \boldsymbol{K}|^2\right)(0)d\boldsymbol{x}d\boldsymbol{y}.
\end{aligned}
$$

Lemma 22. *Denote constants C_4 and C_5 as*

$$
\begin{aligned}
C_4 = \max_{\overline{\Omega}\times\overline{\Xi}} & \left(\left|\frac{\nabla\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right| + \left|\frac{\nabla\mu}{\mu\sqrt{\varepsilon\mu}}\right| + \left|\frac{\omega_{pe}\nabla\varepsilon}{\varepsilon}\right| + \left|\frac{\omega_{pm}\nabla\mu}{\mu}\right|,\right. \\
& \left.\left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right| + |\nabla\Gamma_m|, \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right| + |\nabla\Gamma_e|\right), \\
C_5 = \max_{\overline{\Omega}\times\overline{\Xi}} & \left(\left|\frac{\omega_{pe}\nabla\varepsilon}{\varepsilon}\right| + |\nabla\Gamma_e|, \left|\frac{\omega_{pm}\nabla\mu}{\mu}\right| + |\nabla\Gamma_m|, \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|, \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right).
\end{aligned}
$$

Then for any $t \in [0, T]$ and $i = 1, \cdots, d$, we have

$$
\begin{aligned}
& \int_\Xi \int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla \times \partial_t\boldsymbol{E}|^2 + \mu|\nabla \times \partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \partial_t\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y} \\
\le & e^{C_4 t}\left(1 + \frac{C_5 C_{max1}}{C_4}\right)\left[\int_\Xi \int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla \times \partial_t\boldsymbol{E}|^2 + \mu|\nabla \times \partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \partial_t\boldsymbol{J}|^2\right.\right. \\
& \left. + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t\boldsymbol{K}|^2\right)(0) \\
& \left. + \int_\Xi \int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2 + \varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)\right].
\end{aligned}
$$

Proof. Taking ∂_t of (4.43)-(4.46), we obtain

$$\varepsilon\partial_t(\nabla\times\partial_t\boldsymbol{E})-\nabla\times(\nabla\times\partial_t\boldsymbol{H})+\nabla\times\partial_t\boldsymbol{J}=-\nabla\varepsilon\times\partial_t(\partial_t\boldsymbol{E}), \tag{4.49}$$

$$\mu\partial_t(\nabla\times\partial_t\boldsymbol{H})+\nabla\times(\nabla\times\partial_t\boldsymbol{E})+\nabla\times\partial_t\boldsymbol{K}=-\nabla\mu\times\partial_t(\partial_t\boldsymbol{H}), \tag{4.50}$$

$$\partial_t(\nabla\times\partial_t\boldsymbol{J})+\Gamma_e\nabla\times\partial_t\boldsymbol{J}-\varepsilon\omega_{pe}^2\nabla\times\partial_t\boldsymbol{E}=\nabla(\varepsilon\omega_{pe}^2)\times\partial_t\boldsymbol{E}-\nabla\Gamma_e\times\partial_t\boldsymbol{J}, \tag{4.51}$$

$$\partial_t(\nabla\times\partial_t\boldsymbol{K})+\Gamma_m\nabla\times\partial_t\boldsymbol{K}-\mu\omega_{pm}^2\nabla\times\partial_t\boldsymbol{H}=\nabla(\mu\omega_{pm}^2)\times\partial_t\boldsymbol{H}-\nabla\Gamma_m\times\partial_t\boldsymbol{K}. \tag{4.52}$$

Denote

$$\begin{aligned}ENG_4(t)=\int_\Xi\int_\Omega\rho(\boldsymbol{y})\big(&\varepsilon|\nabla\times\partial_t\boldsymbol{E}|^2+\mu|\nabla\times\partial_t\boldsymbol{H}|^2\\&+\frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2\Big)(t)d\boldsymbol{x}d\boldsymbol{y},\end{aligned}$$

and

$$ENG_5(t)=\int_\Xi\int_\Omega\rho(\boldsymbol{y})\left(\varepsilon|\partial_t\boldsymbol{E}|^2+\mu|\partial_t\boldsymbol{H}|^2+\frac{1}{\varepsilon\omega_{pe}^2}|\partial_t\boldsymbol{J}|^2+\frac{1}{\mu\omega_{pm}^2}|\partial_t\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}.$$

Multiplying (4.49)-(4.52) by $2\rho(\boldsymbol{y})\nabla\times\partial_t\boldsymbol{E}, 2\rho(\boldsymbol{y})\nabla\times\partial_t\boldsymbol{H}$, $\frac{2\rho(\boldsymbol{y})}{\varepsilon\omega_{pe}^2}\nabla\times\partial_t\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\nabla\times\partial_t\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\begin{aligned}&\frac{d}{dt}ENG_4(t)+\int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\nabla\times\partial_t\boldsymbol{J}|^2\right.\\&\qquad\left.+\frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\nabla\times\partial_t\boldsymbol{K}|^2\right)d\boldsymbol{x}d\boldsymbol{y}\\&=\int_\Xi\int_\Omega(\nabla\times\nabla\times\partial_t\boldsymbol{H}\cdot2\rho\nabla\times\partial_t\boldsymbol{E}-\nabla\times\nabla\times\partial_t\boldsymbol{E}\cdot2\rho\nabla\times\partial_t\boldsymbol{H})\\&\quad-\int_\Xi\int_\Omega\nabla\varepsilon\times\partial_t(\partial_t\boldsymbol{E})\cdot2\rho\nabla\times\partial_t\boldsymbol{E}-\int_\Xi\int_\Omega\nabla\mu\times\partial_t(\partial_t\boldsymbol{H})\cdot2\rho\nabla\times\partial_t\boldsymbol{H}\\&\quad-\int_\Xi\int_\Omega(\nabla\Gamma_e\times\partial_t\boldsymbol{J})\cdot\frac{2\rho}{\varepsilon\omega_{pe}^2}\nabla\times\partial_t\boldsymbol{J}+\int_\Xi\int_\Omega\nabla(\varepsilon\omega_{pe}^2)\times\partial_t\boldsymbol{E}\cdot\frac{2\rho}{\varepsilon\omega_{pe}^2}\nabla\times\partial_t\boldsymbol{J}\\&\quad-\int_\Xi\int_\Omega(\nabla\Gamma_m\times\partial_t\boldsymbol{K})\cdot\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\partial_t\boldsymbol{K}+\int_\Xi\int_\Omega\nabla(\mu\omega_{pm}^2)\times\partial_t\boldsymbol{H}\cdot\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\partial_t\boldsymbol{K}\\&:=\sum_{i=1}^7 Err_i.\end{aligned} \tag{4.53}$$

Using integration by parts, (4.1), and boundary conditions $\boldsymbol{n} \times \boldsymbol{E} = 0$ and $\boldsymbol{n} \times \boldsymbol{J} = 0$, we obtain

$$
\begin{aligned}
&\int_\Xi \int_\Omega \nabla \times \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\nabla \times \partial_t \boldsymbol{H}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} + \int_\Xi \int_\Omega \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\varepsilon \partial_{t^2} \boldsymbol{E} + \partial_t \boldsymbol{J}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} + \int_\Xi \int_\Omega \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E} \\
&= \int_\Xi \int_\Omega \nabla \times \partial_t \boldsymbol{H} \cdot 2\rho \nabla \times \nabla \times \partial_t \boldsymbol{E},
\end{aligned}
$$

which leads to $Err_1 = 0$.

Using (4.1) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
Err_2 = &- \int_\Xi \int_\Omega \frac{\nabla \varepsilon}{\varepsilon} \times (\nabla \times \partial_t \boldsymbol{H} - \partial_t \boldsymbol{J}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{E} \\
= &- \int_\Xi \int_\Omega 2 \frac{\nabla \varepsilon}{\varepsilon \sqrt{\varepsilon \mu}} \times (\sqrt{\rho \mu} \nabla \times \partial_t \boldsymbol{H}) \cdot \sqrt{\rho \varepsilon} \nabla \times \partial_t \boldsymbol{E} \\
&+ \int_\Xi \int_\Omega 2 \frac{\omega_{pe} \nabla \varepsilon}{\varepsilon} \times (\sqrt{\frac{\rho}{\varepsilon \omega_{pe}^2}} \partial_t \boldsymbol{J}) \cdot \sqrt{\rho \varepsilon} \nabla \times \partial_t \boldsymbol{E} \\
\leq &\left(\max_{\overline{\Omega} \times \overline{\Xi}} \left| \frac{\nabla \varepsilon}{\varepsilon \sqrt{\varepsilon \mu}} \right| \right) \int_\Xi \int_\Omega \rho \left(\varepsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 \right) \\
&+ \left(\max_{\overline{\Omega} \times \overline{\Xi}} \left| \frac{\omega_{pe} \nabla \varepsilon}{\varepsilon} \right| \right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\varepsilon \omega_{pe}^2} |\partial_t \boldsymbol{J}|^2 + \varepsilon |\nabla \times \partial_t \boldsymbol{E}|^2 \right).
\end{aligned}
$$

Using (4.2) and the Cauchy-Schwarz inequality, we can obtain

$$
\begin{aligned}
Err_3 = &\int_\Xi \int_\Omega \frac{\nabla \mu}{\mu} \times (\nabla \times \partial_t \boldsymbol{E} + \partial_t \boldsymbol{K}) \cdot 2\rho \nabla \times \partial_t \boldsymbol{H} \\
\leq &\left(\max_{\overline{\Omega} \times \overline{\Xi}} \left| \frac{\nabla \mu}{\mu \sqrt{\varepsilon \mu}} \right| \right) \int_\Xi \int_\Omega \rho \left(\varepsilon |\nabla \times \partial_t \boldsymbol{E}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 \right) \\
&+ \left(\max_{\overline{\Omega} \times \overline{\Xi}} \left| \frac{\omega_{pm} \nabla \mu}{\mu} \right| \right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\mu \omega_{pm}^2} |\partial_t \boldsymbol{K}|^2 + \mu |\nabla \times \partial_t \boldsymbol{H}|^2 \right).
\end{aligned}
$$

Similarly, by the Cauchy-Schwarz inequality, we have

$$Err_4 \leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_e|\right) \int_\Xi \int_\Omega \frac{\rho}{\varepsilon\omega_{pe}^2} \left(|\partial_t \boldsymbol{J}|^2 + |\nabla \times \partial_t \boldsymbol{J}|^2\right),$$

$$Err_5 \leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right) \left(\int_\Xi \int_\Omega \rho\varepsilon|\partial_t \boldsymbol{E}|^2 + \int_\Xi \int_\Omega \frac{\rho}{\varepsilon\omega_{pe}^2} |\nabla \times \partial_t \boldsymbol{J}|^2\right),$$

$$Err_6 \leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_m|\right) \int_\Xi \int_\Omega \frac{\rho}{\mu\omega_{pm}^2} \left(|\partial_t \boldsymbol{K}|^2 + |\nabla \times \partial_t \boldsymbol{K}|^2\right),$$

$$Err_7 \leq \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right) \left(\int_\Xi \int_\Omega \rho\mu|\partial_t \boldsymbol{H}|^2 + \int_\Xi \int_\Omega \frac{\rho}{\mu\omega_{pm}^2} |\nabla \times \partial_t \boldsymbol{K}|^2\right).$$

Substituting the above estimates into (4.53) and using the notations $ENG_4(t)$ and $ENG_5(t)$ and Lemma 18, we have

$$\begin{aligned}
&\frac{d}{dt} ENG_4(t) \\
&\leq C_4 \cdot ENG_4(t) + C_5 \cdot ENG_5(t) \\
&\leq C_4 \cdot ENG_4(t) + C_5 C_{max1} \left[ENG_0(0) + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2)\right]. \qquad (4.54)
\end{aligned}$$

Applying the Gronwall inequality (cf. Lemma 19) to (4.54), we have

$$\begin{aligned}
&ENG_4(t) \\
&\leq e^{C_4} \left[ENG_4(0) + \frac{C_5 C_{max1}}{C_4} \left(ENG_0(0) + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2)\right)\right] \\
&\leq e^{C_4} \left(1 + \frac{C_5 C_{max1}}{C_4}\right) \left[ENG_4(0) + ENG_0(0) + \int_\Xi \int_\Omega \rho(\varepsilon|\nabla \times \boldsymbol{E}_0|^2 + \mu|\nabla \times \boldsymbol{H}_0|^2)\right],
\end{aligned}$$

which concludes the proof. □

Remark 7. *If the physical parameters $\varepsilon, \mu, \Gamma_e, \Gamma_m, \omega_{pe}, \omega_{pm}$ are independent of the spatial variable $\boldsymbol{x}$, then the constants $C_4 = C_5 = 0$. In this case, Lemma 22 simply reduces to*

$$\begin{aligned}
&\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\nabla \times \partial_t \boldsymbol{E}|^2 + \mu|\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t \boldsymbol{K}|^2\right)(t) d\boldsymbol{x} d\boldsymbol{y} \\
\leq &\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\nabla \times \partial_t \boldsymbol{E}|^2 + \mu|\nabla \times \partial_t \boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla \times \partial_t \boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla \times \partial_t \boldsymbol{K}|^2\right)(0) d\boldsymbol{x} d\boldsymbol{y}.
\end{aligned}$$

Theorem 23. *Denote constants $C_{6,1}, C_{6,2}, C_{6,3}, C_{6,4}$ and C_6 as*

$$\begin{aligned}
C_{6,1} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{\nabla\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right| + \left|\frac{\omega_{pe}\nabla\varepsilon}{\varepsilon}\right| + \left|\frac{(\partial_{y_i}\varepsilon)\nabla\varepsilon}{\varepsilon^2}\right| + \left|\frac{\nabla(\partial_{y_i}\varepsilon)}{\varepsilon}\right| + \left|\frac{\partial_{y_i}\varepsilon}{\varepsilon}\right| + \left|\frac{\nabla\mu}{\mu\sqrt{\varepsilon\mu}}\right| \right), \\
C_{6,2} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{\nabla\mu}{\mu\sqrt{\varepsilon\mu}}\right| + \left|\frac{\omega_{pm}\nabla\mu}{\mu}\right| + \left|\frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2}\right| + \left|\frac{\nabla(\partial_{y_i}\mu)}{\mu}\right| + \left|\frac{\partial_{y_i}\mu}{\mu}\right| + \left|\frac{\nabla\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right| \right), \\
C_{6,3} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(|\nabla\Gamma_e| + \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right| + \left|\frac{\nabla(\partial_{y_i}(\varepsilon\omega_{pe}^2))}{\varepsilon\omega_{pe}}\right| + \left|\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right| + |\nabla(\partial_{y_i}\Gamma_e)| \right. \\
&\qquad \left. +|\partial_{y_i}\Gamma_e| \right), \\
C_{6,4} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(|\nabla\Gamma_m| + \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right| + \left|\frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}}\right| + \left|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right| \right. \\
&\qquad \left. +|\nabla(\partial_{y_i}\Gamma_m)| + |\partial_{y_i}\Gamma_m| \right), \\
C_6 &= \max\left(C_{6,1}, C_{6,2}, C_{6,3}, C_{6,4}\right), \quad C_{12} = C_6 + C_4 + C_1 + C_1^*.
\end{aligned}$$

Then for any $t \in [0,T]$ and $i = 1, \cdots, d$, we have

$$\begin{aligned}
&\int_\Xi\int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2 \right. \\
&\quad \left. + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2 \right)(t)d\boldsymbol{x}d\boldsymbol{y} \\
&\le e^{C_{12}t}C_{13} \left[\int_\Xi\int_\Omega \rho \left(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2 \right. \right. \\
&\quad \left. + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi\int_\Omega \rho \left(\varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi\int_\Omega \rho \left(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}_0|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi\int_\Omega \rho \left(\varepsilon|\nabla\times\partial_t\boldsymbol{E}|^2 + \mu|\nabla\times\partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2 \right. \\
&\quad \left. + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \\
&\quad \left. + \int_\Xi\int_\Omega \rho \left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \right],
\end{aligned}$$

where constant $C_{13} > 0$ depends on parameters $\varepsilon, \mu, \omega_{pe}, \omega_{pm}, \Gamma_e$ and Γ_m, but is independent of t.

Proof. Taking $\nabla\times$ of (4.32)-(4.35), we obtain

$$
\begin{aligned}
&\varepsilon\partial_t(\nabla\times\partial_{y_i}\boldsymbol{E}) - \nabla\times(\nabla\times\partial_{y_i}\boldsymbol{H}) + \nabla\times\partial_{y_i}\boldsymbol{J}\\
&= -\nabla\varepsilon\times\partial_{ty_i}\boldsymbol{E} - \nabla(\partial_{y_i}\varepsilon)\times\partial_t\boldsymbol{E} - (\partial_{y_i}\varepsilon)\partial_t(\nabla\times\boldsymbol{E}), \qquad (4.55)\\
&\mu\partial_t(\nabla\times\partial_{y_i}\boldsymbol{H}) + \nabla\times(\nabla\times\partial_{y_i}\boldsymbol{E}) + \nabla\times\partial_{y_i}\boldsymbol{K}\\
&= -\nabla\mu\times\partial_{ty_i}\boldsymbol{H} - \nabla(\partial_{y_i}\mu)\times\partial_t\boldsymbol{H} - (\partial_{y_i}\mu)\partial_t(\nabla\times\boldsymbol{H}), \qquad (4.56)\\
&\partial_t(\nabla\times\partial_{y_i}\boldsymbol{J}) + \Gamma_e\nabla\times\partial_{y_i}\boldsymbol{J} - \varepsilon\omega_{pe}^2\nabla\times\partial_{y_i}\boldsymbol{E} = -\nabla\Gamma_e\times\partial_{y_i}\boldsymbol{J} + \nabla(\varepsilon\omega_{pe}^2)\times\partial_{y_i}\boldsymbol{E}\\
&+\nabla(\partial_{y_i}(\varepsilon\omega_{pe}^2))\times\boldsymbol{E} + (\partial_{y_i}(\varepsilon\omega_{pe}^2))\nabla\times\boldsymbol{E} - \nabla(\partial_{y_i}\Gamma_e)\times\boldsymbol{J} - (\partial_{y_i}\Gamma_e)\nabla\times\boldsymbol{J}, \qquad (4.57)\\
&\partial_t(\nabla\times\partial_{y_i}\boldsymbol{K}) + \Gamma_m\nabla\times\partial_{y_i}\boldsymbol{K} - \mu\omega_{pm}^2\nabla\times\partial_{y_i}\boldsymbol{H} = -\nabla\Gamma_m\times\partial_{y_i}\boldsymbol{K} + \nabla(\mu\omega_{pm}^2)\times\partial_{y_i}\boldsymbol{H}\\
&+\nabla(\partial_{y_i}(\mu\omega_{pm}^2))\times\boldsymbol{H} + (\partial_{y_i}(\mu\omega_{pm}^2))\nabla\times\boldsymbol{H} - \nabla(\partial_{y_i}\Gamma_m)\times\boldsymbol{K} - (\partial_{y_i}\Gamma_m)\nabla\times\boldsymbol{K}. \qquad (4.58)
\end{aligned}
$$

Denote

$$
\begin{aligned}
&ENG_6(t)\\
&= \int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right.\\
&\qquad\left.+\frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}.
\end{aligned}
$$

Multiplying (4.55)-(4.58) by $2\rho\nabla\times\partial_{y_i}\boldsymbol{E}, 2\rho\nabla\times\partial_{y_i}\boldsymbol{H}, \frac{2\rho}{\varepsilon\omega_{pe}^2}\nabla\times\partial_{y_i}\boldsymbol{J}$ and $\frac{2\rho}{\mu\omega_{pm}^2}\nabla\times\partial_{y_i}\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$
\begin{aligned}
&\frac{d}{dt}ENG_6(t) + \int_\Xi \int_\Omega 2\rho(\boldsymbol{y}) \left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \partial_{y_i}\boldsymbol{J}|^2 \right.\\
&\left. + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})} |\nabla \times \partial_{y_i}\boldsymbol{K}|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
= &\int_\Xi \int_\Omega (\nabla \times \nabla \times \partial_{y_i}\boldsymbol{H} \cdot 2\rho \nabla \times \partial_{y_i}\boldsymbol{E} - \nabla \times \nabla \times \partial_{y_i}\boldsymbol{E} \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{H}) \\
&- \int_\Xi \int_\Omega \nabla\varepsilon \times \partial_{ty_i}\boldsymbol{E} \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} \\
&- \int_\Xi \int_\Omega [\nabla(\partial_{y_i}\varepsilon) \times \partial_t\boldsymbol{E} + \partial_{y_i}\varepsilon \cdot \partial_t(\nabla \times \boldsymbol{E})] \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} \\
&- \int_\Xi \int_\Omega \nabla\mu \times \partial_{ty_i}\boldsymbol{H} \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{H} \\
&- \int_\Xi \int_\Omega [\nabla(\partial_{y_i}\mu) \times \partial_t\boldsymbol{H} + \partial_{y_i}\mu \cdot \partial_t(\nabla \times \boldsymbol{H})] \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{H} \\
&+ \int_\Xi \int_\Omega [-\nabla\Gamma_e \times \partial_{y_i}\boldsymbol{J} + \nabla(\varepsilon\omega_{pe}^2) \times \partial_{y_i}\boldsymbol{E} + \nabla(\partial_{y_i}(\varepsilon\omega_{pe}^2)) \times \boldsymbol{E} + \partial_{y_i}(\varepsilon\omega_{pe}^2) \cdot \nabla \times \boldsymbol{E} \\
&- \nabla(\partial_{y_i}\Gamma_e) \times \boldsymbol{J} - (\partial_{y_i}\Gamma_e)\nabla \times \boldsymbol{J}] \cdot \frac{2\rho}{\varepsilon\omega_{pe}^2} \nabla \times \partial_{y_i}\boldsymbol{J} \\
&+ \int_\Xi \int_\Omega [-\nabla\Gamma_m \times \partial_{y_i}\boldsymbol{K} + \nabla(\mu\omega_{pm}^2) \times \partial_{y_i}\boldsymbol{H} + \nabla(\partial_{y_i}(\mu\omega_{pm}^2)) \times \boldsymbol{H} \\
&+ \partial_{y_i}(\mu\omega_{pm}^2) \cdot \nabla \times \boldsymbol{H} - \nabla(\partial_{y_i}\Gamma_e) \times \boldsymbol{K} - (\partial_{y_i}\Gamma_m)\nabla \times \boldsymbol{K}] \\
&\cdot \frac{2\rho}{\mu\omega_{pm}^2} \nabla \times \partial_{y_i}\boldsymbol{K} := \sum_{i=1}^{7} Err_i.
\end{aligned} \tag{4.59}
$$

Below we will estimate each Err_i of (4.59). First, using integration by parts, (4.32), and boundary conditions $\boldsymbol{n} \times \boldsymbol{E} = 0$ and $\boldsymbol{n} \times \boldsymbol{J} = 0$, we obtain

$$
\begin{aligned}
&\int_\Xi \int_\Omega \nabla \times \nabla \times \partial_{y_i}\boldsymbol{H} \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\nabla \times \partial_{y_i}\boldsymbol{H}) \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} + \int_\Xi \int_\Omega \nabla \times \partial_{y_i}\boldsymbol{H} \cdot 2\rho\nabla \times \nabla \times \partial_{y_i}\boldsymbol{E} \\
&= \int_\Xi \int_{\partial\Omega} \boldsymbol{n} \times (\varepsilon\partial_{ty_i}\boldsymbol{E} + \partial_{y_i}\boldsymbol{J} + \partial_{y_i}\varepsilon\partial_t\boldsymbol{E}) \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} \\
&\quad + \int_\Xi \int_\Omega \nabla \times \partial_{y_i}\boldsymbol{H} \cdot 2\rho\nabla \times \nabla \times \partial_{y_i}\boldsymbol{E} \\
&= \int_\Xi \int_\Omega \nabla \times \partial_{y_i}\boldsymbol{H} \cdot 2\rho\nabla \times \nabla \times \partial_{y_i}\boldsymbol{E},
\end{aligned}
$$

which leads to $Err_1 = 0$.

Using (4.32) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
Err_2 &= -\int_\Xi \int_\Omega \frac{\nabla\varepsilon}{\varepsilon} \times (\nabla \times \partial_{y_i}\boldsymbol{H} - \partial_{y_i}\boldsymbol{J} - \partial_{y_i}\varepsilon\partial_t\boldsymbol{E}) \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{E} \\
&\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla \times \partial_{y_i}\boldsymbol{H}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\omega_{pe}\nabla\varepsilon}{\varepsilon}\right|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{(\partial_{y_i}\varepsilon)\nabla\varepsilon}{\varepsilon^2}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_t\boldsymbol{E}|^2 + \varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we can obtain

$$\begin{aligned}
Err_3 &= -\int_\Xi \int_\Omega 2\frac{\nabla(\partial_{y_i}\varepsilon)}{\varepsilon} \times \sqrt{\rho\varepsilon}\partial_t\boldsymbol{E} \cdot \sqrt{\rho\varepsilon}\nabla \times \partial_{y_i}\boldsymbol{E} \\
&\quad - \int_\Xi \int_\Omega 2\frac{\partial_{y_i}\varepsilon}{\varepsilon}\sqrt{\rho\varepsilon}\partial_t(\nabla \times \boldsymbol{E}) \cdot \sqrt{\rho\varepsilon}\nabla \times \partial_{y_i}\boldsymbol{E} \\
&\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\partial_{y_i}\varepsilon)}{\varepsilon}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_t\boldsymbol{E}|^2 + \varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i}\varepsilon}{\varepsilon}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_t(\nabla \times \boldsymbol{E})|^2 + \varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2\right).
\end{aligned}$$

Similarly, by (4.33) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
Err_4 &= \int_\Xi \int_\Omega \frac{\nabla\mu}{\mu} \times (\nabla \times \partial_{y_i}\boldsymbol{E} + \partial_{y_i}\boldsymbol{K} + \partial_{y_i}\mu\partial_t\boldsymbol{H}) \cdot 2\rho\nabla \times \partial_{y_i}\boldsymbol{H} \\
&\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla\mu}{\mu\sqrt{\varepsilon\mu}}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla \times \partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla \times \partial_{y_i}\boldsymbol{H}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\omega_{pm}\nabla\mu}{\mu}\right|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 + \mu|\nabla \times \partial_{y_i}\boldsymbol{H}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\partial_t\boldsymbol{H}|^2 + \mu|\nabla \times \partial_{y_i}\boldsymbol{H}|^2\right).
\end{aligned}$$

By similar arguments, we have

$$
\begin{aligned}
Err_5 \leq & \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\partial_{y_i}\mu)}{\mu}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\partial_t \boldsymbol{H}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i}\mu}{\mu}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\partial_t(\nabla\times\boldsymbol{H})|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2\right), \\
Err_6 \leq & \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_e|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\partial_{y_i}(\varepsilon\omega_{pe}^2))}{\varepsilon\omega_{pe}}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\boldsymbol{E}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right) \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla\times\boldsymbol{E}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla(\partial_{y_i}\Gamma_e)|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\partial_{y_i}\Gamma_e|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
Err_7 \leq & \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla\Gamma_m|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right) \int_\Xi \int_\Omega \rho \left(\mu|\nabla\times\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\nabla(\partial_{y_i}\Gamma_m)|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right) \\
& + \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\partial_{y_i}\Gamma_m|\right) \int_\Xi \int_\Omega \rho \left(\frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2\right).
\end{aligned}
$$

Let us introduce the notations:

$$
\begin{aligned}
C_7 &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{(\partial_{y_i}\varepsilon)\nabla\varepsilon}{\varepsilon^2} \right| + \left| \frac{\nabla(\partial_{y_i}\varepsilon)}{\varepsilon} \right|, \left| \frac{(\partial_{y_i}\mu)\nabla\mu}{\mu^2} \right| + \left| \frac{\nabla(\partial_{y_i}\mu)}{\mu} \right| \right), \\
C_8 &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\partial_{y_i}\varepsilon}{\varepsilon} \right|, \left| \frac{\partial_{y_i}\mu}{\mu} \right| \right), \\
C_9 &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\omega_{pe}\nabla\varepsilon}{\varepsilon} \right| + |\nabla\Gamma_e|, \left| \frac{\omega_{pm}\nabla\mu}{\mu} \right| + |\nabla\Gamma_m|, \left| \frac{\nabla(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}} \right|, \left| \frac{\nabla(\mu\omega_{pm}^2)}{\mu\omega_{pm}} \right| \right), \\
C_{10} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}} \right|, \left| \frac{\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}} \right|, |\partial_{y_i}\Gamma_e|, |\partial_{y_i}\Gamma_m| \right), \\
C_{11} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\nabla(\partial_{y_i}(\varepsilon\omega_{pe}^2))}{\varepsilon\omega_{pe}} \right|, \left| \frac{\nabla(\partial_{y_i}(\mu\omega_{pm}^2))}{\mu\omega_{pm}} \right|, |\nabla(\partial_{y_i}\Gamma_e)|, |\nabla(\partial_{y_i}\Gamma_m)| \right).
\end{aligned}
$$

Substituting the above estimates into (4.59) and using the notation $ENG_6(t)$, we have

$$
\begin{aligned}
&\frac{d}{dt}ENG_6(t) \leq C_6 ENG_6(t) + C_7 \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_t \boldsymbol{E}|^2 + \mu|\partial_t \boldsymbol{H}|^2 \right) \\
&+C_8 \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_t(\nabla\times\boldsymbol{E})|^2 + \mu|\partial_t(\nabla\times\boldsymbol{H})|^2 \right) \\
&+C_9 \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 \right) \\
&+C_{10} \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla\times\boldsymbol{E}|^2 + \mu|\nabla\times\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}|^2 \right) \\
&+C_{11} \int_\Xi \int_\Omega \rho \left(\varepsilon|\boldsymbol{E}|^2 + \mu|\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2 \right). \qquad (4.60)
\end{aligned}
$$

Applying Lemma 18, Lemma 22, Theorem 20, Lemma 21 and Lemma 17 to the C_7, C_8, C_9, C_{10}

and C_{11} terms, respectively, we obtain

$$
\begin{aligned}
&\frac{d}{dt}ENG_6(t) \le C_6 ENG_6(t)\\
&\quad + C_7 C_{max1}\int_\Xi\int_\Omega \rho(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2\\
&\quad + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2)d\boldsymbol{x}d\boldsymbol{y} + C_8 e^{C_4 t}(1+\frac{C_5 C_{max1}}{C_4})\left[\int_\Xi\int_\Omega \rho(\varepsilon|\nabla\times\partial_t\boldsymbol{E}|^2\right.\\
&\quad + \mu|\nabla\times\partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2)(0)d\boldsymbol{x}d\boldsymbol{y}\\
&\quad + \int_\Xi\int_\Omega\rho\left(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2\right.\\
&\left.\left. + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)\right]\\
&\quad + C_9 e^{C_2 t}\left[\int_\Xi\int_\Omega\rho\left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right)(0)d\boldsymbol{x}d\boldsymbol{y}\right.\\
&\quad + (1+C_{max1})\int_\Xi\int_\Omega\rho\left(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 +\right.\\
&\quad \left.\left.\frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)\right]\\
&\quad + C_{10}e^{C_1^* t}\left[(2+C_{max1})\int_\Xi\int_\Omega\rho\left(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2\right.\right.\\
&\quad \left. + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}_0|^2\right)\\
&\quad \left. + (1+C_{max1})\int_\Xi\int_\Omega\rho\left(\varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right)\right]\\
&\quad + C_{11}\int_\Xi\int_\Omega\rho\left(\varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2\right).
\end{aligned}
\tag{4.61}
$$

Using Lemma 19 to (4.61) and absorbing those constants in (4.61), we conclude the proof.

□

With Lemmas 21-22 and Theorem 23, we can prove the boundness of the second derivative with respect to the random variables.

Theorem 24. *Denote the following constants:*

$$C_{14,1} = \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\partial_{y_i^2}\varepsilon}{\varepsilon} \right| + \left| \frac{2(\partial_{y_i}\varepsilon)^2}{\varepsilon^2} \right| + \left| \frac{2\partial_{y_i}\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}} \right| + \left| \frac{2\omega_{pe}\partial_{y_i}\varepsilon}{\varepsilon} \right| \right),$$
$$C_{14,2} = \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\partial_{y_i^2}\mu}{\mu} \right| + \left| \frac{2(\partial_{y_i}\mu)^2}{\mu^2} \right| + \left| \frac{2\partial_{y_i}\mu}{\mu\sqrt{\varepsilon\mu}} \right| + \left| \frac{2\omega_{pm}\partial_{y_i}\mu}{\mu} \right| \right),$$
$$C_{14,3} = \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left| \frac{\varepsilon\partial_{y_i^2}\Gamma_e}{\Gamma_e} \right| + \left| \frac{2\varepsilon\partial_{y_i}\Gamma_e}{\Gamma_e} \right| + \left| \frac{\partial_{y_i^2}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}} \right| + \left| \frac{2\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}} \right| \right),$$
$$C_{14,4} = \max_{\overline{\Omega}\times\overline{\Xi}} \left(|\partial_{y_i^2}\Gamma_e| + |2\partial_{y_i}\Gamma_m| + \left| \frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}} \right| + \left| \frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}} \right| \right),$$
$$C_{14} = \max\left(C_{14,1}, C_{14,2}, C_{14,3}, C_{14,4}\right), \quad C_{15} = C_{14} + C_{12} + C_2.$$

Then for any $t \in [0,T]$ and $i = 1, \cdots, d$, we have

$$\begin{aligned}
&\int_\Xi \int_\Omega \rho(\boldsymbol{y}) \left(\varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)(t)d\boldsymbol{x}d\boldsymbol{y} \\
&\leq e^{C_{15}t}C_{20} \left[\int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_{y_i}\boldsymbol{J}|^2 \right.\right. \\
&\quad \left. + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_{y_i}\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi \int_\Omega \rho \left(\varepsilon|\boldsymbol{E}_0|^2 + \mu|\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}_0|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla\times\boldsymbol{E}_0|^2 + \mu|\nabla\times\boldsymbol{H}_0|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\boldsymbol{J}_0|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\boldsymbol{K}_0|^2 \right) d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi \int_\Omega \rho \left(\varepsilon|\nabla\times\partial_t\boldsymbol{E}|^2 + \mu|\nabla\times\partial_t\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\nabla\times\partial_t\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\nabla\times\partial_t\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \\
&\quad + \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \\
&\quad \left. + \int_\Xi \int_\Omega \rho \left(\varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2 \right)(0)d\boldsymbol{x}d\boldsymbol{y} \right],
\end{aligned}$$

where constant $C_{20} > 0$ depends on parameters $\varepsilon, \mu, \omega_{pe}, \omega_{pm}, \Gamma_e$ and Γ_m, but is independent of t.

Proof. Differentiating (4.32)-(4.35) with respect to any y_i $(i = 1, \cdots, d)$, we obtain

$$\varepsilon\partial_t(\partial_{y_i^2}\boldsymbol{E}) - \nabla\times(\partial_{y_i^2}\boldsymbol{H}) + \partial_{y_i^2}\boldsymbol{J} = -(\partial_{y_i^2}\varepsilon)\partial_t\boldsymbol{E} - 2(\partial_{y_i}\varepsilon)\partial_{ty_i}\boldsymbol{E}, \tag{4.62}$$

$$\mu\partial_t(\partial_{y_i^2}\boldsymbol{H}) + \nabla\times(\partial_{y_i^2}\boldsymbol{E}) + \partial_{y_i^2}\boldsymbol{K} = -(\partial_{y_i^2}\mu)\partial_t\boldsymbol{H} - 2(\partial_{y_i}\mu)\partial_{ty_i}\boldsymbol{H}, \tag{4.63}$$

$$\begin{aligned}&\partial_t(\partial_{y_i^2}\boldsymbol{J}) + \Gamma_e\partial_{y_i^2}\boldsymbol{J} - \varepsilon\omega_{pe}^2\partial_{y_i^2}\boldsymbol{E}\\ &\quad= -(\partial_{y_i^2}\Gamma_e)\boldsymbol{J} - 2(\partial_{y_i}\Gamma_e)\partial_{y_i}\boldsymbol{J} + (\partial_{y_i^2}(\varepsilon\omega_{pe}^2))\boldsymbol{E} + 2\partial_{y_i}(\varepsilon\omega_{pe}^2)\partial_{y_i}\boldsymbol{E},\end{aligned} \tag{4.64}$$

$$\begin{aligned}&\partial_t(\partial_{y_i^2}\boldsymbol{K}) + \Gamma_m\partial_{y_i^2}\boldsymbol{K} - \mu\omega_{pm}^2\partial_{y_i^2}\boldsymbol{H}\\ &\quad= -(\partial_{y_i^2}\Gamma_m)\boldsymbol{K} - 2(\partial_{y_i}\Gamma_m)\partial_{y_i}\boldsymbol{K} + (\partial_{y_i^2}(\mu\omega_{pm}^2))\boldsymbol{H} + 2\partial_{y_i}(\mu\omega_{pm}^2)\partial_{y_i}\boldsymbol{H}.\end{aligned} \tag{4.65}$$

Denote

$$ENG_7(t) = \int_\Xi\int_\Omega \rho(\boldsymbol{y})\left(\varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2\right)(t)d\boldsymbol{x}d\boldsymbol{y}.$$

Multiplying (4.62)-(4.65) by $2\rho(\boldsymbol{y})\partial_{y_i^2}\boldsymbol{E}, 2\rho(\boldsymbol{y})\partial_{y_i^2}\boldsymbol{H}, \frac{2\rho(\boldsymbol{y})}{\varepsilon\omega_{pe}^2}\partial_{y_i^2}\boldsymbol{J}$ and $\frac{2\rho(\boldsymbol{y})}{\mu\omega_{pm}^2}\partial_{y_i^2}\boldsymbol{K}$, respectively, then integrating over Ω and Ξ, and adding the resultants, we have

$$\begin{aligned}&\frac{d}{dt}ENG_7(t) + \int_\Xi\int_\Omega 2\rho(\boldsymbol{y})\left(\frac{\Gamma_e(\boldsymbol{x},\boldsymbol{y})}{\varepsilon(\boldsymbol{x},\boldsymbol{y})\omega_{pe}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i^2}\boldsymbol{J}|^2 + \frac{\Gamma_m(\boldsymbol{x},\boldsymbol{y})}{\mu(\boldsymbol{x},\boldsymbol{y})\omega_{pm}^2(\boldsymbol{x},\boldsymbol{y})}|\partial_{y_i^2}\boldsymbol{K}|^2\right)d\boldsymbol{x}d\boldsymbol{y}\\
=& -\int_\Xi\int_\Omega(\partial_{y_i^2}\varepsilon)\partial_t\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{E} - \int_\Xi\int_\Omega(2\partial_{y_i}\varepsilon)\partial_{ty_i}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{E}\\
&- \int_\Xi\int_\Omega(\partial_{y_i^2}\mu)\partial_t\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{H} - \int_\Xi\int_\Omega(2\partial_{y_i}\mu)\partial_{ty_i}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{H}\\
&- \int_\Xi\int_\Omega\frac{\partial_{y_i^2}\Gamma_e}{\varepsilon\omega_{pe}^2}\boldsymbol{J}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J} - \int_\Xi\int_\Omega\frac{2\partial_{y_i}\Gamma_e}{\varepsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{J}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}\\
&+ \int_\Xi\int_\Omega\frac{\partial_{y_i^2}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}^2}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J} + \int_\Xi\int_\Omega\frac{2\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}^2}\partial_{y_i}\boldsymbol{E}\cdot 2\rho\partial_{y_i^2}\boldsymbol{J}\\
&- \int_\Xi\int_\Omega\frac{\partial_{y_i^2}\Gamma_m}{\mu\omega_{pm}^2}\boldsymbol{K}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K} - \int_\Xi\int_\Omega\frac{2\partial_{y_i}\Gamma_m}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{K}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}\\
&+ \int_\Xi\int_\Omega\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K} + \int_\Xi\int_\Omega\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}^2}\partial_{y_i}\boldsymbol{H}\cdot 2\rho\partial_{y_i^2}\boldsymbol{K}.\end{aligned} \tag{4.66}$$

By the Cauchy-Schwarz inequality, we have

$$Err_1 \le \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{\partial_{y_i^2}\varepsilon}{\varepsilon}\right|\right)\int_\Xi\int_\Omega \rho(\varepsilon|\partial_t\boldsymbol{E}|^2 + \varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2).$$

Similarly, by (4.32) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
Err_2 &= -\int_\Xi \int_\Omega 2\frac{\partial_{y_i}\varepsilon}{\varepsilon}(\nabla \times \partial_{y_i}\boldsymbol{H} - \partial_{y_i}\boldsymbol{J} - \partial_{y_i}\varepsilon\partial_t\boldsymbol{E}) \cdot 2\rho\partial_{y_i^2}\boldsymbol{E} \\
&\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2\partial_{y_i}\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right|\right)\int_\Xi\int_\Omega \rho(\mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2 + \varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2\omega_{pe}\partial_{y_i}\varepsilon}{\varepsilon}\right|\right)\int_\Xi\int_\Omega \rho(\frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2(\partial_{y_i}\varepsilon)^2}{\varepsilon^2}\right|\right)\int_\Xi\int_\Omega \rho(\varepsilon|\partial_t\boldsymbol{E}|^2 + \varepsilon|\partial_{y_i^2}\boldsymbol{E}|^2).
\end{aligned}
$$

Similarly, by (4.33) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
Err_4 &= \int_\Xi \int_\Omega 2\frac{\partial_{y_i}\mu}{\mu}(\nabla \times \partial_{y_i}\boldsymbol{E} + \partial_{y_i}\boldsymbol{K} + \partial_{y_i}\mu\partial_t\boldsymbol{H}) \cdot 2\rho\partial_{y_i^2}\boldsymbol{H} \\
&\leq \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2\partial_{y_i}\mu}{\mu\sqrt{\varepsilon\mu}}\right|\right)\int_\Xi\int_\Omega \rho(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2\omega_{pm}\partial_{y_i}\mu}{\mu}\right|\right)\int_\Xi\int_\Omega \rho\left(\frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2\right) \\
&\quad + \left(\max_{\overline{\Omega}\times\overline{\Xi}}\left|\frac{2(\partial_{y_i}\mu)^2}{\mu^2}\right|\right)\int_\Xi\int_\Omega \rho(\mu|\partial_t\boldsymbol{H}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2).
\end{aligned}
$$

Similarly, by the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
Err_3 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i^2}\mu}{\mu}\right|\right) \int_\Xi\int_\Omega \rho(\mu|\partial_t \boldsymbol{H}|^2 + \mu|\partial_{y_i^2}\boldsymbol{H}|^2),\\
Err_5 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\varepsilon\partial_{y_i^2}\Gamma_e}{\Gamma_e}\right|\right) \int_\Xi\int_\Omega \frac{\rho}{\varepsilon\omega_{pe}^2}(|\boldsymbol{J}|^2 + |\partial_{y_i^2}\boldsymbol{J}|^2),\\
Err_6 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{2\varepsilon\partial_{y_i}\Gamma_e}{\Gamma_e}\right|\right) \int_\Xi\int_\Omega \frac{\rho}{\varepsilon\omega_{pe}^2}(|\partial_{y_i}\boldsymbol{J}|^2 + |\partial_{y_i^2}\boldsymbol{J}|^2),\\
Err_7 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i^2}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right) \int_\Xi\int_\Omega \rho\left(\varepsilon|\boldsymbol{E}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2\right),\\
Err_8 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{2\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|\right) \int_\Xi\int_\Omega \rho\left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i^2}\boldsymbol{J}|^2\right),\\
Err_9 &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} |\partial_{y_i^2}\Gamma_m|\right) \int_\Xi\int_\Omega \frac{\rho}{\mu\omega_{pm}^2}(|\boldsymbol{K}|^2 + |\partial_{y_i^2}\boldsymbol{K}|^2),\\
Err_{10} &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} |2\partial_{y_i}\Gamma_m|\right) \int_\Xi\int_\Omega \frac{\rho}{\mu\omega_{pm}^2}(|\partial_{y_i}\boldsymbol{K}|^2 + |\partial_{y_i^2}\boldsymbol{K}|^2),\\
Err_{11} &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right) \int_\Xi\int_\Omega \rho\left(\mu|\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2\right),\\
Err_{12} &\le \left(\max_{\overline{\Omega}\times\overline{\Xi}} \left|\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|\right) \int_\Xi\int_\Omega \rho\left(\mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i^2}\boldsymbol{K}|^2\right).
\end{aligned}
$$

Denote the following constants:

$$
\begin{aligned}
C_{16} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{\partial_{y_i^2}\varepsilon}{\varepsilon}\right| + \left|\frac{2(\partial_{y_i}\varepsilon)^2}{\varepsilon^2}\right|, \left|\frac{\partial_{y_i^2}\mu}{\mu}\right| + \left|\frac{2(\partial_{y_i}\mu)^2}{\mu^2}\right| \right),\\
C_{17} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{\partial_{y_i^2}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|, \left|\frac{\partial_{y_i^2}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|, \left|\frac{\varepsilon\partial_{y_i^2}\Gamma_e}{\Gamma_e}\right|, |\partial_{y_i^2}\Gamma_m| \right),\\
C_{18} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{2\partial_{y_i}(\varepsilon\omega_{pe}^2)}{\varepsilon\omega_{pe}}\right|, \left|\frac{2\partial_{y_i}(\mu\omega_{pm}^2)}{\mu\omega_{pm}}\right|, \left|\frac{2\varepsilon\partial_{y_i}\Gamma_e}{\Gamma_e}\right| + \left|\frac{2\omega_{pe}\partial_{y_i}\varepsilon}{\varepsilon}\right|, |2\partial_{y_i}\Gamma_m| \right),\\
C_{19} &= \max_{\overline{\Omega}\times\overline{\Xi}} \left(\left|\frac{2\partial_{y_i}\varepsilon}{\varepsilon\sqrt{\varepsilon\mu}}\right|, \left|\frac{2\partial_{y_i}\mu}{\mu\sqrt{\varepsilon\mu}}\right| \right).
\end{aligned}
$$

Substituting the above estimates into (4.66), we obtain

$$\begin{aligned}\frac{d}{dt}ENG_7(t) \le C_{14}ENG_7(t) &+ C_{16}\int_\Xi\int_\Omega \rho(\varepsilon|\partial_t\boldsymbol{E}|^2 + \mu|\partial_t\boldsymbol{H}|^2) \\ &+ C_{17}\int_\Xi\int_\Omega \rho\left(\varepsilon|\boldsymbol{E}|^2 + \mu|\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\boldsymbol{K}|^2\right) \\ &+ C_{18}\int_\Xi\int_\Omega \rho\left(\varepsilon|\partial_{y_i}\boldsymbol{E}|^2 + \mu|\partial_{y_i}\boldsymbol{H}|^2 + \frac{1}{\varepsilon\omega_{pe}^2}|\partial_{y_i}\boldsymbol{J}|^2 + \frac{1}{\mu\omega_{pm}^2}|\partial_{y_i}\boldsymbol{K}|^2\right) \\ &+ C_{19}\int_\Xi\int_\Omega \rho(\varepsilon|\nabla\times\partial_{y_i}\boldsymbol{E}|^2 + \mu|\nabla\times\partial_{y_i}\boldsymbol{H}|^2). \end{aligned} \tag{4.67}$$

Applying Lemma 18, Lemma 17, Theorem 20, and Theorem 23 to the C_{16}, C_{17}, C_{18} and C_{19} terms, respectively, then using the Gronwall inequality (cf. Lemma 19) to the resultant, we conclude the proof. □

Remark 8. *By similar techniques, we believe that if the random parameters are smooth enough, then higher derivatives with respect to the random vector* $\boldsymbol{y}$ *can be proved to be bounded similarly as stated in Theorems 20, 23 and 24. Since the proofs will become quite technical and are similar, we skip the proofs for higher derivatives.*

4.2.2 Convergence analysis

To prove the convergence estimate for the stochastic collocation method, let us first recall the following interpolation error estimates.

Lemma 25. *[49, p.289-290] Let* $I_N^y u$ *denote the polynomial of degree* N *that interpolates* u *at the* $(N+1)$ *Gauss, or Gauss-Radau, or Gauss-Lobatto points* $\{y_j\}_{j=0}^N$*, i.e.,* $I_N^y u(y) = \sum_{j=0}^N u(y_j)\mathcal{L}_j(y)$*. Then we have the interpolation error in the* L^2*-norm:*

$$\|u - I_N^y u\|_{L^2(-1,1)} \le CN^{-m}|u|_{H^m(-1,1)}, \quad \forall\, u \in H^m(-1,1) \text{ with } m \ge 1, \tag{4.68}$$

and the interpolation error in the H^l*-norm:*

$$\|u - I_N^y u\|_{H^l(-1,1)} \le CN^{2l-\frac{1}{2}-m}|u|_{H^m(-1,1)}, \quad \forall\, u \in H^m(-1,1) \text{ with } m \ge l \ge 1. \tag{4.69}$$

For the Gauss-Lobatto interpolation, we have the following optimal error estimate:

$$\|(u - I_N^y u)'\|_{L^2(-1,1)} \leq CN^{1-m}|u|_{H^m(-1,1)}, \quad \forall\ u \in H^m(-1,1) \text{ with } m \geq 1. \tag{4.70}$$

Below are the extension of the above interpolation results to tensor product interpolation.

Lemma 26. *Let $I_N u = I_N^{y_1} I_N^{y_2} \cdots I_N^{y_d} u$ denote the d-dimension tensor product polynomial of the 1-D interpolation polynomial of degree N that interpolates u at the $(N+1)$ Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{y_j\}_{j=0}^N$. Then we have the interpolation error in the L^2-norm [49, (5.8.20)]:*

$$\|u - I_N u\|_{L^2(\Xi)} \leq CN^{-m}|u|_{H^m(\Xi)}, \quad \forall\ u \in H^m(\Xi) \text{ with } m > d/2. \tag{4.71}$$

For the Gauss-Lobatto interpolation, we have the following optimal error estimate [49, (5.8.21)]:

$$\|u - I_N u\|_{H^1(\Xi)} \leq CN^{1-m}|u|_{H^m(\Xi)}, \quad \forall\ u \in H^m(\Xi) \text{ with } m > (d+1)/2. \tag{4.72}$$

To present the error estimate, recall that the mean (or expectation) of a function u is defined by

$$\mathbb{E}[u] = \int_\Xi \int_\Omega \rho(\boldsymbol{y}) u(t, \boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y}, \tag{4.73}$$

and its mean square is defined by

$$\mathcal{M}[u] = \left(\int_\Xi \int_\Omega \rho(\boldsymbol{y}) |u(t, \boldsymbol{x}, \boldsymbol{y})|^2 d\boldsymbol{x} d\boldsymbol{y} \right)^{\frac{1}{2}}. \tag{4.74}$$

Theorem 27. *Let $(\boldsymbol{E}, \boldsymbol{H})$ be the solution of (4.1)-(4.7), and $(\boldsymbol{E}^N, \boldsymbol{H}^N)$ be the stochastic collocation solution of (4.15). If the assumptions of Theorems 20 and 23 are satisfied, then* *the following mean and mean square errors hold: For any* $0 < t \leq T$*,*

$$\mathcal{M}[\boldsymbol{E}-\boldsymbol{E}^N]+\mathcal{M}[\boldsymbol{H}-\boldsymbol{H}^N]+\mathcal{M}[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+\mathcal{M}[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)] \leq C_T N^{-1}, \tag{4.75}$$

$$\mathbb{E}[|\boldsymbol{E}-\boldsymbol{E}^N|]+\mathbb{E}[|\boldsymbol{H}-\boldsymbol{H}^N|]+\mathbb{E}[\nabla\times(\boldsymbol{E}-\boldsymbol{E}^N)]+\mathbb{E}[\nabla\times(\boldsymbol{H}-\boldsymbol{H}^N)] \leq C_T N^{-1}. \tag{4.76}$$

Here and below C_T *is a constant depending on* T *but independent of* N*. Furthermore, if the assumptions of Theorem 24 are satisfied, then we have the following higher error estimates: For any* $0 < t \leq T$*,*

$$\mathcal{M}[\boldsymbol{E}-\boldsymbol{E}^N]+\mathcal{M}[\boldsymbol{H}-\boldsymbol{H}^N]+\mathbb{E}[|\boldsymbol{E}-\boldsymbol{E}^N|]+\mathbb{E}[|\boldsymbol{H}-\boldsymbol{H}^N|] \leq C_T N^{-2}. \tag{4.77}$$

Finally, if the assumptions of Theorem 24 are satisfied, for the Gauss-Lobatto interpolation, we have the error estimate for the derivative of the solution with respect to the random variables: For any $0 < t \leq T$*, and* $j = 1, \cdots, d$*,*

$$\mathcal{M}[\partial_{y_j}(\boldsymbol{E}-\boldsymbol{E}^N)]+\mathcal{M}[\partial_{y_j}(\boldsymbol{H}-\boldsymbol{H}^N)]+\mathbb{E}[|\partial_{y_j}(\boldsymbol{E}-\boldsymbol{E}^N)|]+\mathbb{E}[|\partial_{y_j}(\boldsymbol{H}-\boldsymbol{H}^N)|] \leq CN^{-1}. \tag{4.78}$$

Proof. For any fixed $\boldsymbol{x}$, using (4.68) of Lemma 22 for $u = \boldsymbol{E}$ and $u = \boldsymbol{H}$ with $m = 1$, respectively, we have

$$\begin{aligned} & \int_\Xi \big(\rho(\boldsymbol{y})\varepsilon(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{E}^N(t,\boldsymbol{x},\boldsymbol{y})|^2 \\ & \qquad +\rho(\boldsymbol{y})\mu(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{H}^N(t,\boldsymbol{x},\boldsymbol{y})|^2\big)\,d\boldsymbol{y} \\ \leq\ & CN^{-2}\int_\Xi \big(\rho(\boldsymbol{y})\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_y\boldsymbol{E}|^2+\rho(\boldsymbol{y})\mu(\boldsymbol{x},\boldsymbol{y})|\partial_y\boldsymbol{H}|^2\big)\,d\boldsymbol{y}. \end{aligned} \tag{4.79}$$

Similarly, using (4.68) of Lemma 22 for $u = \nabla\times\boldsymbol{E}$ and $u = \nabla\times\boldsymbol{H}$ with $m = 1$, respectively, we have

$$\begin{aligned}
&\int_{\Xi} \big(\rho(\boldsymbol{y})\varepsilon(\boldsymbol{x},\boldsymbol{y})|\nabla\times(\boldsymbol{E}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{E}^N(t,\boldsymbol{x},\boldsymbol{y}))|^2 \\
&+\rho(\boldsymbol{y})\mu(\boldsymbol{x},\boldsymbol{y})|\nabla\times(\boldsymbol{H}(t,\boldsymbol{x},\boldsymbol{y})-\boldsymbol{H}^N(t,\boldsymbol{x},\boldsymbol{y}))|^2\big)\,d\boldsymbol{y} \\
\le\; & CN^{-2}\int_{\Xi}\big(\rho(\boldsymbol{y})\varepsilon(\boldsymbol{x},\boldsymbol{y})|\partial_y(\nabla\times\boldsymbol{E})|^2+\rho(\boldsymbol{y})\mu(\boldsymbol{x},\boldsymbol{y})|\partial_y(\nabla\times\boldsymbol{H})|^2\big)\,d\boldsymbol{y}. \qquad (4.80)
\end{aligned}$$

Adding (4.79) and (4.80) together, then integrating the resultant with respect to $\boldsymbol{x}$ over D and using Theorems 20 and 24, we complete the proof of (4.75).

The estimates (4.77) can be proved similarly by using (4.68) of Lemma 22 with $m = 2$ and the higher regularity obtained in Theorem 24.

Similarly, using (4.70) of Lemma 22 with $m = 2$, and the higher regularity proved in Theorem 24, we obtain the proof of (4.78).

Finally, the mean errors follow from the standard inequality $\|u\|_{L^1} \le C\|u\|_{L^2}$ and the estimates (4.75), (4.77) and (4.78). □

With the above interpolation estimate, we can show that the overall errors for solving the metamaterial Maxwell's equations by the classical Yee scheme (cf. [41]) are estimated as follows. Denote the electric field solution of Yee scheme for any fixed random vector $\boldsymbol{y}$ as $\boldsymbol{E}^N$, and $\boldsymbol{E}^N_{h,\Delta t}$ for the electric field solution of the fully-discrete solution with stochastic collocation method and Yee's scheme imposed. Denote the discrete L^2-norm over the physical space Ω as $|\cdot|_{l^2(\Omega)}$ (cf. [41]). Then we can obtain the discrete mean square error as following:

$$\begin{aligned}
&\left(\int_{\Xi}\rho\varepsilon|\boldsymbol{E}-\boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(\Omega)}d\boldsymbol{y}\right)^{\frac{1}{2}} \le \left(\int_{\Xi}2\rho\varepsilon(|\boldsymbol{E}-\boldsymbol{E}^N|^2_{l^2(\Omega)}+|\boldsymbol{E}^N-\boldsymbol{E}^N_{h,\Delta t}|^2_{l^2(\Omega)})d\boldsymbol{y}\right)^{\frac{1}{2}} \\
&\le C[N^{-m}+(h^2+(\Delta t)^2)], \qquad (4.81)
\end{aligned}$$

where we used the error estimate of Yee scheme and Theorem 27. The error estimate for other variables can be bounded similarly.

4.3 Numerical results

To justify our theoretical analysis, here we present some numerical results carried out for the metamaterial model in TM_z mode, whose governing equations are:

$$\mu_0 \frac{\partial H_{x_1}}{\partial t} = -\frac{\partial E}{\partial x_2} - K_{x_1} + g_1, \tag{4.82}$$

$$\mu_0 \frac{\partial H_{x_2}}{\partial t} = \frac{\partial E}{\partial x_2} - K_{x_2} + g_2, \tag{4.83}$$

$$\varepsilon_0 \frac{\partial E}{\partial t} = \frac{\partial H_{x_2}}{\partial x_1} - \frac{\partial H_{x_1}}{\partial x_2} - J + g_3, \tag{4.84}$$

$$\frac{\partial J}{\partial t} = \varepsilon_0 \omega_e^2 E - \Gamma_e J + g_4, \tag{4.85}$$

$$\frac{\partial K_{x_1}}{\partial t} = \mu_0 \omega_m^2 H_{x_1} - \Gamma_m K_{x_1} + g_5, \tag{4.86}$$

$$\frac{\partial K_{x_2}}{\partial t} = \mu_0 \omega_m^2 H_{x_2} - \Gamma_m K_{x_2} + g_6, \tag{4.87}$$

where g_i $(1 \le i \le 6)$ are added source terms used to construct exact solutions for checking convergence rates. The parameters μ_0, ε_0, Γ_m, Γ_e, ω_m and ω_e are functions of spatial variable $\boldsymbol{x}$ and random vector $\boldsymbol{y}$.

Example 1. In this test, we choose the following parameters:

$$\varepsilon_0(\boldsymbol{x},\boldsymbol{y}) = 1 + 0.01(\sin(\pi(y_1x_1 + y_2x_2 - 1)) + \cos(\pi(y_3x_1 + y_4x_2 - 1)) + \exp(-y_5x_1 - y_6x_2)),$$

$$\mu_0(\boldsymbol{x},\boldsymbol{y}) = 1 + 0.01(\sin(\pi(y_1x_1 + y_2x_2 - 1)) + \exp(-y_3x_1 - y_4x_2) + \cos(\pi(y_5x_1 + y_6x_2 - 1))),$$

$$\Gamma_e(\boldsymbol{x},\boldsymbol{y}) = \pi + 0.01(\cos(\pi(y_1x_1 + y_2x_2 - 1)) + \sin(\pi(y_3x_1 + y_4x_2 - 1)) + \exp(-y_5x_1 - y_6x_2)),$$

$$\Gamma_m(\boldsymbol{x},\boldsymbol{y}) = \pi + 0.01(\cos(\pi(y_1x_1 + y_2x_2 - 1)) + \exp(-y_3x_1 - y_4x_2) + \sin(\pi(y_5x_1 + y_6x_2 - 1))),$$

$$\omega_e(\boldsymbol{x},\boldsymbol{y}) = \pi + 0.01(\exp(-y_1x_1 - y_2x_2) + \cos(\pi(y_3x_1 + y_4x_2 - 1)) + \sin(\pi(y_5x_1 + y_6x_2 - 1))),$$

$$\omega_m(\boldsymbol{x},\boldsymbol{y}) = \pi + 0.01(\exp(-y_1x_1 - y_2x_2) + \sin(\pi(y_3x_1 + y_4x_2 - 1)) + \cos(\pi(y_5x_1 + y_6x_2 - 1))),$$

where y_i $(1 \le i \le 6)$ are uniform independent random variables on $[0, 1]$.

In our tests, we use Yee scheme (cf. [41]) to solve the TM_z model on physical domain

Table 4.1: Errors of the solutions when the analytic solutions are infinitely smooth in both random and spatial variables.

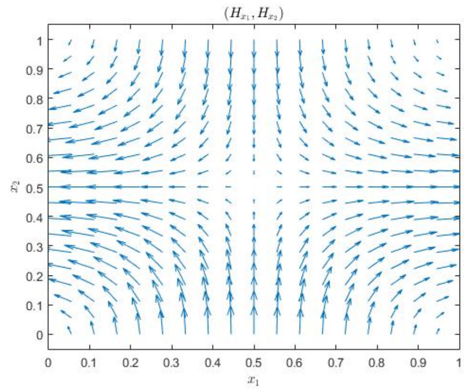
Mesh	1/5	1/10	Rate	1/20	Rate	1/40	Rate
$\mathbb{E}[\|H_{x_1} - H_{x_1}^h\|]$	$9.51281E-03$	$2.25052E-03$	2.0796	$4.90362E-04$	2.1390	$1.16331E-04$	2.1259
$\mathcal{M}[\|H_{x_1} - H_{x_1}^h\|]$	$9.51281E-03$	$2.25090E-03$	2.0794	$4.90442E-04$	2.1389	$1.16353E-04$	2.1258
$\mathbb{E}[\|H_{x_2} - H_{x_2}^h\|]$	$9.51281E-03$	$2.25462E-03$	2.0770	$4.91154E-04$	2.1378	$1.16533E-04$	2.1252
$\mathcal{M}[\|H_{x_2} - H_{x_2}^h\|]$	$9.51281E-03$	$2.25498E-03$	2.0768	$4.91231E-04$	2.1377	$1.16554E-04$	2.1251
$\mathbb{E}[\|E - E_{x_1}^h\|]$	$1.14777E-02$	$2.33418E-03$	2.2978	$5.27242E-04$	2.2221	$1.26109E-04$	2.1670
$\mathcal{M}[\|E - E_{x_1}^h\|]$	$1.14777E-02$	$2.33587E-03$	2.2968	$5.27710E-04$	2.2215	$1.26237E-04$	2.1666
$\mathbb{E}[\|K_{x_1} - K_{x_1}^h\|]$	$8.77292E-03$	$1.71711E-03$	2.3531	$3.72208E-04$	2.2794	$8.73870E-05$	2.2154
$\mathcal{M}[\|K_{x_1} - K_{x_1}^h\|]$	$8.77292E-03$	$1.71767E-03$	2.3526	$3.72326E-04$	2.2792	$8.74154E-05$	2.2153
$\mathbb{E}[\|K_{x_2} - K_{x_2}^h\|]$	$8.77292E-03$	$1.71711E-03$	2.3531	$3.72208E-04$	2.2794	$8.73870E-05$	2.2154
$\mathcal{M}[\|K_{x_2} - K_{x_2}^h\|]$	$8.77292E-03$	$1.71767E-03$	2.3526	$3.72326E-04$	2.2792	$8.74154E-05$	2.2153
$\mathbb{E}[\|J - J_{x_1}^h\|]$	$1.71215E-02$	$3.95199E-03$	2.1152	$9.11318E-04$	2.1159	$2.18246E-04$	2.0998
$\mathcal{M}[\|J - J_{x_1}^h\|]$	$1.71215E-02$	$3.95296E-03$	2.1148	$9.11584E-04$	2.1156	$2.18302E-04$	2.0997

$\Omega = [0,1]^2$ and time domain $[0,1]$ with the exact solution given as

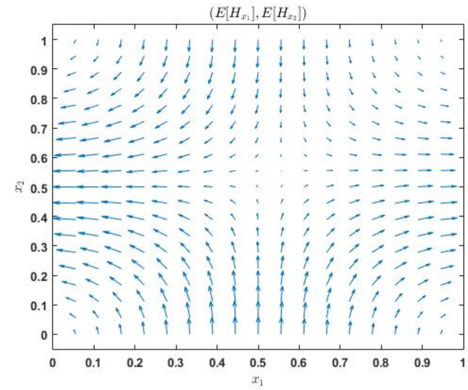
$$
\begin{aligned}
H_{x_1} &= \sin(\pi x_1 + \mu_0)\cos(\pi x_2 + \mu_0)\exp(-\pi t), \\
H_{x_2} &= -\cos(\pi x_1 + \mu_0)\sin(\pi x_2 + \mu_0)\exp(-\pi t), \\
E &= \sin(\pi x_1 + \epsilon_0)\sin(\pi x_2 + \epsilon_0)\exp(-\pi t), \\
K_{x_1} &= \pi^2 t \sin(\pi x_1)\cos(\pi x_2)\exp(-\pi t), \\
K_{x_2} &= -\pi^2 t \cos(\pi x_1)\sin(\pi x_2)\exp(-\pi t), \\
J &= \pi^2 t \sin(\pi x_1)\sin(\pi x_2)\exp(-\pi t).
\end{aligned}
$$

To test the convergence rate, we vary the partition size in the x_1 and x_2 directions $h_{x_1} = h_{x_2} = h$ from 1/5 to 1/40, and time step size from 1/10 to 1/80. We set time partition equals two times of spatial to guarantee the stability. In the same time, the partition numbers in random space vary from 1 to 8. We present the errors of all six components $(H_{x_1}, H_{x_2}, E, K_{x_1}, K_{x_2}, J)$ in the discrete $\mathbb{E}[\cdot]$ and $\mathcal{M}[\cdot]$ in Table 4.1. We can see clearly that all solutions show second order convergence which agrees with our theoretical result, since in this case the exact solution is infinitely smooth in both random and spatial variables and the overall error is dominated by the numerical scheme error.

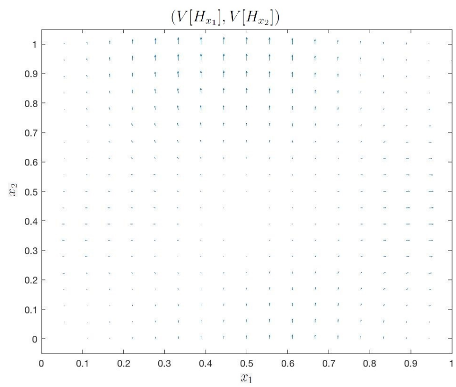
In Figure 4.1, we present one sample magnetic field and its mean and variance obtained by solving the same problem by a 20×20 spatial uniform partition on $[0,1]^2$. We set the initial



(a) The random magnetic field



(b) The mean magnetic field



(c) The variance of magnetic field

Figure 4.1: Comparison of a random sample of magnetic field and its mean and variance obtained with $\boldsymbol{y} = (0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975)$.

conditions and boundary values using the above exact solution and no added source functions. Figure 4.1 is obtained with the random vector $\boldsymbol{y} = (0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975)$, and shows that the mean magnetic field is similar to the sample field in this case.

Example 2

This example is used to test the convergence rate when the solution has limited regularity in the random variables. For simplicity, we use the same exact solution as Example 1 except H_{x_1} being given as:

$$H_{x_1} = \sin(\pi x_1 + \mu_0)\cos(\pi x_2 + \mu_0)\exp(-\pi t) + (y_1 - \frac{\sqrt{2}}{2})^m \mathrm{sgn}\left(y_1 - \frac{\sqrt{2}}{2}\right), \quad m = 1, 2.$$

We choose number $\sqrt{2}/2$ to avoid the case that some interpolation point falls at this cusp

Table 4.2: Errors of the solutions when $H_{x_1} \in H^{3/2-\epsilon}(\Xi)$

N	2	4	Rate	8	Rate	16	Rate
$\mathcal{E}[\|H_{x_1} - H^h_{x_1}\|]$	$2.544284E-01$	$5.899390E-02$	1.8103	$1.560676E-02$	1.6431	$6.047332E-03$	1.3678
$\mathcal{M}[\|H_{x_1} - H^h_{x_1}\|]$	$2.972258E-01$	$7.969676E-02$	1.6161	$2.654922E-02$	1.4797	$1.024674E-02$	1.3735
$\mathcal{E}[\|H_{x_2} - H^h_{x_2}\|]$	$9.506088E-06$	$1.570312E-05$	0.9623	$5.537782E-06$	1.7153	$1.456298E-06$	1.9270
$\mathcal{M}[\|H_{x_2} - H^h_{x_2}\|]$	$1.576924E-05$	$1.125031E-05$	1.1302	$5.549302E-06$	1.4701	$1.465709E-06$	1.9207
$\mathcal{E}[\|K_{x_1} - K^h_{x_1}\|]$	$6.473112E-04$	$1.479130E-04$	2.0982	$3.443082E-05$	2.0817	$8.255164E-06$	2.0603
$\mathcal{M}[\|K_{x_1} - K^h_{x_1}\|]$	$6.473390E-04$	$1.479207E-04$	2.0982	$3.443912E-05$	2.0816	$8.255810E-06$	2.0606
$\mathcal{E}[\|K_{x_2} - K^h_{x_2}\|]$	$5.212914E-04$	$1.306984E-04$	2.0072	$3.209770E-05$	2.0099	$8.057718E-06$	1.9940
$\mathcal{M}[\|K_{x_2} - K^h_{x_2}\|]$	$5.213384E-04$	$1.307129E-04$	2.0070	$3.211124E-05$	2.0095	$9.062296E-06$	1.9938
$\mathcal{E}[\|E - E^h\|]$	$1.291961E-04$	$4.004254E-05$	1.9222	$9.690372E-06$	2.0176	$2.442342E-06$	1.9883
$\mathcal{M}[\|E - E^h\|]$	$1.297785E-04$	$4.008978E-05$	1.9236	$9.704608E-06$	2.0176	$2.445378E-06$	1.9886
$\mathcal{E}[\|J - J^h\|]$	$1.206363E-03$	$3.078828E-04$	1.9984	$7.689780E-05$	2.0120	$1.892578E-05$	2.0226
$\mathcal{M}[\|J - J^h\|]$	$1.206383E-03$	$3.078850E-04$	1.9984	$7.690346E-05$	2.0119	$1.892703E-05$	2.0226

point. The corresponding source terms are obtained by plugging the exact solution into the governing equations. It is easy to check that the exact solutions are infinitely smooth except that $H_{x_1}(\Xi) \in H^{m+1/2-\epsilon}(\Xi)$ when $m = 1, 2$, respectively.

To investigate the convergence rate, we initialize the partition number for x_1, x_2, t and y as 10, 20, 40 and 2 respectively to make a uniform spatial and temporal partition and use a Gauss-Lobatto points for each random space. Then we double all partition numbers three times.

The numerical results of original solutions are given in Tables 4.2 and 4.3 for $m = 1$ and $m = 2$, respectively. Table 4.2 shows that the error of H_{x_1} is about $O(N^{-1.3})$ in both mean and mean square norm defined earlier, and errors of other solutions are still $O(N^{-2})$ due to their infinite smoothness. This is consistent with our theoretical analysis. When $m = 2$, all the solutions have $O(N^{-2})$ convergence, which shows clearly by the results stated in Table 4.3. Notice the rate of H_{x_1} is limited to 2 due to the 2nd convergent rate of the FDTD scheme. We also plotted the variances of the electric fields at variance times in Figure 4.2.

Example 3

In this example, we solve a classic example showing the backward wave propagation in metamaterials (cf. [42, 41]). This example assumes that a metamaterial slab of size $[0.024, 0.054]m \times [0.002, 0.062]m$ is located inside a vacuum of size $[0, 0.07]m \times [0, 0.064]m$. An incident source wave is imposed as E field and excited at line $x_1 = 0.004m$ ranging from

Table 4.3: Errors of the solutions when $H_{x_1} \in H^{5/2-\epsilon}(\Xi)$

N	2	4	Rate	8	Rate	16	Rate
$\mathcal{E}[\lvert H_{x_1} - H^h_{x_1}\rvert]$	$4.707117E-02$	$6.770013E-03$	2.7029	$7.617172E-04$	2.5807	$1.891617E-04$	2.0096
$\mathcal{M}[\lvert H_{x_1} - H^h_{x_1}\rvert]$	$5.827503E-02$	$8.021087E-03$	2.6621	$1.028223E-03$	2.5124	$2.463826E-04$	2.0612
$\mathcal{E}[\lvert H_{x_2} - H^h_{x_2}\rvert]$	$2.053239E-05$	$3.153899E-06$	1.9998	$1.204564E-06$	1.7503	$2.786668E-07$	2.1119
$\mathcal{M}[\lvert H_{x_2} - H^h_{x_2}\rvert]$	$2.361496E-05$	$3.209744E-06$	2.0270	$1.213414E-06$	1.7048	$3.020466E-07$	2.0062
$\mathcal{E}[\lvert K_{x_1} - K^h_{x_1}\rvert]$	$3.563472E-04$	$8.236230E-05$	2.0961	$2.025432E-05$	2.0996	$4.483891E-06$	2.1754
$\mathcal{M}[\lvert K_{x_1} - K^h_{x_1}\rvert]$	$3.563567E-04$	$8.236546E-05$	2.0961	$2.025525E-05$	2.0996	$4.484184E-06$	2.1754
$\mathcal{E}[\lvert K_{x_2} - K^h_{x_2}\rvert]$	$3.563472E-04$	$8.236230E-05$	2.0961	$2.025432E-05$	2.0996	$4.483891E-06$	2.1754
$\mathcal{M}[\lvert K_{x_2} - K^h_{x_2}\rvert]$	$3.563567E-04$	$8.236546E-05$	2.0961	$2.025525E-05$	2.0996	$4.484184E-06$	2.1754
$\mathcal{E}[\lvert E - E^h\rvert]$	$8.567125E-05$	$1.922316E-05$	2.0511	$4.694926E-06$	2.0016	$1.198734E-06$	1.9696
$\mathcal{M}[\lvert E - E^h\rvert]$	$8.570607E-05$	$1.923237E-05$	2.0509	$4.697663E-06$	2.0013	$1.199810E-06$	1.9691
$\mathcal{E}[\lvert J - J^h\rvert]$	$4.566191E-04$	$1.075776E-04$	2.0366	$2.599359E-05$	2.0100	$6.630678E-06$	1.9709
$\mathcal{M}[\lvert J - J^h\rvert]$	$4.566465E-04$	$1.075871E-04$	2.0366	$2.599465E-05$	2.0100	$6.631424E-06$	1.9708

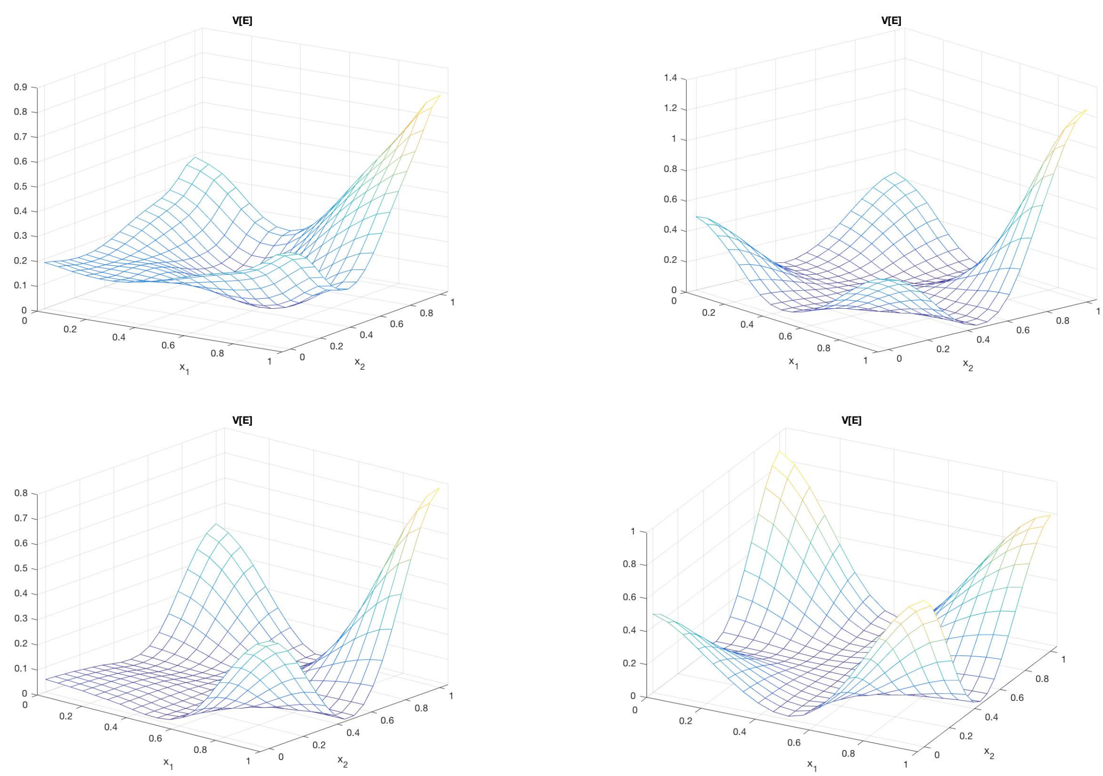


Figure 4.2: Example 2. The variances of electronic fields at $t = 0.25$ (Top left), $t = 0.5$ (Top right), $t = 0.75$ (Bottom left) and $t = 1$ (Bottom right).

$x_2 = 0.025$m to $x_2 = 0.035$m. The source wave varies in space as $\exp(-(x_1 - 0.03)^2/(50h)^2)$ where h is the partition size in space, and in time as:

$$f(t) = \begin{cases} 0 & t < 0 \text{ or } t > (2m+k)T_p \\ g_1(t)\sin(\omega_0 t) & 0 \le t < mT_p \\ \sin(\omega_0 t) & mT_p \le t < (m+k)T_p \\ g_2(t)\sin(\omega_0 t) & (m+k)T_p \le t < (2m+k)T_p \end{cases}$$

where

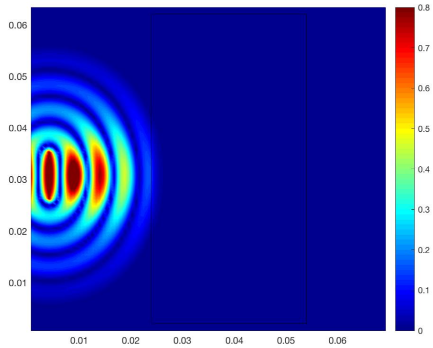
$$g_1(t) = 10a^3 - 15a^4 + 6a^5, \qquad a = t/mT_p$$
$$g_2(t) = 1 - (10b^3 - 15b^4 + 6b^5), \qquad b = (t - (m+k)T_p)/mT_p$$

here $T_p = 1/f_0$ and $\omega_0 = 2\pi f_0$. In this simulation, $m = 2$, $k = 100$, $f_0 = 30$GHz.

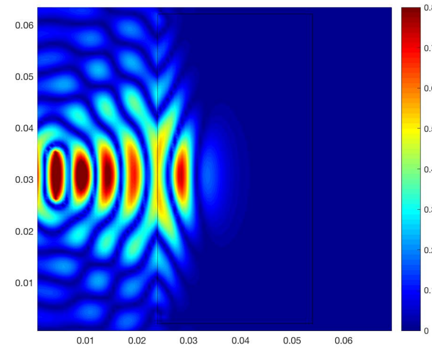
This model is solved on a uniform mesh with time step size $\tau = 10^{-13}$s $= 0.1$ps and 12 perfectly matched layer (PML) imposed around the physical domain. Details can refer to our previous work [41]. We use the following random parameters for our simulation:

$$\begin{aligned} \varepsilon_0(\boldsymbol{x}, \boldsymbol{y}) &= 1.11 \times 10^{-11}(1 + y_1 + y_2), \\ \mu_0(\boldsymbol{x}, \boldsymbol{y}) &= 10^{-6}/(1 + y_1 + y_2), \\ \Gamma_m(\boldsymbol{x}, \boldsymbol{y}) &= 10^8(1 + 10^{-4}(y_3 - 0.5)), \\ \Gamma_e(\boldsymbol{x}, \boldsymbol{y}) &= 10^8(1 + 10^{-4}(y_4 - 0.5)), \\ \omega_m(\boldsymbol{x}, \boldsymbol{y}) &= 2\pi\sqrt{2} \times 3 \times 10^{10}(1 + 10^{-4}(y_5 - 0.5)), \\ \omega_e(\boldsymbol{x}, \boldsymbol{y}) &= 2\pi\sqrt{2} \times 3 \times 10^{10}(1 + 10^{-4}(y_6 - 0.5)). \end{aligned}$$

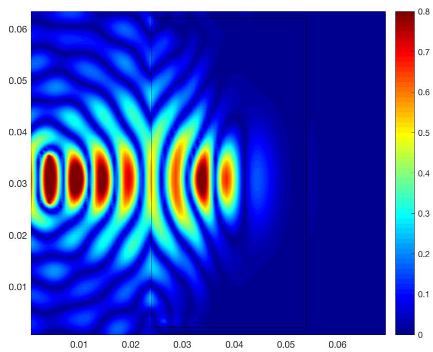
The obtained electric field at various time steps are plotted in Figure 4.3, which shows that as the source wave enters the metamaterial slab, the wave propagates backward due to the negative refractive index of the metamaterial and propagates forward after the wave moves out the metamaterial subdomain. This example shows that the backward wave propagation phenomenon still exists in the random metamaterial.



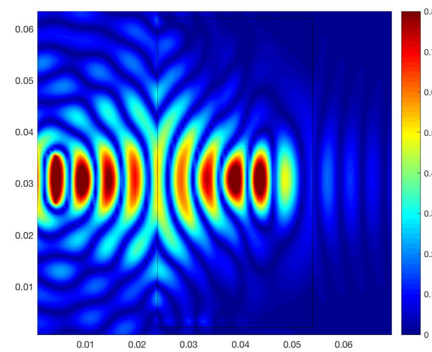
$|E|$ at time step 100



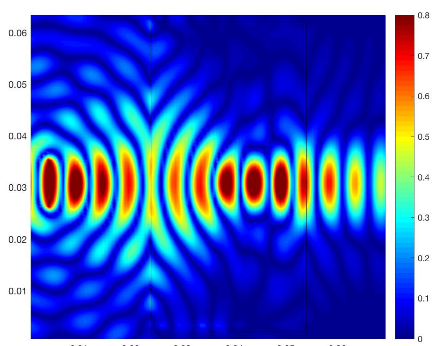
$|E|$ at time step 200



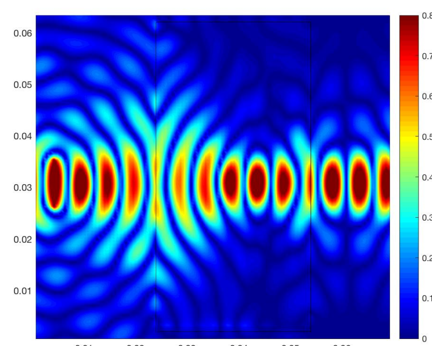
$|E|$ at time step 300



$|E|$ at time step 400



$|E|$ at time step 500



$|E|$ at time step 600

Figure 4.3: The contour plot of electric field $|E|$ at various time steps.

CHAPTER 5

CONCLUSION AND FUTURE WORK

5.1 Conclusion

This dissertation focused on the uncertainty quantification in electromagnetic fields, including the efficient algorithm, regularity and error analysis, and numerical simulation. The stochastic Galerkin method, quasi Monte Carlo method, and stochastic collocation method have been discussed, and the numerical experiments have been shown to verify the theoretical results.

In chapter 2, we use the stochastic Galerkin method to solve the standard Maxwell's equations with random inputs. We first showed that the stochastic Galerkin method is energy preserved. Additionally, we come up with the spatial finite element method and two different time. The modified leap-frog type scheme was designed to further reduce the computational cost. The numerical experiments are used to support our theoretical results.

In chapter 3, we discuss the single and multi-level Monte Carlo methods and quasi Monte Carlo method for Maxwell's equation with random inputs. The advantage of Monte Carlo class methods is that all the governor PDEs for different samples are totally decoupled. Hence, it is super easy to establish the parallel algorithm for stochastic PDEs problems by using Monte Carlo class methods. The convergence results have been proved, and the numerical experiments have been displayed to verify our theoretical results.

In chapter 4, we establish the regularity and error analysis of the time-dependent Maxwell's equations in Drude metamaterial with randomness. In the numerical experiments, we first verify the convergence rates and compare the CPUs' times. And then, we demonstrate the backward wave propagation phenomenon happened when the electromagnetic wave travels in the random metamaterial.

5.2 Future work

The uncertainty quantification is still a popular topic in mathematical and engineering area. Due to curse of dimensionality, it still deserves our efforts to develop more efficient algorithms. Recently, a machine learning method has been developed to solve the stochastic PDEs problems with amazing efficiency. This new idea can be used in stochastic electromagnetic fields.

Another promising idea is the high level quasi Monte Carlo method, for example, multilevel quasi Monte Carlo method. As showed in chapter 3, the Monte Carlo class methods is decoupled but their accuracy is low. Quasi Monte Carlo may conquer this issue, and then balance the computational workload and the accuracy.

I will also keep working on another area I am interested in. In [43], we solved metamaterial design problems by using optimal control method. But the theoretical analysis, such as existence, smoothness of the solution, are still unsolved. These meaningful works worth our best efforts to solve them.

For the classic numerical algorithms, such as finite element method and Yee scheme, there are lot of work should be done. In [44], we shown analysis of Ziolkowskis PML problems. In [45], we presented the analysis of Crank-Nicolson Yee scheme. Those works can be continued for more complicate models.

In [46], we shown a FDTD scheme for Kerr-type nonlinear media. Actually, this algorithm can be more efficient if compact finite difference scheme or hp finite element methods have been used.

Finally, I am also interested in a variational approach for PDEs. In [47], we showed a symmetry breaking result by using bifurcation and critical group. The symmetry preservation problem will be a potential future work.

BIBLIOGRAPHY

[1] Chauviere, C., Hesthaven, J.S. and Lurati, L., 2006. *Computational modeling of uncertainty in time-domain electromagnetics*. SIAM Journal on Scientific Computing, 28(2), pp.751-775.

[2] Benner, P. and Schneider, J., 2015. *Uncertainty quantification for Maxwell's equations using stochastic collocation and model order reduction*. International Journal for Uncertainty Quantification, 5(3).

[3] Hong, J., Ji, L. and Zhang, L., 2014. *A stochastic multi-symplectic scheme for stochastic Maxwell equations with additive noise*. Journal of Computational Physics, 268, pp.255-268.

[4] Horsin, T., Stratis, I.G. and Yannacopoulos, A.N., 2010. *On the approximate controllability of the stochastic Maxwell equations*. IMA Journal of Mathematical Control and Information, 27(1), pp.103-118.

[5] Babuška, I., Nobile, F. and Tempone, R., 2007. *A stochastic collocation method for elliptic partial differential equations with random input data*. SIAM Journal on Numerical Analysis, 45(3), pp.1005-1034.

[6] Babuška, I., Tempone, R. and Zouraris, G.E., 2004. *Galerkin finite element approximations of stochastic elliptic partial differential equations*. SIAM Journal on Numerical Analysis, 42(2), pp.800-825.

[7] Ghanem, R.G. and Spanos, P.D., 2003. *Stochastic finite elements: a spectral approach*. Courier Corporation.

[8] Gunzburger, M.D., Webster, C.G. and Zhang, G., 2014. *Stochastic finite element methods for partial differential equations with random input data*. Acta Numerica, 23, pp.521-650.

[9] Caflisch, R.E., 1998. *Monte carlo and quasi-monte carlo methods*. Acta numerica, 7, pp.1-49.

[10] Barth, A., Schwab, C. and Zollinger, N., 2011. *Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients*. Numerische Mathematik, 119(1), pp.123-161.

[11] Xiu, D. and Karniadakis, G.E., 2002. *The Wiener-Askey polynomial chaos for stochastic differential equations*. SIAM journal on scientific computing, 24(2), pp.619-644.

[12] Fang, Z., Li, J., Tang, T. and Zhou, T., 2019. *Efficient stochastic Galerkin methods for Maxwells equations with random inputs*. Journal of Scientific Computing, 80(1), pp.248-267.

[13] Xiu, D. and Shen, J., 2009. *Efficient stochastic Galerkin methods for random diffusion equations*. Journal of Computational Physics, 228(2), pp.266-281.

[14] Zhang, Z. and Karniadakis, G., 2017. *Numerical methods for stochastic partial differential equations with white noise*. Springer International Publishing.

[15] Monk, P., 2003. *Finite element methods for Maxwell's equations*. Oxford University Press.

[16] Li, J., Huang, Y.: *Time-Domain Finite Element Methods for Maxwells Equations in Metamaterials*. Springer Series in Computational Mathematics, vol. 43. Springer, Berlin (2013)

[17] Cao, Y., 2006. *On convergence rate of Wiener-Ito expansion for generalized random variables*. Stochastics: An International Journal of Probability and Stochastics Processes, 78(3), pp.179-187.

[18] Galvis, J. and Sarkis, M., 2009. *Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity*. SIAM Journal on Numerical Analysis, 47(5), pp.3624-3651.

[19] Balanis, C.A.: *Advanced Engineering Electromagnetics*, 2nd edn. Wiley, Hoboken, NJ (2012)

[20] Cliffe, K.A., Giles, M.B., Scheichl, R. and Teckentrup, A.L., 2011. *Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients*. Computing and Visualization in Science, 14(1), p.3.

[21] Graham, I.G., Kuo, F.Y., Nichols, J.A., Scheichl, R., Schwab, C. and Sloan, I.H., 2015. *Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients*. Numerische Mathematik, 131(2), pp.329-368.

[22] Harbrecht, H., Peters, M. and Siebenmorgen, M., 2016. *Multilevel accelerated quadrature for PDEs with log-normally distributed diffusion coefficient*. SIAM/ASA Journal on Uncertainty Quantification, 4(1), pp.520-551.

[23] Xiang Wang, Jichun Li, and Zhiwei Fang, *Analysis and application of single level, multilevel Monte Carlo and quasi-Monte Carlo Finite element methods for Maxwell's equations with random inputs*. publish soon.

[24] Kuo, F.Y., Schwab, C. and Sloan, I.H., 2012. *Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients*. SIAM Journal on Numerical Analysis, 50(6), pp.3351-3374.

[25] Dick, J., Kuo, F.Y. and Sloan, I.H., 2013. *High-dimensional integration: the quasi-Monte Carlo way*. Acta Numerica, 22, pp.133-288.

[26] Cools, R., Kuo, F.Y. and Nuyens, D., 2006. *Constructing embedded lattice rules for multivariate integration*. SIAM Journal on Scientific Computing, 28(6), pp.2162-2188.

[27] Sloan, I.H. and Woźniakowski, H., 1998. *When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?*. Journal of Complexity, 14(1), pp.1-33.

[28] Dick, J., Sloan, I.H., Wang, X. and Woźniakowski, H., 2004. *Liberating the weights*. Journal of Complexity, 20(5), pp.593-623.

[29] Sloan, I.H., Wang, X. and Woźniakowski, H., 2004. *Finite-order weights imply tractability of multivariate integration*. Journal of Complexity, 20(1), pp.46-74.

[30] Huang, Y., Li, J. and Wu, C., 2013. *Averaging for superconvergence: verification and application of 2D edge elements to Maxwells equations in metamaterials*. Computer Methods in Applied Mechanics and Engineering, 255, pp.121-132.

[31] Xiu, D. and Hesthaven, J.S., 2005. *High-order collocation methods for differential equations with random inputs*. SIAM Journal on Scientific Computing, 27(3), pp.1118-1139.

[32] F. Nobile, R. Tempone and C.G. Webster, *A sparse grid stochastic collocation method for partial differential equations with random input data*, SIAM J. Numer. Anal. 46(5) (2008) 2309-2345.

[33] M. Motamed, F. Nobile and R. Tempone, *A stochastic collocation method for the second order wave equation with a discontinuous random speed*, Numer. Math. 123 (2013) 493-536.

[34] C. Schwab and C.J. Gittelson, *Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs*, Acta Numer. 20 (2011) 291-467.

[35] D. Xiu, *Numerical Methods for Stochastic Computations: A Spectral Method Approach*, Princeton University Press, 2010.

[36] G.J. Lord, C.E. Powell and T. Shardlow, *An Introduction to Computational Stochastic PDEs*, Cambridge University Press, Cambridge, 2014.

[37] Li, J. and Fang, Z., 2018. *Analysis and application of stochastic collocation methods for Maxwells equations with random inputs*. Advances in Applied Mathematics and Mechanics, 10(6), p.1305.

[38] Li, J., Fang, Z. and Lin, G., 2018. *Regularity analysis of metamaterial Maxwells equations with random coefficients and initial conditions*. Computer Methods in Applied Mechanics and Engineering, 335, pp.24-51.

[39] J. Tryoen, O. LeMaitre, M. Ndjinga and A.Ern, *Intrusive Galerkin methods with upwinding for uncertain nonlinear hyperbolic systems*, J. Comput. Phys. 229 (2010) 6485-6511.

[40] X. Wan and G.E. Karniadakis, *Solving elliptic problems with spatially-dependent non-Gaussian random inputs: algorithms, error analysis and applications*, Comput. Methods Appl. Mech. Engrg. 198 (2009) 1985-1995.

[41] J. Li and S. Shields, *Superconvergence analysis of Yee scheme for metamaterial Maxwell's equations on non-uniform rectangular meshes*, Numer. Math. 134 (2016) 741-781.

[42] J. Li and J. Hesthaven, *Analysis and application of the nodal discontinuous Galerkin method for wave propagation in metamaterials*, J. Comput. Phys. 258 (2014) 915-930.

[43] Fang, Z., Li, J. and Wang, X., 2019.*Optimal control for electromagnetic cloaking metamaterial parameters design*. Computers & Mathematics with Applications.

[44] Huang, Y., Li, J. and Fang, Z., 2020. *Mathematical analysis of Ziolkowskis PML model with application for wave propagation in metamaterials*. Journal of Computational and Applied Mathematics, 366, p.112434.

[45] Wang, X., Li, J. and Fang, Z., 2018. *Development and analysis of CrankNicolson scheme for metamaterial Maxwell's equations on nonuniform rectangular grids*. Numerical Methods for Partial Differential Equations, 34(6), pp.2040-2059.

[46] Jia, H., Li, J., Fang, Z. and Li, M., 2018. *A new FDTD scheme for Maxwells equations in Kerr-type nonlinear media*. Numerical Algorithms, pp.1-21.

[47] Costa, D.G. and Fang, Z., 2019. *A note on breaking of symmetry for a class of variational problems*. Applied Mathematics Letters, 98, pp.329-335.

[48] Kuo, F.Y., Schwab, C. and Sloan, I.H., 2011. *Quasi-Monte Carlo methods for high-dimensional integration: the standard (weighted Hilbert space) setting and beyond*. The ANZIAM Journal, 53(1), pp.1-37.

[49] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral Methods Fundamentals in Single Domains*, Springer-Verlag, 2006.

CURRICULUM VITAE

Graduate College
University of Nevada, Las Vegas

Zhiwei Fang

Email: zhiweifang1987@gmail.com

Degrees:

Bachelor of Science, Hospitality Administration, 2011
Tianjin University of Finance and Economics, Tianjin, China

Master of Science, Mathematical Sciences, 2014
Tianjin University of Finance and Economics, Tianjin, China

Doctor of Philosophy, Mathematics, 2020
University of Nevada, Las Vegas

Special Honors and Awards:

UNLV Summer Doctoral Research Fellowship (2018)

Publications (During Ph.D.):

Fang, Z. and Zhan, J., 2019. *A Physics-Informed Neural Network Framework For Partial Differential Equations on 3D Surfaces: Time Independent Problems.* IEEE Access.

Fang, Z. and Zhan, J., 2019. *Deep Physical Informed Neural Networks For Metamaterial Design.* IEEE Access.

Fang, Z., Li, J. and Wang, X., 2019. *Optimal control for electromagnetic cloaking metamaterial parameters design.* Computers & Mathematics with Applications.

Fang, Z., Li, J., Tang, T. and Zhou, T., 2019. *Efficient Stochastic Galerkin Methods for Maxwells Equations with Random Inputs.* Journal of Scientific Computing, pp.1-20.

Huang, Y., Li, J. and Fang, Z., 2019. *Mathematical analysis of Ziolkowskis PML model with application for wave propagation in metamaterials.* Journal of Computational and Applied Mathematics, p.112434.

Costa, D.G. and Fang, Z., 2019. *A note on breaking of symmetry for a class of variational problems.* Applied Mathematics Letters.

Li, J., Fang, Z. and Lin, G., 2018. *Regularity analysis of metamaterial Maxwells equations with random coefficients and initial conditions.* Computer Methods in Applied Mechanics and Engineering, 335, pp.24-51.

Wang, X., Li, J. and Fang, Z., 2018. *Development and analysis of CrankNicolson scheme for metamaterial Maxwell's equations on nonuniform rectangular grids.* Numerical Methods for Partial Differential Equations.

Jia, H., Li, J., Fang, Z. and Li, M., 2018. *A new FDTD scheme for Maxwells equations in Kerr-type nonlinear media.* Numerical Algorithms, pp.1-21.

Li, J. and Fang, Z., 2018. *Analysis and Application of Stochastic Collocation Methods for Maxwells Equations with Random Inputs.* Adv. Appl. Math. Mech., Vol. 10, No. 6, pp. 1305-1326.

Publications (Before Ph.D.):

Wang, S., Zhang, S. and Fang, Z., 2015. *A superconvergent fitted finite volume method for BlackScholes equations governing European and American option valuation.* Numerical Methods for Partial Differential Equations, 31(4), pp.1190-1208.

Chang, S., Fang, Z., Liu, X. and Shaydurov, V., 2014. *A fitted finite volume method for unit-linked policy with surrender option.* Computational Research, 2(3), pp.49-53.

Dissertation Title: Uncertainty Quantification for Maxwell's Equations

Dissertation Examination Committee:
Chairperson, Jichun Li, Ph.D.
Committee Member, Hongtao Yang, Ph.D.
Committee Member, Pengtao Sun, Ph.D.
Committee Member, Monika Neda, Ph.D.
Graduate Faculty Representative, Robert A. Schill Jr., Ph.D.