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Equivalences of Determinacy Between Levels of the Borel Hierarchy and Long Games, and Some Generalizations

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EQUIVALENCES OF DETERMINACY BETWEEN LEVELS OF THE BOREL HIERARCHY AND LONG GAMES, AND SOME GENERALIZATIONS

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Abstract

This thesis will be primarily focused on directly proving that the determinacy of Borel games in X^ω is equivalent to the determinacy of certain long open games, from a fragment of ZFC that's well-known to be insufficient to prove Borel determinacy. The main theorem is a level by level result which shows the equivalence between determinacy of open games in a long tree, $[\Upsilon^\alpha]$, and determinacy of $\mathbf{\Sigma}^0_\alpha$ games in X^ω. In Chapter 9, we mimic the proof used in our main theorem to show that the determinacy of clopen games in the product space $X^\omega \times \omega^\omega$ (i.e., $\prod_{i \in \omega+\omega} X_i$ with $X_i = X$ for $i < \omega$ and $X_i = \omega$ otherwise) is equivalent to Borel determinacy in X^ω. In particular, the determinacy of clopen games from $\omega^{\omega+\omega}$ is equivalent to Borel determinacy in ω^ω.

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First and foremost, I would like to thank my advisor, Dr. Derrick DuBose, for the incredible amount of time and energy he put into helping me finish my thesis. I would not have been able to accomplish this without him. Next, I would like to thank my committee members: Dr. Douglas Burke, Dr. Peter Shiue, and Dr. Pushkin Kachroo. Finally, my family and friends have been a constant source of support and encouragement.

Preface

This thesis will be primarily focused on directly proving that the determinacy of Borel games in X^ω is equivalent to the determinacy of certain long open games, from a fragment of ZFC that's well-known to be insufficient to prove Borel determinacy.

The landscape of our results lies directly below determinacy requiring large cardinals. The determinacy of $\mathbf{\Sigma}^0_1 \restriction \omega^{\omega+\omega}$ cannot be proven without the existence of large cardinals. In fact, the determinacy of $\mathbf{\Sigma}^0_1 \restriction \omega^{\omega+\omega}$ is equivalent to the determinacy of $\mathbf{\Pi}^1_1 \restriction \omega^\omega$, which in turn is equivalent to the existence of $r^\#$ for every $r \in \omega^\omega$. We will be working with particular trees Υ^α (see Definition 3.10) of height $\omega + \omega$, but whose paths all have length strictly less than $\omega + \omega$. We show in Chapter 5 that for all countable limit ordinals α, open determinacy in Υ^α is equivalent to the determinacy of $\mathbf{\Sigma}^0_\alpha \restriction X^\omega$. This is the main theorem of this paper. As a direct corollary to this, $\mathrm{Det}\,(\mathbb{B} \restriction X^\omega)$, i.e. Borel determinacy in X^ω, is equivalent to $\forall$ limit ordinals $\alpha < \omega_1\,(\mathrm{Det}(\mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha]))$. A corollary (which we show in Chapter 9) to the proof of Theorem 5.1, is that

$$\mathrm{Det}\,(\mathbb{B} \restriction X^\omega) \iff \mathrm{Det}\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times \omega^\omega\right).$$

Part I is primarily devoted to proving the main theorem. Chapter 1 will include some basic background in the field of determinacy which will be utilized in the later sections. In Chapter 3 we create the Υ^α tree (for countable limit ordinals α), and in Chapter 2, we establish some functions that allow us to move back and forth between the standard tree X^ω and our new trees Υ^α. Chapter 4 explains how we will use the trees Υ^α for decomposing (or constructing) Borel sets. This will all culminate in Chapter 5 with the equivalence result, Theorem 5.1, described above.

In Part II, we extend the work already done in Part I. We generalize the results of Theorem 5.1, replacing X^ω by the body $[T]$ of certain trees. In fact, X^ω never appears in any definition, proof, or theorem in Chapters 2, 3, or 4 (with the exception of Definition 3.10 which defines the tree to be used in Chapter 5). X^ω only appears to explain the motivation of these results, to remind the readers of the primary goal of proving Theorem 5.1. Furthermore, even in Chapter 5, we do not rely on any particular properties of X^ω to prove Theorem 5.1. Moreover, for γ and α countable limit ordinals, we will see that our construction from $\mathbf{\Sigma}^0_1 \restriction \chi$ to $\mathbf{\Sigma}^0_\alpha \restriction \chi$ and vice versa, with some minor adjustments, take us from $\mathbf{\Sigma}^0_\gamma \restriction \chi$ to $\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$, as will be shown in Chapter 6.

Chapter 7 will mimic Chapter 5 with the adjustments described above taken into account. Chapter 8 will handle the case of $\mathbf{\Sigma}^0_{\alpha+n}$ determinacy for a finite n and α a countable limit ordinal, and as mentioned before, Chapter 9 proves

$$\mathrm{Det}\left(\mathbb{B} \restriction X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Delta}^0_1 \restriction X^\omega \times \omega^\omega\right).$$

As alluded to earlier, we are informally working in a fragment of ZFC insufficient to prove Borel determinacy. By an early result of Friedman [8], Borel determinacy in the tree $\omega^{<\omega}$

requires ω_1 iterations of the Power Set Axiom. Moreover, well-known results of Friedman and Martin have shown that for $\alpha < \omega_1$, Det $\left(\mathbf{\Sigma}^0_{1+\alpha+3}\right)$ requires $\alpha + 1$ iterations of the power set axiom [10], [16]. In particular, for $\alpha < \omega_1$,

$$\text{ZFC}^- + \mathscr{P}^\alpha(\omega) \text{ exists} \nvdash \text{Det}\left(\mathbf{\Sigma}^0_{1+\alpha+3} \restriction \omega^\omega\right).$$

(Here, ZFC$^-$ refers to ZFC without the power set axiom.) Sherwood Hachtman investigates the strength of Det $\left(\mathbf{\Sigma}^0_{1+\alpha+3} \restriction \omega^\omega\right)$ in [10]. Also, $\mathbf{\Sigma}^0_3 \restriction \omega^\omega$ determinacy is known to hold in ZC$^-$ + Σ_1 Replacement. For such optimal results, the best resource is Martin's book on determinacy [16].

As described above, we are proving equivalences of determinacy. In particular, we are proving Theorem 5.1:

$$\forall \text{ limit ordinals } \alpha < \omega_1 \left(\text{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega) \iff \text{Det}(\mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha])\right).$$

In this thesis, we do not iterate the Power Set Axiom as described in the previous paragraph. However, we do make modest use of power sets in order to form the body of trees, even when T is countable. Furthermore, our equivalences involve $X^{<\omega}$, with X arbitrary. For some X, one might need the Power Set Axiom to produce X. However, given X, we only mildly use the Power Set Axiom, and informally work in a fragment of ZFC that is insufficient to prove Borel determinacy.

An excellent resource for Borel and Projective determinacy is Martin's book [16]. In addition to this, the following sources are standard references for the material in this thesis:

1. T. Jech, Descriptive Set Theory, 3rd ed., Springer, Berlin, Germany, 2003.

2. A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, NY, 2010.

3. Y.N. Moschovakis, Descriptive set theory, 2nd ed., American Mathematical Society, Providence, R.I., 2009

4. Itay Neeman, The determinacy of long games: de gruyter series in logic and its applications, Vol. 7, Walter de Gruyter GmbH, Berlin, Germany, 2004.

5. John R. Steel, Long games (Steel J.R. Kechris A.S. Martin D.A., ed.), Lecture Notes in Mathematics, vol. 1333, Springer, Cabal Seminar 81–85, Berlin, Germany, 1988.

We include these along with our other references on listed on page 110.

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Part I

The Main Theorem

Part I is mainly devoted to proving Theorem 5.1, which states:

For $\alpha \in (0, \omega_1)$ a limit ordinal and X a nonempty set,

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega) \iff \mathrm{Det}(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast [T^\alpha_{g.t.}]).^{1}$$

Chapter 1 will include some basic background in the field of determinacy which will be utilized in the later sections. In Chapter 3 we create the Υ^α tree,[2] and in Chapter 2 we establish some functions that allow us to move back and forth between the standard tree X^ω and our new trees Υ^α. Chapter 4 explains how we will use the trees Υ^α for decomposing or constructing Borel sets. This will all culminate in Chapter 5 with the main result, Theorem 5.1, described above.

[1] The operation $\circledast$ is defined on page 6 in Definition 1.9.

[2] $[\Upsilon^\alpha] = X^\omega \circledast [T^\alpha_{g.t.}]$. See Definition 3.5 and Definition 3.10

Chapter 1

Preliminaries

Next we begin with the preliminary information presented in this chapter. Chapter 1 will include some basic background in the field of determinacy which will be utilized in the later sections.

In Section 1.1 and 1.2, we give basic definitions regarding two-player games, strategies, and determinacy. In Section 1.3, we begin by providing definitions regarding topology. In Theorem 1.2, we show that in a tree T with countable height (which are the types of trees that appear in this thesis) $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$, in which case:

- If α is a limit ordinal, $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, $\alpha = \sup_{i \in \omega} \beta_i$, and each $\beta_i < \alpha$, then A can be written as a countable union of sets A_i, where each $A_i \in \mathbf{\Pi}^0_{\beta_i} \restriction [T]$.

- If α is a successor ordinal and $A \in \mathbf{\Sigma}^0_\alpha \restriction [T]$, then A can be written as a countable union of sets from $\mathbf{\Pi}^0_{\alpha-1} \restriction [T]$.

In Comment 1.2.1, we provide a counterexample to demonstrate that there are topological spaces where Theorem 1.2 does not hold.

1.1 Two-Player Games

To start, let us define a sequence:

Definition 1.1 (Sequence)**.** *A sequence x is a function with some ordinal α as its domain. In this case, α is the length of x, and is denoted by $\ell n\,(x)$.*

Commonly the notation $x = \langle x_n \mid n \in \alpha \rangle$ is used for sequences. Due to the convenient nature of this notation, we will also sometimes use this notation for other functions. Specifically, we will refer to x as an A-sequence to mean a function with the domain A and use the notation $x = \langle x_a \mid a \in A \rangle$. Often in this thesis, the A will be a game tree, T, as defined below.

Definition 1.2 (Game Tree)**.** *A game tree T is a set of sequences such that for any sequence q and for any sequence $p \in T (q \subset p \to q \in T)$.*

Here, $x \subset y$ or $y \supset x$ means that the set y properly extends the set x (in particular, $x \neq y$). Thus, a game tree is a set of sequences which is closed under initial segments. Additionally, we have a contrasting idea of incompatible sequences. We say $x \perp y$, i.e. x is incompatible with y, if $\exists n \in \mathrm{Dom}(x) \cap \mathrm{Dom}(y)$ such that $x(n) \neq y(n)$.

Next, we can construct what is known as the body of a game tree:

Definition 1.3 (Body of a Game Tree)**.** *Suppose T is a nonempty game tree, and t is a sequence. Then $t \in [T]$ if, and only if:*

1. $(\ell n\,(t)$ *is a limit ordinal* $) \wedge (\forall \alpha < \ell n\,(t)\,(t \restriction \alpha \in T)) \wedge (\forall x \supset t (x \notin T))$*, or*

2. $(\ell n\,(t)$ *is a successor ordinal* $) \wedge (t \in T) \wedge (\forall x \supset t (x \notin T))$

$[T]$ is known as the body of T.

Next we will discuss the elements of a tree and of its body.

Definition 1.4 (Positions and Plays)**.** *Suppose T is a nonempty game tree. If $p \in T$, then p is a position of T. If $t \in [T]$, then t is a play.*

Definition 1.5 (Finite Partial-Position)**.** *x is a finite partial-position for the nonempty tree T if*

$$\exists p \in T(x \subseteq p \textit{ and } x \textit{ is finite}),$$

i.e., x is a finite subset of a position.[1] *Note that despite its name, x is frequently not a position of the tree T, and this is why we hyphenate the name.*

Note that the tree and its body may not necessarily be disjoint. When p is in their intersection, p is considered both a position and a play. This occurs in this thesis specifically when dealing with well-founded trees, which are trees with only finite plays.

The most frequently encountered example of a game tree is the set $\omega^{<\omega}$. Recall that $\omega = \{0, 1, 2, 3, \ldots\}$, so $\omega^{<\omega} = \bigcup_{k\in\omega} \omega^k$ is the set of all finite, ω-valued sequences. In this example, the corresponding body of the tree is ω^ω, which is the set of ω-valued sequences of length ω. Note that the tree $\omega^{<\omega}$ and its body ω^ω are disjoint. On the other hand, Definition 1.2 allows for the tree $\omega^{<\omega} \cup \omega^\omega$, whose body is also ω^ω. In this case, however, the tree is not disjoint from its body.

Definition 1.6 (Moves or Nodes)**.** *Suppose T is a game tree and $p \in T$ is a position. If $p\,\hat{}\,\langle m\rangle \in T$, then we refer to m as either a move at $p \in T$ or a node at $p \in T$.*[2]

---

[1] We also refer to a finite partial-sequence in the context of a sequence from a space which is not necessarily a game tree. The definition of a finite partial-sequence is identical to a finite partial-position.

[2] Here, $p\,\hat{}\,\langle m\rangle = p \cup \{(\ell n\,(p)\,, m)\}$. We expand upon this more in Definition 1.8.

In the context of a game being played with two players in the game tree $\omega^{<\omega}$, each player would alternate moves from ω to form a play:

$$\begin{array}{llllllll} \text{Player I:} & m_0 & & m_2 & & m_4 & & \\ & & & & & & & \cdots \\ \text{Player II:} & & m_1 & & m_3 & & m_5 & \end{array}$$

Here, $\langle m_0, m_1, m_2, m_3, m_4, m_5, \dots \rangle \in \omega^\omega$ is the play.

When considering an individual play, either player I or player II will have won. The winner is determined by payoff sets. Specifically, every game will have two payoff sets A and B where:

1. $A \cup B = [T]$,
2. $A \cap B = \varnothing$,
3. A is the payoff set for player I, i.e. if a play $p \in A$, player I wins for that play, and
4. B is the payoff set for player II, i.e. if a play $p \in B$, player II wins for that play.

Notice that $B = [T] \setminus A$, which allows us to only specify the payoff set for player I.

Definition 1.7 (Payoff Sets)**.** *Suppose T is a game tree and $A \subseteq [T]$. Then $G(A, T)$ is the game played in the tree T with A as the payoff set for player I.*

In many contexts, the game tree is fixed. In these cases, the notation is frequently simplified to $G(A)$ or G_A.

Occassionally, we will need to extend a position with additional moves. To do so, we use the concatenation operation.

Definition 1.8 (Concatenation of Sequences)**.** *Suppose x and y are sequences (with ordinal domains). Then*

$$x\hat{\ }y = x \cup \{(\mathrm{Dom}(x) + i, y(i)) \mid i \in \mathrm{Dom}(y)\}.$$

Specifically, if $x = \langle x_0, x_1, \ldots, x_n \rangle$ and $y = \langle y_0, y_1, \ldots, y_m \rangle$,

$$x\hat{\ }y = \langle x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m \rangle.^3$$

In the context of game trees, if m is a move at a position p, we will use $p\hat{\ }\langle m \rangle$ to mean

$$p\hat{\ }\{(0, m)\} = \langle p_0, p_1, \ldots, p_n, m \rangle.$$

In some cases, to improve the readability, we will suppress the brackets and instead write $p\hat{\ }m$.

We will also be using the concatenation operation to build longer trees.

Definition 1.9 (Concatenation of Trees)**.** *Suppose T_1 and T_2 are game trees. Then $[T_1] \circledast [T_2] = \{f\hat{\ }g \mid f \in [T_1] \wedge g \in [T_2]\}$.*

1.2 Determinacy

Determinacy is the concept that one of the two players can guarantee he or she will win, regardless of the other player, by following a winning strategy.

[3] Note that the ordered pair $(n+1+i, y_i) \in x\hat{\ }y$. Sometimes we use parenthesis instead of angle brackets so that $x\hat{\ }y$ is also denoted by $(x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m)$. This can obviously create ambiguity. However, the context should make clear any confusion between an ordered pair and a sequence (e.g. $(n+1+i, y_i)$ versus (p_0, p_1)).

Definition 1.10. *A strategy s for player I or II in the tree T is a game tree such that $[s] \subseteq [T]$ and:*

1. *When s is a strategy for player I:*

   (a) *Every move for player I is unique. That is, for any even length k and any $p \in s$ and $\tilde{p} \in s$ with lengths greater than k, if $p \restriction k = \tilde{p} \restriction k$, then $p(k) = \tilde{p}(k)$.*

   (b) *Moves from T for player II are preserved. That is, for any odd length k and any $p \in T$ with length greater than k, if $p \restriction k \in s$, then $p \restriction (k+1) \in s$.*

2. *When s is a strategy for player II:*

   (a) *Every move for player II is unique. That is, for any odd length k and any $p \in s$ and $\tilde{p} \in s$ with lengths greater than k, if $p \restriction k = \tilde{p} \restriction k$, then $p(k) = \tilde{p}(k)$.*

   (b) *Moves from T for player I are preserved. That is, for any even length k and any $p \in T$ with length greater than k, if $p \restriction k \in s$, then $p \restriction (k+1) \in s$.*

Roughly speaking, a strategy for player I will dictate what moves player I must make while player II is always free to choose any available move. Similarly, a strategy for player II will dictate what moves player II must make while player I is free to choose any available move.

As an example, consider the following illustrations of a tree and a strategy for player I in Figure 1.1 and a strategy for player II in Figure 1.2 on that tree:

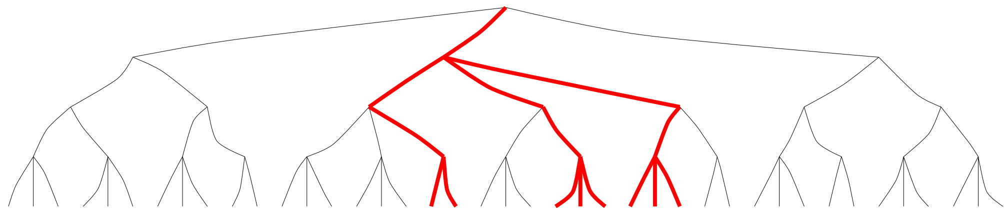

Figure 1.1: A strategy for player I.
Player I's strategy s_I is shown in red.

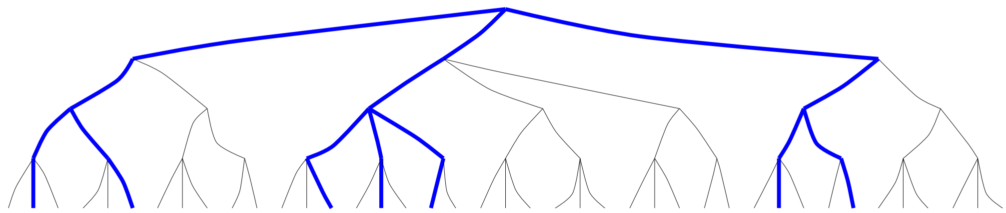

Figure 1.2: A strategy for player II.
Player II's strategy s_{II} is shown in blue.

Notice that in Figure 1.1, on player I's turn there is only one available move. Meanwhile, player II's moves are unrestricted. On the other hand, in Figure 1.2, player I's moves are unrestricted and player II only has one available move on each turn.

Because a strategy s has unique moves for one of the players, we can use it to create a natural function. That is, s induces a function s^*:

$$s^* : T_N \cap s \to T^*$$

$$s^*(p) = m, \text{ for the unique move } m \text{ such that } p\hat{}\langle m \rangle \in s$$

where $T_N \subseteq T$ consists of all positions at which player N has the next move (for $N \in \{I, II\}$) and T^* has all possible moves.

Notice that if we began with s^*, we could also generate the tree s. Because of this, we identify the strategy s with the function s^*, and a strategy can be defined as a subtree, or, equivalently, as a function on positions for either player I or player II. In the later chapters, we will primarily use a function to define our strategies.

Next, we have some terminology relating to strategies.

Definition 1.11 (According to a Strategy). *If s is a strategy, then we say a position p is "according to s" if $p \in s$.*

Definition 1.12 (Winning Strategy). *Let $N \in \{I, II\}$, and let C be the payoff set for player N. A winning strategy s for player N is a strategy such that $[s] \subseteq C$.*

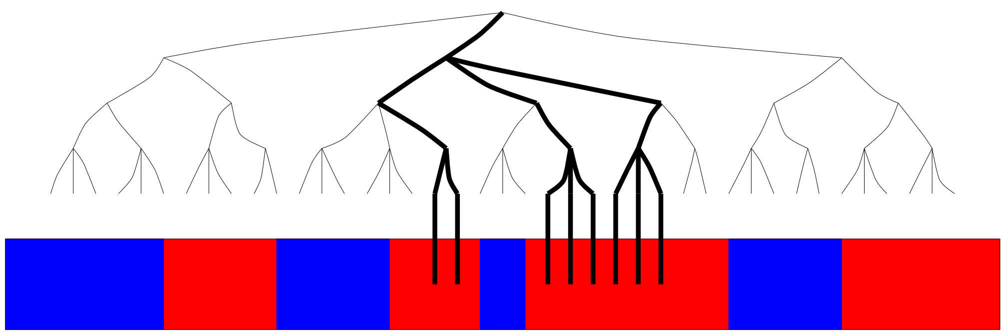

Figure 1.3: A winning strategy for player I.
A tree with a winning strategy for player I. The payoff set for player I is represented by the red segements and the payoff set for player II is represented by the blue segments.

Definition 1.13 (Determined Game). *A game $G(A, T)$ is said to be determined when either player I or player II has a winning strategy.*

In many games, we would not expect such a strategy to exist. In fact, a well-known result is that given the Axiom of Choice, non-determined games on 2^ω must exist [9].

1.3 Topology

In this next section, we will lay out some of the basics of topology which will be necessary for our results. Recall the definition of unary unions and intersections:

$$\bigcup \mathcal{X} = \bigcup_{X \in \mathcal{X}} X \text{ and}$$

$$\bigcap \mathcal{X} = \bigcap_{X \in \mathcal{X}} X.$$

Definition 1.14 (Topology)**.** *For a space χ, a topology τ must satisfy the following conditions:*

1. $\tau \subseteq \mathscr{P}(\chi)$
2. $\chi \in \tau$ *and* $\varnothing \in \tau$
3. $\forall \varsigma \subseteq \tau, \bigcup \varsigma \in \tau$
4. $\forall$ *finite* $\varsigma \subseteq \tau, \bigcap \varsigma \in \tau$

Sets which are elements of τ are said to be *open*. For any $Y \subseteq \chi$, if $\chi \setminus Y \in \tau$, Y is said to be *closed*. Some sets are both open and closed - these are referred to as *clopen*. For example, $\varnothing$ and χ are always clopen sets for any topology τ.

A frequently used technique for showing sets are open uses the concept of an interior point, whose definition follows:

Definition 1.15 (Interior Point)**.** *For a given set $Y \subseteq \chi$, the point $x \in Y$ is said to be an interior point of Y when there exists some open set G where $x \in G \subseteq Y$.*

It is easy to show that G is open if, and only if, every point in G is an interior point.

Next, many topologies are defined using a basis.

Definition 1.16 (Basis of a Topology)**.** *A collection of sets $\mathscr{B}$ is said to form a basis for the space χ if the following two conditions are met:*

1. $\chi = \bigcup \mathscr{B}$
2. *For any sets B_1 and $B_2 \in \mathscr{B}$, and for any $b \in B_1 \cap B_2$, there exists some $B_3 \in \mathscr{B}$ so that $b \in B_3 \subseteq B_1 \cap B_2$.*

The topology $\tau_{\mathscr{B}}$ which is generated by the basis $\mathscr{B}$ is defined as follows:

$$G \in \tau_{\mathscr{B}} \iff \exists \tilde{\mathscr{B}} \subseteq \mathscr{B} \left(G = \bigcup \tilde{\mathscr{B}} \right).$$

Note that we can obtain $\varnothing \in \tau_{\mathscr{B}}$ by allowing $\tilde{\mathscr{B}} = \varnothing$. Next we will describe several standard relevant topologies.

Definition 1.17 (Discrete Topology)**.** *If χ is a space, the discrete topology is $\tau = \mathscr{P}(\chi)$.*

Definition 1.18 (Product Topology)**.** *Suppose for every $i \in \mathscr{I}$, χ_i is a space equipped with the topology τ_i. Then the product space $\prod_{i \in \mathscr{I}} \chi_i$ is equipped with the topology τ with basis sets of the form $\prod_{i \in \mathscr{I}} G_i$, where each G_i is open in τ_i and only finitely many $G_i \neq \chi_i$.*

Note that it is routine to show that every open set can be written as a union of sets of the form $O(r) = \{f \in \prod_{i \in \mathscr{I}} \chi_i \mid f \supseteq r\}$ where r is any finite set. We expand on this further in Definition 1.20.

Definition 1.19 (Subspace Topology)**.** *If χ is a space equipped with a topology τ, and $\Psi \subseteq \chi$, then the subspace topology on Ψ is $\tau_\Psi = \{\Psi \cap G \mid G \in \tau\}$.*

In the context of game trees, we wish to put a topology on the body $[T]$. Next we will describe a very common topology that is placed on trees such as ω^ω, where all plays have equal length.

Suppose a tree has all plays with ordinal length l. Define the χ_i to be the set of all possible ith moves from T. In this case, $[T] \subseteq \prod_{i<j} \chi_i$. Here, each χ_i will be given the discrete topology, and then $[T]$ will be given the subspace topology of the product topology.

However, the trees we are using have variable length plays so that $[T] \not\subseteq \prod_{i<j} \chi_i$, so we need to adjust our definition of openness on $[T]$. Our definition is a natural generalization of the topology described in the previous paragraph.

Definition 1.20 (Open set in a Game Tree)**.** *Suppose x^{finite} is a finite partial-position of a game tree T. Define the basic open sets as follows:*

$$O(x^{finite}) = \{y \in [T] \mid y \supseteq x^{finite}\}.$$

G is open if it can be written as a union of these basic open sets.

This is sometimes referred to as the tree topology.

Comment 1.1. *If $[T] \subseteq \prod_{i<j} \chi_i$, then the topology described prior to Definition 1.20 is equal to our standard topology.*

Comment 1.2. *For any game tree T and any finite partial-position r of T, $O(r)$ is clopen. We show a more general result in Lemma 1.1.*

Once we have an established topology, we can classify certain sets as Borel:

Definition 1.21 (Borel Sets)**.** *The collection of all Borel sets from χ is denoted*

$$\mathbb{B} \restriction \chi,^{4}$$

which is the collection of all subsets of χ that can be built from open sets using the operations of countable unions, countable intersections, and complements.

In fact, the Borel sets can be stratified according to the number of countable unions and countable intersections used to build each set. This classification system is referred to as the Borel hierarchy.

Definition 1.22 (Borel Hierarchy)**.**

1. $A \in \mathbf{\Sigma}^0_1 \restriction \chi$ *if and only if A is open in the topology on χ.*

2. $A \in \mathbf{\Pi}^0_\alpha \restriction \chi$ *if and only if* $\chi \setminus A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$.

3. $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ *if and only if there exist sets* $A_n \in \mathbf{\Pi}^0_{\beta_n} \restriction \chi$, *where each* $\beta_n < \alpha$, *so that* $A = \bigcup_{n\in\omega} A_n$.

4. $\mathbf{\Delta}^0_\alpha \restriction \chi = (\mathbf{\Sigma}^0_\alpha \restriction \chi) \cap (\mathbf{\Pi}^0_\alpha \restriction \chi)$.

Note that it can be shown by induction that if $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$, then $\mathbf{\Sigma}^0_\alpha \restriction \chi$ and $\mathbf{\Pi}^0_\alpha \restriction \chi$ are both contained by $\mathbf{\Delta}^0_{\alpha+1} \restriction \chi$. With $(\mathbf{\Sigma}^0_\alpha \restriction \chi) \cup (\mathbf{\Pi}^0_\alpha \restriction \chi) \subseteq \mathbf{\Delta}^0_{\alpha+1} \restriction \chi$, we can refine (3) of Definition 1.22 to the following:

5. $A \in \mathbf{\Sigma}^0_{\alpha+1} \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Pi}^0_\alpha \restriction \chi$ so that $A = \bigcup_{n\in\omega} A_n$.

[4] We follow the convenient notation in Moschovakis and write $\mathbb{B} \restriction \chi$ rather than $\mathbb{B} \restriction \mathscr{P}(\chi)$. (See page 27 of Descriptive Set Theory [22].)

6. $A \in \mathbf{\Pi}^0_{\alpha+1} \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ so that $A = \bigcap_{n\in\omega} A_n$.

7. If α is a limit ordinal, $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Pi}^0_{\beta_n} \restriction \chi$, where each $\beta_n < \alpha$ so that $A = \bigcup_{n\in\omega} A_n$.

Note that when $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$, we have the following strengthening of (7):

8. If α is a limit ordinal and $\alpha = \sup_{n\in\omega} \beta_n$ where each $\beta_n < \alpha$, then $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ if and only if there exists $A_n \in \mathbf{\Pi}^0_{\beta_n} \restriction \chi$ for each $n \in \omega$ such that $A = \bigcup_{n\in\omega} A_n$.

We will use this for our later results.

If χ is either a separable metric space or $\chi = [T]$ for some tree T which has countable height,[5] it can be shown that $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$, allowing us to make use of (5), (6), and (8). In the following lemma and theorem, we will obtain this result for trees with countable height.

Lemma 1.1. *Suppose T is a tree with countable height. Suppose $G = \bigcup_{i\in\mathcal{g}} O(r_i)$, where each r_i is a finite partial-position with respect to T. If $D = \bigcup_{i\in\mathcal{g}} Dom(r_i)$ is finite, then G is clopen.*

Proof. Clearly G is open, so we only need to show G is closed.

Fix some $x \in [T] \setminus G = \bigcap_{i\in\mathcal{g}} ([T] \setminus O(r_i))$. We will show the complement of G is open by showing that x is an interior point of $[T] \setminus G$.

Define $q_x = x \restriction D$, where $D = \bigcup_{i\in\mathcal{g}} \mathrm{Dom}(r_i)$. Recall D if finite by the hypothesis. q_x is a finite partial-position, as described in Definition 1.20. Clearly, $x \in O(q_x)$. We need to show $O(q_x) \subseteq [T] \setminus G$, so pick $y \in O(q_x)$. We know that $y \restriction D = q_x = x \restriction D$. However, $x \notin O(r_i)$

[5] In this situtation, T need not be countable. Furthermore, recall that $\mathrm{height}(T) = \sup_{p\in T} \ell n\,(p)$.

for any $i \in \mathcal{I}$. This means that for every i, $x \perp r_i$, and therefore $y \perp r_i$ since $r_i \subseteq q_x$. Hence $y \in [T] \setminus G$. Finally, $O(q_x) \subseteq [T] \setminus G$. Consequently, x is an interior point of $[T] \setminus G$. □

Theorem 1.2. *Suppose G is an open set from the tree T with countable height α. Then $G \in \mathbf{\Delta}^0_2 \restriction [T]$. Hence $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$ for a tree T with countable height.*

Proof. First, $G \in \mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Pi}^0_2 \restriction [T]$. To show that $G \in \mathbf{\Delta}^0_2 \restriction [T]$ we need only show that $G \in \mathbf{\Sigma}^0_2 \restriction [T]$.

Since G is open, it can be written as an arbitrary union of basic open sets. Let $G = \bigcup_{i \in \mathcal{I}} O(r_i)$, where each r_i is a finite partial-position.

Let us rewrite G into a countable union. Since α is countable, there is some function $f : \omega \xrightarrow[\text{onto}]{1-1} \alpha$. Define

$$O_n = \bigcup_{\substack{f^{-1}(\mathrm{Dom}(r_i)) \subseteq n, \\ i \in \mathcal{I}}} O(r_i)$$

Then $G = \bigcup_{n \in \omega} O_n$.

By Lemma 1.1, each O_n is clopen so that G is a countable union of closed sets, i.e. $G \in \mathbf{\Sigma}^0_2 \restriction [T]$. □

Comment 1.2.1. *There exists a game tree T such that $\mathbf{\Sigma}^0_1 \restriction [T] \not\subseteq \mathbf{\Delta}^0_1 \restriction [T]$. We show this counterexample next.*

Consider the set

$$G = \{x \in 2^{\omega_1} \mid \exists \alpha < \omega_1 \, (x(\alpha) = 0)\}.$$

In the space 2^{ω_1} we will denote $\langle 0, 0, 0, \dots \rangle$ by $\mathbf{0}$ and $\langle 1, 1, 1, \dots \rangle$ by $\mathbf{1}$.

Clearly, $G \in \mathbf{\Sigma}_1^0 \restriction 2^{\omega_1}$. To see this, for each x define $p_x = \{(\alpha_x, 0)\}$ where α_x is least such that $x(\alpha_x) = 0$. Then $G = \bigcup_{x \in G} O(p_x)$.

However $G \notin \mathbf{\Sigma}_2^0 \restriction 2^{\omega_1}$. To see this, suppose otherwise: $G = \bigcup_{i \in \mathcal{I}} C_i$ where each C_i is closed. Define for each $i \in \omega$, $O_i = 2^{\omega_1} \setminus C_i$. Consider the complement of G:

$$\{\mathbf{1}\} = 2^{\omega_1} \setminus G = \bigcap_{i \in \omega} (2^{\omega_1} \setminus C_i) = \bigcap_{i \in \omega} O_i.$$

Since $\mathbf{1} \notin G$, for every $i \in \omega$, $\mathbf{1} \in O_i$. Additionally, since each O_i is open, $\mathbf{1}$ is an interior point, i.e. there exist finite partial-positions $x_i^{finite} \subseteq \mathbf{1}$ such that for any play $y \supseteq x_i^{finite}$, $y \in O_i$.

Since each $x_i^{finite} \subseteq \mathbf{1}$, they are compatible with one another. Thus, $x = \bigcup_{i \in \omega} x_i^{finite}$ is a partial-position. Additionally, since x is countable, there exists some countable α where $\mathrm{Dom}(x) \subseteq \alpha < \omega_1$. Thus, pick a position $y \in 2^{<\omega_1}$ where $y \supseteq x$.

Here, for every $i \in \omega$, $y\hat{\ }\mathbf{0} \supseteq y \supseteq x \supseteq x_i^{finite}$, so that $y\hat{\ }\mathbf{0} \in O_i$. In other words, $y\hat{\ }\mathbf{0} \in \bigcap_{i \in \omega} O_i = \{\mathbf{1}\}$, a contradiction. Hence $G \notin \mathbf{\Sigma}_2^0 \restriction 2^{\omega_1}$.

1.4 Remarks

There are a number of investigations that have been done on related material. We list some of those here.

In Emi Ikeda's dissertation [12], she investigates the determinacy of games on game trees with variable length paths. In particular, she investigates two types of trees, which she calls Type 1 and Type 2. The length of each path in her Type 1 trees is determined by its first

ω moves, and her Type 2 trees are generalizations of her Type 1 trees. In Lemma 3.4.9 on page 243 of her dissertation, she proves that the game trees $X^\omega \circledast [T^\alpha_{g.t.}]$ used in this thesis are Type 2 trees.

Recall, in our Definition 1.20, we define what is sometimes called the tree topology on $[T]$. In Katlyn Cox's thesis [2], she investigates what conditions on the trees T and S are needed so that the tree topology on $[T \circledast S]$ is homeomorphic to the natural topology on $[T] \times [S]$.

It is well-known that open determinacy in the product space $\omega^{\omega+\omega}$ requires large cardinals, but that open determinacy in the compact product space X^ω for X finite does not. In Deborah Fraker's thesis [6], she shows that one can use the pigeonhole principle to obtain open determinacy for the product space $X^{\omega\cdot n}$ for X finite and $n < \omega$.[6,7]

It is also well-known that for some trees T of height less than or equal to ω, $[T]$ is compact if and only if T is finitely branching. In Andrew DuBose's thesis [3], he develops necessary and sufficient conditions for the body of any game tree to be compact.[8] He provides an example of a compact $[T]$ of height $\omega + 1$ with the last level infinite. One can show open determinacy holds in this $[T]$. We do not know about open determinacy in compact $[T]$ for any game tree T.

It is easy to prove that determinacy fails in three player games on a nonempty set X with at least one move. In McKenna's thesis [21], she examines some conditions on the game tree $X^{<\omega}$ that results in a winning strategy for one of the players in a multiplayer game[9] when all but one payoff set is open.

[6]The proof in Fraker's thesis works for $\prod_{i\in\omega\cdot n} X_i$, where X_i is finite when i is infinite and odd.
[7]The argument is based on a proof which Douglas Burke showed Fraker's thesis advisor for $\{0,1\}^{\omega\cdot 2}$.
[8]In particular, the plays of the game tree need not have uniform length.
[9]Here we mean a game with n players, where $3 \leq n < \omega$.

Chapter 2

Projection/Lifting

The primary result of this thesis is a proof that establishes the equivalence between the determinacy of games with a Borel payoff set in the standard tree X^ω and open games in longer trees with variable length plays. In order to accomplish this, we need to be able to switch between sets in these two spaces. Therefore, in this chapter we establish the Lift operation, which takes a $[T_2]$-sequence of subsets of X^ω to a subset of $X^\omega \circledast [T_2]$, and the Proj operation, which takes a subset of $X^\omega \circledast [T_2]$ to a $[T_2]$-sequence of subsets of X^ω. Recall from the comment after Definition 1.1 on page 3, by a $[T_2]$-sequence we mean a function with domain $[T_2]$. In Theorem 2.1 we show that these are inverse functions.

Definition 2.1 (Projection)**.** *Suppose T_1 and T_2 are trees, and $E \subseteq [T_1] \circledast [T_2]$. Then define*

$$\text{Proj}_{T_1,T_2} : \mathscr{P}\left([T_1] \circledast [T_2]\right) \to \mathscr{P}\left([T_1]\right)^{[T_2]}$$

$$\text{Proj}_{T_1,T_2}(E) = \langle E_g \mid g \in [T_2] \rangle$$

where each $E_g = \{f \in [T_1] \mid f\hat{\ }g \in E\}$.

Below is an illustration of how a typical E_g is formed:

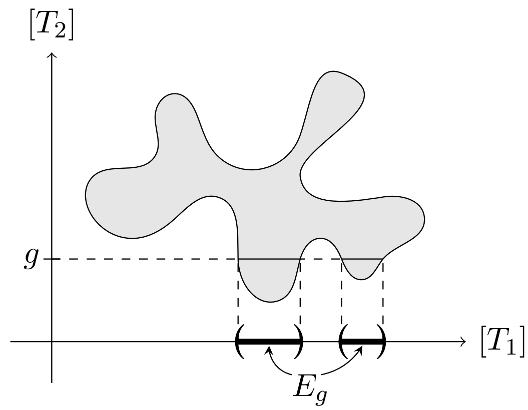

Figure 2.1: Projections.
Construction of a typical E_g.

Definition 2.2 (Lift)**.** *Suppose T_1 and T_2 are trees. Then define*

$$\mathrm{Lift}_{T_1,T_2} : \mathscr{P}\left([T_1]\right)^{[T_2]} \to \mathscr{P}\left([T_1] \circledast [T_2]\right)$$

$$\mathrm{Lift}_{T_1,T_2}(\vec{\mathbf{E}}) = \left\{h \mid \exists f \in [T_1] \exists g \in [T_2] \left(h = f\,\hat{}\,g \wedge f \in \vec{\mathbf{E}}(g)\right)\right\},$$

where each $\vec{\mathbf{E}}$ is a $[T_2]$-sequence of subsets of $[T_1]$ indexed by the elements of $[T_2]$, i.e. $\vec{\mathbf{E}}$ is a function with domain $[T_2]$ and $\mathrm{Im}(\vec{\mathbf{E}}) \subseteq \mathscr{P}\left([T_1]\right)$.

Comment 2.1. *Notice that in the case of projections, the terms of the $[T_2]$-sequence (denoted by E_g) are computed, whereas in the case of lifting, the terms of the $[T_2]$-sequence (denoted by $\vec{\mathbf{E}}(g)$) are given, and then used to compute the Lift.*

Comment 2.2. *When the choice of T_1 and T_2 is clear, the subscripts on the Lift and Proj functions will be suppressed.*

The first property details the inverse relationship between these two functions.

Theorem 2.1. Proj_{T_1,T_2} *and* Lift_{T_1,T_2} *are inverse functions.*

Proof. As mentioned above, we will suppress the subscript notation for the duration of this proof.

Consider an arbitrary $\vec{\mathbf{E}} = \langle E_g \mid g \in [T_2] \rangle$ where each $E_g \subseteq [T_1]$. Then $\mathrm{Proj}(\mathrm{Lift}(\vec{\mathbf{E}})) = \langle \hat{E}_g \mid g \in [T_2] \rangle$ where each $\hat{E}_g = \{f \in [T_1] \mid f\hat{\ }g \in \mathrm{Lift}(\vec{\mathbf{E}})\}$, according to the definition of Proj. However:

$$\begin{aligned} f \in \hat{E}_g &\iff f\hat{\ }g \in \mathrm{Lift}(\vec{\mathbf{E}}) \\ &\iff f \in E_g. \end{aligned}$$

In other words, $\hat{E}_g = E_g$. Thus, $\mathrm{Proj}(\mathrm{Lift}(\vec{\mathbf{E}})) = \langle E_g \mid g \in [T_2] \rangle = \vec{\mathbf{E}}$.

Likewise, consider $E \subseteq [T_1] \circledast [T_2]$. Define $\hat{E} = \mathrm{Lift}(\mathrm{Proj}(E))$ and $E_g = \{f \in [T_1] \mid f\hat{\ }g \in E\}$. Notice that

$$\begin{aligned} \hat{E} &= \mathrm{Lift}(\langle E_g \mid g \in [T_2] \rangle) \\ &= \{h \in [T_1] \circledast [T_2] \mid \exists f \in [T_1]\ \exists g \in [T_2]\ (h = f\hat{\ }g \wedge f \in E_g)\}. \end{aligned}$$

Note that, for $h \in [T_1] \circledast [T_2]$:

$$\begin{aligned} h \in \hat{E} &\iff \exists f \in [T_1]\; \exists g \in [T_2]\, (h = f\hat{\ }g \wedge f \in E_g) \\ &\iff \exists f \in [T_1]\; \exists g \in [T_2]\, (h = f\hat{\ }g \wedge f\hat{\ }g \in E) \\ &\iff h \in E. \end{aligned}$$

Thus, we have that $\mathrm{Lift}(\mathrm{Proj}(E)) = E$. □

Next we would like to expand on how Lift and Proj preserve levels of the Borel hierarchy. In general, projections do not necessarily follow this pattern, as illustrated in the following example:

Example 2.1 (An Open Set Whose Projection is Not Open)**.** *Suppose $A \subseteq [T]$ is such that $A \notin \mathbf{\Sigma}^0_1 \upharpoonright [T]$. In particular, the height of T must be at least ω so that such an A exists. Define the following:*

$$[\hat{T}] = \{(f, n) \mid (f \in A \wedge n = 0) \vee (f \notin A \wedge n = 1)\}$$

$$O = \{h \in [\hat{T}] \;\mid h(\ell n(h) - 1) = 0\}$$

Notice that $O \in \mathbf{\Sigma}^0_1 \upharpoonright [\hat{T}]$ since membership to O is determined entirely by knowing the last move; however, when we project along 0, we have $O_0 = \{f \in [T] \mid (f, 0) \in O\} = A \notin \mathbf{\Sigma}^0_1 \upharpoonright [T]$. Indeed, we could take this further and require that A be a nondetermined set, in which case, assuming the existence of sufficiently large cardinals (e.g. ω Woodin cardinal), O_0 cannot be projective.

Despite Example 2.1, the trees in this thesis will always be simple enough to allow the Borel level to be preserved during Lift and Proj. The proof of this fact will be broken down into steps, beginning with the preservation of open sets.

Lemma 2.2 (Proj and Lift Preserve Openness)**.** *Let T_1 and T_2 be game trees, and suppose all plays in T_1 are of equal length.*

(a) Suppose $E \in \mathbf{\Sigma}^0_1 \restriction ([T_1] \circledast [T_2])$. Then for every $g \in [T_2]$, $(\mathrm{Proj}(E))(g) \in \mathbf{\Sigma}^0_1 \restriction [T_1]$.

(b) Suppose T_2 is well-founded.[1] *If $\vec{\mathbf{E}} \in \mathscr{P}([T_1])^{[T_2]}$, and for every $g \in [T_2]$, $\vec{\mathbf{E}}(g) \in \mathbf{\Sigma}^0_1 \restriction [T_1]$, then $\mathrm{Lift}(\vec{\boldsymbol{E}}) \in \mathbf{\Sigma}^0_1 \restriction ([T_1] \circledast [T_2])$.*

Proof. For both part (a) and part (b), let k be the length of all plays in T_1. We will define the notation $a \overset{k}{\frown} b = a \cup \{(k+n, b(n)) \mid n \in \mathrm{Dom}(b)\}$ when a and b are partial sequences.

(a) Let $\mathrm{Proj}(E) = \langle E_g \mid g \in [T_2] \rangle$. Fix E_g, and pick $f \in E_g$. To show openness, we show f is an interior point of E_g, i.e. we find a basic open set O where $f \in O \subseteq E_g$.

By definition of Proj, $f \frown g \in E$, which is open. Thus, there exists a finite partial-position h^{finite} where $f \frown g \in O_{[T_1]\circledast[T_2]}(h^{finite}) \subseteq E$. From $f \frown g \in O_{[T_1]\circledast[T_2]}(h^{finite})$, we get that $f \frown g \supseteq h^{finite}$.

We shall obtain finite partial positions f^{finite} of T_1 and g^{finite} of T_2 such that

- $h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$,
- $f^{finite} \subseteq f$, and
- $g^{finite} \subseteq g$.

[1] To see that the proof will not go through without T_2 being well-founded, consider the example $T_1 = T_2 = \omega^{<\omega}$ and $\vec{\mathbf{E}}(g) = \begin{cases} \omega^\omega & \text{if } g = \vec{0} \\ \varnothing & \text{otherwise} \end{cases}$. Then $\mathrm{Lift}\left(\vec{\mathbf{E}}\right) = \omega^\omega \circledast \left\{\vec{0}\right\}$, which is not open.

Let $f^{finite} = h^{finite} \restriction \ell n(f) = h^{finite} \restriction k$. After fixing f^{finite}, note that there exists a unique g^{finite} such that $h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$. $f = (f \frown g) \restriction k \supseteq h^{finite} \restriction k = f^{finite}$ so that $f \supseteq f^{finite}$, and hence $f \in O_{[T_1]}\left(f^{finite}\right)$, as desired.

So far, we have not used that other plays of $[T_1]$ also have length $k = \ell n(f)$. We will need this fact to show the second part of this proof. Specifically, next we show that $O_{[T_1]}(f^{finite}) \subseteq E_g$.

Consider some $\hat{f} \in O_{[T_1]}(f^{finite})$. One can verify that $g \supseteq g^{finite}$, using that:

- $h^{finite} \restriction [k, \ell n(f \frown g)) = \{(k+i, x) \mid (i,k) \in g\}$,
- $f \frown g \supseteq h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$, and
- $k = \ell n(f)$.

Since $\hat{f} \supseteq f^{finite}$, $g \supseteq g^{finite}$, and $k = \ell n(f) = \ell n(\hat{f})$, $\hat{f} \frown g \supseteq f^{finite} \overset{k}{\frown} g^{finite} = h^{finite}$. (In Comment 2.3 below, we point out that this step fails if $\ell n(\hat{f}) \neq \ell n(f)$.) Thus, $\hat{f} \frown g \in O_{[T_1] \circledast [T_2]}(h^{finite}) \subseteq E$. By definition of Proj, $\hat{f} \in E_g$.

Thus we have shown $O(f^{finite}) \subseteq E_g$. Hence, $f \in O(f^{finite}) \subseteq E_g$, and therefore E_g is open.

(b) Assume T_2 is well-founded and that all plays in T_1 have the same length k.

Fix $\vec{\mathbf{E}} = \langle E_g \mid g \in [T_2] \rangle$ and define $E = \text{Lift}(\vec{\mathbf{E}})$. We will show every point in E is an interior point.

Pick $f \frown g \in E$. We will find a basic open O so that $f \frown g \in O \subseteq E$. By definition of Lift, $f \in E_g$, and since E_g is open, there is some finite partial-position f^{finite} of T_1 such that $f \in O_{[T_1]}(f^{finite}) \subseteq E_g$. Additionally, T_2 is well-founded, and therefore g is finite.

Let $g^{finite} = g$, and $h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$. The basic open set we will consider is $O_{[T_1]\circledast[T_2]}(h^{finite})$. Clearly, $f^\frown g \in O_{[T_1]\circledast[T_2]}(h^{finite})$.

We next show that $O_{[T_1]\circledast[T_2]}(f^{finite} \overset{k}{\frown} g^{finite}) \subseteq E$. Suppose some $h \in O_{[T_1]\circledast[T_2]}(h^{finite})$. Since $h \in [T_1] \circledast [T_2]$, $\exists \hat{f} \in [T_1]\ \exists \hat{g} \in [T_2]$ such that $h = \hat{f}^\frown \hat{g}$. Since all plays in T_1 have length k, $\ell n\left(\hat{f}\right) = k$ and

$$\hat{f} = h \restriction k \supseteq h^{finite} \restriction k = \left(f^{finite} \overset{k}{\frown} g^{finite}\right) \restriction k = f^{finite}.$$

Hence $\hat{f} \in O_{[T_1]}\left(f^{finite}\right) \subseteq E_g$, and by definition of E_g, $\hat{f}^\frown g \in E$. Also one can verify that $\hat{g} \supseteq g^{finite}$, using that:

- $h^{finite} \restriction [k, \ell n(h)) = \{(k+i, x) \mid (i, x) \in g\}$,
- $\hat{f}^\frown \hat{g} = h \supseteq h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$, and
- $k = \ell n\left(\hat{f}\right)$.

Since $\hat{g} \supseteq g^{finite}$, and $g^{finite} = g$ is a play in T_2, $\hat{g} = g$. Thus, $\hat{f}^\frown \hat{g} = \hat{f}^\frown g \in E$.

$h = \hat{f}^\frown \hat{g} \in O_{[T_1]\circledast[T_2]\left(h^{finite}\right)}$ since $\hat{f} \supseteq f^{finite}$, $\hat{g} \supseteq g^{finite}$, $k = \ell n\left(\hat{f}\right)$, and $h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$.

Consequently, $O_{[T_1]\circledast[T_2]}(h^{finite}) \subseteq E$, h is an interior point of E, and E is open.

□

The proof given in Lemma 2.2 relies on the plays in T_1 all having the same length. If T_1 is allowed to have plays with variable lengths, issues would arise in both directions of the proof. In the following two comments, we discuss potential problems. For additional information regarding how to weaken this requirement, see Cox's thesis [2].

Comment 2.3. *In part (a), if T_1 has plays with variable lengths, a problem would occur while showing that $O(f^{finite}) \subseteq E_g$. To construct an example, one could potentially have an $\hat{f}$ that satisfies the following:*

- *the domain of $\hat{f}$ includes all of the domain of h^{finite},*
- *$f^{finite} \subseteq \hat{f}$, and*
- *$\hat{f}$ differs on at least one move for the shifted g^{finite}.*

In this instance, we would have $\hat{f} \in O(f^{finite})$ but $\hat{f}\hat{\ } g \not\supseteq h^{finite}$.

For example, suppose T_1 and T_2 are such that $f = \langle a, b\rangle \in [T_1]$, $\hat{f} = \langle a, c, c\rangle \in [T_1]$, and $g = \langle a\rangle \in [T_2]$. Also suppose $f^{finite} = g^{finite} = \langle a\rangle = \{(0, a)\}$. Clearly, $\hat{f} \in O(f^{finite})$. However, $k = \ell n(f) = 2$ and $f^{finite} \overset{k}{\hat{\ }} g^{finite} = \{(0, a), (2, a)\}$, whereas $\hat{f}\hat{\ }g = \{(0, a), (1, c), (2, c), (3, a)\}$. Thus $f^{finite} \overset{k}{\hat{\ }} g^{finite} \not\subseteq \hat{f}\hat{\ }g$.

Comment 2.4. *Similarly in part (b), if T_1 has plays with variable lengths, then a problem occurs while showing that $O_{[T_1]\circledast[T_2]}(f^{finite} \overset{k}{\hat{\ }} g^{finite}) \subseteq E$, where $k = \ell n(f)$. One could have $\hat{f}\hat{\ }\hat{g} \supseteq f^{finite} \overset{k}{\hat{\ }} g^{finite}$ and simultaneously have an f^{finite} which extends beyond the length of $\hat{f}$. In that situation, $\hat{f} \not\supseteq f^{finite}$.*

For example, suppose

- *$f = \langle a, a, a\rangle \in [T_1]$,*
- *$\hat{f} = \langle a, b\rangle \in [T_1]$,*

- $\hat{g} = g = \langle a, a \rangle \in [T_2]$,

- $f^{finite} = \{(0,a),(2,a)\} \subseteq f$,

- $g^{finite} = \{(0,a)\} \subseteq g = \hat{g}$, *and*

- $h^{finite} = f^{finite} \overset{k}{\frown} g^{finite}$.

Then $k = \ell n(f) = 3$, $h^{finite} = \{(0,a),(2,a),(3,a)\}$, *and* $\hat{f}\frown\hat{g} = \langle a,b,a,a \rangle \in O\left(h^{finite}\right)$. *However,* $f^{finite} \not\subseteq \hat{f}$.

Refering back to Example 2.1, Lemma 2.2 does not hold because the tree $[\hat{T}]$ was not built as a concatenation. Specifically, in a concatenation, if $f\frown 0 \in [\hat{T}]$, then $f\frown 1 \in [\hat{T}]$.

Lemma 2.2 is the base case of a larger result. Recall that our goal is to show that these functions preserve all levels of the Borel heirarchy, as described in Definition 1.22. These higher levels are constructed from unions, intersections, and complements of lower levels. We will obtain the result for any level by showing Lift and Proj preserve unions and complements. These straighforward calculations are provided in the next two lemmas.

Lemma 2.3 (Preservation of Unions)**.** *Let T_1 and T_2 by any game trees.*

(a) Suppose each $E_i \subseteq [T_1] \circledast [T_2]$. *Then* $\operatorname{Proj}\left(\bigcup_{i\in I} E_i\right) = \left\langle \bigcup_{i\in I} \operatorname{Proj}(E_i)(g) \mid g \in [T_2]\right\rangle$, *i.e. for every* $g \in [T_2]$, $\left(\operatorname{Proj}\left(\bigcup_{i\in I} E_i\right)\right)(g) = \bigcup_{i\in I}\Big(\left(\operatorname{Proj}(E_i)\right)(g)\Big)$.

(b) $\operatorname{Lift}\left(\left\langle \bigcup_{i\in I} E_{(i,g)} \mid g \in [T_2]\right\rangle\right) = \bigcup_{i\in I} \operatorname{Lift}\left(\left\langle E_{(i,g)} \mid g \in [T_2]\right\rangle\right)$.

Proof.

(a) Fix any $g \in [T_2]$. For any $f \in [T_1]$:

$$\begin{aligned} f \in \left(\text{Proj}\left(\bigcup_{i\in I} E_i\right)\right)(g) &\iff f\,\hat{}\,g \in \bigcup_{i\in I} E_i \\ &\iff \exists i \in I\,(f\,\hat{}\,g \in E_i) \\ &\iff \exists i \in I\,\Big(f \in \Big(\text{Proj}(E_i)\Big)(g)\Big) \\ &\iff f \in \bigcup_{i\in I}\left(\Big(\text{Proj}(E_i)\Big)(g)\right). \end{aligned}$$

(b) We could do a similar calculation as in Part (a), but instead we will use the inverse relationship of the Lift and Proj functions. From Part (a), we obtain the fact that for any $\langle F_i \mid i \in I\rangle$ such that each $F_i \subseteq [T_1] \circledast [T_2]$,

$$\text{Proj}\left(\bigcup_{i\in I} F_i\right) = \left\langle \bigcup_{i\in\omega}\left(\Big(\text{Proj}(F_i)\Big)(g)\right) \mid g \in [T_2] \right\rangle.$$

So, we will pick $F_i =_{def} \text{Lift}(\langle E_{(i,g)} \mid g \in [T_2]\rangle)$ and apply the Lift function to both sides:

$$\text{Lift}\left(\text{Proj}\left(\bigcup_{i\in I} F_i\right)\right) = \text{Lift}\left(\left\langle \bigcup_{i\in I}\left(\Big(\text{Proj}(F_i)\Big)(g)\right) \mid g \in [T_2] \right\rangle\right).$$

When we account for Theorem 2.1, both sides of the equation can be simplified:

$$\bigcup_{i\in I} \text{Lift}(\langle E_{(i,g)} \mid g \in [T_2]\rangle) = \text{Lift}\left(\left\langle \bigcup_{i\in I} E_{(i,g)} \mid g \in [T_2] \right\rangle\right).$$

□

Lemma 2.4 (Preservation of Complements)**.** *Let T_1 and T_2 by any game trees.*

(a) $\mathrm{Proj}(([T_1] \circledast [T_2]) \setminus E) = \Big\langle [T_1] \setminus (\mathrm{Proj}(E))\,(g) \mid g \in [T_2] \Big\rangle$, *i.e. for every* $g \in [T_2]$, $\mathrm{Proj}\,(([T_1] \circledast [T_2]) \setminus E)\,(g) = [T_1] \setminus \mathrm{Proj}(E)(g)$.

(b) $\mathrm{Lift}\Big(\langle [T_1] \setminus E_g \mid g \in [T_2] \rangle\Big) = ([T_1] \circledast [T_2]) \setminus \mathrm{Lift}\Big(\langle E_g \mid g \in [T_2] \rangle\Big)$.

Proof.

(a) Fix any $g \in [T_2]$. For any $f \in [T_1]$:

$$\begin{aligned} f \in (\mathrm{Proj}\,(([T_1] \circledast [T_2]) \setminus E))\,(g) &\iff f\,\hat{}\,g \in [T_1] \circledast [T_2] \setminus E \\ &\iff f\,\hat{}\,g \in [T_1] \circledast [T_2] \wedge f\,\hat{}\,g \notin E \\ &\iff f \in [T_1] \wedge f \notin \mathrm{Proj}(E)(g) \\ &\iff f \in [T_1] \setminus \mathrm{Proj}(E)(g). \end{aligned}$$

(b) For every $\hat{f}\,\hat{}\,\hat{g} \in [T_1] \circledast [T_2]$:

$$\begin{aligned} \hat{f}\,\hat{}\,\hat{g} \in \mathrm{Lift}(\langle [T_1] \setminus E_g \mid g \in [T_2] \rangle) &\iff \hat{f} \in [T_1] \setminus E_{\hat{g}} \wedge \hat{g} \in [T_2] \\ &\iff \hat{f} \in [T_1] \wedge \hat{g} \in [T_2] \wedge \hat{f} \notin E_{\hat{g}} \\ &\iff \hat{f}\,\hat{}\,\hat{g} \in [T_1] \circledast [T_2] \wedge \hat{f}\,\hat{}\,\hat{g} \notin \mathrm{Lift}(\langle E_g \mid g \in [T_2] \rangle) \\ &\iff \hat{f}\,\hat{}\,\hat{g} \in ([T_1] \circledast [T_2]) \setminus \mathrm{Lift}(\langle E_g \mid g \in [T_2] \rangle). \end{aligned}$$

□

Finally, we can show that the $\mathrm{Lift}_{[T_1],[T_2]}$ and $\mathrm{Proj}_{[T_1],[T_2]}$ preserve any level of the Borel hierarchy.

Theorem 2.5. *Let α be a nonzero ordinal and fix $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha\}$. Additionally, let T_1 and T_2 be game trees, where all plays in T_1 are of equal length.*

(a) Suppose $E \in \mathbf{\Gamma} \restriction ([T_1] \circledast [T_2])$. Then for every $g \in [T_2]$, $\mathrm{Proj}(E)(g) \in \mathbf{\Gamma} \restriction [T_1]$.

(b) Suppose $\vec{\mathbf{E}} \in \mathscr{P}([T_1])^{[T_2]}$ (i.e. $\vec{\mathbf{E}}$ is a $[T_2]$-sequence of sets $E_g \subseteq [T_1]$) and for every $g \in [T_2]$, $\vec{\mathbf{E}}(g) \in \mathbf{\Gamma} \restriction [T_1]$. If $[T_2]$ is a well-founded tree, then $\mathrm{Lift}(\vec{\mathbf{E}}) \in \mathbf{\Gamma} \restriction ([T_1] \circledast [T_2])$.

Proof. Both results follow by induction.

(a) First we will note that the case where $\mathbf{\Gamma} = \mathbf{\Pi}^0_\alpha$ follows from the case where $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\alpha$. Fix $g \in [T_2]$ and suppose $E \in \mathbf{\Pi}^0_\alpha \restriction ([T_1] \circledast [T_2])$. We know then that $F =_{def} ([T_1] \circledast [T_2]) \setminus E \in \mathbf{\Sigma}^0_\alpha \restriction ([T_1] \circledast [T_2])$ and therefore by assumption $\mathrm{Proj}(F)(g) \in \mathbf{\Sigma}^0_\alpha \restriction [T_1]$. Since $E = ([T_1] \circledast [T_2]) \setminus F$, by applying Lemma 2.4 we get the following:

$$\begin{aligned}\Big(\mathrm{Proj}(E)\Big)(g) &= \Big(\mathrm{Proj}(([T_1] \circledast [T_2]) \setminus F)\Big)(g) \\ &= [T_1] \setminus \Big(\mathrm{Proj}(F)\Big)(g) \in \mathbf{\Pi}^0_\alpha \restriction [T_1].\end{aligned}$$

Due to the preceeding argument, from here on we need only show (a) holds for $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\alpha$.

The base case $\alpha = 1$ has been shown in Lemma 2.2, since we are assuming all plays in $[T_1]$ are of equal length.

Suppose that (a) holds true with $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\beta$ for all $\beta < \alpha$ (i.e. the induction hypothesis) and consider $E \in \mathbf{\Sigma}^0_\alpha \restriction ([T_1] \circledast [T_2])$. By definition, $E = \bigcup_{n\in\omega} E_n$ where each $E_n \in \mathbf{\Pi}^0_{\beta_n} \restriction ([T_1] \circledast [T_2])$ for particular $\beta_n < \alpha$. From the first paragraph, we know that since

(a) holds for each $\mathbf{\Gamma} = \mathbf{\Sigma}^0_{\beta_n}$ (the induction hypothesis), it will also hold for $\mathbf{\Gamma} = \mathbf{\Pi}^0_{\beta_n}$, and therefore we know that $\text{Proj}(E_n)(g) \in \mathbf{\Pi}^0_{\beta_n} \upharpoonright [T_1]$. Applying Lemma 2.3,

$$\Big(\text{Proj}(E)\Big)(g) = \left(\text{Proj}\left(\bigcup_{n\in\omega} E_n\right)\right)(g)$$
$$= \bigcup_{n\in\omega}\left(\Big(\text{Proj}(E_n)\Big)(g)\right) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T_1].$$

Thus, we get the result for all ordinals α.

(b) As before, we show the result with $\mathbf{\Gamma} = \mathbf{\Pi}^0_\alpha$ follows from the result with $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\alpha$. Suppose for every $g \in [T_2]$, $\vec{\mathbf{E}}(g) \in \mathbf{\Pi}^0_\alpha \upharpoonright [T_1]$. Then for each g, $\vec{\mathbf{F}}(g) =_{def} [T_1] \setminus \vec{\mathbf{E}}(g) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T_1]$, and from our assumption, we get $\text{Lift}\left(\langle \vec{\mathbf{F}}(g) \mid g \in [T_2]\rangle\right) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T_1] \circledast [T_2]$. Note that for each g, $\vec{\mathbf{E}}(g) = [T_1] \setminus \vec{\mathbf{F}}(g)$. Thus, using Lemma 2.4,

$$\text{Lift}\left(\vec{\mathbf{E}}\right) = \text{Lift}\left(\left\langle [T_1] \setminus \vec{\mathbf{F}}(g) \mid g \in [T_2]\right\rangle\right)$$
$$= ([T_1] \circledast [T_2]) \setminus \text{Lift}\left(\left\langle \vec{\mathbf{F}}(g) \mid g \in [T_2]\right\rangle\right) \in \mathbf{\Pi}^0_\alpha \upharpoonright ([T_1] \circledast [T_2]).$$

Next, we show the statement (b) for $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\alpha$. It has already been shown in the base case for $\alpha = 1$ in Lemma 2.2. Recall in this case we are assuming that all plays in $[T_1]$ have equal length and $[T_2]$ is well-founded.

We will assume the induction hypothesis (i.e. that (b) is true when $\mathbf{\Gamma} = \mathbf{\Sigma}^0_\beta$ for all $\beta < \alpha$). Fix $\vec{\mathbf{E}}(g) \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T_1]$ for all $g \in [T_2]$. By definition, $\vec{\mathbf{E}}(g) = \bigcup_{n\in\omega} E_{(n,g)}$ where each $E_{(n,g)} \in \mathbf{\Pi}^0_{\beta_n} \upharpoonright [T_1]$ for particular $\beta_n < \alpha$. By the induction hypothesis, we know that for all $n \in \omega$, the statement (b) holds true for each $\mathbf{\Sigma}^0_{\beta_n}$; using the the first paragraph we know it also holds true for each $\Gamma = \mathbf{\Pi}^0_{\beta_n}$. Thus, $\text{Lift}\left(\langle E_{(n,g)} \mid g \in [T_2]\rangle\right) \in \mathbf{\Pi}^0_{\beta_n} \restriction ([T_1] \circledast [T_2])$. From Lemma 2.3,

$$\begin{aligned}\text{Lift}(\vec{\mathbf{E}}) &= \text{Lift}\left(\left\langle \bigcup_{n\in\omega} E_{(n,g)} \mid g \in [T_2]\right\rangle\right)\\ &= \bigcup_{n\in\omega} \text{Lift}\left(\langle E_{(n,g)} \mid g \in [T_2]\rangle\right) \in \mathbf{\Sigma}^0_\alpha \restriction ([T_1] \circledast [T_2]).\end{aligned}$$

□

With this result we will be able to move back and forth between sets in our short and long trees, without corrupting the complexity of the set.

Chapter 3

Ordinal Tree and Decomposition Tree

In this section we will define several trees which we use as we deconstruct or construct a Borel set in a uniform manner. One of the trees, T^α, will be a tree of ordinals corresponding to certain complexities of a Borel decomposition. The other tree, $T^\alpha_{g.t.}$, will be used to create an auxiliary game tree with plays from $X^\omega \circledast [T^\alpha_{g.t.}]$. Determinacy of open games on this auxiliary tree are equivalent to determinacy of $\mathbf{\Sigma}^0_\alpha \restriction X^\omega$ games. This result is shown in Chapter 5.

Our upcoming definitions are motivated by the goal of standardizing the decomposition of $\mathbf{\Sigma}^0_\alpha \restriction X^\omega$ sets. To illustrate how we arrive at these definitions, consider a set $A \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$. Then for each ordinal $\gamma \in (1, \alpha]$ we fix $\delta_\gamma(i)$ in the following manner:

(a) When $\gamma \in (\omega, \alpha]$ is a limit ordinal, choose $\delta_\gamma(i)$ to be an odd successor ordinal so that $\sup_{i\in\omega} \delta_\gamma(i) = \gamma$.

(b) When $\gamma = \omega$, let $\delta_\gamma(i) = 2i + 2$. (Notice that, as in (a), $\sup_{i\in\omega} \delta_\gamma(i) = \gamma$.)

(c) If γ is a successor ordinal, let $\delta_\gamma(i) = \gamma - 1$.

We create a tree $\mathscr{T}^A$ of triples (B, n, γ) which will map out each stage the deconstruction of A. Ultimately the set B is meant to be a Borel component of our fixed Borel set A from the γth level of the hierarchy. We will be able to deduce based on the parity of γ if $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ or $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$. Specifically, $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ when γ is an odd finite or an even infinite countable ordinal. Otherwise, $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$. Furthermore, each stage of the decomposition will involve either a countable union or a countable intersection (i.e. $\bigcup_{i\in\omega} E_i$ or $\bigcap_{i\in\omega} E_i$), and B will be one of the sets from this operation. The n of (B, n, γ) records that the set B is the nth set E_n.

The Top of the Tree: The base node of the tree $\mathscr{T}^A$ is $(A, \varnothing, \alpha)$.

Child Nodes of Limit Nodes: Suppose (B, n, γ) is a node where γ is a limit ordinal and $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$. There exist sets $C_i \in \mathbf{\Pi}^0_{\delta_\gamma(i)} \upharpoonright X^\omega$ such that $B = \bigcup_{i\in\omega} C_i$, so we choose the child nodes of (B, n, γ) to be $(C_i, i, \delta_\gamma(i))$ as i varies through ω.

Child Nodes of Successor Nodes: Suppose (B, n, γ) is a node where γ is a successor ordinal. Note that $\delta_\gamma(i) = \gamma - 1$ for any i. Here, $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ or $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$.[1] If $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$, then there exist sets $C_i \in \mathbf{\Pi}^0_{\gamma-1} \upharpoonright X^\omega = \mathbf{\Pi}^0_{\delta_\gamma(i)} \upharpoonright X^\omega$ where $B = \bigcup_{i\in\omega} C_i$. Using this, we choose the child nodes to be $(C_i, i, \gamma - 1) = (C_i, i, \delta_\gamma(i))$. On the other hand, if $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$, there exist sets $C_i \in \mathbf{\Sigma}^0_{\gamma-1} \upharpoonright X^\omega = \mathbf{\Sigma}^0_{\delta_\gamma(i)} \upharpoonright X^\omega$ where $B = \bigcap_{i\in\omega} C_i$. Thus the child nodes are $(C_i, i, \gamma - 1) = (C_i, i, \delta_\gamma(i))$.

This process is iterated until B is open for a node (B, n, γ); in this case, $\gamma = 1$.

[1]If γ is infinite, then $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ when γ is even and $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$ when γ is odd. When γ is finite, $B \in \mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ when γ is odd and $B \in \mathbf{\Pi}^0_\gamma \upharpoonright X^\omega$ when γ is even.

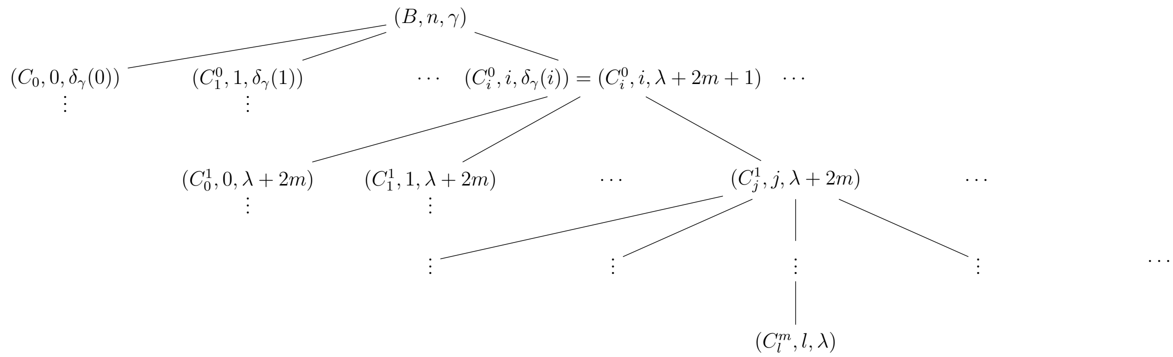

Figure 3.1: Subtree of $\mathscr{T}^A$.
An illustration of a position in $\mathscr{T}^A$ from one limit node (B, n, γ) to another limit node (C_l^m, l, λ) below it. Here, $\delta_\gamma(i) = \lambda + 2m + 1$ for some limit λ and some finite m.

Notice that in this process, the second and third components do not depend directly on the set A. That is, any other set $\hat{A} \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$ would generate an isomorphic tree $\mathscr{T}^{\hat{A}}$ with identical second and third components. In fact, these components depend only on (a), (b) and (c) on page 32. This is the key idea behind the trees defined in this chapter. Our tree of ordinals, T^α, is nothing more than the tree of only third coordinates, and our tree $T^\alpha_{g.t.}$ is nothing more than the tree of the second coordinates (excluding the $\varnothing$ from the base node of the tree). However, we will build these trees directly without relying on a set A or the tree $\mathscr{T}^A$ described above.

On a side note, a set B could show up in multiple nodes throughout the tree. When this occurs, the corresponding complexities in the third component may be different. Indeed, it is also very possible that neither of these complexities are actually the lowest possible Borel complexity for the set B.

Notation for Chapter 3-9: In Chapter 2, we frequently had to differentiate between sequences and ordered pairs. For example, in Comment 2.4 we had both the sequence $\langle a, a\rangle$ and the ordered pair $(0, a)$. As such, in Chapter 2 we consistently used angle brackets for sequences. However, for the rest of the paper, we regularly use parenthesis around sequences for readability, as commented in footnote 2 on page 6 from Chapter 1. In particular, you will see this usage increase drastically in Section 3.2.

3.1 The Ordinal Tree T^α

We now build T^α without reference to the tree $\mathscr{T}^A$. The reader may wish to think of T^α as a recording of complexities for a future Borel set, either being deconstructed (starting from the top of the tree) or being constructed (starting from the bottom of the tree).

Definition 3.1 (Ordinal tree T^α, ω-sequence β_α of successor ordinals and ω-sequence γ_α of limit ordinals or 1; where $\alpha \in (0, \omega_1)$ is a limit ordinal)**.**

Base Step: $\alpha = \omega$,

$$[T^\omega] = \{(\omega, 2i+2, 2i+1, \ldots, 1) \mid i \in \omega\}$$

$$\beta_\omega(i) = 2i + 2$$

$$\gamma_\omega(i) = 1$$

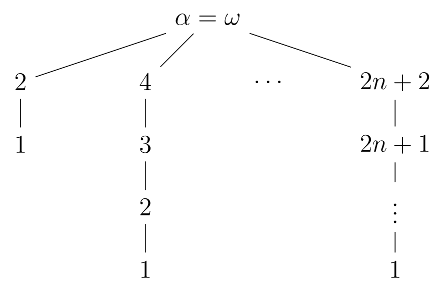

Figure 3.2: The T^ω tree.

Inductive Step: If $\alpha \in (\omega, \omega_1)$ is a limit ordinal, and for all limit ordinals $\lambda < \alpha$ we have T^λ, β_λ, and γ_λ, then fix ω-sequences β_α (of strictly increasing successor ordinals) and γ_α (of limit ordinals) so that:

$$\begin{aligned} &(1) \quad \forall i \in \omega(\beta_\alpha(i) < \alpha) \\ &(2) \quad \sup_{i\in\omega} \beta_\alpha(i) = \alpha \\ &(3) \quad \beta_\alpha(i) = \gamma_\alpha(i) + 2i + 1 \end{aligned} \tag{3.1}$$

Finally, define $[T^\alpha] = \{(\alpha, \beta_\alpha(i), \beta_\alpha(i) - 1, \ldots, \gamma_\alpha(i) + 1)\hat{}f \mid i \in \omega \wedge f \in [T^{\gamma_\alpha(i)}]\}$.

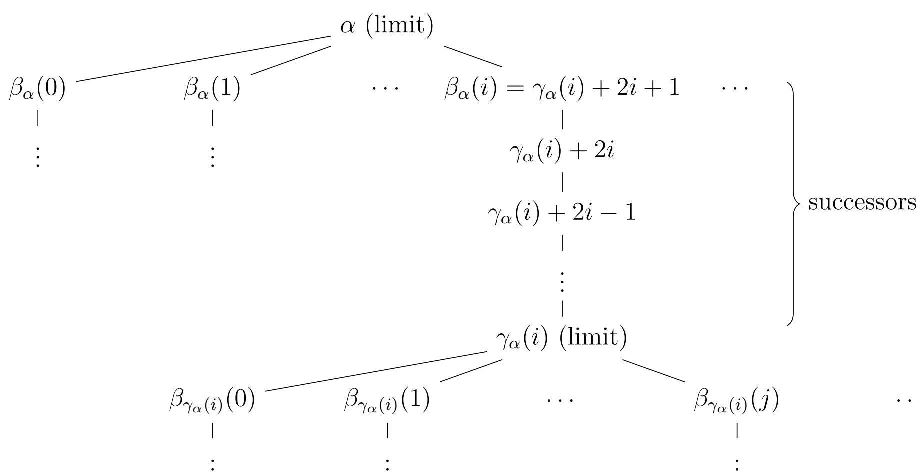

Figure 3.3: The top of the T^α tree.

Comment 3.1. *Note that there will always be β_α and γ_α sequences to satisfy the above* *properties.*

If α is a limit of limit ordinals, say $\alpha = \sup_{i\in\omega} \gamma_i$ (where each $\gamma_i < \alpha$ is a limit and γ_i is strictly increasing), then $\alpha = \sup_{i\in\omega}(\gamma_i + 2i + 1)$.

If α is not a limit of limit ordinals, then fix

$$\gamma = \sup_{\substack{\beta<\alpha \\ \beta \text{ a limit}}} \beta,$$

and note that $\gamma < \alpha$. We wish to show that $\alpha = \gamma + \omega$ by showing we have neither $\gamma + \omega < \alpha$ nor $\gamma + \omega > \alpha$. If the former were true, then $\gamma + \omega$ would be included in the supremum, resulting in $\gamma \geq \gamma + \omega$. In the latter case, α would be a limit ordinal such that $\gamma < \alpha < \gamma + \omega$, yet there are no such limit ordinals. Hence, it must be the case that $\alpha = \gamma + \omega$. Finally, this means that $\alpha = \sup_{i\in\omega}(\gamma + 2i + 1)$. Let $\gamma_i = \gamma$.

Comment 3.2. *Notice that the β_α sequence differs in the base case compared to the inductive case in terms of the parity. This is a consequence of the change in parity of the γ_α sequence in the base case versus the inductive case.*

Comment 3.3. *Due to their construction, these ordinal trees have a certain uniformity. In particular, if there are two trees $T^{\tilde{\alpha}}$ and $T^{\hat{\alpha}}$, and a limit ordinal α is a node in each, then the subtree which begins at that node α is in fact T^α for both trees.*

This process gives us our T^α tree, but it also introduces some messy notation in the form of nested subscripts. The nodes at the bottom of Figure 3.3 begin to demonstrate the issue, which will become more complex as we move further down the tree. Notice, however, that a path through the tree can be coded by a sequence of integers. Thus, for a particular fixed

α, we can inductively provide new names for certain nodes in the corresponding tree in the following manner:

Definition 3.2 (Nice Labeling of T^α)**.** *If $\vec{i} \in \omega^{<\omega}$ and $i \in \omega$,*

$$\alpha_\varnothing =_{def} \alpha$$

$$\beta_{\vec{i}^\frown i} =_{def} \beta_{\alpha_{\vec{i}}}(i)$$

$$\alpha_{\vec{i}^\frown i} =_{def} \gamma_{\alpha_{\vec{i}}}(i)$$

With this new labeling, each $\alpha_{\vec{i}}$ is a limit ordinal and each $\beta_{\vec{i}}$ is a successor ordinal. (1)-(3) described in Properties 3.1 in our relabeled tree become the following:

$$\begin{array}{ll} (1) & \forall i \in \omega(\beta_{\vec{i}^\frown i} < \alpha_{\vec{i}}) \\ (2) & \sup_{i\in\omega} \beta_{\vec{i}^\frown i} = \alpha_{\vec{i}} \\ (3) & \beta_{\vec{i}^\frown i} = \alpha_{\vec{i}^\frown i} + 2i + 1 \end{array} \tag{3.2}$$

The T^α tree with the new notation will look like the following:

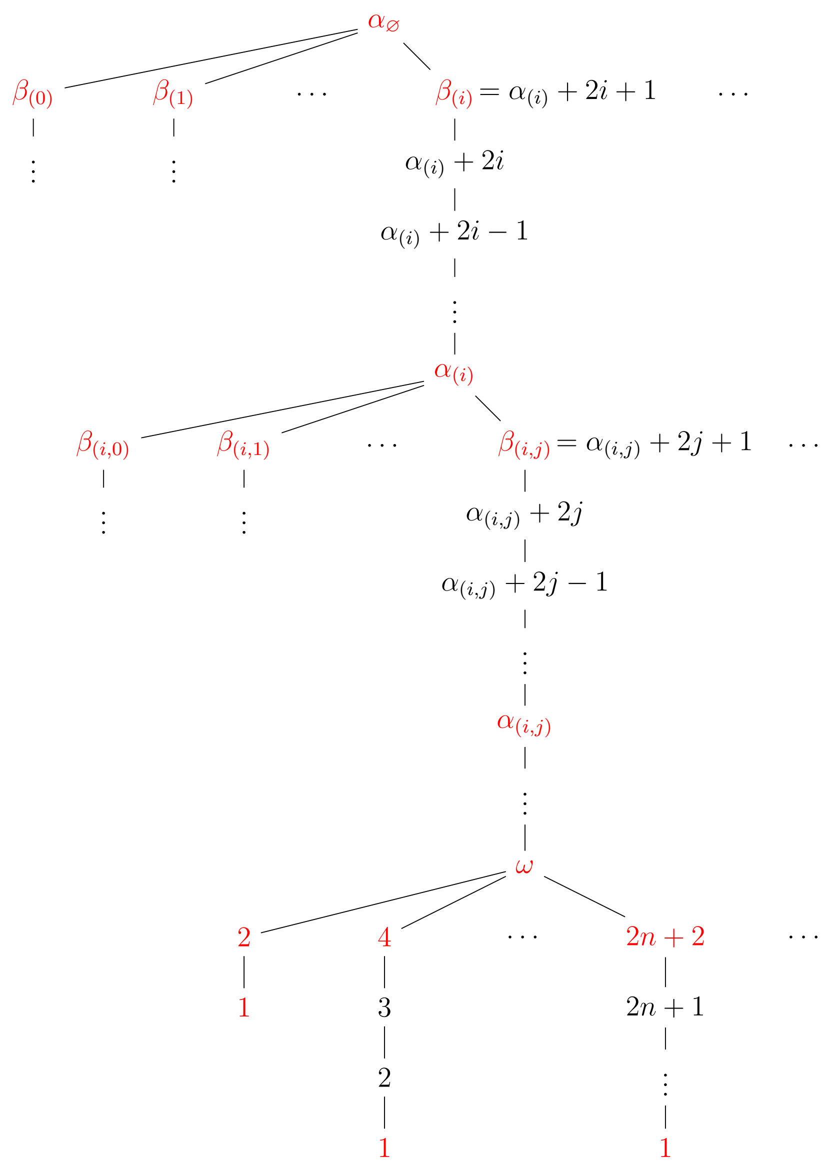

Figure 3.4: Relabeled T^{α} tree.
The nodes at the α and β stages are shown in red.

Comment 3.4. *Each ordinal in the tree T^{α} will ultimately correspond to specific levels in the Borel hierarchy, e.g. the node $\alpha_{(i)}$ corresponds to $\mathbf{\Sigma}^0_{\alpha_{(i)}}$ and the node $\beta_{(i,j)}$ corresponds to $\mathbf{\Pi}^0_{\beta_{(i,j)}}$.*

Comment 3.5. *This new coding only works because the tree T^{α} is well-founded.*

3.2 The Index Tree I^α

In the tree T^α, we have plays of this form:

$$(\alpha_\varnothing, \beta_{(i_0)}, \beta_{(i_0)} - 1, \ldots, \alpha_{(i_0)}, \beta_{(i_0,i_1)}, \beta_{(i_0,i_1)} - 1, \ldots, \alpha_{(i_0,i_1)}, \ldots, \omega, 2i_n + 2, 2i_n + 1, \ldots, 1)$$

where $\alpha_{(i_0,i-1,\ldots,i_{n-1})} = \omega$, $\beta_{(i_0,i_1,\ldots,i_n)} = 2i_n + 2$ and $\alpha_{(i_0,i_1,\ldots,i_n)} = 1$. Notice, for example, that the sequence of ordinals from $\alpha_{(i_0)}$ to $\alpha_{(i_0,i_1)}$,

$$(\alpha_{(i_0)}, \beta_{(i_0,i_1)}, \beta_{(i_0,i_1)} - 1, \beta_{(i_0,i_1)} - 2, \ldots, \alpha_{(i_0,i_1)}),$$

can be constructed by only knowing i_0 and i_1. The $\alpha_{(i_0)}$ and $\beta_{(i_0,i_1)}$ was fixed in the definition of T^α, and the remaining terms are obtained by repeatedly subtracting one until obtaining a limit ordinal (except in the case of a finite β, in which case you stop at 1). In general, if we know an appropriate sequence of indices $\vec{i} = (i_0, i_1, \ldots, i_k)$, then we can construct exactly one corresponding position in T^α.

It is useful to note that the collection of all possible sequences $\vec{i}$ obtained through these subscripts will form a tree. Each i_k will be able to vary through all of ω, since each i_k is introduced by selecting $\beta_{(i_0,i_1,\ldots,i_k)}$, which had ω many choices.

Furthermore, when we construct our game tree $T^\alpha_{g.t.}$, we will be making integer moves. These indices will make up some of the moves in this tree. Because of this, and additionally for convenience later in the chapter, it will be useful to define the tree of such indices.

Definition 3.3 (Index Tree)**.**

$$[I^\alpha] = \{\vec{i} \in \omega^{<\omega} \mid \alpha_{\vec{i}} = 1\}.$$

Here any $\vec{i} = (i_0, i_1, \ldots, i_n) \in [I^\alpha]$ *corresponds to the play*

$$(\alpha_\varnothing, \beta_{(i_0)}, \beta_{(i_0)} - 1, \ldots, \alpha_{(i_0)}, \beta_{(i_0,i_1)}, \beta_{(i_0,i_1)} - 1, \ldots, \alpha_{(i_0,i_1)}, \ldots, \omega, 2i_n + 2, 2i_n + 1, \ldots, 1).$$

where $\alpha_{(i_0,i_1,\ldots,i_{n-1})} = \omega$*,* $\beta_{(i_0,i_1,\ldots,i_n)} = 2i_n + 2$ *and* $\alpha_{(i_0,i_1,\ldots,i_n)} = 1$*.*

Previously in this section we noted how, given a sequence of indices $\vec{i} \in I^\alpha$, we know the corresponding position in T^α. However, we can also start with a position in T^α and extract the indices in I^α as well. To do this, notice that a play in T^α can naturally be partitioned into "blocks" by breaking off a section each time a limit ordinal is reached. For example, we can create the following sequences:

$$\begin{aligned}
p_0 &= (\beta_{(i_0)}, \beta_{(i_0)} - 1, \ldots, \alpha_{(i_0)} + 1, \alpha_{(i_0)}),\\
p_1 &= (\beta_{(i_0,i_1)}, \beta_{(i_0,i_1)} - 1, \ldots, \alpha_{(i_0,i_1)} + 1, \alpha_{(i_0,i_1)}),\\
&\vdots\\
p_{n-1} &= (\omega + 2i_{n-1} + 1, \omega + 2i_{n-1}, \ldots, \omega + 1, \omega),\\
p_n &= (\beta_{(i_0,i_1,\ldots,i_n)} = 2i_n + 2, 2i_n + 1, \ldots, 2, \alpha_{(i_0,i_1,\ldots,i_n)} = 1).
\end{aligned}$$

Each p_k has length $2i_k + 2$. We will use this strategy to calculate the entire sequence of indices.

Definition 3.4 (Extract from the tree T^α). *Suppose $p \in T^\alpha$. Then inductively construct p_k so that*

(1) $p = (\alpha_\varnothing)^\frown p_0{}^\frown p_1{}^\frown \dots {}^\frown p_n$,

(2) *the only limit ordinal in p_k is its final node, $p_k(\ell n(p_k) - 1)$*

Then define i_k as follows:

$$i_k = \frac{\ell n(p_k) - 2}{2} \text{ for } 0 \leq k \leq n$$

Finally,

$$\text{extract}_{T^\alpha}(p) = (i_0, i_1, \dots, i_n).$$

3.3 The Tail Tree $T^\alpha_{g.t.}$

Our goal is to show that the determinacy of games in $\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega$ is equivalent to games in $\mathbf{\Sigma}^0_1 \upharpoonright [\Upsilon^\alpha]$ where Υ^α is a more complex auxiliary game tree. Defining this Υ^α is our final goal for this chapter. This tree will be built from two separate trees: X^ω and a tree we denote by $T^\alpha_{g.t.}$. Here we focus on the second tree, $T^\alpha_{g.t.}$, which we also refer to as the tail tree. $T^\alpha_{g.t.}$ is easily defined from any one of the following:

1. The ordinal tree T^α.

2. A "reduced" T^α tree containing only the $\alpha_{\vec{i}}$ nodes.

3. The index tree I^α.

Here we will focus on the latter approach. Corresponding to each $\vec{i} = (i_0, i_1, \ldots, i_n) \in [I^\alpha]$, there is a collection of plays in $T^\alpha_{g.t.}$ of the form

I	i_0		$g_0(1)$	$\cdots$		i_1		$g_1(1)$	$\cdots$		$\cdots$	i_n		$g_n(1)$	$\cdots$	
II		$g_0(0)$			$g_0(2i_0)$		$g_1(0)$			$g_1(2i_1)$			$g_n(0)$			$g_n(2i_n)$

Figure 3.5: Plays in $T^\alpha_{g.t.}$ "tail" tree

where each $g_j(m) \in \omega$. Notice the above play in $T^\alpha_{g.t.}$ and the play in T^α corresponding to $\vec{i} \in [I^\alpha]$ have lengths which differ by exactly one. (A play in T^α always has an extra $\alpha_\varnothing$ as the first move.)

We give the formal definition of this tree next.

Definition 3.5 (Tail Tree $T^\alpha_{g.t.}$)**.**

$$[T^\alpha_{g.t.}] = \{(i_0)^\frown \vec{g_0}^\frown (i_1)^\frown \vec{g_1}^\frown \ldots {}^\frown (i_n)^\frown \vec{g_n} \in \omega^{<\omega} \mid \forall k \in [0,n](\vec{g_k} \in \omega^{2i_k+1}) \wedge (i_0, i_1, \ldots, i_n) \in [I^\alpha]\}.$$

Comment 3.6. *Since I^α is well-founded, so is $T^\alpha_{g.t.}$.*

Comment 3.7. *Note the importance of each "i_k" move. These moves determine the length of the "$\vec{g_k}$" sequence that follows, and they are the moves that correspond to the subscripts of each $\alpha_{\vec{i}}$ from the T^α tree.*

One can see from the definition that these plays are built by repeating "i" moves followed by a sequence of "g" moves. It will occassionally be useful to discuss an $i^\frown \vec{g}$ separately from the rest of the play, so the next definition establishes what we refer to as rounds.

Definition 3.6 (Rounds in the $T^{\alpha}_{g.t.}$ tree)**.** *A position $t \in T^{\alpha}_{g.t.}$ is said to* ***end at a full round*** *if $t = (i_0)^\frown \vec{g_0}^\frown (i_1)^\frown \vec{g_1}^\frown \ldots ^\frown (i_n)^\frown \vec{g_n}$ where for each k, $\ell n\,(\vec{g_k}) = 2i_k + 1$ (i.e. when t is not a play, the next move is an "i_{k+1}" move), and $(i_0, i_1, \ldots, i_n) \in I^{\alpha}$. Note that t can be empty.*

Additionally, when for each $k < n$, $\ell n\,(\vec{g_k}) = 2i_k + 1$ and $\ell n\,(\vec{g_n}) \leq 2i_n + 1$, each $i_k{}^\frown \vec{g_k}$ is called a ***round of*** $\mathbf{t}$.

In Chapter 4, we will be using $T^{\alpha}_{g.t.}$ to decompose and build $\mathbf{\Sigma}^0_{\alpha}$ sets. Each move m in this tail $T^{\alpha}_{g.t.}$ will correspond to picking a set of the appropriate complexity, which is determined by a matching node $\gamma = \gamma(m)$ in the T^{α} tree. Specifically, the set will be of complexity $\mathbf{\Sigma}^0_{\gamma}$ or $\mathbf{\Pi}^0_{\gamma}$. In particular, suppose $A \in \mathbf{\Sigma}^0_{\alpha} \upharpoonright X^{\omega}$. Since $A = \bigcup_{i \in \omega} B_i$ with appropriately chosen sets $B_i \in \mathbf{\Pi}^0_{\beta_i} \upharpoonright X^{\omega}$, we can use i_0 to pick a particular one; namely, B_{i_0}. This set can further be broken down into sets with lower complexity taken from T^{α}, and each subsequent move in $T^{\alpha}_{g.t.}$ will choose one of those sets.

With the above in mind, we need to be able to move from the $T^{\alpha}_{g.t.}$ back to the T^{α} tree. The definitions that follow allow us to do so.

Definition 3.7 (Extract from the tree $T^{\alpha}_{g.t.}$)**.** *Suppose $t \in T^{\alpha}_{g.t.}$. Then it is possible to write $t = i_0{}^\frown \vec{g_0}^\frown i_1{}^\frown \vec{g_1}^\frown \ldots ^\frown i_l{}^\frown \vec{g_l}$, where each $i_j{}^\frown \vec{g_j}$ is a round of t. Define*

$$\text{extract}_{T^{\alpha}_{g.t.}}(t) = \text{ext}\,(t) = (i_0, i_1, \ldots, i_l).$$

Notice that we have actually built each $t \in T^{\alpha}_{g.t.}$ from its extract (as seen in Definition 3.5).

Next, corresponding to each $t \in T^{\alpha}_{g.t.}$, there's a natural position $p \in T^{\alpha}$. This is easily illustrated by returning to the example at the beginning of the chapter with the tree $\mathscr{T}^A$. In Definition 3.8 below, we define the function which transfers each position in $T^\alpha_{g.t.}$ to its matching position in T^α.

Definition 3.8 (Canonical Tail-Tree to Ordinal-Tree Transfer Function)**.** *Suppose $t \in T^\alpha_{g.t.}$. Let $\vec{i} = (i_0, i_1, \ldots, i_n) = \mathrm{ext}\,(t)$ and $p_t = r_0\hat{\ }r_1\hat{\ }\ldots\hat{\ }r_{n+1} \in T^\alpha$ where*

$$r_k = \begin{cases} (\alpha_\varnothing) & \textit{when } k = 0 \\ (\beta_{\vec{i}\restriction k}, \beta_{\vec{i}\restriction k} - 1, \beta_{\vec{i}\restriction k} - 2, \ldots, \alpha_{\vec{i}\restriction k}) & \textit{when } 0 < k \leq n+1. \end{cases}$$

(Notice that r_k is essentially a "round" in the ordinal tree T^α.) We define the natural transfer function from $T^\alpha_{g.t.}$ to T^α as follows:

$$\mathrm{transfer}(t) = p_t \restriction (\ell n\,(t) + 1).$$

We will refer to this function simply as the Tail-Tree/Ordinal-Tree transfer function. Note that transfer(t) is not one-to-one. There are many positions in $T^\alpha_{g.t.}$ which correspond to a single position in T^α (and also a single play in I^α). This fact should not be surprising if one recalls the tree $\mathscr{T}^A$ described at the beginning of this chapter. Consider a node (B, n, γ) where γ is a successor ordinal. The child nodes are all of the form $(C_i, i, \gamma - 1)$, where i varies through ω. In other words, the third component (which corresponds to a move in the tree T^α) is identical for all child nodes of (B, n, γ), while the second component (which corresponds to a move in the tree $T^\alpha_{g.t.}$) is not.

Lemma 3.1. *If $t \in T^\alpha_{g.t.}$ with $t \neq \varnothing$, $\vec{i} = (i_0, i_1, \ldots, i_n) = \mathrm{ext}\,(t)$, and $i_n\hat{\ }\vec{g}_n$ is the final round* *of t, then:*

$$\text{transfer}(t) = r_0\hat{\ }r_1\hat{\ }\dots\hat{\ }r_n\hat{\ }(\beta_{\vec{i}}, \beta_{\vec{i}} - 1, \beta_{\vec{i}} - 2, \dots, \beta_{\vec{i}} - \ell n\,(\vec{g}_n)),$$

where each r_k is defined as in Definition 3.8.

Proof. The proof uses simple calculations based on the length of t and p_t

Suppose $t = i_0\hat{\ }\vec{g}_0\hat{\ }i_1\hat{\ }\vec{g}_1\hat{\ }\dots\hat{\ }i_n\hat{\ }\vec{g}_n$ where each $i_k\hat{\ }\vec{g}_k$ is a round of t. Then for $0 \leq k \leq n-1$,

$$\ell n\,(i_k\hat{\ }\vec{g}_k) = 2i_k + 2 = \ell n\,(r_{k+1}),$$

since $\ell n\,(\vec{g}_k) = 2i_k + 1$ and $r_{k+1} = (\beta_{(i_0,\dots,i_k)}, \beta_{(i_0,\dots,i_k)} - 1, \beta_{(i_0,\dots,i_k)} - 2, \dots, \alpha_{(i_0,\dots,i_k)})$ where $\beta_{(i_0,\dots,i_k)} = \alpha_{(i_0,\dots,i_k)} + 2i_k + 1$.

Next we compute

$$\begin{aligned}
\ell n\,(t) + 1 &= 1 + \sum_{k=0}^{n} \ell n\,(i_k\hat{\ }\vec{g}_k) \\
&= \left(1 + \sum_{k=0}^{n-1} \ell n\,(i_k\hat{\ }\vec{g}_k)\right) + \ell n\,(i_n\hat{\ }\vec{g}_n) \\
&= \left(\ell n\,(r_0) + \sum_{k=0}^{n-1} \ell n\,(r_{k+1})\right) + \ell n\,(\vec{g}_n) + 1 \\
&= \ell n\,(r_0\hat{\ }r_1\hat{\ }\dots r_n) + \ell n\,(\vec{g}_n) + 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
\text{transfer}(t) &= p_t \restriction (\ell n\,(t) + 1) \\
&= (r_0{}^\frown r_1{}^\frown \dots r_n{}^\frown r_{n+1}) \restriction (\ell n\,(r_0{}^\frown r_1{}^\frown \dots r_n) + \ell n\,(\vec{g}_n) + 1) \\
&= r_0{}^\frown r_1{}^\frown \dots r_n{}^\frown (r_{n+1} \restriction (\ell n\,(\vec{g}_n) + 1)) \\
&= r_0{}^\frown r_1{}^\frown \dots r_n{}^\frown (\beta_{\vec{i}\restriction(n+1)}, \beta_{\vec{i}\restriction(n+1)} - 1, \beta_{\vec{i}\restriction(n+1)} - 2, \dots, \beta_{\vec{i}\restriction(n+1)} - \ell n\,(\vec{g}_n)) \\
&= r_0{}^\frown r_1{}^\frown \dots {}^\frown r_n{}^\frown (\beta_{\vec{i}}, \beta_{\vec{i}} - 1, \beta_{\vec{i}} - 2, \dots, \beta_{\vec{i}} - \ell n\,(\vec{g}_n)).
\end{aligned}$$

□

Next, it will be useful to have each position in $T^{\alpha}_{g.t.}$ correspond to a single Borel complexity, so we define this function below.

Definition 3.9 (Complexity of a Position from $T^{\alpha}_{g.t.}$)**.** *Suppose* $t \in T^{\alpha}_{g.t.}$. *Let* $p_t = \text{transfer}(t)$. *Then define*

$$\text{complexity}(t) = \text{c}\,(t) = p_t\,(\ell n\,(p_t) - 1)\,.$$

In other words, the complexity of $t \in T^{\alpha}_{g.t.}$ *is defined as the last node of the corresponding position in* T^{α} *using the Tail-Tree/Ordinal-Tree transfer function.*

Lemma 3.2. *Suppose* $t \in T^{\alpha}_{g.t.}$ *and* $i_n{}^\frown \vec{g}_n$ *is the last round of* t.[2]

(a) $\text{c}\,(t) = \begin{cases} \beta_{\text{ext}(t)} - \ell n\,(\vec{g}_n) & \text{if } t \neq \varnothing \\ \alpha_{\varnothing} & \text{if } t = \varnothing. \end{cases}$

(b) If $\ell n\,(\vec{g}_n) \geq 1$, *then* $\text{c}\,(t \restriction (\ell n\,(t) - 1)) = \text{c}\,(t) + 1$.

[2] Since t is not necessarily a play, $i_n{}^\frown \vec{g}_n$ is not necessarily a full round. This is consistent with Definition 3.6, which only requires that in the last round, $\ell n\,(\vec{g}_n) \leq 2i_n + 1$.

(c) If t ends at a full round, $\mathrm{c}(t) = \alpha_{\mathrm{ext}(t)}$.

(d) If t is a full play (i.e. $t \in [T^{\alpha}_{g.t.}]$), $\mathrm{c}(t) = 1$.

Proof. For the following proof, let $\vec{i} = \mathrm{ext}(t)$.

Proof of (a): If $t = \varnothing$, transfer$(t) = p_t \restriction 1 = (\alpha_\varnothing)$ (from Definition 3.8). Thus $\mathrm{c}(t) = (\alpha_\varnothing)(0) = \alpha_\varnothing$.

$t \neq \varnothing$ follows immediately from Lemma 3.1, which demonstrates that the last node of transfer(t) is $\beta_{\vec{i}} - \ell n(\vec{g}_n) = \beta_{\mathrm{ext}(t)} - \ell n(\vec{g}_n)$.

Proof of (b): Fix u so that $t = u \hat{}\, i_n \hat{}\, \vec{g}_n$, and let $\tilde{t} = t \restriction (\ell n(t) - 1)$. Since $\ell n(\vec{g}_n) \geq 1$, $\tilde{t} = u \hat{}\, i_n \hat{}\, \vec{h}_n$, where $\vec{h}_n = \vec{g}_n \restriction (\ell n(\vec{g}_n) - 1)$. Next,

$$\mathrm{ext}(t) = \mathrm{ext}(u)\,\hat{}\, i_n = \mathrm{ext}\left(\tilde{t}\right)$$

and

$$\ell n\left(\vec{h}_n\right) = \ell n(\vec{g}_n) - 1.$$

Thus, from part (a),

$$\begin{aligned}\mathrm{c}\left(\tilde{t}\right) &= \beta_{\mathrm{ext}(\tilde{t})} - \ell n\left(\vec{h}_n\right)\\ &= \beta_{\mathrm{ext}(t)} - \ell n(\vec{g}_n) + 1\\ &= \mathrm{c}(t) + 1.\end{aligned}$$

Proof of (c): Note that when t ends in a full round,

$$\ell n\left(g_n\right) = 2i_n + 1.$$

Therefore, from part (a),

$$\begin{aligned} \mathrm{c}\left(t\right) &= \beta_{\mathrm{ext}(t)} - \ell n\left(\vec{g}_n\right) \\ &= \alpha_{\mathrm{ext}(t)} + 2i_n + 1 - \left(2i_n + 1\right) \\ &= \alpha_{\mathrm{ext}(t)}. \end{aligned}$$

Proof of (d): Using the definitions of $[T^{\alpha}_{g.t.}]$ and $[I^{\alpha}]$,

$$\begin{aligned} t \in [T^{\alpha}_{g.t.}] &\implies \mathrm{ext}\left(t\right) \in [I^{\alpha}] \\ &\implies \alpha_{\mathrm{ext}(t)} = 1. \end{aligned}$$

Additionally, when $t \in [T^{\alpha}_{g.t.}]$, t ends at a full round, so from part (c),

$$t \in [T^{\alpha}_{g.t.}] \implies \ell n\left(g_n\right) = 2i_n + 1.$$

Therefore, since $t \in [T^{\alpha}_{g.t.}]$,

$$\mathrm{c}\left(t\right) = \alpha_{\mathrm{ext}(t)} = 1.$$

□

3.4 The Auxiliary Game Tree Υ^α

The final tree we wish to build is the full auxiliary game tree. Fortunately, its definition is simple once we have the tail.

Definition 3.10 (Auxiliary Game Tree)**.**

$$[\Upsilon^\alpha] = X^\omega \circledast [T^\alpha_{g.t.}] = \{f\,\hat{}\,h \mid f \in X^\omega \wedge h \in [T^\alpha_{g.t.}]\}.$$

Although the final result will involve games in this full tree, the tail is the more significant part of the tree for our results in Chapter 4. In the next chapter we explain how the tail is used, in conjunction with the T^α tree, to build and decompose $\mathbf{\Sigma}^0_\alpha$ sets.

Chapter 4

Decomposition/Construction of Borel Set

In this chapter we use the ordinal tree T^α and tail game tree $T^\alpha_{g.t.}$ to provide:

(i) a canonical method of building $\mathbf{\Sigma}^0_\alpha \upharpoonright \chi$ sets from open sets, and

(ii) a canonical decomposition of $\mathbf{\Sigma}^0_\alpha \upharpoonright \chi$ sets into open sets.

(i) and (ii) will be documented in detail in Definition 4.1 and Theorem 4.3. We attempt to first provide a short overview (especially for those who want to skip the details).

Both items will be used to show our main result Theorem 5.1, each corresponding to one direction of the proof. Here χ can be any nonempty set with a topology such that $\mathbf{\Sigma}^0_1 \upharpoonright \chi \subseteq \mathbf{\Sigma}^0_2 \upharpoonright \chi$. Initially, in Theorem 5.1, we will be interested in $\chi = X^\omega$. Plays from the tail game tree $T^\alpha_{g.t.}$ will be used to index the open sets described in (i) and (ii), and moves from the ordinal tree T^α provide an upper bound for the levels of the Borel hierarchy at each stage of the construction in (i) or deconstruction in (ii).

The process of doing either direction results in a $T^{\alpha}_{g.t.}$-sequence of sets B_p. These sets are related by alternating quantifiers. For example, given any even length position $t \in T^{\alpha}_{g.t.}$, $B_t = \bigcup_{i\in\omega}\bigcap_{j\in\omega} B_{t\frown i\frown j}$, i.e.

$$f \in B_t \iff \exists i \in \omega \forall j \in \omega (f \in B_{t\frown i\frown j}).$$

This relationship between the sets naturally provides canonical strategies (one for each player I and player II) for positions in the tail tree $T^{\alpha}_{g.t.}$. If we have some $\tilde{f} \in B_t$, we can use the existential quantifier to follow the strategy for player I that keeps $\tilde{f} \in B_{t\frown i\frown j}$. Similarly, if $\tilde{f} \notin B_t$, for every $i \in \omega$ there is some $j \in \omega$ so that $\tilde{f} \notin B_{t\frown i\frown j}$, implicitly defining a strategy for player II.

4.1 Construction of a Borel Set

For our construction of a set at the $\mathbf{\Sigma}^0_\alpha$ level, we begin with an appropriate $[T^{\alpha}_{g.t.}]$-sequence of open sets, and then we take unions and intersections to construct higher complexity sets.

Definition 4.1 (Canonical Construction of a $\mathbf{\Sigma}^0_\alpha \restriction \chi$ Set)**.** *Given a $[T^{\alpha}_{g.t.}]$-sequence $\langle E_h \subseteq \chi \mid h \in [T^{\alpha}_{g.t.}]\rangle$, where χ is a nonempty space, we define the $T^{\alpha}_{g.t.}$-sequence $\vec{\mathbf{E}} = \langle E_t \subseteq \chi \mid t \in T^{\alpha}_{g.t.}\rangle$. More precisely, each E_t will be defined inductively on the rank of t as follows:*

(1) $E_t = \bigcup_{i\in\omega} E_{t\frown i}$ *if $\ell n(t)$ is even and $t \in T^{\alpha}_{g.t.} \setminus [T^{\alpha}_{g.t.}]$.*

(2) $E_t = \bigcap_{i\in\omega} E_{t\frown i}$ *if $\ell n(t)$ is odd and $t \in T^{\alpha}_{g.t.} \setminus [T^{\alpha}_{g.t.}]$.*

Note that in the above definition, we also have the E_t's defined for $t \in [T^{\alpha}_{g.t.}] \subseteq T^{\alpha}_{g.t.}$, as these were given from the $[T^\alpha_{g.t.}]$-sequence $\langle E_h \subseteq \chi \mid h \in [T^\alpha_{g.t.}]\rangle$. Next we perform a straightforward calculation to compute the complexities of the E_t's when t ends at a full round, as reviewed below. Formally, this calculation is done by induction on the rank of positions $t \in T^\alpha_{g.t.}$.

Readers will want to recall:

- Definition 3.6, i.e. $t = i_0{}^\frown \vec{g_0}{}^\frown i_1{}^\frown \vec{g_1}{}^\frown \ldots {}^\frown i_l{}^\frown \vec{g_l} \in T^\alpha_{g.t.}$ ends at a *full round* when each $\ell n\,(\vec{g_j}) = 2i_j + 1$.

- Definition 3.7, i.e. if $t = i_0{}^\frown \vec{g_0}{}^\frown i_1{}^\frown \vec{g_1}{}^\frown \ldots {}^\frown i_l{}^\frown \vec{g_l}$, extract$(t)$ = ext$\,(t) =_{def} (i_0, i_1, \ldots, i_l)$ as long as each $i_k{}^\frown \vec{g_k}$ is a round of t.

- Definition 3.9, i.e. complexity(t) = c$\,(t)$ which assigns each position in $T^\alpha_{g.t.}$ to an appropriate ordinal move from T^α.

- The inductive definition of the sequence of limit ordinals $\alpha_{(i_0,i_1,\ldots,i_l)}$ in the ordinal tree T^α for $(i_0, i_1, \ldots, i_l) \in I^\alpha$, which can be found in Definition 3.1 and 3.2. These were strictly decreasing as we progress farther down the tree. That is, if $\vec{i} \supset \vec{j}$, then $\alpha_{\vec{i}} < \alpha_{\vec{j}}$.

- The inductive definition of the sequence of successor ordinals $\beta_{(i_0,i_1,\ldots,i_l)}$ in the ordinal tree T^α, which also can be found in Definition 3.1 and 3.2.

Theorem 4.1. *Suppose we are given $\langle E_h \mid h \in [T^\alpha_{g.t.}]\rangle$ where each $E_h \in \mathbf{\Sigma}^0_1 \upharpoonright \chi$. Construct $\vec{\mathbf{E}} = \langle E_t \mid t \in T^\alpha_{g.t.}\rangle$ as in Definition 4.1. Then, for each $t \in T^\alpha_{g.t.}$,*

$$E_t \in \Gamma_t = \begin{cases} \mathbf{\Sigma}^0_{\mathrm{c}(t)} \upharpoonright \chi & \textit{if } \ell n\,(t) \textit{ is even} \\ \mathbf{\Pi}^0_{\mathrm{c}(t)} \upharpoonright \chi & \textit{if } \ell n\,(t) \textit{ is odd.} \end{cases}$$

In particular, $E_\varnothing \in \mathbf{\Sigma}^0_\alpha \restriction \chi$.

Proof. We will do an inductive proof (formally an inductive proof on the rank of positions from the $T^\alpha_{g.t.}$ tail tree).

We start with the base case where $t = h$ for some full length play $h \in [T^\alpha_{g.t.}]$. By Lemma 3.2, $\mathrm{c}(h) = 1$. Immediately from the definition of $\vec{\mathbf{E}}$ and our assumptions in this theorem, each $E_h \in \mathbf{\Sigma}^0_1 \restriction \chi = \mathbf{\Sigma}^0_{\mathrm{c}(t)} \restriction \chi$.

Next is the inductive step. Suppose t ends at a full round. By the induction hypothesis, for any $i \in \omega$, and any $\vec{g} = (j_0, j_1, \ldots, j_{2i}) \in \omega^{2i+1}$, $E_{t\hat{\ }i\hat{\ }\vec{g}} \in \mathbf{\Sigma}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g})} \restriction \chi$. We wish to show that $E_t \in \mathbf{\Sigma}^0_{\mathrm{c}(t)} \restriction \chi$.

For our next set of computations, recall from Lemma 3.2 on page 47, while $\ell n(\vec{g}) \geq 1$,

$$\mathrm{c}\left(t\hat{\ }i\hat{\ }(\vec{g} \restriction (k-1))\right) = \mathrm{c}\left(t\hat{\ }i\hat{\ }(\vec{g} \restriction k)\right) + 1.$$

Then by our Definition 4.1 and since $E_{t\hat{\ }i\hat{\ }\vec{g}} \in \mathbf{\Sigma}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g})} \restriction \chi$, we know

$$(1) \qquad E_{t\hat{\ }i\hat{\ }\vec{g}\restriction(2i)} = \bigcap_{j_{2i}\in\omega} E_{t\hat{\ }i\hat{\ }\vec{g}} \in \mathbf{\Pi}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g})+1} \restriction \chi = \mathbf{\Pi}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g}\restriction(2i))} \restriction \chi.$$

$$(2) \qquad E_{t\hat{\ }i\hat{\ }\vec{g}\restriction(2i-1)} = \bigcup_{j_{2i-1}\in\omega} E_{t\hat{\ }i\hat{\ }\vec{g}\restriction(2i)} \in \mathbf{\Sigma}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g}\restriction(2i))+1} \restriction \chi = \mathbf{\Sigma}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g}\restriction(2i-1))} \restriction \chi.$$

$$\vdots$$

$$(2i{+}1) \qquad E_{t\hat{\ }i} = \bigcap_{j_0\in\omega} E_{t\hat{\ }i\hat{\ }\vec{g}\restriction 1} \in \mathbf{\Pi}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g}\restriction 1)+1} \restriction \chi = \mathbf{\Pi}^0_{\mathrm{c}(t\hat{\ }i\hat{\ }\vec{g}\restriction 0)} \restriction \chi = \mathbf{\Pi}^0_{\mathrm{c}(t\hat{\ }i)} \restriction \chi.$$

Using Lemma 3.2, this gives us that $E_{t^\frown i} \in \mathbf{\Pi}^0_{\beta_{\text{ext}(t^\frown i)}} \restriction \chi = \mathbf{\Pi}^0_{\beta_{\text{ext}(t)^\frown i}} \restriction \chi$. Next,

$$E_t = \bigcup_{i \in \omega} E_{t^\frown i}.$$

$\sup_{i\in\omega} \beta_{\text{ext}(t)^\frown i} = \alpha_{\text{ext}(t)}$ (from Property 3.2 (2) on page 38), so that $E_t \in \mathbf{\Sigma}^0_{\alpha_{\text{ext}(t)}} \restriction \chi = \mathbf{\Sigma}^0_{\text{c}(t)} \restriction \chi$.

This concludes our inductive argument, and so now we can simply consider the case when $t = \varnothing$:

$$E_\varnothing \in \mathbf{\Sigma}^0_{\text{c}(\varnothing)} \restriction \chi = \mathbf{\Sigma}^0_{\alpha_\varnothing} \restriction \chi = \mathbf{\Sigma}^0_\alpha \restriction \chi.$$

□

We will be applying the construction described in Definition 4.1 in our our main result, Theorem 5.1 and our extended result, Theorem 7.1. Theorem 5.1 will require exactly Theorem 4.1, beginning with open sets to build $\mathbf{\Sigma}^0_\alpha$ sets. However, Theorem 7.1 will instead use a very similar result, shown in Chapter 6, which begins with higher complexity sets instead of open sets. Theorem 6.1 follows by using a similar argument to Theorem 4.1; there we will use the underlying rank function.

4.2 Decomposition of a Borel Set

Our next goal is to work in the opposite direction. Given a set $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$, we wish to decompose it to create a $[T^\alpha_{g.t.}]$-sequence of open sets $\langle A_h \mid h \in [T^\alpha_{g.t.}]\rangle$. In fact, we will create Borel components of A, A_t for every $t \in T^\alpha_{g.t.}$. A_t has Borel complexity $\text{c}(t)$ from the T^α tree. We will break off the successor steps in this process into a lemma for additional clarity in the proof of Theorem 4.3.

Lemma 4.2. *Let χ satisfy $\mathbf{\Sigma}^0_1 \upharpoonright \chi \subseteq \mathbf{\Sigma}^0_2 \upharpoonright \chi$. Suppose $F \in \mathbf{\Pi}^0_{\gamma+n} \upharpoonright \chi$, where γ is an ordinal and $n \in \omega$ is odd. Define for $g \in \omega^{\leq n}$:*

$$\Gamma_g = \begin{cases} \mathbf{\Pi}^0_{\gamma+n-\ell n(g)} \upharpoonright \chi & \text{if } \ell n\,(g) \text{ is even.} \\ \mathbf{\Sigma}^0_{\gamma+n-\ell n(g)} \upharpoonright \chi & \text{if } \ell n\,(g) \text{ is odd.} \end{cases}$$

Then there exist sets F_g for every $g \in \omega^{\leq n}$ which satisfy the following properties:

(1) $F_g = \bigcup_{j\in\omega} F_{g\hat{\ }j}$ *if $\ell n(g)$ is odd and $g \in \omega^{<n}$.*

(2) $F_g = \bigcap_{j\in\omega} F_{g\hat{\ }j}$ *if $\ell n(g)$ is even and $g \in \omega^{<n}$.*

(3) $F_g \in \Gamma_g$.

(4) $F_\varnothing = F$.

More specifically, (3) gives us that $F_g \in \mathbf{\Sigma}^0_\gamma \upharpoonright \chi$ when $g \in \omega^n$.

Comment 4.2.1. *$\mathbf{\Sigma}^0_1 \upharpoonright \chi \subseteq \mathbf{\Sigma}^0_2 \upharpoonright \chi$ and its consequences (5) (6) and (8) as listed after Definition 1.22 are used to show property (3) in Lemma 4.2 and property (3) in Lemma 4.3.*

Proof. As in the statement of the lemma, set $F_\varnothing = F \in \mathbf{\Pi}^0_{\gamma+n} \upharpoonright \chi$. Then the remaining sets are formed as a direct result of the definition of the Borel hierarchy, and Properties (1)-(3) immediately follow.

(1) There are sets $F_{(j_0)} \in \mathbf{\Sigma}^0_{\gamma+n-1} \upharpoonright \chi$ so that $F_\varnothing = \bigcap\limits_{j_0\in\omega} F_{(j_0)}$.

(2) For each $F_{(j_0)}$ there are sets $F_{(j_0,j_1)} \in \mathbf{\Pi}^0_{\gamma+n-2} \upharpoonright \chi$ so that $F_{(j_0)} = \bigcup\limits_{j_1\in\omega} F_{(j_0,j_1)}$.

$\vdots$

(*n*) For each $F_{(j_0,j_1,\ldots,j_{n-2})}$ there are sets $F_{(j_0,j_1,\ldots,j_{n-2},j_{n-1})} \in \mathbf{\Sigma}^0_\gamma \restriction \chi$ so that $F_{(j_0,j_1,\ldots,j_{n-2})} = \bigcap_{j_{n-1}\in\omega} F_{(j_0,j_1,\ldots,j_{n-2},j_{n-1})}$.

Thus all four properties have been satisfied. □

Lemma 4.2 extracted part of the proof of Theorem 4.3 below, to simplify the notation of that proof. Specifically, it extracted the decomposition of the successor ordinal Borel complexities down to the level of limit ordinal Borel complexities.

Theorem 4.3 (Canonical Decomposition of $\mathbf{\Sigma}^0_\alpha$ Sets into Open Sets)**.** *Let* χ *satisfy* $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$*. Given* $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$*, there exists a* $T^\alpha_{g.t.}$*-sequence of subsets of* χ*,* $\vec{\mathbf{A}} = \langle A_t \mid t \in T^\alpha_{g.t.}\rangle$*, which satisfies the following properties:*

(1) $A_t = \bigcup_{i\in\omega} A_{t^\frown i}$ *if* $\ell n(t)$ *is even and* $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$.

(2) $A_t = \bigcap_{i\in\omega} A_{t^\frown i}$ *if* $\ell n(t)$ *is odd and* $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$.

(3) $A_t \in \Gamma_t = \begin{cases} \mathbf{\Sigma}^0_{\mathrm{c}(t)} \restriction \chi & \text{if } \ell n\,(t) \text{ is even} \\ \mathbf{\Pi}^0_{\mathrm{c}(t)} \restriction \chi & \text{if } \ell n\,(t) \text{ is odd} \end{cases}$

(4) $A_\varnothing = A$.

In particular, by (3), when $h \in [T^\alpha_{g.t.}]$*,* $A_h \in \mathbf{\Sigma}^0_1 \restriction \chi$*.*

Proof. We will proceed by induction on t. First we define $A_\varnothing = A$ to satisfy (4), so that $A_\varnothing \in \mathbf{\Sigma}^0_\alpha \restriction \chi = \mathbf{\Sigma}^0_{\alpha_\varnothing} \restriction \chi = \Gamma_\varnothing$, which means that (3) holds for the base case.

Suppose we have some $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$ which ends after a full round. Suppose also that $A_t \in \Gamma_t = \mathbf{\Sigma}^0_{\alpha_{\mathrm{ext}(t)}}$, i.e., that property (3) holds for t. Observe from the definition of "ending at a full round", the length of t is even.

Any $\mathbf{\Sigma}^0_{\alpha_{\mathrm{ext}(t)}}$ set can be deconstructed into a union of lower complexity sets. Recall that from our Properties 3.2 on page 38 we have the following:

$$(1)\quad \forall i \in \omega \left(\beta_{\mathrm{ext}(t)\hat{\ }i} < \alpha_{\mathrm{ext}(t)}\right)$$

$$(2)\quad \sup_{i\in\omega} \beta_{\mathrm{ext}(t)\hat{\ }i} = \alpha_{\mathrm{ext}(t)}$$

From this we know we can find appropriately chosen $A_{t\hat{\ }i} \in \mathbf{\Pi}^0_{\beta_{\mathrm{ext}(t)\hat{\ }i}} \restriction \chi = \mathbf{\Pi}^0_{\beta_{\mathrm{ext}(t\hat{\ }i)}} \restriction \chi = \Gamma_{t\hat{\ }i}$ so that

$$A_t = \bigcup_{i\in\omega} A_{t\hat{\ }i}.$$

(This was, in fact, the reason the ordinal tree T^α was constructed with properties (1) and (3).) Next, each $\beta_{\mathrm{ext}(t)\hat{\ }i} = \alpha_{\mathrm{ext}(t)\hat{\ }i} + 2i + 1$ (from Property 3.2 (3)). From here apply Lemma 4.2 for each $i \in \omega$, setting $F = A_{t\hat{\ }i}$, $\gamma = \alpha_{\mathrm{ext}(t)\hat{\ }i}$ and $n = 2i+1$, to obtain F_g for all $g \in \omega^{\leq 2i+1}$. Then for every $g \in \omega^{\leq 2i+1}$, let $A_{t\hat{\ }i\hat{\ }g} =_{def} F_g$. All that remains is to verify that these new sets satisfy the Properties (1)-(3).

Since $\ell n(t)$ is even, $\ell n(t\hat{\ }i\hat{\ }g)$ has the opposite parity as $\ell n(g)$. Hence, the properties (1) and (2) from Lemma 4.2:

$$(1)\quad F_g = \textstyle\bigcup_{j\in\omega} F_{g\hat{\ }j} \quad \text{if } \ell n(g) \text{ is odd and } g \in \omega^{<2i+1}.$$

$$(2)\quad F_g = \textstyle\bigcap_{j\in\omega} F_{g\hat{\ }j} \quad \text{if } \ell n(g) \text{ is even and } g \in \omega^{<2i+1}.$$

immediately become:

$$(1) \quad A_{t^\frown i^\frown g} = \bigcup_{j\in\omega} A_{t^\frown i^\frown g^\frown j} \quad \text{if } \ell n(t^\frown i^\frown g) \text{ is even and } g \in \omega^{<2i+1}.$$

$$(2) \quad A_{t^\frown i^\frown g} = \bigcap_{j\in\omega} A_{t^\frown i^\frown g^\frown j} \quad \text{if } \ell n(t^\frown i^\frown g) \text{ is odd and } g \in \omega^{<2i+1}.$$

which is sufficient for Properties (1) and (2) of this theorem. Next,

$$\begin{aligned}\gamma + n - \ell n\,(g) &= \alpha_{\mathrm{ext}(t)^\frown i} + 2i + 1 - \ell n\,(g) \\ &= \alpha_{\mathrm{ext}(t^\frown i^\frown g)} + 2i + 1 - \ell n\,(g) \\ &= \beta_{\mathrm{ext}(t^\frown i^\frown g)} - \ell n\,(g) \\ &= \mathrm{c}\,(t^\frown i^\frown g)\,,\end{aligned}$$

so that Property (3) of the lemma:

$$F_g \in \begin{cases} \mathbf{\Pi}^0_{\gamma+n-\ell n(g)} \restriction \chi & \text{if } \ell n\,(g) \text{ is even.} \\ \mathbf{\Sigma}^0_{\gamma+n-\ell n(g)} \restriction \chi & \text{if } \ell n\,(g) \text{ is odd.} \end{cases}$$

can be rewritten as:

$$A_{t^\frown i^\frown g} \in \Gamma_{t^\frown i^\frown g} = \begin{cases} \mathbf{\Pi}^0_{\mathrm{c}(t^\frown i^\frown g)} \restriction \chi & \text{if } \ell n\,(t^\frown i^\frown g) \text{ is odd.} \\ \mathbf{\Sigma}^0_{\mathrm{c}(t^\frown i^\frown g)} \restriction \chi & \text{if } \ell n\,(t^\frown i^\frown g) \text{ is even.} \end{cases}$$

This concludes our inductive step.

Finally, in the case when $t = h$ is a full play, $\mathrm{c}\,(h) = 1$, and therefore $A_h \in \Gamma_h = \mathbf{\Sigma}^0_1 \restriction \chi$. □

One point to notice is the properties shared by both the construction and the decomposition of Borel sets. In particular, Properties (1) and (2) from Definition 4.1 and from Theorem 4.3 are identical. These two properties will be extremely important for the following section in the development of canonical strategies in $T^{\alpha}_{g.t.}$.

4.3 Canonical Strategies in $T^{\alpha}_{g.t.}$

In this section we wish to describe natural strategies in the tree $T^{\alpha}_{g.t.}$. Here, we will simply describe them and lay out some of their properties. Ultimately, in the main theorem, we wish for these to be used to build winning strategies in our long game tree $X^{\omega} \circledast [T^{\alpha}_{g.t.}]$.

The definitions for these strategies will rely on having a $T^{\alpha}_{g.t.}$-sequence of sets, $\vec{\mathbf{B}}$. Although we can define our strategies with almost any such "sequence", we will be interested specifically in using a $T^{\alpha}_{g.t.}$-sequence that was formed either from the construction of a Borel set, as in Section 4.1, or from the decomposition of a Borel set, as in Section 4.2. In either of these cases, $\vec{\mathbf{B}}$ will satisfy the properties (1) and (2), wherein the sets from the $T^{\alpha}_{g.t.}$-sequence are related by alternating unions and intersections, which correspond to alternating existential and universal quantifiers. It is these quantifiers that in turn create a natural strategy, where each existential quantifier allows us to select a move.

Definition 4.2 (Canonical Strategies for Player I and Player II)**.** *Let χ be a topological space. Given a limit ordinal $\alpha \in (0, \omega_1)$, $\vec{\mathbf{B}} = \langle B_t \mid t \in T^{\alpha}_{g.t.} \rangle$, with each $B_t \subseteq \chi$, and $f \in \chi$, define the natural canonical strategies $\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}$ for player I and $\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}$ for player II, on positions t in $T^{\alpha}_{g.t.}$.*

Player I's strategy:

$$\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}(t) = \begin{cases} \textit{the least } i \in \omega \textit{ such that } f \in B_{t^\frown i}, & \textit{if such an } i \textit{ exists} \\ 0 & \textit{otherwise} \end{cases}$$

Player II's strategy:

$$\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}(t) = \begin{cases} \textit{the least } i \in \omega \textit{ such that } f \notin B_{t^\frown i}, & \textit{if such an } i \textit{ exists} \\ 0 & \textit{otherwise} \end{cases}$$

Although it was not explicitly specified above, the intended domain for $\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}$ is meant to be only those positions where player I has the next move (t has even length), and for $\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}$, only those positions where player II has the next move (t has odd length).

In situations where the context is clear, some of the indices may be suppressed.

With respect to our main result in Chapter 5, the goal for these strategies in $T^{\alpha}_{g.t.}$ is to use them to formulate strategies in our long game tree $X^{\omega} \circledast [T^{\alpha}_{g.t.}]$. Here, we will be interested in $\chi = X^{\omega}$. The strategies above will be used to generate certain sets that $f \in X^{\omega}$ belongs to. If f is in the Borel set, we use σ^{I} to create h from the tail tree so that $f \in B_h$, one of the open sets. Similarly, when f is not in the Borel set, we can use σ^{II} to generate h so that $f \notin B_h$.

In Chapter 7, we will using the same strategies with the same goal in mind, except with $\chi = X^{\omega} \circledast [T]$ for certain trees T (described there), in which case $f \in X^{\omega} \circledast [T]$.

We show these facts in the next two theorems.

Theorem 4.4. *Suppose $\vec{\mathbf{B}} = \langle B_t \mid t \in T^{\alpha}_{g.t.}\rangle$ is a $[T^{\alpha}_{g.t.}]$-sequence of sets $B_t \subseteq \chi$ which satisfy* *the following properties for $t \in T^{\alpha}_{g.t.} \setminus [T^{\alpha}_{g.t.}]$:*

$$(1) \quad B_t = \bigcup_{i\in\omega} B_{t^\frown i} \quad \textit{if } \ell n(t) \textit{ is even.}$$

$$(2) \quad B_t = \bigcap_{i\in\omega} B_{t^\frown i} \quad \textit{if } \ell n(t) \textit{ is odd.}$$

If h is a play according to $\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}$ and $f \in B_{\varnothing}$, then $f \in B_h$.

Proof. For the duration of this proof, $\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}$ will be denoted simply by σ.

We will proceed by inductively showing that $f \in B_{h\restriction n}$ for every $n \leq \ell n(h)$. The base case, that $f \in B_{\varnothing}$, is given as an assumption in our theorem, and so the inductive step is the only part we need to show.

Suppose $f \in B_{h\restriction n}$, for some $n < \ell n(h)$. We need to show that $f \in B_{h\restriction(n+1)}$. To do this, we will break our proof into two cases.

- If n is even, then $B_{h\restriction n} = \bigcup_{i\in\omega} B_{(h\restriction n)^\frown i}$. Since $f \in B_{h\restriction n}$, we know that $f \in B_{(h\restriction n)^\frown i}$ for at least one $i \in \omega$. This means that when computing $h(n)$ (which will be player I's move), the first part of the definition of σ was used. In other words:

$$h(n) = \sigma\,(h \restriction n) = i, \text{ where } i \in \omega \text{ is the least such that } f \in B_{h\restriction n^\frown i}.$$

  This means that $f \in B_{(h\restriction n)^\frown h(n)} = B_{h\restriction(n+1)}$.

- If n is odd, then $B_{h\restriction n} = \bigcap_{i\in\omega} B_{(h\restriction n)^\frown i}$. Since $f \in B_{h\restriction n}$, we know that $f \in B_{(h\restriction n)^\frown i}$ for every $i \in \omega$. In particular, $f \in B_{(h\restriction n)^\frown h(n)} = B_{h\restriction(n+1)}$.

This completes our induction, and therefore we obtain that for $n = \ell n(h)$, $f \in B_{h\restriction n} = B_h$. $\square$

Theorem 4.5. *Suppose $\vec{\mathbf{B}} = \langle B_t \mid t \in T^{\alpha}_{g.t.}\rangle$ is a $[T^{\alpha}_{g.t.}]$-sequence of sets $B_t \subseteq \chi$ which satisfy the following properties for $t \in T^{\alpha}_{g.t.} \setminus [T^{\alpha}_{g.t.}]$:*

$$(1) \quad B_t = \bigcup_{i\in\omega} B_{t^\frown i} \quad \textit{if } \ell n(t) \textit{ is even.}$$

$$(2) \quad B_t = \bigcap_{i\in\omega} B_{t^\frown i} \quad \textit{if } \ell n(t) \textit{ is odd.}$$

If h is a play according to $\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}$ and $f \notin B_\varnothing$, then $f \notin B_h$.

Proof. For this proof, $\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}$ will be denoted by σ.

As in the proof of the last theorem, we will show using induction that $f \notin B_{h\restriction n}$ for every $n \leq \ell n(h)$. $f \notin B_\varnothing$ is an assumption in our theorem, so we can immediately show the inductive step.

Suppose $f \notin B_{h\restriction n}$, for some $n < \ell n(h)$. We will show that that $f \notin B_{h\restriction(n+1)}$. Again, as in the proof of Theorem 4.4, we will break the proof into even and odd cases.

- If n is even, then $B_{h\restriction n} = \bigcup_{i\in\omega} B_{(h\restriction n)^\frown i}$. Since $f \notin B_{h\restriction n}$, we know that $f \notin B_{(h\restriction n)^\frown i}$ for every $i \in \omega$. In particular, $f \notin B_{(h\restriction n)^\frown h(n)} = B_{h\restriction(n+1)}$.

- If n is odd, then $B_{h\restriction n} = \bigcap_{i\in\omega} B_{(h\restriction n)^\frown i}$. Since $f \notin B_{h\restriction n}$, we know that $f \notin B_{(h\restriction n)^\frown i}$ for at least one $i \in \omega$. This means that when computing $h(n)$ (which will be player II's move), the first part of the definition of σ was used. In other words:

  $$h(n) = \sigma(h \restriction n) = i, \text{ where } i \in \omega \text{ is the least such that } f \notin B_{h\restriction n^\frown i}.$$

  This means that $f \notin B_{(h\restriction n)^\frown h(n)} = B_{h\restriction(n+1)}$.

With our induction finished, we know that for $n = \ell n(h)$, $f \notin B_{h\restriction n} = B_h$. □

Comment 4.5.1. *Notice that in the proofs of both Theorem 4.4 and Theorem 4.5, we relied on player I in* $T^{\alpha}_{g.t.}$ *playing on the odd moves and player II playing on the even moves. We will be using* $T^{\alpha}_{g.t.}$ *as the second half of a tree, as seen in Definition 3.10, where we introduced* $X^{\omega} \circledast [T^{\alpha}_{g.t.}]$*. When we use these strategies, we can only apply these theorems as long as the plays in the first half of the concatenation have even length.*

We will be applying these strategies in Chapter 5. Specifically, if player I has a winning strategy for the game $G(A, X^{\omega})$ where $A \in \mathbf{\Sigma}^0_{\alpha} \restriction X^{\omega}$, we can play according to the strategy to get $\tilde{f} \in A$. Then we can play the strategy $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$ on the tail of the game $G\left(\hat{A}, X^{\omega} \circledast [T^{\alpha}_{g.t.}]\right)$, where $\vec{\mathbf{A}} = \langle A_t \mid t \in T^{\alpha}_{g.t.} \rangle$ is the $[T^{\alpha}_{g.t.}]$-sequence from Theorem 4.3 and $\hat{A} = \mathrm{Lift}\left(\langle A_h \mid h \in [T^{\alpha}_{g.t.}] \rangle\right)$. This will generate $\tilde{h} \in [T^{\alpha}_{g.t.}]$. Since $\tilde{f} \in A$, we can use our above theorems to get $\tilde{f} \in A_{\tilde{h}}$. From this, we obtain that $\tilde{f}^\frown \tilde{h} \in \hat{A}$, i.e. we have discovered a winning strategy for the game $G\left(\hat{A}, X^{\omega} \circledast [T^{\alpha}_{g.t.}]\right)$. A very similar idea will be applied to player II with the strategy $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$.

Chapter 5

Main Theorem: Det $\left(\boldsymbol{\Sigma}^0_\alpha \restriction X^\omega\right) \iff$ Det $\left(\boldsymbol{\Sigma}^0_1 \restriction X^\omega \circledast [T^\alpha_{g.t.}]\right)$ for Countable Limit Ordinals α

In this chapter we obtain our primary result, which establishes the equivalence of the determinacy of $\boldsymbol{\Sigma}^0_\alpha \restriction X^\omega$ sets and open sets from $X^\omega \circledast [T^\alpha_{g.t.}]$, where $\alpha < \omega_1$ is a limit ordinal. We will be extending this result in both Chapter 7 and Chapter 8, as discussed at the end of this chapter.

Notation: Recall from Definition 3.10 from Chapter 3, $[\Upsilon^\alpha] = X^\omega \circledast [T^\alpha_{g.t.}]$. Additionally, it will be useful to have notation to refer to the second half of a position from Υ^α. With this in mind, if p is a position in Υ^α define:

$$\operatorname{tail}(p) = \begin{cases} \varnothing & \text{if } \ell n\,(p) \leq \omega \\ (p(\omega), p(\omega+1), \cdots, p(\ell n(p)-1)) & \text{if } \ell n\,(p) > \omega. \end{cases}$$

Theorem 5.1. *For $\alpha \in (0, \omega_1)$ a limit ordinal and X a nonempty set,*

$$\operatorname{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega) \iff \operatorname{Det}(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast [T^\alpha_{g.t.}]).$$

Proof. The forwards and backwards directions of the proof are done separately in Lemmas 5.2 and 5.3, which are shown below. □

Lemmas 5.2 and 5.3 are relatively short because many of the details of this proof were extracted into results in previous chapters. In particular, we extracted Lemma 2.2(a) and Theorems 4.1, 4.3, 4.4, and 4.5. We will refer back to these as we encounter them.

Lemma 5.2. *For $\alpha \in (0, \omega_1)$ a limit ordinal,* $\operatorname{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega) \implies \operatorname{Det}(\mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha])$.

Proof. Fix a limit ordinal $\alpha \in (0, \omega_1)$. Consider the game $G(E, \Upsilon^\alpha)$, where $E \in \mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha]$. We will construct an auxiliary game in the tree X^ω with complexity $\mathbf{\Sigma}^0_\alpha$ to show that this game is determined. Recall $\operatorname{Proj}_{X^{<\omega}, T^\alpha_{g.t}}(E) = \langle E_h \mid h \in [T^\alpha_{g.t}] \rangle$ from Definition 2.1. Here, each $E_h = \{f \in X^\omega \mid f\hat{\ }h \in E\}$. From Lemma 2.2(a), we know that each of these sets $E_h \in \mathbf{\Sigma}^0_1 \restriction X^\omega$. We will next apply Definition 4.1 to this $[T^\alpha_{g.t.}]$-sequence to create a new

$T^\alpha_{g.t.}$-sequence $\vec{\mathbf{E}} = \langle E_t \mid t \in T^\alpha_{g.t.} \rangle$ with each $E_t \subseteq X^\omega$, such that for $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$:

$$(1) \quad E_t = \bigcup_{i\in\omega} E_{t\hat{\ }i} \quad \text{if } \ell n(t) \text{ is even.}$$

$$(2) \quad E_t = \bigcap_{i\in\omega} E_{t\hat{\ }i} \quad \text{if } \ell n(t) \text{ is odd.}$$

Then from Theorem 4.1, we know that $E_\varnothing \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$. Thus, the auxiliary game that we consider is $G(E_\varnothing, X^{<\omega})$, which, by our assumption in this direction, is determined, and therefore has a winning strategy s^{short} for either player I or player II.

We will use the strategy s^{short} to define a new strategy, s^{long} for the tree Υ^α, and further, we will also show that this is a winning strategy for the same player in the game $G(E, \Upsilon^\alpha)$. To do this, we break down the proof into a separate case for each player.

Case 1: s^{short} is a winning strategy for player I.

Define s^{long} in $G(E, \Upsilon^\alpha)$ in the following way:

$$s^{long}(p) = \begin{cases} s^{short}(p) & \text{if } \ell n(p) < \omega \text{ and } \ell n(p) \text{ is even} \\ \sigma^{\mathrm{I}, p\restriction\omega}_{\vec{\mathbf{E}},\alpha}(\text{tail}(p)) & \text{if } \ell n(p) \geq \omega \text{ and } \ell n(p) \text{ is even.} \end{cases}$$

Note that s^{short} is defined only for player I's moves. Next we show that s^{long} is a winning strategy for player I.

Suppose $\tilde{f}\hat{\ }\tilde{h} \in [\Upsilon^\alpha]$ is a play according to s^{long}, with $\tilde{f} \in X^\omega$ and $\tilde{h} \in [T^\alpha_{g.t.}]$. It is clear from the definition of s^{long} that $\tilde{f}$ is according to s^{short} and $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$.

Since $\tilde{f}$ is according to s^{short}, which is a winning strategy for player I in the game $G(E_\varnothing, X^{<\omega})$, we know that $\tilde{f} \in E_\varnothing$. Hence we can apply Theorem 4.4 to get $\tilde{f} \in E_{\tilde{h}}$, making use of the following:

- $\tilde{f} \in E_\varnothing$

- $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$, and

- Properties (1) and (2) listed in the hypothesis of Theorem 4.4 on page 61 for the $T^\alpha_{g.t.}$-sequence $\vec{\mathbf{E}}$.

Recall from the definition of Proj, that since $\tilde{f} \in E_{\tilde{h}}$, $\tilde{f}\,\hat{}\,\tilde{h} \in E$. In other words, $\tilde{f}\,\hat{}\,\tilde{h}$ is a win for player I in the game $G(E, \Upsilon^\alpha)$. Consequently, s^{long} is a winning strategy for player I.

Case 2: s^{short} is a winning strategy for player II.

Define s^{long} for player II's moves in $G(E, \Upsilon^\alpha)$ in the following way:

$$s^{long}(p) = \begin{cases} s^{short}(p) & \text{if } \ell n(p) < \omega \text{ and } \ell n(p) \text{ is odd} \\ \sigma^{\mathrm{II},p\restriction\omega}_{\vec{\mathbf{E}},\alpha}(\mathrm{tail}(p)) & \text{if } \ell n(p) \geq \omega \text{ and } \ell n(p) \text{ is odd.} \end{cases}$$

Next we show that s^{long} is a winning strategy for player II.

Suppose $\tilde{f}\,\hat{}\,\tilde{h} \in [\Upsilon^\alpha]$ is a play according to s^{long}, with $\tilde{f} \in X^\omega$ and $\tilde{h} \in [T^\alpha_{g.t.}]$. As before, $\tilde{f}$ is according to s^{short} and $\tilde{h}$ is according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$.

Since s^{short} is a winning strategy for player II in the game $G(E_\varnothing, X^{<\omega})$, $\tilde{f} \notin E_\varnothing$. From here we use Theorem 4.5 to get $\tilde{f} \notin E_{\tilde{h}}$ and therefore $\tilde{f}\,\hat{}\,\tilde{h} \notin E$. This means that $\tilde{f}\,\hat{}\,\tilde{h}$ is a win for player II in the game $G(E, \Upsilon^\alpha)$, and hence s^{long} is a winning strategy for player II.

Thus, in all cases, we have a winning strategy for either player I or II in the game $G(E, T)$, which means it is determined. □

Lemma 5.3. *For $\alpha \in (0, \omega_1)$ a limit ordinal,* $\mathrm{Det}(\mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha]) \implies \mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega)$.

Proof. Fix a limit ordinal $\alpha \in (0, \omega_1)$. Consider the game $G(A, X^{<\omega})$, where $A \in \mathbf{\Sigma}^0_\alpha \restriction X^\omega$. We will create a new auxiliary game in $\mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha]$ towards showing this game is determined. By Theorem 1.2, we know $\mathbf{\Sigma}^0_1 \restriction X^\omega \subseteq \mathbf{\Sigma}^0_2 \restriction X^\omega$, and hence Theorem 4.3 applies. Next, using Theorem 4.3, we decompose A to form $\vec{\mathbf{A}} = \langle A_t \mid t \in T^\alpha_{g.t.} \rangle$, where $A_\varnothing =_{def} A$, which satisfies the following properties[1]:

$$
\begin{aligned}
&(1) \quad A_t = \textstyle\bigcup_{i \in \omega} A_{t^\frown i} && \text{if } \ell n(t) \text{ is even and } t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}].\\
&(2) \quad A_t = \textstyle\bigcap_{i \in \omega} A_{t^\frown i} && \text{if } \ell n(t) \text{ is odd and } t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}].\\
&(3) \quad A_h \in \mathbf{\Sigma}^0_1 \restriction X^\omega && \text{if } h \in [T^\alpha_{g.t.}].
\end{aligned}
$$

Define $\hat{A} = \text{Lift}_{X^{<\omega}, T^\alpha_{g.t.}} \left(\langle A_h \mid h \in [T^\alpha_{g.t.}] \rangle \right) = \{ f^\frown h \in [\Upsilon^\alpha] \mid f \in A_h \}$. From Lemma 2.2, we know that $\hat{A} \in \mathbf{\Sigma}^0_1 \restriction [\Upsilon^\alpha]$. By the hypothesis of the Lemma, this means that $G(\hat{A}, X^\omega)$ is determined. Hence, there is a winning strategy s^{long} in the game $G(\hat{A}, \Upsilon^\alpha)$.

Using this strategy, we define the strategy s^{short} for the game $G(A, X^{<\omega})$. Fix

$$s^{short} =_{def} s^{long} \restriction (X^{<\omega} \cap \text{Dom}(s^{long})).$$

To finish, we will show that s^{short} is a winning strategy for the same player as s^{long} in the original game $G(A, X^{<\omega})$. To do this, we will consider two cases.

Case 1: s^{long} is a winning strategy for player I.

To show that s^{short} is a winning strategy for player I in the game $G(A, X^{<\omega})$, suppose $\tilde{f}$ is a play according to s^{short} and, toward a contradiction, suppose that $\tilde{f} \notin A$. We extend $\tilde{f}$ into

[1] The complexity of each A_t depends on T^α and $T^\alpha_{g.t.}$.

$\tilde{f}^\frown\tilde{h} \in [\Upsilon^\alpha]$ by playing s^{long} against $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$. Specifically, $\tilde{h}$ is formed in the following manner:

$$\tilde{h}(n) = \begin{cases} s^{long}\left(\tilde{f}^\frown\left(\tilde{h}\upharpoonright n\right)\right) & \text{if } \tilde{h}(n) \text{ is player I's move} \\ \sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}\left(\tilde{h}\upharpoonright n\right) & \text{if } \tilde{h}(n) \text{ is player II's move.} \end{cases}$$

Since $\tilde{f}$ is according to s^{short}, which is just a restriction of s^{long}, $\tilde{f}$ is also according to s^{long}. Additionally, $\tilde{h}$ was completed according to s^{long} as well, so that $\tilde{f}^\frown\tilde{h}$ is according to s^{long}. s^{long} is a winning strategy for player I in the game $G(\hat{A}, T)$, which gives that $\tilde{f}^\frown\tilde{h} \in \hat{A}$.

However, we also know that $\tilde{h}$ was played according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$. By our assumption toward a contradiction, we have that $\tilde{f} \notin A = A_\varnothing$. Then, applying Theorem 4.5, $\tilde{f} \notin A_{\tilde{h}}$. Recall that $\hat{A} = \mathrm{Lift}_{X^{<\omega},T^\alpha_{g.t.}}\left(\langle A_h \mid h \in [T^\alpha_{g.t.}]\rangle\right)$, and from the definition of Lift, since $\tilde{f} \notin A_{\tilde{h}}$, $\tilde{f}^\frown\tilde{h} \notin \hat{A}$. Comparing this with the conclusion of the previous paragraph, we see that we have reached our desired contradiction.

Thus, we must have $\tilde{f} \in A$ so that s^{short} is a winning strategy for player I.

Case 2: s^{long} is a winning strategy for player II.

We will proceed very similarly to how we did in the first case. Players I and II will jointly produce $\tilde{f}^\frown\tilde{g}$. Player II will play according to s^{long} to land $\tilde{f}^\frown\tilde{g} \notin \hat{A}$ while player I plays according the σ^I to land $\tilde{f}^\frown\tilde{g} \in \hat{A}$.

Now we proceed with the details. Here, s^{short} is a strategy for player II, and we need to show that it is also a winning strategy. Suppose $\tilde{f}$ is a play according to s^{short} and for a contradiction also assume $\tilde{f} \in A$. Next, complete the play in the tree Υ^α to form $\tilde{f}^\frown\tilde{h}$, where

$\tilde{h} \in [T^{\alpha}_{g.t.}]$, as follows:

$$\tilde{h}(n) = \begin{cases} s^{long}\left(\tilde{f}^\frown\left(\tilde{h} \upharpoonright n\right)\right) & \text{if } \tilde{h}(n) \text{ is player II's move} \\ \sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}\left(\tilde{h} \upharpoonright n\right) & \text{if } \tilde{h}(n) \text{ is player I's move} \end{cases}$$

We can see that $\tilde{f}^\frown\tilde{h}$ is according to s^{long}, a winning strategy for player II, so $\tilde{f}^\frown\tilde{h} \notin \hat{A}$.

On the other hand, based on our assumption toward a contradiction, we have that $\tilde{f} \in A = A_{\varnothing}$. We also have that $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$, and from here we apply Theorem 4.4, so that $\tilde{f} \in A_{\tilde{h}}$. From the definition of $\hat{A} = \mathrm{Lift}_{X^{<\omega},T^{\alpha}_{g.t.}}\left(\langle A_h \mid h \in [T^{\alpha}_{g.t.}]\rangle\right)$, since $\tilde{f} \in A_{\tilde{h}}$, $\tilde{f}^\frown\tilde{h} \in \hat{A}$.

We have arrived at our contradiction. Therefore $\tilde{f} \notin A$, making it a win for player II, so that s^{short} is a winning strategy for player II in the game $G(A, X^{<\omega})$.

From these two cases, we see that either player I or player II will always have a winning strategy in the game $G(A, X^{<\omega})$, and therefore this game is determined. □

Comment 5.1. *Both cases of the proof of Lemma 5.3 are shown using a contradiction. It is also possible to give a direct proof by induction, using the rank function on $T^{\alpha}_{g.t.}$.*

Corollary 5.3.1. *Recall that $\mathbb{B} \upharpoonright X^{\omega}$ is the Borel subsets of X^{ω}. Then*

$$\mathrm{Det}\left(\mathbb{B} \upharpoonright X^{\omega}\right) \iff \forall \text{ limit ordinals } \alpha \in (0, \omega_1)\left(\mathrm{Det}(\mathbf{\Sigma}^0_1 \upharpoonright [\Upsilon^{\alpha}])\right).$$

The remaining chapters all complement Theorem 5.1. Specifically:

- Chapters 6 and 7 are particular extensions of results from Chapters 4 and 5.

- Chapter 8 gives a similar result with a successor ordinal in place of α.
- In Chapter 9, we will show $\mathrm{Det}\,(\mathbb{B} \upharpoonright X^\omega) \iff \mathrm{Det}\,(\boldsymbol{\Delta}^0_1 \upharpoonright (X^\omega \circledast \omega^\omega))$. The proofs there depend on recreating the same style of proof as in Theorem 5.1 with only minor adjustments.

Part II

Generalizations

In Part I, our goal was to prove Theorem 5.1 concerning $\mathbf{\Sigma}^0_\alpha$ determinacy of X^ω.

In Part II, we will generalize the earlier results, replacing X^ω by the body $[T]$ of a certain tree. In fact, X^ω never appears in any definition, proof or theorem in Chapters 2, 3, and 4 (with the exception of Definition 3.10 which defines the tree to be used in Chapter 5). X^ω only appears to explain the motivation of these results, in order to remind the readers of the primary goal of proving Theorem 5.1.

Moreover, we will see for countable limit ordinals, that our construction from $\mathbf{\Sigma}^0_1 \restriction \chi$ to $\mathbf{\Sigma}^0_\alpha \restriction \chi$ and vice versa, with some minor adjustments, take us from $\mathbf{\Sigma}^0_\gamma \restriction \chi$ to $\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$, as will be shown in Chapter 6 (for certain χ).

Chapter 7 will mimic Chapter 5 with the adjustments described above taken into account. Chapter 8 will handle the case of $\mathbf{\Sigma}^0_{\alpha+n}$ determinacy for a finite n, and Chapter 9 proves

$$\mathrm{Det}\left(\mathbb{B} \restriction X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Delta}^0_1 \restriction X^\omega \circledast \omega^\omega\right).$$

Chapter 6

Extended Decomposition/Construction of Borel Set

The theorems shown in this chapter are meant to extend the results from Chapter 4. Rather than using open sets to construct $\mathbf{\Sigma}^0_\alpha$ sets, we will use $\mathbf{\Sigma}^0_\gamma$ sets to construct $\mathbf{\Sigma}^0_{\gamma+\alpha}$ sets. Ultimately in Chapter 7, we will prove the same type of theorem as in Chapter 5 but with a tree $[T]$ taking the place of X^ω, for certain appropriately chosen trees T. In this chapter, it will be useful to recall the following:

- $\alpha_{\vec{i}}$ and $\beta_{\vec{i}}$ as defined in Chapter 3 and illustrated in Figure 3.1.

- $\text{extract}(t) = \text{ext}\,(t) = (i_0, i_1, \ldots, i_l)$ when $t = (i_0, \vec{g_0}, i_1, \vec{g_1}, \ldots, i_l, \vec{g_l})$, and for each $j < l$, $\ell n\,(\vec{g_j}) = 2i_j + 1$, and $\ell n\,(\vec{g_l}) \leq 2i_l + 1$, as given in Definition 3.7.

- The definition of $\text{complexity}(t) = \text{c}\,(t)$ from Definition 3.9, as well as the results of

Lemma 3.2

Furthermore, recall our construction from Section 4.1. We were given a $[T^{\alpha}_{g.t.}]$-sequence $\langle E_h \mid h \in [T^{\alpha}_{g.t.}]\rangle$ with each $E_h \in \mathbf{\Sigma}^0_1 \restriction X^\omega$. Then we constructed $\langle E_t \mid t \in T^{\alpha}_{g.t.}\rangle$ in Definition 4.1 such that:

- Each $E_t \in \left(\mathbf{\Sigma}^0_{\mathrm{c}(t)} \restriction X^\omega\right) \cup \left(\mathbf{\Pi}^0_{\mathrm{c}(t)} \restriction X^\omega\right)$ (Theorem 4.1).
- $E_\varnothing \in \mathbf{\Sigma}^0_{\mathrm{c}(\varnothing)} \restriction X^\omega$.

Here, we wish to start with $\langle E_h \mid h \in [T^{\alpha}_{g.t.}]\rangle$, where each $E_h \in \mathbf{\Sigma}^0_\gamma \restriction \chi$, and follow a similar construction as described in the preceeding paragraph, to get $E_\varnothing \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$. Parallel to our work in Part I, we define Γ^γ_t in Definition 6.1 and show $E_t \in \Gamma^\gamma_t$.

For both the construction and deconstruction in this chapter, we will refer to the following definition:

Definition 6.1 (Γ^γ_t)**.** *Let χ be a topological space and let γ be a nonzero ordinal. For each $t \in T^{\alpha}_{g.t.}$:*

$$\Gamma^\gamma_t = \begin{cases} \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)} \restriction \chi & \text{if } \ell n\,(t) \text{ is even and } \mathrm{c}\,(t) \geq \omega. \\ \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)} \restriction \chi & \text{if } \ell n\,(t) \text{ is odd and } \mathrm{c}\,(t) \geq \omega. \\ \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi & \text{if } \ell n\,(t) \text{ is even and } \mathrm{c}\,(t) < \omega. \\ \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi & \text{if } \ell n\,(t) \text{ is odd and } \mathrm{c}\,(t) < \omega. \end{cases}$$

When γ is clear we suppress the superscript γ and instead write Γ_t. Additionally, it depends on χ, which we are already suppressing. In Comments 7.2.1 and 7.3.1, we write $\Gamma^\delta_t \restriction [T]$ to mean Γ^δ_t with $\chi = [T]$.

Note that the formula used to compute the Borel level differs between when $\mathrm{c}\,(t)$ is finite and when $\mathrm{c}(t)$ is infinite. Additionally, note that Γ_t^1 is equal to the Γ_t given in Chapter 4 (in particular, in Theorems 4.1 and 4.3).

6.1 Construction

Theorem 6.1 is the extension of Theorem 4.1. The proof is extremely similar, with only a slight modification near the bottom of the $T^\alpha_{g.t.}$ tree.

Theorem 6.1. *Suppose $\alpha \in (0, \omega_1)$ is a limit ordinal, γ is a nonzero ordinal, and χ is a topological space. Also suppose we are given $\langle E_h \mid h \in [T^\alpha_{g.t.}]\rangle$ where each $E_h \in \mathbf{\Sigma}^0_\gamma \restriction \chi$. Construct $\vec{\mathbf{E}} = \langle E_t \mid t \in T^\alpha_{g.t.}\rangle$ as in Definition 4.1. Then $E_t \in \Gamma^\gamma_t$. In particular, $E_\varnothing \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$.*

Proof. Fix $t \in T^\alpha_{g.t.}$. The proof will proceed inductively on the rank of t to show that $E_t \in \Gamma^\gamma_t$.

For the base case, $\mathrm{rank}(t) = 0$. Here since $t = h$ for some full play h, $\mathrm{c}(h) = 1$ and $\ell n(h)$ is even. Thus, $E_h \in \mathbf{\Sigma}^0_\gamma \restriction \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi = \Gamma^\gamma_t$.

We will separate the inductive step when $\mathrm{rank}(t) > 0$ into several separate cases.

Case 1: $\mathrm{c}(t)$ is finite and $\mathrm{rank}(t) > 0$.

Suppose $k\hat{\ }\vec{g}$ is the final round of t. Since $\mathrm{rank}(t) > 0$ but $\mathrm{c}(t)$ is finite, $k\hat{\ }\vec{g}$ is not a full round.

Notice by Lemma 3.2, that for any $i \in \omega$, since $k\hat{\ }\vec{g}$ is not a full round,

$$\mathrm{c}(t\hat{\ }i) = \mathrm{c}(t) - 1.$$

Subcase 1(a): If $\ell n(t)$ is even, since $\vec{\mathbf{E}}$ satisfies Definition 4.1, $E_t = \bigcup_{i\in\omega} E_{t\hat{\ }i}$. Additionally, by the induction hypothesis, each $E_{t^\frown i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t^\frown i)-1} \restriction \chi = \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-2} \restriction \chi$. Thus, $E_t \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi = \Gamma^\gamma_t$.

Subcase 1(b): If $\ell n\,(t)$ is odd, $E_t = \bigcap_{i\in\omega} E_{t^\frown i}$, where each $E_{t^\frown i} \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t^\frown i)-1} \restriction \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-2} \restriction \chi$. Thus, $E_t \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi = \Gamma^\gamma_t$.[1]

Case 2: $\mathrm{c}\,(t) = \omega$.

When $\mathrm{c}\,(t) = \omega$, $\ell n\,(t)$ is even and $\vec{\mathbf{E}}$ satisfies Definition 4.1, so $E_t = \bigcup_{i\in\omega} E_{t^\frown i}$. Additionally, each $\mathrm{c}\,(t^\frown i) < \omega$, so by the inductive hypothesis, each $E_{t^\frown i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t^\frown i)-1} \restriction \chi = \mathbf{\Pi}^0_{\gamma+2i+1} \restriction \chi$.

Since $\sup_{i\in\omega}(\gamma+2i+1) = \gamma+\omega$, $E_t \in \mathbf{\Sigma}^0_{\gamma+\omega} \restriction \chi = \Gamma^\gamma_t$.

Case 3: $\mathrm{c}\,(t) > \omega$ is a limit ordinal.

When $\mathrm{c}\,(t)$ is a limit ordinal, $\ell n\,(t)$ is even, so $E_t = \bigcup_{i\in\omega} E_{t^\frown i}$.

Additionally, from Lemma 3.2, $\mathrm{c}\,(t) = \beta_{\mathrm{ext}(t)} - \ell n\,(\vec{g}) = \alpha_{\mathrm{ext}(t)} + 2k + 1 - \ell n\,(\vec{g})$, where $k^\frown\vec{g}$ is the last round of t. However, $\ell n\,(\vec{g}) = 2k+1$ since t ends in a full round, and therefore $\mathrm{c}\,(t) = \alpha_{\mathrm{ext}(t)}$. Meanwhile, $\mathrm{c}\,(t^\frown i) = \beta_{\mathrm{ext}(t^\frown i)} - \ell n\,(\varnothing) = \beta_{\mathrm{ext}(t)^\frown i}$.

Next, by the inductive hypothesis, each $E_{t^\frown i} \in \Gamma^\gamma_{t^\frown i} = \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t^\frown i)} \restriction \chi = \mathbf{\Pi}^0_{\gamma+\beta_{\mathrm{ext}(t)^\frown i}} \restriction \chi$.

Recall that $\alpha_{\mathrm{ext}(t)} = \sup_{i\in\omega} \beta_{\mathrm{ext}(t)^\frown i}$, and therefore $E_t = \bigcup_{i\in\omega} E_{t^\frown i} \in \mathbf{\Sigma}^0_{\gamma+\alpha_{\mathrm{ext}(t)}} \restriction \chi = \Gamma^\gamma_t$.

Case 4: $\mathrm{c}\,(t) > \omega$ is a successor ordinal.

Here, for any $i \in \omega$, $\mathrm{c}\,(t^\frown i) = \mathrm{c}\,(t) - 1$, as in Case 1.

Subcase 4(a): If $\ell n\,(t)$ is even, $E_t = \bigcup_{i\in\omega} E_{t^\frown i}$ since $\vec{\mathbf{E}}$ satisfies Definition 4.1. Furthermore, by the induction hypothesis, $E_{t^\frown i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t^\frown i)} \restriction \chi = \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi$. Thus, $E_t \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)} \restriction \chi = \Gamma^\gamma_t$.

[1] Subcase 1(b) will include the case where each $t^\frown i$ is a full play (i.e. the base case), where $\mathrm{c}\,(t^\frown i) = 1$, $\mathrm{c}\,(t) = 2$, $t^\frown i$ has even length, t has odd length, and $E_{t^\frown i} \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t^\frown i)-1} \restriction \chi = \mathbf{\Sigma}^0_{\gamma+1-1} \restriction \chi = \mathbf{\Sigma}^0_\gamma \restriction \chi$.

Subcase 4(b): Similarly, when the $\ell n\,(t)$ is odd, $E_t = \bigcap_{i\in\omega} E_{t^\frown i}$ since $\vec{\mathbf{E}}$ satisfies Definition 4.1. Again by the induction hypothesis, each $E_{t^\frown i} \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t^\frown i)} \restriction \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi$. Thus, $E_t \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)} \restriction \chi = \Gamma^\gamma_t$.

□

Comment 6.1. *Suppose $\delta = \gamma + \alpha = \hat{\gamma} + \alpha$ where $0 < \hat{\gamma} < \gamma$ and α is a nonzero countable limit ordinal. If we have $E_h \in \mathbf{\Sigma}^0_\gamma \restriction \chi$ and $\hat{E}_h \in \mathbf{\Sigma}^0_{\hat{\gamma}} \restriction \chi$ as in Theorem 6.1, our result gives us that $E_\varnothing \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi = \mathbf{\Sigma}^0_\delta \restriction \chi$ and $\hat{E}_\varnothing \in \mathbf{\Sigma}^0_{\hat{\gamma}+\alpha} \restriction \chi = \mathbf{\Sigma}^0_\delta \restriction \chi$. Specifically, we have sets which start at different levels, but through this theorem our final sets are tracked to the same Borel complexity.*

6.2 Decomposition

Theorem 6.2 is the extension of Theorem 4.3. Similar to the last theorem, the proof will differ slightly at the bottom of the $T^\alpha_{g.t.}$ tree, but otherwise it will follow the structure in Theorem 4.3.

Comment 6.2. *In Theorem 4.3, we were working with the topological space X^ω. Since the tree $X^{<\omega}$ has countable height, by Theorem 1.2, $\mathbf{\Sigma}^0_1 \restriction X^\omega \subseteq \mathbf{\Sigma}^0_2 \restriction X^\omega$. In Theorem 6.2 below, we will instead be working with a topological space χ such that $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$. This condition implies the following hold (see page 13):*

5. *$A \in \mathbf{\Sigma}^0_{\alpha+1} \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Pi}^0_\alpha \restriction \chi$ so that $A = \bigcup_{n\in\omega} A_n$.*

6. *$A \in \mathbf{\Pi}^0_{\alpha+1} \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ so that $A = \bigcap_{n\in\omega} A_n$.*

7. *If α is a limit ordinal, $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ if and only if there exist sets $A_n \in \mathbf{\Pi}^0_{\beta_n} \restriction \chi$, where each $\beta_n < \alpha$ so that $A = \bigcup_{n\in\omega} A_n$.*

8. *If α is a limit ordinal and $\alpha = \sup_{n\in\omega} \beta_n$ where each $\beta_n < \alpha$, then $A \in \mathbf{\Sigma}^0_\alpha \restriction \chi$ if and only if there exists $A_n \in \mathbf{\Pi}^0_{\beta_n} \restriction \chi$ for each $n \in \omega$ such that $A = \bigcup_{n\in\omega} A_n$.*

We make use of (5), (6), and (8) and the ordinal tree T^α to have exact Borel levels assigned to the sets we define in the proof of Theorem 6.2.

Theorem 6.2. *Let $\alpha \in (0, \omega_1)$ be a limit ordinal, let γ be a nonzero ordinal, and let χ satisfy $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$, e.g. χ could be the body of a tree of countable height. Suppose $A \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$. Then there exists a $T^\alpha_{g.t.}$-sequence $\vec{\mathbf{A}} = \langle A_t \mid t \in T^\alpha_{g.t.}\rangle$ where each $A_t \subseteq \chi$, which satisfies the following properties:*

$$
\begin{aligned}
&(1) \quad A_t = \textstyle\bigcup_{i\in\omega} A_{t\hat{\ }i} \quad \textit{if } \ell n(t) \textit{ is even and } t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}].\\
&(2) \quad A_t = \textstyle\bigcap_{i\in\omega} A_{t\hat{\ }i} \quad \textit{if } \ell n(t) \textit{ is odd and } t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}].\\
&(3) \quad A_t \in \Gamma^\gamma_t.\\
&(4) \quad A_\varnothing = A.
\end{aligned}
$$

Proof. Fix $A_\varnothing =_{def} A$, satisfying (4). The construction of A_t and the proof of the properties (1)-(3) will proceed inductively.

Suppose $t = \varnothing$. Since $A_\varnothing = A$ and $\mathrm{c}(\varnothing) = \alpha$, $A_\varnothing \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)} \restriction \chi$, satisfying (3).

We will proceed with the induction step by breaking it into four cases. We will assume the induction hypothesis holds at t and show the result holds for $t\hat{\ }i$.

Case 1: $\mathrm{c}(t) > \omega$ is a limit ordinal.

Here, $c(t) = \alpha_{\mathrm{ext}(t)}$, and for any $i \in \omega$, $\mathrm{c}(t\hat{\ }i) = \beta_{\mathrm{ext}(t\hat{\ }i)} = \alpha_{\mathrm{ext}(t\hat{\ }i)} + 2i + 1$. Additionally, we have that $\gamma + \alpha_{\mathrm{ext}(t)} = \sup_{i\in\omega}\left(\gamma + \beta_{\mathrm{ext}(t\hat{\ }i)}\right)$. Since $A_t \in \mathbf{\Sigma}^0_{\gamma+\alpha_{\mathrm{ext}(t)}} \upharpoonright \chi$ by the induction hypothesis, for every $i \in \omega$ there exist sets

$$A_{t\hat{\ }i} \in \mathbf{\Pi}^0_{\gamma+\beta_{\mathrm{ext}(t\hat{\ }i)}} \upharpoonright \chi = \mathbf{\Pi}^0_{\gamma+\alpha_{\mathrm{ext}(t\hat{\ }i)}+2i+1} \upharpoonright \chi$$

so that $A = \bigcup_{i\in\omega} A_{t\hat{\ }i}$.

Case 2: $\mathrm{c}(t) > \omega$ is a successor ordinal.

If $\ell n(t)$ is even, $A_t \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)} \upharpoonright \chi$ by the induction hypothesis. Then for every $i \in \omega$, there exist sets

$$A_{t\hat{\ }i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-1} \upharpoonright \chi = \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t\hat{\ }i)} \upharpoonright \chi$$

such that $A_t = \bigcup_{i\in\omega} A_{t\hat{\ }i}$.

If $\ell n(t)$ is odd, $A_t \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)} \upharpoonright \chi$ by the induction hypothesis. Then for every $i \in \omega$, there exist sets

$$A_{t\hat{\ }i} \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \upharpoonright \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t\hat{\ }i)} \upharpoonright \chi$$

such that $A_t = \bigcap_{i\in\omega} A_{t\hat{\ }i}$.

Case 3: $\mathrm{c}(t) = \omega$.

Here, note that for any $i \in \omega$, $\mathrm{c}(t\hat{\ }i) = 2i+2$. Thus, we have that $\gamma+\omega = \sup_{i\in\omega}(\gamma + \mathrm{c}(t\hat{\ }i) - 1)$.

Since $A_t \in \mathbf{\Sigma}^0_{\gamma+\omega} \restriction \chi$ by the induction hypothesis, for every $i \in \omega$ there exist sets

$$A_{t\hat{}i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t\hat{}i)-1} \restriction \chi$$

so that $A = \bigcup_{i\in\omega} A_{t\hat{}i}$.

Case 4: $\mathrm{c}\,(t) > 1$ is finite.

If $\ell n\,(t)$ is even, $A_t \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi$ by the induction hypothesis. Then for every $i \in \omega$, there exist sets

$$A_{t\hat{}i} \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-2} \restriction \chi = \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t\hat{}i)-1} \restriction \chi$$

such that $A_t = \bigcup_{i\in\omega} A_{t\hat{}i}$.

If $\ell n\,(t)$ is odd, $A_t \in \mathbf{\Pi}^0_{\gamma+\mathrm{c}(t)-1} \restriction \chi$ by the induction hypothesis. Then for every $i \in \omega$, there exist sets

$$A_{t\hat{}i} \in \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t)-2} \restriction \chi = \mathbf{\Sigma}^0_{\gamma+\mathrm{c}(t\hat{}i)-1} \restriction \chi$$

such that $A_t = \bigcap_{i\in\omega} A_{t\hat{}i}$.[2]

□

Comment 6.3. *Theorem 6.1 doesn't require that χ be a countable tree. It works for any topological space, even when $\mathbf{\Sigma}^0_1 \restriction \chi \nsubseteq \mathbf{\Sigma}^0_2 \restriction \chi$.*

Comment 6.4. *Theorem 6.2 uses that $\mathbf{\Sigma}^0_1 \restriction \chi \subseteq \mathbf{\Sigma}^0_2 \restriction \chi$ so as to build A_t such that $A_t \in \Gamma^\gamma_t$. (In particular, $A_h \in \Gamma^\gamma_h$.) Without this condition, a decomposition of a $\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction \chi$ set will correspond to a different $T^\alpha_{g.t.}$. We talk more about this in Theorem 7.5.*

[2] Note that when $t\hat{}i$ is a full play h, $\mathrm{c}\,(t\hat{}i) = 1$ so that $A_{t\hat{}i} \in \mathbf{\Sigma}^0_\gamma \restriction \chi$, that is, $A_h \in \Gamma^\gamma_h$.

Chapter 7

Extended Theorem

Theorem 7.1 is a generalization of the main result, Theorem 5.1, which we restate below:

Theorem 5.1. *For $\alpha \in (0, \omega_1)$ a limit ordinal and X a nonempty set,*

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega) \iff \mathrm{Det}(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast [T^\alpha_{g.t.}]).$$

Similarly, Theorem 7.1 states that for a nonzero ordinal γ and an appropriately chosen T,

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction [T]) \iff \mathrm{Det}(\mathbf{\Sigma}^0_\gamma \restriction [T] \circledast [T^\alpha_{g.t.}]).$$

The proof itself is extremely similar to the proof of Theorem 5.1. Recall that most of the work for Theorem 5.1 was done in Theorems 4.1 and 4.3. Similarly, Theorem 7.1 is mostly done in Theorems 6.1 and 6.2. The roles of X^ω, $\mathbf{\Sigma}^0_1$, and $\mathbf{\Sigma}^0_\alpha$ in 5.1 are replaced by $[T]$, $\mathbf{\Sigma}^0_\gamma$, and $\mathbf{\Sigma}^0_{\gamma+\alpha}$, respectively.

Notation: For the proofs found in this chapter, given a tree T, let $[\Upsilon^{T,\alpha}] = [T] \circledast [T^\alpha_{g.t.}]$. As before in Chapter 5, we will use a tail function to extract the final part of a position from $[\Upsilon^{T,\alpha}]$. Furthermore, we will be using a head function to extract the initial part a position. With this in mind, define the following:

$$\Upsilon_*^{T,\alpha} = \left\{p \in \Upsilon^{T,\alpha} \mid \exists \hat{p} \subseteq p\,(\hat{p} \in [T])\right\}$$

$$\text{head}_{T,\alpha} : \Upsilon_*^{T,\alpha} \to [T]$$

$$\text{head}_{T,\alpha}(f\,\hat{}\,t) = f, \text{ where } f \in [T] \text{ and } t \in T^{\alpha}_{g.t.}$$

$$\text{tail}_{T,\alpha} : \Upsilon_*^{T,\alpha} \to T^{\alpha}_{g.t.}$$

$$\text{tail}_{T,\alpha}(f\,\hat{}\,t) = t, \text{ where } f \in [T] \text{ and } t \in T^{\alpha}_{g.t.}.$$

Note that in the above definitions, the tail is a position from $T^{\alpha}_{g.t.}$, while the head is always a play from T. The reason we wish the separate the first part of a position from the second is found in Chapter 6. Recall that the contruction and deconstruction of a Borel set was performed using induction on the tail.

To complete the proofs of Theorem 7.1, Lemma 7.2, and Lemma 7.3, recall the strategies $\sigma^{\mathrm{I},f}_{\vec{\mathbf{B}},\alpha}$ and $\sigma^{\mathrm{II},f}_{\vec{\mathbf{B}},\alpha}$ defined in Definition 4.2 from Section 4.3. In the definition of these strategies, we depend on the following:

- a limit ordinal $\alpha \in (0, \omega_1)$,
- a topological space χ,

- a $[T^{\alpha}_{g.t.}]$-sequence $\vec{\mathbf{B}} = \langle B_t \mid t \in [T^{\alpha}_{g.t.}]\rangle$, where each $B_t \subseteq \chi$,

- an $f \in \chi$.

In Chapter 5, we used $\chi = X^{\omega}$. Here, we are going to apply these strategies with $\chi = [T]$. Also recall that under the appropriate circumstances from Theorem 4.4,

$$f \in B_h \text{ for some play } h \text{ when } f \in B_{\varnothing} \text{ and } h \text{ is according to } \sigma^{\mathrm{I}}$$

and from Theorem 4.5,

$$f \notin B_h \text{ for some play } h \text{ when } f \notin B_{\varnothing} \text{ and } h \text{ is according to } \sigma^{\mathrm{II}}.$$

Theorem 7.1. *Suppose α is a limit ordinal in $(0, \omega_1)$ and γ is a nonzero ordinal. Let T be a tree which satisfies the following:*

1. $\forall f \in [T](\ell n(f)$ *is even),*

2. $\forall f_1 \in [T] \forall f_2 \in [T]\, (\ell n(f_1) = \ell n(f_2))$, *and*

3. $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$.

Then:

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction [T]) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma} \restriction [T] \circledast [T^{\alpha}_{g.t.}]).$$

Proof. The proofs of the forward and backward directions are done separately in Lemmas 7.2 and 7.3, which are shown below. □

Lemma 7.2. *Suppose α is a limit ordinal in $(0, \omega_1)$ and γ is a nonzero ordinal. Let T be a tree which satisfies the following:*

1. *$\forall f \in [T](\ell n(f)$ is even),*

2. *$\forall f_1 \in [T] \forall f_2 \in [T]\,(\ell n(f_1) = \ell n(f_2))$, and*

Then:

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright [T]) \implies \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma} \upharpoonright [T] \circledast [T^{\alpha}_{g.t.}])$$

Proof. Suppose $G(E, \Upsilon^{T,\alpha})$ is a game with $E \in \mathbf{\Sigma}^0_\gamma \upharpoonright [\Upsilon^{T,\alpha}]$. We will next create a new game in T, and use this to show that $G(E, \Upsilon^{T,\alpha})$ is determined. First, create

$$\mathrm{Proj}_{T, T^{\alpha}_{g.t}}(E) = \langle E_h \mid h \in [T^{\alpha}_{g.t}] \rangle.$$

Recall that each $E_h = \{f \in [T] \mid f\hat{\ }h \in E\}$. From Theorem 2.5, which we can use due to assumption 2 of this lemma, we know that each of these sets $E_h \in \mathbf{\Sigma}^0_\gamma \upharpoonright [T]$. We will next apply Definition 4.1 to this $[T^{\alpha}_{g.t.}]$-sequence to create a new $T^{\alpha}_{g.t.}$-sequence $\vec{\mathbf{E}} = \langle E_t \mid t \in T^{\alpha}_{g.t} \rangle$ which satisfies the following properties, for $t \in T^{\alpha}_{g.t.} \setminus [T^{\alpha}_{g.t.}]$:

$$(1) \quad E_t = \bigcup_{i \in \omega} E_{t\hat{\ }i} \quad \text{if } ln(t) \text{ is even.}$$

$$(2) \quad E_t = \bigcap_{i \in \omega} E_{t\hat{\ }i} \quad \text{if } ln(t) \text{ is odd.}$$

Then from Theorem 6.1, we know that $E_\varnothing \in \mathbf{\Sigma}^0_{\gamma+\alpha} \upharpoonright [T]$. Thus, our auxiliary game which we will consider is $G(E_\varnothing, T)$, which by our assumption is determined.

We will call the winning strategy of $G(E_\varnothing, T)$ s^{short}, and we use s^{short} to define a new strategy, s^{long} for the tree $\Upsilon^{T,\alpha}$. When s^{short} is a strategy for player I, s^{long} is a winning strategy for player I in the game $G(E, \Upsilon^{T,\alpha})$. Similarly when s^{short} is a strategy for player II, s^{long} is a winning strategy for player II in the game $G(E, \Upsilon^{T,\alpha})$. To show this, we break down the proof into two cases.

Case 1: s^{short} is a winning strategy for player I.

Define s^{long} for player I's moves in $G(E, \Upsilon^{T,\alpha})$ in the following way:

$$s^{long}(p) = \begin{cases} s^{short}(p) & \text{if } p \in T \\ \sigma^{\mathrm{I},\mathrm{head}(p)}_{\vec{\mathbf{E}},\alpha}(\mathrm{tail}(p)) & \text{if } p \in \Upsilon^{T,\alpha} \setminus T. \end{cases}$$

Next we show that s^{long} is a winning strategy for player I.

Suppose $\tilde{f}\,\hat{}\,\tilde{h} \in [\Upsilon^{T,\alpha}]$ is a play according to s^{long}, with $\tilde{f} \in [T]$ and $\tilde{h} \in [T^{\alpha}_{g.t.}]$. From the first part of the definition of s^{long}, $\tilde{f}$ is according to s^{short}, and from the second part of the definition, $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$.

Since $\tilde{f}$ is according to s^{short}, which is a winning strategy for player I in the game $G(E_{\varnothing}, T)$, we know that $\tilde{f} \in E_{\varnothing}$. With this fact in addition to those mentioned above, we can apply Theorem 4.4 on page 61 so that $\tilde{f} \in E_{\tilde{h}}$.[1] Recall from the definition of Proj, however, that since $\tilde{f} \in E_{\tilde{h}}$, $\tilde{f}\,\hat{}\,\tilde{h} \in E$. In other words, $\tilde{f}\,\hat{}\,\tilde{h}$ is a win for player I in the game $G(E, \Upsilon^{T,\alpha})$, which means that s^{long} is a winning strategy for player I.

Case 2: s^{short} is a winning strategy for player II.

[1]To satisfy the conditions of Theorem 4.4, we need that $\vec{\mathbf{E}}$ satisfies properties (1) and (2), $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$, and $\tilde{f} \in E_{\varnothing}$. Furthermore, as explained in Comment 4.5.1, we need that $\tilde{f}$ has even length.

Define s^{long} for player II's moves in $G(E, \Upsilon^{T,\alpha})$ as follows:

$$s^{long}(p) = \begin{cases} s^{short}(p) & \text{if } p \in T \\ \sigma^{\mathrm{II},\mathrm{head}(p)}_{\vec{\mathbf{E}},\alpha}(\mathrm{tail}(p)) & \text{if } p \in \Upsilon^{T,\alpha} \setminus T. \end{cases}$$

Next we show that s^{long} is a winning strategy for player II.

Suppose $\tilde{f}^\frown\tilde{h} \in [\Upsilon^{T,\alpha}]$ is a play according to s^{long}, with $\tilde{f} \in [T]$ and $\tilde{h} \in [T^{\alpha}_{g.t.}]$. Similar to Case 1, $\tilde{f}$ is according to s^{short} and $\tilde{h}$ is according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$.

Since $\tilde{f}$ is according to s^{short}, which is a winning strategy for player II in the game $G(E_\varnothing, T)$, $\tilde{f} \notin E_\varnothing$. Next we apply Theorem 4.5 to get $\tilde{f} \notin E_{\tilde{h}}$.[2] Applying the definition of Proj, we know that since $\tilde{f} \notin E_{\tilde{h}}$, $\tilde{f}^\frown\tilde{h} \notin E$, and therefore $\tilde{f}^\frown\tilde{h}$ is a win for player II in the game $G(E, T)$. Finally, s^{long} is a winning strategy for player II.

Thus, in both cases we have a winning strategy for either player I or II in the game $G(E, \Upsilon^{T,\alpha})$, and therefore $\mathbf{\Sigma}^0_\gamma \restriction [\Upsilon^{T,\alpha}]$ is determined. □

Comment 7.2.1. *We wish to call attention to running through the proof of Lemma 7.2 with different γ's.*

In Lemma 7.2, suppose we take $\alpha = \omega$, with γ_1 and γ_2 finite (e.g. $\gamma_1 = 1$ and $\gamma_2 = 2$). If we run through the proof with $E_1 \in \mathbf{\Sigma}^0_{\gamma_1} \restriction [T]$ as well as $E_2 \in \mathbf{\Sigma}^0_{\gamma_2} \restriction [T]$, after applying Definition 4.1 to the $\mathrm{Proj}_{T,T^{\alpha}_{g.t}}(E)$, *we arrive at $\vec{\mathbf{E_1}} = \langle E^1_t \mid t \in T^{\alpha}_{g.t.}\rangle$ and $\vec{\mathbf{E_2}} = \langle E^2_t \mid t \in T^{\alpha}_{g.t.}\rangle$ via Theorem 6.1 with $E^1_t \in \Gamma^{\gamma_1}_t \restriction [T]$ and $E^2_t \in \Gamma^{\gamma_2}_t \restriction [T]$. In particular $E^1_\varnothing \in \mathbf{\Sigma}^0_\omega \restriction [T]$ and $E^2_\varnothing \in \mathbf{\Sigma}^0_\omega \restriction [T]$. We call attention to the fact that in our proof of Lemma 7.2, s^{long}, $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$,*

[2]To satisfy the conditions of Theorem 4.5, we need that $\vec{\mathbf{E}}$ satisfies properties (1) and (2), $\tilde{h}$ is according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$, $\tilde{f} \notin E_\varnothing$, and $\tilde{f}$ has even length.

and $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{E}},\alpha}$ *rely on* $\vec{\mathbf{E}}$*. In the first run,* s^{long} *and each* σ *rely on* $\vec{\mathbf{E_1}}$*. In the second run,* s^{long} *and each* σ *rely on* $\vec{\mathbf{E_2}}$*. Furthermore, the conclusion of the lemma in each case becomes:*

$$\mathrm{Det}(\mathbf{\Sigma}^0_\omega \restriction [T]) \implies \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma_1} \restriction [T] \circledast [T^\alpha_{g.t.}]), \text{ and}$$

$$\mathrm{Det}(\mathbf{\Sigma}^0_\omega \restriction [T]) \implies \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma_2} \restriction [T] \circledast [T^\alpha_{g.t.}]).$$

Comment 7.2.2. *In this lemma we assumed condition 2, which requires that all plays from* T *have equal length. This condition allowed us to apply Theorem 2.5. Rather than using condition 2, we could instead simply require that* T *satisfies the conclusion of Theorem 2.5. That is:*

$$E \in \mathbf{\Sigma}^0_\gamma \restriction \big([T] \circledast [T^\alpha_{g.t.}]\big) \implies \forall g \in [T^\alpha_{g.t.}]\Big(\big(\mathrm{Proj}(E)\big)(g) \in \mathbf{\Sigma}^0_\gamma \restriction [T]\Big).$$

Lemma 7.3. *Suppose* α *is a limit ordinal in* $(0, \omega_1)$ *and* γ *is a nonzero ordinal. Let* T *be a tree which satisfies the following:*

1. $\forall f \in [T](\ell n(f)$ *is even),*
2. $\forall f_1 \in [T] \forall f_2 \in [T]\,(\ell n(f_1) = \ell n(f_2))$*, and*
3. $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$*.*

Then:

$$\mathrm{Det}(\mathbf{\Sigma}^0_\gamma \restriction [T] \circledast [T^\alpha_{g.t.}]) \implies \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction [T]).$$

Proof. Consider a game $G(A, T)$, where $A \in \mathbf{\Sigma}^0_{\gamma+\alpha} \restriction [T]$. Since $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$, Theorem

6.2 holds. Therefore we can decompose $A = A_\varnothing$ to form $\vec{\mathbf{A}} = \langle A_t \mid t \in T^\alpha_{g.t.} \rangle$ which satisfy the following properties:

(1) $A_t = \bigcup_{i\in\omega} A_{t^\frown i}$ if $ln(t)$ is even and $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$.

(2) $A_t = \bigcap_{i\in\omega} A_{t^\frown i}$ if $ln(t)$ is odd and $t \in T^\alpha_{g.t.} \setminus [T^\alpha_{g.t.}]$.

(3) $A_h \in \mathbf{\Sigma}^0_\gamma \restriction [T]$ if $h \in [T^\alpha_{g.t.}]$.

Define $\hat{A} = \mathrm{Lift}_{T,T^\alpha_{g.t.}}\left(\langle A_h \mid h \in [T^\alpha_{g.t.}]\rangle\right)$. Since the lengths of plays from T are uniform by (2) and $T^\alpha_{g.t.}$ is well-founded, we can apply Theorem 2.5 (b). Thus, we know that $\hat{A} \in \mathbf{\Sigma}^0_\gamma \restriction [\Upsilon^{T,\alpha}]$. By our assumption in this direction of our proof, $\hat{A}$ is determined. This gives us a winning strategy, s^{long}, in the long game $G(\hat{A}, \Upsilon^{T,\alpha})$.

Using this strategy, we can define a strategy s^{short} for the game $G(A,T)$. Fix $s^{short} = s^{long} \restriction (T \cap \mathrm{Dom}(s^{long}))$. To finish, we will show that s^{short} is a winning strategy for the same player as s^{long} in the game $G(A,T)$. To do this, we will consider two cases.

Case 1: s^{long} is a winning strategy for player I.

From our definition, s^{short} is also a strategy for player I, but for the game in the game $G(A,T)$.

To show that it is, in fact, a winning strategy, suppose $\tilde{f}$ is a play according to s^{short} and, for a contradiction, suppose that $\tilde{f} \notin A$. Next we will extend $\tilde{f}$ to form the play $\tilde{f}^\frown\tilde{h}$ in the tree $\Upsilon^{T,\alpha}$, where $\tilde{h}$ is formed by playing s^{long} against $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$:

$$\tilde{h}(n) = \begin{cases} s^{long}(\tilde{f}^\frown(\tilde{h} \restriction n)) & \text{if } \tilde{h}(n) \text{ is player I's move} \\ \sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}(\tilde{h} \restriction n) & \text{if } \tilde{h}(n) \text{ is player II's move.} \end{cases}$$

Since $\tilde{f}$ was according to s^{short}, which is just a restriction of s^{long}, $\tilde{f}$ is also according to s^{long}. Additionally, $\tilde{h}$ was completed according to s^{long} as well, so that $\tilde{f}\hat{\ }\tilde{h}$ is according to s^{long}. s^{long} is a winning strategy for player I in the game $G(\hat{A}, T)$, which means that $\tilde{f}\hat{\ }\tilde{h} \in \hat{A}$.

However, we also know that $\tilde{h}$ was played according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$. By our assumption towards a contradiction, we have that $\tilde{f} \notin A$. Recall from Theorem 4.3, $A_\varnothing$ from the $T^\alpha_{g.t.}$-sequence $\vec{\mathbf{A}}$ is defined to be A, so that $\tilde{f} \notin A_\varnothing$. Then, applying Theorem 4.5,[3] $\tilde{f} \notin A_{\tilde{h}}$. Next, $\hat{A} = \mathrm{Lift}_{T,T^\alpha_{g.t.}}\left(\langle A_h \mid h \in [T^\alpha_{g.t.}]\rangle\right)$, and from the definition of Lift, since $\tilde{f} \notin A_{\tilde{h}}$, $\tilde{f}\hat{\ }\tilde{h} \notin \hat{A}$. Using this and the conclusion of the preceeding paragraph, we have a contradiction.

Thus, $\tilde{f} \in A$ so that s^{short} is a winning strategy for player I.

Case 2: s^{long} is a winning strategy for player II.

We will proceed similarly to the first case.

Here, s^{short} will be a strategy for player II, and we need to show that it is also a winning strategy. First, suppose $\tilde{f}$ is a play according to s^{short} and then for a contradiction also assume $\tilde{f} \in A$. Next, complete the play in the tree $\Upsilon^{T,\alpha}$ to form $\tilde{f}\hat{\ }\tilde{h}$, where $\tilde{h} \in [T^\alpha_{g.t.}]$. $\tilde{h}$ will be created in the following manner:

$$\tilde{h}(n) = \begin{cases} s^{long}(\tilde{f}\hat{\ }(\tilde{h} \restriction n)) & \text{if } \tilde{h}(n) \text{ is player II's move} \\ \sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}(\tilde{h} \restriction n) & \text{if } \tilde{h}(n) \text{ is player I's move} \end{cases}$$

We can see that $\tilde{f}\hat{\ }\tilde{h}$ is according to s^{long}, a winning strategy for player II, so $\tilde{f}\hat{\ }\tilde{h} \notin \hat{A}$.

On the other hand, based on our assumption towards a contradiction, we have that $\tilde{f} \in A = A_\varnothing$. We also have that $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$, and from here we apply Theorem

[3]We need that $\vec{\mathbf{A}}$ satisfies properties (1) and (2), $\tilde{h}$ is according to $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$, $\tilde{f} \notin A_\varnothing$, and $\tilde{f}$ has even length.

4.4,[4] so that $\tilde{f} \in A_{\tilde{h}}$. From the definition of $\hat{A} = \text{Lift}_{T,T^{\alpha}_{g.t.}}\left(\langle A_h \mid h \in [T^{\alpha}_{g.t.}]\rangle\right)$, since $\tilde{f} \in A_{\tilde{h}}$, $\tilde{f}^\frown\tilde{h} \in \hat{A}$.

We have arrived at our contradiction, and therefore $\tilde{f} \notin A$, making it a win for player II. s^{short} is a winning strategy for player II in the game $G(A, T)$.

From these two cases, we see that either player I or player II will always have a winning strategy in the game $G(A, T)$, and therefore this game is determined. □

Comment 7.3.1. *We wish to call attention to running through the proof of Lemma 7.3 with different γ's.*

In Lemma 7.3, suppose we take $\alpha = \omega$, with γ_1 and γ_2 finite (e.g. $\gamma_1 = 1$ and $\gamma_2 = 2$). If we run through the proof with $A \in \mathbf{\Sigma}^0_{\gamma_1+\omega} \restriction [T] = \mathbf{\Sigma}^0_{\omega} \restriction [T]$ as well as $A \in \mathbf{\Sigma}^0_{\gamma_2+\omega} \restriction [T] = \mathbf{\Sigma}^0_{\omega} \restriction [T]$, we arrive at $\vec{\mathbf{A_1}} = \langle A^1_t \mid t \in T^{\alpha}_{g.t.}\rangle$ and $\vec{\mathbf{A_2}} = \langle A^2_t \mid t \in T^{\alpha}_{g.t.}\rangle$ via Theorem 6.2 with $A^1_t \in \Gamma^{\gamma_1}_t \restriction [T]$ and $A^2_t \in \Gamma^{\gamma_2}_t \restriction [T]$. In particular, for each $h \in [T^{\alpha}_{g.t.}]$, $A^1_h \in \mathbf{\Sigma}^0_1 \restriction [T]$ whereas each $A^2_h \in \mathbf{\Sigma}^0_2 \restriction [T]$. We call attention to the fact that in our proof of Lemma 7.3, s^{long}, $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$, and $\sigma^{\mathrm{II},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$ rely on $\vec{\mathbf{A}}$. In the first run, s^{long} and each σ rely on $\vec{\mathbf{A_1}}$. In the second run, s^{long} and each σ rely on $\vec{\mathbf{A_2}}$. Furthermore, the conclusion of the lemma in each case becomes:

$$\text{Det}(\mathbf{\Sigma}^0_{\gamma_1} \restriction [T] \circledast [T^{\alpha}_{g.t.}]) \implies \text{Det}(\mathbf{\Sigma}^0_{\omega} \restriction [T]), \text{ and}$$

$$\text{Det}(\mathbf{\Sigma}^0_{\gamma_2} \restriction [T] \circledast [T^{\alpha}_{g.t.}]) \implies \text{Det}(\mathbf{\Sigma}^0_{\omega} \restriction [T]).$$

The next corollary will be useful for results presented in Chapter 8.

[4]We need that $\vec{\mathbf{A}}$ satisfies properties (1) and (2), $\tilde{h}$ is according to $\sigma^{\mathrm{I},\tilde{f}}_{\vec{\mathbf{A}},\alpha}$, $\tilde{f} \in A_{\varnothing}$, and $\tilde{f}$ has even length.

Corollary 7.3.1. *Suppose α is a limit ordinal in $(0, \omega_1)$, $n \in \omega$, γ is a nonzero ordinal, and X is a nonempty set. Then:*

$$\mathrm{Det}(\mathbf{\Sigma}^0_\gamma \restriction X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\gamma+\alpha} \restriction X^\omega \circledast \omega^{2n})$$

Proof. Apply Theorem 7.1 by setting $[T] = X^\omega \circledast \omega^{2n}$.

Notice that the conditions (1), (2), and (3) for Theorem 7.1 are satisfied. For (1) and (2), $\forall f \in [T]\ \ell n\,(f) = \omega + 2n$, which is even. For (3), since the height of $[T]$ is $\omega + 2n$ which is countable, by Theorem 1.2 on page 15, $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$. □

One can show the natural counterpart to Theorem 7.1 by using the same methods as presented in this thesis.

Theorem 7.4. *Suppose α is a limit ordinal in $(0, \omega_1)$ and γ is a nonzero ordinal. Let T be a tree which satisfies the following:*

1. $\forall f \in [T](\ell n(f)$ *is odd*$)$,

2. $\forall f_1 \in [T] \forall f_2 \in [T]\,(\ell n(f_1) = \ell n(f_2))$, *and*

3. $\mathbf{\Sigma}^0_1 \restriction [T] \subseteq \mathbf{\Sigma}^0_2 \restriction [T]$.

Then:

$$\mathrm{Det}(\mathbf{\Pi}^0_{\gamma+\alpha} \restriction [T]) \iff \mathrm{Det}(\mathbf{\Pi}^0_\gamma \restriction [T] \circledast [T^\alpha_{g.t.}]).$$

Similar to how we approached the results in this thesis, for each game $G(A)$, with $A \in \mathbf{\Pi}^0_{\gamma+\alpha} \restriction [T]$, there is a parallel game $G(\tilde{A})$, with $\tilde{A} \in \mathbf{\Pi}^0_\gamma \restriction [T] \circledast [T^\alpha_{g.t.}]$. The set $A = \bigcap_{i\in\omega} B_{(i)}$, where each $B_{(i)} \in \mathbf{\Sigma}^0_{\gamma+\beta_{(i)}} \restriction [T]$ with $\beta_{(i)}$ as defined in Chapter 3.

The first move in $T^\alpha_{g.t.}$ from the game $G(\tilde{A})$ is made by player II. This move, i_0, "corresponds" to selecting the set $B_{(i_0)}$ from this intersection. Next, $B_{(i_0)}$ is a union of sets of lower Borel complexity, and so the second move will correspond to selecting a set from this union. In fact, all of the integer moves from $T^\alpha_{g.t.}$ will similarly code up which set is selected at each stage of the decomposition on A. For any play in $[T] \circledast [T^\alpha_{g.t.}]$, the final move is for player II, and it will "correspond" to selecting a set of complexity $\mathbf{\Pi}^0_\gamma \upharpoonright [T] \circledast [T^\alpha_{g.t.}]$.

Finally, given Theorem 7.4.1, we can obtain a corresponding corollary to pair with Corollary 7.3.1.

Corollary 7.4.1. *Suppose α is a limit ordinal in $(0, \omega_1)$, $n \in \omega$, γ is a nonzero ordinal, and X is a nonempty set. Then:*

$$\mathrm{Det}(\mathbf{\Pi}^0_\gamma \upharpoonright X^\omega \circledast \omega^{2n+1} \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\gamma+\alpha} \upharpoonright X^\omega \circledast \omega^{2n+1}).$$

One can use use the methods in this thesis to prove a result similar to Theorem 7.1 which does not require the condition that $\mathbf{\Sigma}^0_1 \upharpoonright [T] \subseteq \mathbf{\Sigma}^0_2 \upharpoonright [T]$. One can build a well-founded tree $\hat{T}^\alpha_{g.t.}$ of integers, similar to $T^\alpha_{g.t.}$, such that the following holds:

Theorem 7.5. *Suppose α is a limit ordinal in $(0, \omega_1)$ is an ordinal. Let T be a tree which satisfies the following:*

1. *$\forall f \in [T](\ell n(f)$ is even) and*

2. *$\forall f_1 \in [T] \forall f_2 \in [T]\, (\ell n(f_1) = \ell n(f_2))$.*

Then:

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright [T]) \iff \mathrm{Det}(\mathbf{\Delta}^0_1 \upharpoonright [T] \circledast [\hat{T}^\alpha_{g.t.}]).$$

Recall that in Section 3.1, we built the ordinal tree T^α for $\alpha \in (0, \omega_1)$ a limit ordinal that corresponded to a decomposition of any $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$. The child nodes below α all had the same parity (odd if $\alpha > \omega$, and even when $\alpha = \omega$). To obtain Theorem 7.5, we will need to allow for both parities in the child nodes. In particular, if $A \in \mathbf{\Sigma}^0_\alpha \upharpoonright [T]$, for $\alpha > \omega$ and T as in the hypothesis of Theorem 7.5, then $A = \bigcup_{i\in\omega} A_i$, such that each $A_i \in \mathbf{\Pi}^0_{\beta_\alpha(i)} \upharpoonright [T]$, for some successor ordinals $\beta_\alpha(i)$ as in Definition 3.1 on page 35, except we cannot guarantee the parity of each $\beta_\alpha(i)$ is odd, since we are not assuming that $\mathbf{\Sigma}^0_1 \upharpoonright [T] \subseteq \mathbf{\Sigma}^0_2 \upharpoonright [T]$. As we map out the Borel complexities (in a new ordinal tree) that correspond to the decomposition of A_i, we, as before, get to limit ordinals $\gamma_\alpha(i)$ as in Figure 3.3 on page 36, but with some nodes $\gamma_\alpha(i)$ corresponding to $\mathbf{\Sigma}^0_{\gamma_\alpha(i)} \upharpoonright [T]$, and with other nodes $\gamma_\alpha(i)$ corresponding to $\mathbf{\Pi}^0_{\gamma_\alpha(i)} \upharpoonright [T]$. As we continue mapping out the complexity of the decomposition of A through the new ordinal tree, eventually every branch reaches a node 1, with some nodes 1 corresponding to $\mathbf{\Sigma}^0_1 \upharpoonright [T]$ and other nodes 1 corresponding to $\mathbf{\Pi}^0_1 \upharpoonright [T]$. In this new ordinal tree, we will add last nodes 0, corresponding to $\mathbf{\Delta}^0_1 \upharpoonright [T]$. For any node δ in the new ordinal tree, either

(a) the node δ corresponds to $\mathbf{\Sigma}^0_\delta \upharpoonright [T]$, or

(b) the node δ corresponds to $\mathbf{\Pi}^0_\delta \upharpoonright [T]$.

We will want to build $\hat{T}^\alpha_{g.t.}$ so that any move that corresponds to a node δ in case (a) will be a move for player I and any move that corresponds to a node δ in the case (b) will be a move for player II.

Chapter 8

Sucessor Levels

In this section, we discuss how to deal with the case of successor ordinals. Recall from Theorem 5.1 that for a countable limit ordinal α,

$$\mathrm{Det}(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega)$$

Here, we would like to give a similar result when α is replaced by a successor ordinal. Specifically, we will show the following:

Corollary 8.4.1. *Suppose $n \in \omega$ and $\alpha \in (0, \omega_1)$ is a limit ordinal. Then*

(a) $\mathrm{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n} \upharpoonright X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]\right)$*, and*

(b) $\mathrm{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n+1} \upharpoonright X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Pi}^0_1 \upharpoonright X^\omega \circledast \omega^{2n+1} \circledast [T^\alpha_{g.t.}]\right)$.

We will see that Corollary 8.4.1(a) follows from Corollary 7.3.1 and Theorem 8.3 below, while Corollary 8.4.1(b) follows from Proposition 7.4.1 and Theorem 8.4. We will focus on part (a). Corollary 7.3.1, setting $\gamma = 1$, states

$$\mathrm{Det}(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n}).$$

To prove Corollary 8.4.1(a) using with Corollary 7.3.1, it is enough to prove the following:

Theorem 8.3. *Suppose $n \in \omega$ and α is a nonzero ordinal. Then*

$$\mathrm{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n} \restriction X^\omega\right) \iff \mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n}).$$

Note that in Theorem 8.3 and in several other Chapter 8 results, α need not be a countable limit ordinal.

The proof of Theorem 8.3 is based on the proof of the following well-known fact:[1]

$$\mathrm{Det}\left(\mathbf{\Sigma}^0_{2n+1} \restriction X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast \omega^{2n}\right).$$

Theorem 8.3 is shown inductively, and so the following two lemmas contain the inductive step.

Lemma 8.1. *Suppose α is a nonzero ordinal, and $n \in \omega$. Then*

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+1} \restriction X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}).$$

Proof.

($\Longrightarrow$) First we show the forward implication. Pick $A \in \mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}$. Define $\langle A_t \mid$

[1]Notice this well-known fact is just Theorem 8.3 where $\alpha = 1$, as noted in Comment 8.3.2.

$t \in \omega\rangle = \text{Proj}(A)$. Then by Theorem 2.5(a) on page 29, for each $t \in \omega$, $A_t \in \mathbf{\Pi}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}$. Define $\hat{A} = \bigcup_{t\in\omega} A_t$, which by definition, is $\mathbf{\Sigma}^0_{\alpha+1} \upharpoonright X^\omega \circledast \omega^{2n}$. By our assumption, the game $G(\hat{A}, X^\omega \circledast \omega^{2n})$ is determined. Thus, let s^{short} be the winning strategy for this game.

If s^{short} is a strategy for player I, then we will define a new strategy s^{long} for player I in the game $G(A, X^\omega \circledast \omega^{2n+1})$ as follows:

$$s^{long}(p) = \begin{cases} s^{short}(p), & \text{if } \ell n(p) < \omega + 2n \\ \text{least } t \text{ such that } p \in A_t, & \text{if } \ell n(p) = \omega + 2n \text{ and such a } t \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

Note that when $\ell n(p) = \omega + 2n$, it is player I's move, i.e. $s^{long}(p)$ is player I's move.

We will show this is actually a winning strategy for player I. Suppose $\tilde{f}\hat{\ }\tilde{t}$ is according to s^{long}, where $\tilde{f} \in X^\omega \circledast \omega^{2n}$ and $\tilde{t} \in \omega$. Then $\tilde{f}$ is according to s^{short}, so $\tilde{f} \in \hat{A} = \bigcup_{t\in\omega} A_t$. Thus, there exists some t such that $\tilde{f} \in A_t$. By the definition of s^{long}, this means that $\tilde{f} \in A_{\tilde{t}} = \big(\text{Proj}(A)\big)(\tilde{t})$. Then, using the definition of $\text{Proj}(A)$, $\tilde{f}\hat{\ }\tilde{t} \in A$, making it a win for player I.

On the other hand, if s^{short} is a strategy for player II, we will define s^{long} as a strategy for player II in the game $G(A, X^\omega \circledast \omega^{2n+1})$ by $s^{long} = s^{short}$. (Note that since $2n+1$ is odd, the moves of player II in the two game trees are the same.) Again, we will show that s^{long} is a winning strategy for player II. Suppose $\tilde{f}\hat{\ }\tilde{t}$ is according to s^{long}. Then $\tilde{f}$ is according to s^{short}, so $\tilde{f} \notin \hat{A}$. Thus, for every $t \in \omega$, $\tilde{f} \notin A_t$. In particular, $\tilde{f} \notin A_{\tilde{t}}$. Thus, by the definition of $\text{Proj}(A)$, $\tilde{f}\hat{\ }\tilde{t} \notin A$, making it a win for player II.

$(\Longleftarrow)$ Next we complete the other direction of the proof. Pick $A \in \mathbf{\Sigma}^0_{\alpha+1} \upharpoonright X^\omega \circledast \omega^{2n}$.

Then, there exist sets $A_t \in \mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n}$ so that $A = \bigcup_{t\in\omega} A_t$. Define $\hat{A} = \mathrm{Lift}\langle A_t \mid t \in \omega\rangle$. By Theorem 2.5(b) on page 29, $\hat{A} \in \mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}$. From our assumption, the game $G(\hat{A}, X^\omega \circledast \omega^{2n+1})$ is determined, so there exists some winning strategy s^{long}. We define a new strategy s^{short} in the game $G(A, X^\omega \circledast \omega^{2n})$ by $s^{short}(p) = s^{long}(p)$ when $\ell n(p) < \omega + 2n$.

If s^{long} is a strategy for player I, then s^{short} is as well, and in fact it is a winning strategy. Suppose $\tilde{f}$ is according to s^{short}. Then define $\tilde{t} = s^{long}(\tilde{f})$. The play $\tilde{f}{}^\frown\tilde{t}$ is according to s^{long} and therefore must be a win for player I. Hence $\tilde{f}{}^\frown\tilde{t} \in \hat{A}$. But then, $\tilde{f} \in A_{\tilde{t}}$ by definition of the Lift function, so $\tilde{f} \in \bigcup_{t\in\omega} A_t = A$. Thus, $\tilde{f}$ is a win for player I.

However, if s^{long} is a strategy for player II, then s^{short} is as well. Suppose $\tilde{f}$ is according to s^{short}. Then, for any $t \in \omega$, $\tilde{f}{}^\frown t$ is according to s^{long}, so that $\tilde{f}{}^\frown t \notin \hat{A}$. Hence, for every $t \in \omega$, $\tilde{f} \notin A_t$, giving us the result that $\tilde{f} \notin \bigcup_{t\in\omega} A_t = A$. Thus, $\tilde{f}$ is a win for player II. $\square$

Lemma 8.2. *Suppose α is a nonzero ordinal, and $n \in \omega$. Then*

$$\mathrm{Det}(\mathbf{\Pi}^0_{\alpha+1} \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n+2}).$$

Proof. For this proof, we will denote the tree $X^\omega \circledast \omega^{2n+1}$ by $[T^{short}]$ and $X^\omega \circledast \omega^{2n+2}$ by $[T^{long}]$.

($\implies$) First we will complete the forward direction of the proof. Pick $A \in \mathbf{\Sigma}^0_\alpha \restriction [T^{long}]$. Define $\langle A_t \mid t \in \omega\rangle = \mathrm{Proj}(A)$. By Theorem 2.5(a) on page 29, for every $t \in \omega$, $A_t \in \mathbf{\Sigma}^0_\alpha \restriction [T^{short}]$. Finally, define $\hat{A} = \bigcap_{t\in\omega} A_t$, so that $\hat{A} \in \mathbf{\Pi}^0_{\alpha+1} \restriction [T^{short}]$. Thus the game $G(\hat{A}, [T^{short}])$ is determined by assumption. Let s^{short} be a winning strategy for this game.

If s^{short} is a winning strategy for player I, we will create the strategy $s^{long} = s^{short}$ for player I in the game $G(A, [T^{long}])$.[2] Suppose $\tilde{f}\hat{\ }\tilde{t}$ is according to s^{long}. Then $\tilde{f}$ is according to s^{short}, so that $\tilde{f} \in \hat{A} = \bigcap_{t\in\omega} A_t$. Then $\tilde{f} \in A_{\tilde{t}}$ and from the definition of Proj(A), $\tilde{f}\hat{\ }\tilde{t} \in A$, meaning $\tilde{f}\hat{\ }\tilde{t}$ is a win for player I.

Next, if s^{short} is a strategy for player II, we will define the strategy s^{long} as follows:

$$s^{long}(p) = \begin{cases} s^{short}(p), & \text{if } \ell n(p) < \omega + 2n + 1 \\ \text{least } t \text{ such that } p \notin A_t, & \text{if } \ell n(p) = \omega + 2n + 1 \text{ and such a } t \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

Note that when $\ell n(p) = \omega + 2n$, it is player II's move, i.e. $s^{long}(p)$ is player II's move.

To show s^{long} is a winning strategy, suppose $\tilde{f}\hat{\ }\tilde{t}$ is according to s^{long}. Then $\tilde{f}$ is according to s^{short}, necessitating that $\tilde{f} \notin \hat{A} = \bigcap_{t\in\omega} A_t$. Thus, there must exist some $t \in \omega$ with $\tilde{f} \notin A_t$. By definition of $\tilde{t} = s^{long}(\tilde{f})$, this means that $\tilde{f} \notin A_{\tilde{t}}$. Finally, by the definition of Proj, $\tilde{f}\hat{\ }\tilde{t} \notin A$.

($\Longleftarrow$) Now we proceed to the backwards implication. Pick $A \in \mathbf{\Pi}^0_{\alpha+1} \restriction [T^{short}]$. We are able to decompose A to obtain $A_t \in \mathbf{\Sigma}^0_\alpha \restriction [T^{short}]$ for every $t \in \omega$, where $A = \bigcap_{t\in\omega} A_t$. Next, define $\hat{A} = \text{Lift}\langle A_t \mid t \in \omega\rangle$. By Theorem 2.5(b) on page 29, $\hat{A} \in \mathbf{\Sigma}^0_\alpha \restriction [T^{long}]$. For this direction, we assume then that the game $G(\hat{A}, [T^{long}])$ is determined, and we have a winning strategy s^{long} for this game. Define the strategy $s^{short} = s^{long} \restriction T^{short}$ for the game $G(A, [T^{short}])$. We will show that this is a winning strategy.

If s^{long} is a strategy for player I, then s^{short} will be a strategy for player I. Suppose $\tilde{f}$ is according to s^{short}. Then, for any move t player II makes, $\tilde{f}\hat{\ }t$ will be according to s^{long},

[2] Notice that the last move in T^{long} is for player II and the moves of player II in T^{long} and T^{short} are the same.

so that $\tilde{f}\,\hat{}\,t \in \hat{A}$. But then, for any $t \in \omega$, $\tilde{f} \in A_t$, by definition of Lift. In other words, $\tilde{f} \in \bigcap_{t\in\omega} A_t = A$. Thus, $\tilde{f}$ is a win for player I. □

Corollary 8.2.1. *Notice that an immediate corollary to Lemmas 8.1 and 8.2 is that for any nonzero ordinal* α

$$\begin{aligned}\mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2} \restriction X^\omega \circledast \omega^{2n}) &\iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+1} \restriction X^\omega \circledast \omega^{2n+1}) \\ &\iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha} \restriction X^\omega \circledast \omega^{2n+2}).\end{aligned}$$

$$\begin{aligned}\mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2} \restriction X^\omega \circledast \omega^{2n+1}) &\iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+1} \restriction X^\omega \circledast \omega^{2n}) \\ &\iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha} \restriction X^\omega \circledast \omega^{2n-1}).\end{aligned}$$

Now we use Lemmas 8.1 and 8.2 to prove the primary result of this chapter.

Theorem 8.3. *Suppose* $n \in \omega$ *and* α *is a nonzero ordinal. Then*

$$\mathrm{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n} \restriction X^\omega\right) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha} \restriction X^\omega \circledast \omega^{2n}).$$

Proof. The proof will consist of repeatedly applying Lemmas 8.1 and 8.2 to get $\mathbf{\Sigma}^0_{\alpha+2n} \restriction X^\omega$ down to $\mathbf{\Sigma}^0_{\alpha} \restriction X^\omega \circledast \omega^{2n}$. The result is obvious when $n = 0$. We will be proving the theorem for $n \geq 1$.

We will show by induction on i that:

(a) $\mathrm{Det}(\mathbf{\Sigma}^0_{\alpha} \restriction X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i})$

(b) $\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i+1} \upharpoonright X^\omega \circledast \omega^{2n-2i-1})$.

For (a), $i = 0$ is true. For (b), $i = 0$ follows from Lemma 8.2 by replacing n with $n-1$.

For the inductive step, suppose these statments (a) and (b) hold true at $i-1$.

Then, from (b), we know

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i-1} \upharpoonright X^\omega \circledast \omega^{2n-2i+1}).$$

Next, we apply Lemma 8.1, replacing α with $\alpha + 2i - 1$ and n with $n - i$.

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i} \upharpoonright X^\omega \circledast \omega^{2n-2i}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i-1} \upharpoonright X^\omega \circledast \omega^{2n-2i+1}).$$

Hence by combining these two,

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i} \upharpoonright X^\omega \circledast \omega^{2n-2i}).$$

Thus we have finished the inductive step for (a).

Finally, we apply Lemma 8.2, replacing k with $2i$ and n with $n - i - 1$:

$$\mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i+1} \upharpoonright X^\omega \circledast \omega^{2n-2i-1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i} \upharpoonright X^\omega \circledast \omega^{2n-2i}).$$

Combining this with (a) gives us

$$\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i+1} \upharpoonright X^\omega \circledast \omega^{2n-2i-1}),$$

finishing the induction for part (b).

Finally, when $i = n$ in part (a), $\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2n} \restriction X^\omega)$. □

Comment 8.3.1. *The proof of the Theorem 8.3 has actually shown a stronger result. Specifically, for $n \in \omega$:*

(a) $\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i})$ *for $i \leq n$ and*

(b) $\mathrm{Det}(\mathbf{\Sigma}^0_\alpha \restriction X^\omega \circledast \omega^{2n}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i+1} \restriction X^\omega \circledast \omega^{2n-2i-1})$ *for $i < n$.*

Comment 8.3.2. *Note that when $\alpha = 1$ we have shown the finite case that was mentioned at the start of this chapter. Namely, for any nonzero ordinal α:*

$$\mathrm{Det}\left(\mathbf{\Sigma}^0_{2n+1} \restriction X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Sigma}^0_1 \restriction X^\omega \circledast \omega^{2n}\right).$$

We will next show a similar proof for odd parities.

Theorem 8.4. *Suppose $n \in \omega$ and α is a nonzero ordinal. Then*

$$\mathrm{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n+1} \restriction X^\omega\right) \iff \mathrm{Det}\left(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}\right).$$

Proof. We will show by induction:

(a) $\mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i+1})$

(b) $\mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i+1} \restriction X^\omega \circledast \omega^{2n-2i})$.

Note that the theorem is actually (b) when $i = n$.

For the base case when $i = 0$, (a) is trivially true. For (b), use Lemma 8.1.

For the inductive step, suppose these statements are true at $i-1$. Using (b), this means

$$\mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i-1} \restriction X^\omega \circledast \omega^{2n-2i+2}).$$

We use Lemma 8.2 by replacing n with $n-i$ and α with $\alpha+2i-1$,

$$\mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i-1} \restriction X^\omega \circledast \omega^{2n-2i+2}).$$

Combining these, we get

$$\mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i+1}),$$

finishing the inductive step for (a).

Next, use Lemma 8.1 by replacing n with $n-i$ and α with $\alpha+2i$.

$$\mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i+1} \restriction X^\omega \circledast \omega^{2n-2i}) \iff \mathrm{Det}(\mathbf{\Pi}^0_{\alpha+2i} \restriction X^\omega \circledast \omega^{2n-2i+1}).$$

Combining this with (a), we get

$$\mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2i+1} \restriction X^\omega \circledast \omega^{2n-2i}),$$

which completes the inductive step for (b).

When $i = n$ in (b), we obtain that

$$\text{Det}\left(\mathbf{\Pi}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n+1}\right) \iff \text{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n+1} \upharpoonright X^\omega\right).$$

□

Theorem 8.3 discusses the determinacy of $\mathbf{\Sigma}^0_{\alpha+2n} \upharpoonright X^\omega$, whereas Theorem 8.4 discusses the determinacy of $\mathbf{\Sigma}^0_{\alpha+2n+1} \upharpoonright X^\omega$. No assumption in either theorem was made about α being a limit ordinal. Next we do want to consider the case when α is a limit ordinal, because every $\mathbf{\Sigma}^0_\gamma \upharpoonright X^\omega$ can be expressed in one of those two forms.

Corollary 8.4.1. *Suppose $n \in \omega$ and $\alpha \in (0, \omega_1)$ is a limit ordinal. Then*

(a) $\text{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n} \upharpoonright X^\omega\right) \iff \text{Det}\left(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]\right)$*, and*

(b) $\text{Det}\left(\mathbf{\Sigma}^0_{\alpha+2n+1} \upharpoonright X^\omega\right) \iff \text{Det}\left(\mathbf{\Pi}^0_1 \upharpoonright X^\omega \circledast \omega^{2n+1} \circledast [T^\alpha_{g.t.}]\right)$.

Proof.

(a) First, we use Corollary 7.3.1 with $\gamma = 1$ to get

$$\text{Det}(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]) \iff \text{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}).$$

Using this in conjunction with Theorem 8.3,

$$\text{Det}(\mathbf{\Sigma}^0_1 \upharpoonright X^\omega \circledast \omega^{2n} \circledast [T^\alpha_{g.t.}]) \iff \text{Det}(\mathbf{\Sigma}^0_\alpha \upharpoonright X^\omega \circledast \omega^{2n}) \iff \text{Det}(\mathbf{\Sigma}^0_{\alpha+2n} \upharpoonright X^\omega).$$

(b) First, we use Proposition 7.4.1 with $\gamma = 1$ to get

$$\mathrm{Det}(\mathbf{\Pi}^0_1 \restriction X^\omega \circledast \omega^{2n+1} \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}).$$

Using this in conjunction with Theorem 8.4,

$$\mathrm{Det}(\mathbf{\Pi}^0_1 \restriction X^\omega \circledast \omega^{2n+1} \circledast [T^\alpha_{g.t.}]) \iff \mathrm{Det}(\mathbf{\Pi}^0_\alpha \restriction X^\omega \circledast \omega^{2n+1}) \iff \mathrm{Det}(\mathbf{\Sigma}^0_{\alpha+2n+1} \restriction X^\omega).$$

$\square$

Chapter 9

$\mathbf{Det}\left(\mathbb{B} \upharpoonright X^{\omega}\right) \iff \mathbf{Det}\left(\boldsymbol{\Delta}^0_1 \upharpoonright X^{\omega} \circledast \omega^{\omega}\right)$

Here we discuss the equivalence of determinacy of clopen games in $X^{\omega} \circledast \omega^{\omega}$ (e.g. $\omega^{\omega+\omega}$) and Borel determinacy in X^{ω}. It is well-known that determinacy of open games in $\omega^{\omega+\omega}$ requires large cardinals:

$$\mathrm{Det}\left(\boldsymbol{\Sigma}^0_1 \upharpoonright \omega^{\omega+\omega}\right) \iff \mathrm{Det}\left(\boldsymbol{\Pi}^1_1 \upharpoonright \omega^{\omega}\right) \iff \forall x \in \omega^{\omega}\left(x^{\#} \text{ exists}\right).$$

However $\mathrm{Det}\left(\boldsymbol{\Delta}^0_1 \upharpoonright \omega^{\omega+\omega}\right)$ does not require large cardinals.

Theorem 9.1. *Recall that $\mathbb{B} \upharpoonright X^{\omega}$ denotes the Borel subsets of X^{ω}. Then*

$$\mathrm{Det}(\boldsymbol{\Delta}^0_1 \upharpoonright X^{\omega} \circledast \omega^{\omega}) \iff \mathrm{Det}\left(\mathbb{B} \upharpoonright X^{\omega}\right).$$

We will be presenting some details of the proof of the theorem in the following two lemmas, but the full proof will not be shown here.

Lemma 9.2.

$$\mathrm{Det}(\boldsymbol{\Delta}^0_1 \restriction X^\omega \circledast \omega^\omega) \implies \mathrm{Det}\left(\mathbb{B} \restriction X^\omega\right).$$

Outline of Proof. This direction of the proof can be obtained by showing the following implications:

$$\mathrm{Det}(\boldsymbol{\Delta}^0_1 \restriction X^\omega \circledast \omega^\omega) \implies \mathrm{Det}(\boldsymbol{\Delta}^0_1 \restriction X^\omega \circledast [\tilde{T}^\alpha_{g.t}]) \implies \mathrm{Det}(\boldsymbol{\Sigma}^0_\alpha \restriction X^\omega)$$

where the tree $\tilde{T}^\alpha_{g.t}$ is a modification of the $T^\alpha_{g.t}$ tree with a single extra move at the end to allow for decomposing open sets into clopen sets. The first implication is "routine" since every position $p \in \tilde{T}^\alpha_{g.t}$ is a finite sequence of moves from ω and so has extensions in ω^ω. The second implication can be shown by recreating the proof of Lemma 5.3, making minor adjustments that account for the extra move. □

Next we will approach the backwards direction.

Lemma 9.3.

$$\mathrm{Det}\left(\mathbb{B} \restriction X^\omega\right) \implies \mathrm{Det}(\boldsymbol{\Delta}^0_1 \restriction X^\omega \circledast \omega^\omega).$$

Outline of Proof. Consider the game tree $[T] = X^\omega \circledast \omega^\omega$, and fix an arbitrary clopen set $C \subseteq [T]$. This clopen set will help us to create another game tree, T^{cut}, using the following functions to do so:

$$n(x) : [T] \to \omega$$

$$n(x) = \begin{cases} \min\left\{i \in \omega \mid \forall y \in [T](y \supseteq x \restriction (\omega + i) \implies y \in C)\right\} & \text{if } x \in C \\ \min\left\{i \in \omega \mid \forall y \in [T](y \supseteq x \restriction (\omega + i) \implies y \notin C)\right\} & \text{if } x \notin C \end{cases}$$

$$\alpha(x) = \omega + n(x)$$

Finally, define:

$$[T^{cut}] = \{x \restriction \alpha(x) \mid x \in [T]\}$$

$$[T^{tail}] = \left\{h \in \omega^{<\omega} \mid \exists f \in [T^{cut}] \exists g \in X^{\omega} (f = g \char`^ h)\right\}$$

Let

$$O = O_C = \{f \in [T^{cut}] \mid \exists x \in C (x \supseteq f)\}.$$

Based on the definitions of $n(x)$ and $\alpha(x)$, it is routine to show:

- O is open in $[T^{cut}]$, since C is open in $[T]$
- $O = \{f \in [T^{cut}] \mid \forall x \in [T] (x \supseteq f \implies x \in C)\}$

Next, create the projections

$$E_h = \{g \in X^{\omega} \mid g \char`^ h \in O\}$$

for each $h \in [T^{tail}]$. Notice that since $[T^{tail}]$ is well-founded, we can define a rank function on $[T^{tail}]$, and so we can use Definition 4.1 to create the Borel set $E_{\varnothing} \subseteq X^{\omega}$.

Next, mimicking the proof of Theorem 5.1 gives

$$\text{Det}\left(G(E_{\varnothing}, X^{<\omega})\right) \iff \text{Det}\left(G(O, T^{cut})\right).$$

Furthermore, it is routine to show that $\text{Det}\left(G(O, T^{cut})\right) \iff \text{Det}\left(G(C, T)\right)$. To show the forwards direction, if s^{short} is a winning strategy for one of the players in $G(O, T^{cut})$, define the winning strategy s^{long} in $G(C,T)$, which follows s^{short} until we leave the tree T^{cut}, after which the appropriate player will play 0 for every move. To show the backwards direction, if s^{long} is a winning strategy for one of the players in $G(C,T)$, define the strategy s^{short} in $G(O,T^{cut})$, which follows s^{long} as long as we remain in T^{cut}. To show that s^{short} is a winning strategy, use the following fact:

$$\forall x \in [T] \forall f \in [T^{cut}] \text{ such that } x \supseteq f, (f \in O \iff x \in C)$$

Hence, $\mathrm{Det}\,(G(C,T)) \iff \mathrm{Det}\,(G(E_{\varnothing}, X^{<\omega}))$.

Finally, since $E_{\varnothing}$ is Borel, $\mathrm{Det}\,(\mathbb{B} \upharpoonright X^{\omega}) \implies \mathrm{Det}\,(G(E_{\varnothing}, X^{<\omega})) \implies \mathrm{Det}\,(G(C,T))$, and furthermore, since C was arbitrary,

$$\mathrm{Det}\,(\mathbb{B} \upharpoonright X^{\omega}) \implies \mathrm{Det}\left(\boldsymbol{\Delta}^0_1 \upharpoonright X^{\omega} \circledast \omega^{\omega}\right).$$

□

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