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A Survey of the Br´Ezis-Nirenberg Problem and Related Theorems

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A SURVEY OF THE BRÉZIS-NIRENBERG PROBLEM AND RELATED THEOREMS

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ABSTRACT

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Nonlinear elliptic partial differential equations on bounded domains arise in several different areas of mathematics that include geometry, mathematical physics, and the calculus of variations. The Brézis-Nirenberg problem is concerned with a boundary-value problem that is intimately connected to the existence of positive solutions of the Yamabe problem, of non-minimal solutions to Yang-Mills functionals, and of extremal functions to several important inequalities. Results on existence and uniqueness have been obtained in cases when the exponent is sub-critical, but such results have not been obtained when the exponent is critical due to a lack of compactness. The earliest results obtained in this direction were obtained by Brézis and Nirenberg. The goal of this thesis is to serve as a survey of the various results regarding this variational problem.

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Introduction

1.1 Overview of the Brézis-Nirenberg Problem

We will study the problem

$$
\begin{aligned}
-\Delta u = u^p + f(x,u) \quad \text{on} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$, $p = (n+2)/(n-2)$, and f is a lower-order perturbation, i.e. $\lim_{u\to\infty} f(x,u)/u^p = 0$. This problem appears in various parts of mathematics including analysis, differential geometry (Aubin, 1998), and mathematical physics (Taubes, 1982). In particular, when $f(x,u) = \lambda u$, $\lambda \in \mathbb{R}$, and one is looking for positive solutions, i.e. $u > 0$ in Ω, then (1.1) becomes a special case of the Yamabe problem.

(1.1) has been studied previously in cases when $p < (n+2)/(n-2)$, i.e. the exponent is sub-critical. There, the existence and uniqueness of solutions was exhibited for various non-linear f. However, when the exponent is critical, the corresponding variational problem lacks a compactness that was previously assumed. As a consequence, the existence and uniqueness of solutions becomes a non-trivial issue. In fact, when one is looking for positive solutions to (1.1), it can be shown that if Ω is star-shaped, then the problem does not admit a solution for $n \geq 3$. The approach of this thesis is to explain the results made in this area based on the original paper by Brézis and Nirenberg (Brézis, 1983), as well as an approach to show existence of infinitely many sign-changing solutions by Schecter and Zou (Schechter, 2010) when $f(x, u) = \lambda u$, $\lambda \in \mathbb{R}$. To this end, we sketch the proofs and ideas found in these papers.

This thesis will be divided into the following chapters: Chapter 2 will introduce the origins of the Brézis-Nirenberg problem — from the Poincare conjecture to the Yamabe problem and its derivation. To this end, basic differential-geometric definitions will be presented. Chapter 3 will cover the original paper of Brézis and Nirenberg, presented in 1983, which deals with a version of the boundary value problem given by Yamabe in the case when the manifold is $\mathbb{R}^n$, $n \geq 3$. Here, the variational approach will be introduced. While Chapter 3 covers the case of positive solutions, Chapter 4 will cover the existence of sign-changing solutions through an application of Morse theory. Finally, Chapter 5 will conclude this thesis with a reflection on the implications and consequences of the Yamabe problem and the work of Brézis and Nirenberg.

CHAPTER 2

Basic Differential-Geometric Concepts and Introduction to the Yamabe Problem

One of the origins of the Brézis-Nirenberg problem come from geometry. In particular, from an approach to solving one of the seven millennium problems – the Poincaré Conjecture. The statement of the theorem is:

The Poincaré Conjecture.

A compact simply-connected Riemannian manifold (M, g) of dimension $n = 3$ is diffeomorphic to S_3.

The proof for this result would require tools (e.g. Ricci flow) developed by Richard Hamilton and later expanded on by Grigori Perelmann. Using new techniques, Perelman proved the conjecture in the affirmative. For this achievement, Perelman was awarded the Fields medal in 2006.

Prior to its proof exhibited by Russian mathematician Grigori Perelman, mathematicians were eager to solve this problem. Among these mathematicians was Japanese mathematician Hidehiko Yamabe. Before going through Yamabe's approach, we introduce preliminary definitions in differential geometry.

2.1 Preliminaries

Definition 1. A topological manifold is a Hausdorff, second countable, locally Euclidean topological space. A C^∞ manifold is a pair consisting of a topological manifold M and a C^∞ maximal atlas $\{U_\alpha, \phi_\alpha\}$ on M.

To define a notion of distance and angles, one can endow the manifold with a special metric called the Riemannian metric.

Definition 2. Let M be a C^∞-manifold and $p \in M$. A bilinear function $g : T_p(M) \times T_p(M) \to \mathbb{R}$ is called a Riemannian metric on M if it satisfies the following properties:

1. $g(X_p, X_p) \geq 0$ and $g(X_p, X_p) = 0$ iff $X_p \equiv 0$.

2. $g(X_p, Y_p) = g(Y_p, X_p)$.

Furthermore, one is interested in being able to define derivatives on a manifold that generalize the directional derivative in $\mathbb{R}^n$. Let $\mathfrak{X}(M)$ denote the set of C^∞ vector fields on a smooth manifold M and $\mathcal{F}$ denote the ring $C^\infty(M)$ of C^∞ functions on M. A function $F : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is called $\mathcal{F}$-linear if it is linear map and preserves scalar multiplication with respect to $\mathcal{F}$, i.e. $F(fX + gY) = fF(X) + gF(Y)$, $\forall f, g \in \mathcal{F}$ and $X, Y \in \mathfrak{X}(M)$.

Definition 3. An affine connection on a manifold M is an $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

written $\nabla_X Y = \nabla(X, Y)$ satisfying the following two properties:

1. $\nabla_X Y$ is $\mathcal{F}$-linear in X,

2. $\nabla_X Y$ satisfies the Leibniz rule: for $f \in \mathcal{F}$

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y$$

Definition 4. Given two smooth vector fields X, Y on U and $p \in U$, the Lie bracket $[X, Y]$ at p is defined to be

$$[X, Y]_p f = (X_p Y - Y_p X)f$$

for every C^∞ function f at p.

Using the Lie bracket, a canonical connection on a Riemannian manifold that is frequently used is the Levi-Cevita connection which has zero torsion and is compatible with the metric. Moreover, this connection is the ordinary directional derivative in $\mathbb{R}^n$.

Two fundamental structures of analysis that appear frequently in differential geometry are the torsion and curvature tensors.

Definition 5. Suppose ∇ is an affine connection on a manifold M. Then the torsion T of the connection is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Definition 6. Suppose ∇ is an affine connection on a manifold M. Then the curvature R of the connection is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$
$$= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Special quantities are associated with any connection defined on a manifold, and these frequently appear in the torsion and curvature tensors when considering them in local coordinates.

Definition 7. Let ∇ be an affine connection on a manifold M and let $(U, x^1, x^2, ..., x^n)$ denote a coordinate open set in M. Let $\partial_i := \dfrac{\partial}{\partial x^i}$. Then $\nabla_{\partial_i}\partial_j$ can be written as a linear combination of $\{\partial_1, ..., \partial_n\}$, thus there exist Γ_{ij}^k such that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k := \Gamma_{ij}^k \partial_k, \quad 1 \leq i, j \leq n.$$

The coefficients Γ_{ij}^k are called the Christoffel symbols of the connection ∇ on the coordinate set $(U, x^1, ..., x^n)$.

From the definition of the curvature tensor, one can define a $(3, 1)$ (or $(4, 0)$ if contracted with the metric) tensor from which one can analyze various aspects of curvature:

Definition 8. Let ∇ be a connection on a manifold M and $R(X, Y)$ the curvature. The Riemann curvature tensor Rm is defined such that

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle, \quad \forall X, Y, Z, W \in \mathfrak{X}(M).$$

Note that in local coordinates, a component of the tensor can be written as

$$Rm_{abcd} = R_{abcd}.$$

From the Riemann tensor, we may also define new summaries of curvature:

Definition 9. Suppose Rm is the Riemann curvature tensor on M. Then the Ricci curvature $Ric(X,Y)$ is defined to be the contraction of Rm over the first and fourth components:

$$Ric(X,Y) = tr_g Rm.$$

In local coordinates, the components are written as

$$R_{ab} = g^{ij} R_{iabj}.$$

From this quantity, we may define the scalar curvature S as the trace of the Ricci curvature, i.e.

$$S = tr_g Ric.$$

Since the Yamabe problem is a question of finding a conformal metric of constant scalar curvature, we state the definition here:

Definition 10. Let (M,g) be a Riemannian manifold. We say that g' is a conformal metric of g if there exists a function f such that

$$g' = e^{2f} g.$$

Note that g' defines a Riemannian metric over M.

2.2 Derivation of the Yamabe Problem

Yamabe (Yamabe, 1960) was interested in solving the Poincaré conjecture, and he believed the first step was to find a metric of constant scalar curvature that is conformal to the given Riemannian metric g. Formally, the question is

The Yamabe Problem.

Let (M_n, g) be a compact C^∞ Riemannian manifold of dimension $n \geq 3$ and R its respective scalar curvature. Yamabe attempts to answer the following question: Does there exist a metric g' conformal to g, such that the scalar curvature R' of the metric g' is constant?

We follow the derivation in (Aubin, 1998). Let (M_n, g) be a compact C^∞ Riemannian manifold with dimension $n \geq 3$ and let R denote the scalar curvature. To begin with, let $g' = e^f g$, where $f \in C^\infty$. Using the conformal relationship between g and g', the scalar curvatures are related by

$$R' = e^f \left(R - (n-1)\nabla_\nu \nabla^\nu f - \frac{(n-1)(n-2)}{4} \nabla^\nu f \nabla_\nu f \right). \tag{2.1}$$

We will now compute these expressions locally. If we now suppose that $g' = \varphi^{4/(n-2)} g$ such that $e^f = \varphi^{4/(n-2)}$, then $f = \ln\left(\varphi^{4/(n-2)}\right)$, and so

$$\nabla f = \frac{4}{n-2} \frac{\nabla \varphi}{\varphi}$$

$$\nabla^2 f = \nabla \nabla f = \frac{4}{n-2} \frac{\nabla^2 \varphi}{\varphi} - \frac{4}{n-2} \frac{\nabla \varphi \otimes \nabla \varphi}{\varphi^2}$$

Substituting these expressions into (2.1) yields

$$e^{-f}R' = \varphi^{-4/(n-2)}R' = R - (n-1)\left(\frac{4}{n-2}\frac{-\Delta\varphi}{\varphi} - \frac{4}{n-2}\frac{|\nabla\varphi|^2}{\varphi^2}\right) - \frac{(n-1)(n-2)}{4}\left(\frac{4}{n-2}\frac{|\nabla\varphi|}{\varphi}\right)^2$$
$$= R + 4\frac{n-1}{n-2}\Delta\varphi.$$

Multiplying each side by φ, we finally obtain the equation

$$4\frac{n-1}{n-2}\Delta\varphi + R\varphi = R'\varphi^{\frac{n+2}{n-2}}, \tag{2.2}$$

where $\varphi \in C^\infty,\ \varphi > 0$, and $\Delta\varphi = -\nabla_\nu\nabla^\nu\varphi$ denotes the Laplace-Beltrami operator with respect to g. Thus, since the problem is to find a metric of constant scalar curvature, we have $R' = \lambda$ where λ is a constant. Rewriting, we obtain the following differential equation:

$$-4\frac{n-1}{n-2}\Delta\varphi + R\varphi = \lambda\varphi^{\frac{n+2}{n-2}}.$$

Note that in the case of bounded domains $\Omega \subset \mathbb{R}^n$, then from the above we have

$$-\Delta u = u^{\frac{n+2}{n-2}}$$

$$u|_{\partial\Omega} = 0.$$

Note also that the value $\dfrac{n+2}{n-2}$ is special due to the fact that for exponents less than this value, the equation readily admits a solution; in contrast, if the exponent is greater than this exponent, then the resulting equation may be unsolvable.

CHAPTER 3

The Brézis-Nirenberg Problem: Positive Solutions

Yamabe's problem is interesting for many reasons — both geometrically and analytically. Brézis and Nirenberg focus on the case when the domain is bounded in $\mathbb{R}^n$ — the Brézis-Nirenberg problem. It appears the problem is clearly inspired from (2.2).

3.1 Preliminary Definitions and Theorems

The method for solving this problem is built on the variational formulation for differential equations. The approach is based on techniques developed from the calculus of variations. We start by providing several definitions for this approach. The following definitions are due to Chang (Chang, 2005) and Brézis (Brézis, 2010). First, let X and Y denote Banach spaces with corresponding norms $||\cdot||_X$ and $||\cdot||_Y$. We now give a definition for differentiability of functionals.

Definition 11. Let $\varphi : X \to Y$. We say that φ is Gatêaux-differentiable at $u_0 \in X$ if for all $h \in X$ the limit

$$\lim_{\epsilon \to 0} \frac{\varphi(u_0 + \epsilon h) - \varphi(u_0)}{\epsilon}$$

exists in Y with $u_0 + \epsilon h \in X$ for all ϵ suitably small and is a linear function of h. Let us denote this limit as $\varphi'(u_0; h)$. We could equivalently reformulate this as saying that there exists $\varphi'(u_0; h) \in Y$

such that

$$\lim_{\epsilon \to 0} \frac{||\varphi(u_0 + \epsilon h) - \varphi(u_0) - \epsilon \varphi'(u_0; h)||_Y}{\epsilon} = 0$$

The definition of differentiability given by Definition 1. is similar to the directional derivative on $\mathbb{R}^n$. In parallel to this, there exists an analogue to the gradient.

Definition 12. Let $\varphi : X \to \mathbb{R}$. We say that φ is Fréchet-differentiable at $u_0 \in X$ if there exists a linear operator $A : X \to Y$ such that

$$\lim_{n \to \infty} \frac{||\varphi(u_n) - \varphi(u_0) - A(u_n - u_0)||_Y}{||u_n - u_0||_X} = 0.$$

For such a linear operator, which depends on u_0, we let $A := \nabla\varphi(u_0)$. If φ is Fréchet-differentiable for every $u \in X$ and $\nabla\varphi(u)$ is continuous for every u, then we say $\varphi \in C^1(X; \mathbb{R})$.

It is a well-known fact that if a functional is Fréchet-differentiable at u, then it is also Gatêaux-differentiable at u. The key fact to the variational method is that functionals on certain function spaces behave similarly to functions on $\mathbb{R}^n$, namely if u_0 is a minimizer for φ on X, then

$$\nabla\varphi(u_0) = 0.$$

As a result, the methods of the calculus of variations can be applied to such problems.

We define derivatives for a function in $L^p(\Omega)$ through the weak sense, i.e. using integration by parts.

Definition 13. Let $\Omega \subset \mathbb{R}^n$ be bounded and $u \in L^p(\Omega)$. We say v is a weak derivative of u if for any $h \in C^{\infty}(\Omega)$

$$\int_{\Omega} u \nabla h \, dx = -\int_{\Omega} v h \, dx.$$

Higher order derivatives are defined inductively.

We want to discuss the appropriate space to work with for the Brézis-Nirenberg problem. Recall the definition of the Sobolev space $W^{m,p}(\Omega)$:

Definition 14. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $p > 0$. The Sobolev space $W^{m,p}(\Omega)$ is defined as

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega) : \frac{\partial^k f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}} \in L^p(\Omega),\ 1 \leq k \leq m, \sum_i \alpha_i = k,\ \alpha_i \in \mathbb{N} \cup \{0\} \right\}.$$

In other words, $W^{m,p}(\Omega)$ contains all L^p integrable functions on Ω and such that their weak derivatives up to order m are in L^p.

In the special case when $p = 2$ and $m = 1$, then we have $W^{1,2}(\Omega)$ is a Hilbert space and we let $W_0^{1,2}(\Omega) := H_0^1(\Omega)$. It is equipped with the inner product $\langle \cdot, \cdot \rangle : H_0^1 \times H_0^1 \to \mathbb{R}$ defined by

$$\langle u, v \rangle_{H_0^1(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}.$$

We have the following result due to Poincaré:

Theorem 1. *(Poincaré's Inequality)*

Suppose $u \in H_0^1(\Omega)$. Then there exists $C > 0$ such that

$$||u||_{L^2(\Omega)} \leq C ||\nabla u||_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

By Poincaré's inequality, the inner product $\langle u, v\rangle_{H_0^1(\Omega)}$ is equivalent to the inner product $\langle \nabla u, \nabla v\rangle_{L^2(\Omega)}$. The latter inner product is more useful to work with compared to the original one. From here on, we consider the inner product on $H_0^1(\Omega)$ to be the one such that for all $u, v \in H_0^1(\Omega)$, we have

$$\langle u, v\rangle_{H_0^1(\Omega)} := \langle \nabla u, \nabla v\rangle_{L^2(\Omega)}.$$

Poincaré's inequality is a consequence of the Sobolev Embedding Theorem (Brézis, 2010):

Theorem 2. *(Sobolev Embedding Theorem)*

Let $1 \leq p \leq \infty$ and assume $\Omega \subset \mathbb{R}^n$ is bounded. We have

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \quad \textit{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad \textit{if } p < n$$

$$W^{1,p}(\Omega) \subset L^{q}(\Omega), \quad \forall q \in [p, \infty), \quad \textit{if } p = n$$

$$W^{1,p}(\Omega) \subset L^{\infty}(\Omega), \quad \textit{if } p > n$$

where these injections are continuous. Moreover, if Ω is of class C^1, then the following are compactly embedded

$$W^{1,p}(\Omega) \subset L^{q}(\Omega), \quad \forall q \in [1, p^*), \quad \textit{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad \textit{if } p < n$$

$$W^{1,p}(\Omega) \subset L^{q}(\Omega), \quad \forall q \in [p, \infty), \quad \textit{if } p = n$$

$$W^{1,p}(\Omega) \subset C(\bar{\Omega}), \quad \textit{if } p > n.$$

Due to the infinite-dimensional nature of Sobolev spaces, compactness cannot be characterized by closed and bounded sets. Instead, we require a certain compactness condition to hold for functionals. We now define the Palais-Smale condition.

Definition 15. Let φ be a $C^1(X, \mathbb{R})$. φ is said to satisfy the Palais-Smale condition at level c, denoted $(PS)_c$, if for any sequence $\{u_n\} \subset X$ such that $\varphi(u_n) \to c$ and $\nabla\varphi(u_n) \to 0$, then there exists a convergent subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u \in X$.

3.2 Existence of Positive Solutions

Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be bounded. We study

$$\begin{aligned} -\Delta u = u^p + f(x,u) \quad &\text{on} \quad \Omega \\ u > 0 \quad &\text{on} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{3.1}$$

with $p = (n+2)/(n-2)$ and $f(x,u)$ is a lower-order perturbation. Such a p is called the critical Sobolev exponent. We study this problem in the context of (Brézis, 1983). The weak solutions to this problem are critical points to the functional

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} \, dx - \int_\Omega F(x,u) \, dx$$

where $F(x,u) = \displaystyle\int_0^u f(x,t) \, dt$. Since $p+1 = \dfrac{2n}{n-2}$, then by the Sobolev Embedding Theorem [*See* (Brézis, 2010)] $H_0^1(\Omega)$ is not compactly embedded in $L^{p+1}(\Omega)$. As a consequence, the (PS) condition does not hold for this functional. As a result, they rely on a mountain pass theorem without the (PS) condition:

Theorem 3. (Brézis, 1983) *Let Φ be a C^1 function on a Banach space E. Suppose there exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every u in the boundary of U*

with

$$\Phi(0) < \rho \quad \textit{and} \quad \Phi(v) < \rho \quad \textit{for some } v \notin U.$$

Set

$$c = \inf_{A \in \mathcal{A}} \max_{a \in A} \Phi(a) \geq \rho,$$

where $\mathcal{A}$ denotes the class of paths from 0 to v.

Then there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset E$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in E^*.

Along with this theorem, they also analyze a family of constants that proves pivotal in exhibiting existence of a solution for (3.1). Let

$$S_\lambda := \inf_{\substack{u \in H_0^1(\Omega) \\ ||u||_{p+1}=1}} \{||\nabla u||^2 - \lambda ||u||^2\}, \quad \lambda \in \mathbb{R},$$

with

$$S_0 = S = \inf_{\substack{u \in H_0^1(\Omega) \\ ||u||_{p+1}=1}} ||\nabla u||^2.$$

Note that S is the best constant in the Sobolev embedding of $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. Brézis and Nirenberg consider several facts concerning S:

(i) S only depends on n, i.e. it is independent of Ω;

(ii) S is never achieved when Ω is a bounded domain;

(iii) When $\Omega = \mathbb{R}^n$, then S is achieved by

$$U(x) = C(1 + |x|^2)^{-(n-2)/2}$$

or by

$$U_\epsilon(x) = C_\epsilon(1 + |x|^2)^{-(n-2)/2}, \quad \epsilon > 0,$$

where C and C_ϵ are normalization constants.

3.2.1 A Related Problem

To study (3.1), Brézis and Nirenberg began with studying the positive solutions of

$$\begin{aligned} -\Delta u = u^p + \lambda u \quad &\text{on} \quad \Omega \\ u > 0 \quad &\text{on} \quad \Omega \\ u = 0 \quad &\text{on} \quad \partial\Omega \end{aligned} \tag{3.2}$$

with $\lambda \in \mathbb{R}$. They subsequently analyze the cases when $n = 3$ and $n \geq 4$. These two cases give different results. They prove the following theorems.

Theorem 4. (Brézis, 1983) *Assume $n \geq 4$. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of (3.2).*

Here, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary conditions.

Theorem 5. (Brézis, 1983) *Assume $n = 3$ and Ω is a ball. Then there exists a solution of (3.2) if and only if $\lambda \in \left(\frac{1}{4}\lambda_1, \lambda_1\right)$.*

To begin, there is no solution to (3.2) (for $n \geq 3$) if $\lambda \geq \lambda_1$. To show this, Brézis and Nirenberg (Brézis, 1983) let φ_1 denote the eigenfunction corresponding to λ_1, the first eigenvalue of $-\Delta$ with $\varphi_1 > 0$ on Ω. Then if u is a solution to (3.2), we have

$$\lambda_1 \int_\Omega u \cdot \varphi_1 \, dx = -\int_\Omega \Delta u \cdot \varphi_1 \, dx = \int_\Omega u^p \cdot \varphi_1 \, dx + \lambda \int_\Omega u \cdot \varphi_1 \, dx > \lambda \int_\Omega u \cdot \varphi_1 \, dx.$$

This above expression only holds if $\lambda < \lambda_1$.

Recall Pohozaev's Identity:

Theorem 6. *Pohozaev's Identity* (Pokhozhaev, 1965) *Suppose u is a smooth function such that*

$$-\Delta u = g(u) \quad on \quad \Omega$$

$$u = 0 \quad on \quad \partial\Omega,$$

where g is a continuous function on $\mathbb{R}$. Then

$$\left(1 - \frac{1}{2}n\right) \int_\Omega g(u) \cdot u \, dx + n \int_\Omega G(u) \, dx = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2 \, d\sigma,$$

where $G(u) = \int_0^u g(s) \, ds$ and ν is the outward normal vector to $\partial\Omega$.

Using Theorem 6, (Brézis, 1983) show that for $\lambda \leq 0$ and Ω starshaped, then there is no solution to (3.2). Letting $g(u) = u^p + \lambda u$, then $G(u) = \frac{1}{p+1} u^{p+1} + \frac{\lambda}{2} u^2$, and so applying Pohozaev's Identity yields

$$\begin{aligned}\left(1 - \frac{1}{2}n\right) \int_\Omega u^{p+1} + \lambda u^2 \, dx + n \int_\Omega \frac{1}{p+1} u^{p+1} + \frac{\lambda}{2} u^2 \, dx = \lambda \int_\Omega u^2 \, dx \\ = \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu}\right)^2 \, d\sigma.\end{aligned}$$

Recall that under a change of coordinates, we can assume that Ω is star-shaped with respect to the origin. As a consequence, we have $x \cdot \nu > 0$ for a.e. $x \in \partial\Omega$. If $\lambda < 0$, then this implies that $u \equiv 0$ for the above expression to make sense. Next, if $\lambda = 0$, then the above expression implies that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. Using (3.2) and integrating both sides of the equation over Ω, we have

$$\int_\Omega u^p \, dx = -\int_\Omega \Delta u \, dx.$$

Using Green's identity on the right-hand side of this equation yields

$$\int_\Omega u^p \, dx = 0$$

which implies $u \equiv 0$.

Brézis and Nirenberg prove a key lemma

Lemma 7. (Brézis, 1983)

$$S_\lambda < S, \quad \forall \lambda > 0.$$

Proof: We assume that $0 \in \Omega$; otherwise, one may perform a change of coordinates so that this holds. We will estimate the ratio

$$Q_\lambda(u) = \frac{||\nabla u||_2^2 - \lambda ||u||_2^2}{||u||_{p+1}^2}.$$

To prove the lemma, we use the family of functions given by

$$u_\epsilon(x) = \frac{\varphi_\epsilon(x)}{(1+|x|^2)^{(n-2)/2}}, \quad \epsilon > 0,$$

with $\varphi(x)$ a fixed, smooth (infinitely differentiable) function on Ω such $\varphi(x) = 1$ in a neighborhood of 0 that contains Ω. Brézis and Nirenberg show the following as $\epsilon \to 0$:

(i) $||\nabla u_\epsilon||_2^2 = \dfrac{K_1}{\epsilon^{(n-2)/2}} + O(1)$;

(ii) $||u_\epsilon||_{p+1}^2 = \dfrac{K_2}{\epsilon^{(n-2)/2}} + O(\epsilon)$;

(iii)

$$||u_\epsilon||_2^2 = \begin{cases} \dfrac{K_3}{\epsilon^{(n-4)/2}} + O(1) & n \geq 5 \\ \\ K_3|\log(\epsilon)| + O(1) & n = 4, \end{cases}$$

with K_1, K_2, and K_3 denoting positive constants that depend on the dimension n and such that $K_1/K_2 = S$.

Firstly, we have that $\nabla u_\epsilon(x) = \dfrac{\nabla \varphi_\epsilon(x)}{(1+|x|^2)^{(n-2)/2}} - \dfrac{(n-2)\varphi_\epsilon(x)x}{(1+|x|^2)^{n/2}}$. For $x \in \Omega$, we have $\varphi(x) = 1$ so that

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon|^2 \, dx &= (n-2)^2 \int_\Omega \frac{|x|^2}{(1+|x|^2)^n} \, dx + O(1) \\ &= (n-2)^2 \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} \, dx + O(1) \\ &= \frac{K_1}{\epsilon^{(n-2)/2}} + O(1), \end{aligned}$$

where the last equality comes from using the dilation invariance property of the Lebesgue integral. Here, we have $K_1 = (n-2)^2 \displaystyle\int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^n} \, dx$. This shows (i).

Next, we have

$$\int_\Omega |u_\epsilon|^{p+1}\ dx = \int_\Omega \frac{\varphi_\epsilon^{p+1}(x)}{(1+|x|^2)^n}\ dx = \int_\Omega \frac{(\varphi_\epsilon^{p+1}(x)-1)}{(1+|x|^2)^n}\ dx + \int_\Omega \frac{1}{(1+|x|^2)^n}\ dx$$
$$= O(1) + \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n}\ dx = \frac{K_2'}{\epsilon^{n/2}} + O(1),$$

with $K_2' = \displaystyle\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n}\ dx$. If we let $K_2 = (K_2')^{2/(p+1)}$, then we have $K_1/K_2 = S$. Thus, (ii) is verified.

We note

$$\int_\Omega |u_\epsilon|^2\ dx = \int_\Omega \frac{(\varphi_\epsilon^2(x)-1)}{(1+|x|^2)^{n-2}}\ dx + \int_\Omega \frac{1}{(1+|x|^2)^{n-2}}\ dx = O(1) + \int_\Omega \frac{1}{(1+|x|^2)^{n-2}}\ dx.$$

If $n \geq 5$, then

$$\int_\Omega \frac{1}{(1+|x|^2)^{n-2}}\ dx = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{n-2}}\ dx + O(1)$$
$$= \frac{K_3}{\epsilon^{(n-4)/2}} + O(1)$$

where $K_3 = \displaystyle\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{n-2}}\ dx$.

On the other hand, if $n = 4$, there exist constants R_1 and R_2 such that

$$\int_{|x|\leq R_1} \frac{1}{(1+|x|^2)^2}\ dx \leq \int_\Omega \frac{1}{(1+|x|^2)^2}\ dx \leq \int_{|x|\leq R_2} \frac{1}{(1+|x|^2)^2}\ dx$$

and using polar coordinates, we have

$$\int_{|x|\leq R} \frac{1}{(1+|x|^2)^2}\ dx = \omega \int_0^R \frac{r^3}{(1+r^2)^2}\ dr = \frac{1}{2}\omega|\log(\epsilon)| + O(1),$$

where ω is the surface area of S^3. Letting $K_3 = \frac{1}{2}\omega$ yields (iii).

In any case, combining (i), (ii), and (iii) together shows

$$Q_\lambda(u_\epsilon) = \begin{cases} S + O(\epsilon^{(n-2)/2}) - \lambda \dfrac{K_3}{K_2}\epsilon & n \geq 5 \\ S + O(\epsilon) - \lambda \dfrac{K_3}{K_2}\epsilon |\log(\epsilon)| & n = 4. \end{cases}$$

We have $Q_\lambda(u_\epsilon) < S$ for ϵ sufficiently small.

■

Due to Lieb along with the authors, they also obtain the following result:

Lemma 8. *If $S_\lambda < S$, then the infimum in S_λ is achieved.*

These two lemmas allow Brézis and Nirenberg prove Theorem 4.

Proof of Theorem 3: Let $u \in H_0^1(\Omega)$ be such that u satisfies the conclusion of Lemma 8, i.e. $S_\lambda = ||\nabla u||_2^2 - \lambda ||u||_2^2$ with $||u||^{p+1} = 1$. We assume $u > 0$ on Ω (if not, use $u_1 = |u|$). Now since u is where the constrained infimum is achieved, then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta u - \lambda u = \mu u^p, \quad \text{for } x \in \Omega.$$

In particular, if we multiply each side by u and integrate over Ω, we have

$$S_\lambda = \int_\Omega ||\nabla u||^2 - \lambda |u|^2 \, dx = \mu \int_\Omega |u|^{p+1} \, dx = \mu.$$

Hence, $S_\lambda = \mu$. We also note that since $\lambda < \lambda_1$, then $S_\lambda > 0$. From this, we observe that a solution to (3.2) is $v = Cu$ for a special choice of C. In particular, if $C = S_\lambda^{1/(p-1)}$, then

$$\begin{aligned} v^p &= \left(S_\lambda^{1/(p-1)} u\right)^p = S_\lambda^{p/(p-1)} u^p = S_\lambda^{1/(p-1)} \mu u^p \\ &= S_\lambda^{1/(p-1)} (-\Delta u - \lambda u) = -\Delta(Cu) - \lambda(Cu) = -\Delta v - \lambda v. \end{aligned}$$

The proof is finished by invoking the strong maximum principle to show that $u > 0$ on Ω. ■

To prove Theorem 4, we remark that the proof relies on the observation made earlier using Pohozaev's Identity, i.e. there is no solution when $\lambda > \lambda_1$ nor for $\lambda \leq 0$. Under the hypotheses of the theorem, we have $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$. Similar to Lemma 6, Brézis and Nirenberg prove

Lemma 9.

$$S_\lambda < S, \quad \textit{for all } \lambda > \frac{1}{4}\lambda_1.$$

The proof for this result is done similarly to the proof of Lemma 6 using asymptotics and the family of functions given by

$$u_\epsilon(r) = \frac{\varphi_\epsilon(r)}{(1 + r^2)^{1/2}}, \quad r = |x|, \ \epsilon > 0,$$

where φ is some smooth function such that $\varphi(0) = 1$, $\varphi'(0) = 0$, and $\varphi(1) = 0$.

Further, they prove another lemma:

Lemma 10. *There is no solution of (3.2) if* $\lambda \leq \frac{1}{4}\lambda_1$.

Firstly, it is observed that there is no solution for $\lambda \leq 0$ using the arguments given through Pohozaev's identity. The proof for this lemma relies on the fact that u is spherically symmetric using a result from Gidas-Ni-Nirenberg (Gidas, 1979). Using this property, they rewrite (3.2) with

$n = 3$ as

$$-u'' - \frac{2}{r}u' = u^5 + \lambda u, \quad r \in (0,1),$$

subject to

$$u'(0) = u(1) = 0.$$

Brézis and Nirenberg then prove the identity

$$\int_0^1 u^2(\lambda\psi' + \frac{1}{4}\psi''')r^2\ dr = \frac{2}{3}\int_0^1 u^6(r\psi - r^2\psi')\ dr + \frac{1}{2}|u'(1)|^2\psi(1)$$

holds for every smooth function ψ with $\psi(0) = 0$. Now, if $\lambda \in (0, \frac{1}{4}\lambda_1]$, then by using a special ψ, one can show that $\lambda\psi' + \frac{1}{4}\psi''' = 0$, $\psi(1) \geq 0$, and $r\psi - r^2\psi' > 0$ on $(0,1]$. By the proved identity, this leads to a contradiction.

By using these two lemmas, the proof of Theorem 4 is completed similarly to Theorem 3 by using a "stretching" argument.

3.2.2 Existence of Solutions for (3.1)

Brézis and Nirenberg turn to finding sufficient conditions for existence of a solution to (3.1). Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be bounded. We assume $f(x,u) : \Omega \times [0,\infty) \to \mathbb{R}$ is measurable in x, continuous in u, and $\sup_{x\in\Omega,\ 0\leq u\leq M} |f(x,u)| < \infty$ for every $M > 0$. Let $p = \frac{n+2}{n-2}$ and assume that $f(x,0) = 0$

and f is a lower-order perturbation, i.e

$$\lim_{u\to\infty} \frac{f(x,u)}{u^p} = 0.$$

They further put the following extra assumptions on f:

$$f(x,u) = a(x)u + g(x,u) \tag{3.3}$$

and

$$a(x) \in L^\infty(\Omega) \tag{3.4}$$

$$g(x,u) = o(u) \quad \text{as} \quad u \to 0^+, \quad \text{uniformly in} \quad x, \tag{3.5}$$

$$g(x,u) = o(u^p) \quad \text{as} \quad u \to \infty, \quad \text{uniformly in} \quad x. \tag{3.6}$$

They also assume that the operator $-\Delta - a(x)$ has a positive first eigenvalue. This means

$$\int_\Omega |\nabla v|^2 \; - av^2 \; dx \geq \alpha \int_\Omega v^2 \; dx, \quad \forall v \in H_0^1(\Omega),\; \alpha > 0. \tag{3.7}$$

It does not matter what the value of f is when $u < 0$, so we set

$$f(x,u) = 0 \quad \text{for } x \in \Omega, \quad u \leq 0.$$

They let

$$F(x,u) = \int_0^u f(x,t)\; dt \quad \text{for } x \in \Omega,\; u \in \mathbb{R}$$

and

$$\Psi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} - F(x,u)\ dx, \quad \text{for } u \in H_0^1.$$

They then prove

Theorem 11. (Brézis, 1983) *Assume (3.3) - (3.7) hold. Suppose that there exists some* $v_0 \in H_0^1(\Omega)$, $v_0 \geq 0$ *on* Ω, $v_0 \neq 0$, *such that*

$$\sup_{t\geq 0} \Psi(tv_0) < \frac{1}{n} S^{n/2}.$$

Then there exists a solution to (3.1).

In fact, this theorem generalizes Theorems 4 and 5 when $f(x,u) = \lambda u$. Moreover, the proof of this result requires Theorem 3.

Sketch of Proof of Lemma 10: Using (3.3) - (3.7), we fix a constant $\mu \geq 0$ sufficiently large such that

$$-f(x,u) \leq \mu u + u^p \quad \text{for a.e.} \quad x \in \Omega, \quad \text{and for all} \quad u \geq 0. \tag{3.8}$$

Define on $E = H_0^1$ a functional

$$\Phi(u) = \int \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\mu u^2 - \frac{1}{p+1}(u^+)^{p+1} - F(x,u^+) - \frac{1}{2}\mu(u^+)^2 \right)$$

This functional is C^1, so Brézis and Nirenberg show that this also satisfies the hypotheses of Theorem 3. Once this is done, one may apply the theorem to obtain a sequence $\{u_n\} \subset H_0^1$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ in H^{-1}, where $c < \frac{1}{n}S^{n/2}$ by assumption. In particular, this allows them to write

$$-\Delta u_n + \mu u_n - (u_n^+)^p - f(x, u_n^+) - \mu u_n^+ = \zeta_n \tag{3.9}$$

where $\zeta_n \to 0$ in H^{-1}. Next, they claim that

$$||u_n||_{H_0^1} \leq C.$$

Once this has been shown, this implies that there exists a subsequence which we denote by u_j such that

$$u_j \rightharpoonup u \quad \text{weakly in} \quad H_0^1$$

$$u_j \to u \quad \text{strongly in} \quad L^q \quad \text{for all} \quad q < p+1$$

$$u_j \to u \quad \text{a.e. in} \quad \Omega$$

$$(u_j^+)^p \rightharpoonup (u^+)^p \quad \text{weakly in} \quad (L^{p+1})^*$$

$$f(x, u_j^+) \rightharpoonup f(x, u^+) \quad \text{weakly in} \quad (L^{p+1})^*.$$

Using these, we can pass the limit to the expression (3.9) to obtain

$$-\Delta u + \mu u = (u^+)^p + f(x, u^+) + \mu u^+ \quad \text{in} \quad H^{-1}.$$

Using (3.8), we may deduce the right-hand side is greater than zero. By Stampacchia's maximum principle, we have either $u \equiv 0$ in Ω or $u > 0$ in Ω and

$$-\Delta u = u^p + f(x, u)$$

holds. Brézis and Nirenberg proceed to show that $u \not\equiv 0$. They do this using a proof by contradiction such that they eventually show the c from Theorem 3 satisfies

$$c \geq \frac{1}{n} S^{n/2}.$$

This proves Theorem 11.

Finally, Brézis and Nirenberg prove a sufficient condition for the hypotheses of Theorem 11:

Lemma 12. *Assume $f(x, u)$ satisfies (3.3)-(3.7). Suppose also that there is a function $f(u)$ such that*

$$f(x, u) \geq f(u) \geq 0 \quad \textit{for a.e.} \quad x \in \omega, \quad \textit{and for all} \quad u \geq 0,$$

where ω is some nonempty open set in Ω and the primitive $F(u) = \int_0^u f(t)\ dt$ satisfies

$$\lim_{\epsilon \to 0} \int_0^{\epsilon^{-1/2}} F\left[\left(\frac{\epsilon^{-1/2}}{1+s^2}\right)^{n-2/2}\right] s^{n-1}\ ds = \infty$$

then the hypotheses of Theorem 11 hold.

Using this lemma in conjunction with Theorem 11 allows Brézis and Nirenberg to prove theorems for existence of solutions to (3.1) for different conditions imposed on f and for different dimensions. In particular, they prove separate cases for $n \geq 5$, $n = 4$, and $n = 3$.

CHAPTER 4

The Brézis-Nirenberg Problem: Sign-Changing Solutions

In addition to searching for positive solutions, mathematicians were also interested in finding sign-changing solutions. Generally speaking, the Brézis-Nirenberg problem is emblematic of a semilinear elliptic problem on bounded domains which admit very strong geometric and topological content. As a result, topological methods have been popular in this field, such as fixed point theorems, Ljusternik-Schnirelmann theory, degree theory, etc. For Schechter and Zou, they develop a topological theorem and combine it with analysis of the critical points to exhibit infinitely many sign-changing solutions for the Brézis-Nirenberg problem for dimensions $n \geq 7$, i.e. for

$$-\Delta u = \lambda u + u^p, \quad u \in H_0^1(\Omega), \tag{4.1}$$

where $p = \dfrac{n+2}{n-2}$. We follow their work in (Schechter, 2010).

4.1 Preliminaries

The following definitions and theorems may be found in (Mawhin, 1989) and (Chang, 2005).

Definition 16. Let $\varphi \in C^2(X)$. We define the Morse index at a critical point u to be the dimension of the maximal subspace where $D^2\varphi$ (the Hessian matrix) is negative definite.

Note that this definition holds whether u is either non-degenerate or otherwise. In fact, the Hessian matrix is symmetric, so it can be represented as a bilinear form. Thus, it admits a spectral decomposition of E^+, E^-, and E^0 such that

$$D^2\varphi(u) = E^+ \bigoplus E^0 \bigoplus E^-.$$

Here, we have $Morse(u) = dim(E^-)$. The nullity of a critical point u is defined to be $nullity(u) = dim(E^0)$. A critical point u is degenerate if $dim(E^0) > 0$. We may define the augmented Morse index of a critical point u as $Morse(u) + nullity(u)$.

Theorem 3 is a generalization of a variational theorem due to Ambrosetti and Rabinowitz called the Mountain Pass Theorem (see (Costa, 2007)), which is a special case of a so-called linking.

Definition 17. Let X be a Banach space. Let $Q \subset X$ be a compact manifold with boundary ∂Q and let $S \subset X$ be a closed subset of X. ∂Q is said to be linking with S if

- $\partial Q \cap S = \emptyset$,
- $\forall \varphi : Q \to X$ continuous with $\varphi|_{\partial Q} = id|_{\partial Q}$, we have $\varphi(Q) \cap S \neq \emptyset$.

Linkings are helpful in proving minimax theorems like the more general Saddle Point Theorem of Rabinowitz (see (Costa, 2007)). On the other hand, some well-known topological theorems are needed to prove important results such as

Theorem 13. *(The Borsuk-Ulam Theorem) For every odd continuous map $f : S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = 0$.*

One of the hallmarks of Morse theory that is useful for analysis is the Morse lemma (see (Mawhin, 1989)):

Theorem 14. *Let M be a Riemannian manifold, $f \in C^2(M, \mathbb{R})$, and $p \in M$ is a non-degenerate critical point; then there exists a neighborhood U of p and a local diffeomorphism $\Phi : U \to T_p(M)$*

with $\Phi(p) = 0$, such that

$$f \circ \Phi^{-1}(\xi) = f(p) + \frac{1}{2}\langle f''(p)\xi, \xi\rangle_p, \quad \forall \xi \in \Phi(U)$$

where $\langle \cdot, \cdot \rangle_p$ is the Riemannian metric at p.

Moreover, this means that in a neighborhood of p, the function behaves like a quadratic function. So if p is a non-degenerate critical point such that $f(p) = c$, then in some neighborhood of p:

$$f(x) = c + \frac{1}{2}(||x_+||^2 - ||x_-||^2),$$

where $x = x_+ + x_-$, $dim\ H_- = Morse(p)$, $\mathbb{R}^n = H_+ \bigoplus H_-$, and $dim\ M = n$.

4.2 Existence of Sign-Changing Solutions

In contrast to the previous section, we will consider sign-changing solutions to 3.2. Much work has been done in this area with many results proven for various cases of n and domains Ω. We will focus mainly on the approach of (Schechter, 2010) which uses algebraic topological arguments to show there exist infinitely many sign-changing solutions. For more about this approach, see (Mawhin, 1989). In particular, Schechter and Zou prove

Theorem 15. (Schechter, 2010) *If $n \geq 7$, then (4.1) has infinitely many sign-changing solutions for $\lambda > 0$.*

To prove Theorem 15, they prove a key theorem based on linkage which bounds the Morse index of solutions from below. To faciliate the proof of this theorem, we must introduce some notations and assumptions found in (Schechter, 2010):

Let $E = \overline{\cup_{k=1}^{\infty} E_k}$ be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$ with norm $||\cdot||$ such that dim $E_k = k$, $E_k \subset E_{k+1}$. Let $G \in C^2((E, ||\cdot||), \mathbb{R})$ be an even functional which maps bounded sets to bounded sets in terms of the norm $||\cdot||$. Assume that $G''(u_0)$ is Fredholm for any critical point u_0 of G. The gradient G' is of the form $G'(u) = u - K_G(u)$, where $K_G : E \to E$ is a continuous operator. Let $\mathcal{K} := \{u \in E : G'(u) = 0\}$ and $\bar{E} := E \setminus \mathcal{K}$, $\mathcal{K}[a, b] := \{u \in \mathcal{K} : G(u) \in [a, b]\}$. Let $\mathcal{P}$ be the positive cone of E that is closed convex and weakly closed. Any element outside $\pm\mathcal{P}$ is called sign-changing. Assume that $\pm\mathcal{P} \cap (E_k^{\perp} \setminus \{0\}) = \emptyset$ for all $k \geq 2$, that is, any nonzero element of $E_k^{\perp}$ is sign-changing. For each $\mu > 0$, define $\mathcal{D} := \{u \in E : dist(u, \mathcal{P}) < \mu\}$. Then, $\mathcal{D}(\mu)$ is an open convex set containing the positive cone $\mathcal{P}$ in its interior. Set $\mathcal{D}^* = \mathcal{D}^*(\mu) = \mathcal{D}(\mu) \cup (-\mathcal{D}(\mu))$, $S^* = E \setminus D^*(\mu)$. We also consider the following assumptions:

(A0) There is another norm $||\cdot||_*$ of E such that $||u||_* \leq C_0||u||$ for all $u \in E$, where $C_0 > 0$ is a constant. Moreover, we assume that $||u_n - u^*||_* \to 0$ whenever $u_n \rightharpoonup u^*$ weakly in $(E, ||\cdot||)$.

(A1) For any $\mu_0 > 0$ small enough, we have that $K_G(\pm\mathcal{D}(\mu_0)) \subset \pm\mathcal{D}(\mu) \subset \pm\mathcal{D}(\mu_0)$ for some $\mu \in (0, \mu_0)$. Moreover, $\pm\mathcal{D}(\mu_0) \cap \mathcal{K} \subset \pm\mathcal{P}$.

(A2) For each k, $\lim\limits_{||u|| \to \infty, u \in E_k} G(u) = -\infty$.

(A3) Assume that for any $\alpha_1, \alpha_2 > 0$ there exists an α_3 depending on α_1 and α_2 such that $||u|| \leq \alpha_3$ for all $u \in G^{\alpha_1} \cap \{u \in E : ||u||_* \leq \alpha_2\}$, where $G^{\alpha_1} = \{u \in E : G(u) \leq \alpha_1\}$.

Define $C_{k+1}^{**} := \sup\limits_{E_{k+1}} G$. This is well-defined in light of $(A2)$. Next, we write $E = E_k \bigoplus E_k^{\perp}$ and consider the function

$$\beta^*(u) = \begin{cases} \dfrac{||u|| ||u||_*}{||u|| + ||u||_*}, & u \neq 0 \\ 0, & u = 0. \end{cases}$$

We note that $\beta^* : E \to E$ is continuous, so we set $S_0(k) := \{u \in E_k^\perp : \beta^*(u) = 1\}$. Note that for $u \in S_0(k)$, then

$$\beta^*(u) = 1 \Rightarrow ||u||_* = 1 + \frac{||u||_*}{||u||} \leq 1 + C_0 := \alpha_4.$$

by $(A0)$.

Define $S(k) := S_0(k) \cap G^{C_{k+1}^{**}}$. One can then prove the following lemma:

Lemma 16. *By assumption (A3), we have a constant $\alpha_5 = \alpha(\alpha_4, C_{k+1}^{**}) > 0$ such that $||u|| \leq \alpha_5$ for all $u \in S(k)$. Hence, there is a $\Lambda_0 = \Lambda_0(\alpha_5) > 0$ such that $\inf_{u \in S(k)} G \geq -\Lambda_0$.*

To prove the second statement, we have for $u \in S(k)$

$$|G| \leq C||u|| \leq C\alpha_5 := \Lambda_0.$$

This implies

$$G \geq -\Lambda_0 \Rightarrow \inf_{u \in S(k)} G \geq -\Lambda_0.$$

Next, Schechter and Zou prove another lemma:

Lemma 17. *There is a constant $\delta > 0$ such that dist$(S(k), \pm\mathcal{P}) = \delta > 0$.*

To show this, we consider two sequences $\{u_n\} \subset S(k)$ and $\{p_n\} \subset \mathcal{P}$ such that $||u_n - p_n|| \to 0$. By Lemma 16, we know that $\{u_n\}$ is bounded under both $||\cdot||$ and $||\cdot||_*$ by assumption (A0). Given $\epsilon > 0$, there exists N_0 such that for all $n > N_0$:

$$||u_n - p_n|| \leq \epsilon.$$

This implies

$$||p_n|| \leq \epsilon + ||u_n|| \leq \epsilon + \alpha_5.$$

Then taking $C := \max\{p_1, p_2, ..., p_{N_0 - 1}, \epsilon + \alpha_5\}$, we have that $\{p_n\}$ is bounded in both norms as well. As a result, there exist weak limits u^* and $p^* \in \mathcal{P}$ such that $u_n \rightharpoonup u^*$ and $p_n \rightharpoonup p^*$. weakly under $|| \cdot ||$. By assumption (A0), this means strong convergence under $|| \cdot ||_*$. By closure and the strong convergence under $|| \cdot ||_*$, we have $u^* \in E_k^\perp$ and $u^* \neq 0$ since $\dfrac{||u||||u||_*}{||u|| + ||u||_*} = 1$. However, since $u^* = p^*$, $p^* \in \mathcal{P}$, and $\mathcal{P} \cap E_k^\perp = \{0\}$, then there is a contradiction, as this would imply $\mathcal{P}$ contains a sign-changing element.

As a consequence of $|| \cdot ||$ and $|| \cdot ||_*$ being equivalent in E_{k+1}, there exists a constant, say ρ_{k+1} such that $||u|| \leq \rho_{k+1}||u||_*$ for all $u \in E_{k+1}$. Next define $\Gamma^*_{k+1} = \{h : h \in C(\Theta_{k+1}, E), h|_{\partial\Theta_{k+1}} = I, h \text{ is odd}\}$, where $\Theta_{k+1} := \{u \in E_{k+1} : ||u|| \leq R_{k+1}\}$ and I is the identity operator. By Lemma 16 and (A2), we may choose R_{k+1} such that

$$\partial\Theta_{k+1} \cap S(k) = \emptyset, \quad \sup_{\partial\Theta_{k+1}} G << -\Lambda_0 \leq \inf_{S(k)} G, \quad R_{k+1} \geq \rho_{k+1} + 2.$$

We may assume that $R_{k+1} > R_k$ for each k. We may now prove another lemma:

Lemma 18. *$h(\Theta_{k+1} \cap S_0(k)) \neq \emptyset$, $\forall h \in \Gamma^*_{k+1}$.*

Proof. Let $h \in \Gamma^*_{k+1}$. Define $U := \{u \in E_{k+1} : \beta^*(h(u)) < 1\} \cap \{u \in E_{k+1} : ||u|| < R_{k+1}\}$. It is clear that U is a symmetric neighborhood of zero in E_{k+1}. Consider the orthogonal projection $P : E \to E_k$ and the composition $P \circ h : \partial U \to E_k$ which is odd and continuous. By the Borsuk-Ulam theorem (Theorem 13), there exists $u \in \partial U$ such that $P \circ h(u) = 0$. Consequently, $h(u) \in E_k^\perp$. Next, Schechter and Zou prove that $u \notin \partial\{u \in E_{k+1} : ||u|| < R_{k+1}\}$. The proof then proceeds by contradiction: otherwise $||u|| = R_{k+1}$ and $h(u) = u$ with $P(u) = 0$. Since we know $\frac{||h(u)||||h(u)||_*}{||h(u)|| + ||h(u)||_*} \leq 1$, we have $R_{k+1} = ||u|| \leq 1 + \frac{||u||}{||u||_*} \leq 1 + \rho_{k+1}$. This is a contradiction to the fact that R_{k+1} was chosen to be greater than $2 + \rho_{k+1}$. Hence, u cannot be in the boundary of the aforementioned set. Consequently: $u \in \partial\{u \in E_{k+1} : \beta^*(h(u)) < 1\}$, $||u|| \leq R_{k+1}$, $u \in E_{k+1}$. This implies $h(u) \in E_k^\perp$, $\frac{||h(u)||||h(u)||_*}{||h(u)|| + ||h(u)||_*} = 1$. This means $h(u) \in S_0(k)$. Hence, it is in $h(\Theta_{k+1}) \cap S_0(k)$. $\square$

By using Lemma 18 and the definition $\mathcal{S}^*$, then we may choose μ_0 by (A1) so that $S(k) \subset \mathcal{S}^* = E \setminus \mathcal{D}^*(\mu_0)$. Next, Schechter and Zou use another definition of linkage to prove their theorem:

Definition 18. A compact symmetric subset A of E with $\partial\Theta_{k+1} \subset A$ is said to be linked to $S_0(k)$ if, for any continuous mapping $h \in C([0,1] \times A, E)$ satisfying

- $h(t,u)$ is odd in $u \in A$
- $h(t,u) = u$ for all $u \in \partial\Theta_{k+1}$,

there holds $h(1,A) \cap S_0(k) \neq \emptyset$.

Define the set $\mathcal{L} := \{A \subset G^{C^{**}_{k+1}} : A \quad \text{is linked to} \quad S_0(k)\}$. Then Lemma 18 and the fact that $\Theta_{k+1} \in \mathcal{L}$ shows that $\mathcal{L}$ is nonempty. We can now prove the following key theorem:

Theorem 19. (Schechter, 2010) *Suppose that G satisfies the (PS) condition and assumptions (A1)-(A3). Define*

$$C^* = \inf_{A \in \mathcal{L}} \sup_{A \cap \mathcal{S}^*} G(u). \tag{4.2}$$

Then $C^* \in [-\Lambda_0, C^{**}_{k+1}]$ and G has a sign-changing critical point $u^* \in \mathcal{S}^*$ $(u^* \neq 0)$ at level C^* and the augmented Morse index $m^*(u^*)$ of u^* is greater than or equal to k.

This theorem is the most important part of (Schechter, 2010) and allows them to prove Theorem 15. The proof is also somewhat long, so we will provide an abbreviated sketch.

Sketch of Proof. Firstly, for $A \in \mathcal{L}$, then A links with $S_0(k)$. This implies that $A \cap S_0(k) \neq \emptyset$. Since $S(k) = G^{C^{**}_{k+1}} \cap S_0(k)$, then also $A \cap S(k) \neq \emptyset$. This implies $A \cap \mathcal{S}^* \neq \emptyset$ and hence C^* is well defined. Furthermore, since $A \cap \mathcal{S}^* \supset A \cap S(k)$ and $A \cap S(k) \subset S(k)$, then we have the following inequalities:

$$\sup_{A \cap S(k)} G \leq \sup_{A \cap \mathcal{S}*} G$$

$$\inf_{S(k)} G \leq \inf_{A \cap S(k)} G$$

which combined with our previous result yields

$$\sup_{\partial \Theta_{k+1}} G << -\Lambda_0 \leq \inf_{S(k)} G \leq \sup_{A \cap \mathcal{S}^*} G.$$

We also have $C^* \leq \sup G(A) \leq C^{**}_{k+1}$. From here, the proof is divided into 6 steps, which we summarize briefly:

Step 1. Exhibiting the existence of a sign-changing critical point $u^* \in \mathcal{S}^*$ with $u^* \neq 0$ and $G(u^*) = C^*$. To show this is equivalent to showing $\mathcal{K}[C^* - \epsilon, C^* + \epsilon] \cap \mathcal{S}^* \neq \emptyset$ for all small enough $\epsilon > 0$. The proof then proceeds by contradiction by exhibiting a set in $\mathcal{L}$ that is contained in $G^{C^* - \epsilon_1/4}$. This would contradict the definition of C^* as a result.

Steps 2-5 will assume that $\mathcal{K}_{C^*} \cap \mathcal{S}^*$ has finitely many nondegenerate critical points. Step 6 will address the case of degeneracy.

Step 2. The next step is to prove that any $u \in \mathcal{K}_{C^*} \cap \mathcal{S}^*$ with a Morse index m^*, then one can find a closed neighborhood $N_1(u)$ of u such that $N_1(u) \cap \partial \Theta_{k+1} = \emptyset$ and a subset $N_2(u)$ such that $N_2(u) \subset N_1(u) \subset \mathcal{S}^*$ with $N_2(u)$ being homeomorphic to a ball B_{m^*} of $\mathcal{R}^{m^*}$. To do this, they write

$$\mathcal{K}_{C^*} \cap \mathcal{S}^* = \{u_1, ..., u_m\} \cup \{-u_1, ..., -u_m\}.$$

From here, they choose $\delta > 0$ such that $B_\delta(\pm u_i) \cap \partial\Theta_{k+1} = \emptyset$ for all $i = 1, ..., m$. Next, for each u_i or $-u_i$, one can find closed subspaces of E — denoted by E^- and E^+ with $E^+ = (E^-)^\perp$ — that are invariant under $G''(u_i)$. Here, $G''(u_i)$ is negative definite on E^- and positive definite on E^+. By applying the Morse lemma (Theorem 14), then for $z = z^+ + z^-$ with $z \in E$ and $z^\pm \in E^\pm$, then one can find a Lipschitz homeomorphism H_i from a a neighborhood $U_i(0)$ of 0 in E onto a neighborhood of $U(u_i)$ of u_i in E with $H_i(0) = u_i$ and

$$G(H_i(z)) = G(u_i) + ||z^+||^2 - ||z^-||^2, \quad \forall z \in U_i(0).$$

Afterwards, we choose radii $r_i^-, r_i^+ > 0$ such that B_i^- and B_i^+ denote respective balls centered at the origin in E^- and E^+ respectively. Thus, we set $N_1(u_i) = H_i(B_i^- \bigoplus B_i^+)$ and this is a closed neighborhood that is disjoint with $\partial\Theta_{k+1}$. Schechter and Zou also define a special function $h(t, z)$ inspired by Lazer and Solimini (Lazer, 1988) such that $h(t, z)$ is odd in $z \in E$ and continuous on a set dependent on B_i^- and B_i^+. Using this, they set $N_2(u_i) = h(1, G^{C^*+\epsilon} \cap H_i(\frac{1}{3}B_i^- \bigoplus \frac{1}{3}B_i^+))$ for $i = 1, 2, , ..., m$. It is clear that $N_2(u_i) \subset N_1(u_i)$, so all that is left to prove is that N_2 is homeomorphic to the ball in $\mathbb{R}^{m_*}$ with radius equal to the Morse index. This is done by using topological arguments.

Steps 3-4 will be about finding a set A^* such that

$$\sup_{A^* \cap \mathcal{S}^*} G = C^* \tag{4.3}$$

and

$$G(u) < C^*, \quad \forall u \in (A^* \setminus \mathcal{K}_{C^*}) \cap \mathcal{S}^*. \tag{4.4}$$

In other words, there exists $u \in \mathcal{K}_{C^*} \cap \mathcal{S}^* \cap A^*$.

Step 3. This step will focus on finding the particular set $A^* \in \mathcal{L}$. This is done by constructing a flow $\eta_0 : [0,1] \times E \to E$ from any two open sets $\mathcal{O}_1 \subset \mathcal{O}$ in $\mathcal{K}_{C^*}$. This flow in particular is odd and continuous and can be constructed such that

$$\eta_0(1, G^{C^*+\epsilon} \setminus \mathcal{O}) \subset G^{C^*-\epsilon}.$$

Then by using the special function $h(t,z)$ used in Step 2, then by choosing $A \in \mathcal{L}$ such that $\sup_{A \cap \mathcal{S}^*} G \leq C^* + \epsilon$, then let

$$A^* = \eta_0(1, h(1, A)).$$

One can then proceed to show that $A^* \in \mathcal{L}$. This completes Step 3.

Step 4. Next, the set obtained in Step 3 will be shown to satisfy (4.3) and (4.4). The proof for this step requires using the definition of the flow and considering the flow of $v \in \mathcal{S}^*$ into $u \in A^* \cap \mathcal{S}^*$ under the flow, i.e. $u = \eta_0(1, v)$, with $v = h(1, a)$, $a \in \mathcal{S}^* \cap A$. Then by using set-theoretic arguments, one can show that, for any $u \in (A^* \setminus \mathcal{K}_{C^*}) \cap \mathcal{S}^*$, (4.4) must hold. Furthermore, it can be shown otherwise that v must obtain the supremum in (4.3) and that $u \equiv v$. As a result, this exhibits the existence of a sign-changing critical point $u \in A^* \cap \mathcal{S}^*$ at level C^*.

Step 5. In this step, one can find a lower bound k on the Morse index of the sign-changing critical point $u \in \mathcal{K}_{C^*} \cap \mathcal{S}^* \cap A^*$ such that $Morse(u) \geq k$. To see this, note that

$$\mathcal{K}_{C^*} \cap \mathcal{S}^* = \{u_1, u_2, ..., u_m\} \cup \{-u_1, -u_2, ..., -u_m\}.$$

The proof then proceeds by contradiction by assuming that $Morse(\pm u_i) < k$ for $i = 1, 2, ..., m$. To get the contradiction, they construct a set that is based off of several of the sets in constructed in Step 1 and show that it is in $\mathcal{L}$. By the construction, this would contradict (4.3) and (4.4). Consequently, they show this set is in $\mathcal{L}$ by another argument by contradiction. This proof proceeds using topological and critical point theory methods. At this point, the proof is finished for the case when the critical points are non-degenerate.

Step 6. It is important to note that Steps 2-5 prove the theorem in the case of nondegenerate critical points. It is therefore important to show that the result holds if there are degenerate critical points. The proof requires assuming the augmented Morse index of the set of critical points is less than k and using the Morse lemma to obtain a decomposition of the set of critical points at level C^*. Using techniques inspired by Marino and Prodi (Marino, 1975), as well as Lazer and Solimini (Lazer, 1988), they show that G in fact admits a sign-changing critical point and the set of critical points are non-degenerate. Therefore, they apply Steps 2-5 to obtain a critical point with Morse index not less than k. This contradicts with the fact that the augmented Morse index is less than k and so the proof holds for degnerate critical points.

Schechter and Zou proceed to applying Theorem 19 to prove Theorem 15.

Proof of Theorem 15: We must first define the norms and functionals so that we are in the setting of Theorem 19. Fix $p_0 \in (2, 2^*)$, with $2^* = \dfrac{2N}{N-2}$, and next we choose a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $(p_0, 2^*)$ such that $p_n \to 2^*$. Next, we shall consider the functional given by

$$G_{n,\lambda}(u) = \frac{1}{2}||u||^2 - \frac{\lambda}{2}\int_\Omega |u|^2\ dx - \frac{1}{p_n}\int_\Omega |u|^{p_n}\ dx, \quad u \in H_0^1(\Omega),$$

with $H_0^1(\Omega)$ denoting the Hilbert space consisting of functions and their weak derivatives (both L^2-integrable) equipped with the inner product $\langle u, v\rangle = \displaystyle\int_\Omega \nabla u \cdot \nabla v\ dx$. Note that for each n, $G_{n,\lambda}$ satisfies (PS). By the usual theory, we let $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_k \leq ...$ be the eigenvalues for $-\Delta$ on Ω. We also let $\phi_k(x)$ denote the respective eigenfunctions for λ_k, $k \geq 1$. Let $E_k := span\{\phi_1, \phi_2, ..., \phi_k\}$. Next, for each p_n, we let $||\cdot||_* = ||\cdot||_{p_n}$. We note that $G'_{n,\lambda}(u) = u - K_{n,\lambda}(u)$, where $K_{n,\lambda} : E \to E$ is a continuous operator. We let Let $\mathcal{P} := \{u \in H_0^1(\Omega) : u \geq 0\}$ and for each $\mu > 0$, we define $\mathcal{D}(\mu) := \{u \in E : dist(u, \mathcal{P}) < \mu\}$. Therefore, we let $\mathcal{D}^* = \mathcal{D}^*(\mu) := \mathcal{D}(\mu) \cup (-\mathcal{D}(\mu))$ and $\mathcal{S}^* = E \setminus \mathcal{D}^*$. Thus, assumptions (A0)-(A3) are satisfied.

Next, define

$$C_{k+1}^{**}(n,\lambda) := \sup_{E_{k+1}} G_{n,\lambda}.$$

Schechter and Zou proceed to show that this value is bounded by the corresponding eigenvalue multiplied by a constant independent of both k and n.

Lemma 20. *There exists a constant $T_1 > 0$ independent of k and n such that*

$$C_{k+1}^{**}(n,\lambda) \leq T_1 \lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}.$$

Proof. By definition of E_{k+1}, we have

$$||u||^2 \leq \lambda_{k+1}||u||_2^2$$

For any $p_n > p_0$, we have that there exists $D_1 > 0$ independent of n and k such that $||u||_{p_0} \leq D_1||u||_{p_n}$. Thus, we have

$$G_{n,\lambda}(u) \leq \frac{1}{2}||u||^2 - D_2 \int_\Omega |u|^{p_0}\ dx + D_3$$

where $D_2, D_3 > 0$ are constant and independent of either k and n.

There also exists a constant $D_4 > 0$ such that $||u||_2 \leq D_4||u||_{p_0}$, thus we have

$$||u||^2 \leq \lambda_{k+1}||u||_2^2 \leq D_4^2\lambda_{k+1}||u||_{p_0}^2, \quad \forall u \in E_{k+1}.$$

This suggests that there exists $D_5 > 0$ such that

$$||u||^{p_0} \leq D_5\lambda_{k+1}^{p_0/2}||u||_{p_0}^{p_0}, \quad \forall u \in E_{k+1}.$$

Combining these inequalities together, we have

$$\begin{aligned} G_{n,\lambda}(u) &\leq \frac{1}{2}||u||^2 - D_6\lambda_{k+1}^{-p_0/2}||u||^{p_0} + D_3 \\ &\leq D_7\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}} + D_3 \\ &\leq T_1\lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}, \end{aligned}$$

with D_i $(i = 1, ..., 7)$ and T_1 are positive constants that are independent of k and n. $\square$

We apply Theorem 19 to exhibit the existence of a sign-changing critical point $u^*(n, \lambda, k) \neq 0$ of $G_{n,\lambda}$ that satisfies

$$G_{n,\lambda}(u^*) = C^*(n, \lambda, k) \leq C^{**}_{k+1} \leq T_1 \lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}$$

and the augmented Morse index $m^*(u^*(n, \lambda, k)) \geq k$. The next thing is to show that there exists a lower bound of C^* that is independent of n. This is done by applying the definitions given in the notations following Lemma 16. Thus,

$$C^*(n, \lambda, k) \in [-\Lambda_0, T_1 \lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}].$$

Hence, for every fixed λ and k, $\{u^*(n, \lambda, k)\}_{n \in \mathbb{N}}$ is a sequence of solutions that satisfy

$$-\Delta u = \lambda u + |u|^{p_n - 2} u, \quad \forall u \in H_0^1(\Omega),$$

with p_n in $[p_0, 2^*]$. Furthermore, the sequence is bounded in $H_0^1(\Omega)$ as a consequence of Lemma 20.

We now cite a theorem due to Devillanova and Solimini (Devillanova, 2002):

Theorem 21. *Let $n \geq 7$ and U be a bounded set in $H_0^1(\Omega)$ whose elements are solutions, for a fixed $\lambda > 0$, to the problem*

$$-\Delta u = \lambda u + |u|^{p-2} u, \quad u \in H_0^1(\Omega),$$

for p varying in $[2, 2^*]$. Then U is uniformly bounded, that is, there exists a constant $C > 0$ such *that*

$$\sup_{u \in U} \sup_{x \in \Omega} |u(x)| \leq C.$$

Using this theorem, then the set of solutions $\{u^*(n, \lambda, k\}_{n \in \mathbb{N}}$ is uniformly bounded. We may apply standard compactness arguments to obtain a convergent subsequence with limit $u^*(\lambda, k)$ that solves

$$-\Delta u = \lambda u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega)$$

at level C^*. We know that this problem does not have a solution with negative energy, therefore $C^*(\lambda, k) \geq 0$. Hence,

$$0 \leq \lim_{n \to \infty} C^*(n, \lambda, k) = C^*(\lambda, k) \leq T_1 \lambda_{k+1}^{\frac{p_0}{2(p_0-2)}}.$$

The next thing is to show that $u^*(\lambda, k)$ is sign-changing. Since $\{u^*(n, \lambda, k)\}$ is a sequence of sign-changing solutions, then we let

$$u^*(n, \lambda, k)^{\pm} := \max\{\pm u^*(n, \lambda, k), 0\}.$$

Thus, we have from $G'_{n,\lambda}(u)$:

$$||u^*(n, \lambda, k)^{\pm}||^2 = \lambda ||u^*(n, \lambda, k)^{\pm}||_2^2 + ||u^*(n, \lambda, k)^{\pm}||_{p_n}^{p_n}.$$

This implies that there exists $\epsilon_0 > 0$ and independent of n such that $||u^*(n, \lambda, k)^{\pm}||^2 \leq \epsilon_0 ||u^*(n, \lambda, k)^{\pm}||_{p_n}^{p_n}$.

Moreover, this means there exists a constant s_0 independent of n such that $||u^*(n, \lambda, k)^{\pm}|| \geq s_0 > 0$. This implies the limit $u^*(\lambda, k)$ satisfies these inequalities, and hence is sign-changing. To see this, note that the above inequality implies $u^*(\lambda, k)^{\pm} \neq 0$ almost everywhere. Hence, $\pm u^*(\lambda, k) \neq 0$ almost everywhere. This of course leads to the existence of positive and negative parts that are non-zero, and hence u^* is sign-changing. The final part of the proof is to show that $C^*(\lambda, k) \to \infty$ as $k \to \infty$. The proof proceeds by contradiction. If not, then $\{C^*(\lambda, k)\}_{k \in \mathbb{N}}$ is bounded and so $\lim_{k \to \infty} C^*(\lambda, k) = c' < \infty$. For each $k \in \mathbb{N}$, there exists an $n_k > k$ such that $|C^*(n_k, \lambda, k) - C^*(\lambda, k)| < 1/k$. Of course, this means $\lim_{k \to \infty} C^*(n_k, \lambda, k) = c'$. By assumption, we know that each $u^*(n_k, \lambda, k)$ are sign-changing critical points of $G_{n_k, \lambda}$ with augmented Morse indices $m^*(u^*(n_k, \lambda, k)) \geq k$. Similar to before, one can show that $\{u^*(n_k, \lambda, k)\}$ are bounded in $H_0^1(\Omega)$ using the fact that c' is independent of k. By Theorem 21, $u^*(n_k, \lambda, k)_{k \in \mathbb{N}}$ is uniformly bounded. By the results of Bahri and Lion (Bahri, 1992), this is equivalent to the Morse indices of $u^*(n_k, \lambda, k)_{k \in \mathbb{N}}$ being bounded. However, this is a contradiction since we know that the Morse index is increasing for each $u^*(n_k, \lambda, k)$ over k. Thus, $C^*(\lambda, k) \to \infty$. This of course implies infinitely many sign-changing solutions corresponding to distinct critical levels C^*.

CHAPTER 5

Conclusions

5.1 Conclusion

The Brézis-Nirenberg problem appeared as an interesting analytical problem that arose from various problems in mathematics, including geometry. The method used by Yamabe to obtain a solution would later inspire the approach of Brézis and Nirenberg (Brézis, 1983). Originally, the proof of existence of a solution by Yamabe (Yamabe, 1960) had a flaw which would not be redressed until later by Schoen, Yau, and Aubin (Schoen, 1979) (Aubin, 1976).

After the landmark paper by Brézis and Nirenberg (Brézis, 1983), it was clear that the topology of the domain made a difference in understanding the qualitative characteristics of the problem. For example, Cerami et al. (Cerami, 1986) found that when $n \geq 6$, then (3.2) has two pairs of solutions on any smooth bounded domain, whereas Atkinson et al. (Atkinson, 1990) proves that for $4 \leq n \leq 6$ and Ω is a ball, then for certain values of λ, (3.2) has no sign-changing radial solutions. Thus, the field has seen many proofs that approach this problem using topological methods. A survey of some of these techniques may be found in (Chang, 2005).

In addition, the problem is interesting due to the fact that it lacks compactness. The boundary value problem (3.2) loses a compactness property that would make exhibiting a solution much easier — a consequence of the exponent being critical. This observation opened the field of PDEs to consider boundary value problems which did not satisfy the (PS) condition and hence nice compactness properties – especially when working at the critical exponents. Similar problems include working with

$$-\Delta u = \lambda u e^{\alpha_n u^2}, \quad x \in \Omega \subset \mathbb{R}^2$$

$$u = 0, \quad x \in \partial\Omega,$$

which has been explored by Adimurthi (Adimurthi, 1990) and Figueredo et al. (de Figueiredo, 2011). It is related to an analogue to the Sobolev Embedding Theorem in $\mathbb{R}^2$ by Moser (Moser, 1971) and Trudinger (Trudinger, 1967). Here, α_n is considered the critical exponent where $\alpha_n = n\left[\frac{2\pi^{n/2}}{\Gamma(n/2)}\right]^{1/(n-1)}$.

BIBLIOGRAPHY

Yamabe, H. 1960. *On a Deformation of Riemannian Structures on Compact Manifolds*, Osaka Mathematical Journal **12**, no. 1, 21 – 37.

Brézis, H. and Nirenberg, L. 1983. *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents*, SMR **398**, 2.

Schechter, M and Zou, Wenming. 2010. *On the Brézis–Nirenberg Problem*, Archive for rational mechanics and analysis **197**, no. 1.

Brézis, H. 2010. *Functional Analysis, Sobolev spaces and Partial Differential Equations*, Springer Science & Business Media.

Costa, D. 2007. *An Invitation to Variational Methods in Differential Equations*, Birkhauser Boston.

Mawhin, J. and Willem, M. 1989. *Critical Point Theory and Hamiltonian Systems*, Vol. 74, Springer Science & Business Media.

Chang, K.C. 2005. *Methods in Nonlinear Analysis*, Springer Monographs in Mathematics.

Pokhozhaev, Stanislav Ivanovich. 1965. *Eigenfunctions of the Equation $\Delta u+\lambda f(u)=0$*, Doklady Akademii Nauk, pp. 36–39.

Lazer, A. C. and Solimini, S. 1988. *Nontrivial Solutions of Operator Equations and Morse Indices of Critical Points of Min-Max Type*, Nonlinear Analysis **12**, no. 8, 761–775.

Marino, A. and Prodi, G. 1975. *Metodi perturbativi nella teoria di Morse*, Boll. Un. Mat. Ital. **11**, no. 3.

Devillanova, G. and Solimini, S. 2002. *Concentration Estimates and Multiple Solutions to Elliptic Growth Problems at Critical Growth*, Advances in Differential Equations **7**, no. 10, 1257–1280.

Bahri, A. and Lions, P. L. 1992. *Solutions of Superlinear Elliptic Equations and Their Morse Indices*, Communications on Pure and Applied Mathematics **45**, no. 9, 1205-1215.

Aubin, T. 1998. *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics.

Gidas, B., Ni. 1979. *Symmetry and Related properties via the Maximum Principle*, Communications in Mathematical Physics **68**, 209-243.

Aubin, T. 1976. *Equations differentielles non lineaires et probleme de Yamabe concernant la courbure scalaire*, Journal de Mathématiques Pures et Appliquées **55**, 269 – 296.

Schoen, R. and Yau, S. 1979. *On the Proof of the Positive Mass Conjecture in General Relativity*, Communications in Mathematical Physics **65**, no. 1, 45 – 76.

Cerami, G., Solimini. 1986. *Some Existence Results for Superlinear Elliptic Boundary Value Problems Involving Critical Exponents*, Journal of Functional Analysis **69**, 289-306.

Atkinson, F.V., Brezis. 1990. *Nodal solutions of Elliptic Equations with Critical Sobolev Exponents*, Journal of Differential Equations **85**, no. 1, 151-170.

Adimurthi. 1990. *Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical growth for the n-Laplacian*, Ann. Sc. Norm. Sup. Pisa **17**, 3.

de Figueiredo, J., Do O. 2011. *Elliptic Equations and Systems with Critical Trudinger-Moser Nonlinearities*, Discrete and Continuous Dynamical Systems **30**, 2.

Trudinger, N. 1967. *On Imbeddings into Orlicz Spaces and Some Applications*, Indiana University Mathematics Journal **17**, 473–483.

Moser, J. 1971. *A Sharp Form of an Inequality by N. Trudinger*, Indiana University Mathematics Journal **20**, no. 11, 1077–1092.

Taubes, Clifford Henry. 1982. *The existence of a non-minimal solution to the SU (2) Yang-Mills-Higgs equations on &R;^{3}. Part I*, Communications in Mathematical Physics **86**, no. 2, 257-298.

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