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Positive Solutions to Semilinear Elliptic Equations with Logistic-Type Nonlinearities and Harvesting in Exterior Domains

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POSITIVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH LOGISTIC-TYPE NONLINEARITIES AND HARVESTING IN EXTERIOR DOMAINS

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Abstract

POSITIVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH LOGISTIC-TYPE NONLINEARITIES AND HARVESTING IN EXTERIOR DOMAINS

by

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Existing results provide the existence of positive solutions to a class of semilinear elliptic PDEs with logistic-type nonlinearities and harvesting terms both in $\mathbb{R}^N$ and in bounded domains $U \subset \mathbb{R}^N$ with $N \geq 3$, when the carrying capacity of the environment is not constant. We consider these same equations in the exterior domain Ω, defined as the complement of the closed unit ball in $\mathbb{R}^N$, $N \geq 3$, now with a Dirichlet boundary condition. We first show that the existing techniques for solving these equations in the whole space $\mathbb{R}^N$ can be applied to the exterior domain with some modifications. Then, as a second approach, we use the Kelvin transform to move the equation inside the unit ball, solve it there, using the techniques for bounded domains, and then re-apply the Kelvin transform to obtain a solution to the original equation. We are then confronted with the natural question of whether the two different approaches provide a multiplicity result for positive solutions in our exterior domain. As part of this work we prove a uniqueness result under further assumptions on the data. Finally, we briefly show that the Kelvin transform method can also be applied to the case of $N = 2$ with some slight adjustments, and that the solution obtained in this case also satisfies a similar uniqueness property.

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Starting this degree in August 2016 seems like a lifetime ago; the world is inexplicably different, and I believe I have come a long way from the person I was when I first began. This is in no small part thanks to the community of people that have helped and supported me throughout my journey. Now that I have reached this journey's end, I would like to take some time to thank these people.

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Chapter 1

Introduction and Background

In this thesis we study the existence and uniqueness of *positive* solutions to the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ is the complement of the closed unit ball in $\mathbb{R}^N$. We will consider the cases $N \geq 3$ and $N = 2$ separately, and take note of the key differences therein. Both the functions a, b, h and the parameters λ, c are nonnegative, and we impose additional assumptions on these functions and parameters in each of our different approaches, that will be stated as they are needed.

This problem can be thought of as the steady state of the reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty). \end{cases}$$

In this sense, we can interpret this equation as the evolution equation arising from the population biology of one species, with the function u representing the population density of the species. Throughout, we assume that

$$\lim_{s\to 0} \frac{g(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s} = \infty,$$

so that the nonlinearity $\lambda a(x)u - b(x)g(u)$ represents a so-called *logistic-type* growth condition. The coefficient functions $a(x)$ and $b(x)$ depend on the spatial variable, indicating variable linear growth and competition rates in the environment. The function $h(x)$ is interpreted as the harvesting distribution, and $ch(x)$ is the harvesting rate. Such equations have been used to model fishery or hunting management problems, see [17] for further historical background and references. Intuitively, survival of a species, i.e. existence of a positive solution, is only expected for small values of c. With the presence of the harvesting term, the right-hand side of the equation is negative at $u = 0$, and therefore the problem belongs to the class of so-called *semi-positone* problems (see [8]). This prevents a direct application of the maximum principle.

The main inspiration for our work here is [14]. There the authors consider problem (1.1) in $\mathbb{R}^N$, as well as bounded domains $U \subset \mathbb{R}^N$ for $N \geq 3$. The approach presented in [14] is to relate the equation (1.1) to an auxiliary equation of the form

$$\begin{cases} -\Delta u = \lambda a(x) u \left[1 - k\left(\dfrac{u}{\ell d(x)}\right)\right] - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where the term $b(x)g(u)$ is replaced by $a(x)$ multiplied by a power of u (see Section 2.2).

We have also benefited from the works of [5] and [9], in which the authors employ a variational approach, again working in the whole space $\mathbb{R}^N$, $n \geq 3$. Equation (1.1) was also considered by Du and Ma in [10] and [11] for $g(u) = u^p$ in the absence of the harvesting term.

The main reason for our choice of the exterior domain Ω is to make use of results related to the Kelvin transform presented in [4] for the case $N = 2$. There the authors show that there is a one-to-one correspondence between solutions of our problem in the exterior domain Ω and the solutions of the associated transformed problem in the bounded domain B. We will extend this idea to the case of $N \geq 3$ in Section 1.3 below.

In order to prove a relevant uniqueness result for the solution we obtain, we make use of the work [19], where under certain stronger assumptions, the authors prove that equation (1.1) (with $g(u) = u^2$ and in a bounded domain U) has a unique positive solution in the absence of harvesting, i.e. when $c = 0$.

1.1 Outline

In the remainder of Chapter 1, we present some background material which will be used throughout our work here and also extend some of the results of [4] to the case of $N \geq 3$. We show the relationship between solutions of exterior domain problems such as (1.1) and a transformed problem in the bounded domain B for $N \geq 3$ in Lemma 1.10. We additionally show that the Kelvin transform is a still an order-preserving, isometric isomorphism between $H_0^1(B)$ and $\mathcal{D}^{1,2}(\Omega)$ in this case (cf. Theorem 1.11). Finally, we derive the Green's functions for B and Ω for both $N = 2$ and $N \geq 3$ in Section 1.4 for later use.

In Chapter 2, we show that the technique applied in [14] to show the existence of positive solutions to (1.1) in the whole space $\mathbb{R}^N$, $N \geq 3$ can be adapted to our case to show existence of positive solutions in the exterior domain Ω (still in the case of $N \geq 3$). This method will be referred to as the direct approach. Using the assumptions presented in Section 2.1, we first relate (1.1) to the auxiliary problem (1.2), and find a positive weak solution to this auxiliary problem via a basic minimization argument. Then, on a set bounded below by the weak solution to (1.2), we minimize the corresponding functional to (1.1) to obtain what we show is a positive weak solution to (1.1). This result is summarized in Theorem 2.7.

In Chapter 3, we show the existence of positive solutions to (1.1) for $N \geq 3$ using what we call the Kelvin transform approach. In this approach, we first transform (1.1) to the unit ball B, making use of Lemma 1.10. Then, using the technique outlined in [14] for bounded domains (which requires slightly different assumptions on the data), we find a positive weak solution to the transformed equation in the ball. By Lemma 1.10, the Kelvin transform of this solution is then a positive weak solution to (1.1). This result is summarized in Theorem 3.6.

The dual nature of our approaches to solve (1.1) for the case of $N \geq 3$ raises the question of whether the two solutions obtained through the direct approach of Chapter 2 and the Kelvin transform approach of Chapter 3 are in fact the same. To answer this question, in Chapter 4, we define the notion of a stable solution and show, using existing results (cf. [10, 11, 19, 20]), that the transformed equation in the ball has a unique positive stable solution under some slightly stronger assumptions on the data. We then show that the Kelvin transforms of both the solutions obtained in Chapters 2 and 3 are indeed stable in the ball. In this way, if we have a set of assumptions on the data satisfying the requirements for the direct approach as well as the Kelvin approach simultaneously, then the two solutions coincide. This result is summarized in Theorem 4.11.

In a short Chapter 5, we consider the same equation (1.1) now in the case of $N = 2$. Although the direct approach is not suitable in $\mathbb{R}^2$, we show that the Kelvin transform approach can in fact be applied with minimal changes. The existence of a positive solution is summarized in Theorem 5.2. We then prove stability in the same way as in Chapter 4, showing under some slightly stronger assumption on the data that the Kelvin transform of the solution given in Theorem 5.2 is the unique positive stable solution of the transformed equation in the ball.

Finally, in Chapter 6, we give some concluding remarks and possible opportunities for future work in this area. Specifically, some further applications of the Kelvin transform approach are discussed, as well as other types of equations and some possible difficulties in dealing with those equations.

1.2 Notation and Basic Results

We start by defining the spaces we will be working in throughout this thesis, in the whole space $\mathbb{R}^N$, the unit ball $B := B(0,1)$ and the exterior domain $\Omega := \mathbb{R}^N \setminus \overline{B}$. Let $C_0^\infty(U)$ denote the set of all infinitely differentiable functions with compact support in the domain U, for any $U \subseteq \mathbb{R}^N$. For any $1 \le p < \infty$, we use the notation $W^{k,p}(U)$ to indicate the usual Sobolev space, i.e.

$$W^{k,p}(U) := \{u \in L^p(U) : D^\alpha u \in L^p(U) \text{ for all } |\alpha| \le k\}$$

where α is a multiindex, and the derivative $D^\alpha u$ is understood weakly. We additionally define $W_0^{k,p}(U)$ as the closure of $C_0^\infty(U)$ in $W^{k,p}(U)$. In the case $p = 2$, we use the notation $H^k(U)$ (or $H_0^k(U)$) instead. $\mathcal{D}^{1,p}(U)$ is the Beppo-Levi space, defined as the completion of $C_0^\infty(U)$ with respect to the norm

$$\|u\|^p_{\mathcal{D}^{1,p}(U)} = \int_U |\nabla u|^p \,\mathrm{d}x.$$

Some key inequalities related to Sobolev embeddings and basic results in the theory of elliptic partial differential equations are also recorded here for later reference. The first of these is the so-called Hardy Inequality. Proofs of the following results may be found in the listed references.

Lemma 1.1 (Hardy Inequality (cf. [2, 15])). Let $N \ge 3$, and assume $1 < p < N$. Then, if

$u \in W^{1,p}(\mathbb{R}^N)$,

(i) $\dfrac{u}{|x|} \in L^p(\mathbb{R}^N)$,

(ii) $\displaystyle\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\,\mathrm{d}x \leq \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x$, and

(iii) The above constant is optimal.

In this thesis, we will be particularly interested in the case of $p = 2$, for which we have

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,\mathrm{d}x \leq \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x$$

for all $u \in H^1(\mathbb{R}^N)$.

Since we will often be working in the bounded domain B, we also use a version of the Hardy inequality shown to hold for arbitary convex domains in $\mathbb{R}^N$, $N \geq 1$ (cf. [16]):

Lemma 1.2 (Hardy Inequality for Bounded Domains (cf. [3, 16])). Let U be a convex domain in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary. Assume $1 < p < \infty$, then if $u \in W_0^{1,p}(U)$,

(i) $\dfrac{u}{\operatorname{dist}(x, \partial U)} \in L^p(U)$, and

(ii) $\displaystyle\int_U \frac{|u|^p}{|\operatorname{dist}(x, \partial U)|^p}\,\mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_U |\nabla u|^p\,\mathrm{d}x$, and

(iii) This constant is optimal.

Again in the case of $p = 2$, if $U = B \subset \mathbb{R}^N, N \geq 2$, then

$$\int_B \frac{|u|^2}{(1-|x|)^2}\,\mathrm{d}x \leq 4 \int_B |\nabla u|^2\,\mathrm{d}x$$

for all $u \in H_0^1(B)$.

We also have the following result for exterior domains in $\mathbb{R}^N$ where now $N \geq 2$:

Lemma 1.3 (Hardy-Type Inequalities for Exterior Domains (cf. [4, 5, 21])). Let $N \geq 2$, $R > 0$ and $G := \mathbb{R}^N \setminus B(0, 2R)$. Then for $u \in \mathcal{D}^{1,p}(G)$,

(i)
$$\int_G \frac{|u|^p}{|x|^p}\,\mathrm{d}x \le \left(\frac{p}{|N-p|}\right)^p \int_\Omega |\nabla u|^p\,\mathrm{d}x, \qquad \text{for } 1<p<\infty,\ p\neq N \tag{1.3}$$

(ii)
$$\int_G \frac{|u|^N}{|x|^N\left(\ln\frac{|x|}{R}\right)^N}\,\mathrm{d}x \le \left(\frac{N}{N-1}\right)^N \int_G |\nabla u|^N\,\mathrm{d}x. \tag{1.4}$$

For our exterior domain $\Omega\subset\mathbb{R}^2$, we have that $R=\frac{1}{2}$ and so (1.4) can be written as

$$\int_\Omega \frac{|u|^2}{|x|^2(\ln 2|x|)^2}\,\mathrm{d}x \le 4\int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{1.5}$$

Next, we take a quick look at some key results from the theory of elliptic partial differential equations (cf. [12]) to get a maximum principle related to the positive solutions of problems such as (1.1). The first result is the Hopf lemma:

Lemma 1.4 (Hopf Lemma (cf. [12]))**.** Let U be a bounded domain in $\mathbb{R}^N$ and L be an elliptic operator of the form

$$Lu = -\sum_{i,j=1}^N a^{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^N b^i\frac{\partial u}{\partial x_i} + c(x)u, \quad x\in U. \tag{1.6}$$

Assume $u\in C^2(U)\cap C^1(\overline{U})$ and $c\equiv 0$ in U. Suppose further that $Lu\le 0$ in U and there exists a point $x_0\in\partial U$ such that

$$u(x_0)>u(x) \quad \text{ for all } x\in U.$$

Assume finally that there exists an open ball $B_{x_0}\subset U$ with $x_0\in\partial B_{x_0}$. Then:

(i) $\dfrac{\partial u}{\partial\nu}(x_0)>0$, where ν is the outward unit normal to B_{x_0} at x_0, and

(ii) If $c\ge 0$ in U, the same conclusion holds provided $u(x_0)\ge 0$.

The Hopf lemma is a key component of the proof of the so-called strong maximum principle:

Lemma 1.5 (Strong Maximum Principle (cf. [12]))**.** Let L be as in (1.6) and suppose that U is a connected, open and bounded domain in $\mathbb{R}^N$. Assume $u\in C^2(U)\cap C(\overline{U})$ and $c\equiv 0$ in U. Then:

(i) If $Lu \leq 0$ in U and u attains its maximum over $\overline{U}$ at an interior point, then u is constant within U.

(ii) Similarly, if $Lu \geq 0$ in U and u attains its minimum over $\overline{U}$ at an interior point, then u is constant within U.

1.3 Kelvin Transform in $\mathbb{R}^N$

The following results extend those from Carl et al. (cf. [4]) to the case $N \geq 3$. Let $B \subset \mathbb{R}^N$, $N \geq 3$. The mapping $x \mapsto \dfrac{x}{|x|^2} =: \hat{x}$ is the inversion through the sphere ∂B, which provides a bijection from $\mathbb{R}^N \setminus B$ onto $\overline{B} \setminus \{0\}$, and vice versa, since $\hat{\hat{x}} = x$. The Kelvin transform is based on the inversion mapping and defined as follows:

Definition 1.6. Let $u : \overline{B} \setminus \{0\} \to \mathbb{R}$. The Kelvin transform of u, denoted by $(Ku)(x) = \hat{u}(x)$, is defined by

$$(Ku)(x) = \frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big).$$

We now state (with proofs provided for completeness) some calculus results related to the inversion mapping.

Lemma 1.7. Let $\hat{x}(x) = \dfrac{x}{|x|^2}$ be the inversion mapping. Then the Fréchet derivative $x \mapsto D\hat{x}(x)$ is given by

$$D\hat{x}(x) = \frac{1}{|x|^2} I - \frac{2}{|x|^4} T, \tag{1.7}$$

where I is the $N \times N$ identity matrix, T is the matrix

$$T = xx^t = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & \cdots & x_1x_N \\ x_2x_1 & x_2^2 & x_2x_3 & \cdots & x_2x_N \\ x_3x_1 & x_3x_2 & x_3^2 & \cdots & x_3x_N \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ x_Nx_1 & x_Nx_2 & x_Nx_3 & \cdots & x_N^2 \end{bmatrix}, \tag{1.8}$$

and the absolute value of the determinant of $D\hat{x}$, i.e. $|\det(D\hat{x}(x))|$, is equal to

$$|\det(D\hat{x}(x))| = \frac{1}{|x|^{2N}}. \tag{1.9}$$

Moreover, for any $\xi, \eta \in \mathbb{R}^N$, we get

$$\langle D\hat{x}(x)\xi, D\hat{x}(x)\eta\rangle = \frac{1}{|x|^4}\langle \xi, \eta\rangle, \tag{1.10}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$.

Proof. Since the Fréchet derivative $D\hat{x}(x)$ coincides with the Jacobian matrix, we need only compute the partial derivatives of

$$\hat{x}(x) = \frac{x}{|x|^2} = \begin{bmatrix} \dfrac{x_1}{x_1^2 + x_2^2 + \cdots + x_N^2} & \dfrac{x_2}{x_1^2 + x_2^2 + \cdots + x_N^2} & \cdots & \dfrac{x_N}{x_1^2 + x_2^2 + \cdots + x_N^2} \end{bmatrix}.$$

When $i \neq j$, we have that

$$\frac{\partial f_i}{\partial x_j} = \frac{(x_1^2 + x_2^2 + \cdots + x_N^2) \cdot 0 - x_i(2x_j)}{(x_1^2 + x_2^2 + \cdots + x_n^2)^2} = -\frac{2x_i x_j}{|x|^4}.$$

Similarly, when $i = j$, we have

$$\frac{\partial f_i}{\partial x_i} = \frac{(x_1^2 + x_2^2 + \cdots + x_N^2) \cdot 1 - x_i(2x_i)}{(x_1^2 + x_2^2 + \cdots + x_N^2)^2} = \frac{|x|^2 - 2x_i^2}{|x|^4} = \frac{1}{|x|^2} - \frac{2x_i^2}{|x|^4}.$$

Therefore

$$\big[D\hat{x}(x)\big]_{ij} = \frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4}, \quad \text{and so} \quad D\hat{x}(x) = \frac{1}{|x|^2}I - \frac{2}{|x|^4}T.$$

Note that $D\hat{x}(x)$ is a symmetric matrix, and so

$$D\hat{x}(x)D\hat{x}(x) = \frac{1}{|x|^4}I - \frac{4}{|x|^6}T + \frac{4}{|x|^8}T^2 = \frac{1}{|x|^4}I,$$

since $T^2 = |x|^2 T$. Therefore we have that

$$|\det(D\hat{x}(x))| = \sqrt{\det(D\hat{x}(x)D\hat{x}(x))} = \sqrt{\det\left(\frac{1}{|x|^4} I\right)} = \sqrt{\frac{1}{|x|^{4N}}} = \frac{1}{|x|^{2N}}.$$

Finally,

$$\langle D\hat{x}(x)\xi, D\hat{x}(x)\eta\rangle = \langle D\hat{x}(x)D\hat{x}(x)\xi, \eta\rangle = \frac{1}{|x|^4}\langle \xi, \eta\rangle.$$

□

Equation (1.9) will be used extensively in the change-of-variables between the exterior domain Ω and the ball B (or vice versa). In [4], for the case $N = 2$, there is shown to be a nice relationship between $|\nabla\varphi|$ and $|\nabla\hat{\varphi}|$ for $\varphi \in C_0^\infty(B)$. However, the relationship is not as straightforward for the case of $N \geq 3$:

Lemma 1.8. Let $\varphi \in C_0^\infty(B)$, $N \geq 2$, and let $\hat{\varphi}(x) = \dfrac{1}{|x|^{N-2}}\varphi\Big(\dfrac{x}{|x|^2}\Big)$ be its Kelvin transform. Then the gradient of $\hat{\varphi}(x)$ can be calculated by

$$\nabla\hat{\varphi}(x) = \frac{1}{|x|^N}\nabla\varphi\Big(\frac{x}{|x|^2}\Big) - \frac{2}{|x|^{N+2}}\Big\langle \nabla\varphi\Big(\frac{x}{|x|^2}\Big), x\Big\rangle x - \frac{(N-2)}{|x|^N}\varphi\Big(\frac{x}{|x|^2}\Big)x, \qquad \text{for all } x \in \Omega. \quad (1.11)$$

In particular,

$$|\nabla\hat{\varphi}(x)| = \frac{1}{|x|^N}\left|\nabla\varphi\Big(\frac{x}{|x|^2}\Big) + (N-2)\varphi\Big(\frac{x}{|x|^2}\Big)x\right|. \quad (1.12)$$

Proof. Note that $\hat{\varphi} \in C^\infty(\Omega)$ and that $\hat{\varphi} = 0$ in a neighborhood of ∂B. Then, applying the product rule for the gradient operator as well as the chain rule, we get

$$\begin{aligned}\nabla\hat{\varphi}(x) = \nabla\Big(\frac{1}{|x|^{N-2}}\varphi(\hat{x}(x))\Big) &= \frac{1}{|x|^{N-2}}\nabla\varphi(\hat{x}(x))D\hat{x}(x) - \frac{(N-2)}{|x|^N}\varphi(\hat{x}(x))x \\ &= \frac{1}{|x|^{N-2}}\nabla\varphi(\hat{x}(x))\Big(\frac{1}{|x|^2}I - \frac{2}{|x|^4}T\Big) - \frac{(N-2)}{|x|^N}\varphi(\hat{x}(x))x.\end{aligned}$$

Note that

$$\nabla\varphi(\hat{x}(x))T = \begin{bmatrix} \frac{\partial\varphi(\hat{x}(x))}{\partial x_1} & \frac{\partial\varphi(\hat{x}(x))}{\partial x_2} & \cdots & \frac{\partial\varphi(\hat{x}(x))}{\partial x_N} \end{bmatrix} \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & \cdots & x_1x_N \\ x_2x_1 & x_2^2 & x_2x_3 & \cdots & x_2x_N \\ x_3x_1 & x_3x_2 & x_3^2 & \cdots & x_3x_N \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ x_Nx_1 & x_Nx_2 & x_Nx_3 & \cdots & x_N^2 \end{bmatrix}$$

$$= \begin{bmatrix} \varphi_1 x_1^2 + \cdots + \varphi_N x_N x_1 & \cdots & \varphi_1 x_1 x_N + \cdots + \varphi_N x_N^2 \end{bmatrix}$$

$$= \Big(\varphi_1 x_1 + \varphi_2 x_2 + \cdots + \varphi_N x_N\Big) \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix} = \langle \nabla\varphi(\hat{x}(x)), x\rangle x,$$

where $\varphi_i = \dfrac{\partial}{\partial x_i}\varphi(\hat{x}(x))$, and so

$$\nabla\hat{\varphi}(x) = \frac{1}{|x|^N}\nabla\varphi(\hat{x}(x)) - \frac{2}{|x|^{N+2}}\langle\nabla\varphi(\hat{x}(x)), x\rangle x - \frac{(n-2)}{|x|^N}\varphi(\hat{x}(x))x.$$

Therefore

$$\begin{aligned}
|\nabla\hat{\varphi}(x)|^2 &= \langle\nabla\hat{\varphi}(x), \nabla\hat{\varphi}(x)\rangle \\
&= \Big\langle \frac{1}{|x|^N}\nabla\varphi(\hat{x}(x)) - \frac{2}{|x|^{N+2}}\langle\nabla\varphi(\hat{x}(x)), x\rangle x - \frac{(N-2)}{|x|^N}\varphi(\hat{x}(x))x, \\
&\qquad\qquad \frac{1}{|x|^N}\nabla\varphi(\hat{x}(x)) - \frac{2}{|x|^{N+2}}\langle\nabla\varphi(\hat{x}(x)), x\rangle x - \frac{(N-2)}{|x|^N}\varphi(\hat{x}(x))x \Big\rangle \\
&= \frac{1}{|x|^{2N}}\langle\nabla\varphi(\hat{x}(x)), \nabla\varphi(\hat{x}(x))\rangle - \frac{4}{|x|^{2N+2}}\langle\nabla\varphi(\hat{x}(x)), x\rangle^2 \\
&\quad + \frac{4}{|x|^{2N+4}}\langle\nabla\varphi(\hat{x}(x)), x\rangle^2\langle x, x\rangle - \frac{2(N-2)}{|x|^{2N}}\langle\nabla\varphi(\hat{x}(x)), \varphi(\hat{x}(x))x\rangle \\
&\quad + \frac{4(N-2)}{|x|^{2N+2}}\langle\nabla\varphi(\hat{x}(x)), x\rangle\langle\varphi(\hat{x}(x))x, x\rangle + \frac{(N-2)^2}{|x|^{2N}}\langle\varphi(\hat{x}(x))x, \varphi(\hat{x}(x))x\rangle \\
&= \frac{1}{|x|^{2N}}\langle\nabla\varphi(\hat{x}(x)), \nabla\varphi(\hat{x}(x))\rangle + \frac{2N-4}{|x|^{2N}}\langle\nabla\varphi(\hat{x}(x)), \varphi(\hat{x}(x))x\rangle \\
&\quad + \frac{(N-2)^2}{|x|^{2N-2}}\langle\varphi(\hat{x}(x))x, \varphi(\hat{x}(x))x\rangle \\
&= \frac{1}{|x|^{2N}}\langle\nabla\varphi(\hat{x}(x)) + (N-2)\varphi(\hat{x}(x))x, \nabla\varphi(\hat{x}(x)) + (N-2)\varphi(\hat{x}(x))x\rangle
\end{aligned}$$

$$= \frac{1}{|x|^{2N}} \left| \nabla\varphi(\hat{x}(x)) + (N-2)\varphi(\hat{x}(x))x \right|^2,$$

and so (1.12) follows. $\square$

Remark 1.9. As $\hat{\hat{\varphi}} = \varphi \in C_0^\infty(B)$, we also have:

$$\nabla\varphi(x) = \frac{1}{|x|^N} \nabla\hat{\varphi}\Big(\frac{x}{|x|^2}\Big) - \frac{2}{|x|^{N+2}} \Big\langle \nabla\hat{\varphi}\Big(\frac{x}{|x|^2}\Big), x \Big\rangle x - \frac{(N-2)}{|x|^N} \hat{\varphi}\Big(\frac{x}{|x|^2}\Big) x,$$

for all $x \in B \setminus \{0\}$.

Note that for $N = 2$, (1.12) simplifies to

$$|\nabla\hat{\varphi}(x)| = \frac{1}{|x|^2} \left| \nabla\varphi\Big(\frac{x}{|x|^2}\Big) \right|. \tag{1.13}$$

This is used in [4] to prove that the Kelvin transform is an order-preserving isometric isomorphism between the spaces $H_0^1(B)$ and $\mathcal{D}^{1,2}(\Omega)$. However, for $N \geq 3$, to deal with the term involving $N-2$ in (1.12), we need the following well-known result (whose proof is also provided here for completeness of the presentation).

Lemma 1.10. Let $v(x) = \frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big)$ be the Kelvin transform of u. Then

$$\Delta v(x) = \frac{1}{|x|^{N+2}} (\Delta u)\Big(\frac{x}{|x|^2}\Big).$$

Proof. From the product rule for the Laplacian operator, i.e.

$$\Delta(fg) = \Delta f \cdot g + 2\langle \nabla f, \nabla g \rangle + f \cdot \Delta g,$$

we have

$$\begin{aligned}
\Delta v(x) &= \Delta\Big(\frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big) \Big) \\
&= \Delta\Big(\frac{1}{|x|^{N-2}} \Big) u\Big(\frac{x}{|x|^2}\Big) + 2\Big\langle \nabla\Big(\frac{1}{|x|^{N-2}} \Big), \nabla\Big(u\Big(\frac{x}{|x|^2}\Big) \Big) \Big\rangle + \frac{1}{|x|^{N-2}} \Delta\Big(u\Big(\frac{x}{|x|^2}\Big) \Big).
\end{aligned}$$

Now

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x|^{N-2}}\right) = \frac{(2-N)x_i}{|x|^N}$$

and so

$$\frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|x|^{N-2}}\right) = \frac{(2-N)}{|x|^N} - \frac{(2N-N^2)x_i^2}{|x|^{N+2}},$$

Summing over all $1 \le i \le N$ gives

$$\Delta\left(\frac{1}{|x|^{N-2}}\right) = \sum_{i=1}^{N} \frac{(2-N)}{|x|^N} - \frac{(2N-N^2)x_i^2}{|x|^{N+2}} = \frac{N(2-N)}{|x|^N} - \frac{(2N-N^2)|x|^2}{|x|^{N+2}} = 0. \tag{1.14}$$

Next, by the same calculation as in Lemma 1.8, we have

$$\nabla\left(u\left(\frac{x}{|x|^2}\right)\right) = \nabla u\left(\frac{x}{|x|^2}\right)\left(\frac{1}{|x|^2}I - \frac{2}{|x|^4}T\right) = \frac{1}{|x|^2}\nabla u\left(\frac{x}{|x|^2}\right) - \frac{2x}{|x|^4}\left\langle \nabla u\left(\frac{x}{|x|^2}\right), x\right\rangle,$$

and so

$$\begin{aligned}
\left\langle \nabla\left(\frac{1}{|x|^{N-2}}\right), \nabla\left(u\left(\frac{x}{|x|^2}\right)\right)\right\rangle &= \left\langle \frac{(2-N)}{|x|^N}x, \frac{1}{|x|^2}\nabla u\left(\frac{x}{|x|^2}\right) - \frac{2x}{|x|^4}\left\langle \nabla u\left(\frac{x}{|x|^2}\right), x\right\rangle\right\rangle \\
&= \frac{(2-N)}{|x|^{N+2}}\left\langle x, \nabla u\left(\frac{x}{|x|^2}\right)\right\rangle - \frac{2(2-N)}{|x|^{N+4}}\left\langle \nabla u\left(\frac{x}{|x|^2}\right), x\right\rangle\langle x, x\rangle \\
&= -\frac{(2-N)}{|x|^{N+2}}\left\langle x, \nabla u\left(\frac{x}{|x|^2}\right)\right\rangle. \qquad (1.15)
\end{aligned}$$

Next, we must compute $\Delta\left(u\left(\frac{x}{|x|^2}\right)\right)$. Using the formula for the Laplacian of a composition, i.e.

$$\Delta(f \circ g) = \nabla f \cdot \Delta g + \operatorname{trace}\Big[\nabla g^t \cdot \Delta f \cdot \nabla g\Big],$$

we have, using the notation of Lemma 1.8,

$$\begin{aligned}
\Delta\left(u\left(\frac{x}{|x|^2}\right)\right) &= \nabla u\left(\frac{x}{|x|^2}\right) \cdot \Delta\left(\frac{x}{|x|^2}\right) + \operatorname{trace}\left[\nabla\left(\frac{x}{|x|^2}\right)^t \Delta u\left(\frac{x}{|x|^2}\right) \cdot \nabla\left(\frac{x}{|x|^2}\right)\right] \\
&= \nabla u\left(\frac{x}{|x|^2}\right) \cdot D(D\hat{x}(x)) + D\hat{x}(x)D\hat{x}(x)\Delta u\left(\frac{x}{|x|^2}\right) \\
&= \nabla u\left(\frac{x}{|x|^2}\right) \cdot D(D\hat{x}(x)) + \frac{1}{|x|^4}\Delta u\left(\frac{x}{|x|^2}\right).
\end{aligned}$$

Now,

$$\frac{\partial}{\partial x_k}\big[D\hat{x}(x)\big]_{ij} = \frac{\partial}{\partial x_k}\bigg(\frac{\delta_{ij}}{|x|^2} - \frac{2x_ix_j}{|x|^4}\bigg) = \frac{-2(\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i)}{|x|^4} + \frac{8x_ix_jx_k}{|x|^6},$$

and so

$$\begin{aligned}\big[D(D\hat{x}(x))\big]_i &= \sum_{j=1}^{N}\frac{\partial}{\partial x_j}\big[D\hat{x}(x)\big]_{ij} = \sum_{j=1}^{N}\frac{-4\delta_{ij}x_j - 2x_i}{|x|^4} + \frac{8x_ix_j^2}{|x|^6}\\ &= \frac{-4x_i}{|x|^4} - \frac{2nx_i}{|x|^4} + \frac{8x_i|x|^2}{|x|^6} = \frac{2(2-N)x_i}{|x|^4}.\end{aligned}$$

Therefore

$$\Delta\bigg(u\Big(\frac{x}{|x|^2}\Big)\bigg) = \bigg\langle \nabla u\Big(\frac{x}{|x|^2}\Big), \frac{2(2-n)x}{|x|^4}\bigg\rangle + \frac{1}{|x|^4}\Delta u\Big(\frac{x}{|x|^2}\Big). \tag{1.16}$$

We then have

$$\begin{aligned}\Delta v(x) = 0 &- \frac{2(2-N)}{|x|^{N+2}}\bigg\langle x, \nabla u\Big(\frac{x}{|x|^2}\Big)\bigg\rangle\\ &+ \frac{1}{|x|^{N-2}}\Bigg(\bigg\langle \nabla u\Big(\frac{x}{|x|^2}\Big), \frac{2(2-N)x}{|x|^4}\bigg\rangle + \frac{1}{|x|^4}\Delta u\Big(\frac{x}{|x|^2}\Big)\Bigg)\\ = &-\frac{2(2-N)}{|x|^{N+2}}\bigg\langle x, \nabla u\Big(\frac{x}{|x|^2}\Big)\bigg\rangle + \frac{2(2-N)}{|x|^{N+2}}\bigg\langle \nabla u\Big(\frac{x}{|x|^2}\Big), x\bigg\rangle + \frac{1}{|x|^{N+2}}\Delta u\Big(\frac{x}{|x|^2}\Big)\\ = &\frac{1}{|x|^{N+2}}(\Delta u)\Big(\frac{x}{|x|^2}\Big),\end{aligned}$$

as desired. □

Finally, we are ready to generalize the result in [4] and prove that for $N \geq 3$ the Kelvin transform does define an isometric isomorphism between the spaces $H_0^1(B)$ and $\mathcal{D}^{1,2}(\Omega)$. Explicitly:

Theorem 1.11. The Kelvin transform K defined by

$$\hat{u}(x) = (Ku)(x) = \frac{1}{|x|^{N-2}}u\Big(\frac{x}{|x|^2}\Big)$$

provides an order-preserving, isometric isomorphism from $H_0^1(B)$ to $\mathcal{D}^{1,2}(\Omega)$, and $K^{-1} = K$.

Proof. Suppose $\varphi, \psi \in C_0^\infty(B)$. Then their respective Kelvin transforms $\hat{\varphi}(x), \hat{\psi}(x)$ belong to $C^\infty(\Omega)$ with $\hat{\varphi} = \hat{\psi} = 0$ in a neighborhood of ∂B. Let $B_R = B(0, R)$, $R > 1$. We then have

$$\int_{\Omega \cap B_R} |\hat{\varphi}(y)|^2 \,\mathrm{d}y = \int_{\Omega \cap B_R} \left| \frac{1}{|y|^{N-2}} \varphi\Big(\frac{y}{|y|^2}\Big) \right|^2 \mathrm{d}y = \int_{B \cap \{|x| > \frac{1}{R}\}} |x|^{N-2} |\varphi(x)|^2 \frac{1}{|x|^{2N}} \,\mathrm{d}x$$
$$= \int_{B \cap \{|x| > \frac{1}{R}\}} |\varphi(x)|^2 \frac{1}{|x|^{N+2}} \,\mathrm{d}x \le R^{N+2} \int_B |\varphi(x)|^2 \,\mathrm{d}x, \tag{1.17}$$

Next, using Green's first identity, we have that

$$\int_B \nabla\varphi(x) \nabla\psi(x) \,\mathrm{d}x = -\int_B \big(\Delta\varphi(x)\big)\psi(x) \,\mathrm{d}x.$$

Making the change of variables $x \mapsto \dfrac{y}{|y|^2}$, we then have

$$\int_B \nabla\varphi(x)\nabla\psi(x) \,\mathrm{d}x = -\int_\Omega (\Delta\varphi)\Big(\frac{y}{|y|^2}\Big)\psi\Big(\frac{y}{|y|^2}\Big)\frac{1}{|y|^{2N}} \,\mathrm{d}y$$
$$= -\int_\Omega \frac{1}{|y|^{N+2}}(\Delta\varphi)\Big(\frac{y}{|y|^2}\Big)\frac{1}{|y|^{N-2}}\psi\Big(\frac{y}{|y|^2}\Big) \,\mathrm{d}y = -\int_\Omega \big(\Delta\hat{\varphi}(y)\big)\hat{\psi}(y) \,\mathrm{d}y.$$

Here we use Lemma 1.10 and the definition of the Kelvin transform. Now, however, we cannot directly apply Green's first identity as Ω is an unbounded domain. Therefore we write:

$$-\int_{|y|>1} \big(\Delta\hat{\varphi}(y)\big)\hat{\psi}(y) \,\mathrm{d}y = \lim_{R\to\infty} -\int_{1<|y|<R} \big(\Delta\hat{\varphi}(y)\big)\hat{\psi}(y) \,\mathrm{d}y$$
$$= \lim_{R\to\infty} \int_{1<|y|<R} \nabla\hat{\varphi}(y)\nabla\hat{\psi}(y) \,\mathrm{d}y - \int_{|y|=R} \frac{\partial\hat{\varphi}}{\partial\nu}(y)\hat{\psi}(y) \,\mathrm{dS}(y). \tag{1.18}$$

Here $\partial\hat{\varphi}/\partial\nu$ is the normal derivative of $\hat{\varphi}$ in the direction of the outward normal at $|y| = R$ and $\mathrm{S}(y)$ is the surface element. We now show that the second integral vanishes as $R \to \infty$.

First we transition to spherical coordinates and let $r = |x|$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_{N-1})$ be the angular components of the point x. Then for $\varphi(x) = \varphi(r, \omega)$ and $\psi(x) = \psi(r, \omega)$, we have

$$\hat{\varphi}(x) = \frac{1}{|x|^{N-2}}\varphi\Big(\frac{x}{|x|^2}\Big) = r^{2-N}\varphi\Big(\frac{1}{r}, \omega\Big) \quad \text{and} \quad \hat{\psi}(x) = r^{2-N}\psi\Big(\frac{1}{r}, \omega\Big),$$

and so

$$\begin{aligned}\frac{\partial\hat{\varphi}}{\partial\nu}(x) = \frac{\partial}{\partial r}\hat{\varphi}(r,\omega) = \frac{\partial}{\partial r}\Big[r^{2-N}\varphi\Big(\frac{1}{r},\omega\Big)\Big] \\ = (2-N)r^{1-N}\varphi\Big(\frac{1}{r},\omega\Big) + r^{2-N}\frac{\partial\varphi}{\partial r}\Big(\frac{1}{r},\omega\Big)\Big(-\frac{1}{r^2}\Big).\end{aligned}$$

Therefore, the second integral in (1.18) can be written as

$$\begin{aligned}&\int_{|\omega|=1}\Big[(2-N)R^{1-N}\varphi\Big(\frac{1}{R},\omega\Big) + R^{2-N}\frac{\partial\varphi}{\partial r}\Big(\frac{1}{R},\omega\Big)\Big(-\frac{1}{R^2}\Big)\Big]R^{2-N}\psi\Big(\frac{1}{R},\omega\Big)R^{N-1}\,\mathrm{d}\omega \\ &\quad = (2-N)R^{2-N}\int_{|\omega|=1}\varphi\Big(\frac{1}{R},\omega\Big)\psi\Big(\frac{1}{R},\omega\Big)\,\mathrm{d}\omega - R^{1-N}\int_{|\omega|=1}\frac{\partial\varphi}{\partial r}\Big(\frac{1}{R},\omega\Big)\psi\Big(\frac{1}{R},\omega\Big)\,\mathrm{d}\omega.\end{aligned}$$

Now, since $\varphi, \psi \in C_0^\infty(B)$, we have that, as $R \to \infty$, $\varphi\Big(\frac{1}{R},\omega\Big)$, $\psi\Big(\frac{1}{R},\omega\Big)$, and $\frac{\partial\varphi}{\partial r}\Big(\frac{1}{R},\omega\Big)$ all remain bounded, and so as $R \to \infty$, both of those integrals will go to zero. Explicitly,

$$\begin{aligned}&\lim_{R\to\infty}\int_{1<|y|<R}\nabla\hat{\varphi}(y)\nabla\hat{\psi}(y)\,\mathrm{d}y - \int_{|y|=R}\frac{\partial\hat{\varphi}}{\partial\nu}(y)\hat{\varphi}(y)\,\mathrm{dS}(y) \\ &\quad = \lim_{R\to\infty}\int_{1<|y|<R}\nabla\hat{\varphi}(y)\nabla\hat{\psi}(y)\,\mathrm{d}y - (2-N)R^{2-N}\int_{|\omega|=1}\varphi\Big(\frac{1}{R},\omega\Big)\psi\Big(\frac{1}{R},\omega\Big)\,\mathrm{d}\omega \\ &\qquad + R^{1-N}\int_{|\omega|=1}\frac{\partial\varphi}{\partial r}\Big(\frac{1}{R},\omega\Big)\varphi\Big(\frac{1}{R},\omega\Big)\,\mathrm{d}\omega \\ &\quad = \int_{|y|>1}\nabla\hat{\varphi}(y)\nabla\hat{\psi}(y)\,\mathrm{d}y.\end{aligned}$$

Therefore

$$\int_B \nabla\varphi(x)\nabla\psi(x)\,\mathrm{d}x = \int_\Omega \nabla\hat{\varphi}(y)\nabla\hat{\psi}(y)\,\mathrm{d}y, \tag{1.19}$$

for all $\varphi, \psi \in C_0^\infty(B)$. In particular, if $\varphi = \psi$, we have

$$\int_B |\nabla\varphi(x)|^2\,\mathrm{d}x = \int_\Omega |\nabla\hat{\varphi}(y)|^2\,\mathrm{d}y \tag{1.20}$$

for all $\varphi \in C_0^\infty(B)$. From (1.17) and (1.20), we see that $\hat{\varphi} \in X$, where X is the space

$$X = \{u : \Omega \to \mathbb{R} : u \in L^2_{\mathrm{loc}}(\Omega), \nabla u \in L^2(\Omega)\}.$$

Note that $\mathcal{D}^{1,2}(\Omega) \subsetneq X$. By Simader and Sohr (cf. [21, Theorem 2.8]), we have that for $u \in X$, $u \in \mathcal{D}^{1,2}(\Omega)$ if and only if $u \in L^{2^*}(\Omega)$, where 2^* is the Sobolev conjugate $2N/(N-2)$. We therefore claim that $\hat{\varphi} \in L^{2^*}(\Omega)$. Indeed, we have

$$\begin{aligned}\int_\Omega |\hat{\varphi}(y)|^{2N/(N-2)}\,\mathrm{d}y &= \int_\Omega \left|\frac{1}{|y|^{N-2}}\varphi\Big(\frac{y}{|y|^2}\Big)\right|^{2N/(N-2)}\mathrm{d}y = \int_\Omega \frac{1}{|y|^{2N}}\left|\varphi\Big(\frac{y}{|y|^2}\Big)\right|^{2N/(N-2)}\mathrm{d}y \\ &= \int_B |x|^{2N}|\varphi(x)|^{2N/(N-2)}\frac{1}{|x|^{2N}}\,\mathrm{d}x = \int_B |\varphi(x)|^{2N/(N-2)}\,\mathrm{d}x.\end{aligned}$$

Since $\varphi \in C_0^\infty(B) \subset H_0^1(B)$, we have by Sobolev embedding that $\varphi \in L^{2^*}(B)$, and so $\hat{\varphi} \in L^{2^*}(\Omega)$.

Next, if $\varphi, \psi \in C_0^\infty(B)$ and $\alpha, \beta \in \mathbb{R}$, we have

$$K(\alpha\varphi + \beta\psi) = \frac{1}{|x|^{N-2}}(\alpha\varphi+\beta\psi)\Big(\frac{x}{|x|^2}\Big) = \frac{\alpha}{|x|^{N-2}}\varphi\Big(\frac{x}{|x|^2}\Big) + \frac{\beta}{|x|^{N-2}}\psi\Big(\frac{x}{|x|^2}\Big) = \alpha K\varphi + \beta K\psi.$$

Thus $K : C_0^\infty(B) \to \mathcal{D}^{1,2}(\Omega)$ is a linear isometry.

K therefore has a unique extension to $H_0^1(B)$ due to the density of $C_0^\infty(B)$ in $H_0^1(B)$, which is denoted by $\tilde{K}$. Moreover, by (1.20), we get $\|K\| = \|\tilde{K}\| = 1$, which shows that the extension $\tilde{K} : H_0^1(B) \to \mathcal{D}^{1,2}(\Omega)$ is an isometric, linear operator. Next we must show that the extension allows for the same characterization as K, i.e. that the following holds true:

$$(\tilde{K}u)(x) = \hat{u}(x) = \frac{1}{|x|^{N-2}}u\Big(\frac{x}{|x|^2}\Big), \qquad \text{for all } u \in H_0^1(B). \tag{1.21}$$

The proof is by approximation. To this end let $\{\varphi_n\} \subset C_0^\infty(B)$ and $\varphi_n \to u$ in $H_0^1(B)$. Since $\{\varphi_n\}$ is a Cauchy sequence in $H_0^1(B)$, from (1.20), it follows that the sequence of Kelvin transforms $\hat{\varphi}_n = K\varphi_n = \tilde{K}\varphi_n$ is a Cauchy sequence in X, and thus

$$\hat{\varphi}_n \to v = \tilde{K}u \quad \text{in } \mathcal{D}^{1,2}(\Omega),$$

which in particular yields

$$\lim_{n\to\infty}\int_\Omega |\nabla(\hat{\varphi}_n - v)|^2\,\mathrm{d}y = 0, \quad \lim_{n\to\infty}\int_{\Omega\cap B_R} |\hat{\varphi}_n - v|^2\,\mathrm{d}y = 0 \quad \text{for all } R > 1.$$

From (1.17) with φ replaced by $\varphi_n - u$ and $\hat{\varphi}_n$ replaced by $\hat{\varphi}_n - \hat{u}$, respectively, we deduce for any $R > 1$,

$$\int_{\Omega \cap B_R} |\hat{\varphi}_n(y) - \hat{u}(y)|^2 \, \mathrm{d}y = \int_{B \cap \{|x| > \frac{1}{R}\}} |\varphi_n(x) - u(x)|^2 \frac{1}{|x|^{N+2}} \, \mathrm{d}x$$
$$\leq R^{N+2} \int_B |\varphi_n(x) - u(x)|^2 \, \mathrm{d}x,$$

which implies

$$\lim_{n \to \infty} \int_{\Omega \cap B_R} |\hat{\varphi}_n(y) - \hat{u}(y)|^2 \, \mathrm{d}y = 0 \qquad \text{for all } R > 1.$$

So, we have that $\hat{u}(y) = v(y)$ for almost every $y \in \Omega$, which proves (1.21). Therefore $\tilde{K}u = Ku = \hat{u}$. So far, we have shown that $K : H_0^1(B) \to \mathcal{D}^{1,2}(\Omega)$ is a linear, bounded, isometric and injective operator. We now show that K is surjective, i.e. $K(H_0^1(B)) = \mathcal{D}^{1,2}(\Omega)$. Let $v \in \mathcal{D}^{1,2}(\Omega)$, then there is a sequence $\psi_n \in C_0^\infty(\Omega)$ such that $\psi_n \to v$ in $\mathcal{D}^{1,2}(\Omega)$, that is,

$$\lim_{n \to \infty} \int_\Omega |\nabla(\psi_n(y) - v(y))|^2 \, \mathrm{d}y = 0 \tag{1.22}$$

Since $\hat{\psi}_n \in C_0^\infty(B)$, we have that $\{\hat{\psi}_n\}$ is a Cauchy sequence in $H_0^1(B)$, and thus $\hat{\psi}_n \to u$ in $H_0^1(B)$ for some $u \in H_0^1(B)$. From this, it follows that for the corresponding Kelvin transforms $K\hat{\psi}_n - Ku = \psi_n - \hat{u}$ we have using (1.20) that

$$\lim_{n \to \infty} \int_\Omega |\nabla(\psi_n(y) - \hat{u}(y))| \, \mathrm{d}y = 0. \tag{1.23}$$

Therefore $\hat{u} = v$ in $\mathcal{D}^{1,2}(\Omega)$. Finally, we readily verify that $K(Ku) = u$ for all $u \in H_0^1(\Omega)$, and thus $K = K^{-1}$. This completes the proof of the theorem. $\square$

Using the results we have established thus far, namely Lemma 1.7 and Theorem 1.11, we are able manipulate the Hardy inequalities for exterior domains (cf. Lemma 1.3) to introduce the following Hardy-type inequalities for the unit ball B for specific values of N and p:

Lemma 1.12 (Hardy-Type Inequalities for B)**.** Let $B = B(0,1) \subset \mathbb{R}^N$. Then for $u \in H_0^1(B)$,

(i) If $N \geq 3$, then

$$\int_B \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leq \left(\frac{2}{N-2}\right)^2 \int_B |\nabla u|^2\,\mathrm{d}x \tag{1.24}$$

(ii) If $N = 2$, then

$$\int_B \frac{|u|^2}{|x|^2\big(\ln\frac{|x|}{2}\big)^2}\,\mathrm{d}x \leq 4 \int_B |\nabla u|^2\,\mathrm{d}x. \tag{1.25}$$

Proof. Note that since $u \in H_0^1(B)$, we have $\hat{u} \in \mathcal{D}^{1,2}(\Omega)$, and so we can apply Lemma 1.3 for both cases:

(i) Considering (1.3), note that

$$\int_\Omega \frac{|\hat{u}|^2}{|x|^2}\,\mathrm{d}x = \int_B \left|\hat{u}\Big(\frac{x}{|x|^2}\Big)\right|^2 \frac{|x|^4}{|x|^2}\frac{1}{|x|^{2N}}\,\mathrm{d}x = \int_B \left|\frac{1}{|x|^{N-2}}\hat{u}\Big(\frac{x}{|x|^2}\Big)\right|^2 \frac{1}{|x|^2}\,\mathrm{d}x = \int_B \frac{|u|^2}{|x|^2}\,\mathrm{d}x.$$

Also, by (1.20), we have that $\int_\Omega |\nabla\hat{u}|^2 = \int_B |\nabla u|^2\,\mathrm{d}x$, and so (1.3) with $p = 2$ gives (1.24).

(ii) Considering (1.5), note that

$$\int_\Omega \frac{|\hat{u}|^2}{|x|^2(\ln 2|x|)^2}\,\mathrm{d}x = \int_B \left|\hat{u}\Big(\frac{x}{|x|^2}\Big)\right|^2 \frac{|x|^4}{|x|^2(\ln 2/|x|)^2}\frac{1}{|x|^4}\,\mathrm{d}x = \int_B \frac{|u|^2}{|x|^2(\ln|x|/2)^2}\,\mathrm{d}x$$

Now by the corresponding result to (1.20) in [4], we have that $\int_B |\nabla u|^2\,\mathrm{d}x = \int_\Omega |\nabla\hat{u}|^2\,\mathrm{d}x$ in the case of $N = 2$ as well, and (1.25) follows.

□

1.4 Green's Functions

We now compute the Green's functions of $\Omega = \mathbb{R}^N \setminus \overline{B}$ in the case that $N = 2$ as well as $N \geq 3$. We recall the fundamental solution $\Phi(x)$ for the Laplace equation $\Delta u = 0$:

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\ln|x|, & N = 2, \\[2ex] \dfrac{1}{N(N-2)\omega_N}\dfrac{1}{|x|^{N-2}}, & N \geq 3, \end{cases} \tag{1.26}$$

where ω_N is the surface area of the unit ball in $\mathbb{R}^N$. Recall that a function $\widetilde{G}_\Omega(x,y)$ is called a Green's function for a domain Ω if for $y \in \Omega$,

$$\begin{cases} -\Delta_x \widetilde{G}_\Omega(x,y) = \delta(x-y), & x \in \Omega, \\ \widetilde{G}_\Omega(x,y) = 0, & x \in \partial\Omega. \end{cases} \tag{1.27}$$

We know that $\widetilde{G}_\Omega(x,y) = \Phi(x-y) - \phi_y(x)$, where $\phi_y(x)$ solves the Dirichlet problem

$$\begin{cases} -\Delta_x \phi_y(x) = 0, & x \in \Omega, \\ \phi_y(x) = \Phi(x-y), & x \in \partial\Omega. \end{cases} \tag{1.28}$$

1.4.1 Green's Function of B

We first recall the derivation of the Green's function of B. Consider the mapping $y \mapsto y/|y|^2 =: \hat{y}$, i.e. the inversion through the unit sphere in $\mathbb{R}^N$. First, we assume $N \geq 3$. The mapping $x \mapsto \Phi(x-\hat{y})$ is harmonic for $x \neq \hat{y}$. Thus $x \mapsto |y|^{2-N}\Phi(x-\hat{y})$ is harmonic for $x \neq \hat{y}$, and so for $y \in B$,

$$\phi_y(x) := \Phi(|y|(x-\hat{y}))$$

is harmonic in B. Furthermore, if $x \in \partial B$ and $y \neq 0$,

$$|y|^2|x-\hat{y}|^2 = |y|^2\left(|x|^2 - \frac{2x \cdot y}{|y|^2} + \frac{1}{|y|^2}\right) = |y|^2 - 2y \cdot x + 1 = |y-x|^2.$$

Thus $(|y||x-\hat{y}|)^{-(N-2)} = |y-x|^{-(N-2)}$. Consequently,

$$\phi_y(x) = \Phi(x-y) \text{ for } x \in \partial B,$$

as required. Therefore Green's function for the unit ball is

$$\widetilde{G}_B(x,y) = \Phi(x-y) - \Phi(|y|(x-\hat{y})), \quad x, y \in B, x \neq y \tag{1.29}$$

The same formula is valid for $N = 2$ as well (cf. [12]). Explicitly, we have

$$\widetilde{G}_B(x,y) = -\frac{1}{2\pi}\ln|x-y| + \frac{1}{2\pi}\ln\left(|y|\Big|x - \frac{y}{|y|^2}\Big|\right) \tag{1.30}$$

for $B \subset \mathbb{R}^2$ and

$$\widetilde{G}_B(x,y) = \frac{1}{N(N-2)\omega_N}\left(\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2}\big|x - \frac{y}{|y|^2}\big|^{N-2}}\right) \tag{1.31}$$

for $B \subset \mathbb{R}^N$, $N \geq 3$.

1.4.2 A Derivation of the Green's Function of $\Omega = \mathbb{R}^N \setminus \overline{B}$

The motivation for our approach to derivation of the Green's function $\widetilde{G}_\Omega(x,y)$ of Ω is the fact that it will provide a representation formula for solutions to the Poisson equation

$$\begin{cases} -\Delta u = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.32}$$

of the form

$$u(x) = \int_\Omega \widetilde{G}_\Omega(x,y) f(y)\, dy \tag{1.33}$$

$\left(\text{informally } -\Delta u(x) = \int_\Omega (-\Delta_x \widetilde{G}_\Omega(x,y)) f(y)\, dy = \int_\Omega \delta(y-x) f(y)\, dy = f(x)\right).$

But $\hat{u}$, the Kelvin transform of the solution of (1.32), will then solve:

$$-\Delta \hat{u} = \frac{1}{|x|^{N+2}} f\Big(\frac{x}{|x|^2}\Big), \quad x \in B \setminus \{0\}.$$

Therefore

$$\hat{u}(x) = \frac{1}{|x|^{N-2}} u\Big(\frac{x}{|x|^2}\Big) = \int_B \widetilde{G}_B(x,y) \frac{1}{|y|^{N+2}} f\Big(\frac{y}{|y|^2}\Big)\, \mathrm{d}y, \quad x \in B,$$

and so

$$u(x) = \int_B \frac{1}{|x|^{N-2}} \widetilde{G}_B\Big(\frac{x}{|x|^2}, y\Big) \frac{1}{|y|^{N+2}} f\Big(\frac{y}{|y|^2}\Big)\, \mathrm{d}y, \quad x \in \Omega.$$

Next, using the change of variables $y \mapsto \dfrac{y}{|y|^2}$ gives

$$\begin{aligned} u(x) &= \int_\Omega \frac{1}{|x|^{N-2}} \widetilde{G}_B\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big) |y|^{N+2} f(y) \cdot \frac{1}{|y|^{2N}}\, \mathrm{d}y \\ &= \int_\Omega \frac{1}{|x|^{N-2}} \frac{1}{|y|^{N-2}} \widetilde{G}_B\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big) f(y)\, \mathrm{d}y, \quad x \in \Omega. \end{aligned}$$

So, we claim that

$$\widetilde{G}_\Omega(x,y) = \frac{1}{|x|^{N-2}} \cdot \frac{1}{|y|^{N-2}} \widetilde{G}_B\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big),$$

where $\widetilde{G}_B$ given in (1.30) for $N=2$ or (1.31) for $N \geq 3$ is the Green's function for the unit ball in $\mathbb{R}^N$:

Lemma 1.13. The Green's function for $\Omega = B^C \subset \mathbb{R}^2$ is

$$\widetilde{G}_\Omega(x,y) = \widetilde{G}_B\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big) = -\frac{1}{2\pi} \ln\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right| + \frac{1}{2\pi} \ln\left(\frac{1}{|y|}\left|\frac{x}{|x|^2} - y\right|\right), \; x, y \in \Omega. \tag{1.34}$$

Lemma 1.14. The Green's function for $\Omega = B^C \subset \mathbb{R}^N$, $N \geq 3$ is

$$\begin{aligned} \widetilde{G}_\Omega(x,y) &= \frac{1}{|x|^{N-2}} \frac{1}{|y|^{N-2}} \widetilde{G}_B\Big(\frac{x}{|x|^2}, \frac{y}{|y|^2}\Big) \\ &= \frac{1}{N(N-2)\omega_N |x|^{N-2}|y|^{N-2}} \left(\frac{1}{|\frac{x}{|x|^2} - \frac{y}{|y|^2}|^{N-2}} - \frac{|y|^{N-2}}{|\frac{x}{|x|^2} - y|^{N-2}}\right), \; x, y \in \Omega. \end{aligned} \tag{1.35}$$

It can now be shown using simple calculations that equations (1.34) and (1.35) can be reduced to the same formulas as (1.30) and (1.31), respectively. In other words, formulas (1.30) and (1.31) provide Green's functions of the unit ball B (if $(x,y) \in B \times B$) as well as $\Omega = \mathbb{R}^N \setminus \overline{B}$ (if $(x,y) \in \Omega \times \Omega$).

Chapter 2

Solution in $\Omega \subset \mathbb{R}^N$, $N \geq 3$, by Direct Approach

In this chapter, we show the existence of a positive weak solution to the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega = \partial B, \end{cases} \tag{2.1}$$

for $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$, $N \geq 3$. We show the existence of a solution by adapting a direct approach, employed in [14], where the authors consider such an equation in the whole space $\mathbb{R}^N$. The main steps are as follows:

1. After stating the needed assumptions on the functions $a(x), b(x), h(x)$ and $g(u)$, we derive a related problem, (cf. (2.12)), to problem (2.1).

2. We show the existence of a positive weak solution $\underline{u}_c$ (cf. Theorem 2.2) for the related problem using a minimization argument. This solution $\underline{u}_c$ is then shown to be a subsolution to problem (2.1).

3. Using this subsolution $\underline{u}_c$, we minimize the corresponding functional to problem (2.1) on a set of the form $M = \{\underline{u}_c \leq u \text{ a.e. in } \Omega\}$. Then, showing that the functional is differentiable in certain directions, we conclude that the minimizer is in fact a solution to (2.1).

Although many proofs are unchanged from [14], there are several new contributions here, as will be pointed out below.

2.1 Assumptions

We start by considering $\mathcal{D}^{1,2}(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}.$$

A weak solution of (2.1) is defined as a function $u \in \mathcal{D}^{1,2}(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega a(x)uv \, \mathrm{d}x - \int_\Omega b(x)g(u)v \, \mathrm{d}x - c \int_\Omega h(x)v \, \mathrm{d}x \tag{2.2}$$

for all $v \in C_0^\infty(\Omega)$. The following assumptions are similar to those given in [14] with $\mathbb{R}^N$ replaced by the exterior domain Ω. We will show below that they are indeed sufficient to prove the existence of a positive solution to our exterior domain problem.

(Ha) The function $a : \Omega \to \mathbb{R}$ is positive and belongs to $L^{N/2}(\Omega) \cap L^\infty(\Omega)$.

We call

$$\lambda_1 = \inf_{u \in \mathcal{D}^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int au^2}$$

(Hg) The function $g : \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(s) = 0$ for $s \le 0$. Furthermore, it satisfies

$$\limsup_{s\to 0} \frac{g(s)}{s^{1+\beta}} < \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s} = \infty, \tag{2.3}$$

where $0 < \beta \le 1$ is a fixed constant.

(Hb) The measurable function $b : \Omega \to \mathbb{R}$ is non-negative, not identically equal to zero, and satisfies

$$b(x) \le C_1 a(x) d(x)^{-\beta} \tag{2.4}$$

for some $C_1 > 0$ and $x \in \Omega$, where $d : \mathbb{R}^N \to \mathbb{R}$ is the Aubin-Talenti instanton defined by

$$d(x) = (1 + |x|^2)^{-(N-2)/2} \tag{2.5}$$

Let $\Omega_0 = \{x \in \Omega : b(x) = 0\}$. We assume either Ω_0 has measure zero or $\Omega_0 = \overline{\operatorname{int}\Omega_0}$ (closure in Ω_0) with $\partial\Omega_0$ Lipschitz.

In the former case we set $\lambda_* = +\infty$, and in the latter case

$$\lambda_* = \inf_{u \in \mathcal{D}^{1,2}(\operatorname{int}\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} \|\nabla u\|^2}{\int_{\Omega_0} au^2}$$

By the unique continuation principle (cf. [22]), we have $\lambda_1 < \lambda_*$.

(Hλ) The value λ is such that $\lambda_1 < \lambda < \lambda_*$.

(Hh) The non-negative and not identically equal to zero function $h(x)$ belongs to the space $h \in L^1(\Omega) \cap L^s(\Omega)$ for some $s > N$, and there exists a constant $C_2 > 0$ such that for some $q > N/2$,

$$R^{N/r}\|h\|_{L^q(\Omega\setminus B_R(0))} \leq C_2 \quad \text{for all } R \in \mathbb{R}^+, \tag{2.6}$$

with $\frac{1}{q} + \frac{1}{r} = 1$. Here $B_R(0)$ denotes the ball centered at zero with radius R.

(Hc) The parameter c is nonnegative.

We note that any function $h(x) \cong 1/|x|^m$ for $|x|$ large, where $m > N$, satisfies the hypothesis **(Hh)** (see [6, 7] for details).

Remark 2.1. Under the above hypotheses, any positive weak solution u of (2.1) belongs to $C^{1,\alpha}_{\text{loc}}(\Omega)$. Furthermore, $\lim_{|x|\to\infty} u(x) = 0$.

Indeed, u satisfies

$$-\Delta u - \lambda a(x)u \leq 0$$

Therefore, by [13, Theorem 8.17], for any $x \in \Omega$ such that $|x| > 3$, we have for some $C_1 = C_1(\|a\|_{L^\infty(\Omega)}, \lambda)$,

$$\sup_{B_1(x)} u \leq C_1(\|u\|_{L^{2N/(N-2)}(B_2(x))} + 1) \leq C\|u\| \leq \tilde{C}_1.$$

Next, on the bounded domain $S = \{x : 1 < |x| < 4\}$, let $M = \sup_{|x|\geq 3} u$ (which is finite by above) and $v = u - M$. Then $v \leq 0$ on ∂S and v satisfies

$$-\Delta v - \lambda a(x)v \leq \lambda a(x)M$$

So, by [13, Theorem 8.15], we have for some $C_2 = C_2(M, \|a\|_{L^\infty(\Omega)}, \lambda)$,

$$\sup_S v \leq C_2(\|v\|_{L^2(S)} + 1) = \tilde{C}_2$$

Therefore $u \le \max\{\tilde{C}_1, \tilde{C}_2 + M\}$ on Ω, i.e. $u \in L^\infty(\Omega)$, and $\lim_{|x|\to\infty} u(x) = 0$. From elliptic regularity theory, it follows $u \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$.

2.2 Related Problem

Following the approach of [14], we first introduce the auxiliary problem (2.12) below. By assumption **(Hg)**, there exist $0 < s_0 \le 1$ and $C_3 > 1$ such that

$$\frac{g(s)}{s} \le \lambda \frac{C_3}{C_1} s^\beta \qquad \text{for } s \le s_0.$$

We may assume $C_3 \ge 1/s_0^\beta$. We take

$$\ell = \left(\frac{1}{C_3}\right)^{1/\beta} \tag{2.7}$$

so that

$$\ell \le s_0. \tag{2.8}$$

Using (2.4),

$$b(x)\frac{g(s)}{s} \le \lambda a(x)\left(\frac{s}{\ell d(x)}\right)^\beta \qquad \text{for } s \le s_0. \tag{2.9}$$

where d is as in (2.5). We define

$$k(s) = s^\beta \tag{2.10}$$

for $s > 0$, $k(s) = 0$ for $s \le 0$. Then

$$b(x)g(s) \le \lambda a(x) s k\left(\frac{s}{\ell d(x)}\right) \qquad \text{for } s \le s_0. \tag{2.11}$$

We consider the equation

$$\begin{cases} -\Delta u = \lambda a(x) u \left[1 - k\left(\dfrac{u}{\ell d(x)}\right)\right] - ch(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.12}$$

Now, instead of **(Hλ)**, we assume

(Hλ)$'$ The value λ is such that $\lambda > \lambda_1$.

Theorem 2.2. Under **(Ha)**, **(Hλ)$'$**, and **(Hh)**, there exists $c_0 > 0$ such that for all $0 \leq c \leq c_0$, equation (2.12) has a positive weak solution $\underline{u}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$. Furthermore, there exists $C_4 > 0$ such that for all $0 \leq c \leq c_0$, this weak solution $\underline{u}_c$ satisfies

$$\underline{u}_c(x) \geq \frac{C_4}{|x|^{N-2}} \quad \text{for large } |x|. \tag{2.13}$$

In this section, by adapting the arguments used in [14] to our present case, we first prove existence of a solution to (2.16) below. This solution will be used in the next section to establish Theorem 2.2. First, observe that the function $\ell d(x)$ is a super solution of (2.12). Indeed, this easily follows from $-\Delta d = N(N-2)d^{2^*-1} > 0$ ($2^* = 2N/(N-2)$), and $d > 0$ on $\partial\Omega$. Consider $\overline{G} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ with $\overline{G}(x,u) := \lambda a(x) \int_0^u sk\big(\frac{s}{\ell d(x)}\big)\,\mathrm{d}s$ and the functional $\tilde{I}_c : \mathcal{D}^{1,2}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\tilde{I}_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega a(x)(u^+)^2\,\mathrm{d}x + \int_\Omega \overline{G}(x,u)\,\mathrm{d}x + c\int_\Omega h(x)u\,\mathrm{d}x \tag{2.14}$$

if $\int_\Omega \overline{G}(x,u)\,\mathrm{d}x < \infty$ and $\tilde{I}_c(u) = +\infty$ otherwise. We have used the standard notation $u^+ = \max\{0,u\}$.

We define the set

$$N_\ell = \{u \in \mathcal{D}^{1,2}(\Omega) : u \leq \ell d(x) \text{ a.e. in } \Omega\}, \tag{2.15}$$

which is weakly closed in $\mathcal{D}^{1,2}(\Omega)$. The following results are proved in [14] for $\Omega = \mathbb{R}^N$, but their proofs carry over to our case with minimal change.

Lemma 2.3. Let $L \geq 0$. The functional $\tilde{I}_c$ is coercive on N_ℓ, uniformly in c with $0 \leq c \leq L$, i.e. for each $C > 0$, there exists $R > 0$ such that for all $0 \leq c \leq L$ and $u \in N_\ell$, if $\|u\| > R$ then $\tilde{I}_c(u) > C$.

Since the functional $\tilde{I}_c$ is weakly lower-semicontinuous and coercive on N_ℓ, it admits a minimizer $\tilde{u}_c$ in N_ℓ for each $c \geq 0$. It is not difficult to see that the derivative $\tilde{I}_c'(\tilde{u}_c)\varphi$ is well defined for any $\varphi \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega)$ with compact support, as $\sup \tilde{u}_c$ is uniformly bounded by ℓd (look to Lemma 2.10 below, where a similar result is proved). We therefore have:

Lemma 2.4. The function $\tilde{u}_c$ is a solution to the equation

$$-\Delta u = \lambda a(x)u^+ - \lambda a(x)uk\left(\frac{u}{\ell d}\right) - ch(x) \tag{2.16}$$

We finish this section by stating the following simple but useful result.

Lemma 2.5. There exists $c_1, C_5 > 0$ such that for $0 \le c \le c_1$, $\inf_{N_\ell} \tilde{I}_c \le -C_5 < 0$, which implies that there exist $0 < r_0 < R_0$ such that

$$0 \le c \le c_1 \qquad \Longrightarrow \qquad r_0 \le \|\tilde{u}_c\| \le R_0 \tag{2.17}$$

Proof. The fact that $\inf_{N_\ell} \tilde{I}_c \le -C_5 < 0$ for $0 \le c \le c_1$ for $0 < c_1$ sufficiently small is easily seen. Now (2.17) follows from Lemma 2.3 and by observing that since

$$\tilde{I}_c(u) \ge -C\|u\|^2 + \int_\Omega G(\cdot, u) - C\|u\| \ge -C\|u\|^2 - C\|u\|,$$

we have

$$\liminf_{u \to 0} \tilde{I}_c(u) \ge 0.$$

$\square$

2.3 A Positive Solution for the Related Problem

The minimizers $\tilde{u}_c$ of $\tilde{I}_c$ obtained above, and Lemmas 2.3 and 2.5 and (2.17) are now used to complete the proof of Theorem 2.2 as in [14]. The first step is to consider the behavior of w, the unique solution to $-\Delta w = h(x)$ in $\mathcal{D}^{1,2}(\Omega)$ as $|x| \to \infty$. As we are now in an exterior domain and not the whole space $\mathbb{R}^N$, the proof of [14, Lemma 4.1] (which is [1, Lemma 4]) has to be adopted to our present case. We have:

Lemma 2.6. Let $h \in L^1(\Omega)$ and suppose (2.6) holds. Then the equation $-\Delta w = h(x)$, $w \in \mathcal{D}^{1,2}(\Omega)$

has a unique solution, and there exists $c > 0$ such that

$$w(x) \leq \frac{C}{|x|^{N-2}} \quad \text{for all } x \in \Omega.$$

Proof. If $-\Delta w = h(x)$, by the Green's formula in Section 1.4, we have that

$$w(x) = C_N \int_\Omega \left(\frac{1}{|x-y|^{N-2}} - \frac{1}{|y|^{N-2} \left| x - \frac{y}{|y|^2} \right|^{N-2}} \right) h(y)\, \mathrm{d}y,$$

where C_N is a constant depending only on N. We first write

$$w(x) = C_N \int_\Omega \frac{1}{|x-y|^{N-2}} h(y)\, \mathrm{d}y - C_N \int_\Omega \frac{1}{|y|^{N-2} \left| x - \frac{y}{|y|^2} \right|^{N-2}} h(y)\, \mathrm{d}y = J_1 - J_2.$$

We have that

$$(N-2) \int_{|x-y|}^\infty \frac{1}{t^{N-1}}\, \mathrm{d}t = \frac{(N-2)}{(2-N)t^{N-2}} \Bigg|_{|x-y|}^\infty = \frac{1}{|x-y|^{N-2}},$$

and so, using Fubini's Theorem, and letting

$$\tilde{h}(x) = \begin{cases} h(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

we have

$$\begin{aligned} J_1 &= C \int_\Omega \left(\int_{|x-y|}^\infty \frac{1}{t^{N-1}}\, \mathrm{d}t \right) h(y)\, \mathrm{d}y = C \int_0^\infty \left(\int_{\{|x-y| \leq t\}} \frac{1}{t^{N-1}} \tilde{h}(y)\, \mathrm{d}y \right) \mathrm{d}t \\ &= C \int_0^\infty \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t(x))}\, \mathrm{d}t. \end{aligned}$$

Using the fact that $\tilde{h}$ is nonnegative, we may write

$$\int_0^\infty \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t(x))}\, \mathrm{d}t = \int_{|x|/2}^\infty \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t(x))}\, \mathrm{d}t + \int_0^{|x|/2} \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t(x))}\, \mathrm{d}x = K_1 + K_2.$$

Note first that

$$K_1 = \int_{|x|/2}^{\infty} \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t)} \,\mathrm{d}t \le \|h\|_{L^1(\Omega)} \int_{|x|/2}^{\infty} \frac{1}{t^{N-1}} \,\mathrm{d}t = C\|h\|_{L^1(\Omega)} \frac{1}{|x|^{N-2}} = \frac{C}{|x|^{N-2}},$$

while

$$\begin{aligned} K_2 &= \int_0^{|x|/2} \frac{1}{t^{N-1}} \|\tilde{h}\|_{L^1(B_t(x))} \,\mathrm{d}t \le C \int_0^{|x|/2} t^{1-N+N/r} \|\tilde{h}\|_{L^q(B_t(x))} \,\mathrm{d}t \\ &\le C\|\tilde{h}\|_{L^q(B_{|x|/2}(x))} \left(\frac{|x|}{2}\right)^{2-N+N/r} \le C\|\tilde{h}\|_{L^q(\mathbb{R}^N \setminus B_{|x|/2}(0))} \left(\frac{|x|}{2}\right)^{N/q} \cdot |x|^{2-N} \le \frac{C}{|x|^{N-2}} \end{aligned}$$

by (2.6) and so

$$J_1 \le \frac{C}{|x|^{N-2}}.$$

Next, note that $f(x,y) = |y|^{2-N} \left| x - \frac{y}{|y|^2} \right|^{2-N}$ is a continuous function in $\Omega \times \Omega$, since for $x, y \in \Omega$, $y \ne 0$ and $\frac{y}{|y|^2} \in B$, so $\left| x - \frac{y}{|y|^2} \right| \ne 0$. Thus for $|x| > 2$,

$$\left| x - \frac{y}{|y|^2} \right| \ge |x| - 1 \ge \frac{|x|}{2},$$

and therefore

$$J_2 \lesssim \frac{C}{|x|^{N-2}} \int_\Omega \frac{h(y)}{|y|^{N-2}} \,\mathrm{d}y \le \frac{C}{|x|^{N-2}}$$

since $h \in L^1(\Omega)$. Therefore

$$w(x) \le \frac{C}{|x|^{N-2}} \qquad \text{for all } x \in \Omega.$$

The proof is complete. $\square$

Now we can rewrite (2.16) as:

$$-\Delta(\tilde{u}_c + cw) = \lambda a \tilde{u}_c^+ \left[1 - k\left(\frac{\tilde{u}_c}{\ell d(x)} \right) \right]. \tag{2.18}$$

The remaining steps are as follows:

1. Observing that $\tilde{u}_{c_1}$ is a sub solution of (2.16) for $0 \le c \le c_1$, we minimize the functional $\tilde{I}_c$

over the set

$$M = \{u \in \mathcal{D}^{1,2}(\Omega) : \tilde{u}_{c_1} \le u \le \ell d(x) \text{ a.e. in } \Omega\} \tag{2.19}$$

to obtain a new solution $\underline{u}_c$ of (2.18) for $0 \le c \le c_1$ with $\underline{u}_c \ge \tilde{u}_{c_1}$, i.e.

$$-\Delta \underline{u}_c = \lambda a(x)\underline{u}_c^+ - \lambda a(x)\underline{u}_c k\left(\frac{\underline{u}_c}{\ell d(x)}\right) - ch(x). \tag{2.20}$$

Note that as $\underline{u}_c^+ \le \ell d(x)$, $k(s) \equiv 0$ for $s \le 0$ and $(\mathbf{H}h)$, standard elliptic regularity theory (see the argument in Remark 2.1 above) implies that $\underline{u}_c \in C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$.

2. Using Lemma 2.5 and (2.17), we have that $\tilde{I}_c(\underline{u}_c) < 0$ for $0 \le c \le c_1$. This in particular implies that $\underline{u}_{c_1} = \tilde{u}_{c_1}$ cannot be nonpositive in Ω, and therefore there exists $x_0 = x_0(c_1)$ and $\rho > 0$ such that $\overline{B_\rho(x_0)} \subset \Omega$ and

$$\inf_{B_\rho(x_0)} \tilde{u}_{c_1} > 0.$$

We then fix ϵ_0 sufficiently small satisfying

$$\epsilon_0 \widetilde{G}_\Omega(x, x_0) < \tilde{u}_{c_1}(x) \le u_c(x) \quad \text{ if } x \in \partial B_\rho(x_0), \text{ and } 0 \le c \le c_1,$$

where $\widetilde{G}_\Omega(x, x_0)$ is the Green's function for Ω (cf. Section 1.4).

3. Using (2.19) and (2.20), we get

$$\int_\Omega \nabla(\underline{u}_c + cw) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \lambda a(x)\underline{u}_c^+\left[1 - k\left(\frac{\underline{u}_c}{\ell d(x)}\right)\right]\varphi \,\mathrm{d}x \quad \text{ for all } \varphi \in \mathcal{D}^{1,2}(\Omega)$$

and

$$\int_\Omega \nabla\big(\epsilon \widetilde{G}_\Omega(x, x_0)\big) \cdot \nabla\varphi \,\mathrm{d}x = 0.$$

for all $\varphi \in \mathcal{D}^{1,2}(\Omega)$ such that $\varphi(x) = 0$ on $B_\rho(x_0)$. Subtracting these two equations yields

$$\int_\Omega \nabla\big(\underline{u}_c + cw - \epsilon_0 \widetilde{G}_\Omega(x, x_0)\big) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \lambda a(x)\underline{u}_c^+\left[1 - k\left(\frac{\underline{u}_c}{\ell d(x)}\right)\right]\varphi \,\mathrm{d}x \tag{2.21}$$

for all $\varphi \in \mathcal{D}^{1,2}(\Omega)$ such that $\varphi(x) = 0$ on $B_\rho(x_0)$.

4. Considering the set

$$S_c = \{x \in \Omega \setminus \overline{B_\rho(x_0)} : \epsilon_0 \widetilde{G}_\Omega(x, x_0) > (\underline{u}_c + cw)(x)\},$$

we let $\varphi = (\underline{u}_c + cw - \epsilon_0 \widetilde{G}_\Omega(x, x_0))\chi_{S_c} \in \mathcal{D}^{1,2}(\Omega)$. Note that $S_c \cap \partial\Omega = \emptyset$ since $\widetilde{G}_\Omega(x, x_0) = 0$ for $x \in \partial\Omega$, and so φ is less than or equal to zero and has support in $\Omega \setminus \overline{B_\rho(x_0)}$. Using this function φ in (2.21) gives

$$\int_{S_c} \left|\nabla(\underline{u}_c + cw - \epsilon_0 \widetilde{G}_\Omega(x, x_0))\right|^2 \leq 0,$$

meaning that $S_c = \emptyset$ and so $\epsilon_0 \widetilde{G}_\Omega(x, x_0) \leq (\underline{u}_c + cw)(x)$ for all $x \in \Omega \setminus B_\rho(x_0)$.

By the Hopf lemma (cf. Lemma 1.4), $\inf_{x\in\partial\Omega} \frac{\partial}{\partial\nu} \widetilde{G}_\Omega(x, x_0) > 0$. Additionally, since $\widetilde{G}_\Omega(x, x_0) \geq C/|x|^{N-2}$ for $|x|$ large, we can combine these steps with Lemma 2.6 to conclude that here exists $0 < c_0 \leq c_1$ such that for all $0 \leq c \leq c_0$ the function $\underline{u}_c$ is positive and $\underline{u}_c(x) \geq C_3/|x|^{N-2}$ for all $x \in \Omega$ large. This completes the proof of Theorem 2.2.

2.4 Positive Solution for Ω

We now turn to equation (3.1). The outline of the proof is similar to that of the proof of [14, Theorem 5.1], but with one key difference. In [14], the nonlinearity $g(u)$ is approximated by a series of cutoff functions, defined as polynomials of the form $j(s) = s^p$ for some $1 < p \leq (N+2)/(N-2)$. This is used to show that the corresponding functionals to the original problem with $g(u)$ replaced with $j(u)$ are coercive and differentiable in the direction of functions $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ with compact support.

The approach we take here shows that this is not needed. We instead work with the original nonlinearity $g(u)$ and show that the corresponding functional I_c (see (2.24) below) is coercive and differentiable in slightly different directions, as shown in Lemmas 2.8 and 2.10. Our results are summarized as:

Theorem 2.7. Under **(H**a**)**, **(H**b**)**, **(H**g**)**, **(H**λ**)** and **(H**h**)**, there exists $c_0 > 0$ such that for all $0 \leq c \leq c_0$, equation (2.1) has a positive weak solution $u_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$. Furthermore, there exists $C_6 > 0$ such that for all $0 \leq c \leq c_0$ this weak solution satisfies

$$u_c(x) \geq \frac{C_6}{|x|^{N-2}} \quad \text{for large } |x|. \tag{2.22}$$

Proof. We take the function k as in (2.10) and apply Theorem 2.2 to obtain a positive solution $\underline{u}_c$ of (2.12) for $0 \leq c \leq c_0$. Using (2.11) and

$$\underline{u}_c \leq \ell d(x) \leq \ell \leq s_0 \tag{2.23}$$

(see (2.8)), the function $\underline{u}_c$ therefore satisfies

$$-\Delta \underline{u}_c \leq \lambda a(x)\underline{u}_c - b(x)g(\underline{u}_c) - ch(x),$$

and so is a sub solution to (2.1). For each $0 \leq c \leq c_0$ we define the weakly closed set

$$M_c = \{u \in \mathcal{D}^{1,2}(\Omega) : \underline{u}_c \leq u \text{ a.e. in } \Omega\}.$$

Let $G(u) := \int_0^u g(s)\,\mathrm{d}s$ and define the functional $I_c : M_c \to \mathbb{R} \cup \{+\infty\}$ by

$$I_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x + \int_\Omega b(x)G(u)\,\mathrm{d}x + c\int_\Omega h(x)u\,\mathrm{d}x \tag{2.24}$$

if $\int_\Omega b(x)G(u)\,\mathrm{d}x < \infty$ and $I_c(u) = +\infty$ otherwise. Again, one can easily see (look at the proof of Theorem 6 in [11]) that:

Lemma 2.8. The functional I_c is coercive on M_c, uniformly in c with $0 \leq c \leq c_0$, i.e. for each $L > 0$, there exists $R > 0$ such that for all $0 \leq c \leq c_0$ and $u \in M_c$, if $\|u\| > R$, then $I_c(u) > L$.

As I_c is weakly lower semi-continuous on the weakly closed set M_c, $\inf\{I_c(u) : u \in M_c\}$ is achieved as some $u_c \in M_c$ with $u_c \geq \underline{u}_c$. To show that u_c is a weak solution of our problem, we need the differentiability of

$$J(u) := \int_\Omega b(x)G(u)\,\mathrm{d}x$$

in certain directions. First we show:

Lemma 2.9. The minimizer u_c of I_c on M_c is in $L^\infty(\Omega)$.

Proof. Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \le 0$. Motivated by the proof of [23, Theorem 2.4], let $\epsilon > 0$, and define $v_\epsilon = \max\{\underline{u}_c, u_c + \epsilon\varphi\}$. Note that v_ϵ can be written as $u_c + \epsilon\varphi + \varphi^\epsilon$, where $\varphi^\epsilon = \big[\underline{u}_c - (u_c + \epsilon\varphi)\big]^+$. Therefore $v_\epsilon \in M_c$, and so $I_c(v_\epsilon) \ge I_c(u_c)$, since u_c is the minimizer of I_c on M_c.

Since the set M_c is convex, for all $0 \le t \le 1$, the function $(1-t)u_c + tv_\epsilon = u_c + t(v_\epsilon - u_c)$ is also in M_c, and $I_c(u_c + t(v_\epsilon - u_c)) \ge I_c(u_c)$ for all $0 \le t \le 1$. Let $w_\epsilon = v_\epsilon - u_c$. We then have that

$$\begin{aligned}
I(u_c + tw_\epsilon) &= \frac{1}{2}\int_\Omega |\nabla(u_c + tw_\epsilon)|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)(u_c + tw_\epsilon)^2 \,\mathrm{d}x + \int_\Omega b(x)G(u_c + tw_\epsilon)\,\mathrm{d}x \\
&\qquad + c\int_\Omega h(x)(u_c + tw_\epsilon)\,\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega |\nabla u_c|^2 \,\mathrm{d}x + t\int_\Omega \nabla u_c \nabla w_\epsilon \,\mathrm{d}x + \frac{t^2}{2}\int_\Omega |\nabla w_\epsilon|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega a(x)u_c^2 \,\mathrm{d}x \\
&\qquad - \lambda t\int_\Omega a(x)u_c w_\epsilon - \frac{\lambda t^2}{2}\int_\Omega a(x)w_\epsilon^2 \,\mathrm{d}x + \int_\Omega b(x)G(u_c + tw_\epsilon)\,\mathrm{d}x \\
&\qquad + c\int_\Omega h(x)u_c \,\mathrm{d}x + ct\int_\Omega h(x)w_\epsilon \,\mathrm{d}x \\
&\ge \frac{1}{2}\int_\Omega |\nabla u_c|^2 - \frac{\lambda}{2}\int_\Omega a(x)u_c^2 \,\mathrm{d}x + \int_\Omega b(x)G(u_c)\,\mathrm{d}x + c\int_\Omega h(x)u_c \,\mathrm{d}x = I(u_c)
\end{aligned}$$

Canceling out the identical terms from both sides and rearranging, we have that

$$\begin{aligned}
t\int_\Omega &\nabla u_c \nabla w_\epsilon \,\mathrm{d}x - \lambda t\int_\Omega a(x)u_c w_\epsilon \,\mathrm{d}x \\
&\ge E(t^2) + \int_\Omega b(x)\big[G(u_c) - G(u_c + tw_\epsilon)\big]\,\mathrm{d}x - ct\int_\Omega h(x)w_\epsilon \,\mathrm{d}x. \qquad (2.25)
\end{aligned}$$

Here $E(t^2)$ represents all terms with a t^2 in front. Note that $G(u_c) - G(u_c + tw_\epsilon)$ is nonnegative. Indeed, $w_\epsilon = v_\epsilon - u_c \le 0$ (because $\varphi \le 0$), and so $u_c + tw_\epsilon \le u_0$, and G is increasing since $g \ge 0$. Similarly, $-ct\int_\Omega h(x)w_\epsilon \ge 0$. Dividing both sides of (2.25) by t and taking the limit as $t \to 0$, we then have

$$\int_\Omega \nabla u_c \nabla w_\epsilon \,\mathrm{d}x - \lambda\int_\Omega a(x)u_c w_\epsilon \,\mathrm{d}x$$

$$\geq \lim_{t\to 0}\frac{E(t^2)}{t}+\lim_{t\to 0}\frac{\int_\Omega b(x)\big[G(u_c)-G(u_c+tw_\epsilon)\big]\,\mathrm{d}x}{t}-c\int_\Omega h(x)w_\epsilon\,\mathrm{d}x\geq 0$$

for all $\epsilon > 0$. Writing $w_\epsilon = \epsilon\varphi + \varphi^\epsilon$, we get

$$\int_\Omega \nabla u_c\big(\epsilon\nabla\varphi+\nabla\varphi^\epsilon\big)\,\mathrm{d}x-\lambda\int_\Omega a(x)u_c(\epsilon\varphi+\varphi^\epsilon)\geq 0,$$

or, equivalently,

$$\epsilon\int_\Omega \nabla u_c\nabla\varphi\,\mathrm{d}x-\epsilon\lambda\int_\Omega a(x)u_c\varphi\,\mathrm{d}x\geq -\int_\Omega \nabla u_c\nabla\varphi^\epsilon\,\mathrm{d}x+\lambda\int_\Omega a(x)u_c\varphi^\epsilon\,\mathrm{d}x \tag{2.26}$$

Dividing both sides of (2.26) by ϵ, we obtain

$$\int_\Omega \nabla u_c\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega a(x)u_c\varphi\,\mathrm{d}x\geq \frac{1}{\epsilon}\Bigg[-\int_\Omega \nabla u_c\nabla\varphi^\epsilon\,\mathrm{d}x+\lambda\int_\Omega a(x)u_c\varphi^\epsilon\,\mathrm{d}x\Bigg]. \tag{2.27}$$

Since $\underline{u}_c$ is a sub solution to (2.1), we have (using $\varphi^\epsilon = \big[\underline{u}_c-(u_c+\epsilon\varphi)\big]^+\geq 0$):

$$\int_\Omega \nabla\underline{u}_c\nabla\varphi^\epsilon\,\mathrm{d}x-\lambda\int_\Omega a(x)\underline{u}_c\varphi^\epsilon\,\mathrm{d}x+\int_\Omega b(x)g(\underline{u}_c)\varphi^\epsilon\,\mathrm{d}x+c\int_\Omega h(x)\varphi^\epsilon\,\mathrm{d}x\leq 0,$$

and since the last two terms are positive, this implies that

$$\int_\Omega \nabla\underline{u}_c\nabla\varphi^\epsilon\,\mathrm{d}x-\lambda\int_\Omega a(x)\underline{u}_c\varphi^\epsilon\,\mathrm{d}x\leq 0. \tag{2.28}$$

Dividing the left side of (2.28) by ϵ and adding it to the right side of (2.27) yields

$$\begin{aligned}\int_\Omega \nabla u_c\nabla\varphi\,\mathrm{d}x-\lambda\int_\Omega a(x)u_c\varphi\,\mathrm{d}x &\geq \frac{1}{\epsilon}\Bigg[-\int_\Omega \nabla u_c\nabla\varphi^\epsilon\,\mathrm{d}x+\lambda\int_\Omega a(x)u_c\varphi^\epsilon\,\mathrm{d}x\\ &\qquad\qquad+\int_\Omega \nabla\underline{u}_c\nabla\varphi^\epsilon\,\mathrm{d}x-\lambda\int_\Omega a(x)\underline{u}_c\varphi^\epsilon\,\mathrm{d}x\Bigg]\\ &=\frac{1}{\epsilon}\Bigg[\int_\Omega \nabla(\underline{u}_c-u_c)\nabla\varphi^\epsilon\,\mathrm{d}x-\lambda\int_\Omega a(x)(\underline{u}_c-u_c)\varphi^\epsilon\,\mathrm{d}x\Bigg].\end{aligned} \tag{2.29}$$

Now, since $\varphi^\epsilon = \big[\underline{u}_c - (u+\epsilon\varphi)\big]^+$,

$$\nabla\varphi^\epsilon = \nabla\big[\underline{u}_c - (u+\epsilon\varphi)\big]^+ = \big[\nabla(\underline{u}_c - u) - \epsilon\nabla\varphi\big]\chi_{\Omega_\epsilon},$$

where $\Omega_\epsilon = \{x \in \Omega : u_c(x) + \epsilon\varphi(x) < \underline{u_c}(x) \leq u_c(x)\}$. Therefore (2.29) can be written as

$$\frac{1}{\epsilon}\int_{\Omega_\epsilon} |\nabla(\underline{u}_c - u_c)|^2 \,\mathrm{d}x - \int_{\Omega_\epsilon} \nabla(\underline{u}_c - u_c)\nabla\varphi \,\mathrm{d}x - \frac{\lambda}{\epsilon}\int_{\Omega_\epsilon} a(x)(\underline{u}_c - u_c)\big[\underline{u}_c - (u+\epsilon\varphi)\big]^+ \,\mathrm{d}x. \quad (2.30)$$

Both the first and third terms of (2.30) are positive, so subtracting them from the above inequality gives

$$\int_\Omega \nabla u_c\nabla\varphi \,\mathrm{d}x - \lambda\int_\Omega a(x)u_c\varphi \,\mathrm{d}x \geq -\int_{\Omega_\epsilon} \nabla(\underline{u}_c - u_c)\nabla\varphi \,\mathrm{d}x.$$

Note that $|\Omega_\epsilon| \to 0$ as $\epsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we finally obtain

$$\int_\Omega \nabla u_c\nabla\varphi \,\mathrm{d}x - \lambda\int_\Omega a(x)u_c\varphi \,\mathrm{d}x \geq 0$$

for $\varphi \leq 0$. Then replacing φ with $\psi = -\varphi$, we have that

$$\int_\Omega \nabla u_c\nabla\psi \,\mathrm{d}x - \lambda\int_\Omega a(x)u_c\psi \,\mathrm{d}x \leq 0$$

for all $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, i.e.

$$-\Delta u_c - \lambda a(x)u_c \leq 0.$$

Then, by the same argument as immediately following Remark 2.1, we have that $u_c \in L^\infty(\Omega)$. $\square$

Lemma 2.10. Suppose $v \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega)$ with compact support, and let u_c be the minimizer of the functional I_c on the set M_c. Then I_c is differentiable at u_c in the direction v as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b(x)G(u_c + tv)\,\mathrm{d}x\bigg|_{t=0} = \int_\Omega b(x)g(u_c)\,\mathrm{d}x.$$

Proof. By Lemma 2.9, we have that $u_c \in L^\infty(\Omega)$. We must first show that $J(u_c + tv) =$

$\int_\Omega b(x)G(u_c+tv)\,\mathrm{d}x$ is defined for $|t|<\delta$ with δ sufficiently small. Now we see that

$$G(u_c+tv)-G(u_c)=g(u_c+stv)tv \qquad (0\le s=s(t,x)\le 1)$$

and

$$\int_\Omega b(x)g(u_c+stv)|v|\,\mathrm{d}x \le \int_\Omega C_1 a(x)d(x)^{-\beta}g(u_c+stv)|v|\,\mathrm{d}x$$

using (2.4) and d as in (2.5). Since $u_c, v\in L^\infty(\Omega)$ and g is continuous, we have that $g(u_c+stv)<\infty$, say $g(u_c+stv)\le M=M(\|u\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)})$. So

$$\int_\Omega b(x)g(u_c+tv)|v|\,\mathrm{d}x \le MC_1\int_\Omega a(x)d(x)^{-\beta}|v|\,\mathrm{d}x. \tag{2.31}$$

Since v has compact support, say $v\ne 0$ on $\Omega_v \subsetneq \Omega$, we can then use that $a(x)\in L^\infty(\Omega)$ and $d(x)$ is continuous (and nonzero) to get

$$\begin{aligned}\int_\Omega a(x)(1+|x|^2)^{\beta(N+2)/2}|v|\,\mathrm{d}x &= \|a\|_{L^\infty(\Omega)}\int_{\Omega_v} d(x)^{-\beta}|v|\,\mathrm{d}x\\ &\le C=C(\beta,\|a\|_{L^\infty(\Omega)},|\Omega_v|,\|v\|_{L^\infty(\Omega)}) \qquad (2.32)\end{aligned}$$

Combining (2.31) and (2.32), we have that $J(u_c+tv)$ is well-defined for any $|t|<\delta$ for δ sufficiently small. Now, for the differentiability of J at u_c in the direction of v, we have

$$\lim_{t\to 0}\frac{J(u_c+tv)-J(u_c)}{t}=\lim_{t\to 0}\frac{1}{t}\int_\Omega b(x)g(u_c+stv)tv\,\mathrm{d}x=\lim_{t\to 0}\int_\Omega b(x)g(u_c+stv)v\,\mathrm{d}x.$$

We need to show that $b(x)g(u_c+stv)v(x)\le k(x)\in L^1(\Omega)$. As shown above,

$$|b(x)g(u_c+tv)v(x)|\le Cd(x)^{-\beta}|v(x)|$$

where C depends on $\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}$, and $\|a\|_{L^\infty(\Omega)}$. Since v has compact support, $d(x)^{-\beta}v(x)\in L^1(\Omega)$. So, Lebesgue's Dominated Convergence Theorem implies that J is differentiable at u_c in the direction of $v\in\mathcal{D}^{1,2}(\Omega)\cap L^\infty(\Omega)$ with compact support, and the lemma is proved. $\square$

Combining Lemmas 2.9 and 2.10, we can now argue as in the proof of [23, Theorem 2.4] to see that u_c is a critical point of the functional I_c and so is a weak solution to (2.1). $\square$

Chapter 3

Solution in $\Omega \subset \mathbb{R}^N$, $N \geq 3$, by Kelvin Transform

In this chapter, we prove the existence of a positive solution to equation (2.1) in $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ for $N \geq 3$ using a different approach than the one employed in Chapter 2. Our approach in this chapter is to transform the equation from the exterior domain to the unit ball $B = B(0,1)$ using the Kelvin transform with Lemma 1.10 in mind, and solve the equation there. The technique for solving the transformed equation (3.4) below in B is adapted from [14], with one major difference, which will be expanded on below.

3.1 Application of the Kelvin Transform

Again let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $N \geq 3$, and let $\Omega = \mathbb{R}^N \setminus \overline{B}$. We seek conditions to prove the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega = \partial B, \end{cases} \tag{3.1}$$

where, as in Chapter 2, the nonlinearity $g(s)$ satisfies:

(Hg) The function $g : \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(s) = 0$ for $s \leq 0$. Furthermore, it satisfies

$$\limsup_{s \to 0} \frac{g(s)}{s^{1+\beta}} < \infty \quad \text{and} \quad \lim_{s \to \infty} \frac{g(s)}{s} = \infty, \tag{3.2}$$

where $0 < \beta \leq 1$ is a fixed constant.

By Lemma 1.10, If u is a weak $\mathcal{D}^{1,2}(\Omega)$ solution of (3.1), then its Kelvin transform $\hat{u}$ is a weak $H_0^1(B)$ solution to the boundary value problem

$$\begin{cases} -\Delta \hat{u} = \dfrac{\lambda}{|y|^{N+2}} a\Big(\dfrac{y}{|y|^2}\Big) u\Big(\dfrac{y}{|y|^2}\Big) - \dfrac{1}{|y|^{N+2}} b\Big(\dfrac{y}{|y|^2}\Big) g\Big(u\Big(\dfrac{y}{|y|^2}\Big)\Big) - \dfrac{c}{|y|^{N+2}} h\Big(\dfrac{y}{|y|^2}\Big), & y \in B \\ \hat{u} = 0, & y \in \partial B, \end{cases}$$

or

$$\begin{cases} -\Delta \hat{u} = \lambda \tilde{a}(y)\hat{u} - \tilde{b}(y) g(|y|^{N-2}\hat{u}) - c\tilde{h}(y), & y \in B \\ \hat{u} = 0, & y \in \partial B, \end{cases} \tag{3.3}$$

where $\tilde{a}(y) = |y|^{-4} a\big(\frac{y}{|y|^2}\big)$, $\tilde{b}(y) = |y|^{-N-2} b\big(\frac{y}{|y|^2}\big)$, and $\tilde{h}(y) = |y|^{-N-2} h\big(\frac{y}{|y|^2}\big)$.

3.2 Direct Method in the Ball

We consider the problem of existence of positive weak solutions for the equation

$$\begin{cases} -\Delta u = \lambda \tilde{a}(x)u - \tilde{b}(x) g(|x|^{N-2}u) - c\tilde{h}(x), & x \in B \\ u = 0, & x \in \partial B \end{cases} \tag{3.4}$$

in $H_0^1(B)$. Note that, considering (3.2) and (3.3) above, we have:

$$\limsup_{s\to 0} \frac{g(|x|^{N-2}s)}{s^{1+\beta}} \le C|x|^{(N-2)(1+\beta)} \quad \text{and} \quad \lim_{s\to\infty} \frac{g(|x|^{N-2}s)}{s} = \infty \tag{3.5}$$

for all $x \in B \setminus \{0\}$, where $0 < \beta \le 1$ is a fixed constant.

Other assumptions on the heterogeneous coefficients will be stated below. For reference, we state below the following result (cf. [14, Lemma 7.1], part (i)) concerning the existence of suitable superharmonic functions in B:

Lemma 3.1 ([14, Lemma 7.1])**.** Let $r > 0$ and $y_0 \in B$ such that $1 - |y_0| > 3r$, and $\widetilde{G}_B$ be the Green's function for B (cf. Section 1.4). Then there exists a function $z \in C^2(\overline{B})$, such that $0 < z$ in B, $z = 0$ on ∂B, z is superharmonic in B and harmonic in $B \setminus B_r(y_0)$. The function $z(x)$ further satisfies

$$c\widetilde{G}_B(x, y_0) \le z(x) \le C\widetilde{G}_B(x, y_0) \quad \text{for } \overline{B} \setminus B_{2r}(y_0) \tag{3.6}$$

for some constants $c, C > 0$, and therefore

$$\tilde{c} \cdot \text{dist}(x, \partial B) \leq z(x) \leq \tilde{C} \cdot \text{dist}(x, \partial B) \tag{3.7}$$

for some constants $\tilde{c}, \tilde{C} > 0$.

Our approach here is similar to the one taken in Chapter 2 to deal with equation (2.1) in the exterior domain Ω, and similarly involves the following steps:

1. First we consider an auxiliary equation using the same ideas as in Section 2.2, where now the Aubin-Talenti instanton is replaced by the function $z(x)$ given in Lemma 3.1 above. We then prove the existence of positive solutions $\underline{u}_c$ for the auxiliary equation for $0 \leq c \leq c_0$ (with some $c_0 > 0$).

2. Now, arguing as in Section 2.4, using $\underline{u}_c$ as sub solutions to (3.4), we obtain a solution to our problem by minimizing the corresponding energy functional on the set $M_c = \{u \in H_0^1(B) : \underline{u}_c \leq u \text{ a.e. in } B\}$.

To start this process, we first state our assumptions:

(H$\tilde{a}$) The function $\tilde{a} : B \to \mathbb{R}$ is positive and belongs to $L^\infty(B)$.

We call

$$\tilde{\lambda}_1 = \inf_{u \in H_0^1(B) \setminus \{0\}} \frac{\|u\|^2}{\int_B \tilde{a} u^2}. \tag{3.8}$$

Furthermore, as stated above, $g(|x|^{N-2}u)$ satisfies (3.5).

The assumption on $\tilde{b}(x)$ is related to the construction of the auxiliary equation, whose positive weak solutions will provide sub solutions to (3.4), as in Chapter 2. In fact, we need

$$\tilde{b}(x) g(|x|^{N-2}s) \leq C \tilde{a}(x) s \left(\frac{s}{z(x)} \right)^\beta$$

for all $0 < s$ sufficiently small and z as in Lemma 3.1. Considering (3.5), we now have

$$\tilde{b}(x) g(|x|^{N-2}s) \leq \tilde{b}(x) |x|^{(N-2)(1+\beta)} s \cdot s^\beta, \quad \text{for } 0 < s \text{ sufficiently small.}$$

Thus, taking (3.6) and (3.7) into account, we are going to assume:

(H$\tilde{b}$) The measurable function $\tilde{b} : B \to \mathbb{R}$ is nonnegative, not identically equal to zero, and satisfies

$$\tilde{b}(x) \le C_1 \tilde{a}(x)(1-|x|)^{-\beta}|x|^{-(N-2)(1+\beta)}. \tag{3.9}$$

for some $C_1 > 0$. Note that **(H$\tilde{b}$)** allows $\tilde{b}(x)$ to blow up as $|x| \to 0$ or as $|x| \to 1^-$. Let $B_0 = \{x \in B : \tilde{b}(x) = 0\}$. We assume either B_0 has measure zero of $B_0 = \overline{\operatorname{int} B_0}$ (closure in B_0) with ∂B_0 Lipschitz.

In the former case we set $\tilde{\lambda}_* = +\infty$, and in the latter case

$$\tilde{\lambda}_* = \inf_{u \in H_0^1(\operatorname{int} B_0)\setminus\{0\}} \frac{\int_{B_0} |\nabla u|^2}{\int_{B_0} \tilde{a} u^2}. \tag{3.10}$$

(H$\tilde{h}$) The nonnegative and not identically equal to zero function $\tilde{h}$ belongs to the space $L^s(B)$ for some $s > N$.

(Hλ) The value λ is such that $\tilde{\lambda}_1 < \lambda < \tilde{\lambda}_*$.

(Hc) The parameter c is nonnegative.

Similar to Chapter 2, the auxiliary equation is given as

$$\begin{cases} -\Delta u = \lambda \tilde{a}(x) u \left[1 - k\left(\dfrac{u}{\ell z(x)}\right)\right] - c\tilde{h}(x), & x \in B \\ u = 0, & x \in \partial B, \end{cases} \tag{3.11}$$

where, as mentioned before, we make use of the function $z(x)$ as given in Lemma 3.1(i). Using similar arguments as in Chapter 2, we get positive solutions $\underline{z}_c$ for $0 < c \le c_1$. These will be our sub solutions to (3.4). Now, as in Section 2.4 and mentioned above, we consider $\tilde{M}_c = \{u \in H_0^1(B) : \underline{z}_c \le u$ a.e. in $B\}$ and show that the corresponding functional to equation (3.4) is coercive on $\tilde{M}_c$ and that it is differentiable in certain directions. Since this takes on a slightly different form than in both [14] and Section 2.4, we provide a proof below. To this end, we let $G(u) := \int_0^u g(s)\,\mathrm{d}s$ and define the functional $\tilde{I}_c : \tilde{M}_c \to \mathbb{R} \cup \{+\infty\}$ by

$$\tilde{I}_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_B \tilde{a}(x)u^2 \,\mathrm{d}x + \int_B \tilde{b}(x)\frac{G(|x|^{N-2}u)}{|x|^{N-2}} \,\mathrm{d}x + c\int_B \tilde{h}(x)u \,\mathrm{d}x, \tag{3.12}$$

if $\int_B \tilde{b}(x)G(|x|^{N-2}u)/|x|^{N-2} \,\mathrm{d}x < \infty$ and $\tilde{I}_c(u) = +\infty$ otherwise. We then claim the following:

Lemma 3.2. The functional $\tilde{I}_c$ is coercive in $\tilde{M}_c$, uniformly in c with $0 \le c \le c_1$, i.e. for each $L > 0$, there exists $R > 0$ such that for all $0 \le c \le c_1$ and $u \in \tilde{M}_c$, if $\|u\| > R$, then $\tilde{I}_c(u) > L$.

Proof. The argument is similar to the one in [11, proof of Theorem 6]. Suppose by contradiction there exists $c_n \in [0, c_0]$ and $u_n \in M_{c_n}$ with $\|u_n\| \to \infty$, such that $I_{c_n}(u_n) \le C$. Then we would have

$$p_n^2 := \int_B \tilde{a}(x)u_n^2 \,\mathrm{d}x \to +\infty$$

since G is nonnegative and $\int_B \tilde{h}(x)u \ge 0$ for all $u \in \tilde{M}_c$. We define a sequence of functions, (v_n), with $v_n = u_n/p_n$, so that $\int_B \tilde{a}(x)v_n^2 \,\mathrm{d}x = 1$ and

$$\frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2} + \frac{1}{p_n^2}\int_B \tilde{b}(x)\frac{G(|x|^{N-2}p_n v_n)}{|x|^{N-2}} \,\mathrm{d}x + \frac{c_n}{p_n}\int_B \tilde{h}(x)v_n \,\mathrm{d}x \le \frac{C}{p_n^2} \tag{3.13}$$

Inequality (3.13) implies $\|v_n\|$ is uniformly bounded in n. Up to a subsequence, $v_n \rightharpoonup v$ in $H_0^1(B)$ and $v_n \to v$ a.e. in B. The function v is nonnegative. From the assumption (3.2), we have that $\lim_{s\to+\infty} G(s)/s^2 = +\infty$. Taking the limit inferior of both sides of (3.13), and using Fatou's lemma,

$$\frac{1}{2}\|v\|^2 - \frac{\lambda}{2} + \int_{\{x\in B: v(x)>0\}} b(x) \times (+\infty)v^2 \,\mathrm{d}x \le 0.$$

Therefore the function v must be zero almost everywhere on the set where the function $b(x)$ is positive, i.e. (aside from a set of measure zero) v must have support in B_0. We also obtain $\|v\|^2 \le \lambda$. On the other hand, since $\int_\Omega a(x)v_n^2 \,\mathrm{d}x = 1$ and $\int_\Omega a(x)v_n^2 \,\mathrm{d}x \to \int_\Omega a(x)v^2 \,\mathrm{d}x$, the function $v \not\equiv 0$ and $\int_\Omega a(x)v^2 \,\mathrm{d}x = 1$. If B_0 has measure zero, then we are done. Otherwise, **(H$\tilde{b}$)** implies $v \in H_0^1(\operatorname{int} B_0)$ and

$$\tilde{\lambda}_* \le \frac{\|v\|^2}{\int_B a(x)v^2 \,\mathrm{d}x} \le \lambda$$

which contracts $\lambda < \tilde{\lambda}_*$. The lemma is proved. $\square$

Next, since $\tilde{I}_c$ is coercive on the weakly closed set $\tilde{M}_c$, for $0 \le c \le c_1$ it has a minimizer z_c. As in Section 2.4, we seek to prove differentiability of

$$J(u) := \int_B \tilde{b}(x) \frac{G(|x|^{N-2}u)}{|x|^{N-2}} \, \mathrm{d}x, \tag{3.14}$$

which requires the following result:

Lemma 3.3. The minimizer z_c of $\tilde{I}_c$ on $\tilde{M}_c$ is in $L^\infty(B)$.

The proof follows the same structure as the proof of Lemma 2.9. With these results at hand, we are then able to prove the following:

Lemma 3.4. Suppose $v \in H_0^1(B) \cap L^\infty(B)$, and let z_c be the minimizer of the functional $\tilde{I}_c$ on the set $\tilde{M}_c$. Then $\tilde{I}_c$ is differentiable at z_c in the direction v as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \frac{\tilde{b}(x) G(|x|^{N-2}(z_c + tv))}{|x|^{N-2}} \, \mathrm{d}x \bigg|_{t=0} = \int_B \tilde{b}(x) g(|x|^{N-2} z_c) \, \mathrm{d}x.$$

Proof. We show first that $J(z_c + tv)$ is defined for $|t| < \delta$ with δ sufficiently small and $v \in H_0^1(B) \cap L^\infty(B)$. We have:

$$G(|x|^{N-2}(z_c + tv)) - G(|x|^{N-2} z_c) = |x|^{N-2} g(|x|^{N-2}(z_c + stv)) tv.$$

and

$$\int_B \tilde{b}(x) g(|x|^{N-2}(z_c + stv))|v| \, \mathrm{d}x \le \int_B \tilde{a}(x)(1 - |x|)^{-\beta} |x|^{-(N-2)(1+\beta)} g(|x|^{N-2}(z_c + stv))|v| \, \mathrm{d}x$$

using (3.5) and (5.8). Now, since $z_c, v \in L^\infty(B)$, for $|x| < \delta = \delta(\|z_c\|_{L^\infty(B)}, \|v\|_{L^\infty(B)})$, we have

$$g(|x|^{N-2}(z_c + stv)) \le C(|x|^{N-2})^{1+\beta}.$$

So, using $\tilde{a}(x) \in L^\infty(B)$ and recalling that $|x| \le 1$, we have

$$\begin{aligned}
\int_B \tilde{b}(x) g(|x|^{N-2}(z_c + stv))|v| \,\mathrm{d}x &\le \int_B \tilde{a}(x)(1-|x|)^{-\beta}|x|^{-(N-2)(1+\beta)} g(|x|^{N-2}(z_c + stv))|v| \,\mathrm{d}x \\
&\le C_1 \int_{|x|<\delta} |v| + C(\delta) \int_{|x|>\delta} g(|x|^{N-2}(z_c + stv))|v|(1-|x|)^{-\beta} \,\mathrm{d}x \\
&\le C(\delta)\bigg(\|v\|_{L^2(B)} + \int_B \frac{|v|^2}{(1-|x|)^{2\beta}} \,\mathrm{d}x \bigg)
\end{aligned}$$

But since we are assuming $\beta \le 1$, using the Hardy inequality for bounded domains (cf. Lemma 1.2) implies

$$\int_B \frac{|v|^2}{(1-|x|)^{2\beta}} \,\mathrm{d}x \le \int_B \frac{|v|^2}{(1-|x|)^2} \,\mathrm{d}x \le C \int_B |\nabla v|^2 \,\mathrm{d}x.$$

Therefore $J(z_c + tv)$ is well-defined for any $|t| < \delta$ for δ sufficiently small. Now, for the differentiability of J at z_c in the direction of v, we have

$$\lim_{t\to 0} \frac{J(z_c + tv) - J(z_c)}{t} = \lim_{t\to 0} \frac{1}{t} \int_B \tilde{b}(x) g(|x|^{N-2}(z_c + stv)) tv \,\mathrm{d}x = \lim_{t\to 0} \int_B \tilde{b}(x) g(|x|^{N-2}(z_c + stv)) v.$$

We need to show that $\tilde{b}(x) g(|x|^{N-2}(z_c + stv)) v(x) \le k(x) \in L^1(B)$. As shown above,

$$\tilde{b}(x) g(|x|^{N-2}(z_c + stv)) v(x) \le C_1 \chi_{\{|x|<\delta\}} v(x) + C_2 \chi_{\{|x|>\delta\}} v(x)(1-|x|)^{-\beta},$$

where C_1 and C_2 depend on δ, $\|z_c\|_{L^\infty}$, and $\|v\|_{L^\infty(B)}$. Again since $\beta \le 1$, by the Hardy inequality for bounded domains, we have that $v(x)(1-|x|)^{-\beta} \in L^1(B)$. Therefore J is differentiable at z_c in the direction of $v \in H_0^1(B) \cap L^\infty(B)$. $\square$

Combining all of the above, we then have the following result (cf. [14, Theorem 7.3]):

Theorem 3.5. Under **(H$\tilde{a}$)**, **(Hg)**, **(H$\tilde{b}$)**, **(Hλ)** and **(H$\tilde{h}$)**, there exists $c_1 > 0$ such that for all $0 \le c \le c_1$, equation (3.4) has a positive weak solution $z_c \in H_0^1(B) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{B})$.

3.3 Return to External Domain

Now that we have a positive solution z_c to equation (3.4), we consider $\hat{z}_c$, the Kelvin transform of z_c, which will provide a solution to (3.1) in the exterior domain Ω. From Theorem 1.11, we know that the Kelvin transform maps $H_0^1(B)$ to the space $\mathcal{D}^{1,2}(\Omega)$, and so $\hat{z}_c$ will be in $\mathcal{D}^{1,2}(\Omega)$. Throughout this section, we use the variable y inside the ball B and x in the exterior domain Ω.

We first state the assumption in Theorem 3.5 in terms of the original data: $a(x)$, $b(x)$, and $h(x)$. Since by $(\mathbf{H}\tilde{a})$, $\tilde{a} \in L^\infty(B)$, we have

$$\sup_{y\in B} |\tilde{a}(y)| = \sup_{y\in B} \left| \frac{1}{|y|^4} a\Big(\frac{y}{|y|^2}\Big) \right| < \infty.$$

Therefore we require:

$(\mathbf{H}a)'$ The function $a : \Omega \to \mathbb{R}$ is positive and belongs to $L^\infty(\Omega; |x|^4)$.

Recall that

$$\lambda_1 = \inf_{u\in\mathcal{D}^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int_\Omega au^2},$$

which, since by Theorem 1.11, $\|u\|^2_{H_0^1(B)} = \|\hat{u}\|^2_{\mathcal{D}^{1,2}(\Omega)}$ and $\int_B \tilde{a}u^2 = \int_\Omega a\hat{u}^2$, satisfies $\lambda_1 = \tilde{\lambda}_1$.

From (5.8), we have that

$$\frac{1}{|y|^{N+2}} b\Big(\frac{y}{|y|^2}\Big) \le C_1 \frac{1}{|y|^4} a\Big(\frac{y}{|y|^2}\Big)(1-|y|)^{-\beta}|y|^{-(N-2)(1+\beta)}$$

for some constant $C_1 > 0$ and $y \in B$. Again with $x = \dfrac{y}{|y|^2} \in \Omega$, this becomes

$$b(x) \le C_1|x|^{\beta(N-2)}a(x)\left(1 - \left|\frac{x}{|x|^2}\right|\right)^{-\beta} = C_1|x|^{\beta(N-2)}a(x)\left(\frac{|x|-1}{|x|}\right)^{-\beta} = C_1 a(x)\frac{|x|^{\beta(N-1)}}{(|x|-1)^\beta}$$

In particular, since $\beta > 0$, taking noted of $(\mathbf{H}a)'$ we observe that $b(x)$ can be unbounded as $|x| \to 1^+$ as well as $|x| \to \infty$ (at least for N sufficiently large). We therefore require

$(\mathbf{H}b)'$ The function $b : \Omega \to \mathbb{R}$ is non-negative, not identically equal to zero, and satisfies

$$b(x) \le C_1 a(x)\frac{|x|^{\beta(N-1)}}{(|x|-1)^\beta} \tag{3.15}$$

for some $0 < \beta \le 1$, $C_1 > 0$ and all $x \in \Omega$. Now, with $\Omega_0 = \{x \in \Omega : b(x) = 0\} \subsetneq \Omega$, we require either Ω_0 has measure zero or $\Omega_0 = \overline{\operatorname{int}\Omega_0}$ (closure in Ω_0) with $\partial\Omega_0$ Lipschitz. Note that Ω_0 may now be unbounded as well.

In the former case we set $\lambda_* = +\infty$, and in the latter case

$$\lambda_* = \inf_{u \in \mathcal{D}^{1,2}(\operatorname{int}\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla u|^2}{\int_{\Omega_0} a u^2} = \inf_{\hat{u} \in H_0^1(\operatorname{int} B_0)\setminus\{0\}} \frac{\int_{B_0} |\nabla \hat{u}|^2}{\int_{B_0} \tilde{a} \hat{u}^2} = \tilde{\lambda}_*$$

For the function h, we needed only that $\tilde{h} \in L^s(B)$ for some $s > N$, meaning that

$$\int_B \left| \frac{1}{|y|^{N+2}} h\Big(\frac{y}{|y|^2}\Big) \right|^s \mathrm{d}y < \infty.$$

Using the properties of the Kelvin transform (cf. Section 1.3), we have

$$\int_B \left| \frac{1}{|y|^{N+2}} h\Big(\frac{y}{|y|^2}\Big) \right|^s \mathrm{d}y = \int_B \frac{1}{|y|^{(N+2)s}} h\Big(\frac{y}{|y|^2}\Big)^s \mathrm{d}y = \int_\Omega |x|^{(N+2)s} h(x)^s \frac{1}{|x|^{2N}} \,\mathrm{d}x < \infty$$

Therefore, we must have

(Hh)′ The non-negative and not identically equal to zero function $h(x)$ belongs to the weighted L^s space $L^s(\Omega; |x|^{(N+2)s-2N})$ for some $s > N$.

Finally, since $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda}_* = \lambda_*$, we still have:

(Hλ) The value λ is such that $\lambda_1 < \lambda < \lambda_*$.

(Hc) The parameter c is nonnegative.

We can now state:

Theorem 3.6. Under **(Ha)′**, **(Hb)′**, **(Hg)**, **(Hλ)** and **(Hh)′**, there exists $c_1 > 0$ such that for all $0 \le c \le c_1$, (3.1) has a positive weak solution $\hat{z}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$.

Remark 3.7. Since the solution described in Theorem 3.6 is the Kelvin transform of $z_c \in H_0^1(B) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{B})$, we observe that there exists $C_3 > 0$ such that the solution $\hat{z}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ satisfies

$$\lim_{|x|\to\infty} \hat{z}_c(x)\, |x|^{N-2} = C_3. \tag{3.16}$$

Chapter 4

Comparison of Results and Stability

In this chapter, we first compare the assumptions on our coefficient functions from Chapters 2 and 3, and then seek to answer the following question: Do the solutions obtained for the equation

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{4.1}$$

in Chapters 2 and 3 satisfy some uniqueness property? To answer this question, we will first show that if the assumptions given in Chapters 2 and 3 are slightly strengthened in each case, then we do in fact have uniqueness in a sense that will be made precise below. Additionally, under a new set of assumptions on the data that satisfies both sets of hypotheses adopted in Chapters 2 and 3, respectively, we will show that the solutions to (4.1) obtained through the two different approaches are actually the same.

4.1 Comparison of Assumptions

We first compare the stated assumptions on the data for the two different approaches above.

In Chapter 3, we assumed that the positive function $a : \Omega \to \mathbb{R}$ belongs to the space $L^\infty(\Omega; |x|^4)$, whereas the direct method of Chapter 2 required $a \in L^{N/2}(\Omega) \cap L^\infty(\Omega)$.

We observe that Kelvin transform approach requires a stronger assumptions on $a(x)$. Indeed, $a \in L^\infty(\Omega; |x|^4)$ implies that $a \in L^\infty(\Omega)$, and furthermore,

$$\int_\Omega |a(x)|^{N/2} \, \mathrm{d}x \leq \int_\Omega \frac{C}{|x|^{2N}} \, \mathrm{d}x < \infty.$$

As for the nonlinearity, in both approaches we assume the same hypothesis $(\mathbf{H}g)$.

Next, we take up the assumptions on $b(x)$. Under the Kelvin transform approach of Chapter 3, we required that the nonnegative and not identically zero function $b : \Omega \to \mathbb{R}$ satisfy

$$b(x) \leq C_1 a(x) \frac{|x|^{\beta(N-1)}}{(|x|-1)^{\beta}}, \quad x \in \Omega \tag{4.2}$$

for some constant $C_1 > 0$.

On the other hand, in the direct method employed in Chapter 2, the above inequality is replaced by

$$b(x) \leq C_2 a(x)(1+|x|^2)^{\beta(N-2)/2}, \quad x \in \Omega \tag{4.3}$$

for some $C_2 > 0$. Here we can see that (4.2) allows $b(x)$ to become unbounded both as $|x| \to \infty$ and $|x| \to 1^+$, whereas (4.3) provides the same relation between $b(x)$ and $a(x)$ as $|x| \to \infty$ but will only allow $b(x) \leq Ca(x)$ as $|x| \to 1^+$. This is the key difference in the approaches here.

The direct approach of Chapter 2 imposes a stronger hypothesis on $b(x)$, and therefore assumption (4.3) is needed to fulfill the restrictions needed for both approaches.

As shown in Section 3.3, the values $\tilde{\lambda}_1$ and $\tilde{\lambda}_*$ in the ball B are equal to the values λ_1 and λ_*, defined as

$$\lambda_1 = \inf_{u \in \mathcal{D}^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\int au^2} \tag{4.4}$$

and $\lambda_* = +\infty$ if $\Omega_0 := \{x \in \Omega : b(x) = 0\}$ has measure zero or

$$\lambda_* = \inf_{u \in \mathcal{D}^{1,2}(\operatorname{int}\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} \|\nabla u\|^2}{\int_{\Omega_0} au^2}, \tag{4.5}$$

otherwise.

Finally, under the Kelvin transform approach of Chapter 3, we assumed that the non-negative and not identically equal to zero function $h(x)$ belongs to the weighted L^s space $V := L^s(\Omega; |x|^{(N+2)s-2N})$ for some $s > N$. In the direct approach of Chapter 2, we needed for $h(x) \in Z$ where $Z := L^1(\Omega) \cap L^s(\Omega)$ for some $s > N$, and additionally required the existence of a constant $C_3 > 0$ such that

$$R^{N/r}\|h\|_{L^q(\Omega\setminus B_R(0))} \leq C_3 \quad \text{for all } R \in \mathbb{R}^+ \tag{4.6}$$

with $\frac{1}{q} + \frac{1}{r} = 1$. We recall that (4.6) is satisfied, for example, if $|h(x)| \leq C/|x|^{\alpha}$ for some $\alpha > N$ (cf.

[6, 7]). We then make the following claim:

Lemma 4.1. If there exists $C_4 > 0$ such that

$$h(x) \leq \frac{C_4}{|x|^m}$$

for all $x \in \Omega$ and $m > N+2$, then $h(x) \in L^s(\Omega; |x|^{(N+2)s-2N}) \cap L^1(\Omega) \cap L^s(\Omega)$ for some $s > N$, and additionally satisfies (4.6).

Proof. For $h(x)$ to be in the space $L^s(\Omega; |x|^{(N+2)s-2N})$, with $s > N$, we must consider the integral

$$\int_\Omega |h(x)|^s |x|^{(N+2)s-2N} \, \mathrm{d}x \leq C \int_\Omega \frac{|x|^{(N+2)s}}{|x|^{2N+sm}} \, \mathrm{d}x.$$

For this integral to be finite, we need $2N + sm - s(N+2) > N$, i.e.

$$s(m - N - 2) > -N \qquad \Longleftrightarrow \qquad m > N + 2 - \frac{N}{s},$$

which is certainly satisfied for $m > N+2$. Similarly, since $m > N$, we have that the integrals

$$\int_\Omega |h(x)| \, \mathrm{d}x \leq \int_\Omega \frac{C_4}{|x|^m} \, \mathrm{d}x \quad \text{ and } \quad \int_\Omega |h(x)|^s \, \mathrm{d}x \leq C \int_\Omega \frac{1}{|x|^{sm}} \, \mathrm{d}x$$

are both finite; therefore $h(x) \in L^1(\Omega) \cap L^s(\Omega)$. Finally, since $h \leq C/|x|^m$ and $m > N+2$, it satisfies the sufficient conditions to satisfy (4.6) as shown in [6, 7]. □

Under a set of assumptions on the data that satisfies what is required for both approaches simultaneously, we then obtain two solutions – one via the direct approach of Chapter 2 and one via the Kelvin transform approach of Chapter 3. Explicitly:

Theorem 4.2. Consider equation (4.1) with the following assumptions:

1. The function $a : \Omega \to \mathbb{R}$ is positive and belongs to the space $L^\infty(\Omega; |x|^4)$.

2. The function $g : \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(s) = 0$ for $s \leq 0$. Furthermore, it satisfies

$$\limsup_{s\to 0} \frac{g(s)}{s^{1+\beta}} < \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s} = \infty, \tag{4.7}$$

where $0 < \beta \leq 1$ is a fixed constant.

3. The measurable function $b : \Omega \to \mathbb{R}$ is non-negative, not identically equal to zero, and satisfies

$$b(x) \leq C_1 a(x)(1 + |x|^2)^{\beta(N-2)/2} \tag{4.8}$$

for some $C_1 > 0$ and all $x \in \Omega$.

4. The value λ is such that $\lambda_1 < \lambda < \lambda_*$, where λ_1 and λ_* are defined above in (4.4) and (4.5), respectively.

5. There exists $C_4 > 0$ such that the non-negative and not identically zero function $h(x)$ satisfies

$$h(x) \leq \frac{C_4}{|x|^m} \tag{4.9}$$

for all $x \in \Omega$ and $m > N + 2$, and

6. The parameter c is nonnegative.

Then there exists $c_0 > 0$ such that for all $0 \leq c \leq c_0$, we obtain two (possibly identical) positive weak solutions to (4.1), namely $u_c, \hat{z}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$, from the direct approach of Chapter 2 and the Kelvin transform approach of Chapter 3, respectively. Furthermore, there exists $C_5, C_6 > 0$ such that for all $0 \leq c \leq c_0$, these weak solutions satisfy

$$u_c(x) \geq \frac{C_5}{|x|^{N-2}} \quad \text{for large } |x| \quad \text{and} \quad \lim_{|x|\to\infty} \hat{z}_c(x)\,|x|^{N-2} = C_6.$$

4.2 Uniqueness of Stable Solution in the Ball

We now wish to show that the solution $z_c \in H_0^1(B)$ obtained via Theorem 3.5 is the unique positive and stable solution of (3.4). Let $\lambda_i(\phi, B)$ denote the i-th eigenvalue of $-\Delta + \phi$ over the region B

with Dirichlet boundary condition. We omit the potential ϕ and write $\lambda_i(B)$ if $\phi = 0$. Furthermore, for a solution u of (3.4), let $\mu_i(u)$ denote the i-th eigenvalue of the linearization of (3.4) at u, that is $\mu_i(u) = \lambda_i(-\lambda\tilde{a} + \tilde{b}(x)|x|^{N-2}g'(|x|^{N-2}u), B)$. Following the classical terminology, u will be called *stable* if $\mu_1(u) > 0$, and *unstable* if $\mu_1(u) < 0$. We also recall that

$$\lambda_i(\phi_1, B) < \lambda_i(\phi_2, B) \quad \text{ if } \phi_1 \geq \phi_2 \text{ and } \phi_1 \not\equiv \phi_2.$$

Note that $\lambda_1(B)$ and $\lambda_1(B_0)$ are the same as $\tilde{\lambda}_1$ and $\tilde{\lambda}_*$ in Section 3.2. Based on the results in Du & Ma (cf. [10, 11]), Ouyang (cf. [18]), and Shabani & Tehrani (cf. [19, 20]), for the case $c = 0$, that is, in the absence of harvesting, the equation

$$\begin{cases} -\Delta u = \lambda\tilde{a}(x)u - \tilde{b}(x)g(|x|^{N-2}u), & x \in B \\ u = 0, & x \in \partial B. \end{cases} \tag{4.10}$$

has a unique positive solution:

Theorem 4.3 ([20, Theorem 1.1])**.** Assume $\tilde{b}(x) \not\equiv 0$ in B.

(i) If $B_0 = \emptyset$, then for every $\lambda > \lambda_1(B)$, equation (4.10) has a unique positive solution u_λ.

(ii) If $B_0 \neq \emptyset$, then for any $\lambda \in (\lambda_1(B), \lambda_1(B_0))$, equation (4.10) has a unique positive solution u_λ. In addition, if $\lambda \geq \lambda_1(B_0)$, then (4.10) has no nonnegative solution except zero.

Furthermore, in either case the curve $\lambda \to u_\lambda$ is continuous and increasing and the positive solution u_λ is stable, i.e. $\mu_1(u_\lambda) > 0$.

This result requires a classical comparison principle (see Du & Ma [11]), which we state below in a version suitable for our purpose here:

Lemma 4.4 (Comparison Principle)**.** Suppose that $\tilde{a}(x), \tilde{b}(x) \in L^\infty(B)$ and $0 \leq \tilde{b}(x)$ is not identically zero. Let $u_1, u_2 \in C^2(B)$ be positive in B and satisfy

$$\Delta u_1 + \lambda\tilde{a}(x)u_1 - \tilde{b}(x)g(|x|^{N-2}u_1) \leq 0 \leq \Delta u_2 + \lambda\tilde{a}(x)u_2 - \tilde{b}(x)g(|x|^{N-2}u_2), \quad x \in B \tag{4.11}$$

and $\limsup_{x\to\partial B}(u_2 - u_1) \leq 0$, where g is continuous and such that $g(s)/s$ is strictly increasing in the range $\inf_B\{u_1, u_2\} < u < \sup_B\{u_1, u_2\}$. Then $u_2 \leq u_1$ in B.

For Lemma 4.4 to hold as well as in the following proofs, we need some slightly stronger assumptions on the nonlinearity g and the function h, and so in addition to the hypothesis listed in Section 3.3, we also assume:

(H) $h(x)$ is positive in Ω and the function $g : \mathbb{R} \to \mathbb{R}_0^+$ belongs to $C^1(\mathbb{R})$ with $g(s) = 0$ for $s \leq 0$ and $g'(s)$ strictly increasing for $s > 0$. Additionally, g satisfies

$$\limsup_{s\to 0^+} \frac{g'(s)}{s^\beta} < \infty \tag{4.12}$$

where $0 < \beta < 1$ is a fixed constant.

One can easily see that g' strictly increasing and $g(0) = 0$ implies that $g(s)/s$ is strictly increasing as well. We then claim the following:

Theorem 4.5. For $0 \leq c < c_1$ (with c_1 as in Theorem 3.5) the solution z_c to problem (3.4) given in Theorem 3.5 is stable, i.e. $\mu_1(z_c) \geq 0$.

Proof. As described above, $\mu_1(z_c)$ is the first eigenvalue of the linearization of (3.4) at z_c, i.e.

$$\mu_1(z_c) = \inf_{w\in H_0^1(B), w\neq 0} \frac{\int_B |\nabla w|^2 + (\tilde{b}(x)|x|^{N-2}g'(|x|^{N-2}z_c) - \lambda\tilde{a}(x))w^2}{\int_B w^2} \tag{4.13}$$

Note that the solution z_c is found by minimizing the functional

$$\tilde{I}_c(u) = \frac{1}{2}\int_B |\nabla u|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_B \tilde{a}(x)u^2 \,\mathrm{d}x + \int_B \tilde{b}(x)\frac{G(|x|^{N-2}u)}{|x|^{N-2}} \,\mathrm{d}x + c\int_B \tilde{h}(x)u \,\mathrm{d}x$$

on the set $\tilde{M}_c = \{u \in H_0^1(B) : \underline{z}_c \leq u \text{ a.e. in } B\}$, where $G(u) = \int_0^u g(s)\,\mathrm{d}s$ and $\underline{z}_c$ is the solution to the auxiliary problem (3.11). Thus we expect that $\langle \tilde{I}_c''(z_c)w, w\rangle \geq 0$ for any directions w where it is defined. We have previously shown that $\tilde{I}_c$ is differentiable in the direction of $w \in H_0^1(B) \cap L^\infty(B)$

(cf. Lemma 3.4), and for $w \in H_0^1(B) \cap L^\infty(B)$, we have

$$\tilde{I}_c'(z_c)w = \int_B \nabla z_c \cdot \nabla w \,\mathrm{d}x - \lambda \int_B \tilde{a}(x) z_c w \,\mathrm{d}x + \int_B \tilde{b}(x) g(|x|^{N-2} z_c) w \,\mathrm{d}x + \int_B h(x) w \,\mathrm{d}x.$$

From this, we would like to prove that

$$\langle \tilde{I}_c''(z_c)w, w\rangle = \int_B |\nabla w|^2 \,\mathrm{d}x + \int_B (\tilde{b}(x)|x|^{N-2} g'(|x|^{N-2} z_c) - \lambda \tilde{a}(x)) w^2 \,\mathrm{d}x$$

for all $w \in H_0^1(B) \cap L^\infty(B)$. We first have to show that $\langle \tilde{I}_c''(z_c)w, w\rangle$ is defined for all such w. It is enough to consider twice-differentiability (at $t = 0$) of

$$K_w(t) := J(z_c + tw) = \int_B \frac{\tilde{b}(x) G(|x|^{N-2}(z_c + tw))}{|x|^{N-2}} \,\mathrm{d}x.$$

By the same arguments as in the proofs of Lemmas 2.10 and 3.4, $K_w(t)$ can be seen to be well-defined and differentiable for $|t|$ sufficiently small, with

$$K_w'(t) = \int_B \tilde{b}(x) g(|x|^{N-2}(z_c + tw)) w \,\mathrm{d}x.$$

Now

$$g(|x|^{N-2}(z_c + tw)) - g(|x|^{N-2} z_c) = |x|^{N-2} g'(|x|^{N-2}(z_c + stw))(tw) \ \text{ for some } s = s(t,x),\ 0 \le s \le 1,$$

and so we have to show that

$$\begin{aligned} \lim_{t\to 0} \frac{K_w'(t) - K_w'(0)}{t} &= \lim_{t\to 0} \int_B \tilde{b}(x)|x|^{N-2} g'(|x|^{N-2}(z_c + stw)) w^2 \,\mathrm{d}x \\ &= \int_B \tilde{b}(x)|x|^{N-2} g'(|x|^{N-2} z_c) w^2 \,\mathrm{d}x. \end{aligned} \tag{4.14}$$

We claim that this follows from (4.12) for $\beta \le 1$. Indeed, by (4.12), for $|x| < \delta = \delta(\|z_c\|_{L^\infty(B)}, \|w\|_{L^\infty(B)})$, we have

$$0 \le g'(|x|^{N-2}(z_c + stw)) \le C(|x|^{N-2})^\beta.$$

Then from (5.8) and the fact that $\tilde{a}(x) \in L^\infty(B)$, we have

$$
\begin{aligned}
\int_B \tilde{b}(x)|x|^{N-2} g'(|x|^{N-2}(z_c + stw))w^2 \, \mathrm{d}x & \\
&\le C_1 \int_B \tilde{a}(x)(1-|x|)^{-\beta}|x|^{-\beta(N-2)} g'(|x|^{N-2}(z_c+stw))w^2 \, \mathrm{d}x \\
&\le C \int_{|x|<\delta} w^2 \, \mathrm{d}x + C(\delta) \int_{|x|>\delta} g'(|x|^{N-2}(z_c+stw))w^2(1-|x|)^{-\beta} \, \mathrm{d}x \\
&\le C(\delta)\bigg(\|w\|_{L^2(B)} + \int_B \frac{|w|^2}{(1-|x|)^\beta} \, \mathrm{d}x \bigg).
\end{aligned}
$$

So, as in the proof of Lemma 3.4, the Hardy inequality for bounded domain implies (since $\beta \le 1$):

$$
\int_B \frac{|w|^2}{(1-|x|)^\beta} \, \mathrm{d}x \le \int_B \frac{|w|^2}{(1-|x|)^2} \, \mathrm{d}x \le C \int_B |\nabla w|^2 \, \mathrm{d}x.
$$

Therefore (4.14) follows by Lebesgue's Dominated Convergence Theorem. Thus $\langle \tilde{I}_c''(z_c)w, w\rangle$ is well-defined, and we have that

$$
K_w''(0) = \langle \tilde{I}_c''(z_c)w, w\rangle = \int_B |\nabla w|^2 \, \mathrm{d}x + \int_B \big(\tilde{b}(x)|x|^{N-2} g'(|x|^{N-2} z_c) - \lambda \tilde{a}(x)\big) w^2 \, \mathrm{d}x.
$$

Now recall that the weak solution $z_c \in L^\infty(B)$, $\tilde{a}(x) \in L^\infty(B)$, $\tilde{b}(x)g(|x|^{N-2}z_c) \in L^p$ for some $p > N$ (since $\beta < 1$), and $\tilde{h}(x) \in L^\infty(B)$, so $z_c \in C^{1,\alpha}(B)$, and so by the strong maximum principle (cf. Lemma 1.5) $\underline{z}_c < z_c$ in B. Therefore for $w_0 \in C_0^\infty(B)$ we have: $z_c + tw_0 \in \tilde{M}_c$ for $|t|$ small, hence as z_c is the minimizer of $\tilde{I}_c$ on $\tilde{M}_c$, we have $K_{w_0}''(0) = \langle \tilde{I}_c''(z_c)w_0, w_0\rangle \ge 0$. Now by a density argument we finally have that

$$
K_w''(0) = \langle \tilde{I}_c''(z_c)w, w\rangle \ge 0
$$

for all $w \in H_0^1(B) \cap L^\infty(B)$, i.e. $\mu_1(z_c) \ge 0$. □

Next note that if u_1 is a positive solution of (3.4) for $c = c_1$, then for $c_2 < c_1$, we have that u_1 is a sub solution of (3.4) for $c = c_2$. Also, $u_1 \le u_\lambda$, where u_λ is the solution for $c = 0$ (see Lemma 4.4 above). Thus u_1 and u_λ form an ordered sub/super solution pair of (3.4). Therefore there exists a solution $\bar{u}$ of (3.4) for $c = c_2$ such that $u_1 \le \bar{u} \le u_\lambda$ in B. Now considering $u_1 - \bar{u}$ and applying

Hopf's strong maximum principle (cf. Lemmas 1.4, 1.5), we have that $u_1(x) < \bar{u}(x)$ on B, and in particular $\bar{u} > 0$ in B.

In this way, for $0 < c < c_1$, we get a new family of solutions, which we still denote by z_c, such that $z_c > z_{c_1}$. As g' is strictly increasing we now have $\mu_1(z_c) > 0$ for $0 < c < c_1$, and so the solution z_c is stable.

Next, for $q > N$, let $X = \{u \in W^{2,q}(B), u = 0 \text{ on } \partial B\}$, $Y = L^q(B)$, and consider $F : \mathbb{R}\times X \to Y$, given by:

$$F(c,u) = \Delta u + \lambda \tilde{a}(x)u - \tilde{b}(x)g(|x|^{N-2}u) - c\tilde{h}(x).$$

The stability of z_c for $0 \le c \le c_1$ implies that for a fixed $0 < c^* < c_1$, the implicit function theorem is applicable to $F(c,u)$ near the point (c^*, z_{c^*}), and the solution set of (3.4) near (c^*, z_c) forms a curve, which we denote by $\gamma(c)$, for $|c - c^*|$ small. So $\gamma(c) = z_{c^*} + \gamma'(c^*)(c-c^*) + o(|c-c^*|)$, where $\gamma'(c^*) = w^*$ solves

$$-\Delta w^* - \left[\lambda \tilde{a}(x) z_{c^*} - \tilde{b}(x) g(|x|^{N-2} z_{c^*})\right] w^* = -\tilde{h}(x).$$

Since $\mu_1(z_{c^*}) > 0$ we have $w^* < 0$ in B which implies that $\gamma(c) > z_{c^*}$ for $c < c^*$. Therefore $\mu_1(\gamma(c)) > 0$ for $c < c^*$ and $|c - c^*|$ small, and hence $\gamma(c)$ can be continued for $c < c^*$ all the way to $c = 0$. Now the uniqueness result of Theorem 4.3 above implies that $\gamma(0) = u_\lambda$. Hence z_c for $0 < c < c^*$ are on the curve $\gamma(c)$ and are the unique stable solutions of (3.4). See Figure 4.1 for a

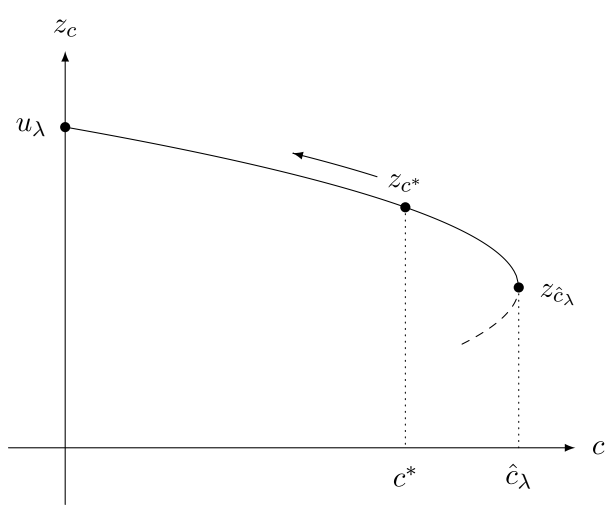

Figure 4.1: An application of the implicit function theorem to show stability.

visual representation of the ideas presented here. These considerations provide the sketch of the proof of the following result, which was first given in [19] for a similar equation:

Theorem 4.6 ([19, Theorem 2.6])**.** Suppose $\lambda_1(B) < \lambda < \lambda_1(B_0)$ as in **(H**λ**)**, $b(x) \not\equiv 0$ and $h(x) \not\equiv 0$. Then there exists $\hat{c}_\lambda > 0$ such that:

(i) If $0 \le c < \hat{c}_\lambda$, equation (3.4) has a positive solution z_c. If $c > \hat{c}_\lambda$ then no solution of (3.4) stays positive in B.

(ii) The curve $c \to z_c$ is decreasing with respect to the parameter c for $c \in [0, \hat{c}_\lambda)$ and z_c is stable, that is, $\mu_1(z_c) > 0$. Furthermore, z_c is the unique positive stable solution of (3.4).

We can now state:

Theorem 4.7. Consider equation (4.1).

(i) Under the hypotheses **(H**a**)**$'$, **(H**b**)**$'$, **(H**g**)**, **(H**λ**)** and **(H**h**)**$'$ presented in Chapter 3, there exists $c_1 > 0$ such that for all $0 \le c < c_1$, (4.1) has a positive weak solution $\hat{z}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$. In addition, for each $0 < c < c_1$, there exists $L_c > 0$ such that

$$\lim_{|x| \to \infty} \hat{z}_c(x)\, |x|^{N-2} = L_c. \tag{4.15}$$

(ii) Furthermore, under the additional hypothesis **(H)**, for $0 \le c < c_1$, z_c, the Kelvin transform of $\hat{z}_c$, is the unique stable positive solution of the transformed equation (3.4) in the ball.

4.3 Stability through the Direct Approach

We now show that the positive weak solution u_c obtained by the direct approach of Chapter 2 (cf. Theorem 2.7) is stable in the sense that its Kelvin transform $\hat{u}_c$ is stable in the ball. We start by adding the additional assumptions **(H)** mentioned in Section 4.2.

We first want to show that $\hat{u}_c$ is a weak solution to the equation

$$\begin{cases} -\Delta \hat{u}_c(y) = \lambda \tilde{a}(y)\hat{u}_c - \tilde{b}(y) g(|y|^{N-2}\hat{u}_c) - c\tilde{h}(y), & y \in B \\ \hat{u}_c = 0, & y \in \partial B, \end{cases} \tag{4.16}$$

where $\tilde{a}(y) = \dfrac{1}{|y|^4} a\Big(\dfrac{y}{|y|^2}\Big)$, $\tilde{b}(y) = \dfrac{1}{|y|^{N+2}} b\Big(\dfrac{y}{|y|^2}\Big)$, and $\tilde{h}(y) = \dfrac{1}{|y|^{N+2}} h\Big(\dfrac{y}{|y|^2}\Big)$, as in Section 3.2. In other words, with

$$\tilde{I}_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_B \tilde{a}(x)u^2 \,\mathrm{d}x + \int_B \tilde{b}(x)\frac{G(|x|^{N-2}u)}{|x|^{N-2}}\,\mathrm{d}x + c\int_B \tilde{h}(x)u\,\mathrm{d}x, \tag{4.17}$$

we want to show that

$$\langle \tilde{I}_c'(\hat{u}_c), \varphi\rangle = \int_B \nabla\hat{u}_c\nabla\varphi\,\mathrm{d}x - \lambda\int_B \tilde{a}(x)\hat{u}_c\varphi\,\mathrm{d}x + \int_B \tilde{b}(x)g(|x|^{N-2}\hat{u}_c)\varphi\,\mathrm{d}x + c\int_B \tilde{h}(x)\varphi\,\mathrm{d}x = 0$$

for all $\varphi \in H_0^1(B)$. Using (1.19) and the properties of the Kelvin transform, we have that

$$\begin{aligned}
\int_B \nabla\hat{u}_c\nabla\varphi\,\mathrm{d}x - \lambda\int_B \tilde{a}(x)\hat{u}_c\varphi\,\mathrm{d}x &= \int_\Omega \nabla u_c\nabla\hat{\varphi}\,\mathrm{d}x - \lambda\int_B \frac{1}{|x|^4}a\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{N-2}}u_c\Big(\frac{x}{|x|^2}\Big)\varphi(x)\,\mathrm{d}x \\
&= \int_\Omega \nabla u_c\nabla\hat{\varphi}\,\mathrm{d}x - \lambda\int_\Omega |x|^{N+2}a(x)u_c(x)\varphi\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{2N}}\,\mathrm{d}x \\
&= \int_\Omega \nabla u_c\nabla\hat{\varphi}\,\mathrm{d}x - \lambda\int_\Omega a(x)u_c\hat{\varphi}\,\mathrm{d}x.
\end{aligned}$$

We would like to say that this is equal to

$$-\int_\Omega b(x)g(u_c)\hat{\varphi}\,\mathrm{d}x - c\int_\Omega h(x)\hat{\varphi}\,\mathrm{d}x$$

as u_c solves equation (4.1), but the integral $\int_\Omega b(x)g(u_c)\hat{\varphi}\,\mathrm{d}x$ might not make sense. We have only shown thus far (cf. Lemma 2.10) that the term

$$J(u) = \int_\Omega b(x)G(u)\,\mathrm{d}x \tag{4.18}$$

is differentiable at u_c in the direction of $v \in \mathcal{D}^{1,2}(\Omega)\cap L^\infty(\Omega)$ with compact support. Since $\varphi \in H_0^1(B)$, its Kelvin transform $\hat{\varphi}$ does not necessarily have compact support, and instead belongs to the space $\mathcal{D}^{1,2}(\Omega)\cap L^\infty(\Omega; |x|^{N-2})$. See Figure 4.2 for an illustration.

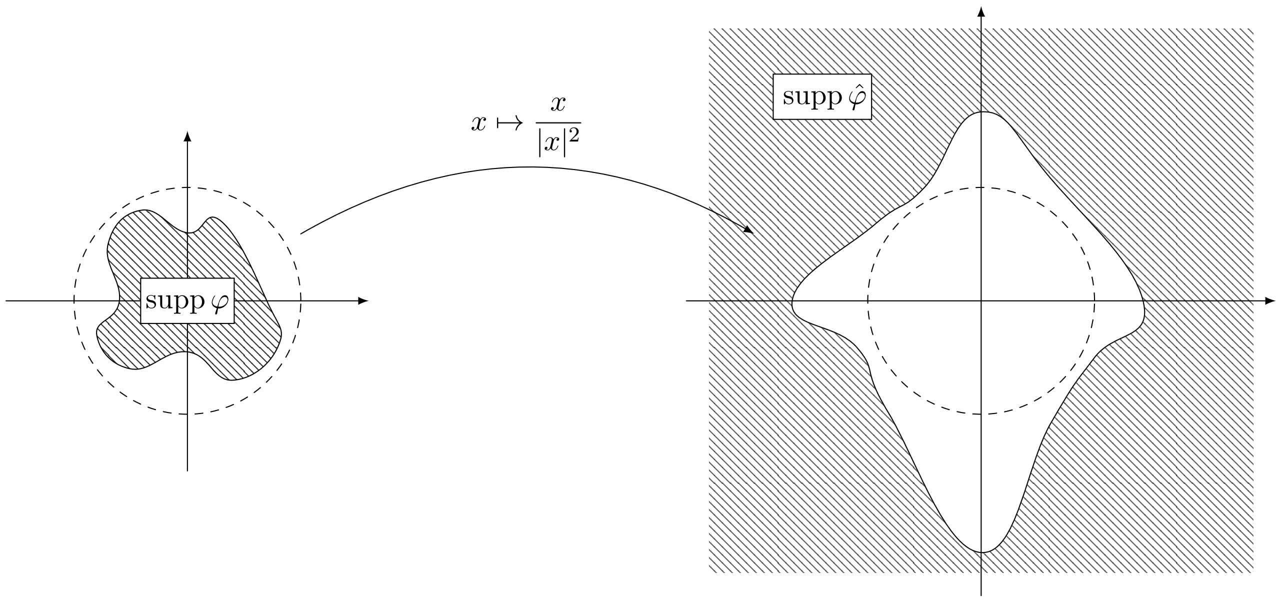

Figure 4.2: The support of the Kelvin transform of a function $\varphi \in H_0^1(B)$.

We must introduce additional assumptions in order to prove differentiability of $J(u)$ in the direction of $\hat{\varphi}$:

Lemma 4.8. Suppose that $\hat{\varphi} \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega; |x|^{N-2})$, and let u_c be the solution of equation (4.1) obtained through the direct method of Chapter 2. Assume that either

- $\beta < \frac{1}{2}$ and $a(x) \in L^{2N/(N+2)}(\Omega)$

or

- $\beta < 1$ and $a(x) \in L^{p'}(\Omega)$, where p' is the Hölder conjugate of

$$p = \frac{N + \epsilon}{(1-\beta)(N-2)} \tag{4.19}$$

for some $\epsilon > 0$.

Then the functional

$$I_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega a(x)u^2\,\mathrm{d}x + \int_\Omega b(x)G(u)\,\mathrm{d}x + c\int_\Omega h(x)u\,\mathrm{d}x$$

is twice-differentiable at u_c in the direction of $\hat{\varphi}$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b(x)G(u_c+t\hat{\varphi})\,\mathrm{d}x\bigg|_{t=0} = \int_\Omega b(x)g(u_c)\hat{\varphi}\,\mathrm{d}x. \tag{4.20}$$

as well as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega b(x)G(u_c+t\hat{\varphi})\,\mathrm{d}x\bigg|_{t=0} = \int_\Omega b(x)g'(u_c)\hat{\varphi}^2\,\mathrm{d}x. \tag{4.21}$$

Proof. We will first show that $J(u_c+t\hat{\varphi})$ is well-defined in either case. We know that $u_c \in L^\infty(\Omega)$ from Remark 2.1 and Lemma 2.9, and since $L^\infty(\Omega;|x|^{N-2}) \subset L^\infty(\Omega)$, we also know that $\hat{\varphi} \in L^\infty(\Omega)$. Thus by the same argument as in Lemma 2.10, we get that

$$G(u_c+t\hat{\varphi}) - G(u_c) = g(u_c+st\hat{\varphi})t\hat{\varphi} \;\text{ for some } s = s(t,x),\ 0 \le s \le 1.$$

Assume first that $\beta < \frac{1}{2}$ and $a(x) \in L^{2N/(N+2)}(\Omega)$. Then

$$\int_\Omega b(x)g(u_c+t\hat{\varphi})|\hat{\varphi}|\,\mathrm{d}x \le C\int_\Omega a(x)(1+|x|^2)^{\beta(N-2)/2}|\hat{\varphi}|\,\mathrm{d}x. \tag{4.22}$$

Since $\hat{\varphi} \in L^\infty(\Omega;|x|^{N-2})$ and by applying Hölder's inequality, we then have

$$\begin{aligned}\int_\Omega b(x)g(u_c+t\hat{\varphi})|\hat{\varphi}|\,\mathrm{d}x &\le C\int_\Omega a(x)\frac{(1+|x|^2)^{\beta(N-2)/2}}{|x|^{N-2}}\,\mathrm{d}x\\ &\le C\Big(\int_\Omega |a(x)|^{2N/(N+2)}\,\mathrm{d}x\Big)^{(N+2)/2N}\Big(\int_\Omega \frac{(1+|x|^2)^{\beta N}}{|x|^{2N}}\,\mathrm{d}x\Big)^{(N-2)/2N}.\end{aligned}$$

The first integral on the right hand side is finite since $a \in L^{2N/(N+2)}(\Omega)$. As for the second integral, since $\beta < \frac{1}{2}$, we have that

$$\frac{(1+|x|^2)^{\beta N}}{|x|^{2N}} \approx \frac{1}{|x|^{N+\epsilon}} \qquad \text{for large } |x|.$$

Therefore the right hand side of (4.22) is bounded, and so $J(u_c+t\hat{\varphi})$ is well-defined for any $|t| < \delta$ for δ sufficiently small.

Now assume that $\beta < 1$ and $a(x) \in L^{p'}(\Omega)$, with p as in (4.19). By (2.4) and (2.5) (and since

$a(x) \in L^\infty(\Omega)$),

$$b(x) \le C_1 a(x)(1+|x|^2)^{\beta(N-2)/2} \le C a(x)|x|^{\beta(N-2)}, \quad x \in \Omega,$$

and so

$$\begin{aligned}\int_\Omega b(x) g(u_c + t\hat{\varphi})\hat{\varphi}\,\mathrm{d}x &\le \int_\Omega a(x)|x|^{\beta(N-2)} g(u_c + t\hat{\varphi})\hat{\varphi}\,\mathrm{d}x \le C\int_\Omega a(x)|x|^{\beta(N-2)}\frac{1}{|x|^{N-2}}\,\mathrm{d}x \\ &= C\int_\Omega \frac{a(x)}{|x|^{(N-2)(1-\beta)}}\,\mathrm{d}x.\end{aligned}$$

Now, since $a \in L^{p'}(\Omega)$,

$$\int_\Omega a(x)|x|^{(\beta-1)(N-2)}\,\mathrm{d}x \le \left(\int_\Omega a(x)^{p'}\,\mathrm{d}x\right)^{1/p'}\left(\int_\Omega \frac{1}{|x|^{N+\epsilon}}\,\mathrm{d}x\right)^{1/p} < \infty.$$

and so $J(u_c + t\hat{\varphi})$ is well-defined for any $|t| < \delta$ for δ sufficiently small.

In either case, as in the proof of Lemma 2.10, we then have

$$\lim_{t\to 0}\frac{J(u_c + t\hat{\varphi}) - J(u_c)}{t} = \lim_{t\to 0}\frac{1}{t}\int_\Omega b(x) g(u_c + st\hat{\varphi}) t\hat{\varphi}\,\mathrm{d}x = \lim_{t\to 0}\int_\Omega b(x) g(u_c + st\hat{\varphi})\hat{\varphi}\,\mathrm{d}x.$$

We can now apply Lebesgue's Dominated Convergence Theorem to see that J is differentiable at u_c in the direction of $\hat{\varphi} \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega; |x|^{N-2})$ and (4.20) is proved.

We have shown that

$$I_c'(u_c)\hat{\varphi} = \int_\Omega \nabla u_c \cdot \nabla\hat{\varphi}\,\mathrm{d}x - \lambda\int_\Omega a(x) u_c\hat{\varphi}\,\mathrm{d}x + \int_\Omega b(x) g(u_c)\hat{\varphi}\,\mathrm{d}x + \int_\Omega h(x)\hat{\varphi}\,\mathrm{d}x,$$

for all $\hat{\varphi} \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega; |x|^{N-2})$, and from this we would like to prove that

$$\langle I_c''(u_c)\hat{\varphi}, \hat{\varphi}\rangle = \int_\Omega |\nabla\hat{\varphi}|^2\,\mathrm{d}x + \int_\Omega \big(b(x) g'(u_c) - \lambda a(x)\big)\hat{\varphi}^2\,\mathrm{d}x.$$

As for the second derivative, we see that

$$\int_\Omega b(x)g'(u_c+st\hat{\varphi})t\hat{\varphi}^2\,\mathrm{d}x \le C\int_\Omega a(x)|x|^{(N+2)\beta}\hat{\varphi}^2\,\mathrm{d}x \le C\int_\Omega \frac{a(x)}{|x|^{(N-2)(2-\beta)}}\,\mathrm{d}x$$
$$\le \left(\int_\Omega |a(x)|^{N/2}\,\mathrm{d}x\right)^{2/N}\left(\int_\Omega \frac{1}{|x|^{N(2-\beta)}}\,\mathrm{d}x\right)^{(N-2)/N} < \infty$$

using the facts that g' is bounded for bounded inputs, $t \le 1$, and $\hat{\varphi} \le C/|x|^{N-2}$. The final expression being finite relies on the facts that $a \in L^{N/2}(\Omega)$ and $N(2-\beta) > N$ for $\beta < 1$. Thus $\langle I_c''(u_c)\hat{\varphi},\hat{\varphi}\rangle$ is well-defined for $\hat{\varphi} \in \mathcal{D}^{1,2}(\Omega)\cap L^\infty(\Omega;|x|^{N-2})$, and (4.21) is proved. □

We have shown that under additional assumptions on the function $a(x)$ and the number β that

$$\begin{aligned}\int_B \nabla\hat{u}_c\nabla\varphi\,\mathrm{d}x - \lambda\int_B \tilde{a}(x)\hat{u}_c\varphi\,\mathrm{d}x &= \int_\Omega \nabla u_c\nabla\hat{\varphi}\,\mathrm{d}x - \lambda\int_\Omega a(x)u_c\hat{\varphi}\,\mathrm{d}x\\
&= -\int_\Omega b(x)g(u_c)\hat{\varphi}\,\mathrm{d}x - c\int_\Omega h(x)\hat{\varphi}\,\mathrm{d}x\\
&= -\int_\Omega |x|^{N+2}b(x)g(u_c)|x|^{N-2}\hat{\varphi}\frac{1}{|x|^{2N}}\,\mathrm{d}x - c\int_\Omega |x|^{N+2}h(x)|x|^{N-2}\hat{\varphi}\frac{1}{|x|^{2N}}\,\mathrm{d}x.\end{aligned}$$

So, making the change of variables $x \mapsto \dfrac{x}{|x|^2}$, we have that

$$\begin{aligned}\int_B \nabla\hat{u}_c\nabla\varphi\,\mathrm{d}x &= -\int_B \frac{1}{|x|^{N+2}}b\Big(\frac{x}{|x|^2}\Big)g\bigg(\frac{|x|^{N-2}}{|x|^{N-2}}u_c\Big(\frac{x}{|x|^2}\Big)\bigg)\frac{1}{|x|^{N-2}}\hat{\varphi}\Big(\frac{x}{|x|^2}\Big)\frac{|x|^{2N}}{|x|^{2N}}\,\mathrm{d}x\\
&\qquad - c\int_B \frac{1}{|x|^{N+2}}h\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{N-2}}\hat{\varphi}\Big(\frac{x}{|x|^2}\Big)\frac{|x|^{2N}}{|x|^{2N}}\,\mathrm{d}x\\
&= -\int_B \tilde{b}(x)g(|x|^{N-2}\hat{u}_c)\varphi\,\mathrm{d}x - c\int_B \tilde{h}(x)\varphi\,\mathrm{d}x\end{aligned}$$

for all $\varphi \in H_0^1(B)$. Therefore we do indeed see that $\langle \tilde{I}_c'(\hat{u}_c),\varphi\rangle = 0$ for all $\varphi \in H_0^1(B)$, and so $\hat{u}_c$ is a weak solution to (4.16) under these strengthened assumptions. Then $\hat{u}_c$ satisfies

$$\begin{cases} -\Delta\hat{u}_c - \lambda\tilde{a}(x)\hat{u}_c \le 0, & x \in B\\ \hat{u}_c = 0, & x \in \partial B,\end{cases}$$

and so standard elliptic regularity results give that $\hat{u}_c \in L^\infty(B)$.

We then make the following claim:

Theorem 4.9. Let u_c be the positive solution to equation (4.1) obtained via the direct method in $\Omega \subset \mathbb{R}^N$ for $N \geq 3$ and $0 \leq c \leq c_6$. Under either set of assumptions on the coefficient functions a, b, and h, the Kelvin transform $\hat{u}_c$ of u_c is a positive and stable solution to the transformed equation (4.16), i.e. $\mu_1(\hat{u}_c) \geq 0$.

Proof. The solution $u_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ is the minimizer of the corresponding functional

$$I_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_\Omega a(x)u^2 \,\mathrm{d}x + \int_\Omega b(x)G(u)\,\mathrm{d}x + c\int_\Omega h(x)u\,\mathrm{d}x$$

on the set $M_c = \{u \in \mathcal{D}^{1,2}(\Omega) : \underline{u}_c \leq u \text{ a.e. in } \Omega\}$, where $\underline{u}_c$ is given by Theorem 2.2. This means that we have

$$\langle I_c''(u_c)w, w\rangle = \int_\Omega |\nabla w|^2 \,\mathrm{d}x + \int_\Omega (b(x)g'(u_c) - \lambda a(x))w^2 \,\mathrm{d}x \geq 0 \tag{4.23}$$

for all $w \in \mathcal{D}^{1,2}(\Omega) \cap L^\infty(\Omega; |x|^{N-2})$, since we have shown (cf. Lemma 4.8) that I_c is twice-differentiable in the direction of all such functions w. Now, considering the transformed equation's corresponding functional $\tilde{I}_c$, we would like to show that

$$\langle \tilde{I}_c''(\hat{u}_c)v, v\rangle = \int_B |\nabla\varphi|^2 \,\mathrm{d}y + \int_B (\tilde{b}(y)|y|^{N-2}g'(|y|^{N-2}\hat{u}_c) - \lambda\tilde{a}(y))v^2 \,\mathrm{d}y \geq 0$$

for all $v \in H_0^1(B) \cap L^\infty(B)$. We first have to show that $\langle \tilde{I}_c''(\hat{u}_c)v, v\rangle$ is defined for all such v. The only portion of concern is whether

$$\langle J'(\hat{u}_c + tv), v\rangle =: K(t)$$

is differentiable at $t = 0$. The argument is similar to that in Theorem 4.5. We have

$$\lim_{t\to 0}\frac{K(t) - K(0)}{t} = \lim_{\ell\to 0}\frac{\int_B \tilde{b}(y)\left[g(|y|^{N-2}(\hat{u}_c + tv)) - g(|y|^{N-2}\hat{u}_c)\right]v\,\mathrm{d}y}{\ell}$$

Now

$$g(|y|^{N-2}(\hat{u}_c + tv) - g(|y|^{N-2}\hat{u}_c) = |y|^{N-2}g'(|y|^{N-2}(\hat{u}_c + stv))(tv)$$

and so we have to show that

$$\lim_{\ell\to 0}\int_B \tilde{b}(y)|y|^{N-2}g'(|y|^{N-2}(\hat{u}_c+stv)v^2\,\mathrm{d}y=\int_B \tilde{b}(y)|y|^{N-2}g'(|y|^{N-2}\hat{u}_c)v^2\,\mathrm{d}y,$$

which is true by (4.12) as shown in the proof of Theorem 4.5.

Now that we have that $\langle \tilde{I}_c''(\hat{u}_c)v,v\rangle$ is well-defined, we replace $\tilde{a}(y)$, $\tilde{b}(y)$, and $\hat{u}_c(y)$ with their explicit definitions:

$$\begin{aligned}\langle \tilde{I}_c''(\hat{u}_c)v,v\rangle &= \int_B |\nabla v|^2\,\mathrm{d}y+\int_B\left(\frac{1}{|y|^{N+2}}b\Big(\frac{y}{|y|^2}\Big)|y|^{N-2}g'\Big(u_c\Big(\frac{y}{|y|^2}\Big)\Big)-\frac{\lambda}{|y|^4}a\Big(\frac{y}{|y|^2}\Big)\right)v^2\,\mathrm{d}y\\ &= \int_B |\nabla v|^2\,\mathrm{d}y+\int_B\frac{1}{|y|^4}\left(b\Big(\frac{y}{|y|^2}\Big)g'\Big(u_c\Big(\frac{y}{|y|^2}\Big)\Big)-\lambda a\Big(\frac{y}{|y|^2}\Big)\right)v^2\,\mathrm{d}y.\end{aligned}$$

By making the change of variables $y\mapsto \dfrac{x}{|x|^2}$ in the second integral, we then have

$$\begin{aligned}\langle \tilde{I}_c''(\hat{u}_c)v,v\rangle &= \int_B |\nabla v|^2\,\mathrm{d}y+\int_\Omega |x|^4\big(b(x)g'(u_c)-\lambda a(x)\big)v^2\Big(\frac{x}{|x|^2}\Big)\frac{1}{|x|^{2N}}\,\mathrm{d}x\\ &= \int_B |\nabla v|^2\,\mathrm{d}y+\int_\Omega \big(b(x)g'(u_c)-\lambda a(x)\big)\frac{1}{|x|^{2(N-2)}}v^2\Big(\frac{x}{|x|^2}\Big)\,\mathrm{d}x.\end{aligned}$$

Then, using Theorem 1.11 and the definition of the Kelvin transform of v, we have that

$$\langle \tilde{I}_c''(\hat{u}_c)v,v\rangle=\int_\Omega |\nabla \hat{v}|^2\,\mathrm{d}x+\int_\Omega (b(x)g'(u_c)-\lambda a(x))\hat{v}^2(x)\,\mathrm{d}x=\langle I_c''(u_c)\hat{v},\hat{v}\rangle$$

for all $v\in H_0^1(B)\cap L^\infty(B)$, or equivalently, for all $\hat{v}\in\mathcal{D}^{1,2}(\Omega)\cap L^\infty(\Omega;|x|^{N-2})$.

Therefore

$$\langle I''(u_c)\hat{v},\hat{v}\rangle=\langle \tilde{I}_c''(\hat{u}_c)v,v\rangle\geq 0,$$

i.e. $\mu_1(\hat{u}_c)\geq 0$. □

Now, by the argument following Theorem 4.5 as well as Theorem 4.6, we can now state:

Theorem 4.10. Consider equation (4.1).

(i) Under the hypotheses **(Ha)**, **(Hb)**, **(Hg)**, **(Hλ)** and **(Hh)**, presented in Chapter 2, there exists $c_0>0$ such that for all $0\leq c\leq c_0$, (4.1) has a positive weak solution $u_c\in\mathcal{D}^{1,2}(\Omega)\cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$.

In addition, there exists $C_6 > 0$ such that for all $0 \le c \le c_0$, u_c satisfies

$$u_c(x) \ge \frac{C_6}{|x|^{N-2}} \qquad \text{for large } |x|.$$

(ii) Furthermore, under the additional hypothesis **(H)** and either set of assumptions presented in Lemma 4.8, for $0 \le c < c_0$, $\hat{u}_c$, the Kelvin transform of u_c, is the unique stable positive solution of the transformed equation (4.16) in the ball.

Theorem 4.2 stated that under a preliminary set of assumptions, we obtain two solutions to (4.1), namely u_c (obtained through the direct approach of Chapter 2) and $\hat{z}_c$ (obtained through the Kelvin transform approach of Chapter 3). However, if we strengthen the assumptions to include both **(H)** and either set of assumptions presented in Lemma 4.8, since we have shown uniqueness in both cases, the solutions we obtain must be the same solution. Explicitly:

Theorem 4.11. Consider equation (4.1) with the following assumptions:

1. The function $a : \Omega \to \mathbb{R}$ is positive and belongs to the space $L^\infty(\Omega; |x|^4)$.

2. The function $g : \mathbb{R} \to \mathbb{R}_0^+$ belongs to $C^1(\mathbb{R})$, with $g(s) = 0$ for $s \le 0$ and $g'(s)$ increasing for all $s > 0$. Furthermore, it satisfies

$$\limsup_{s\to 0} \frac{g(s)}{s^{1+\beta}} < \infty \qquad \text{and} \qquad \lim_{s\to\infty} \frac{g(s)}{s} = \infty,$$

   where $0 < \beta < 1$ is a fixed constant.

3. One of the following is true:

   - $\beta < \frac{1}{2}$ and $a(x) \in L^{2N/(N+2)}(\Omega)$, or
   - $\beta < 1$ and $a(x) \in L^{p'}(\Omega)$, where p' is the Hölder conjugate of p, defined in (4.19).

4. The measurable function $b : \Omega \to \mathbb{R}$ is non-negative, not identically equal to zero, and satisfies

$$b(x) \le C_1 a(x)(1 + |x|^2)^{\beta(N-2)}$$

for some $C_1 > 0$ and $x \in \Omega$.

5. The value λ is such that $\lambda_1 < \lambda < \lambda_*$, where λ_1 and λ_* are defined above in (4.4) and (4.5), respectively.

6. $h(x)$ is positive in Ω, and there exists $C_4 > 0$ such that

$$h(x) \leq \frac{C_4}{|x|^m}$$

for all $x \in \Omega$ and $m > N + 2$, and

7. The parameter c is nonnegative.

Then there exists $c_0 > 0$ such that for all $0 \leq c < c_0$, the two solutions $u_c, \hat{z}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega})$, obtained from the direct approach of Chapter 2 and the Kelvin transform approach of Chapter 3, respectively, are in fact the same solution $\breve{u}_c$. Furthermore, for $0 < c < c_0$, there exists $L_c > 0$ such that

$$\lim_{|x| \to \infty} \breve{u}_c(x)|x|^{N-2} = L_c. \tag{4.24}$$

Chapter 5

Solution in $\Omega \subset \mathbb{R}^2$

In this chapter, we wish to discuss the existence and uniqueness of stable solutions for the problem

$$\begin{cases} -\Delta u = \lambda a(x)u - b(x)g(u) - ch(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega = \partial B, \end{cases} \tag{5.1}$$

now in $\Omega = \mathbb{R}^2 \setminus \overline{B(0,1)}$. We use some of the same techniques employed in Chapters 2 and 3, but will require some slight differences along the way.

The solution space is again $\mathcal{D}^{1,2}(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}.$$

In [4] and [21], this space is shown to coincide with the spaces

$$Y = \{u \in L^2_{\mathrm{loc}}(\Omega) : \nabla u \in [L^2(\Omega)]^2, u|_{|x|=1} = 0\}$$

and

$$\hat{\mathcal{D}}^{1,2}(\Omega) = \{u \in L^{1,2}(\Omega) : u \in L^2(\Omega \cap B_R),\ \text{for all } R > 1,\ \text{and } \eta u \in H_0^1(\Omega) \text{ for any } \eta \in C_0^\infty(\mathbb{R}^2)\},$$

where $B_R = B(0,R)$ is the unit ball of radius R, and

$$L^{1,2}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : \nabla u \in [L^2(\Omega)]^2\}.$$

In particular, the Kelvin transform is shown in [4] to be an isometric isomorphism between $\mathcal{D}^{1,2}(\Omega)$ and $H_0^1(B)$.

As before, we consider nonlinearities g satisfying:

(Hg) The function $g : \mathbb{R} \to \mathbb{R}_0^+$ is continuous, with $g(s) = 0$ for $s \le 0$. Furthermore, it satisfies

$$\limsup_{s\to 0} \frac{g(s)}{s^{1+\beta}} < \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{g(s)}{s} = \infty, \tag{5.2}$$

where $0 < \beta \le 1$ is a fixed constant.

The following result, adapted from [14, Lemma 7.1] and Lemma 3.1 is essential in proving the existence of a positive solution to (5.1) in the case of $N = 2$:

Lemma 5.1. Let U be a smooth domain in $\mathbb{R}^2$, $r > 0$, $y_0 \in U$ with $\mathrm{dist}(y_0, \partial U) > 3r$, and $\widetilde{G}_U$ be the Green's function for U. Then there exists a function $d \in C^2(\overline{U})$, such that $0 < d$ in U, $d = 0$ on ∂U, d is superharmonic in U and harmonic in $U \setminus B_r(y_0)$, satisfying

$$c\widetilde{G}_U(x, y_0) \le d(x) \le C\widetilde{G}_U(x, y_0) \quad \text{for } x \in \overline{U} \setminus B_{2r}(y_0) \tag{5.3}$$

for some constants $c, C > 0$. If the domain U is bounded, we additionally have that

$$\tilde{c} \cdot \mathrm{dist}(x, \partial U) \le d(x) \le \tilde{C} \cdot \mathrm{dist}(x, \partial U) \tag{5.4}$$

for some constants $\tilde{c}, \tilde{C} > 0$.

Proof. Since the proof of (5.3) depends on the fundamental solution of the Laplace equation in $\mathbb{R}^2$, it is shown here for completion. The reader is referred to [14] for a proof of (5.4). Let

$$\Phi(x) = -\frac{1}{2\pi} \ln |x|.$$

The function Φ is uniformly continuous in $\mathbb{R}^2 \setminus B_r(0)$. This means for each $\epsilon > 0$ there exists $0 < \delta < r$ such that $y_1, y_2 \in B_r(0)^c$ and $|y_1 - y_2| < 2\delta$ implies $|\Phi(y_1) - \Phi(y_2)| < \epsilon$. If $y_1, y_2 \in B_\delta(y_0)$ and $|x - y_1| \ge r$, $|x - y_2| \ge r$, then $|\Phi(x - y_1) - \Phi(x - y_2)| < \epsilon$. Hence,

$$y_1, y_2 \in B_\delta(y_0) \text{ and } x \in \overline{U} \setminus B_{r+\delta}(y_0) \quad \Longrightarrow \quad |\Phi(x - y_1) - \Phi(x - y_2)| < \epsilon.$$

Note that the definition of the Green's function for U, just as in Section 1.4, is $\widetilde{G}_U(x, y) =$

$\Phi(x-y)-\phi_y(x)$, where

$$\begin{cases} -\Delta\phi_y(x)=0, & x\in U \\ \phi_y(x)=\Phi(x-y), & x\in\partial U. \end{cases}$$

For $y_1,y_2\in B_\delta(y_0)$ and $x\in\partial U$, we have $|\phi_{y_1}(x)-\phi_{y_2}(x)|<\epsilon$, so by the maximum principle

$$y_1,y_2\in B_\delta(y_0) \text{ and } x\in\overline{U}\setminus B_{r+\delta}(y_0) \quad\Longrightarrow\quad |\widetilde{G}_U(x,y_1)-\widetilde{G}_U(x,y_2)|<2\epsilon.$$

One easily obtains $x\in\partial B_{r+\delta}(y_0)$ implies

$$\widetilde{G}_U(x,y_0)=-\frac{1}{2\pi}\ln|x-y_0|+\phi_{y_0}(x)=:c>0,$$

where the value c depends only on r. Let

$$C=\max_{x\in\partial B_{r+\delta}(y_0)}\widetilde{G}_U(x,y_0)$$

and choose $\epsilon=c/4$. We have

$$y\in B_\delta(y_0) \text{ and } x\in\partial B_{r+\delta}(y_0) \quad\Longrightarrow\quad \frac{c}{2}\le\widetilde{G}_U(x,y)\le C+\frac{c}{2}.$$

So $y\in B_\delta(y_0)$ and $x\in\partial B_{r+\delta}(y_0)$ implies

$$\frac{c}{2C}\widetilde{G}_U(x,y_0)\le\widetilde{G}_U(x,y)\le\left(\frac{C}{c}+\frac{1}{2}\right)\widetilde{G}_U(x,y_0). \tag{5.5}$$

By the maximum principle, the two inequalities of the last previous line also hold for $x\in\overline{U}\setminus B_{r+\delta}(y_0)$. Let $\eta\in C_0^\infty(B_\delta(y_0))$, $\eta\ge0$ and $\int_U\eta=\rho>0$ and consider the function $z\in C_0^\infty(\overline{U})$ defined by

$$z(x)=\int_U\widetilde{G}_U(x,y)\eta(y)\,\mathrm{d}y. \tag{5.6}$$

Multiplying (5.5) by $\eta(y)$ and integrating, for $x \in \overline{U} \setminus B_{r+\delta}(y_0)$, we then have

$$\rho \frac{c}{2C} \widetilde{G}_U(x, y_0) \leq z(x) \leq \rho \left(\frac{C}{c} + \frac{1}{2} \right) \widetilde{G}_U(x, y_0).$$

Obviously, $-\Delta z = \eta$ in U and $z = 0$ on ∂U. $\square$

5.1 Direct Approach

When applying the direct approach to the case of $N = 2$, there are a few difficulties that arise, specifically related to the behavior of the solution at infinity. In particular, functions $u \in \mathcal{D}^{1,2}(\Omega)$ in the case of $N = 2$ do not belong to any L^p space (see the definition of the space Y above), and so many of the estimates used in Chapter 2 are not applicable in this case. Remark 2.1 does not hold directly due to the L^p estimates that are used, and the main result that establishes the behavior of our solution at infinity, namely Lemma 2.6, also does not hold in the case of $N = 2$, since the Green's function for Ω takes a different form.

For these reasons, we do not consider the direct approach in this dissertation, and instead choose to focus on the Kelvin transform approach.

5.2 Kelvin Transform Approach

The Kelvin transform approach in $\mathbb{R}^2$ is nearly identical to that presented in Chapter 3, namely we first transform the equation from the exterior domain to the unit ball $B = B(0, 1)$ using the Kelvin transform, and solve the equation there. The format of the proofs presented in Chapter 3 are largely unchanged.

By Lemma 1.10, if u is a weak $\mathcal{D}^{1,2}(\Omega)$ solution of (5.1) if and only if its Kelvin transform $\hat{u}$ is a weak $H_0^1(B)$ solution to the boundary value problem

$$\begin{cases} -\Delta \hat{u} = \dfrac{\lambda}{|y|^4} a\Big(\dfrac{y}{|y|^2}\Big) u\Big(\dfrac{y}{|y|^2}\Big) - \dfrac{1}{|y|^4} b\Big(\dfrac{y}{|y|^2}\Big) g\bigg(u\Big(\dfrac{y}{|y|^2}\Big)\bigg) - \dfrac{c}{|y|^4} h\Big(\dfrac{y}{|y|^2}\Big), & y \in B \\ \hat{u} = 0, & y \in \partial B, \end{cases}$$

or

$$\begin{cases} -\Delta\hat{u} = \lambda\tilde{a}(y)\hat{x} - \tilde{b}(x)g(\hat{u}) - c\tilde{h}(x), & x \in B \\ \hat{u} = 0, & x \in \partial B, \end{cases} \tag{5.7}$$

where $\tilde{a}(y) = |y|^{-4}a\big(\frac{y}{|y|^2}\big)$, $\tilde{b}(y) = |y|^{-4}b\big(\frac{y}{|y|^2}\big)$, and $\tilde{h}(y) = |y|^{-4}h\big(\frac{y}{|y|^2}\big)$. Notice the slight difference between (5.7) and (3.3) due to $N = 2$. We then can state the assumptions on our data:

(H$\tilde{a}$) The function $\tilde{a} : B \to \mathbb{R}$ is positive and belongs to $L^\infty(B)$.

(H$\tilde{b}$) The measurable function $\tilde{b} : B \to \mathbb{R}$ is nonnegative, not identically equal to zero, and satisfies

$$\tilde{b}(x) \leq C_1\tilde{a}(x)(1 - |x|)^{-\beta} \tag{5.8}$$

for some $C_1 > 0$ and all $x \in B$.

(H$\tilde{h}$) The nonnegative and not identically equal to zero function $\tilde{h}$ belongs to the space $L^s(B)$ for some $s > 2$.

(Hλ) The value λ is such that $\tilde{\lambda}_1 < \lambda < \tilde{\lambda}_*$, where $\tilde{\lambda}_1$ and $\tilde{\lambda}_*$ are defined as in (3.8) and (3.10).

(Hc) The parameter c is nonnegative.

Now that we have transformed our equation, we can continue the proof of existence using Kelvin transform approach given in Chapter 3:

1. First we consider the auxiliary equation

$$\begin{cases} -\Delta u = \lambda\tilde{a}(x)\left[1 - k\left(\dfrac{u}{\ell d(x)}\right)\right] - c\tilde{h}(x), & x \in B \\ u = 0, & x \in \partial B, \end{cases} \tag{5.9}$$

where now the function $d(x)$ is given by Lemma 5.1 and $k(s) = s^\beta$ for $s > 0$, $k(s) = 0$ for $s \leq 0$. We then prove the existence of positive solutions $\underline{u}_c$ for the auxiliary equation for $0 \leq c \leq c_0$ (for some $c_0 > 0$) by the same process as in Section 2.3. Some small differences arise in the proof of Lemma 2.3, as we have $h \in L^1(B)$ (as opposed to $L^{2N/(N+2)}(B)$ in [14]).

Additionally, we are not using Lemma 2.6 since we are in a bounded domain, but the sketch of the proof immediately following Lemma 2.6 holds, using the Green's function for the ball $\widetilde{G}_B$ instead of $\widetilde{G}_\Omega$.

2. Now, arguing as in Section 2.4 and using $\underline{u}_c$ as sub solutions to (5.7), we obtain a solution to our problem by minimizing the corresponding energy functional on the set

$$M_c = \{u \in H_0^1(B) : \underline{u}_c \leq u \text{ a.e. in } B\}.$$

The energy functional is now given as

$$I_c(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int_B \tilde{a}(x)u^2 \, \mathrm{d}x + \int_B \tilde{b}(x)G(u)\, \mathrm{d}x + c\int_B \tilde{h}(x)u \, \mathrm{d}x,$$

where $G(u) = \int_0^u g(s)\, \mathrm{d}s$ (as the $|x|^{N-2}$ terms in (3.12) are gone). The minimizer we obtain, u_c, is in $L^\infty(B)$ by Lemma 2.9, since the proof there does not depend on N. Both Lemma 3.4 and the final proof of existence (cf. [14, Theorem 7.3]) also do not depend on N, and so the existence of a positive weak solution $u_c \in H_0^1(B) \cap C^{1,\alpha}_{\text{loc}}(\overline{B})$ to (5.7) follows just as in Section 3.2.

Lastly, just as in Section 3.3, we transform the assumptions needed on the data in the ball back in terms of the external domain Ω now with $N = 2$ fixed:

(Ha**)′** The function $a : \Omega \to \mathbb{R}$ is positive and belongs to $L^\infty(\Omega; |x|^4)$.

(Hb**)′** The function $b : \Omega \to \mathbb{R}$ is non-negative, not identically equal to zero, and satisfies

$$b(x) \leq C_1 a(x)\left(\frac{|x|}{|x|-1}\right)^\beta \tag{5.10}$$

for some $0 < \beta \leq 1$, $C_1 > 0$ and all $x \in \Omega$. Now, with $\Omega_0 = \{x \in \Omega : b(x) = 0\} \subsetneq \Omega$, we require either Ω_0 has measure zero or $\Omega_0 = \overline{\operatorname{int}\Omega_0}$ (closure in Ω_0) with $\partial\Omega_0$ Lipschitz.

(Hh**)′** The non-negative and not identically equal to zero function $h(x)$ belongs to the weighted L^s space $L^s(\Omega; |x|^{4(s-1)})$ for some $s > 2$.

(Hλ) The value λ is such that $\lambda_1 < \lambda < \lambda_*$, where λ_1 and λ_* are defined as in (3.8) and (3.10) (since $\tilde{\lambda}_1 = \lambda_1$ and $\tilde{\lambda_*} = \lambda_*$).

(Hc) The parameter c is nonnegative.

We can therefore state:

Theorem 5.2. Under **(Ha)$'$**, **(Hb)$'$**, **(Hg)**, **(Hλ)** and **(Hh)$'$**, there exists $c_1 > 0$ such that for all $0 \leq c \leq c_1$, (3.1) has a positive weak solution $\hat{u}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$. Furthermore, there exists a positive constant C_∞ such that

$$\lim_{|x|\to\infty} \hat{u}_c(x) = \lim_{x\to 0} u_c(x) = C_\infty > 0. \tag{5.11}$$

5.3 Uniqueness of Stable Solution

As in Section 4.2, we now wish to show that the solution $u_c \in H^1_0(B) \cap C^{1,\alpha}_{\text{loc}}(\overline{B})$ obtained above is the unique positive and stable solution of (5.7). Following the definitions given in Section 4.2, we must show that

$$\mu_1(u_c) := \inf_{w \in H^1_0(B), w \neq 0} \frac{\int_B |\nabla w|^2 + (\tilde{b}(x)g'(u_c) - \lambda\tilde{a}(x))w^2}{\int_B w^2} > 0.$$

We will also assume the additional hypothesis **(H)**, listed here for convenience:

(H) $h(x)$ is positive in Ω and $g \in C^2(\mathbb{R})$ is strictly convex, i.e. $g''(s) > 0$. Additionally, g satisfies

$$\limsup_{s\to 0^+} \frac{g'(s)}{s^\beta} < 0. \tag{5.12}$$

The procedure to prove $\mu_1(u_c) > 0$ is nearly identical to that presented in Section 4.2, summarized briefly here:

1. Based on the results in Du & Ma (cf. [10, 11]), Ouyang (cf. [18]), and Shabani & Tehrani (cf. [19, 20]), obtain the unique positive solution u_λ for (5.7) for the case $c = 0$, that is, in the absence of harvesting (similar to Theorem 4.3). This result requires the comparison principle given in Lemma 4.4, now with $N = 2$.

2. Show directly that the function I_c is twice-differentiable at u_c in the direction of $v \in H_0^1(B) \cap L^\infty(B)$. This proof again makes use of the Hardy inequality for bounded domains (cf. Lemma 1.2) and Lebesgue's Dominated Convergence Theorem, similar to Theorem 4.5. Since $\langle I_c''(u_c)v, v\rangle$ is the same as the numerator of $\mu_1(u_c)$ and u_c was obtained through minimization, we obtain that $\mu_1(u_c) \geq 0$.

3. Finally, using the solution to the equation with no harvesting term u_λ and the implicit function theorem, follow the argument given following the proof of Theorem 4.5 and [19] to obtain that $\mu_1(u_c) > 0$, i.e. u_c is in fact the unique positive stable solution to (5.7).

We can then state our final result, analogous to Theorem 4.7:

Theorem 5.3. Consider equation (5.1).

(i) Under the hypotheses $(\mathbf{H}a)'$, $(\mathbf{H}b)'$, $(\mathbf{H}g)$, $(\mathbf{H}\lambda)$ and $(\mathbf{H}h)'$, there exists $c_3 > 0$ such that for all $0 \leq c \leq c_3$, (5.1) has a positive weak solution $\hat{u}_c \in \mathcal{D}^{1,2}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\overline{\Omega})$, satisfying, for some $C_\infty > 0$,

$$\lim_{|x| \to \infty} \hat{u}_c(x) = C_\infty. \tag{5.13}$$

(ii) Furthermore, under the additional hypothesis $(\mathbf{H})$, for $0 \leq c < c_3$, u_c, the Kelvin transform of $\hat{u}_c$ is the unique stable positive solution of the transformed equation (5.7) in the ball.

Chapter 6

Conclusion

In this thesis, we have shown the existence of positive weak solutions to equation (1.1) in the exterior domain Ω in both $\mathbb{R}^2$ and $\mathbb{R}^N$, $N \geq 3$. For $N \geq 3$, we have been able prove the existence of a positive weak solution to (1.1) using both the direct approach shown in Chapter 2 as well as the Kelvin transform approach shown in Chapter 3, under some slight different assumptions on the coefficient functions. Additionally, under some slightly stronger assumptions, we have shown that the solutions obtained through each method are the unique stable solutions for the transformed equation in the ball. In this way, if we have a set of assumptions satisfying both requirements of the direct approach as well as the Kelvin transform approach simultaneously, then the solutions obtained through the two approaches are in fact the same. Finally, we have shown that the Kelvin transform approach can be employed in the case of $\mathbb{R}^2$ and that we obtain existence and uniqueness results for a positive solution in this case as well.

The key contributions of this work are as follows:

1. We extend the results of [4] related to the Kelvin transform to the case of $N \geq 3$, specifically Theorem 1.11.

2. We show that the method presented in [14] to solve equation (1.1) in the whole space $\mathbb{R}^N$ can be adapted to the exterior domain Ω, summarized in Theorem 2.7. Additionally, it is shown that the differentiability of the corresponding energy functional of problem (1.1) can be proven directly, rather than using polynomial approximations, as in [14, Lemma 5.4] (cf. Lemma 2.10).

3. We apply the Kelvin transform to equation (1.1), and making use of the corresponding results for bounded domains in [14], show the existence of a positive solution to (1.1) in the exterior domain through the use of Lemma 1.10.

4. We relate the notion of stability to equation (1.1) and show that the solutions obtained via both methods are in fact stable (cf. Theorems 4.5, 4.9), and therefore are the same.

5. Finally, we show that [14, Lemma 7.1] also holds in the case of $N = 2$ and that the same ideas presented for the Kelvin transform approach in Chapter 3 can also be applied to show existence and uniqueness of positive solutions to (1.1) in this case (cf. Theorem 5.3).

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