5-1-2022

Some Graph Laplacians and Variational Methods Applied to Partial Differential Equations on Graphs

Daniel Anthony Corral

SOME GRAPH LAPLACIANS AND VARIATIONAL METHODS APPLIED TO PARTIAL DIFFERENTIAL EQUATIONS ON GRAPHS

By

Daniel Anthony Corral

Bachelor of Science - Mathematics
University of Nevada, Las Vegas
2006

Master of Science - Mathematics
University of Nevada, Las Vegas
2007

Master of Arts - Mathematics
University of Kentucky
2011

A dissertation submitted in partial fulfillment
of the requirements for the

Doctor of Philosophy - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
May 2022

UNLV | GRADUATE COLLEGE

Dissertation Approval

The Graduate College
The University of Nevada, Las Vegas

April 8, 2022

This dissertation prepared by

Daniel Anthony Corral

entitled

Some Graph Laplacians and Variational Methods Applied to Partial Differential Equations on Graphs

is approved in partial fulfillment of the requirements for the degree of

Doctor of Philosophy - Mathematical Sciences
Department of Mathematical Sciences

David Costa, Ph.D.
Examination Committee Chair

Ebrahim Salehi, Ph.D.
Examination Committee Member

Hossein Tahrani, Ph.D.
Examination Committee Member

Pushkin Kachroo, Ph.D.
Graduate College Faculty Representative

Kathryn Hausbeck Korgan, Ph.D.
Vice Provost for Graduate Education &
Dean of the Graduate College

ABSTRACT

SOME GRAPH LAPLACIANS AND VARIATIONAL METHODS APPLIED TO PARTIAL DIFFERENTIAL EQUATIONS ON GRAPHS

by

Daniel Anthony Corral

Dr. David Costa, Examination Committee Chair
Professor of Mathematical Sciences
University of Nevada, Las Vegas, USA

In this dissertation we will be examining partial differential equations on graphs. We start by presenting some basic graph theory topics and graph Laplacians with some minor original results. We move on to computing original Jost graph Laplacians of friendly labelings of various finite graphs. We then continue on to a host of original variational problems on a finite graph. The first variational problem is an original basic minimization problem. Next, we use the Lagrange multiplier approach to the Kazdan-Warner equation on a finite graph, our original results generalize those of Dr. Grigor'yan, Dr. Yang, and Dr. Lin. Then we do an original saddle point approach to the Ahmad, Lazer, and Paul resonant problem on a finite graph. Finally, we tackle an original Schrödinger operator variational problem on a locally finite graph inspired by some papers written by Dr. Zhang and Dr. Pankov. The main keys to handling this difficult breakthrough Schrödinger problem on a locally finite graph are Dr. Costa's definition of uniformly locally finite graph and the locally finite graph analog Dr. Zhang and Dr. Pankov's compact embedding theorem when a coercive potential function is used in the energy functional. It should also be noted that Dr. Zhang and Dr. Pankov's deeply insightful Palais-Smale and linking arguments are used to inspire the bulk of our original linking proof.

ACKNOWLEDGEMENTS

I would like to start by thanking my PhD advisor Dr. David Costa for being so inspirational and supportive during the entire process of producing this PhD dissertation. Dr. Costa has a keen level of insight into difficult mathematical problems that he shares with all those around him. I appreciate his saint-like patience with me as I take a long time to understand mathematical concepts.

I also am deeply thankful for Dr. Costa for sending me to the 1141st AMS Meeting at University of Delaware in September of 2018 to present my talk "Remarks on Lagrange Multiplier Approch to Kadzan-Warner Equations on a Finite Graph" which is the result of joint work between him and I. I would also like to thank Dr. Alexander Pankov (Morgan State University, USA), Dr. Junping Shi (College of William and Merry, USA) and Dr. Jun Wang (Jiangsu University, China) for inviting me to the AMS sectional in Delaware.

I also want to thank my PhD committee for their help in this process. My committee members are chair Dr. David Costa, Dr. Ebrahim Salehi, Dr. Hossein Tehrani, outside committee member Dr. Pushkin Kachroo. Serving on a PhD committee takes a lot of time from their already busy schedules: and I truly appreciate their service and help.

I would like to extend a special thank you to Dr. Hossein Tehrani for his extremely helpful questions and edit suggestions prior to my defense. These edit suggestions have substantially improved this dissertation to its final form. Dr. Tehrani's questions both before and during the defense helped to make the defense stronger. I am very grateful for the time it took him to prepare these helpful comments.

I would like to give a special thank you to Dr. Ebrahim Salehi for his introducing me to the field of graph labeling while he was chair of my master's thesis and for his generous working with me on our research project on fully cordial trees in our talk "Fully Cordial Trees" presented at the 28th Midwest Conference on Combinatorics and Combinatorial Computing at University of Nevada, Las Vegas in October of 2014 and our paper "Fully Cordial Trees" published in 2016. I would also like to thank Dr. Salehi for providing the figures used in this dissertation.

Dr. Costa and I both wish to thank Dr. Alexander Pankov, who has sadly passed away, for his suggesting working on a Schrödinger operator variational problem on a locally finite graph inspired by Dr. Zhang and Dr. Pankov papers. We are deeply saddened that he did not get to see the completion of this project. I would personally like to thank both Dr. Costa and and Dr. Pankov for working on this problem with me.

I need to give a big thank you to my Dad for being the best Dad in the world and being so supportive of me in all my endeavors. I appreciate him doing all the household chores in order to give me the free time to devote to working on my graduate school work and PhD training activities. Also, my Dad drives me everywhere as I cannot drive.

I give a giant shout out to my close friend Sean Trendell for all of his help with the typing of some this dissertation in LaTeX and formatting of this dissertation to comply with the graduate college regulations. Without Sean, typing of this dissertation would never have been completed in time.

I have a lot of friends I wish to thank by name, however you know who you are. Thank you for putting up with me throughout this PhD experience.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

In this section we introduce some very basic definitions and concepts from both graph theory and partial differential equation theory on graphs. The definitions and theorems in this chapter are standard graph theory definitions and theorems. They are taken from [27] and [55] unless stated otherwise.

Definition 1.1. *A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called endpoints.*

In general, we will simply say a graph G has a vertex set V and an edge set E.

Definition 1.2. *Let G be a graph. Two vertices $x, y \in G$ are said to be adjacent if there is an edge connecting x to y.*

In a graph two vertices being adjacent is denoted by $x \sim y$.

Definition 1.3. *A weighted graph is a couple (Γ, μ) where $\Gamma = (V, E)$ is a graph with vertex set V and edge set E and μ_{xy} is a non-negative function on $V \times V$ such that*

1. $\mu_{xy} = \mu_{yx}$,
2. $\mu_{xy} > 0$ *if and only if* $x \sim y$,
3. $\mu(x) = \sum_{y \sim x} \mu_{xy}$,

where, μ is called the weight (or measure) of the graph Γ.

It should be noted that this definition implies that $\mu_{xy} = 0$ if and only if x and y are not adjacent.

Note that μ_{xy} can be considered as a positive function on the set E of edges that is extended to be 0 on non-edge pairs (x, y). This means that $\mu : E \to \mathbb{R}_{\geq 0}$ can be thought of as what graph theorists usually refer to as a graph labeling. Now we are ready to state some more useful definitions.

Definition 1.4. *Given a weighted graph* (Γ, μ) *then* μ *is said to be simple if and only if* $x \sim y$ *implies that* $\mu_{xy} = 1$.

A word of caution should be noted at this point. Notice that Alexander Grigoryan is talking about a simple measure here. Where the typical notion of simple in graph theory usually refers to a simple graph. A simple graph is a graph where there are no loops.

There are numerous different graph laplacians. We now give a definition of the kind of Laplace operator studied by Jost, Banerjee, Bauer, and Hao Chen. The following definition is from Alexander Grigoryan text Analysis on Graphs. We begin by defining what it means for a graph to be locally finite. However, before we can define what it means for a graph to be locally finite we define the notion of the degree of a vertex. We adapt the definition of degree which Douglas B. West gives in his text "Introduction to Graph Theory" (not a direct citation). The adaptation is more suited to notation of this manuscript.

Definition 1.5. *Given a graph* $G = (V, E)$ *and* $x \in V$*. The degree of* x*,* $\deg(x)$*, is the number of edges incident to* x*, except that each loop at* x *counts twice. Keep in mind when graph theorists refer to a simple graph that* G *has no loops.*

Definition 1.6. *A graph* (V, E) *is said to be a locally finite graph if and only if* $0 < \deg(x) < \infty \ \forall x \in V$*.*

Definition 1.7. [55] *If* G *has a* x, y*-path, then the distance form* x *to* y*, written* $d_G(x, y)$ *or simply* $d(x, y)$*, is the least length of all* x, y*-paths.*

Note if there are no x, y-paths, then $d(x, y) = \infty$ [55].

Definition 1.8. [55] *The diameter of a graph* G*, (diam* G*), is the* $\max\limits_{x,y \in V} d(x, y)$*.*

Now we may proceed in defining the Laplace operators studied by Jost and Grigoryan.

Definition 1.9. *(Degree Laplace Operator) Let* (V, E) *be a locally finite connected graph. For any function* $f : V \to \mathbb{R}$*, define the Laplace operator,* Δf*, by*

$$\Delta f(x) = \frac{1}{deg(x)} \sum_{y \sim x} f(y) - f(x). \tag{1.1}$$

The Laplace operator studied by Jost is $\mathcal{L} = -\Delta f$*.*

Unless it is said otherwise when we mention Laplace operator we are referring to the Jost Laplace operator.

Definition 1.10. *(Weighted Laplace Operator) Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. For any function $f : V \to \mathbb{R}$, define the weighted μ-Laplace operator, $\Delta_\mu f$, by*

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x). \tag{1.2}$$

The normalized graph Laplace operator studied by Grigoryan is $\mathcal{L}_\mu = -\Delta_\mu f$.

It is well known that the Laplace operators studied by Jost and Grigoryan are linear. We will just demonstrate the linearity of Δ_μ and a couple of other well known results.

Theorem 1.1. *Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. Then Δ_μ is linear.*

Proof. Let $\mathcal{F}$ be the set of all real-valued functions on V. Suppose that $f, g \in \mathcal{F}$. Let $\lambda \in \mathbb{R}$. Note that

$$\begin{aligned}
\Delta_\mu (\lambda f + g)(x) &= \frac{1}{\mu(x)} \sum_{y \sim x} (\lambda f(y) + g(y)) \mu_{xy} - (\lambda f(x) + g(x)) \\
&= \lambda \left(\frac{1}{\mu(x)} \sum_{y \sim x} f(y) - f(x) \right) + \frac{1}{\mu(x)} \sum_{y \sim x} g(y) - g(x) \\
&= \lambda \Delta_\mu f(x) + \Delta_\mu g(x)
\end{aligned}$$

Therefore Δ_μ is linear. □

Theorem 1.2. *Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. Let $c \in \mathbb{R}$. Then $\Delta_\mu(c) = 0$.*

Proof. Let $c \in \mathbb{R}$. Observe that $\Delta_\mu c = \frac{1}{\mu(x)} \sum_{y \sim x} c - c = c\frac{1}{\mu(x)} \sum_{y \sim x} 1 - c = c\frac{\mu(x)}{\mu(x)} - c = c - c = 0$.

□

As was mentioned earlier, there are numerous different graph laplacians. We will now define a few more of them.

The first Laplace operator that we will discuss was studied by Manfredi, Obermann, and Sviridov in their paper "Nonlinear Elliptic Partial Differential Equations And P-Harmonic Functions On Graphs"[44]. We will call this Laplace operator the Manfredi operator and denote it by $\Delta_{man} f(x)$.

Definition 1.11. *(Laplace Operator Studied by Manfredi) Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. For any function $f : V \to \mathbb{R}$, define the Manfredi Laplace operator, $\Delta_{man} f$, by*

$$\Delta_{man} f(x) = \sum_{y \sim x} (f(y) - f(x))\mu_{xy}. \tag{1.3}$$

We show that the Manfredi operator is also linear. This result is fundamental enough that I suspect it is proven elsewhere. However, I could not find a proof anywhere while doing research.

Theorem 1.3. *Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. Then $\Delta_{man} f$ is linear.*

Proof. Let $\mathcal{F}$ be the set of all real-valued functions on V. Suppose that $f, g \in \mathcal{F}$. Let $\lambda \in \mathbb{R}$. Note that

$$\begin{aligned}
\Delta_{man}\left(\lambda f + g\right)(x) &= \sum_{y\sim x}\left((\lambda f + g)(y) - (\lambda f + g)(x)\right)\mu_{xy} \\
&= \sum_{y\sim x}\left((\lambda f(y) + g(y)) - (\lambda f(x) + g(x))\right)\mu_{xy} \\
&= \sum_{y\sim x}\left((\lambda f(y) - \lambda f(x)) + (g(y) - g(x))\right)\mu_{xy} \\
&= \sum_{y\sim x}\left((\lambda f(y) - \lambda f(x))\mu_{xy} + (g(y) - g(x))\mu_{xy}\right) \\
&= \lambda\sum_{y\sim x}(f(y) - f(x))\mu_{xy} + \sum_{y\sim x}(g(y) - g(x))\mu_{xy} \\
&= \lambda\Delta_{man} f + \Delta_{man} g
\end{aligned}$$

Therefore $\Delta_{man} f$ is linear. □

Let $x \in \mathbb{R}^d$. The gradient studied by Manfredi is

$$\nabla u(x) = (w(e_{xy_1})(u(y_1) - u(x)), ..., w(e_{xy_d})(u(y_d) - u(x))). \tag{1.4}$$

Note that the Laplace operator studied by Manfredi may be written as $\Delta_{man} f(x) = \overrightarrow{1} \cdot \nabla u(x)$ instead of (1.3).

The next we will state a number of Laplace operators that are defined by Manfredi, Obermann, and Sviridov in [44]. The Laplace operators we star are positive and negative eikonal operators, the infinity Laplacian, the homogeneous Dirichlet problem for the positive eikonal operator, and the 1-Laplacian.

Definition 1.12. *Let $G = (V, E)$ be a locally finite connected graph. For $x \in \mathbb{R}^d$ let $\min(x) = \min\{x_1, ..., x_d\}$ and $\max(x) = \max\{x_1, ..., x_d\}$. The positive and negative eikonal operators on a graph are*

$$\begin{aligned}
|\nabla u(x)|^+ &= \max(\nabla u(x)) = \max_{y\sim x}(f(y) - f(x)))\mu_{xy}; \\
|\nabla u(x)|^- &= \min(\nabla u(x)) = \min_{y\sim x}(f(y) - f(x)))\mu_{xy}.
\end{aligned} \tag{1.5}$$

Definition 1.13. *Let $G = (V, E)$ be a locally finite connected graph and let $x \in \mathbb{R}^d$. The Infinity Laplacian is defined by $\Delta_\infty u(x) = \frac{|\nabla u(x)|^+ + |\nabla u(x)|^-}{2}$.*

Definition 1.14. *The homogeneous Dirichlet problem (D) for the positive eikonal operator is defined by*

$$|\nabla u(x)|^{+} - 1 = 0, x \in V - \partial V;$$
$$u(x) = 0, x \in \partial V. \tag{1.6}$$

Definition 1.15. *The 1-Laplacian is defined by*

$$\Delta_1 u(x) = median(\nabla u(x)). \tag{1.7}$$

We will now discuss a relationship between the graph Laplacians studied by Grigoryan and Manfredi. To our knowledge the following theorem is original work.

Theorem 1.4. *Let (V, E) be a locally finite connected graph. Suppose that G has a weight μ. The relationship between the graph Laplacians studied by Grigoryan and Manfredi may be described by:*

$$\begin{aligned} \Delta_\mu f(x) &= \frac{1}{\mu(x)} \Delta_{man} f(x); \\ \mathcal{L}_\mu f(x) &= -\frac{1}{\mu(x)} \Delta_{man} f(x).; \end{aligned} \tag{1.8}$$

Proof. Note that,

$$\begin{aligned} \Delta_\mu f(x) &= \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x) \\ &= \frac{1}{\mu(x)} (\sum_{y \sim x} f(y)\mu_{xy} - f(x)\mu(x)) \\ &= \frac{1}{\mu(x)} (\sum_{y \sim x} f(y)\mu_{xy} - f(x) \sum_{y \sim x} \mu_{xy})) \\ &= \frac{1}{\mu(x)} \sum_{y \sim x} (f(y) - f(x))\mu_{xy} \\ &= \frac{1}{\mu(x)} \Delta_{man} f(x). \end{aligned} \tag{1.9}$$

Hence, $\mathcal{L}_\mu f(x) = -\frac{1}{\mu(x)} \Delta_{man} f(x)$.

□

The following theorem shows that the sum of the graph Laplacian studied by Manfredi over every vertex is zero. To our knowledge the statement of this theorem, the corollary that follows it and their proofs are original. The theorem may have applications in network flow theory.

Theorem 1.5. *Let (V, E) be a connected finite graph. Then*

$$\sum_{x \in V} \Delta_{man} f(x) = 0. \tag{1.10}$$

Furthermore,

$$\sum_{x \in V} \sum_{y \sim x} (\nabla_{xy} f)\, \mu_{xy} = 0. \tag{1.11}$$

Proof. Consider an arbitrary edge $ab \in E$ on the graph. Notice that when $x = a$ then $b \sim a$ and $(f(b) - f(a))\,\mu_{ab}$ is in the sum $\sum_{x \in V} \Delta_{man} f(x) = \sum_{x \in V} \sum_{y \sim x} (f(y) - f(x))\,\mu_{xy}$. Also when $x = b$ then $a \sim b$ and $(f(a) - f(b))\,\mu_{ba}$ is in the sum $\sum_{x \in V} \Delta_{man} f(x) = \sum_{x \in V} \sum_{y \sim x} (f(y) - f(x))\,\mu_{xy}$. It is important to note that $\mu_{ab} = \mu_{ba}$ because the adjacency operation is symmetric.

Hence,

$$
\begin{aligned}
\sum_{x \in V} \Delta_{man} f(x) &= \sum_{x \in V} \sum_{y \sim x} (f(y) - f(x))\,\mu_{xy} \\
&= \sum_{ab \in E} \left((f(b) - f(a))\,\mu_{ab} + (f(a) - f(b))\,\mu_{ba} \right) \\
&= \sum_{ab \in E} \left(f(b)\mu_{ab} - f(a)\mu_{ab} + f(a)\mu_{ba} - f(b)\mu_{ba} \right) \\
&= \sum_{ab \in E} \left(f(b)\mu_{ab} - f(b)\mu_{ba} + f(a)\mu_{ba} - f(a)\mu_{ab} \right) \\
&= \sum_{ab \in E} \left(f(b)\mu_{ab} - f(b)\mu_{ab} + f(a)\mu_{ab} - f(a)\mu_{ab} \right) \\
&= \sum_{ab \in E} 0 \\
&= 0 \cdot |E| \\
&= 0
\end{aligned}
$$

Furthermore since, $\nabla_{xy} f = f(y) - f(x)$, then $\sum_{x \in V} \sum_{y \sim x} (\nabla_{xy} f)\,\mu_{xy} = 0$.

□

The following corollary shows that the discrete integral of the Grigoryan graph laplacian over every vertex is also zero. It will be very useful in our proof of the Lagrange multiplier approach to the Kadzan Warner equation on a finite graph later in this dissertation. We will again arrive at this corollary later in the dissertation by using a discrete version of Green's Theorem on a graph (integration by parts).

Corollary 1.6. *Let (V, E) be a connected finite graph. Then*

$$\sum_{x \in V} \Delta_{\mu} f(x) \mu(x) = 0 \tag{1.12}$$

and

$$\sum_{x \in V} \mathcal{L}_{\mu} f(x) \mu(x) = 0 \tag{1.13}$$

Proof. Observe that,

$$\sum_{x \in V} \Delta_{\mu} f(x) \mu(x) = \sum_{x \in V} \frac{1}{\mu(x)} \Delta_{man} f(x) \mu(x) = \sum_{x \in V} \Delta_{man} f(x) = 0 \tag{1.14}$$

and

$$\sum_{x\in V}\mathcal{L}_\mu f(x)\mu(x)=\sum_{x\in V}-\frac{1}{\mu(x)}\Delta_{man}f(x)\mu(x)=\sum_{x\in V}-\Delta_{man}f(x)=0. \tag{1.15}$$

□

It should be noted that Grigoryan, in his text Analysis on Graphs [27], defines the Dirichlet problem for the Grigoryan graph Laplacian and proves the existence of its solution. We start by giving the definitions for boundary, interior, subharmonic and superharmonic.

Definition 1.16. *Let (V,μ) be a connected locally finite weighted graph and let Ω be a nonempty subset of V. Define $\Omega^c = V - \Omega$ (in other words $V = \Omega \cup \Omega^c$). The boundary of Ω is defined to be $\partial\Omega = \{y \in \Omega^c : y \sim x \text{ for some } x \in \Omega\}$. The interior of Ω is defined by $\Omega^o = \Omega - \partial\Omega$.*

Definition 1.17. *Let (V,μ) be a connected locally finite weighted graph and let Ω be a nonempty subset of V. A function $u : V \to R$ is called subharmonic in Ω if $\Delta_\mu u(x) \geq 0\ \forall x \in \Omega$, and superharmonic in Ω if $\Delta_\mu u(x) \leq 0\ \forall x \in \Omega$. A function u is called harmonic in Ω if it is both subharmonic and superharmonic, that is, if it satisfies the Laplace equation: $\Delta_\mu u(x) = 0$.*

Next, we prove a helpful result which leads to the maximum/minimum principle.

Theorem 1.7. *Let $G = (V,E)$ be a connected graph with vertex set V and edge set E. Suppose H is a non-empty subset of V. Suppose further that H has the property that $\forall x \in H$, if $y \in V$ and $y \sim x$ then $y \in H$. Then, $H = V$.*

Proof. Let $x, y \in H$ be arbitrary. Since $G = (V,E)$ is connected, there exists a path $\{x_k\}_{k=0}^n \subseteq V$ from x to y such that $x = x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_n = y$. One proceeds by the Principle of Mathematical Induction. Notice that, $x_1 \in V$ and $x_1 \sim x_0 = x \in H$ implies that $x_1 \in H$, and this serves as our basis case for the induction. For the sake of induction suppose that $x_k \in H$ and $x_k \sim x_{k+1}$, then by the property $x_{k+1} \in H$. By the Principal of Mathematical Induction: $x_m \in H\ \forall m$ such that $1 \leq m \leq n$. In particular, $y = x_n \in H$. Since $y \in V$ was arbitrary, then $V \subseteq H \subseteq V$. Hence, $H = V$. □

Theorem 1.8. *(A maximum/minimum principle from Grigoryan Analysis on Graphs) Let (V,μ) be a connected locally finite weighted graph. Let $\Omega \subseteq V$ such that Ω is finite and $\Omega^c \neq \emptyset$. Then, for any function $u : V \to \mathbb{R}$, that is subharmonic in Ω, we have*

$$\max_{\Omega} u \leq \sup_{\Omega^c} u \tag{1.16}$$

and for any function $u : V \to \mathbb{R}$, that is superharmonic in Ω, we have

$$\min_{\Omega} u \geq \inf_{\Omega^c} u. \tag{1.17}$$

Proof. It suffices to prove the first claim. First note that if $\sup_{\Omega^c} u = +\infty$ then the result already holds. So let's suppose that $\sup_{\Omega^c} u < \infty$. Note that u can be replaced by $u - c$ where c is some constant. So without loss of generality take $\sup_{\Omega^c} u(x) = 0$. Observe that, $\sup_{\Omega^c} u(x) = 0$ implies $u(x) \leq 0 \ \forall \ x \in \Omega^c$.

Let $M = \max_{\Omega} u$.

Suppose to the contrary that, $M > 0$. Define $X = \{x \in V : u(x) = M\}$. By definition: $X \subseteq V$. Note that, $|\Omega| < \infty$ implies that $\max_{\Omega} u$ is obtained for some $w \in \Omega$. In particular, there is a $w \in \Omega$ such that $u(w) = M$. Thus, $w \in X$. Hence, $X \neq \emptyset$.

Let $x \in X$ be arbitrary and $y \in V$ such that $y \sim x$ is arbitrary. By definition: $u(x) = M$. Note that, $\sup_{\Omega^c} u = 0$ implies that $u(y) \leq 0 < M \ \forall y \in \Omega^c$. From $M = \max_{\Omega} u$ it is seen that $u(y) \leq M \ \forall y \in \Omega$. Combining the previous two sentences results in: $u(y) \leq M \ \forall \ y \in V$ as $V = \Omega \cup \Omega^c$. Since $u : V \to \mathbb{R}$ is subharmonic in Ω, then $\Delta_{\mu}(x) \geq 0$. So then,

$$\begin{aligned} M = u(x) &\leq \frac{1}{\mu(x)} \sum_{y \sim x} u(y)\mu_{xy} \\ &\leq \frac{1}{\mu(x)} \sum_{y \sim x} u(y)\mu_{xy} \\ &\leq \frac{1}{\mu(x)} \sum_{y \sim x} M\mu_{xy} \\ &= M\frac{1}{\mu(x)} \sum_{y \sim x} \mu_{xy} \\ &= M\frac{\mu(x)}{\mu(x)} \\ &= M. \end{aligned} \tag{1.18}$$

Consequently, $\frac{1}{\mu(x)} \sum\limits_{y \sim x} u(y)\mu_{x,y} = M$ with $u(y) \leq M \ \forall \ y \in V$ such that $y \sim x$. Hence, $u(y) = M$. Thus, X has the property that $\forall x \in X$, if $y \in V$ is such that $y \sim x$ then $y \in X$. Now by the previous theorem, $X = V$. Therefore, $u(x) = M > 0 \ \forall \ x \in V$. However, this contradicts the fact that $u(x) \leq 0 \ \forall x \in \Omega^c$. □

Now the Dirichlet problem is stated. In the following theorem we show a uniqueness result using the min/max principle. We will also prove the existence of a solution to this Dirichlet problem.

Theorem 1.9. [27] *Let (V, μ) be a connected locally finite weighted graph and let Ω be a finite, connected,* *proper subset of V. Consider the following Dirichlet problem*

$$\begin{aligned} \Delta_\mu u(x) &= f(x)\ \forall x \in \Omega \\ u(x) &= g(x)\ \forall x \in \Omega^c, \end{aligned} \tag{1.19}$$

where $u : V \to \mathbb{R}$ is an unknown function while the functions $f : \Omega \to \mathbb{R}$ and $g : \Omega^c \to \mathbb{R}$ are given. If Ω is finite and $\Omega^c \neq \emptyset$ then, $\forall f, g$ as above, the Dirichlet equation (1.19) has a unique solution.

Proof. The easy part is to prove the uniqueness of the solution if one exists. Suppose that there are two solutions u_1 and u_2 of the Dirichlet problem above. Let $u = u_1 - u_2$. Note that,

$$\begin{aligned} \Delta_\mu u(x) &= 0\ \forall x \in \Omega, \\ u(x) &= 0\ \forall x \in \Omega^c. \end{aligned} \tag{1.20}$$

This implies that, u is harmonic in Ω (i.e. u is both subharmonic and superharmonic in Ω). Now by the previous maximum/minimum principle,

$$0 = \inf_{\Omega^c} u \leq \min_{\Omega} u \leq \max_{\Omega} u \leq \sup_{\Omega^c} u = 0. \tag{1.21}$$

Hence, $u_1(x) - u_2(x) = u(x) = 0\ \forall x \in V = \Omega \cup \Omega^c$. Thus, $u_1 = u_2$. Therefore, if the above Dirichlet problem has a solution, then that solution is unique.

Now the existence of a solution to the Dirichlet problem above the equation is proven. By definition, $\Delta_\mu u(x) = f(x)$ reads

$$\frac{1}{\mu(x)} \sum_{y \sim x, y \in V} u(y)\mu_{xy} - u(x) = f(x). \tag{1.22}$$

Using $V = \Omega \cup \Omega^c$, the previous equation becomes

$$\frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega} u(y)\mu_{xy} + \frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega^c} u(y)\mu_{xy} - u(x) = f(x). \tag{1.23}$$

Hence,

$$\frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega} u(y)\mu_{xy} - u(x) = f(x) - \frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega^c} u(y)\mu_{xy}. \tag{1.24}$$

Note that $u(y) = g(y)$ on Ω^c and the previous equation implies that,

$$\frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega} u(y)\mu_{xy} - u(x) = f(x) - \frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega^c} g(y)\mu_{xy}. \tag{1.25}$$

Let X be the set of all real-valued functions u on Ω and define $L : X \to X$ by

$$Lu(x) = \frac{1}{\mu(x)} \sum_{y \sim x, y \in \Omega} u(y)\mu_{xy} - u(x). \tag{1.26}$$

Note that $L : X \to X$ is an operator. By setting $z = f(x) - \frac{1}{\mu(x)} \sum_{y\sim x, y\in\Omega^c} g(y)\mu_{xy}$ it is seen that $Lu = z$. Since X is a linear space with $\{\mathbf{1}_{\{x\}}\}_{x\in\Omega}$ as a basis, then $\dim X = |\omega| < \infty$. This implies that $L : X \to X$ is a linear operator in a finite dimensional space which is injective (L being injective comes from the uniqueness part of the proof). Since any injective operator mapping spaces of equal dimensions must be bijective, then $L : X \to X$ is subjective. Therefore, $\forall\ z \in X$, there is a solution $u = L^{-1}z$.

□

Given a graph $G = (V, E)$, [27] defines $\mathcal{F}$ to be the set of real-valued functions on V. In [27] it is stated that $\mathcal{F}$ is a vector space and provides an inner product for this space defined for any two functions $f, g \in \mathcal{F}$ by:

$$(f, g) = \sum_{x\in V} f(x)g(x)\mu(x). \tag{1.27}$$

Note that the above equation is the integration of fg against the measure μ on V. As usual the inner product induces a norm on $\mathcal{F}$ defined for $f \in \mathcal{F}$ by:

$$\|f\|_G = \sqrt{(f, f)} = \sqrt{\sum_{x\in V} (f(x))^2\mu(x)}. \tag{1.28}$$

With this in mind it is seen that $\mathcal{F}$ equipped with the norm $\|\cdot\|_G$ gives the Hilbert space $l^2(\mathcal{F})$ with measure of integration $\mu(x)$.

Grigoryan defines the difference operator for $x, y \in V$, by $\nabla_{xy} f = f(y) - f(x)$. Note that

$$\begin{aligned}
\Delta_\mu f(x) &= \frac{1}{\mu(x)} \sum_{y\sim x} f(y)\mu_{xy} - f(x) \\
&= \frac{1}{\mu(x)} \sum_{y\sim x} f(y)\mu_{xy} - f(x)\frac{1}{\mu(x)} \sum_{y\sim x} \mu_{x,y} \ (\text{as } \mu(x) = \sum_{y\sim x} \mu_{x,y}) \\
&= \frac{1}{\mu(x)} \sum_{y\sim x} f(y)\mu_{xy} - \frac{1}{\mu(x)} \sum_{y\sim x} f(x)\mu_{x,y} \\
&= \frac{1}{\mu(x)} \sum_{y\sim x} (f(y) - f(x))\mu_{xy} \\
&= \frac{1}{\mu(x)} \sum_{y\sim x} (\nabla_{xy} f)\mu_{xy}.
\end{aligned} \tag{1.29}$$

Now we states a discrete analogue of Green's formula on connected locally finite weighted graph.

Theorem 1.10. [27] *Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. Let $\Omega \subseteq V$ be such that $\Omega \neq \emptyset$ and Ω is finite. Then for any two functions f, g on V:*

$$\sum_{x\in\Omega} \Delta_\mu f(x)g(x)\mu(x) = -\frac{1}{2} \sum_{x,y\in\Omega} (\nabla_{xy} f)(\nabla_{xy} g)\mu_{xy} + \sum_{x\in\Omega} \sum_{y\in\Omega^c} (\nabla_{xy} f)g(x)\mu_{xy}. \tag{1.30}$$

If V is finite and $\Omega = V$ then Ω^c is empty so that the last "boundary" term in the equation above vanishes yielding the following equation:

$$\sum_{x\in\Omega} \Delta_\mu f(x) g(x)\mu(x) = -\frac{1}{2}\sum_{x,y\in\Omega} (\nabla_{xy} f)(\nabla_{xy} g)\mu_{xy}. \tag{1.31}$$

We now provide a proof of Corollary 1.6 using Green's Theorem (integration by parts).

Corollary 1.11. *Let (V, E) be a finite connected graph. Then*

$$\sum_{x\in V} \Delta_\mu f(x)\mu(x) = 0 \tag{1.32}$$

and

$$\sum_{x\in V} \mathcal{L}_\mu f(x)\mu(x) = 0. \tag{1.33}$$

Proof. Apply Green's Formula with $g(x) = 1$ for every vertex $x \in V$.

□

Next we state the result that Δ_μ is symmetric with respect to the inner product discussed at the beginning of this section.

Theorem 1.12. [27] *Let $G = (V, E)$ be a finite connected graph. Suppose that G has a weight μ. The operator Δ_μ is symmetric with respect to $(\cdot,\cdot)$, that is,*

$$(\Delta_\mu f, g) = (f, \Delta_\mu g). \tag{1.34}$$

$\forall f, g \in \mathcal{F}$.

Proof. Using the discrete Green's formula (1.31) yields:

$$\begin{aligned}(\Delta_\mu f, g) &= \sum_{x\in V} \Delta_\mu f(x) g(x)\mu(x) \\ &= -\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} f)(\nabla_{xy} g)\mu_{xy} \\ &= -\frac{1}{2}\sum_{x,y\in V} (\nabla_{xy} g)(\nabla_{xy} f)\mu_{xy} \\ &= (f, \Delta_\mu g).\end{aligned} \tag{1.35}$$

□

Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. We earlier noted that Grigoryan has found it useful to study the positive definite Laplacian: $\mathcal{L}_\mu = -\Delta_\mu f$. Using the Green's formula, It is noted that the Rayleigh quotient of $\mathcal{L}$ is:

$$\mathcal{R} = \frac{(\mathcal{L}_\mu f, f)}{(f, f)} = -\frac{(\Delta_\mu f, f)}{(f, f)} = \frac{1}{2}\frac{\sum_{x,y\in V} (\nabla_{xy} f)(\nabla_{xy} f)\mu_{xy}}{\sum_{x\in V} (f(x))^2\mu(x)}. \tag{1.36}$$

Theorem 1.13. [27]

For any finite, connected, weighted graph (V, μ) with $|V| > 1$, the following is true:

(a) Zero is a simple eigenvalue of $\mathcal{L}$.

(b) All the eigenvalues of $\mathcal{L}$ are contained in $[0, 2]$.

(c) If (V, μ) is not bipartite then all the eigenvalues of $\mathcal{L}$ are in $[0, 2)$.

Proof. (a) First recall that $\mathcal{L}c = 0 \; \forall c \in \mathbb{R}$. Consequently, the constant function is an eigenfunction with the eigenvalue 0. On the other-hand, suppose that $\mathcal{L}f = 0$. Hence,

$$\mathcal{R} = \frac{(\mathcal{L}f, f)}{(f, f)} = -\frac{(\Delta_\mu f, f)}{(f, f)} = \frac{1}{2}\frac{\sum_{x,y\in V}(\nabla_{xy}f)(\nabla_{xy}f)\mu_{xy}}{\sum_{x\in V}(f(x))^2\mu(x)} = 0. \tag{1.37}$$

Thus, $\sum_{\{x,y\in V: x\sim y\}}(f(y) - f(x))^2\mu_{xy} = 0$. So then, $f(x) = f(y)$ if and only if $x \sim y$. Let $x, y \in V$ be arbitrary. Since $G = (V, E)$ is connected, then there exists a path $\{x_k\}_{k=0}^m$ where $x = x_0 \sim x_1 \sim \cdots \sim x_N = y$. Hence, $f(x) = f(x_0) = f(x_1) = \cdots = f(x_N) = f(y)$. Therefore, f is constant.

It should be noted that you can prove that zero is a simple eigenvalue by appealing to the max/min principle discussed earlier. Pick a single vertex $a \in V$ and define $\Omega = V - \{a\}$. Since Ω is finite and f is harmonic, then the max/min principle implies:

$$f(a) = \inf_{\Omega^c} f \le \min_{\Omega} f \le \max_{\Omega} f \le \sup_{\Omega^c} f = f(a). \tag{1.38}$$

Therefore, f is constant.

(b) Suppose that λ is an eigenvalue of $\mathcal{L}$ with an eigenfunction f. Then by definition, $\mathcal{L}f = \lambda f$. Hence,

$$\begin{aligned}\lambda\sum_{x\in V}(f(x))^2\mu(x) &= \sum_{x\in V}(\mathcal{L}f)(x)f(x)\mu(x) \\ &= \frac{1}{2}\sum_{x,y\in V}(\nabla_{xy}f)(\nabla_{xy}f)\mu_{xy} \\ &= \frac{1}{2}\sum_{\{x,y\in V: x\sim y\}}(f(y) - f(x))^2\mu_{xy}.\end{aligned} \tag{1.39}$$

Using the inequality $(a + b)^2 \le 2(a^2 + b^2)$, the previous equation yields the following inequality:

$$\begin{aligned}\lambda\sum_{x\in V}(f(x))^2\mu(x) &\le \sum_{\{x,y\in V: x\sim y\}}((f(x))^2 + (f(y))^2)\mu_{xy} \\ &= \sum_{x,y\in V}(f(y))^2\mu_{xy} + \sum_{x,y\in V}(f(x))^2\mu_{xy} \\ &= \sum_{y\in V}(f(y))^2\mu(x) + \sum_{x\in V}(f(x))^2\mu(x) \\ &= 2\sum_{x\in V}(f(x))^2\mu(x).\end{aligned} \tag{1.40}$$

Dividing the previous equation by $\sum_{x\in V}(f(x))^2\mu(x)$ yields: $\lambda \le 2$.

(c) We proceed by way of the contrapositive. Suppose $\lambda = 2$ is an eigenvalue of $\mathcal{L}$ with an eigenfunction f. Replacing $\lambda = 2$ in the previous inequalities yields: $\frac{1}{2}(f(y) - f(x))^2 = ((f(x))^2 + (f(y))^2)$. Thus, $(f(y)-f(x))^2 = 2((f(x))^2+(f(y))^2)$. This in turn implies, $(f(x)+f(y))^2 = (f(x))^2+2f(x)f(y)+(f(y))^2 = 0$. So then, $f(x) + f(y) = 0$. If $f(v) = 0$ for some $v \in V$ then, $f(w) = 0$ for any $w \in V$ such that $w \sim v$. By the connectedness of $G = (V, E)$, f would be identically zero contradicting that f is an eigenfunction. This implies that, $f(x) \neq 0\ \forall x \in V$. It is seen now that $V = A \cup B$ where $A = \{x \in V : f(x) > 0\}$ and $B = \{x \in V : f(x) < 0\}$. Therefore, V is bipartite.

□

Note that the eigenvalues of $\mathcal{L}$ can be placed in increasing the ordering: $0 = \lambda_0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_{N-1}$. Next we state the following additional theorem about eigenvalues of $\mathcal{L}$. The proof is not provided here.

Theorem 1.14. [27] *For any finite, connected, weighted graph (V, μ) with $|V| > 1$ and suppose the notation in the previous sentence, the following is true:*

(a) $\lambda_1+\lambda_2+\cdots+\lambda_{N-1} \le N$. *Furthermore,* $\lambda_1 \le \frac{N}{N-1}$. *If (V, μ) has no loops then,* $\lambda_1+\lambda_2+\cdots+\lambda_{N-1} = N$. *and* $\lambda_{N-1} \ge \frac{N}{N-1}$.

(b) If (V, μ) is a complete graph with a simple weight μ, then $\lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = \frac{N}{N-1}$.

(c) If (V, μ) is non-complete then $\lambda_1 \le 1$.

(d) If (V, μ) is a bipartite and λ is an eigenvalue of $\mathcal{L}$, then $2 - \lambda$ is also an eigenvalue of $\mathcal{L}$ with the same multiplicity as λ. In particular, 2 is a simple eigenvalue of $\mathcal{L}$.

The original Laplacian of a graph from traditional graph theory obeys similar properties concerning its eigenvalues. A great reference for this Laplacian is ”Spectral Graph Theory” by Fan R. K. Chung [15].

CHAPTER 2

GRAPH LAPLACIANS APPLIED TO FRIENDLY LABELINGS ON FINITE GRAPHS

2.1

Introduction

In this chapter we introduce potential applications of the concept of the graph Laplacian to friendly labelings on weighted graphs. Hopefully, we may use insight gained from studying such concepts and problems and apply this insight to social network theory, partial differential equations on graphs, or graph theory in general.

Section 2.2 gives a brief history about friendly labelings and friendly index sets.

Section 2.3 is original work between Dr. Ebrahim Salehi and me. We give the definition of fully cordial graph and show some trees are fully cordial. The main original result of this section is that trees with near perfect matching are fully cordial. We also determine if and only if conditions for caterpillars of dimeter 4 to be fully cordial and show $ST(n, k)$ is fully cordial if and only if k is odd.

In Section 2.4 we provide original computations of the Jost graph Laplacian of friendly labelings on various finite connected simple graphs. These computations are done for the complete graph K_n, the complete bipartite graph $K(m, n)$, the star $ST(n)$, and a particular caterpillar of diameter 4. The most interesting result in this section is when K_n has an even number of vertices, then the Jost graph Laplacian of any friendly labeling is a constant multiple of the eigenvalue of the Jost graph Laplacian operator on K_n.

In Section 2.5 we provide an original definition for a wieghted Laplacian like operator. We provide an original computation of the weighted Laplacian like operator of friendly labelings on the complete graph K_n. The goal of this section is to provide an operator that may take into consideration edge labelings that may occur in the study of friendly index set theory.

Section 2.6 deals with future research objectives dealing with graph Laplacians of friendly labelings. The main original result of this section shows, if $q \in FI(G)$, then $q - 2 \in FI(G)$, when you have an odd number of vertices and a leaf. In this section we suggest potential ideas that may help in the search for if and only if conditions for a tree to be fully cordial and other potential research ideas.

2.2

Friendly Labelings and Friendly Index Set

We begin this section with a discussion of the concept of friendly labelings of finite simple connected graphs.

If we have a graph G with set of vertices $V(G)$, set of edges $E(G)$, and an abelian group A, then a labeling $f : V(G) \rightarrow A$ creates an edge labeling f^* defined by $f^*(xy) = f(x) + f(y)$. For $i \in A$, let $v_f(i) = |f^{-1}(i)|$, $e_f(i) = |f^{*-1}(i)|$, and let $X = \{|e_f(i) - e_f(j)| : i, j \in A\}$. The graph G is called A-cordial if it has a graph labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Graph theorists have examined cordial graphs quite a bit. The concept of cordial graphs was introduced by Cahit in [10][11][12]. Hovey studied A-cordial labelings where A is a Abelian group [35]. The labeling f is called A-friendly if $|v_f(i) - v_f(j)| \leq 1$ for every $i, j \in A$.

Given a graph $G = (V, E)$ with a coloring $f : V(G) \rightarrow \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. When $|v_f(1) - v_f(0)| \leq 1$, the coloring f is said to be friendly. This coloring produces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(xy) \equiv f(x) + f(y) \pmod 2$ for every $x, y \in E(G)$. Also let $e_f(i) = |f^{*-1}(i)|$. For a graph G, the friendly index set of the graph G is denoted by $FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly labeling}\}$. The concept of a friendly index set was first introduced by Chartrand, Lee, and Zhang in [16].

It should be noted that a graph G is cordial if 0 or 1 is in the friendly index set $FI(G)$ [41]. Thus, the study of friendly index sets is just a generalization of cordiality. Our goal will be to investigate A-friendly labelings where $A = \mathbb{Z}_2$ and $FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly labeling of G}\}$ will denote the friendly index set. Note we will often drop the subscript f in the case where the context is clear.

Sin-Min Lee and Ho Kuen Ng made the first calculations of friendly index sets of finite graphs in [41]. Cairne and Edwards proved that computational complexity in determining whether a graph has a cordial labeling is NP-complete in [13]. In fact the question on if a connected graph of diameter 2 admits a cordial labeling, which is a smaller scale question is also NP-complete. Thus, Cairne and Edwards concluded that determining the friendly index sets of graphs is difficult.

We now state a few known results in this area from the literature of friendly indices of finite graphs. From now on we assume G has p vertices and q edges, i.e. $|V(G)| = p$ and $|E(G)| = q$.

Theorem 2.1. [41] *For any finite graph G with q edges:* $FI(G) \subseteq \left\{q - 2i : i = 0, 1, 2, ..., \left\lfloor \frac{q}{2} \right\rfloor \right\}$.

Theorem 2.2. [41] *Suppose that,* $1 \leq m \leq n$. *For the complete bipartite graph* $K_{m,n}$*:*

$$FI(K_{m,n}) = \begin{cases} \left\{(m-2i)^2 : 0 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \right\}, & \text{if } m+n \text{ is even} \\ \{i(i+1) : 0 \leq i \leq m\}, & \text{if } m+n \text{ is odd.} \end{cases} \tag{2.1}$$

Note that for $n \geq 2$ that the complete bipartite graph $K(1, n)$ is called a star and in the literature is usually denoted by $ST(n)$. It should also be noted that stars are the trees of diameter 2. Now we state the friendly index set of a star from the literature.

Theorem 2.3. [41]

$$FI(ST(n)) = \begin{cases} \{0, 2\}, & \text{if } n \text{ is even} \\ \{1\}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Theorem 2.4. [41] *The friendly index set of a full binary tree with depth $d > 1$ is* $\{0, 2, 4, ..., 2^{d+1} - 4\}$.

Ebrahim Salehi and Sin Min Li in [54], found the friendly index sets of trees with perfect matching, Fibonacci trees, and Lucas Trees. We state a few of these results below. Also the definition of the concepts of matchings and perfect matchings are given.

Definition 2.1. [54] *A* matching *in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a* perfect matching *if every vertex of G is incident with an edge in M.*

Definition 2.2. [54][55] *A matching M in a graph G is said to be a* near perfect matching *if it covers all the vertices of G but one. G is called a* near perfect matching *graph if any maximal matching is a near perfect matching.*

Theorem 2.5. [54] *If $T = (p, q)$ is a tree with perfect matching, then* $FI(T) = \{1, 3, 5, \cdots, q\}$.

Corollary 2.6. [54] *Given a tree T with q edges, the coronation $T \copyright K_1$ is a tree with perfect matching and its friendly index set is* $\{1, 3, 5, \cdots, 2q + 1\}$.

Theorem 2.7. [54] *For* $n \geq 3$, *the friendly index set of* FT_n *is* $\left\{ |E_n| - 2i \, : \, i = 0, 1, 2, \cdots, \left\lfloor \frac{q_n}{2} \right\rfloor \right\}$. *Moreover, every element of the index set can be obtained by a friendly coloring* $f : V_n \to \mathbb{Z}_2$ *with* $N(f) = e_f(1) - e_f(0)$, *and the color of root is* 1.

Theorem 2.8. [54] *For* $n \geq 3$, *the friendly index set of the Lucas tree* $LT_n = (V_n, E_n)$ *is* $\{ 0, 2, 4, \cdots, |E_n|\}$. *Moreover, every element* b *of this index set can be obtained by a friendly coloring* $f : V_n \to \mathbb{Z}_2$ *with the following properties:*

(a) $f(r_n) = 1$;

(b) $b = N(f) = e_f(1) - e_f(0)$;

(c) If $b \equiv 0 \pmod 4$, *then* $v_f(1) = v_f(0) + 1$; *and*

(d) If $b \equiv 2 \pmod 4$, *then* $v_f(0) = v_f(1) + 1$.

2.3

Fully Cordial Trees

Notice from the last 4 theorems in the previous section that every theoretical possible friendly index from minimal index to the number of edges was obtained, i.e. $FI(G) = \left\{q - 2i : i = 0, 1, 2, ..., \left\lfloor \frac{q}{2} \right\rfloor \right\}$. This led to a definition of the concept of a fully cordial graph. The work in this section is original work I did with Dr. Ebrahim Salehi and was published as the paper "Fully Cordial Trees" [52].

Definition 2.3. *A graph* G *is said to be fully cordial if*

$$FI(G) = \{q - 2i : i = 0, 1, 2, \ldots, \lfloor q/2 \rfloor\}.$$

The next leema we present shows that $e_f(1) \neq 0$.

Lemma 2.9. *Let* G *be a non trivial connected graph and* $f : V(G) \to \mathbb{Z}_2$ *any friendly coloring of* G. *Then* $e_f(1) \geq 1$.

Proof. The two sets $A = \{u \in V(G) : f(u) = 0\}$ and $B = \{v \in V(G) : f(v) = 1\}$ partition $V(G)$. Since G is connected, there are vertices $u \in A$ and $v \in B$ that are adjacent. The label of edge uv is 1. Therefore, $e_f(1) \geq 1$. □

Corollary 2.10. *For any graph G, $q \in FI(G)$ if and only if $e_f(0) = 0$ for some friendly coloring $f : V(G) \to \mathbb{Z}_2$.*

Lemma 2.11. *A tree T with near perfect matching M contains at least a P_3 pendent $u_1 \sim u_2 \sim u_3$ such that $\deg u_1 = 1$, $\deg u_2 = 2$ and $u_1u_2 \in M$.*

Proof. Let $P : u_1 \sim u_2 \sim u_3 \sim \cdots \sim u_{k-2} \sim u_{k-1} \sim u_k$ be the longest path in T. Clearly, $\deg u_1 = \deg u_k = 1$. Also, $\deg u_2 = 2$ or $\deg u_{k-1} = 2$. Otherwise, any maximum matching of T would miss at least two vertices. If $\deg u_2 = \deg u_{k-1} = 2$, then $u_1u_2 \in M$ or $u_ku_{k-1} \in M$. Suppose (WLOG) $\deg u_2 = 2$ and $\deg u_{k-1} > 2$. Then $u_1u_2 \in M$. Otherwise, any maximum matching of T would miss at least two vertices. □

Next we state the main result of [52].

Theorem 2.12. *Any near perfect matching tree is fully cordial.*

Proof. Note that T is a tree of odd order, $|T| = 2n+1$. We proceed by induction on n. Clearly, the statement of theorem is true for $n = 1$. Suppose the statement is true for any tree of order $2n + 1$ and let T be a tree of order $2n + 3$ with near-perfect matching M. By Lemma 2.11, T contains vertices $u \sim v \sim w$ such that $\deg u = 1$, $\deg v = 2$ and the edge uv is in M. Now consider the tree $S = T - \{u, v\}$ which has order $2n + 1$ and has near perfect matching $M^{'} = M - \{uv\}$. Therefore, by the induction hypothesis

$$FI(S) = \{0, 2, 4, \cdots, 2n\}.$$

We need to show that $FI(T) = \{0, 2, \cdots, 2n, 2n + 2\}$. Consider a friendly coloring $f : V(S) \to \mathbb{Z}_2$ of S and extend it to $g : V(T) \to \mathbb{Z}_2$ by defining $g(v) = f(w)$, $g(u) = 1 - f(w)$. Then g is a friendly coloring of T with $e_g(1) = e_f(1) + 1$, $e_g(0) = e_f(0) + 1$. Therefore, $N(g) = N(f)$. This implies that

$$FI(S) = \{0, 2, 4, \cdots, 2n\} \subseteq FI(T).$$

It only remains to show that $2n + 2 \in FI(T)$. Let $\phi : V(S) \to \mathbb{Z}_2$ be a friendly coloring of S with index $2n$. We may assume that $e(1) = 2n$, $e(0) = 0$, and extend ϕ to $\psi : V(T) \to \mathbb{Z}_2$ by defining $\psi(v) = 1 - \phi(w)$, $\psi(u) = \phi(w)$. Then ψ is a friendly coloring of T with $e_\psi(1) = e_\phi(1) + 2$, $e_\psi(0) = e_\phi(0) = 0$. Therefore, $N(\psi) = N(\phi) + 2 = 2n + 2$. □

Corollary 2.13. *For $n \geq 2$, the path of order n is fully cordial.*

Proof. This is an immediate consequence of Theorems 2.5 and 2.12. Because, any path P_n is either near perfect matching or is a perfect matching tree. Therefore, it is fully cordial. □

By Theorem 2.7 it is shown that any Fibonacci tree is fully cordial [54]. We now present a different proof of Theorem 2.7 using the concepts of perfect matching and near perfect matching.

Theorem 2.14. *For $n \geq 1$, every Fibonacci tree FT_n is fully cordial.*

Proof. Note that every Fibonacci tree has either a perfect matching or is a near perfect matching tree. In fact, if $n \equiv 1 \pmod 3$, then FT_n is a near perfect matching tree; otherwise, it has a perfect matching. We prove this statement by induction on n. Clearly the statement is true for $n = 1, 2, 3$. Now suppose the statement is true for all positive integers less than n $(3 < n)$ and let FT_n be the Fibonacci tree of order n. We consider the following cases:

(A) $n \equiv 1 \pmod 3$. In this case, by the induction hypothesis, both the left and right children have perfect matchings. Let M_1 and M_2 be perfect matchings of FT_{n-1} and FT_{n-2}, respectively. Then $M_1 \cup M_2$ is a maximum matching of FT_n that covers all the vertices but its root. Therefore, FT_n is a near perfect matching tree.

(B) $n \equiv 2 \pmod 3$. In this case, by the induction hypothesis, the left child FT_{n-1} is near perfect matching while the right child FT_{n-2} has a perfect matching. Let M_1 be a maximum matching of FT_{n-1} (we may assume that M_1 leaves the root r_{n-1} out) and M_2 be a perfect matching of FT_{n-2}. Then $M_1 \cup M_2 \cup \{r_n r_{n-1}\}$ will form a perfect matching of FT_n.

(C) $n \equiv 0 \pmod 3$. The argument is similar to the previous case.

□

Definition 2.4. Star-like tree, denoted by $ST(n, k)$, is a graph formed by n copies of P_k when all of them share exactly an end-vertex. This common end-vertex is clearly the center of the graph.

We observe that $ST(1, k) \simeq P_k$, $ST(2, k) \simeq P_{2k-1}$, $ST(n, 2) \simeq K_{1,n}$ and $ST(n, 1)$ the trivial graph having just one vertex. The friendly index sets of these graphs have been determined. Therefore, from now on we assume that $n, k \geq 3$.

Corollary 2.15. *For any $n, k \geq 3$ the star-like tree $ST(n, k)$ is fully cordial if and only if k is odd.*

Proof. If k is odd, then $ST(n,k)$ is a near perfect matching tree and by Theorem 2.12 it is fully cordial. When k is even, then $e(0) \neq 0$ holds for any friendly coloring of the graph. Hence, by Corollary 2.10, the maximum possible friendly index cannot be achieved. In fact, when k is even, then $e(0) \geq \lfloor (n-1)/2 \rfloor$. □

Next we give two definitions a particular type of graph called a caterpillar. We will then give some notation we use to describe caterpillar. Then we give visual examples of caterpillars.

Definition 2.5. [52][20] *A caterpillar is a tree having the property that the removal of its end-vertices results in a path called the spine.*

Definition 2.6. [55] *A caterpillar is a tree in which a single path called the spine contains every edge.*

We use the notation $CR(a_1, a_2, a_3, \ldots, a_n)$ to represent a caterpillar with a P_n-spine. We label the ith vertex of P_n as u_i and say $\deg(u_i) = a_i$. We will assume that $a_i \geq 2$. Note that a caterpillar with a P_n-spine has a diameter of $n+1$.

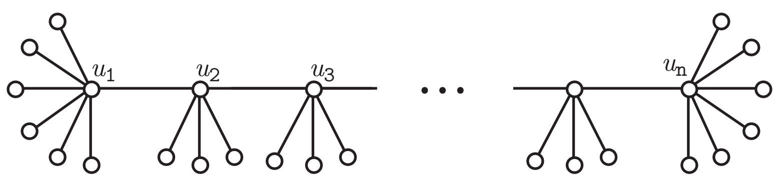

Figure 2.1: A caterpillar of diameter $n+1$ (P_n-spine)[52][20].

Definition 2.7. *A double star is a tree of diameter 3. Double stars have two central vertices u and v and are denoted by $DS(a,b)$, where $\deg u = a$ and $\deg v = b$.*

Double star $DS(a,b)$ has $a+b$ vertices and its friendly index set is known and stated below.

Theorem 2.16. [53][20] *Let $a \leq b$. Then*

$$FI(DS(a,b)) = \begin{cases} \{1, 3, \ldots, 2a-1\} & \text{if } a+b \text{ is even;} \\ \{0, 2, \ldots, 2a\} & \text{if } a+b \text{ is odd.} \end{cases}$$

Corollary 2.17. *Double star $DS(a,b)$ is fully cordial if and only if $|a-b| \leq 1$.*

For a caterpillar of diameter 4, $CR(a,b,c)$ we call the vertices on the spine u, v, and w. We calculate the number of vertices by $|V| = a+b+c-1$ and the number of edges by $|E| = a+b+c-2$.

The friendly index set of $G = CR(a,b,c)$, when $a+b+c$ is odd, is determined and stated below

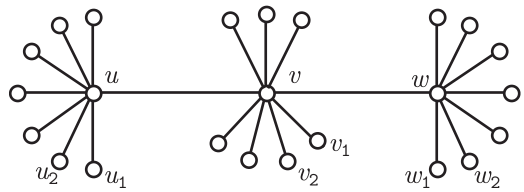

Figure 2.2: A caterpillar of diameter 4, $CR(8, 9, 8)$[52][20].

Theorem 2.18. [53][20] *Let $a, b, c \geq 2$ and $a + b + c$ be odd. Then $FI(CR(a, b, c)) = A \cup B \cup C$, where*

$$\begin{aligned} A &= \{|\, 2a - 4i - 1 \,| : m_A \leq i \leq M_A\}; \\ B &= \{|\, 2b - 4j - 1 \,| : m_B \leq j \leq M_B\}; \\ C &= \{|\, 2c - 4k - 1 \,| : m_C \leq k \leq M_C\}; \end{aligned}$$

and

$$\begin{aligned} m_A &= \max\left\{0, \tfrac{a-b-c+3}{2}\right\}; & M_A &= \min\left\{a-1, \tfrac{a+b+c-3}{2}\right\}; \\ m_B &= \max\left\{0, \tfrac{-a+b-c+1}{2}\right\}; & M_B &= \min\left\{b-2, \tfrac{a+b+c-3}{2}\right\}; \\ m_C &= \max\left\{0, \tfrac{-a-b+c+3}{2}\right\}; & M_C &= \min\left\{c-1, \tfrac{a+b+c-3}{2}\right\}. \end{aligned}$$

Lemma 2.19. *Let $a+b+c$ be odd. The caterpillar $G = CR(a, b, c)$ has the maximum possible friendly index if and only if $|b - a - c + 1| \leq 1$.*

Proof. By the Corollary 2.10, such a friendly labeling f exists if and only if all edges are labeled 1. Let $f(u) = f(w) = 1$ and $f(v) = 0$. Then all the end-vertices adjacent to u and w are labeled 0, and all end-vertices adjacent to v are labeled 1. That is, $a + c - 1$ vertices are labeled 0 and b vertices are labeled 1. But for this labeling to be friendly we require $|b - a - c + 1| \leq 1$. □

Theorem 2.20. *Let $a + b + c$ be odd. Then $G = CR(a, b, c)$ is fully cordial if and only if $b = a + c - 1$ and $a = 2$ or $c = 2$.*

Proof. Suppose G is fully cordial. Then by Lemma 2.19, $b = a + c - 1$. Also, $a = 2$ or $c = 2$. Otherwise, using the notation of Theorem 2.18, the sets A and C are subsets of B and

$$FI(G) = B = \{|\, 2b - 4j - 1 \,| : 0 \leq j \leq b - 2\}.$$

However, this set has $b - 1$ odd numbers; the smallest is 1 and the largest element is $2b - 1$. Therefore, one odd number between 1 and $2b - 1$ is missing. In fact, $2b - 3$ is not in $FI(G)$.

Conversely, let $b = a + c - 1$ and $a = 2$ or $c = 2$. Without loss of generality, we may assume $a = 2$. In this case $G = CR(2, c + 1, c)$. Using Theorem 2.18, one can easily see that $FI(G) = \{1, 3, \ldots, 2c + 1\}$ which shows that G is fully cordial. □

In what follows, we consider the caterpillar $G = CR(a, b, c)$, when $a + b + c$ is even. First we determine its friendly index set, then we completely identify those that are fully cordial. As mentioned before, G has $a + b + c - 1$ vertices and $a + b + c - 2$ edges.

We observe that any friendly coloring $f : G \to \mathbb{Z}_2$ that labels the central vertices the same will result in either index $N(f) = 0$ or $N(f) = 2$, which are not very interesting. Therefore, we consider the cases in which the central vertices are labeled differently.

Case 1. Suppose we label the central vertices by $f(u) = 0$, and $f(v) = f(w) = 1$ and label all other vertices by 1 except

$$\begin{array}{llll} f(u_1) & = f(u_2) & = \cdots & = f(u_i) = 0; \\ f(v_1) & = f(v_2) & = \cdots & = f(v_j) = 0; \\ f(w_1) & = f(w_2) & = \cdots & = f(w_k) = 0. \end{array} \tag{2.3}$$

Then $v(0) = i + j + k + 1$ and $e(1) = a - i + j + k$. For this labeling to be friendly we require either

$$i + j + k + 1 = \frac{a + b + c}{2}, \tag{2.4}$$

or

$$i + j + k + 1 = \frac{a + b + c - 2}{2}, \tag{2.5}$$

Equation (2.4) yields $N(f) = |e(1) - e(0)| = |2a - 4i|$. In this situation,

$$i + 1 \leq \frac{a + b + c}{2} \text{ and } a - i + 1 \leq \frac{a + b + c - 2}{2},$$

which provide the inequalities

$$\frac{a - b - c + 4}{2} \leq i \leq \frac{a + b + c - 2}{2}.$$

Therefore, the friendly indices obtained in this case would be

$$A = \{|2a - 4i| : m_A \leq i \leq M_A\},$$

where

$$m_A = \max\left\{0, \frac{a - b - c + 4}{2}\right\} \text{ and } M_A = \min\left\{a - 1, \frac{a + b + c - 2}{2}\right\}.$$

Equation (2.5) gives us $|e(1) - e(0)| = |2a - 4i - 2|$. In this situation,

$$i + 1 \leq \frac{a + b + c - 2}{2} \text{ and } a - i + 1 \leq \frac{a + b + c}{2},$$

which provide the inequalities

$$\frac{a - b - c + 2}{2} \leq i \leq \frac{a + b + c - 4}{2}.$$

Therefore, the friendly indices obtained in this situation would be

$$D = \{|2a - 4i - 2| : m_D \leq i \leq M_D\},$$

where

$$m_D = \max\left\{0, \frac{a-b-c+2}{2}\right\} \text{ and } M_D = \min\left\{a-1, \frac{a+b+c-4}{2}\right\}.$$

Case 2. Let $f(v) = 0$, and $f(u) = f(w) = 1$ be the labeling of the central vertices and all other vertices be labeled 1 except for those specified in equation (2.3). In this case, $v(0) = i+j+k+1$ and $e(1) = b-j+i+k$. Again, for this labeling to be friendly we require either (2.4) or (2.5).

The equation (2.4) gives us $N(f) = |e(1) - e(0)| = |2b - 4j|$. In this instance,

$$j + 1 \leq \frac{a+b+c}{2} \text{ and } b - j \leq \frac{a+b+c-2}{2},$$

which provide the inequalities

$$\frac{b-a-c+2}{2} \leq j \leq \frac{a+b+c-2}{2}.$$

Therefore, the friendly indices obtained in this case would be

$$B = \{|2b - 4j| : m_B \leq j \leq M_B\},$$

where

$$m_B = \max\left\{0, \frac{b-a-c+2}{2}\right\} \text{ and } M_B = \min\left\{b-2, \frac{a+b+c-2}{2}\right\}.$$

The equation (2.5) yields $N(f) = |e(1) - e(0)| = |2b - 4j - 2|$. In this instance,

$$j + 1 \leq \frac{a+b+c-2}{2} \text{ and } b - j \leq \frac{a+b+c}{2},$$

which provide the inequalities

$$\frac{b-a-c}{2} \leq j \leq \frac{a+b+c-4}{2}.$$

Therefore, the friendly indices obtained in this subcase would be

$$E = \{|2b - 4j - 2| : m_E \leq j \leq M_E\},$$

where

$$m_E = \max\left\{0, \frac{b-a-c}{2}\right\} \text{ and } M_E = \min\left\{b-2, \frac{a+b+c-4}{2}\right\}.$$

Case 3. Suppose we label the central vertices by $f(w) = 0$, $f(u) = f(v) = 1$ and label all other vertices by 1 except for those specified in equation (2.3). Then $v(0) = i + j + k + 1$ and $e(1) = b - k + i + j$. Again, for this labeling to be friendly we require either (2.4) or (2.5).

The equation (2.4) gives us $N(f) = |e(1) - e(0)| = |2c - 4k|$. In this situation,

$$k + 1 \leq \frac{a+b+c}{2} \text{ and } c - k + 1 \leq \frac{a+b+c-2}{2},$$

which provide the inequalities

$$\frac{c-a-b+4}{2} \leq k \leq \frac{a+b+c-2}{2}.$$

Therefore, the friendly indices obtained in this subcase would be

$$C = \{|2c - 4k| : m_C \leq k \leq M_C\},$$

where

$$m_C = \max\left\{0, \frac{c-a-b+4}{2}\right\} \text{ and } M_C = \min\left\{c-1, \frac{a+b+c-2}{2}\right\}.$$

The equation (2.5) gives us $N(f) = |e(1) - e(0)| = |2c - 4k - 2|$. In this situation,

$$k + 1 \leq \frac{a+b+c-2}{2} \text{ and } c - k + 1 \leq \frac{a+b+c}{2},$$

which provide the inequalities

$$\frac{c-a-b+2}{2} \leq k \leq \frac{a+b+c-4}{2}.$$

Therefore, the friendly indices obtained in this subcase would be

$$F = \{|2c - 4k| : m_F \leq k \leq M_F\},$$

where

$$m_F = \max\left\{0, \frac{c-a-b+2}{2}\right\} \text{ and } M_F = \min\left\{c-1, \frac{a+b+c-4}{2}\right\}.$$

We summarize the above discussion in the following theorem.

Theorem 2.21. *Suppose $a + b + c$ is even and $a, b, c \geq 2$. Then*

$$FI(CR(a, b, c)) = A \cup B \cup C \cup D \cup E \cup F,$$

where

$$\begin{array}{ll} A = \{|2a - 4i| : m_A \leq i \leq M_A\}; & D = \{|2a - 4i - 2| : m_D \leq i \leq M_D\}; \\ B = \{|2b - 4j| : m_B \leq j \leq M_B\}; & E = \{|2b - 4j - 2| : m_E \leq j \leq M_E\}; \\ C = \{|2c - 4k| : m_C \leq k \leq M_C\}; & F = \{|2c - 4k - 2| : m_F \leq k \leq M_F\}; \end{array}$$

Lemma 2.22. *Let $a+b+c$ be even. Then the caterpillar $G = CR(a,b,c)$ has a maximum friendly index if and only if $|b-a-c+1| = 1$.*

Proof. By Corollary 2.10, such a friendly labeling f exists if and only if all edges are labeled 1. Without loss of generality we may assume that $f(u) = f(w) = 0$ and $f(v) = 1$. Then all the end-vertices adjacent to u and w are labeled 1, and all end-vertices adjacent to v are labeled 0. That is, $v(0) = b$. However this labeling is friendly if and only if either $2b = a+b+c$ or $2b = a+b+c-2$ which proves the lemma. □

Lemma 2.23. *Let $|b-a-c+1| = 1$. Then $FI(CR(a,b,c)) = \Delta \cup \Omega$ where*

$$\Delta = \left\{|2b-4j| : 1 \le j \le \lfloor \tfrac{b}{2} \rfloor\right\}; \quad \Omega = \left\{|2b-4j-2| : 0 \le j \le \lfloor \tfrac{b-1}{2} \rfloor\right\}.$$

Proof. We utilize Theorem 2.21 and note that

$$\begin{array}{ll} A = \{|2a-4i| : 0 \le i \le a-1\}; & D = \{|2a-4i-2| : 0 \le i \le a-1\}; \\ B = \{|2b-4j| : 1 \le j \le b-2\}; & E = \{|2b-4j-2| : 0 \le j \le b-2\}; \\ C = \{|2c-4k| : 0 \le k \le c-1\}; & F = \{|2c-4k-2| : 0 \le k \le c-1\}. \end{array}$$

Since $A \cup D$, $C \cup F \subset B \cup E$ then by 2.21, $FI(G) = B \cup E$. We also observe that $|2b-4j|$ produces the same number for j and $b-j$. Similarly, $|2b-4k-2|$ produces the same number for k and $b-k+1$. Therefore, $B = \Delta$ and $E = \Omega$. □

Theorem 2.24. *Let $a+b+c$ be even. Then $G = CR(a,b,c)$ is fully cordial if and only if $|b-a-c+1| = 1$.*

Proof. Suppose G is fully cordial. Then G achieves its maximum friendly index and by Lemma 2.22, $|b-a-c+1| = 1$.

Conversely, let $|b-a-c+1| = 1$. Then by Lemma 2.23, $FI(G) = \Delta \cup \Omega$. Also, we observe that G has either $2b-2$ or $2b$ edges and the set $\Delta \cup \Omega$ generates exactly either $\{0,2,4,\dots,2b-2\}$ or $\{0,2,4,\dots,2b\}$. These numbers are the full spectrum of friendly indices of G. □

2.4

Jost Graph Laplacian Applied to Friendly Labelings on Various Finite Graphs

I was once asked by Dr. Ebrahim Salehi what are if and only if conditions for a tree to be fully cordial. It was also recently brought to my attention by Dr. Pushkin Kachroo that there are potential applications of the concepts of friendly index sets of finite graphs and the concept of fully cordial graphs to network theory and the study of social media. In light of these ideas, a research goal is to compute graph Laplacians of friendly labelings of certain graphs with various types of edge labelings. In particular we focus on the Jost Graph Laplacian of friendly labelings on various finite graphs. The goal will be to discover what combinatorial properties continually arise and see if these new properties can be applied to partial differential equation theory on graphs, the study of network theory, the study of fully cordial graphs, or graph theory in general.

We will recall the definition of the weighted graph Laplacian operators given earlier in this manuscript.

Definition 2.8. [27]*(Degree Laplace Operator) Let (V, E) be a locally finite connected graph. For any function $f : V \to \mathbb{R}$, define the Laplace operator, Δf, by*

$$\Delta_{deg(x)} f(x) = \frac{1}{deg(x)} \sum_{y \sim x} f(y) - f(x). \tag{2.6}$$

The Laplace operator studied by Jost is $\mathcal{L} = -\Delta f$.

Definition 2.9. [27]*(Weighted Laplace Operator) Let $G = (V, E)$ be a connected, locally finite graph. Suppose that G has a weight μ. For any function $f : V \to \mathbb{R}$, define the weighted μ-Laplace operator, $\Delta_\mu f$, by*

$$\Delta_\mu f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x). \tag{2.7}$$

The normalized graph Laplace operator studied by Grigoryan is $\mathcal{L}_\mu = -\Delta_\mu f$.

We first compute the graph Laplacian studied by Jost for various friendly labelings for various finite graphs. This means we are setting $\mu_{xy} = 1 \ \forall \{x, y\} \in E(G)$. We first do such a computation using the complete graph $G = K_n$.

Definition 2.10. *A graph G is complete if every two distinct vertices are joined by an edge (adjacent).*

Note that a complete graph of order n is denoted by K_n and has $\frac{n(n-1)}{2}$ edges.

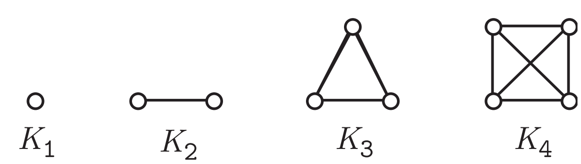

Figure 2.3: The complete graphs K_1, K_2 ,K_3, K_4[20].

Theorem 2.25. *Let $f : K_n \to \mathbb{Z}_2$ be a friendly labeling. Then*

$$\Delta f(K_n) = \begin{cases} \left\{-\frac{n}{2(n-1)}, \frac{n}{2(n-1)}\right\}, & \text{if } n \text{ is even} \\ \left\{\frac{1}{2}, -\frac{n+1}{2(n-1)}\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n+1}{2} \text{ and } v_f(1) = \frac{n-1}{2} \\ \left\{-\frac{1}{2}, \frac{n+1}{2(n-1)}\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n-1}{2} \text{ and } v_f(1) = \frac{n+1}{2}. \end{cases}$$

Consequently,

$$\mathcal{L}f(K_n) = \begin{cases} \left\{-\frac{n}{2(n-1)}, \frac{n}{2(n-1)}\right\}, & \textit{if } n \textit{ is even} \\ \left\{-\frac{1}{2}, \frac{n+1}{2(n-1)}\right\}, & \textit{if } n \textit{ is odd, } v_f(0) = \frac{n+1}{2} \textit{ and } v_f(1) = \frac{n-1}{2} \\ \left\{\frac{1}{2}, -\frac{n+1}{2(n-1)}\right\}, & \textit{if } n \textit{ is odd, } v_f(0) = \frac{n-1}{2} \textit{ and } v_f(1) = \frac{n+1}{2}. \end{cases}$$

Proof. Suppose that $f : K_n \to \mathbb{Z}_2$ be a friendly labeling. Let $x \in V(K_n)$. We have a three cases to consider each with two subcases depending on if $f(x) = 0$ or $f(x) = 1$.

Case 1: Suppose that n is even. This means $v_f(0) = \frac{n}{2}$ and $v_f(1) = \frac{n}{2}$.

Subcase 1: Let $f(x) = 0$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1}\left(0 \cdot \left(\frac{n}{2} - 1\right) + 1 \cdot \frac{n}{2}\right) - 0 = \frac{n}{2(n-1)}. \tag{2.8}$$

Thus, $\mathcal{L}f(x) = -\frac{n}{2(n-1)}$.

Subcase 2: Let $f(x) = 1$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1}\left(0 \cdot \frac{n}{2} + 1 \cdot \left(\frac{n}{2} - 1\right)\right) - 1 = -\frac{n}{2(n-1)}. \tag{2.9}$$

Thus, $\mathcal{L}f(x) = \frac{n}{2(n-1)}$.

Case 2: Suppose that n is odd, $v_f(0) = \frac{n+1}{2}$ and $v_f(1) = \frac{n-1}{2}$.

Subcase 1: Let $f(x) = 0$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1}\left(0 \cdot \left(\frac{n+1}{2} - 1\right) + 1 \cdot \frac{n-1}{2}\right) - 0 = \frac{1}{2}. \tag{2.10}$$

Thus, $\mathcal{L}f(x) = -\frac{1}{2}$.

Subcase 2: Let $f(x) = 1$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1}\left(0 \cdot \left(\frac{n+1}{2}\right) + 1 \cdot \left(\frac{n-1}{2} - 1\right)\right) - 1 = -\frac{n+1}{2(n-1)}. \tag{2.11}$$

Thus, $\mathcal{L}f(x) = \frac{n+1}{2(n-1)}$.

Case 3: Suppose that n is odd, $v_f(0) = \frac{n-1}{2}$ and $v_f(1) = \frac{n+1}{2}$.

Subcase 1: Let $f(x) = 0$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1}\left(0 \cdot \left(\frac{n-1}{2} - 1\right) + 1 \cdot \frac{n+1}{2}\right) - 0 = \frac{n+1}{2(n-1)}. \tag{2.12}$$

Thus, $\mathcal{L}f(x) = -\frac{n+1}{2(n-1)}$.

Subcase 2: Let $f(x) = 1$. Then,

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n-1} \left(0 \cdot \frac{n-1}{2} + 1 \cdot \left(\frac{n+1}{2} - 1 \right) \right) - 1 = -\frac{1}{2}. \tag{2.13}$$

Thus, $\mathcal{L}f(x) = \frac{1}{2}$.

□

In view of Theorem 1.14 part (b) concerning the eigenvalues of a complete graph with simple weight μ the previous theorem is stated as the following corollary.

Corollary 2.26. *Let $f : K_n \to \mathbb{Z}_2$ be a friendly labeling. Suppose that λ is an eigenvalue of Laplace operator Δ over K_n. Then*

$$\Delta f(K_n) = \begin{cases} \left\{-\frac{1}{2}\lambda, \frac{1}{2}\lambda\right\}, & \text{if } n \text{ is even} \\ \left\{\frac{(n-1)}{2n}\lambda, -\frac{(n+1)}{2n}\lambda\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n+1}{2} \text{ and } v_f(1) = \frac{n-1}{2} \\ \left\{-\frac{(n-1)}{2n}\lambda, \frac{(n+1)}{2n}\lambda\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n-1}{2} \text{ and } v_f(1) = \frac{n+1}{2}. \end{cases}$$

Consequently,

$$\mathcal{L}f(K_n) = \begin{cases} \left\{\frac{1}{2}\lambda, -\frac{1}{2}\lambda\right\}, & \text{if } n \text{ is even} \\ \left\{-\frac{(n-1)}{2n}\lambda, \frac{(n+1)}{2n}\lambda\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n+1}{2} \text{ and } v_f(1) = \frac{n-1}{2} \\ \left\{\frac{(n-1)}{2n}\lambda, -\frac{(n+1)}{2n}\lambda\right\}, & \text{if } n \text{ is odd, } v_f(0) = \frac{n-1}{2} \text{ and } v_f(1) = \frac{n+1}{2}. \end{cases}$$

Proof. From Theorem 1.14 part (b) we recall that for the complete graph K_n with a simple weight μ, that the Laplace operator Δ has eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \frac{n}{n-1}$. So for any given eigenvalue λ of the Laplace operator Δ: $\lambda = \frac{n}{n-1}$. Comparing this with the previous theorem yields the corollary.

□

Note Corollary 2.26 shows when K_n has an even number of vertices, then the Jost graph Laplacian of any friendly labeling is a constant multiple of the eigenvalue of the Jost graph Laplacian operator on K_n.

Now we recall the definitions of bipartite and complete bipartite finite graphs in order to compute the graph laplacian. Then we give an example complete bipartite graph.

Definition 2.11. [55] *A graph G is called bipartite if $V(G)$ can be partitioned into two disjoint subsets S and T, called partite sets, such that every edge of G joins a vertex of S and a vertex of T. A graph that is both complete and bipartite is called a complete bipartite graph. A complete bipartite graph such that $|S| = m$ and $|T| = n$ is denoted by $K(m, n)$.*

In the following theorem we present sets that have the range of the Jost graph Laplace operator on a friendly labeling of a complete bipartite graph as a subset.

Figure 2.4: Two complete bipartite graphs $K(2,3)$ is on the left and $K(3,3)$ is on the right [20].

Theorem 2.27. *Let $m \leq n$. Let $f : K(m,n) \to \mathbb{Z}_2$ be a friendly labeling. Then*

$$\Delta f(K(m,n)) \subseteq \begin{cases} \left\{\frac{n-i}{n}, -\frac{i}{n}, \frac{m-j}{m}, -\frac{j}{m} : \frac{n-m}{2} \leq i \leq \frac{m+n}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n}{2}\right\}, & \text{if } m+n \text{ is even} \\ \left\{\frac{n-i}{n}, -\frac{i}{n}, \frac{m-j}{m}, -\frac{j}{m} : \frac{n-m+1}{2} \leq i \leq \frac{m+n+1}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n+1}{2}\right\}, & \text{if } m+n \text{ is odd and } v_f(0) = v_f(1)+1 \\ \left\{\frac{n-i}{n}, -\frac{i}{n}, \frac{m-j}{m}, -\frac{j}{m} : \frac{n-m-1}{2} \leq i \leq \frac{m+n-1}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n-1}{2}\right\}, & \text{if } m+n \text{ is odd and } v_f(1) = v_f(0)+1. \end{cases}$$

Consequently,

$$\mathcal{L}f(K(m,n)) \subseteq \begin{cases} \left\{-\frac{n-i}{n}, \frac{i}{n}, -\frac{m-j}{m}, \frac{j}{m} : \frac{n-m}{2} \leq i \leq \frac{m+n}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n}{2}\right\}, & \text{if } m+n \text{ is even} \\ \left\{-\frac{n-i}{n}, \frac{i}{n}, -\frac{m-j}{m}, \frac{j}{m} : \frac{n-m+1}{2} \leq i \leq \frac{m+n+1}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n+1}{2}\right\}, & \text{if } m+n \text{ is odd and } v_f(0) = v_f(1)+1 \\ \left\{-\frac{n-i}{n}, \frac{i}{n}, -\frac{m-j}{m}, \frac{j}{m} : \frac{n-m-1}{2} \leq i \leq \frac{m+n-1}{2}, 0 \leq j \leq m, \text{ and } i+j = \frac{m+n-1}{2}\right\}, & \text{if } m+n \text{ is odd and } v_f(1) = v_f(0)+1. \end{cases}$$

Proof. Let $m \leq n$. Suppose that $f : K(m,n) \to \mathbb{Z}_2$ be a friendly labeling. Let i be the number of vertices in $K(m,n)$ of degree m such that $f(x) = 0$ and j be the number of vertices in $K(m,n)$ of degree n such that $f(x) = 0$. Let $x \in V(K(m,n))$.

Regardless of the parity of $m+n$ we have the following 4 cases for computing the Laplace operator at a vertex x:

Case 1: Let $\deg(x) = n$ and $f(x) = 0$. Then

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n}(0 \cdot i + 1 \cdot (n-i)) - 0 = \frac{n-i}{n}. \tag{2.14}$$

Thus, $\mathcal{L}f(x) = -\frac{n-i}{n}$.

Case 2: Let $\deg(x) = n$ and $f(x) = 1$. Then

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{n}(0 \cdot i + 1 \cdot (n-i)) - 1 = -\frac{i}{n}. \tag{2.15}$$

Thus, $\mathcal{L}f(x) = \frac{i}{n}$.

Case 3: Let $\deg(x) = m$ and $f(x) = 0$. Then

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{m}(0 \cdot j + 1 \cdot (m - j)) - 0 = \frac{m - j}{m}. \tag{2.16}$$

Thus, $\mathcal{L}f(x) = -\frac{m-j}{m}$.

Case 4: Let $\deg(x) = m$ and $f(x) = 1$. Then

$$\Delta f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y) - f(x) = \frac{1}{m}(0 \cdot i + 1 \cdot (m - j)) - 1 = -\frac{j}{m}. \tag{2.17}$$

Thus, $\mathcal{L}f(x) = \frac{j}{m}$.

Now we need to determine which values of i, j are allowed under the friendly labeling f on $K(m, n)$. We have a three cases to consider.

Case 1: Suppose that $m + n$ is even. Here $v_f(0) = \frac{m+n}{2}$ and $v_f(1) = \frac{m+n}{2}$. Note that, $0 \leq i \leq n$ and $\frac{n-m}{2} \leq i \leq \frac{m+n}{2}$. Hence, $\frac{n-m}{2} = \max\{0, \frac{n-m}{2}\} \leq i \leq \min\{n, \frac{m+n}{2}\} = \frac{m+n}{2}$. Observe that, $0 \leq j \leq m$ and $\frac{m-n}{2} \leq j \leq \frac{m+n}{2}$. Thus, $0 = \max\{0, \frac{m-n}{2}\} \leq j \leq \min\{m, \frac{m+n}{2}\} = m$. Also since for a friendly labeling we have a given max number of zeros overall we see $i + j = v_f(0) = \frac{m+n}{2}$.

Case 2: Suppose that $m + n$ is odd, $v_f(0) = \frac{m+n+1}{2}$ and $v_f(1) = \frac{m+n-1}{2}$. Note that, $0 \leq i \leq n$ and $\frac{n-m+1}{2} \leq i \leq \frac{m+n+1}{2}$. Hence, $\frac{n-m+1}{2} = \max\{0, \frac{n-m+1}{2}\} \leq i \leq \min\{n, \frac{m+n+1}{2}\} = \frac{m+n+1}{2}$ (as $m < n$ implies $n - m \geq 1$ and this in turn implies $n \geq m + 1$). Observe that, $0 \leq j \leq m$ and $\frac{m-n+1}{2} \leq j \leq \frac{m+n+1}{2}$. Thus, $0 = \max\{0, \frac{m-n+1}{2}\} \leq j \leq \min\{m, \frac{m+n+1}{2}\} = m$. Also since for a friendly labeling we have a given max number of zeros overall we see $i + j = v_f(0) = \frac{m+n+1}{2}$.

Case 3: Suppose that $m + n$ is odd, $v_f(0) = \frac{m+n-1}{2}$ and $v_f(1) = \frac{m+n+1}{2}$. Note that, $0 \leq i \leq n$ and $\frac{n-m-1}{2} \leq i \leq \frac{m+n-1}{2}$. Hence, $\frac{n-m-1}{2} = \max\{0, \frac{n-m-1}{2}\} \leq i \leq \min\{n, \frac{m+n-1}{2}\} = \frac{m+n+1}{2}$ (as $m < n$ implies $n - m \geq 1$ and this in turn implies $n \geq m + 1$). Observe that, $0 \leq j \leq m$ and $\frac{m-n-1}{2} \leq j \leq \frac{m+n-1}{2}$. Thus, $0 = \max\{0, \frac{m-n-1}{2}\} \leq j \leq \min\{m, \frac{m+n-1}{2}\} = m$ (as $n - m \geq 1$ implies $m - n \leq -1$ which in turn implies $m \leq n - 1$). Also since for a friendly labeling we have a given max number of zeros overall we see $i + j = v_f(0) = \frac{m+n-1}{2}$.

$\square$

It is important to note in Theorem 2.27 we get "$\Delta f(K(m,n)) \subseteq$" because depending on the labeling of the graph some of the cases in the proof do not occur. To be more explicit you could have a graph with no vertices x such that $\deg(x) = n$ and $f(x) = 0$ or you could have a graph with no vertices x such that $\deg(x) = n$ and $f(x) = 1$. For example, given $K(2,4)$ there is a friendly labeling where both vertices of degree 4 are labeled 1. This means for this graph there are no vertices x such that $\deg(x) = 4$ and $f(x) = 0$.

Next we have way to calculate the Jost Graph Laplace operator for a vertex of a given friendly labeling on a complete bipartite graph. Note this is just a way to shortening the calculation so we do not have to find $\frac{1}{\deg(x)} \sum\limits_{y \sim x} f(y) - f(x)$ for each vertex.

Corollary 2.28. *Let $m \leq n$. Let $x \in V(K(m,n))$. Let $f : K(m,n) \to \mathbb{Z}_2$ be a friendly labeling with i vertices, x_i, of degree n such that $f(x_i) = 0$ and j, x_j, vertices of degree m such that $f(x_i) = 0$. Then*

$$\Delta f(x) = \begin{cases} \frac{n-i}{n}, & \text{when } \deg(x) = n \text{ and } f(x) = 0 \\ -\frac{i}{n}, & \text{when } \deg(x) = n \text{ and } f(x) = 1 \\ \frac{m-j}{m}, & \text{when } \deg(x) = m \text{ and } f(x) = 0 \\ -\frac{j}{m}, & \text{when } \deg(x) = m \text{ and } f(x) = 1. \end{cases}$$

Proof. These cases follow directly from the proof of Theorem 2.27. □

Here we introduce a special type of complete bipartite graph called a star and then display an example of a star. We denote a star with n edges by $ST(n)$ and has $n+1$ vertices. It is the complete bipartite graph $K(1,n)$.

Definition 2.12. *A star is a graph where every exterior vertex is connected by an edge to a single common point called the center.*

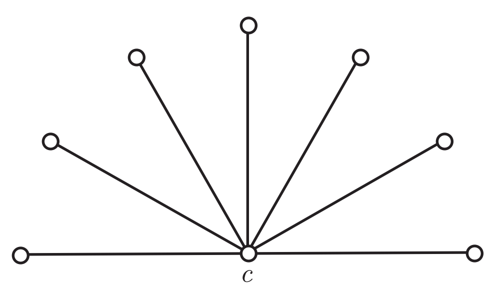

Figure 2.5: The star with 7 edges [20].

Theorem 2.29. *Let $f : ST(n) \to \mathbb{Z}_2$ be a friendly labeling for star with n edges. Let $x, c \in V(ST(n))$,* *where c is the center vertex. Then*

$$\Delta f(x) = \begin{cases} 0, & \textit{if } x \textit{ is an exterior vertex, } f(x)=0, \textit{ and } f(c)=0 \textit{ or} \\ & \textit{if } x \textit{ is an exterior vertex, } f(x)=1, \textit{ and } f(c)=1 \\ 1, & \textit{if } x \textit{ is an exterior vertex, } f(x)=0, \textit{ and } f(c)=1 \\ -1, & \textit{if } x \textit{ is an exterior vertex, } f(x)=1, \textit{ and } f(c)=0 \\ \frac{1}{2}, & \textit{if } x=c,\ f(c)=0,\ n \textit{ is even, and } v_f(0)=v_f(1)+1 \\ \frac{n+2}{2n}, & \textit{if } x=c,\ f(c)=0,\ n \textit{ is even, and } v_f(1)=v_f(0)+1 \\ \frac{n+1}{2n}, & \textit{if } x=c,\ f(c)=0, \textit{ and } n \textit{ is odd} \\ -\frac{n+2}{2n}, & \textit{if } x=c,\ f(c)=1,\ n \textit{ is even, and } v_f(0)=v_f(1)+1 \\ -\frac{1}{2}, & \textit{if } x=c,\ f(c)=1,\ n \textit{ is even, and } v_f(1)=v_f(0)+1 \\ -\frac{n+1}{2n}, & \textit{if } x=c,\ f(c)=1, \textit{ and } n \textit{ is odd.} \end{cases}$$

Proof. Let $f : ST(n) \to \mathbb{Z}_2$ be a friendly labeling for star with n edges. Let $x, c \in V(ST(n))$, where c is the center vertex. We break this proof into to cases. The first case is when $f(c) = 0$ ad the second is when $f(c) = 1$.

Case 1: Let $f(c) = 0$.

Subcase 1: Let x be an exterior vertex such that $f(x) = 0$. Note that for this calculation it does not matter if n is odd or even. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(0) - 0 = 0.$$

Subcase 2: Let x be an exterior vertex such that $f(x) = 1$. Note that for this calculation it does not matter if n is odd or even. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(0) - 1 = -1.$$

Subcase 3: Let $f(c) = 0$, n be even, and $v_f(0) = v_f(1) + 1$. Since n is even and $v_f(0) = v_f(1) + 1$, then $v_f(0) = \frac{n}{2} + 1$ and $v_f(1) = \frac{n}{2}$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c} \sum_{y \sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right)\right) - 0 = \frac{1}{2}.$$

Subcase 4: Let $f(c) = 0$, n be even, and $v_f(1) = v_f(0) + 1$. Since n is even and $v_f(1) = v_f(0) + 1$, then $v_f(0) = \frac{n}{2}$ and $v_f(1) = \frac{n}{2} + 1$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c} \sum_{y \sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n}{2} - 1\right) + 1\left(\frac{n}{2} + 1\right)\right) - 0 = \frac{n+2}{2n}.$$

Subcase 5: Let $f(c) = 0$ and n be odd. Since n is odd, then $v_f(0) = v_f(1) = \frac{n+1}{2}$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c}\sum_{y\sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n+1}{2} - 1\right) + 1\left(\frac{n+1}{2}\right)\right) - 0 = \frac{n+1}{2n}.$$

Case 2: Let $f(c) = 1$.

Subcase 1: Let x be an exterior vertex such that $f(x) = 0$. Note that for this calculation it does not matter if n is odd or even. Then

$$\Delta f(x) = \frac{1}{\deg x}\sum_{y\sim x} f(y) - f(x) = \frac{1}{1}(1) - 0 = 1.$$

Subcase 2: Let x be an exterior vertex such that $f(x) = 1$. Note that for this calculation it does not matter if n is odd or even. Then

$$\Delta f(x) = \frac{1}{\deg x}\sum_{y\sim x} f(y) - f(x) = \frac{1}{1}(1) - 1 = 0.$$

Subcase 3: Let $f(c) = 1$, n be even, and $v_f(0) = v_f(1) + 1$. Since n is even and $v_f(0) = v_f(1) + 1$, then $v_f(0) = \frac{n}{2} + 1$ and $v_f(1) = \frac{n}{2}$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c}\sum_{y\sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n}{2} + 1\right) + 1\left(\frac{n}{2} - 1\right)\right) - 1 = -\frac{n+2}{2n}.$$

Subcase 4: Let $f(c) = 1$, n be even, and $v_f(1) = v_f(0) + 1$. Since n is even and $v_f(1) = v_f(0) + 1$, then $v_f(0) = \frac{n}{2}$ and $v_f(1) = \frac{n}{2} + 1$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c}\sum_{y\sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n}{2}\right) + 1\left(\frac{n}{2}\right)\right) - 1 = -\frac{1}{2}.$$

Subcase 5: Let $f(c) = 1$ and n be odd. Since n is odd, then $v_f(0) = v_f(1) = \frac{n+1}{2}$ because f is a friendly labeling. Then

$$\Delta f(c) = \frac{1}{\deg c}\sum_{y\sim c} f(y) - f(c) = \frac{1}{n}\left(0\left(\frac{n+1}{2}\right) + 1\left(\frac{n+1}{2} - 1\right)\right) - 1 = -\frac{n+1}{2n}.$$

□

Corollary 2.30. *Let $f : ST(n) \to \mathbb{Z}_2$ be a friendly labeling for star with n edges. Let $x, c \in V(ST(n))$,* *where c is the center vertex. Then*

$$\mathcal{L}f(x) = \begin{cases} 0, & \text{if } x \text{ is an exterior vertex, } f(x)=0, \text{ and } f(c)=0 \text{ or} \\ & \text{if } x \text{ is an exterior vertex, } f(x)=1, \text{ and } f(c)=1 \\ -1, & \text{if } x \text{ is an exterior vertex, } f(x)=0, \text{ and } f(c)=1 \\ 1, & \text{if } x \text{ is an exterior vertex, } f(x)=1, \text{ and } f(c)=0 \\ -\frac{1}{2}, & \text{if } x=c,\ f(c)=0,\ n \text{ is even, and } v_f(0)=v_f(1)+1 \\ -\frac{n+2}{2n}, & \text{if } x=c,\ f(c)=0,\ n \text{ is even, and } v_f(1)=v_f(0)+1 \\ -\frac{n+1}{2n}, & \text{if } x=c,\ f(c)=0, \text{ and } n \text{ is odd} \\ \frac{n+2}{2n}, & \text{if } x=c,\ f(c)=0,\ n \text{ is even, and } v_f(0)=v_f(1)+1 \\ \frac{1}{2}, & \text{if } x=c,\ f(c)=1,\ n \text{ is even, and } v_f(1)=v_f(0)+1 \\ \frac{n+1}{2n}, & \text{if } x=c,\ f(c)=1, \text{ and } n \text{ is odd.} \end{cases}$$

Proof. Since $\mathcal{L}f(x) = -\Delta f(x)$ we get the desired result. □

Theorem 2.31. *Let $f : ST(n) \to \mathbb{Z}_2$ be a friendly labeling for star with n edges. Let i be the number of zeros of degree m as in the proof of Theorem 2.27. Then $i = 0$ if and only if $n = 2$, $f(c) = 0$ and both the exterior vertices are all labeled by a 1 i.e. it is the graph $ST(2) = K(1, 2)$.*

Proof. $\Longleftarrow$ Consider the graph $ST(2) = K(1, 2)$ with the center labeled by a zero and both the exterior vertices are all labeled by a 1. In this case observe that $i = 0$.

$\Longrightarrow$ Conversely let $i = 0$, $v_f(1) = n$ and $v_f(0) = n + 1 - n = 1$. Then $|v_f(1) - v_f(0)| = n - 1$. Since $f : ST(n) \to \mathbb{Z}_2$ is a friendly labeling, $n - 1 \leq 1 \Longrightarrow n \leq 2$. Note if $n = 1$ we have the path $P_1 = K_2$ and for f to be friendly means $i = 1 \neq 0$. So $n = 2$. □

Reall for a caterpillar of diameter 4, $CR(a, b, c)$ we call the vertices on the spine u, v, and w. See figure 2.6 for an example. We calculate the number of vertices by $|V| = a + b + c - 1$ and the number of edges by $|E| = a + b + c - 2$.

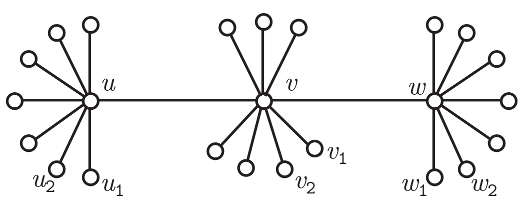

Figure 2.6: A caterpillar of diameter 4, $CR(8, 9, 8)$[52][20].

We now compute the Jost graph Laplacian for all the vertices of a caterpillar of diameter 4, $CR(a, b, c)$ with a specific labeling applied to the vertices on the spine.

Theorem 2.32. *Let $f : CR(a,b,c) \to \mathbb{Z}_2$ be a friendly labeling for a caterpillar of diameter 4. Let $x, u, v, w \in V(CR(a,b,c))$ where u is the vertex of degree a, v is the vertex of degree b, and w is the vertex of degree c. Let $f(u) = 1$, $f(v) = 0$, and $f(w) = 1$. Let i be the number of vertices adjacent to u such that $f(x) = 0$, j be the number of vertices adjacent to v such that $f(x) = 0$, and k be the number of vertices adjacent to w such that $f(x) = 0$. Then*

$$\Delta f(x) = \begin{cases} 0, & \text{if } x \text{ is an exterior vertex of } u \text{ or } w \text{ and } f(x) = 1 \text{ or} \\ & \text{if } x \text{ is an exterior vertex of } v \text{ and } f(x) = 0 \\ 1, & \text{if } x \text{ is an exterior vertex of } u \text{ or } w \text{ and } f(x) = 0 \\ -1, & \text{if } x \text{ is an exterior vertex of } v \text{ and } f(x) = 1 \\ \frac{a-i}{a} - 1, & \text{if } x = u \\ \frac{b-j}{b}, & \text{if } x = v \\ \frac{c-k}{c} - 1, & \text{if } x = w. \end{cases}$$

Proof. Let $f : CR(a,b,c) \to \mathbb{Z}_2$ be a friendly labeling for a caterpillar of diameter 4. Let $x, u, v, w \in V(CR(a,b,c))$ where u is the vertex of degree a, v is the vertex of degree b, and w is the vertex of degree c. Let $f(u) = 1$, $f(v) = 0$, and $f(w) = 1$. Let i be the number of vertices adjacent to u such that $f(x) = 0$, j be the number of vertices adjacent to v such that $f(x) = 0$, and k be the number of vertices adjacent to w such that $f(x) = 0$.

Case 1: Let x be an exterior vertex of u or w such that $f(x) = 0$. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(1) - 0 = 1.$$

Case 2: Let x be an exterior vertex of u or w such that $f(x) = 1$. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(1) - 1 = 0.$$

Case 3: Let x be an exterior vertex of v such that $f(x) = 0$. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(0) - 0 = 0.$$

Case 4: Let x be an exterior vertex of v such that $f(x) = 1$. Then

$$\Delta f(x) = \frac{1}{\deg x} \sum_{y \sim x} f(y) - f(x) = \frac{1}{1}(0) - 1 = -1.$$

Case 5: Let x be the vertex u. Then

$$\Delta f(u) = \frac{1}{\deg u} \sum_{y \sim u} f(y) - f(u) = \frac{1}{a}(i \cdot 0 + (a-i)1) - 1 = \frac{a-i}{a} - 1.$$

Case 6: Let x be the vertex v. Then

$$\Delta f(v) = \frac{1}{\deg v} \sum_{y \sim v} f(y) - f(v) = \frac{1}{b}(j \cdot 0 + (b-j)1) - 0 = \frac{b-j}{b}.$$

Case 7: Let x be the vertex w. Then

$$\Delta f(w) = \frac{1}{\deg w} \sum_{y \sim w} f(y) - f(w) = \frac{1}{c}(k \cdot 0 + (c-k)1) - 1 = \frac{c-k}{c} - 1.$$

□

Corollary 2.33. *Let $f : CR(a,b,c) \to \mathbb{Z}_2$ be a friendly labeling for a caterpillar of diameter 4. Let $x, u, v, w \in V(CR(a,b,c))$ where u is the vertex of degree a, v is the vertex of degree b, and w is the vertex of degree c. Let $f(u) = 1$, $f(v) = 0$, and $f(w) = 1$. Let i be the number of vertices adjacent to u such that $f(x) = 0$, j be the number of vertices adjacent to v such that $f(x) = 0$, and k be the number of vertices adjacent to w such that $f(x) = 0$. Then*

$$\mathcal{L}f(x) = \begin{cases} 0, & \text{if } x \text{ is an exterior vertex of } u \text{ or } w \text{ and } f(x) = 1 \text{ or} \\ & \text{if } x \text{ is an exterior vertex of } v \text{ and } f(x) = 0 \\ -1, & \text{if } x \text{ is an exterior vertex of } u \text{ or } w \text{ and } f(x) = 0 \\ 1, & \text{if } x \text{ is an exterior vertex of } v \text{ and } f(x) = 1 \\ -\left(\frac{a-i}{a} - 1\right), & \text{if } x = u \\ -\left(\frac{b-j}{b}\right), & \text{if } x = v \\ -\left(\frac{c-k}{c} - 1\right), & \text{if } x = w. \end{cases}$$

Proof. Since $\mathcal{L}f(x) = -\Delta f(x)$ we get the desired result. □

In Theorem 2.32 we only calculated the graph Laplacian for a friendly labeling when spine had the labeling $f(u) = f(w) = 1$ and $f(v) = 0$. Note that there are 7 more possible spine labelings for a caterpillar of diameter 4. Theses cases are

(a) $f(u) = f(v) = f(w) = 1$,

(b) $f(u) = f(v) = f(w) = 0$,

(c) $f(u) = f(v) = 0$ and $f(w) = 1$,

(d) $f(u) = f(w) = 0$ and $f(v) = 1$,

(e) $f(v) = f(w) = 0$ and $f(w) = 1$,

(f) $f(u) = f(v) = 1$ and $f(w) = 0$, and

(g) $f(v) = f(w) = 1$ and $f(u) = 0$.

We can preform similar calculations for each case. We leave these calculations for later work outside of this dissertation.

2.5

Weighted Laplacian-Like Operator Applied to Friendly Labelings on Various Finite Graphs

Now we consider a graph operator on friendly labelings of finite graphs that is graph Laplacian like. The following original result I proved below supposed the weighted edge labeling arises from adding the vertex labelings in $\mathbb{Z}_2$, and allowing for $\mu_{xy} = 0$ for some edges. Of course in such problems one loses the classical notion of the graph Laplacian identifying specific edges between given vertices. It should be noted that the only type of finite graph that I have encountered where this operator is well defined at every vertex is the complete graph K_n. Indeed it is easy for one to generate examples of finite graphs G (complete bipartite graphs, caterpillars of diameter 3, and caterpillars diameter 4) where $\mu(x) = 0$ at some of the vertices $x \in V(G)$. Let us now give a more formal definition for this operator.

Definition 2.13. *(Graph Weight Allowing For $\mu_{xy} = 0$ For Some Edges) Here we consider a couple (Γ, μ) where $\Gamma = (V, E)$ is a graph with vertex set V and edge set E and μ_{xy} is a non-negative function on $V \times V$ such that*

1. $\mu_{xy} = \mu_{yx}$,
2. $\mu_{xy} \geq 0$,
3. $\mu(x) = \sum_{y \sim x} \mu_{xy}$,
4. $\mu(x) \neq 0$ *(i.e. there exists adjacent vertices u, v in the graph such that $u_{uv} \neq 0$)*

where, μ is called the weight (or measure) of the graph Γ.

Definition 2.14. *(Weighted Laplacian-Like Operator) Let $G = (V, E)$ be a connected, locally finite graph. Suppose that G has a generalized weight μ. Let A be an Abelian group. For any function $f : V \to A$, define the weighted Laplace-Like (LL) operator, $\Delta_{LL:\mu} f$, by*

$$\Delta_{LL:\mu} f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x). \tag{2.18}$$

The normalized graph Laplace-Like (LL) operator is $\mathcal{L}_{LL:\mu} = -\Delta_{LL:\mu} f$.

It should be noted that one loses the classical notion of the graph Laplacian identifying specific edges between given vertices when one chooses to study weighted Laplacian-Like Operators.

Now we compute the weighted graph Laplace-Like (LL) of a friendly labeling of K_n where the weighted measure is determined by an induced edge labeling is achieved by addition of the vertex labelings of a given edge in $\mathbb{Z}_2$ and we allow for $\mu_{xy} = 0$ for some edges.

Theorem 2.34. *Let* $f : K_n \to \mathbb{Z}_2$ *be a friendly labeling. Then* $\Delta_{LL:\mu} f(K_n) = \{-1, 1\}$. *Consequently,* $\mathcal{L}_{LL:\mu} f(K_n) = \{-1, 1\}$.

Proof. Suppose that $f : K_n \to \mathbb{Z}_2$ be a friendly labeling. Let $x \in V(K_n)$. We have a couple of cases to consider.

Case A: Suppose that n is even. Here $v_f(0) = \frac{n}{2}$ and $v_f(1) = \frac{n}{2}$.

Subcase A1: Let $f(x) = 0$.

Note that x connects to $\frac{n}{2} - 1$ other "zero" vertices and $\frac{n}{2}$ "one" vertices yielding $\frac{n}{2} - 1$ induced labeled "zero" edges and $\frac{n}{2}$ induced labeled "one" edges. Then $\mu(x) = \sum_{y \sim x} \mu_{xy} = \frac{n}{2}$. Hence,

$$\Delta_{LL:\mu} f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x) = \frac{2}{n}\frac{n}{2} - 0 = 1. \tag{2.19}$$

Thus, $\mathcal{L}_{LL:\mu} f(x) = -1$.

Subcase A2: Let $f(x) = 1$.

Note that x connects to $\frac{n}{2} - 1$ other "one" vertices and $\frac{n}{2}$ "zero" vertices yielding $\frac{n}{2} - 1$ induced labeled "zero" edges and $\frac{n}{2}$ induced labeled "one" edges. Then $\mu(x) = \sum_{y \sim x} \mu_{xy} = \frac{n}{2}$. Hence,

$$\Delta_{LL:\mu} f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x) = \frac{2}{n}(0) - 1 = -1. \tag{2.20}$$

Thus, $\mathcal{L}_{LL:\mu} f(x) = 1$.

Case B: Suppose that n is odd.

Subcase B1: Let $f(x) = 0$. Without loss of generality, $v_f(0) = \frac{n}{2} + 1 = \frac{n+2}{2}$ and $v_f(1) = \frac{n}{2}$.

Note that x connects to $\frac{n}{2}$ other "zero" vertices and $\frac{n}{2}$ "one" vertices yielding $\frac{n}{2}$ induced labeled "zero" edges and $\frac{n}{2}$ induced labeled "one" edges. Then $\mu(x) = \sum_{y \sim x} \mu_{xy} = \frac{n}{2}$. Hence,

$$\Delta_{LL:\mu} f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x) = \frac{2}{n}\frac{n}{2} - 0 = 1. \tag{2.21}$$

Thus, $\mathcal{L}_{LL:\mu} f(x) = -1$.

Subcase B2: Let $f(x) = 1$.

Note that x connects to $\frac{n}{2} - 1 = \frac{n-2}{2}$ other "one" vertices and $\frac{n+2}{2}$ "zero" vertices yielding $\frac{n-2}{2}$ induced labeled "zero" edges and $\frac{n+2}{2}$ induced labeled "one" edges. Then $\mu(x) = \sum_{y \sim x} \mu_{xy} = \frac{n+2}{2}$. Hence,

$$\Delta_{LL:\mu} f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} f(y)\mu_{xy} - f(x) = \frac{2}{n+2}(0) - 1 = -1. \tag{2.22}$$

Thus, $\mathcal{L}_{LL:\mu} f(x) = 1$.

□

2.6

Future Research Objectives Dealing with Graph Laplacians of Friendly Labelings

To my knowledge it remains an open problem to determine what are if and only if conditions for a tree to be fully cordial. We now state a known theorem from the literature that is useful in the study of fully cordial trees.

Theorem 2.35. [45] *A graph G has q as a friendly index if and only if G is isomorphic to a spanning subgraph of $K(m, m)$ or $K(m, m+1)$.*

Perhaps Theorem 2.27 and Corollary 2.28 can be used to study the possible Jost graph Laplacians of friendly labelings of fully cordial trees.

Another tool that may be of aid in the search for this fully cordial tree condition may be the weighted laplacian-like operator considered in section 2.5 of this chapter. The reason we were studying this operator is that it would allow one to have a "Laplaicain-like" operator that would allow for $\mu(x)$ created by induced edge labelings caused by adding vertex labelings modulo 2 when studying friendly index sets for some graphs.

In the course of searching for fully cordial caterpillars of diameter 4 in [52], finding an if and only if condition for a fully cordial caterpillar of diameter 4 to achieve its highest theoretical index q was needed. We state both of these lemmas now.

Lemma 2.36. [52] *Let a+b+c be odd. Then the caterpillar G=CR(a,b,c) has the maximum possible friendly index if and only if* $|b-a-c+1| \leq 1$.

Lemma 2.37. [52] *Let* $a+b+c$ *be even. Then the caterpillar G=CR(a,b,c) has the maximum possible friendly index if and only if* $|b-a-c+1| = 1$.

These two lemmas were proven in [52] by forming a friendly labeling which produced the maximum possible index q in these two cases. These lemmas and their proofs in [52] were presented at the 28th Midwest Conference on Combinatorics and Combinatorial Computing at University of Nevada, Las Vegas on October 22-24, 2014 in the talk "Fully Cordial Trees." It is joint work between Dr. Sahlehi and I. It should be noted that these two lemmas can now be more efficiently achieved by using [45] and Theorem 2.35.

Recall earlier we mentioned Theorems 2.20 and 2.24 from [52]. We restate these two theorems next for the convenience of the reader.

Theorem 2.38. [52] *Let* $a+b+c$ *be odd. The caterpillar* $G = CR(a,b,c)$ *is fully cordial if and only if* $b = a+c-1$ *and* $a = 2$ *or* $c = 2$.

Theorem 2.39. [52] *Let* $a+b+c$ *be even. The caterpillar* $G = CR(a,b,c)$ *is fully cordial if and only if* $|b-a-c+1| = 1$.

The interesting thing in the situation of the caterpillar $G = CR(a,b,c)$ of diameter 4 is that there are way more instances of being fully cordial in the $a+b+c$ even case than in the $a+b+c$ odd case. What happens in the $a+b+c$ even case is that when we get the highest possible friendly index q we also get the second highest possible friendly index $q-2$. On the other hand, in the $a+b+c$ odd case, there are lots of situations where you get the highest possible friendly index q but fail to get the second highest possible friendly index $q-2$. This makes one wonder what caused the second highest possible friendly index $q-2$ to be missed in the $a+b+c$, $|b-a-c+1| \leq 1$ case. Or in other words, what is so special in the $a+b+c$ even,$|b-a-c+1| = 1$ case that allowed us to get the second highest possible friendly index $q-2$. A partial answer to this question is that in the $a+b+c$ even,$|b-a-c+1| = 1$ case the caterpillar $G = CR(a,b,c)$ has an odd number of vertices. The next theorem, which we believe to be original, takes a first step at this partial answer.

Theorem 2.40. *Suppose that* $G(V,E)$ *is a connected finite graph having at least one leaf that has terminal vertex* w*. Let* $|V(G)| = n$ *be odd and* $|E(G)| = q \geq 2$*. If* $q \in FI(G)$*, then* $q-2 \in FI(G)$.

Proof. Suppose that $G(V, E)$ is a connected finite graph having at least one leaf that has terminal vertex w. Let $|V(G)| = n$ be odd and $|E(G)| = q \geq 2$. Suppose further that $q \in FI(G)$.

Since n is odd then $n = 2k + 1$ for some positive integer k.

Note that $q \in FI(G)$ and Corollary 2.10 implies that there exists a friendly labeling $f : V(G) \to \mathbb{Z}_2$ such that $e_f(1) = q$ and $e_f(0) = 0$. Let $v_f(i) = \max\{v_f(0), v_f(1)\}$. Either $f(w) = i$ or $f(w) \neq i$. If $f(w) \neq i$, then relabel $V(G)$ by the inverse friendly labeling $g = 1 - f$ to get $g(w) = i$. So without loss of generality we will take $f(w) = i$.

Also without loss of generality let $v_f(i) = v_f(0)$ (i.e. $v_f(0) \geq v_f(1)$). So the $v_f(0) = k+1$ and $v_f(1) = k$.

Now consider the relabeling of $V(G)$ by $h : V(G) \to \mathbb{Z}_2$ defined by

$$h(v) = \begin{cases} f(v), & \text{if } v \neq w \\ 1 - f(w) = 1 - i = 1 - 0 = 1, & \text{if } v = w. \end{cases} \tag{2.23}$$

Note that $v_h(0) = k$ and $v_h(1) = k + 1$. Thus, $|v_h(1) - v_h(0)| = k + 1 - k = 1 \leq 1$. Hence, $h : V(G) \to \mathbb{Z}_2$ is friendly. Observe that, $e_h(0) = 1$ and $e_h(1) = q - 1$. Consequently, $|e_h(1) - e_h(1)| = q - 1 - 1 = q - 2$. Therefore, $q - 2 \in FI(G)$. □

Note that Theorem 2.40 gives a partial answer or reason why the caterpillar $G = CR(a, b, c)$ in the $a+b+c$ even,$|b-a-c+1| = 1$ case can have the second highest possible friendly index $q-2$. What matters is this leaf with terminal vertex w. It is also worth to note that the fully cordial caterpillars in Theorem 2.20 are merely formed by adding a leaf to the fully cordial caterpillars of diameter 3, $DS(a, b)$, in Corollary 4.2 of [52] by adding the terminal vertex w and one edge. Jost and Banerjee refer to the terminal vertex w as a pending vertex [5]. In [5] Jost and Banerjee use pending vertices in processes such as Motif doubling, graph splitting, and joining. In fact they state in [5] that they start by studying Lemma 1.1 from their paper. We paraphrase this Lemma concerning the eigenfunctions for the eigenvalue 1 of the Jost graph laplacian operator below.

Lemma 2.41. [5] *Let w be a pending vertex of G. Then any eigenfunction f_1 for the eigenvalue 1 vanishes at the unique neighbor of w.*

The importance of the pending vertex of Jost and Banerjee from [5] in our study of fully cordial Caterpillars of diameter 4 lead me to believe that processes such as Motif doubling, graph splitting, and joining may be of use in finding an if and only if condition for fully cordial trees. In our case, this pending vertex actually showed in some fashion when $G = CR(a, b, c)$ is fully cordial. Perhaps in future research we can use eigenfunctions for the eigenvalue 1 of the Jost graph laplacian operator and these techniques to further study this problem. More support for this line of thinking is the paper [14], where Chen and Jost study the minimum vertex cover for these pending vertices. We state a result from [14] concerning these pending vertices below.

Theorem 2.42. [14] *For a tree, let L be the set of its leaves (vertices of degree 1). Then L is not a subset of any minimum vertex cover.*

Chen and Jost also list numerous theorems concerning minimum vertex covers and matchings from graph theory that they use in the study of such pending vertices. In section 5 of [14], Chen and Jost study the minimum vertex cover and the eigenvectors associated with the eigenvalue 1 of the Jost graph laplacian operator.

Note that in [54] Salehi and Lee demonstrated that trees with perfect matching are fully cordial. Recall we discussed this in Theorem 2.5 and Corollary 2.6. In [52] Salehi and Corral demonstrated that trees with near perfect matching are fully cordial. We discussed this in Theorem 2.12. The proofs of these come down to having a pendent P_3 path with specific conditions. These pendants have a pending vertex.

This concludes our discussion of fully cordial trees. It is still a research objective to find an application of friendly labelings and fully cordial trees to social media problems. At this point, I am uncertain how I would proceed in such a study.

CHAPTER 3

VARIATIONAL METHODS ON FINITE GRAPHS

3.1

Introduction

In this chapter we study variational methods on finite graphs.

Section 3.2 provide preliminary definitions and concepts.

Section 3.3 provides an original minimization result. The main result of this section is Theorem 3.5 which shows there exists a weak solution to a particular nonlinear Dirichlet problem. Lemma 3.4 is a minor original technical lemma.

Section 3.4 is original joint work between Dr. David Costa and me. It was presented at the 1141st AMS Meeting at the University of Delaware on September 29-30, 2018 as the talk "Remarks on Lagrange Multiplier Approach to Kadzan-Warner Equations on a Finite Graph". The main result of this section is Theorem 3.12 which gives an if an only if condition for the Kazdan-Warner equation, $\Delta_\mu u = h(x) - k(x)e^u$, to have a weak solution for a non-constant function h on a finite graph. Lemmas 3.9, 3.10, and 3.11 are original work used to prove the main result. Lemma 3.11 in particular applies the Lagrange multipliers to the corresponding energy functional.

Section 3.5 is original joint work between Dr. David Costa and me. This section is an original application of the Saddle Point theorem of Rabinowitz to the analogue of Ahmad, Lazer, and Paul resonant problem on a finite graph. The main result is Theorem 3.18 which shows that the analogue of Ahmad, Lazer, and Paul resonant problem on a finite graph has weak solution and thus a strong solution. Theorems 3.16 and 3.17 are minor original results used to prove the main result.

3.2

Definitions

The classical approach of calculus of variations was to find critical points of Φ by solving some partial differential equation $\mathcal{D}(x) = 0$ (i.e. arriving at the Euler-Lagrange Equations and solving them). Now the modern approach is to find critical points of some functional Φ and classify them in order to solve some partial differential equation $\mathcal{D}(x) = 0$. These modern variational techniques consider an associated real-valued function $\Phi : X \to \mathbb{R}$, whose derivative is equal to $\mathcal{D}(x)$, and by looking for points of minimum, maximum, or minimax of Φ. In these situations, $\Phi'(x) = 0$.

We begin with a few basic definitions. Recall from earlier, definition 1.10 of the weighted Laplace operator from Alexander Grigoryan's text Analysis on Graphs [27].

Next we state some definitions by Grigoryan, Lin, and Yang in their papers "Yamabe type equations on finite graphs" [30] and "Kazdan-Warner equations on graphs" [29].

Definition 3.1. [29] [30] *(Corresponding Gradient Form For Weighted Laplace Operator) Let $G = (V, E)$ be a connected finite graph. Suppose that G has a weight μ. For any functions $f, g : V \to \mathbb{R}$, define the corresponding gradient form for the weighted Laplace operator by*

$$\nabla f \cdot \nabla g = \Gamma_\mu(f, g)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} (f(y) - f(x))(g(y) - g(x))\mu_{xy}. \tag{3.1}$$

In the literature: $\Gamma_\mu(f) = \Gamma_\mu(f, f)$.

Definition 3.2. [29] [30] *(Corresponding Length Of Gradient) Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ. For any function $f : V \to \mathbb{R}$, define the corresponding norm for*

$$|\nabla f|(x) = \sqrt{\Gamma(f)(x)} = \sqrt{\frac{1}{2\mu(x)} \sum_{y \sim x} (f(y) - f(x))^2 \mu_{xy}}. \tag{3.2}$$

We need to define the relevant Rayleigh Quotient and first eigenvalue to be involved with our variational problems we wish to study. Again consider the graph $G = (V, E)$ which is finite or locally finite. Let Ω be a finite subgraph of V. Recall that for any function $f : V \to \mathbb{R}$,

$$\int_\Omega f d\mu = \sum_{x \in \Omega} f(x)\mu(x). \tag{3.3}$$

The first eigenvalue of the Laplacian with respect to the Dirichlet boundary condition reads

$$\lambda_1(\Omega) = \inf_{u \neq 0, u|_{\partial\Omega} = 0} \frac{\int_\Omega |\nabla f|^2 d\mu}{\int_\Omega f^2} = \frac{1}{2} \frac{\sum_{x,y \in \Omega} (\nabla_{xy} f)(\nabla_{xy} f)\mu_{xy}}{\sum_{x \in \Omega} f^2(x)\mu(x)}. \tag{3.4}$$

Definition 3.3. [29] [30] *(Norm For Sobolev Space) Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ, $\Omega \subseteq V$ such that $\Omega \neq \emptyset$, $\partial\Omega$ is the boundary of Ω and Ω^o is the interior of Ω. For any function $f : V \to \mathbb{R}$, the norm for $W^{1,2}(\Omega) = H^1(\Omega)$ is*

$$\|f\|_{W^{1,2}(\Omega)} = \sqrt{\int_\Omega |f|^2 d\mu + \int_\Omega |\nabla f|^2 d\mu} = \sqrt{\sum_\Omega |f|^2 \mu(x) + \sum_\Omega |\nabla f|^2 \mu(x)}. \tag{3.5}$$

Recall $\partial\Omega$ and Ω^o are defined in Definition 1.16. The space $W^{1,2}(\Omega) = H^1(\Omega)$ is defined to be the functions f such that $\|f\|_{W^{1,2}(\Omega)} < \infty$.

Following the classical tradition of the field of PDE's one defines $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Definition 3.4. [29] [30] *($H_0^1(\Omega) = W_0^{1,2}(\Omega)$) Let $G = (V, E)$ be a locally finite connected graph. Suppose that G has a weight μ, $\Omega \subseteq V$ such that $\Omega \neq \emptyset$, $\partial\Omega$ is the boundary of Ω and Ω^o is the interior of Ω. $C_0^1(\Omega)$ is defined to be the set of all functions $f : \Omega \to \mathbb{R}$ with $f = |\nabla f| = 0$ on $\partial\Omega$. Now $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ is defined to be the completion of $C_0^1(\Omega)$ by the norm in the previous definition.*

3.3

A Minimization Problem

Before we may proceed in our discussion of minimization type critical point theory for graphs we must state a couple of classical results. The first is an abstract minimization property for compact topological spaces.

Theorem 3.1. [21] *Suppose X is a compact topological space. Let $\Phi : X \to \mathbb{R}$ be a lower-semicontinuous functional. Then Φ is bounded from below and there exists $a \in X$ such that $\Phi(a) = \inf_X \Phi(x)$.*

Proof. (Classical proof from literature) The lower semi-continuity of Φ implies that $\Phi^{-1}(-n, \infty)$ is open. Now note that compactness of X and $X = \cup_{n=1}^{\infty} \Phi^{-1}(-n, \infty)$ yields that $X = \cup_{n=1}^{m} \Phi^{-1}(-n, \infty)$ for some $m \in \mathbb{N}$. Consequently, $\Phi(x) > -m$ for every $x \in X$, this means that Φ is bounded from below.

Let $b = \inf_X \Phi(x)$. Suppose to the contrary that $\Phi(x) > b$ for every $x \in X$. Observe that, $X = \cup_{n=1}^{\infty} \Phi^{-1}(b + \frac{1}{n}, \infty)$. Since X is compact, there is a $t \in \mathbb{N}$ such that $\Phi(x) > b + \frac{1}{t}$ for every $x \in X$. Thus, $b = \inf_X \Phi(x) \geq b + \frac{1}{t}$, which implies the contradiction $0 \geq \frac{1}{t}$. Therefore, there exists $a \in X$ such that $\Phi(a) = \inf_X \Phi(x)$. $\square$

Another well-known theorem is as follows.

Theorem 3.2. [21] *Suppose X is a reflexive Banach space. Let $\Phi : X \to \mathbb{R}$ be*

1. *weakly lower-semi-continuous,*
2. *coercive.*

Then Φ is bounded from below and there exists $a \in X$ such that $\Phi(a) = \inf_X \Phi(x)$.

Proof. (Classical proof from literature) Since Φ is coercive, then one may pick a radius $R > 0$ where $\Phi(x) > \Phi(0)$ for every $x \in X$ satisfying $\|x\| \geq R$. A consequence of Alaoglu's Theorem gives us that, B, the closed ball of radius R centered at 0 is compact in the weak topology. Now since $\Phi : B \to \mathbb{R}$ is lower-semi-continuous and B is compact in the weak topology, then the previous theorem says there exists $a \in B$ such that $\Phi(a) = \inf_B \Phi(x) = \inf_X \Phi$ where last equality comes from how R was picked. □

Now we present some more recent results in the literature that will help us in our study of non-linear operators on graphs. Alexander Grigoryan, Yong Lin, and Yunyan Yang in [30] [29] "Yamabe type equations on finite graphs" gives the following version of the Sobolev embedding theorem.

Theorem 3.3. [29] [30] *Let $G = (V, E)$ be a finite graph, Ω be a nonempty subset of V such that $\Omega^o \neq \emptyset$. Let m by any positive integer and $p > 1$. Then $W_0^{m,p}$ is embedded in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$. In particular, there exists a constant C depending only on m, p, and Ω such that*

$$\left(\int_\Omega |u|^q d\mu\right)^{\frac{1}{q}} \leq C \left(\int_\Omega |\nabla^m u|^p d\mu\right)^{\frac{1}{p}}. \tag{3.6}$$

for all $1 \leq q \leq +\infty$ and for all $u \in W_0^{m,p}(\Omega)$. Moreover, $W_0^{m,p}(\Omega)$ is pre-compact, namely if u_k is bounded in $W_0^{m,p}(\Omega)$, then up to a subsequence, there exists some $u \in W_0^{m,p}(\Omega)$ such that $u_k \to u$ in $W_0^{m,p}(\Omega)$.

Applying this theorem to the Sobolev space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ tells us that H_0^1 is embedded in l^s for every $1 \leq s \leq \infty$ and provides that there exists a c_s such that $\|h\|_{l^s} \leq c_s \|h\|$.

Now we are prepared to present a lemma which will be useful in proving the first non-linear Dirichlet problem we discuss has weak solutions.

Lemma 3.4. *Let $G = (V, E)$ be a finite graph with weight μ. Suppose that $\Omega \subseteq V$. Let $f : V \times \mathbb{R} \to \mathbb{R}$ be continuous in the second variable and $\frac{1}{r} + \frac{1}{s} = 1$. Suppose that $F(x, y) = \int_0^y f(x, z)dz$. Then the functional*

$$\begin{aligned} \Phi(u) &= \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - F(x, u)\right) dx \\ &= \sum_{x \in \Omega} \left(\frac{1}{4}\frac{1}{\mu(x)} \sum_{y \sim x} (u(y) - u(x))^2 \mu_{xy} - F(x, u)\right) \mu(x), u \in H_0^1(\Omega) \end{aligned} \tag{3.7}$$

is well defined. Furthermore, $\Phi \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$ *and*

$$\begin{aligned}
\Phi'(u) \cdot h &= \int_\Omega (\nabla u \cdot \nabla h - f(x,u)h)dx \\
&= \sum_{x \in \Omega} \left(\frac{1}{2\mu(x)} (u(y) - u(x))(h(y) - h(x)) - f(x,u)h \right) \mu(x), \forall u, h \in H_0^1(\Omega).
\end{aligned} \tag{3.8}$$

Proof. Note that $\Phi(u) = q(u) - \Psi(u)$ where,

$$q(u) = \frac{1}{2}\|u\|^2$$

and

$$\Psi(u) = \sum_{x \in \Omega} F(x,u)\mu(x) = \sum_{x \in \Omega} \left(\int_0^{u(x)} f(x,z)dz \right) \mu(x).$$

Note that,

$$\begin{aligned}
\lim_{\|h\| \to 0} \frac{|q(u+h) - q(u) - \langle u, h \rangle|}{\|h\|} &= \lim_{\|h\| \to 0} \frac{|q(u+h) - q(u) - \langle u, h \rangle|}{\|h\|} \\
&= \lim_{\|h\| \to 0} \frac{|\frac{1}{2}\|u\|^2 + \langle u, h \rangle + \frac{1}{2}\|h\|^2 - \frac{1}{2}\|u\|^2 - \langle u, h \rangle|}{\|h\|} \\
&= \frac{1}{2} \lim_{\|h\| \to 0} \frac{|\|h\|^2}{\|h\|} \\
&= \frac{1}{2} \lim_{\|h\| \to 0} \|h\| = 0.
\end{aligned} \tag{3.9}$$

Hence, $q \in C^\infty(H_0^1(\Omega), \mathbb{R})$. Thus, $q \in C^1(H_0^1(\Omega), \mathbb{R})$. Observe that,

$$\begin{aligned}
\delta(h) &= \Psi(u+h) - \Psi(u) - \sum_{x \in \Omega} f(x,u)h\mu(x) \\
&= \sum_{x \in \Omega} (F(x, u+h) - F(x,u))\mu(x) - \sum_{x \in \Omega} f(x,u)h\mu(x) \\
&= \sum_{x \in \Omega} \left(\int_0^1 \frac{d}{dt} F(x, u+th)dt \right) \mu(x) - \int_0^1 \sum_{x \in \Omega} f(x,u)h\mu(x)dt \\
&= \sum_{x \in \Omega} \left(\int_0^1 f(x, u+th)hdt \right) \mu(x) - \int_0^1 \sum_{x \in \Omega} f(x,u)h\mu(x)dt) \\
&= \int_0^1 \sum_{x \in \Omega} f(x, u+th)h\mu(x)dt - \int_0^1 \sum_{x \in \Omega} f(x,u)h\mu(x)dt \\
&= \int_0^1 \sum_{x \in \Omega} (f(x, u+th) - f(x,u))h\mu(x)dt.
\end{aligned} \tag{3.10}$$

By Holder's inequality,

$$|\delta(h)| \leq \int_0^1 \|f(x, u+th) - f(x,u)\|_{L^r} \|h\|_{l^s} dt \tag{3.11}$$

By the continuous Sobolev Embedding $H_0^1(\Omega)$ is embedded in l^s, so $u + th \to u$ in l^s as $h \to 0$ in $H_0^1(\Omega)$. From the continuity of f in the second variable, $f(\cdot, u+th) \to f(\cdot, u)$ in L^r as all norms are equivalent on a finite dimensional vector space. By Lebesgue Dominated Convergence Theorem and the Sobolev inequality, $\|h\|_{l^s} \leq c_s\|h\|$, we get

$$\frac{|\delta(h)|}{\|h\|} \leq c_s \frac{|\delta(h)|}{\|h\|_{l^s}} \leq \int_0^1 \|f(\cdot, u+th) - f(\cdot, u)\|_{L^r} dt \to 0. \tag{3.12}$$

Consequently, $\Psi : H_0^1 \to \mathbb{R}$ is Frechet differentiable at any $u \in H_0^1$ with Frechet derivative

$$\Psi'(u) \cdot h = \sum_{x\in\Omega} f(x,u)h\mu(x) \tag{3.13}$$

for every $u, h \in H_0^1(\Omega)$.

Now the continuity of $\Psi' : H_0^1 \to (H_0^1)^*$ will be demonstrated. By again appealing to Holder's inequality and Sobolev embedding ($\|h\|_{l^s} \leq c\|h\|$) and L^r continuity of f, we observe the following computation:

$$\begin{aligned}
\|\Psi'(u+v) - \Psi'(u)\|_{H^{-1}} &= \sup_{\|h\|\leq 1} |(\Psi'(u+v) - \Psi'(u)) \cdot h| \\
&= \sup_{\|h\|\leq 1} \left| \sum_{x\in V} (f(x,u+v) - f(x,u))h\mu(x) \right| \\
&\leq \sup_{\|h\|\leq 1} \sum_{x\in V} |(f(x,u+v) - f(x,u))| \, \|h\|\mu(x) \\
&\leq \sup_{\|h\|\leq 1} \sum_{x\in V} \|(f(x,u+v) - f(x,u))\|_{L^r} \|h\|_{l^s} \\
&\leq c_s \sup_{\|h\|\leq 1} \sum_{x\in V} \|(f(x,u+v) - f(x,u))\|_{L^r} \|h\| \\
&\leq c_s \sup_{\|h\|\leq 1} \sum_{x\in V} \|(f(x,u+v) - f(x,u))\|_{L^r} \to 0 \text{ as } v \to 0 \text{ in } H_0^1.
\end{aligned} \tag{3.14}$$

Therefore, $\Phi \in C^1(H_0^1(\Omega), \mathbb{R})$ with Frechet derivative

$$\begin{aligned}
\Phi'(u) \cdot h &= \int_\Omega (\nabla u \cdot \nabla h - f(x,u)h) dx \\
&= \sum_{x\in\Omega} \left(\frac{1}{2\mu(x)} (u(y) - u(x))(h(y) - h(x)) - f(x,u)h \right) \mu(x), \forall u, h \in H_0^1(\Omega).
\end{aligned}$$

□

We now study a nonlinear Dirichlet problem on a graph which will be solved using the technique of minimization.

Theorem 3.5. *Let $G = (V, E)$ be a finite graph with weight μ. Suppose that $\Omega \subseteq V$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous in the second variable and $\frac{1}{r} + \frac{1}{s} = 1$. Suppose further that there exists $\alpha < \lambda_1$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta u = \lambda u$ in Ω such that $u = 0$ on $\partial\Omega$, such that $\limsup_{|y|\to\infty} \frac{f(x,y)}{y} \leq \alpha$ uniformly* *for $x \in \Omega$. Suppose also that, there exists $c, d > 0$ and $0 \leq \sigma < \infty$ such that $|f(x,y)| \leq c|y|^\sigma + d$. Then there exists a weak solution to the following nonlinear Dirichlet problem:*

$$\begin{aligned} \mathcal{L}_\mu u(x) &= f(x) \ \forall x \in \Omega \\ u(x) &= 0 \ \forall x \in \partial\Omega. \end{aligned} \tag{3.15}$$

Proof. Note that to find solutions of the weak form of this Dirichlet problem is to look for functions $u \in H_0^1(\Omega)$ such that $\int_\Omega (\nabla u \cdot \nabla h - f(x,u)h)dx = \sum_{x\in\Omega} \left(\frac{1}{2}(u(y)-u(x))(h(y)-h(x)) - f(x,u)h\right)\mu(x) = 0 \ \forall h \in H_0^1(\Omega)$. By the previous theorem finding such a weak solution is the same as finding a critical point of the functional $\Phi \in C^1\left(H_0^1(\Omega), \mathbb{R}\right)$ given by $\Phi(u) = q(u) - \Psi(u)$ where $q(u) = \frac{1}{2}\|u\|^2$ and $\Psi(u) = \sum_{x\in\Omega} F(x,u)\mu(x)$.

First notice that q is weakly lower semi continuous (as $q \in C^\infty(H_0^1(\Omega), \mathbb{R})$ and so q is continuous) and $\Psi(u) = \sum_{x\in\Omega} F(x,u)\mu(x)$ is weakly continuous. Hence, Φ is weakly lower semi continuous.

Since there exists $\alpha < \lambda_1$ such that $\limsup_{|y|\to\infty} \frac{f(x,y)}{y} \leq \alpha$ uniformly for $x \in \Omega$, then by l'Hopital's Rule

$$\limsup_{|y|\to\infty} \frac{2F(x,y)}{y^2} = \limsup_{|y|\to\infty} \frac{f(x,y)}{y} \leq \alpha \text{ uniformly } \forall x \in \Omega. \tag{3.16}$$

This implies that there exists α_1 where $\alpha < \alpha_1 < \lambda_1(\Omega)$ and $R_1 > 0$ such that $F(x,y) \leq \frac{1}{2}\alpha_1 y^2 \ \forall x \in \Omega$ and $|y| \geq R_1$. Now, in view of the fact that, there exists $c, d > 0$ and $0 \leq \sigma < \infty$ such that $|f(x,y)| \leq c|y|^\sigma + d$ then, there is a γ such that $F(x,y) \leq \gamma \ \forall x \in \Omega$ and $|y| \leq R_1$. Thus $F(x,y) \leq \frac{1}{2}\alpha_1 y^2 + \gamma \ \forall x \in \Omega$, $\forall y \in \mathbb{R}$. Hence,

$$\Phi(u) \geq \frac{1}{2}\sum_\Omega (u(y)-u(x))^2\mu(x) - \frac{1}{2}\alpha_1 \sum_\Omega u^2\mu(x) - \gamma|\Omega| \text{ as } \Omega \text{ is bounded.} \tag{3.17}$$

An application of the Poincaré inequality $\lambda_1 \sum_\Omega u^2\mu(x) \leq \sum_\Omega (u(y)-u(x))^2\mu(x)$ results in:

$$\Phi(u) \geq \frac{1}{2}\left(1 - \frac{\alpha_1}{\lambda_1}\right)\sum_\Omega (u(y)-u(x))^2\mu(x) - C = \frac{1}{2}a\|u\|^2 - C, \tag{3.18}$$

where $C = \gamma|\Omega|$ and $a = 1 - \frac{\alpha_1}{\lambda_1} > 0$. The previous sentence gives us that Φ is coercive on H_0^1.

Now since Φ is both weakly lower semi continuous and coercive on H_0^1 and H_0^1 is a reflexive Banach space, then by Theorem 3.2, Φ is bounded from below and there exists $v \in X$ such that $\Phi(v) = \inf_X \Phi(u)$. Therefore, v is a critical point of Φ. Now from the first paragraph of this proof we see that v is a weak solution to equation (3.15). □

3.4

Lagrange Multiplier Approach to Kazdan-Warner Equation on a Finite Graph

In 2016 [29] Grigoryan, Lin, and Yang studied the finite graph analogue of the Kazdan-Warner equation. Given a finite graph V and c a constant, they studied the equations:

$$\Delta_\mu u = c - k(t)e^u \text{ in } V, \tag{3.19}$$

and

$$\Delta_\mu u = -k(t)e^u \text{ in } V. \tag{3.20}$$

They determined if and only if conditions for these equations to have a solution. They followed closely the technique used by J.Kazdan and F. Warner in their 1974 paper "Curvature functions for compact 2-manifolds" [38] for proving existence of solution in the continuous analogue of this equation. The results that Grigoryan, Yang, and Yang found in the paper include:

Theorem 3.6. [29] *Let $G = (V, E)$ be a finite graph, and k ($\not\equiv 0$) be a function on V. Then the equation (3.20) above has a solution u if and only if k changes sign and $\int_V k d\mu < 0$.*

Theorem 3.7. [29] *Let $G = (V, E)$ be a finite graph, and k ($\not\equiv 0$) be a function on V and c be a positive constant. Then the equation (3.19) above has a solution u if and only if k is positive somewhere.*

We now provide an extension of the aforementioned results of the Kazdan-Warner equation on a finite graph by replacing the constant c with a function $h : V \to \mathbb{R}$. Our technique is based upon the concept of Lagrange Multipliers. The work in this section was presented at the 1141st AMS Meeting at the University of Delaware on September 29-30, 2018 as the talk "Remarks on Lagrange Multiplier Approch to Kadzan-Warner Equations on a Finite Graph". The work is joint work between myself and Dr. Costa. We will start by stating the variational principle of Lagrange Multipliers.

Theorem 3.8. [21] *Let $\Phi \in C^1(X, \mathbb{R})$ and suppose $M \subseteq X$ is a C^1-sub-manifold of codimension k, say $M = \{u \in U : \Psi_j(u) = 0, j = 1, ..., k\}$ where $\psi_j \in C^1(U, \mathbb{R})$, $j = 1, .., k$, $U \subseteq X$ is an open set and $\Psi_1'(u), ..., \Psi_k'(u)$ are linearly independent for each $u \in U$. If $u \in M$ is a critical point of $\Phi|M$, then there exists $\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$ such that:*

$$\Phi'(u) = \lambda \cdot \Psi'(u) = \sum_{j=1}^{k} \lambda_j \Psi_j'(u). \tag{3.21}$$

The λ_j's are called Lagrange multipliers.

Again let V be a finite graph. The version of Kazdan-Warner equation for a graph that we study is

$$\Delta_\mu u = h(x) - k(x)e^u \text{ in } V \tag{3.22}$$

where both $h, k : V \to \mathbb{R}$ and $\sum_{x\in V} h(x)\mu(x) = 0$ and $\sum_{x\in V} k(x)\mu(x) < 0$. The weak form of this graph PDE is: $\sum_\Omega \frac{1}{2} \sum_{y\sim x} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy}\mu(x) + \sum_\Omega hv\mu(x) = \sum_\Omega k(x)e^u v\mu(x)$. The corresponding functional $\Phi : H^1(V) \to \mathbb{R}$ needed is:

$$\begin{aligned}\Phi(u) &= \int_V \left(\frac{1}{2}|\nabla u|^2 + hu\right) dx \\ &= \sum_V \left(\frac{1}{4}\frac{1}{\mu(x)} \sum_{y\sim x}(u(y) - u(x))^2\mu_{xy} + hu\right)\mu(x) \\ &= \sum_V \frac{1}{4}\sum_{y\sim x}(\nabla_{xy}u)^2\mu_{xy} + \sum_V hu\mu(x)\ \forall\ u, h \in H^1(V)\ ,\end{aligned} \tag{3.23}$$

and it's Fréchet derivative is

$$\begin{aligned}\Phi'(u)\cdot v &= \int_V (\nabla u \cdot \nabla v + hv)dx \\ &= \sum_V \left(\frac{1}{2}\frac{1}{\mu(x)}\sum_{y\sim x}(u(y) - u(x))(v(y) - v(x))\mu_{xy} + hv\right)\mu(x) \\ &= \sum_V \frac{1}{2}\sum_{y\sim x}(\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy} + \sum_V hv\mu(x)\ \forall\ u, h \in H^1(V)\ .\end{aligned} \tag{3.24}$$

We begin by providing the existence of a non-trivial "basic solution", u_0, for the equation $\Delta_\mu u = h$ in $H_1(V)$. The fact that $\Delta_\mu u : V \to \mathbb{R}^n$ is symmetric with respect to $(\cdot,\cdot)$ on $H^1(V)$ by spectral decomposition assures there exists an orthonormal basis of eigenvectors $\{e_1, e_2, e_3, \cdots, e_n\}$ for Δ_μ over $H^1(V)$ with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$ respectively. Our proof of a non-trivial solution is essentially a version of the Fredholm alternative for our finite-dimensional situation.

Lemma 3.9. *Let $G = (V, E)$ be a finite graph such that $|V| = n$. Then the equation $\Delta_\mu u = h$ has a solution $u_0 \neq 0$ if and only if $h \in \operatorname{Span}\{e_i : \lambda_i \neq 0\}$.*

Proof. For the first direction suppose that $h \in \operatorname{Span}\{e_i : \lambda_i \neq 0\}$. So then $h = \sum_{\lambda_i\neq 0} \alpha_i e_i = \sum_{\lambda_i\neq 0} \left(\frac{\alpha_i}{\lambda_i}\right)(\lambda_i e_i)$. Note that $u_0 = \sum_{\lambda_i\neq 0} \left(\frac{\alpha_i}{\lambda_i}\right) e_i \in H^1(V)$ and

$$\Delta_\mu u_0 = \Delta_\mu \left(\sum_{\lambda_i\neq 0}\left(\frac{\alpha_i}{\lambda_i}\right)e_i\right) = \sum_{\lambda_i\neq 0}\left(\frac{\alpha_i}{\lambda_i}\right)\Delta_\mu e_i = \sum_{\lambda_i\neq 0}\left(\frac{\alpha_i}{\lambda_i}\right)\lambda_i e_i = \sum_{\lambda_i\neq 0}\alpha_i e_i = h.$$

Hence, $u_0 \neq 0$ is a classical solution to $\Delta_\mu u = h$. Conversely, suppose that $\Delta_\mu u = h$ has a solution $u_0 \neq 0$ in $H^1(V)$. Since $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal basis of basis of eigenvectors for Δ_μ over $H^1(V)$ with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$ (we only need the fact that there is a basis for $H^1(V)$), then $u_0 = \sum_{\lambda_i \neq 0} \beta_i e_i$. Therefore,

$$h = \Delta_\mu u_0 = \Delta_\mu \left(\sum_{\lambda_i \neq 0} \beta_i e_i \right) = \sum_{\lambda_i \neq 0} \beta_i \Delta_\mu e_i = \sum_{\lambda_i \neq 0} \beta_i \lambda_i e_i \in \mathrm{Span}\{e_i : \lambda_i \neq 0\}.$$

□

The next lemma shows that every solution to $\Delta_\mu u = h$ is a constant off of $u_0 \neq 0$ defined in the proof of the previous lemma. This is why we refer to this $u_0 \neq 0$ as a basic solution to $\Delta_\mu u = h$.

Lemma 3.10. *Let $G = (V, E)$ be a finite graph. Then any solution of the equation $\Delta_\mu u = h$ has the form $u = u_0 + c$ where $c \in \mathbb{R}$ is a constant.*

Proof. Suppose $u \in V$ is an arbitrary solution to $\Delta_\mu u = h$. By the previous lemma, $\Delta_\mu u_0(x) = h(x)$. Now $\Delta_\mu u = h$ becomes $\Delta_\mu u = \Delta_\mu u_0$. By the linearity of Δ_μ shown in Theorem 1.1 we see $\Delta_\mu(u - u_0) = 0$. Since zero is a simple eigenvalue of Δ_μ by the proof of Theorem 1.13, then $u - u_0 = c$ for some constant $c \in \mathbb{R}$. Therefore, $u = u_0 + c$.

□

Now define the following set

$$M = \left\{ u \in H^1(V) : \sum_{x \in V} k(x) e^{u(x)} \mu(x) = 0, \sum_{x \in V} u(x)\mu(x) = 0 \right\}. \tag{3.25}$$

In the following lemma, we show that M is a C^1 sub-manifold of codimension 2.

Lemma 3.11. *Let $G = (V, E)$ be a finite graph. Suppose that $h, k : V \to \mathbb{R}$ satisfy $\sum_{x \in V} h(x)\mu(x) = 0$ and $\sum_{x \in V} k(x) e^{u_0(x)} \mu(x) < 0$. Then we have the following:*

1. *$M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 2;*
2. *Furthermore, any critical point u of $\Phi | M$ is (up to a constant) a weak solution to $\Delta_\mu u = h(x) - k(x)e^u$ in $H^1(V)$, and this in turn implies that u is a solution of $\Delta_\mu u = h(x) - k(x)e^u$.*

Proof. We begin by proving $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 2. Note that, Grigoryan, Lin, and Yang showed $M \neq \emptyset$ [29]. Observe that the functionals $\psi_1, \psi_2 : H^1(V) \to \mathbb{R}$ defined by

$$\Psi_1(u) = \sum_{x \in V} k(x) e^{u(x)} \mu(x) \ , \ \Psi_2(u) = \sum_{x \in V} u(x)\mu(x),$$

are of class C^∞, hence, C^1. Furthermore, we now show that $\Psi_1'(u)$ and $\Psi_2'(u)$ are linearly independent $\forall u \in M$. Indeed, observe that,

$$(\alpha\Psi_1'(u) + \beta\Psi_2'(u)) \cdot v = \alpha \sum_{x\in V} k(x)e^{u(x)}v(x)\mu(x) + \beta \sum_{x\in V} v(x)\mu(x) = 0 \ \forall v \in H^1(V),$$

implies that $\beta = 0$ (by the choice of $v(x) = 1$) and then $\alpha = 0$ (by the choice of $v(x) = e^{u_0(x)-u(x)}$). Hence, $\Psi_1'(u)$ and $\Psi_2'(u)$ are linearly independent $\forall u \in M$. Therefore, $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 2.

Now we prove for any critical point of $\Phi|M$ is (up to a constant) a solution to $\Delta_\mu u = h(x) - k(x)e^u$. Suppose that $u \in M$ is a critical point of $\Phi|M$. Since $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 2 then by Theorem 3.8 (Lagrange Multipliers), there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned} \Phi'(u) \cdot v &= \int_V (\nabla u \cdot \nabla v + hv)dx \\ &= \sum_V \frac{1}{2\mu(x)} \sum_{y\sim x} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy} + \sum_V hv\mu(x) \\ &= \lambda_1 \sum_{x\in V} k(x)e^{u(x)}v(x)\mu(x) + \lambda_2 \sum_{x\in V} v(x)\mu(x) \ \forall v \in H^1(V). \end{aligned} \tag{3.26}$$

Therefore, $u \in H^1(V)$ is a weak solution to $-\Delta_\mu u + h(x) = \lambda_1 k(x)e^{u(x)} + \lambda_2$. In view of Theorem 1.10 (Integration By Parts) and $V = \Omega \cup \Omega^c$ and $\Omega^c = \emptyset$ (i.e. $\Omega = V$) then

$$\sum_{x\in V} \Delta_\mu u(x)v(x)\mu(x) = -\frac{1}{2} \sum_{x,y\in V} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{x,y} \ \forall v \in H^1(V).$$

This means

$$\sum_{x\in V} ((-\Delta_\mu u) + h - \lambda_1 ke^u - \lambda_2)v\mu(x) = 0 \ \forall v \in H^1(V).$$

Consequently, u satisfies $(-\Delta_\mu u) + h - \lambda_1 ke^u - \lambda_2 = 0$ in $H^1(V)$ (by choosing $v = \mathrm{sgn}((-\Delta_\mu u) + h - \lambda_1 ke^u - \lambda_2)))$. Hence, u satisfies

$$-\Delta_\mu u + h(x) = \lambda_1 k(x)e^{u(x)} + \lambda_2$$

in $H^1(V)$ (in other words u is a solution in $H^1(V)$). The previous equation becomes:

$$\sum_{x\in V} (-\Delta_\mu u + h(x))\mu(x) = \sum_{x\in V} (\lambda_1 k(x)e^{u(x)} + \lambda_2)\mu(x). \tag{3.27}$$

Since $\sum\limits_{x\in V} h(x)\mu(x) = 0$, $\sum\limits_{x\in V} k(x)e^{u(x)}\mu(x) = 0$ by equation (3.25) and $\sum\limits_{x\in V} (-\Delta_\mu u)\mu(x) = 0$ by Corollaries 1.6 and 1.11 then, equation (3.27) implies that $\lambda_2 = 0$ and hence

$$-\Delta_\mu u + h(x) = \lambda_1 k(x)e^{u(x)}. \tag{3.28}$$

Note that, $\lambda_1 \neq 0$, for if $\lambda_1 = 0$ then $\Delta_\mu u = h(x)$ would imply that $u = u_0 + c$ for some constant $c \in \mathbb{R}$ by Lemma 3.10. Now $\sum_{x\in V} k(x)e^{u(x)}\mu(x) = 0$ would become

$$0 = \sum_{x\in V} k(x)e^{u(x)}\mu(x) = e^c \sum_{x\in V} k(x)e^{u_0(x)}\mu(x) < 0,$$

which is a contradiction. Consequently, $\lambda_1 \neq 0$. Now let $u = u_0 + w$ (i.e. $w = u - u_0$). Then equation (3.28) becomes:

$$-\Delta_\mu(u_0 + w) + h = \lambda_1 k e^{u_0+w}.$$

The previous equation becomes:

$$-\Delta_\mu w = \lambda_1 k e^{u_0+w}$$

as $\Delta_\mu u_0 = h$. Multiplying the previous equation by e^{-w} and integrating over V yields:

$$-\sum_{x\in V}(\Delta_\mu w)e^{-w}\mu(x) = \lambda_1 \sum_{x\in V} k e^{u_0}\mu(x).$$

Applying Theorem 1.10 (Integration By Parts) to the left hand side of the previous equation yields:

$$\frac{1}{2}\sum_{x,y\in V}(\nabla_{xy} w)(\nabla_{xy} e^{-w})\mu_{xy} = \lambda_1 \sum_{x\in V} k e^{u_0}\mu(x).$$

The previous equation is equivalent to:

$$\frac{1}{2}\sum_{x,y\in V}(w(y) - w(x))(e^{-w(y)} - e^{-w(x)})\mu_{xy} = \lambda_1 \sum_{x\in V} k e^{u_0}\mu(x).$$

The previous equation directly implies that $\lambda_1 > 0$ as

$$\frac{1}{2}\sum_{x,y\in V}(w(y) - w(x))(e^{-w(y)} - e^{-w(x)})\mu_{xy} < 0$$

and

$$\sum_{x\in V} k e^{u_0}\mu(x) < 0.$$

Since $\lambda_1 > 0$ then there exists a constant $d \in \mathbb{R}$ such that $\lambda_1 = e^d$. Letting $\hat{u} = u + d$ we observe that:

$$\begin{aligned}
\Delta_\mu \hat{u} + k e^{\hat{u}} &= \Delta_\mu(u + d) + k e^{u+d} \\
&= \Delta_\mu u + \Delta_\mu d + k e^u e^d \\
&= \Delta_\mu u + 0 + \lambda_1 k e^u \\
&= \Delta_\mu u + \lambda_1 k e^u \\
&= h,
\end{aligned}$$

in view of equation (3.28). Therefore, $\hat{u}$ is a solution to $\Delta_\mu u + ke^u = h$.

□

Now we present a theorem concerning a necessary and sufficient condition for the existence of solutions to $\Delta_\mu u = h(x) - k(x)e^u$ in $H^1(V)$.

Theorem 3.12. *Let $G = (V, E)$ be a finite graph. Suppose that $h : V \to \mathbb{R}$ where $h \in span\{e_i : \lambda_i \neq 0\}$ and $k(x)$ is not the zero function. Then $\Delta_\mu u = h(x) - k(x)e^u$ has a solution if and only if $\sum_{x\in V} k(x)e^{u_0(x)}\mu(x) < 0$ and $k(x)$ changes sign.*

Proof. First suppose that u is a solution to $\Delta_\mu u + ke^u = h$ in V. Integrating both sides of the previous equation over V yields:

$$\sum_{x\in V}(\Delta_\mu u)\mu(x) + \sum_{x\in V} ke^u\mu(x) = \sum_{x\in V} h\mu(x). \tag{3.29}$$

Note that $\sum_{x\in V} h(x)\mu(x) = 0$ as $h = \sum_{\lambda_i\neq 0}\alpha_i e_i$ and $\sum_{x\in V} e_i(x)\mu(x) = \frac{1}{\lambda_i}\sum_{x\in V}(\Delta_\mu e_i)(x) = 0$. Now since $\sum_{x\in V} h(x)\mu(x) = 0$ and $\sum_{x\in V}(\Delta_\mu u)\mu(x) = 0$ then equation (3.29) becomes:

$$\sum_{x\in V} ke^u\mu(x) = 0.$$

The previous equation implies that $k(x)$ must change sign. Letting $u = u_0 + w$ makes $\Delta_\mu u + ke^u = h$ become $\Delta_\mu(u_0 + w) + ke^{u_0+w} = h$ in V. Thus, $\Delta_\mu u_0 + \Delta_\mu w + ke^{u_0}e^w = h$ in V. The previous equation becomes $h + \Delta_\mu w + ke^{u_0}e^w = h$ as $\Delta_\mu u_0 = h$ by Lemma 3.10. Hence, $\Delta_\mu w + ke^{u_0}e^w = 0$. Multiplying both sides by e^{-w} and integrating both sides of the previous equation over V yields:

$$\sum_{x\in V}(\Delta_\mu w)e^{-w}\mu(x) + \sum_{x\in V} ke^{u_0}\mu(x) = 0.$$

Applying Theorem 1.24 (Integration By Parts) to the left hand side of the previous equation yields:

$$-\frac{1}{2}\sum_{x,y\in\Omega}(\nabla_{xy}w)(\nabla_{xy}e^{-w})\mu_{x,y} + \sum_{x\in V} ke^{u_0}\mu(x) = 0.$$

The previous equation along with ($k(x) \neq 0$ implying $\nabla_{xy}w \neq 0$) implies that

$$\sum_{x\in V} ke^{u_0}\mu(x) < 0.$$

Conversely, suppose that $\sum_{x\in V} ke^{u_0}\mu(x) < 0$. Then, in view of Lemma 3.13, in order to demonstrate that $\Delta_\mu u = h(x) - k(x)e^u$ has a solution, it suffices to find a critical point of $\Phi|M$. Note that, $M \subseteq X_0 = \left\{u \in H^1(V) : \sum_{x\in V} u\mu(x) = 0\right\}$ and on X_0 we have the following Wirtinger inequality:

$$\sum_{x\in V}|u|^2\mu(x) \leq C\sum_{x\in V}(u(y) - u(x))^2\mu(x).$$

Hence,

$$\Phi(u) = \sum_{x\in V}\left(\frac{1}{2}(u(y)-u(x))^2 + hu\right)\mu(x) \geq \hat{C}\|u\|^2 - \|h\|_{l^2}\|u\| \to +\infty$$

as $\|u\| \to +\infty$ for $u \in X_0$. Consequently, since Φ is weakly lower semi continuous and M is weakly closed in X, Theorem 3.2 shows the existence of $v \in M$ such that $\Phi(v) = \inf_M \Phi$. Hence, $v \in M$ is a critical point of $\Phi|M$. Therefore, by Lemma 3.11, v is a solution to $\Delta_\mu u = h(x) - k(x)e^u$ in V.

□

Our non-constant $h(x)$ style of Lagrange Multiplier approach also agrees with the Grigoryan, Yang, Lin result when $h(x)$ is the zero function. Here, we take the fundamental solution to the PDE $\Delta_\mu u = h$ to be the zero-vector (or constant zero graph labeling over the vertex set) denoted by $u_0 = 0$. A further note is that using this notation gives us $e^{u_0(x)} = 1$ is the graph labeling each vertex by 1. Note in this situation that $\sum_{x\in V} h(x)\mu(x) = 0$ as $h(x)$ is the constant zero graph labeling. Using the same manifold of codimension 2 in the previous argument and replacing the non-zero u_0 with $u_0 = 0$ in all the previous arguments results in the result of Grigoryan, Yang, Lin:

Theorem 3.13. [29] *Let $G = (V, E)$ be a finite graph, and k ($\not\equiv 0$) be a function on V. Then the equation (3.20) has a solution u if and only if k changes sign and $\int_V k d\mu < 0$.*

To arrive at the result of Grigoryan, Yang, Lin regarding $h(x) = c$ where $|c| \neq 0$ by a Lagrange Multiplier approach requires use of a manifold of codimension 1. This is what we discuss in the next proof. First we paraphrase the result of Grigoryan, Yang, Lin that we wish to prove via Lagrange Multipliers:

Theorem 3.14. [29] *Let $G = (V, E)$ be a finite graph, and k ($\not\equiv 0$) be a function on V. Suppose that $h(x) = c$ for every vertex $x \in V$ and $|c| \neq 0$. Then the equation (3.20) has a solution.*

Proof. We start by considering the manifold of codimension 1 given by

$$M = \left\{ u \in H^1(V) : \sum_{x\in V} k(x)e^{u(x)}\mu(x) = c\text{vol}(V) \right\}. \tag{3.30}$$

Note that, Grigoryan, Lin, and Yang showed $M \neq \emptyset$ [29].

We now prove $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 1. Note that, $M \neq \emptyset$. Observe that, the functional $\psi_1 : H^1(V) \to \mathbb{R}$ defined by

$$\Psi_1(u) = \sum_{x\in V} k(x)e^{u(x)}\mu(x),$$

is of class C^∞, hence, C^1. Furthermore, we now show that $\{\Psi_1'(u)\}$ is linearly independent $\forall u \in M$. Now we observe that,

$$(\alpha\Psi_1'(u)) \cdot v = \alpha \sum_{x\in V} k(x)e^{u(x)}v(x)\mu(x) = 0 \ \forall v \in H^1(V),$$

implies that $\alpha = 0$. Hence, $\{\Psi_1'(u)\}$ is linearly independent $\forall u \in M$. Therefore, $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 1. Now we prove for any critical point of $\Phi|M$ is (up to a constant) a solution to $\Delta_\mu u = h(x) - k(x)e^u$. Suppose that $u \in M$ is a critical point of $\Phi|M$. Since $M \subseteq H^1(V)$ is a C^1 sub-manifold of codimension 1 then by Theorem 3.8 (Lagrange Multipliers): there exists $\lambda \in \mathbb{R}$ such that

$$\sum_V \frac{1}{2\mu(x)} \sum_{y\sim x} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy} + \sum_V cv\mu(x) = \lambda \sum_{x\in V} k(x)e^{u(x)}v(x)\mu(x).$$

Using integration by parts the previous equation yields:

$$-\Delta_\mu u + c = \lambda k(x)e^{u(x)}.$$

We sum both sides of previous equation over V to get:

$$\sum_{x\in V}(-\Delta_\mu u)\mu(x) + \sum_{x\in V} c\mu(x) = \lambda \sum_{x\in V} k(x)e^{u(x)}\mu(x).$$

The previous equation gives us

$$c\sum_{x\in V}\mu(x) = \lambda \sum_{x\in V} k(x)e^{u(x)}\mu(x).$$

as $\sum_{x\in V}(-\Delta_\mu u)\mu(x) = 0$. From the previous equation we get:

$$c\text{vol}(V) = \lambda c\text{vol}(V).$$

as $\sum_{x\in V} k(x)e^{u(x)}\mu(x) = c\text{vol}(V)$. Hence, $(\lambda - 1)c\text{vol}(V) = 0$. Now since $(\lambda - 1)c\text{vol}(V) = 0$ and $c\text{vol}(V) \neq 0$, then $\lambda - 1 = 0$. Thus, $\lambda = 1$. Consequently,

$$-\Delta_\mu u + c = k(x)e^{u(x)}.$$

Note that Grigoryan, Yang, and Lin successfully proved that the functional

$$\Phi(u) = \sum_{x\in V}(\frac{1}{2}|\nabla u|^2 + cu)\mu(x)$$

is coercive on M [29]. Consequently, since Φ is weakly lower semi continuous and M is weakly closed in X, Theorem 3.7 shows the existence of $v \in M$ such that $\Phi(v) = \inf_M \Phi$. Hence, $v \in M$ is a critical point of $\Phi|M$. Therefore, by Lemma 3.16, v is a solution to $\Delta_\mu u = h(x) - k(x)e^u$ in V.

□

3.5

Saddle Point Approach to Ahmad, Lazer, and Paul Resonant Problem on a Finite Graph

Now we are about to embark on an original application of the Saddle Point theorem of Rabinowitz to the analogue of Ahmad, Lazer, and Paul resonant problem on a finite graph. We start with the definition of the Palais-Smale condition.

Definition 3.5. *(Palais-Smale condition)* [21] *A functional $\Phi \in C^1(X, \mathbb{R})$ is said to satisfy the Palais-Smale condition if and only if any sequence $\{u_n\}_0^\infty$ such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ possesses a convergent sub-sequence.*

Now we state the famous Saddle Point Theorem of Rabinowitz.

Theorem 3.15. [21][50] *Let $X = V \oplus W$ be a Banach space, where $\dim V < \infty$, and let $\Phi \in C^1(X, \mathbb{R})$ be a functional satisfying the Palais-Smale condition. If D is a bounded neighborhood of 0 on V such that*

$$a = \max_{\partial D} \Phi < \inf_{W} \Phi = b,$$

then

$$c = \inf_{h \in \Gamma} \max_{u \in \overline{D}} \Phi(h(u))$$

is a critical value of Φ with $c \geq b$. (Here Γ is the class of deformations of $\overline{D}$ in X which fix ∂D point-wise, that is, $\Gamma = \{h \in C(\overline{D}, X) : h(u) = u \ \forall u \in \partial D\}$.)

We now consider the finite graph analogue of Ahmad, Lazer, and Paul resonant problem [1]:

$$\begin{aligned} -\Delta_\mu u &= \lambda_k u + g(x,u) \ \forall x \in \Omega \subseteq V \\ u(x) &= 0 \ \forall x \in \Omega^c, \end{aligned} \tag{3.31}$$

where V is a finite graph, $u : V \to \mathbb{R}$ is an unknown function while the function $g : V \times \mathbb{R} \to \mathbb{R}$ is continuous and uniformly bounded and given, say $|g(x,s)| \leq M \ \forall x \in \Omega, \forall s \in \mathbb{R}$. Also we impose the following condition:

$$\sum_{x \in \Omega} \int_0^{v(x)} g(x,t) dt \mu(x) \to \pm\infty \text{ as } \|v\| \to \infty, v \in N_k.$$

where N_k the null-space of the operator $L : H_0^1 \to H_0^1$ defined by $< Lu, u >= \sum_{x \in \Omega} \frac{1}{2}(u(y) - u(x))^2 - \lambda_k u^2$. The functional we study is $\Phi(u) = \frac{1}{2} < Lu, u > - \sum_{x \in \Omega} \int_0^{v(x)} g(x,t) dt \mu(x)$. Observe that $H_0^1(\Omega) = X_- \oplus X_0 \oplus X_+$ such that $X_0 = N_k$, X_+ is the subspace where the operator $L : H_0^1(\Omega) \to H_0^1(\Omega)$ is positive definite,

X_- is the subspace where the operator $L : H_0^1(\Omega) \to H_0^1(\Omega)$ is negative definite. The respective orthogonal projections are denoted by P_0, P_+, and P_-.

Theorem 3.16. *Let $G = V$ be a finite graph and let $\Omega \subseteq V$. Suppose $|g(x,s)| \leq M$ $\forall x \in \Omega, \forall s \in \mathbb{R}$ and*

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to -\infty \text{ as } \|v\| \to \infty, v \in N_k,$$

then

(a). $\Phi(u) \to -\infty$ *as* $\|u\| \to \infty$ *and* $u \in X_-$,

(b). $\Phi(u) \to \infty$ *as* $\|u\| \to \infty$ *and* $u \in X_0 \oplus X_+$.

Proof. (a). Set $u = P_-u \in X_-$ and observe

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle LP_-u, P_-u\rangle - \sum_{x\in\Omega} \int_0^{P_-u(x)} g(x,t)dt\mu(x) \\
&\leq -\frac{1}{2}\alpha\|P_-u\|^2 - \sum_{x\in\Omega} MP_-u\mu(x) \\
&= -\frac{1}{2}\alpha\|P_-u\|^2 - M\sum_{x\in\Omega} P_-u\mu(x) \\
&\leq -\frac{1}{2}\alpha\|P_-u\|^2 + M\sum_{x\in\Omega} |P_-u|\mu(x) \\
&= -\frac{1}{2}\alpha\|P_-u\|^2 + M\|P_-u\|_{L^1} \\
&\leq -\frac{1}{2}\alpha\|P_-u\|^2 + M\|P_-u\| \to -\infty
\end{aligned}$$

as $\|u\| = \|P_-u\| \to \infty$.

(b). Set $u = P_0u + P_+u \in X_0 \oplus X_+$ and observe

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\langle LP_+u, P_+u\rangle - \sum_{x\in\Omega} \int_0^{u(x)} g(x,t)dt\mu(x) \\
&\geq \frac{1}{2}\alpha\|P_+u\|^2 - \sum_{x\in\Omega} \int_0^{u(x)} g(x,t)dt\mu(x) \\
&\geq \frac{1}{2}\alpha\|P_+u\|^2 - \sum_{x\in\Omega} \left(\int_0^{u(x)} g(x,t)dt - \int_0^{P_0u(x)} g(x,t)dt\right)\mu(x) - \sum_{x\in\Omega} \int_0^{P_0u(x)} g(x,t)dt \\
&\geq \frac{1}{2}\alpha\|P_+u\|^2 - M\sum_{x\in\Omega} |P_+u|\mu(x) - \sum_{x\in\Omega} \int_0^{P_0u(x)} g(x,t)dt \\
&\geq \frac{1}{2}\alpha\|P_+u\|^2 - M\|P_+u\|_{L^1} - \sum_{x\in\Omega} \int_0^{P_0u(x)} g(x,t)dt \\
&\geq \frac{1}{2}\alpha\|P_+u\|^2 - A\|P_+u\| - \sum_{x\in\Omega} \int_0^{P_0u(x)} g(x,t)dt \to \infty
\end{aligned}$$

as $\|u\| \to \infty$ where $u \in X_0 \oplus X_+$.

□

Note that in the previous theorem if one replaces the assumption

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to -\infty \text{ as } \|v\| \to \infty, v \in N_k$$

with

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to \infty \text{ as } \|v\| \to \infty, v \in N_k$$

one clearly gets the following theorem.

Theorem 3.17. *Let $G = V$ be a finite graph and let $\Omega \subseteq V$. Suppose $|g(x,s)| \leq M$ $\forall x \in \Omega, \forall s \in \mathbb{R}$ and*

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to \infty \text{ as } \|v\| \to \infty, v \in N_k,$$

then

(a). $\Phi(u) \to -\infty$ *as* $\|u\| \to \infty$ *and* $u \in X_- \oplus X_0$,

(b). $\Phi(u) \to \infty$ *as* $\|u\| \to \infty$ *and* $u \in X_+$.

We now state and prove the discrete analog of the Ahmad, Lazer, and Paul [1] resonant problem.

Theorem 3.18. *Suppose that V is a finite graph, $u : V \to \mathbb{R}$ is an unknown function while the function $g : V \times \mathbb{R} \to \mathbb{R}$ is continuous and uniformly bounded and given, say $|g(x,s)| \leq M$ $\forall x \in \Omega, \forall s \in \mathbb{R}$. Also we impose one of the following conditions:*

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to \infty \text{ as } \|v\| \to \infty, v \in N_k$$

or

$$\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to -\infty \text{ as } \|v\| \to \infty, v \in N_k.$$

where N_k the null-space of the operator $L : H_0^1 \to H_0^1$ defined by $\langle Lu, u\rangle = \sum_{x\in\Omega} \frac{1}{2}(u(y) - u(x))^2 - \lambda_k u^2$. Then our Ahmad, Lazer, and Paul resonant problem equation (3.31) has a weak solution and thus a strong solution.

Proof. Let us show that the functional Φ satisfies the Palais-Smale condition. Suppose that $|\Phi(u_n)| \leq C$ and $\Phi'(u_n) \to 0$ while $\sum_{x\in\Omega} \int_0^{v(x)} g(x,t)dt\mu(x) \to -\infty$ as $\|v\| \to \infty, v \in N_k$. Then for every n large enough:

$$|\Phi'(u_n) \cdot h| = |\langle Lu_n, h\rangle - \sum_{x\in\Omega} \int_0^{u_n(x)} g(x,t)hdt\mu(x)| \leq \|h\| \quad \forall h \in H_0^1(\Omega).$$

This implies that

$$\|P_+u_n\| \geq |\Phi'(u_n)\cdot P_+u_n| \geq \alpha\|P_+u_n\|^2 - M\|P_+u_n\|_{L^1} \geq \alpha\|P_+u_n\|^2 - A\|P_+u_n\|.$$

Hence, $\|P_+u_n\|$ is bounded. Note that for $\|u_n\|$ sufficiently large we have

$$|\Phi'(u_n)\cdot h| \leq \|h\| \quad \forall h \in H_0^1(\Omega)$$

also implies that

$$-\|P_-u_n\| \leq |\Phi'(u_n)\cdot P_-u_n| \leq -\alpha\|P_-u_n\|^2 + M\|P_-u_n\|_{L^1} \leq -\alpha\|P_-u_n\|^2 + A\|P_-u_n\|.$$

Thus, $\|P_-u_n\|$ is bounded. Therefore, $\|(P_+ + P_-)u_n\| = \|u_n - P_0u_n\|$ is bounded. Note that

$$\begin{aligned}\Phi(u_n) &= \frac{1}{2}\langle L(u_n - P_0u_n), u_n - P_0u_n\rangle \\ &\quad - \sum_{x\in\Omega}\left(\int_0^{u_n(x)} g(x,t)dt - \int_0^{P_0u_n(x)} g(x,t)dt\right)\mu(x) - \sum_{x\in\Omega}\int_0^{P_0u_n(x)} g(x,t)dt.\end{aligned}$$

Since $\Phi(u_n)$ is bounded together with the first two terms on the right side of the previous equation, we conclude that $\sum_{x\in\Omega}\int_0^{P_0u_n(x)} g(x,t)dt$ is bounded. Now since $\sum_{x\in\Omega}\int_0^{P_0u_n(x)} g(x,t)dt$ is bounded and

$$\sum_{x\in\Omega}\int_0^{u_n(x)} g(x,t)dt\mu(x) \to -\infty \text{ as } \|u_n\| \to \infty, u_n \in N_k,$$

then $\|P_0u_n\|$ is necessarily bounded. Since $\|(P_+ + P_-)u_n\|$ and $\|P_0u_n\|$ are bounded, then $\|u_n\|$ is bounded. Now since $H_0^1(\Omega)$ is a reflexive Banach space, and u_n is bounded, then u_n has a weakly convergent subsequence which we may pass to, so we may write that u_k converges weakly to u. Applying the compact operator $T(u) = u - \nabla\Phi(u)$ to this weakly convergence subsequence u_k we get that $u_k = \nabla\Phi(u_k) + T(u_k) \to 0 + T(u)$ strongly. Consequently, $u_k \to u$ strongly in $H_0^1(\Omega)$. Hence, u_k is a $H_0^1(\Omega)$ convergent subsequence of u_n. Therefore, Φ satisfies the Palais-Smale condition.

Now by Theorem 3.17, we may apply the saddle-point theorem with $V = X_-, W = X_+ \oplus X_0$ to conclude the existence of a critical point of Φ. Therefore, our Ahmad, Lazer, and Paul resonant problem equation (3.31) has a weak solution. Since on a finite graph a solution is weak if and only if it is a strong solution, our Ahmad, Lazer, and Paul resonant problem equation (3.31) has a strong solution.

□

CHAPTER 4

VARIATIONAL METHODS ON LOCALLY FINITE GRAPHS

4.1

Introduction

In this chapter we study a nonlinear Schrödinger partial differential equations on uniformly locally finite graphs. The particular equation we study is equation (4.16).

Section 4.2 gives definitions and defines the energy norm we will be working with.

Section 4.3 is original joint work between Dr. David Costa and me. In this section we give an original compact embedding from E into $l^p(V)$ for $2 \leq p \leq \infty$ inspired by Theorem 2.1 in [47] by Zhang and Pankov. This breakthrough compact embedding is necessary for our proofs in Section 4.6.

Section 4.4 goes over some Schrödinger applications and defines the nonlinear Schrödinger partial differential equation that we are studying in this chapter.

In Section 4.5 we present the energy functional that corresponds to our Schrödinger partial differential equation. Then we present its Fréchet derivative. This is an original application.

Section 4.6 is original joint work between Dr. David Costa and me. In this section we study an original nonlinear Schrödinger problem on a locally finite graph using a equation (4.16). We do this by using and original linking agument. The main original result in this section is Theorem 4.15. Theorem 4.8 along with Lemmas 4.10, 4.11, 4.12, and 4.14 are original results used to help prove this main result. In particular Lemmas 4.10 shows our functional satisfies the Palais-Smale condition and Lemma 4.12 shows our functional satisfies the linking geometry.

4.2

Definitions and Energy Norm

Recall that we defined locally finite graphs in definition 1.6. Now we will define what it means for a graph to be uniformly locally finite.

Definition 4.1. *A graph G(V,E) is uniformly locally finite if there exist a positive integer, U, such that* $0 < \deg(x) \leq U\ \forall x \in V$.

For example in the graph of $\mathbb{Z}\times\mathbb{Z}$, which can be visualized as the Cartesian coordinate plane, each vertex has degree 4 and thus $U = 4$ in this case. Also, in the graph of $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$, which can be visualized as the 3D Cartesian coordinate plane, each vertex has degree 6 and thus $U = 6$ in this case. The vertex set V may be infinite for both locally finite graphs and uniformly locally finite graphs. For a locally finite graph the vertex set V may be uncountably infinite. However, for a uniformly locally finite graph the vertex set v is at most countably infinite. Note this is drastically different from the graphs discussed in Chapter 3 where all graphs have a finite vertex set.

In the next problem we consider some nonlinear Schrödinger partial differential equations on locally finite graphs. So we will first define Schrödinger operators on locally finite graphs. Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E and suppose that there are no loops and multiple edges. Suppose further that the graph is infinite, locally finite and connected. The distance function $d(\cdot,\cdot)$ on V is defined in the standard way. Now we fix a function $m : V \to (0,\infty)$. Note that m generates a measure on V that is a more general measure compared to $\mu(x)$ that we used in previous chapters. We suppose further that m is bounded below by a positive constant m_0. Also we fix a function $\mu_{xy} : E \to (0,\infty)$ where the value μ_{xy} can be considered as the weight of the edge xy as shown in definition 1.3. Now we consider a more general Laplacian (more precisely, the m-Laplacian) defined formally by

$$\Delta_m f(x) = \frac{1}{m(x)} \sum_{y\sim x} (f(y) - f(x))\mu_{xy}.$$

where f is a function on V. Let $\mathcal{V} : V \to \mathbb{R}$ be a function that is bounded below. In particular we take $\mathcal{V} \geq 1$. We consider the Schrödinger operator formally defined by

$$Lf(x) = -\Delta_m f(x) + \mathcal{V}(x)f(x). \tag{4.1}$$

Recall, by definition 1.3, $\mu_{xy} > 0$ if and only if $x \sim y$ and $\mu_{xy} = 0$ if and only if x and y are not adjacent.

Let $l^p(V)$ stand for the Banach space of all $u : V \to \mathbb{R}$ such that $\|u\|^p_{l^p(V)} = \sum\limits_{x\in V} |u(x)|^p m(x) < \infty$, where $p \in [1, \infty)$. $l^2(V)$ is a Hilbert space with the inner product $(u, v) = \sum\limits_{x\in V} u(x)v(x)m(x)$. $l^\infty(V)$ stands for the space of bounded functions on V endowed with the sup-norm. We also denote by c_0 the vector space of finitely supported functions on V (note we use a nonstandard notation). Obviously, $l^q(V) \subset l^p(V)$ if $1 \le q \le p \le \infty$, and this embedding is continuous. The inequality from this embeding is

$$\|u\|_{l^p(V)} \le \|u\|_{l^q(V)}. \tag{4.2}$$

The operator L is well-defined, symmetric operator on c_0 with respect to the inner product $(\cdot, \cdot)$. Furthermore, under our assumptions this operator is essentially self-adjoint in $l^2(V)$. Its unique self-adjoint extension is still denoted by L. The (self-adjoint) operator L can be defined in terms of quadratic forms as follows. The quadratic form

$$q(u) = \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} + \sum_{x\in V} \mathcal{V}(x)u^2(x)m(x), \tag{4.3}$$

with the domain

$$\left\{ u \in l^2(V) : \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} + \sum_{x\in V} \mathcal{V}(x)|u^2(x)m(x) < \infty \right\}$$

is closed, and the domain is a Hilbert space E with respect to the norm

$$\|u\|^2_E = \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} + \sum_{x\in V} \mathcal{V}(x)u^2(x)m(x). \tag{4.4}$$

The inner product associated with $\|u\|^2_E$ is

$$(u, v)_E = \sum_{x,y\in V, x\sim y} (u(x) - u(y))(v(x) - v(y))\mu_{xy} + \sum_{x\in V} \mathcal{V}(x)(x)u(x)v(x)m(x). \tag{4.5}$$

It is important to point out that [40] uses $\frac{1}{2}q(u)$ instead of $q(u)$ in equation (4.3) as do many other papers using this kind of quadratic form including but not limited to [28], [29] and [30]. We denote the vector space of finitely supported functions on V by c_0. Furthermore, c_0 is dense in E. Hence, q generates a self-adjoint operator which is exactly the operator L.

If q is a quadratic form, then its associated bilinear form is denoted by $q(u, v)$.

Our energy space can be defined as

$$E = \left\{ u \in l^2(V) : \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} < \infty \text{ and } \mathcal{V}^{1/2}u \in l^2(V) \right\} \tag{4.6}$$

where $\mathcal{V} \geq 1$. Note that $\mathcal{V}^{1/2}u \in l^2(V)$ means $\left\|\mathcal{V}^{1/2}u\right\|_{l^2(V)} < \infty$. In terms of the inner product of $l^2(V)$ this is equivalent to saying

$$\left\|\mathcal{V}^{1/2}u\right\|_{l^2(V)}^2 = \left(\mathcal{V}^{1/2}u, \mathcal{V}^{1/2}u\right)_{l^2(V)} < \infty.$$

Since $\mathcal{V}^{1/2}$ is the square root of $\mathcal{V}$ and $\mathcal{V}^{1/2}$ is the adjoint of $\mathcal{V}^{1/2}$, we have

$$\left(\mathcal{V}^{1/2}u, \mathcal{V}^{1/2}u\right)_{l^2(V)} = (\mathcal{V}u, u)_{l^2(V)}.$$

Thus $\mathcal{V}^{1/2}u \in l^2(V)$ is equivalent to

$$\sum_{x\in V}(\mathcal{V}u)(x)u(x)m(x) = \sum_{x\in V}(\mathcal{V})(x)u(x)u(x)m(x) < \infty.$$

Therefore, $\mathcal{V}^{1/2}u \in l^2(V)$ is exactly

$$\sum_{x\in V}\mathcal{V}(x)(u(x))^2m(x) < \infty. \tag{4.7}$$

So we can also define the energy space as

$$E = \left\{u \in l^2(V) : u \in \text{dom}(\Delta_m) \text{ and } \sum_{x\in V}\mathcal{V}(x)(u(x))^2m(x) < \infty\right\} \tag{4.8}$$

where $\mathcal{V} \geq 1$. Recall that E is a Hilbert space. We introduce an equivalent version of the norm

$$\|u\|_E = \left\|(-\Delta + \mathcal{V})^{1/2}u\right\|_{l^2(V)} = \left\|L^{1/2}u\right\|_{l^2(V)}. \tag{4.9}$$

We will be using this version of the energy norms in some later proofs of the Palais-Smale condition and linking geometry. The majority of the theorems and proofs in this chapter are motivated by theorems and proofs in [48] by Zhang and Pankov.

4.3

Spectrum and Compact Embedding

Throughout the rest of the text we make the following assumption: "The spectrum $\sigma(L)$ of the operator L is purely discrete, i.e., consists of isolated eigenvalues of finite multiplicity." Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$ be distinct eigenvalues of L. Due to our assumption on $\mathcal{V}$, $1 \leq \lambda_1$.

A necessary and sufficient condition (not that efficient) for the discreteness of the spectrum is given in [40]. Note also that the discreteness of the spectrum is equivalent to the compactness of the embedding $E \subset l^2_m$ (hence, $E \subset l^p_m$ for $2 \leq p$). Therefore, Lemma 2.2 of [28] provides a sufficient condition for our assumption given above. We should also mention that Lemma 2.1 of [28] can be useful to the study of bifurcation of solutions.

Now we define what it means for an function to be coercive on a graph. We will need this definition to prove some theorems on the compactness of the embedding.

Definition 4.2. *An function $h : V \to \mathbb{R}$ is said to be coercive on a graph if for all real numbers $M > 0$, there exists a real number $R_M > 0$ such that for all x where $d(x_0, x) > R_M$, where $x_0 \in V$ is a fixed vertex, normally called the origin, then $h(x) > M$.*

A similar definition holds for a self-adjoint operator $A : H \to H$, where H is a real Hilbert Space, is called coercive if there exists a constant $c > 0$ such that $(Ax, x) \geq c\|x\|^2$ for all $x \in H$. A mapping $f : X \to Y$ between two normed vector space is coercive if and only if $\|f(x)\| \to \infty$ as $\|x\| \to \infty$. Note that for the rest of this dissertation when we say $\mathcal{V}$ is coercive we mean all three of the definitions hold.

Lemma 4.1. *If A is coercive, then $A^{1/2}$ is coercive.*

Proof. Let A be coercive. Since A is coercive, $A^{1/2}$ is the square root of A, and $A^{1/2}$ is self-adjoint, then

$$\left\|A^{1/2}u\right\|^2 = \left(A^{1/2}u, A^{1/2}u\right) = (Au, u) \geq c\|u\|^2.$$

Thus

$$\left\|A^{1/2}u\right\| \to \infty \text{ as } \|u\| \to \infty.$$

Therefore, $A^{1/2}$ is coercive. □

Lemma 4.2. *If A is coercive, then A^{-1} is compact.*

Note we stated Lemma 4.2 without proof as it is a well-known property.

Next we prove an original compact embedding from E into $l^p(V)$ for $2 \leq p \leq \infty$ inspired by Theorem 2.1 in [47] by Zhang and Pankov.

Theorem 4.3. *If $\mathcal{V}$ is coercive, then*

(1) for any $2 \leq p \leq \infty$, the embedding map form E into $l^p(V)$ is compact. Denote the best embedding constant $C_p = \max\limits_{\|u\|_{l^p(v)}=1} \frac{1}{\|u\|_E}$.

(2) the spectrum $\sigma(L)$ is discrete, i.e., it consists of eigenvalues of finite multiplicity.

Proof. Note that $L^{-1/2}$ is a bounded operator form $l^2(V)$ to E and $\mathcal{V}^{1/2}$ is a operator for E to $l^2(V)$. Hence $L^{-1/2}\mathcal{V}^{1/2}$ is an operator from E to E. For any $u \in E$ we have

$$\left|\left(L^{-1/2}\mathcal{V}^{1/2}u, u\right)\right| = \left|\left(\mathcal{V}^{1/2}u, L^{-1/2}u\right)\right| \leq \left\|\mathcal{V}^{1/2}u\right\|_{l^2(V)} \left\|L^{-1/2}u\right\|_{l^2(V)} \leq C\|u\|_E^2.$$

The first inequality comes form the Cauchy–Schwarz inequality. The second inequality uses the definition of E and the boundedness of $L^{-1/2}$. Thus $L^{-1/2}\mathcal{V}^{1/2}$ is a bounded operator from E to E. Since $\mathcal{V}$ is coercive, then $\mathcal{V}^{1/2}$ is coercive by Lemma 4.1. Since $\mathcal{V}^{1/2}$ is coercive, then $\mathcal{V}^{-1/2}$ is compact by Lemma 4.2. Therefore, $L^{-1/2} = \left(L^{-1/2}\mathcal{V}^{1/2}\right)\mathcal{V}^{-1/2}$ is a compact from $l^2(V)$ to E. The compact embedding from E into $l^2(V)$ follows from this fact. The embedding from $l^2(V)$ into $l^P(V)$ is continuous. Thus part (1) holds.

Note that $L^{-1} = (I + L^{-1}\Delta)\mathcal{V}^{-1}$ or $L^{-1} = L^{-1/2}L^{-1/2}$. Then by part (1) we have L^{-1} is compact. Note that part (2) is a consequence of the usual spectral theory of a compact operator. □

4.4

Schrödinger Applications

In [48] Pankov and Zhang study the one-dimensional discrete nonlinear Schrödinger equation

$$i\dot{\psi}_n + \Delta\psi_n - v_n\psi_n + \sigma\gamma_n f(\psi_n) = 0,\ n \in \mathbb{Z} \tag{4.10}$$

where $\sigma = \pm 1$ and

$$\Delta\psi_n = \psi_{n+1} - 2\psi_n + \psi_{n-1} \tag{4.11}$$

is the discrete Laplacian operator. In equation (4.10) σ characterizes the focusing property of the equation. When $\sigma = 1$, the equation is self-focusing and when $\sigma = -1$ the equation is defocusing [48]. Also in equation (4.10) the nonlinearity f is to be a gauge invariant complex valued function of complex variables [48]. Note this means $f(e^{i\omega}u) = e^{i\omega}f(u)$ for any $\omega \in \mathbb{R}$ [48].

Then Pankov looked at special solutions, called standing wave solutions or breather solutions, to equation (4.10) where $\psi_n = e^{-it\omega}u_n$, where u_n is a sequence of complex numbers such that

$$\lim_{n \to \pm\infty} u_n = 0.$$

Inserting this into equation (4.10) we get the infinite nonlinear system of equations

$$-(\Delta u_n) + v_n u_n - \omega u_n - \sigma\gamma_n f(u_n) = 0,\ n \in \mathbb{Z}. \tag{4.12}$$

By letting $H = -\Delta + v$ equations (4.12). Becomes

$$Hu_n - \omega u_n - \sigma\gamma_n f(u_n) = 0. \tag{4.13}$$

Next we prove the standing wave equation $\psi_n = e^{-it\omega}u_n$ is also a solution to equation (4.12) using equation (4.10). Pankov stated this in [48] but did not prove it. However, we prove it here for educational purposes and for the sake of completion. By substituting $\psi_n = e^{-it\omega}u_n$ into equation (4.10) we see it becomes

$$i\dot{e^{-it\omega}}u_n + \Delta e^{-it\omega}u_n - v_n e^{-it\omega}u_n + \sigma\gamma_n f(e^{-it\omega}u_n) = 0.$$

Taking the derivative of the first term we get

$$i(-i\omega)e^{-it\omega}u_n + e^{-it\omega}\dot{u_n} + \Delta e^{-it\omega}u_n - v_n e^{-it\omega}u_n + \sigma\gamma_n f(e^{-it\omega}u_n) = 0.$$

Since u_n is a sequence of real number $\dot{u_n} = 0\ \forall n \in \mathbb{Z}$ and $-i^2 = 1$. So we get

$$\omega e^{-it\omega}u_n + \Delta e^{-it\omega}u_n - v_n e^{-it\omega}u_n + \sigma\gamma_n f(e^{-it\omega}u_n) = 0.$$

Note that the nonlinearity of f is supposed to be a gauge invariant complex valued function of complex variables. This means $f(e^{-it\omega}u_n) = e^{-it\omega}f(u_n)$. So we have

$$\omega e^{-it\omega}u_n + \Delta e^{-it\omega}u_n - v_n e^{-it\omega}u_n + \sigma\gamma_n e^{-it\omega}f(u_n) = 0.$$

Finally by factoring out $-e^{-it\omega}$ and rearranging the terms we get.

$$-e^{-it\omega}\left(-(\Delta u_n) + v_n u_n - \omega u_n - \sigma\gamma_n f(u_n)\right) = 0,\ \ n \in \mathbb{Z}.$$

Therefore, $\psi_n = e^{-it\omega}u_n$ is also a solution to equation (4.12).

In [46] Pankov studies the discrete nonlinear Schrödinger equation

$$i\dot{\psi_n} + \Delta\psi_n - v_n\psi_n + \gamma_n|\psi_n|^{p-2}\psi_n = 0,\ \ n \in \mathbb{Z} \tag{4.14}$$

where

$$\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n \tag{4.15}$$

is the discrete one-dimensional Laplacian operator. In equation (4.14) v_n and γ_n are real valued sequences, v_n is the potential parameter and γ_n is the anharmonic parameter.

Equation (4.10) is a very important discrete linear model that can be used for a number of topics ranging from solid-state and condensed-matter physics to biology [3][22][23][24][25][31].

Now we will state the nonlinear Schrödinger equation that we will be studying which is a generalization of equation (4.12). The equation we will be studying is

$$-\Delta_m u(x) + \mathcal{V}(x)u(x) - \lambda u(x) = \kappa f(x, u(x)),\ x \in V. \tag{4.16}$$

using equation (4.1) we get

$$Lu(x) - \lambda u(x) = \kappa f(x, u(x)),\ x \in V. \tag{4.17}$$

Here, $\kappa = \pm 1$ is responsible for self-focusing when $\kappa = 1$ or defocusing when $\kappa = -1$ character of the nonlinearity, and λ is a real parameter. Note that the κ in equation (4.16) corresponds to σ in equation (4.12) and λ in equation (4.16) corresponds to ω in equation (4.12). Due to the presence of λ, the previously imposed assumption $1 \leq \mathcal{V}(x)$ is not restrictive. Also note that L in equation (4.17) corresponds to H in equation (4.13).

4.5

Energy Functional

We assume that, for every $x \in V$, $f(x, u)$ is a continuous function of $u \in \mathbb{R}$ and $f(x, 0) = 0$. Let

$$F(x, u) = \int_0^u f(x, s)ds. \tag{4.18}$$

Observe that the functional we will use is $\Phi \in C^1(E, \mathbb{R})$ defined by

$$\Phi(u) = \frac{1}{2}((L - \lambda)u, u)_{l^2(V)} - \kappa \sum_{x \in V} F(x, u(x))m(x). \tag{4.19}$$

Hence,

$$\Phi'(u) \cdot v = \frac{1}{2}((L - \lambda)u, v)_{l^2(V)} - \kappa \sum_{x \in V} f(x, u(x))v(x)m(x). \tag{4.20}$$

One can take the inner product with respect to $\|u\|_E$ of both sides of equation (4.16) with $u(x)$ we get

$$(-\Delta_m u(x), u(x))_E + (\mathcal{V}(x)u(x), u(x))_E - (\lambda u(x), u(x))_E = \kappa(f(x, u(x)), u(x))_E. \tag{4.21}$$

Hence,

$$(Lu, u)_E - \lambda \|u\|_E^2 = \kappa(f(x, u(x)), u(x))_E. \tag{4.22}$$

Instead we take the inner product with respect to $\|u\|_{l^2(V)}$ of both sides of equation (4.16) with $u(x)$ we get

$$(-\Delta_m u(x), u(x))_{l^2(V)} + (\mathcal{V}(x)u(x), u(x))_{l^2(V)} - (\lambda u(x), u(x))_{l^2(V)} = \kappa(f(x, u(x)), u(x))_{l^2(V)}. \tag{4.23}$$

Motivated by the idea of weak solutions we use this inner product in equation (4.23) for computations in the remainder of this dissertation. Equation (4.23) defines what it means to be a weak solution in our space. For the remainder of this dissertation we will only be concerned with the self-focusing situation when $\kappa = 1$.

Now consider,

$$\Phi(u) - \frac{1}{q}\Phi'(u) \cdot u = \left(\frac{1}{2} - \frac{1}{q}\right) ((L - \lambda I)u, u)_{l^2(V)} + \kappa \sum_{x \in V} \left(\frac{1}{q} f(x, u)u - F(x, u)\right) m(x). \tag{4.24}$$

This is equivalent to,

$$\Phi(u) - \frac{1}{q}\Phi'(u) \cdot u = \left(\frac{1}{2} - \frac{1}{q}\right) \left((Lu, u)_{l^2(V)} - \lambda \|u\|^2_{l^2(V)}\right) + \kappa \sum_{x \in V} \left(\frac{1}{q} f(x, u)u - F(x, u)\right) m(x). \tag{4.25}$$

Letting $\kappa = 1$ this becomes

$$\Phi(u) - \frac{1}{q}\Phi'(u) \cdot u = \left(\frac{1}{2} - \frac{1}{q}\right) \left((Lu, u)_{l^2(V)} - \lambda \|u\|^2_{l^2(V)}\right) + \sum_{x \in V} \left(\frac{1}{q} f(x, u)u - F(x, u)\right) m(x). \tag{4.26}$$

Using the (AR) condition that there exists a constant $2 < q$ such that $0 < qF(x, u) \leq f(x, u)u,\ s \neq 0$ for all $x \in V$, yields,

$$\Phi(u) - \frac{1}{q}\Phi'(u) \cdot u \geq \left(\frac{1}{2} - \frac{1}{q}\right) \left((Lu, u)_{l^2(V)} - \lambda \|u\|^2_{l^2(V)}\right). \tag{4.27}$$

Since $(Lu, u)_{l^2(V)} \geq \lambda_1 \|u\|^2_{l^2(V)}$ with $\lambda_1 > 0$ then

$$\Phi(u) - \frac{1}{q}\Phi'(u) \cdot u \geq \left(\frac{1}{2} - \frac{1}{q}\right) \left(\lambda_1 \|u\|^2_{l^2(V)} - \lambda \|u\|^2_{l^2(V)}\right) = \left(\frac{1}{2} - \frac{1}{q}\right) (\lambda_1 - \lambda) \|u\|^2_{l^2(V)}. \tag{4.28}$$

4.6

A Nonlinear Schrödinger Problem on a Locally Finite Graph

Here we list some basic assumptions on the nonlinearity of $f(x, u)$.

Assumption 4.4. *Assume that $f(x, s) \in \mathcal{C}(\mathbb{R})$.*

Assumption 4.5. *There exists positive constant C_1 and $2 < p < \infty$ such that*

$$|f(x, s)| \leq C_1 \left(|s| + |s|^{p-1}\right). \tag{4.29}$$

Assumption 4.6. *Assume that f is superlinear near zero, this means*

$$\lim_{u \to 0} \frac{f(s)}{|s|} = 0. \tag{4.30}$$

Assumption 4.7. *There is a constant $2 < q < \infty$ such that*

$$0 \leq qF(x,s) \leq f(x,s)s, \ \forall s \neq 0$$

where

$$F(x,u(x)) = \int_0^{u(x)} f(x,s)ds.$$

Note that Assumption 4.7 is the (AR) condition form Theorem 4.8.

From equations (4.29) and (4.30) it is easy to show that for any $\epsilon > 0$, There exists $A = A(\epsilon) > 0$ such that for any $u(x) \in \mathbb{R}$

$$f(x,u(x))u(x) \leq \epsilon|u(x)|^2 + A|u(x)|^p \tag{4.31}$$

and

$$F(x,u(x)) \leq \frac{\epsilon}{2}|u(x)|^2 + \frac{A}{p}|u(x)|^p. \tag{4.32}$$

An elementary differential inequality argument argument shows for every $\epsilon > 0$ there exists $C_2 > 0$ such that

$$F(x,u(x))u(x) \geq C_2|u(x)|^q - \epsilon|u(x)|^2, \ \forall u(x) \in \mathbb{R}. \tag{4.33}$$

Combining equations (4.29) and (4.30), we can conclude that $2 < q \leq p$.

Theorem 4.8. *Suppose that V is a locally finite graph, $u : V \to \mathbb{R}$ is an unknown function $f(x,u)$ is a continuous function of $u \in \mathbb{R}$ and $f(x,0) = 0$. Assume Assumptions 4.4, 4.5, 4.6, and 4.7 are satisfied. Also suppose that $\kappa = 1$ and $\lambda \leq 0$.*

Then $\Phi(u)$ satisfies the Palais-Smale condition.

Proof. Suppose that u_n is a sequence in E such that $|\Phi(u_n)| \leq C$ for some $C > 0$ and $\lim_{n\to\infty} \Phi'(u_n) = 0$. So then for all n sufficiently large enough, we have $|\Phi'(u_n) \cdot u_n| \leq \|u_n\|_E$. Then

$$\begin{aligned}
\Phi(u_n) - \frac{1}{q}\Phi'(u_n) \cdot u_n &= \left(\frac{1}{2} - \frac{1}{q}\right)\left(\|u_n\|_E^2 - \lambda\|u_n\|_{l^2(V)}^2\right) + \sum_{x\in V}\left(\frac{1}{q}f(x,u)u - F(x,u)\right)m(x) \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)\left(\|u_n\|_E^2 - \lambda\|u_n\|_{l^2(V)}^2\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|_E^2
\end{aligned}$$

Hence,

$$\|u_n\|_E^2 \leq \Phi(u_n) - \frac{1}{q}\Phi'(u_n) \cdot u_n \leq C + \frac{1}{q}\|u_n\|_E. \tag{4.34}$$

Thus, u_n is bounded in E. Since u_n is bounded in E and E is a reflexive Banach space, then u_n has a subsequence u_k weakly converging to u in E (a theorem of Mazur). By compact embedding we have E is compactly embedded in $l^p(V)$. So for any $2 \le p \le \infty$

$$u_k \to u \text{ in } l^p(V). \tag{4.35}$$

We compute

$$\|u_k - u\|_E^2 = \Phi'(u_k - u) \cdot (u_k - u) + \lambda \|u_k - u\|_{l^2(V)}^2 + \kappa \sum_{x \in V} f(x, u_k(x) - u(x))(u_k(x) - u(x))m(x).$$

The Palais-Smale assumptions makes the first term on the right-hand side of the above equation converges to zero. The compact embedding of E into $l^2(V)$ ensures that the second term on the right-hand side of the above equation converges to zero. All that remains is to show that the third term on the right-hand side of the above equation converges to zero. By equation (4.31) and Hölder's inequality we know there exists $\tilde{C}_5 > 0$ and $\tilde{C}_6 > 0$ such that

$$\begin{aligned}\left| \sum_{x \in V} f(u_n)(u_n - u)m(x) \right| &\le \sum_{x \in V} \left(\epsilon |u_k| + A|u_n|^{p-1} \right) |(u_n - u)|\, m(x) \\ &\le \tilde{C}_5 \|u_k\|_{l^2(V)} \|u_k - u\|_{l^2(V)} + \tilde{C}_6 \|u_k\|_{l^p(V)}^{p-1} \|u_k - u\|_{l^p(V)}\end{aligned}$$

So by the compact embedding of E in both $l^2(v)$ and $l^p(V)$ we have $\|u_n\|_{l^2(V)}^2 \le C_{2E} \|u_n\|_E^2$ and $\|u_n\|_{l^p(V)}^2 \le C_{pE} \|u_n\|_E^2$. Thus u_k is bounded in both $l^2(V)$ and $l^p(V)$. Using this and equation (4.44) the third term on the right hand side converges to zero. Hence $u_k \to u$ in E. Thus u_k is a convergent subsequence of u_n in E. Therefore, $\Phi(u)$ satisfies the Palis-Smale condition. □

Theorem 4.8 is as much as we can prove using a Palais-Smale argument without using the fact that the function $\mathcal{V} : V \to \mathbb{R}$ used in equation (4.1) and equation (4.16) is coercive.

Definition 4.3. *Fix $x_0 \in V$, normally the origin. Let $R \in \mathbb{R}$. The ball*

$$B(x_0) = \{x \in V : d(x, x_0) \le R\}.$$

In the following Palais-Smale and linking arguments we will assume $\mathcal{V} : V \to \mathbb{R}$ is coercive. Recall that $\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty$ be distinct eigenvalues of the function L defined in equation (4.1), due to our assumption on $\mathcal{V}$, $1 \le \lambda_1$. Let ϕ_k be the associated normalized eigenfunction with λ_k for each k, That is

$$L\phi_k = \lambda_k \phi_k, \ \|\phi_k\|_{l^2(V)} = 1. \tag{4.36}$$

Further note that, $\{\phi_k : k = 1, 2, 3, \ldots\}$ is an orthonormal basis of $l^2(V)$.

For any $\lambda \geq \lambda_1$ there exists a unique k such that $\lambda \in [\lambda_k, \lambda_{k+1})$. Let

$$Y = \text{Span}\{\phi_1, \phi_2, \phi_3, \ldots, \phi_k\}, \ \dim Y = k < \infty,$$

for $\lambda < \lambda_1$, we set $Y = \{0\}$. Then the Hilbert space E can be decomposed into the direct sum

$$E = Y \oplus Z, \qquad Z = Y^{\perp} = \overline{\text{Span}\{\phi_j : j \geq k+1\}}^{\|\cdot\|_E}.$$

Observe that the linking geometry becomes the mountain pass geometry when $Y = \{0\}$. This means the Mountain Pass Theorem is one of the cases of the Linking Theorem.

Next we follow the typical Linking geometry setup used in [36] and [57]. Let $\rho > r > 0$ and $z \in Z$ such that $\|z\|_E = 1$. Then we define the following sets,

$$M = \{u = y + \lambda z : \|u\|_E \leq \rho, \lambda \geq 0, y \in Y\}$$

$$\partial M = M_0 = \{u = y + \lambda z : y \in Y, \|u\|_E = \rho \text{ and } \lambda \geq 0 \text{ or } \|u\|_E \leq \rho \text{ and } \lambda = 0\}$$

$$N = \{u \in Z : \|u\|_E = r\}$$

Next we state the relevant linking theorem for the problem at hand.

Theorem 4.9. [48][50] *Let $\Phi(u) \in C^1(E, \mathbb{R})$ and assume that Φ satisfies the Palais-Smale condition. Assume also that Φ possesses the following linking geometry*

$$\beta = \inf_{u \in N} \Phi(u) > \sup_{u \in \partial M} \Phi(u) = \alpha. \tag{4.37}$$

Let $\Gamma = \{\gamma \in C(M, E) : \gamma = id \text{ on } \partial M\}$. Then

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in M} \Phi(\gamma(u)) \tag{4.38}$$

is a critical value of Φ and

$$\beta \leq c \leq \sup_{u \in M} \Phi(u). \tag{4.39}$$

In order to show $\Phi(u)$ satisfies the hypothesis of Theorem 4.9 we need to prove two lemmas. The first lemma will show that $\Phi(u)$ satisfies the Palais-Smale condition when $\kappa = 1$. The second lemma, Lemma 4.12, will show that $\Phi(u)$ possesses the linking geometry when $\kappa = 1$.

Lemma 4.10. *Assume Assumptions 4.4, 4.5, 4.6, and 4.7 are satisfied and the function $\mathcal{V}$ is coercive. For $\kappa = 1$ and $\lambda \in \mathbb{R}$, $\Phi(u)$ satisfies the Palis-Smale condition.*

Proof. Suppose that u_n is a sequence in E such that $|\Phi(u_n)| \leq B$ for some $B > 0$ and $\lim_{n\to\infty} \Phi'(u_n) = 0$. Then, for all n sufficiently large enough, we have

$$\Phi(u_n) - \beta\Phi'(u_n) \cdot u_n \leq B + \beta\|u_n\|. \tag{4.40}$$

The coercivity of $\mathcal{V}(x)$ implies there exists and $R > 0$ such that $\mathcal{V}(x) \geq 2\lambda$ whenever $d(x, x_0) > R$. Thus

$$\|u\|^2_{l^2(v)} = \sum_{x\in B(x_0)} (u(x))^2 m(x) + \sum_{x\in V\setminus B(x_0)} (u(x))^2 m(x). \tag{4.41}$$

Recall equation (4.4)

$$\|u\|^2_E = \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} + \sum_{x\in V} \mathcal{V}(x)u^2(x)m(x).$$

Since $\sum_{x,y\in V,x\sim y} (u(x) - u(y))^2\mu_{xy}$ is positive we can see

$$\begin{aligned}\|u\|^2_E &= \sum_{x,y\in V, x\sim y} (u(x) - u(y))^2 \mu_{xy} + \sum_{x\in V} \mathcal{V}(x)u^2(x)m(x)\\ &\geq \sum_{x\in V} \mathcal{V}(x)u^2(x)m(x).\end{aligned}$$

Then using equation (4.41) we get

$$\|u\|^2_E \geq \sum_{x\in B(x_0)} \mathcal{V}(x)(u(x))^2 m(x) + \sum_{x\in V\setminus B(x_0)} \mathcal{V}(x)(u(x))^2 m(x).$$

Since $\sum_{x\in B(x_0)} \mathcal{V}(x)(u(x))^2m(x)$ is positive we get

$$\|u\|^2_E \geq \sum_{x\in V\setminus B(x_0)} \mathcal{V}(x)(u(x))^2 m(x).$$

Then since $\mathcal{V}(x) \geq 2\lambda$ whenever $d(x, x_0) > R$ we get

$$\|u\|^2_E \geq 2\lambda \sum_{x\in V\setminus B(x_0)} (u(x))^2 m(x) \implies \frac{1}{2\lambda}\|u\|^2_E \geq \sum_{x\in V\setminus B(x_0)} (u(x))^2 m(x).$$

Hence equation (4.41) can be written as

$$\|u\|^2_{l^2(v)} \leq \sum_{x\in B(x_0)} (u(x))^2 m(x) + \frac{1}{2\lambda}\|u\|^2_E. \tag{4.42}$$

We start by showing $\|u_n\|_E$ is bounded.

Let $\kappa = 1$. Note that the case for $\lambda \leq 0$ was handled in Theorem 4.8. So this proof will only talk about the case when $\lambda > 0$. Take $\frac{1}{q} < \beta < \frac{1}{2}$.

For any $\epsilon > 0$

$$0 < K(\epsilon) = \left\|(u(x))^2 - \epsilon|(u(x))|^q\right\|_{l^\infty(B(x_0))} < \infty. \tag{4.43}$$

Thus

$$\begin{aligned}
\sum_{x\in B(x_0)} (u(x))^2 m(x) &= \sum_{x\in B(x_0)} \left((u(x))^2 - \epsilon|(u(x))|^q + \epsilon|(u(x))|^q\right) m(x) \\
&= \sum_{x\in B(x_0)} \left((u(x))^2 - \epsilon|(u(x))|^q\right) m(x) + \sum_{x\in B(x_0)} \left(\epsilon|(u(x))|^q\right) m(x) \\
&\le \sum_{x\in B(x_0)} \left\|(u(x))^2 - \epsilon|(u(x))|^q\right\|_{l^\infty(B(x_0))} m(x) + \sum_{x\in B(x_0)} \left(\epsilon|(u(x))|^q\right) m(x) \\
&\le \left\|(u(x))^2 - \epsilon|(u(x))|^q\right\|_{l^\infty(B(x_0))} \sum_{x\in B(x_0)} m(x) + \sum_{x\in B(x_0)} \left(\epsilon|(u(x))|^q\right) m(x) \\
&= CK(\epsilon) + \sum_{x\in B(x_0)} \left(\epsilon|(u(x))|^q\right) m(x) \\
&\le CK(\epsilon) + \epsilon\|u(x)\|_{l^q(V)}^q,
\end{aligned}$$

where $0 < C = \sum\limits_{x\in B(x_0)} m(x) < \infty$ because $B(x_0)$ has finitely many vertices.
Let $T_\epsilon = \left(\frac{1}{2} - \beta\right)\lambda + (\beta q - 1)\,\epsilon$. Then

$$\begin{aligned}
\Phi(u) - \beta\Phi'(u)\cdot u &= \left(\frac{1}{2} - \beta\right)\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) + \sum_{x\in V} \left(\beta f(x,u)u - F(x,u)\right) m(x) \\
&\ge \left(\frac{1}{2} - \beta\right)\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) + (\beta q - 1)\sum_{x\in V} F(x,u)m(x) \\
&\ge \left(\frac{1}{2} - \beta\right)\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) + (\beta q - 1)\sum_{x\in V} \left(C_2|u|^q - \epsilon|u|^2\right) m(x) \\
&\ge \left(\frac{1}{2} - \beta\right)\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) + (\beta q - 1)\left(C_2\|u\|_{l^q(V)}^q - \epsilon\|u\|_{l^2(V)}^2\right) \\
&= \left(\frac{1}{2} - \beta\right)\|u\|_E^2 - \left(\left(\frac{1}{2} - \beta\right)\lambda + (\beta q - 1)\,\epsilon\right)\|u\|_{l^2(V)}^2 + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&= \left(\frac{1}{2} - \beta\right)\|u\|_E^2 - T_\epsilon\|u\|_{l^2(V)}^2 + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&\ge \left(\frac{1}{2} - \beta\right)\|u\|_E^2 - T_\epsilon\left(\sum_{x\in B(x_0)} (u(x))^2 m(x) + \frac{1}{2\lambda}\|u\|_E^2\right) + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&= \left(\frac{1}{2} - \beta\right)\|u\|_E^2 - \left(\frac{1}{2} - \beta\right)\lambda\sum_{x\in B(x_0)} (u(x))^2 m(x) - \frac{\left(\frac{1}{2} - \beta\right)\lambda}{2\lambda}\|u\|_E^2 \\
&\quad - (\beta q - 1)\,\epsilon\sum_{x\in B(x_0)} (u(x))^2 m(x) - \frac{(\beta q - 1)\,\epsilon}{2\lambda}\|u\|_E^2 + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&= \left(\frac{1 - 2\beta}{4} - \frac{(\beta q - 1)\epsilon}{2\lambda}\right)\|u\|_E^2 - \left(\left(\frac{1}{2} - \beta\right)\lambda + (\beta q - 1)\,\epsilon\right)\sum_{x\in B(x_0)} (u(x))^2 m(x) \\
&\quad + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&= \left(\frac{1 - 2\beta}{4} - \frac{(\beta q - 1)\epsilon}{2\lambda}\right)\|u\|_E^2 - T_\epsilon\sum_{x\in B(x_0)} (u(x))^2 m(x) + (\beta q - 1)C_2\|u\|_{l^q(V)}^q \\
&\ge \left(\frac{1 - 2\beta}{4} - \frac{(\beta q - 1)\epsilon}{2\lambda}\right)\|u\|_E^2 - T_\epsilon\left(CK(\epsilon) + \epsilon\|u(x)\|_{l^q(V)}^q\right) + (\beta q - 1)C_2\|u\|_{l^q(V)}^q
\end{aligned}$$

$$= \left(\frac{1-2\beta}{4} - \frac{(\beta q-1)\epsilon}{2\lambda}\right)\|u\|_E^2 - T_\epsilon C K(\epsilon) + ((\beta q - 1)C_2 - T_\epsilon \epsilon)\|u\|_{l^q(V)}^q.$$

For small enough $\epsilon > 0$ there exists $\tilde{C_1} > 0$ and $\tilde{C_2} > 0$ such that

$$\begin{aligned}\Phi(u) - \beta\Phi'(u)\cdot u &\geq \left(\frac{1-2\beta}{4}\right)\|u\|_E^2 + \tilde{C_1}\|u\|_{l^q(V)}^q - \tilde{C_2}\\ &\geq \left(\frac{1-2\beta}{4}\right)\|u\|_E^2 - \tilde{C_2}.\end{aligned}$$

Using this and equation (4.40) when n is sufficiently large we get

$$B + \beta\|u_n\| \geq \Phi(u_n) - \beta\Phi'(u_n)\cdot u_n \geq \left(\frac{1-2\beta}{4}\right)\|u_n\|_E^2 - \tilde{C_2}.$$

Therefore $\|u_n\|_E$ is bounded.

Next we will show $\|u_n\|_E$ has a convergent subsequences. Since u_n is bounded in E and E is a reflexive Banach space, then u_n has a subsequence u_k weakly converging to u in E (a theorem of Mazur). By compact embedding we have E is compactly embedded in $l^p(V)$. So for any $2 \leq p \leq \infty$

$$u_k \to u \text{ in } l^p(V). \tag{4.44}$$

We compute

$$\|u_k - u\|_E^2 = \Phi'(u_k - u)\cdot(u_k - u) + \lambda\|u_k - u\|_{l^2(V)}^2 + \kappa\sum_{x\in V} f(x, u_k(x) - u(x))(u_k(x) - u(x))m(x).$$

The Palais-Smale assumptions makes the first term on the right-hand side of the above equation converges to zero. The compact embedding of E into $l^2(V)$ ensures that the second term on the right-hand side of the above equation converges to zero. All that remains is to show that the third term on the right-hand side of the above equation converges to zero. By equation (4.31) and Hölder's inequality we know there exists $\tilde{C_5} > 0$ and $\tilde{C_6} > 0$ such that

$$\begin{aligned}\left|\sum_{x\in V} f(u_n)(u_n - u)m(x)\right| &\leq \sum_{x\in V}\left(\epsilon|u_k| + A|u_n|^{p-1}\right)|(u_n - u)|\,m(x)\\ &\leq \tilde{C_5}\|u_k\|_{l^2(V)}\|u_k - u\|_{l^2(V)} + \tilde{C_6}\|u_k\|_{l^p(V)}^{p-1}\|u_k - u\|_{l^p(V)}\end{aligned}$$

So by the compact embedding of E in both $l^2(v)$ and $l^p(V)$ we have $\|u_n\|_{l^2(V)}^2 \leq C_{2E}\|u_n\|_E^2$ and $\|u_n\|_{l^p(V)}^2 \leq C_{pE}\|u_n\|_E^2$. Thus u_k is bounded in both $l^2(V)$ and $l^p(V)$. Using this and equation (4.44) the third term on the right hand side converges to zero. Hence $u_k \to u$ in E. Thus u_k is a convergent subsequence of u_n in E. Therefore, $\Phi(u)$ satisfies the Palais-Smale condition. □

Next we state the definition of the Nehari manifold as given in [57] and [47]. We need the Nehari manifold to help prove our next lemma which handles the $\lambda < \lambda_1$ case needed for the proof of Lemma 4.12. Take Φ as defined by equation (4.19) with $\Phi'(0) = 0$. A necessary condition for $u \in E$ to be a critical point of Φ is that $\Phi'(u) \cdot u = 0$. We use this condition to define the Nehari manifold

$$\mathcal{N} = \{u \in E : \Phi'(u) \cdot u = 0, u \neq 0\}. \tag{4.45}$$

Note that a critical point $u \neq 0$ of Φ is called a ground state critical point or least energy critical point if

$$\Phi(u) = \inf_{\mathcal{N}} \Phi.$$

To see a problem using Nehari Manifold look at [47] by Zhang and Pankov. The locally finite graph analog of this problem remains open and is a future research objective.

Lemma 4.11. *Suppose $\kappa = 1$ and $\lambda < \lambda_1$. Then there are constants $C, D > 0$ so that for any $u \in \mathcal{N}$*

$$\|u\|_E \geq C \text{ and } \Phi(u) \geq D.$$

Proof. Let $\kappa = 1$. For any $u \in \mathcal{N}$ with $\Phi'(u) \cdot u = 0$ observe that

$$\begin{aligned}
0 = \Phi'(u) \cdot u &= \frac{1}{2}((L-\lambda)u, u)_{l^2(V)} - \sum_{x \in V} f(x,u)um(x) \\
&= \frac{1}{2}(Lu, u)_{l^2(V)} - \frac{\lambda}{2}\|u\|^2_{l^2(V)} - \sum_{x \in V} f(x,u)um(x) \\
&= \frac{1}{2}\|u\|^2_E - \frac{\lambda}{2}\|u\|^2_{l^2(V)} - \sum_{x \in V} f(x,u)um(x) \\
&= \frac{1}{2}\left(\|u\|^2_E - \lambda\|u\|^2_{l^2(V)}\right) - \sum_{x \in V} f(x,u)um(x).
\end{aligned}$$

Thus

$$\sum_{x \in V} f(x,u)um(x) = \frac{1}{2}\left(\|u\|^2_E - \lambda\|u\|^2_{l^2(V)}\right).$$

By solving for $\|u\|^2_E$ we get

$$\|u\|^2_E = \lambda\|u\|^2_{l^2(V)} + 2\sum_{x \in V} f(x,u)um(x).$$

So we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}((L-\lambda)u, u)_{l^2(V)} - \sum_{x \in V} F(x,u)m(x) \\
&= \frac{1}{2}(Lu, u)_{l^2(V)} - \frac{\lambda}{2}\|u\|^2_{l^2(V)} - \sum_{x \in V} F(x,u)m(x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|u\|_E^2 - \frac{\lambda}{2}\|u\|_{l^2(V)}^2 - \sum_{x\in V} F(x,u)m(x) \\
&= \frac{1}{2}\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) - \sum_{x\in V} F(x,u)m(x) \\
&\geq \frac{1}{2}\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) - \sum_{x\in V} \frac{1}{q} f(x,u)um(x) \\
&= \frac{1}{2}\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) - \frac{1}{q}\sum_{x\in V} f(x,u)um(x) \\
&= \frac{1}{2}\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) - \frac{1}{2q}\left(\|u\|_E^2 - \lambda\|u\|_{l^2(V)}^2\right) \\
&= \left(\frac{1}{2} - \frac{1}{2q}\right)\|u\|_E^2 - \left(\frac{1}{2} - \frac{1}{2q}\right)\lambda\|u\|_{l^2(V)}^2.
\end{aligned}
$$

Now consider two cases.

Case 1: Let $\lambda \leq 0$. Since $\lambda \leq 0$, then we have

$$
\begin{aligned}
\|u\|_E^2 &= \lambda\|u\|_{l^2(V)}^2 - 2\sum_{x\in V} f(x,u)um(x) \\
&\leq 2\sum_{x\in V}\left(\epsilon|u|^2 + A|u|^p\right)m(x) \\
&= 2\epsilon\|u\|_{l^2(V)}^2 + 2A\|u\|_{l^p(V)}^p \\
&\leq 2\epsilon C_2^2\|u\|_E^2 + 2AC_P^P\|u\|_E^p.
\end{aligned}
$$

Thus we see

$$
\left(1 - 2\epsilon C_2^2\right)\|u\|_E^2 \leq 2AC_P^P\|u\|_E^p.
$$

Let $\epsilon = \frac{1}{4C_2^2}$. Hence

$$
\begin{aligned}
\left(1 - 2\epsilon C_2^2\right)\|u\|_E^2 \leq 2AC_P^P\|u\|_E^p &\Longrightarrow \left(1 - 2\left(\frac{1}{4C_2^2}\right)C_2^2\right)\|u\|_E^2 \leq 2AC_P^P\|u\|_E^p \\
&\Longrightarrow \frac{1}{2}\|u\|_E^2 \leq 2AC_P^P\|u\|_E^p \\
&\Longrightarrow \frac{1}{2} \leq 2AC_P^P\|u\|_E^{p-2} \\
&\Longrightarrow \frac{1}{4AC_P^P} \leq \|u\|_E^{p-2} \\
&\Longrightarrow \left(\frac{1}{4AC_P^P}\right)^{\frac{1}{p-2}} \leq \|u\|_E \\
&\Longrightarrow \left(4AC_P^P\right)^{\frac{1}{2-p}} \leq \|u\|_E
\end{aligned}
$$

Thus

$$
\|u\|_E \geq \left(4AC_P^P\right)^{\frac{1}{2-p}} = C > 0.
$$

Therefore,

$$\begin{aligned}\Phi(u) &\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\|u\|_E^2-\left(\frac{1}{2}-\frac{1}{2q}\right)\lambda\|u\|_{l^2(V)}^2\\ &\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\|u\|_E^2\\ &\geq \left(\frac{1}{2}-\frac{1}{2q}\right)C^2\\ &= D>0.\end{aligned}$$

Case 2: Let $0<\lambda<\lambda_1$. Since λ_1 is the smallest eigenvalue of L, then by the definition of C_p we get $\lambda_1=\frac{1}{C_2^2}$. Then for any $\epsilon>0$ we have

$$\begin{aligned}\|u\|_E^2 &= \lambda\|u\|_{l^2(V)}^2-2\sum_{x\in V}f(x,u)um(x)\\ &\leq \lambda\|u\|_{l^2(V)}^2+2\sum_{x\in V}\left(\epsilon|u|^2+A|u|^p\right)m(x)\\ &= \lambda\|u\|_{l^2(V)}^2+2\epsilon\|u\|_{l^2(V)}^2+2A\|u\|_{l^p(V)}^p\\ &\leq \lambda C_2^2\|u\|_E^2+2\epsilon C_2^2\|u\|_E^2+2AC_p^p\|u\|_E^p.\end{aligned}$$

By letting $\lambda_1=\frac{1}{C_2^2}$ we see

$$\begin{aligned}\|u\|_E^2\leq \lambda C_2^2\|u\|_E^2+2\epsilon C_2^2\|u\|_E^2+2AC_p^p\|u\|_E^p &\Longrightarrow \|u\|_E^2\leq \lambda C_2^2\|u\|_E^2+2AC_p^p\|u\|_E^p\\ &\Longrightarrow \|u\|_E^2-\lambda C_2^2\|u\|_E^2\leq 2AC_p^p\|u\|_E^p\\ &\Longrightarrow \left(1-\lambda C_2^2\right)\|u\|_E^2\leq 2AC_p^p\|u\|_E^p\\ &\Longrightarrow 1-\lambda C_2^2\leq 2AC_p^p\|u\|_E^{p-2}\\ &\Longrightarrow \frac{1-\lambda C_2^2}{2AC_p^p}\leq \|u\|_E^{p-2}\\ &\Longrightarrow \left(\frac{1-\lambda C_2^2}{2AC_p^p}\right)^{\frac{1}{p-2}}\leq \|u\|_E\\ &\Longrightarrow \left(\frac{2AC_p^p}{1-\lambda C_2^2}\right)^{\frac{1}{2-p}}\leq \|u\|_E\\ &\Longrightarrow \left(\frac{2AC_p^p}{1-\frac{\lambda}{\lambda_1}}\right)^{\frac{1}{2-p}}\leq \|u\|_E\end{aligned}$$

Thus

$$\|u\|_E\geq \left(\frac{2AC_p^p}{1-\frac{\lambda}{\lambda_1}}\right)^{\frac{1}{2-p}}=C>0.$$

Therefore,

$$\Phi(u)\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\|u\|_E^2-\left(\frac{1}{2}-\frac{1}{2q}\right)\lambda\|u\|_{l^2(V)}^2$$

$$
\begin{aligned}
&\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\|u\|_E^2-\left(\frac{1}{2}-\frac{1}{2q}\right)\lambda C_2^2\|u\|_E^2 \\
&= \left(\frac{1}{2}-\frac{1}{2q}\right)\left(1-\lambda C_2^2\right)\|u\|_E^2 \\
&\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\left(1-\lambda C_2^2\right)C^2 \\
&\geq \left(\frac{1}{2}-\frac{1}{2q}\right)\left(1-\frac{\lambda}{\lambda_1}\right)C^2 \\
&= D > 0.
\end{aligned}
$$

□

The next lemma will show that $\Phi(u)$ satisfies the linking geometry when $\kappa = 1$ needed for Theorem 4.9 . The proof of this lemma makes heavy use of the Pythagorean formula from functional analysis. The Pythagorean formula states that if $x_1, x_2, x_3, \ldots, x_n$ are orthogonal vectors in an inner product space, then

$$
\left\|\sum_{k=1}^{n} x_k\right\|^2 = \sum_{k=1}^{n}\|x_k\|^2 . \tag{4.46}
$$

Lemma 4.12. *There exists two positive constants $\rho > r > 0$ such that*

$$
\inf_{u\in N}\Phi(u) > 0 \geq \sup_{u\in\partial M}\Phi(u). \tag{4.47}
$$

Proof. Let $\kappa = 1$. Since the case for $\lambda < \lambda_1$ has been demonstrated by Lemma 4.11, so we will consider $\lambda_k \leq \lambda < \lambda_{k+1}$ for some $k \geq 1$. Let $y = \sum\limits_{i=1}^{k} a_i\phi_i \in Y$ and $z = \sum\limits_{i=k+1}^{\infty} b_i\phi_i \in Z$ with $\|z\|_E = 1$, that is

$$
\left\|L^{1/2}z\right\|_{l^2(V)} = 1 \Longleftrightarrow \sum_{i=k+1}^{\infty}\lambda_i b_i^2 = 1.
$$

By using equation (4.46) we can see

$$
\|y\|_E^2 = \sum_{i=1}^{k}\lambda_i a_i^2
$$

and

$$
\|y+\eta z\|_E^2 = \sum_{i=1}^{k}\lambda_i a_i^2 + \eta^2.
$$

Let $u = \sum\limits_{i=k+1}^{\infty}\beta_i\phi_i \in Z$, and recall

$$
\Phi(u) = \frac{1}{2}\left(\left(L-\lambda I\right)u,u\right) - \kappa\sum_{x\in V}F(x,u)m(x). \tag{4.48}
$$

Since we are ony considering the case when $\kappa = 1$, equation (4.48) becomes

$$
\Phi(u) = \frac{1}{2}\left(\left(L-\lambda I\right)u,u\right) - \sum_{x\in V}F(x,u)m(x). \tag{4.49}
$$

Since $u = \sum_{i=k+1}^{\infty} \beta_i \phi_i \in Z$, we can see

$$\begin{aligned}
\frac{1}{2}\left(\left(L-\lambda I\right)u,u\right) &= \frac{1}{2}\left(\left(L-\lambda I\right)\sum_{i=k+1}^{\infty}\beta_i\phi_i, \sum_{i=k+1}^{\infty}\beta_i\phi_i\right)\\
&= \frac{1}{2}\left(\sum_{i=k+1}^{\infty}\left(\beta_i L\phi_i - \lambda\beta_i\phi_i\right), \sum_{i=k+1}^{\infty}\beta_i\phi_i\right)\\
&= \frac{1}{2}\left(\sum_{i=k+1}^{\infty}\left(\beta_i \lambda_i\phi_i - \lambda\beta_i\phi_i\right), \sum_{i=k+1}^{\infty}\beta_i\phi_i\right)\\
&= \frac{1}{2}\sum_{i=k+1}^{\infty}\left(\beta_i^2\lambda_i - \lambda\beta_i^2\right)\\
&= \frac{1}{2}\sum_{i=k+1}^{\infty}\left(\lambda_i - \lambda\right)\beta_i^2.
\end{aligned}$$

Thus equation (4.49) becomes

$$\Phi(u) = \frac{1}{2}\sum_{i=k+1}^{\infty}\left(\lambda_i - \lambda\right)\beta_i^2 - \sum_{x\in V} F(x,u)m(x). \tag{4.50}$$

For any $\epsilon > 0$, there exists $A = A(\epsilon) > 0$ such that for any $u \in \mathbb{R}$

$$0 \le F(x,u) \le \epsilon|u|^2 + A|u|^p \Longrightarrow -F(x,u) \ge -\left(\epsilon|u|^2 + A|u|^p\right)$$

Thus equation (4.50) becomes

$$\Phi(u) \ge \frac{1}{2}\sum_{i=k+1}^{\infty}\left(\lambda_i - \lambda\right)\beta_i^2 - \sum_{x\in V}\left(\epsilon|u|^2 + A|u|^p\right)m(x). \tag{4.51}$$

Since, $\|u\|_{l^2(V)}^2 = \sum_{x\in V}|u|^2 m(x)$ and $\|u\|_{l^p(V)}^p = \sum_{x\in V}|u|^p m(x)$, we can see

$$\sum_{x\in V}\left(\epsilon|u|^2 + A|u|^p\right)m(x) = \epsilon\|u\|_{l^2(V)}^2 + A\|u\|_{l^p(V)}^p.$$

Then, becuase $u = \sum_{i=k+1}^{\infty}\beta_i\phi_i \in Z$, we get

$$\epsilon\|u\|_{l^2(V)}^2 + A\|u\|_{l^p(V)}^p = \epsilon\left\|\sum_{i=k+1}^{\infty}\beta_i\phi_i\right\|_{l^2(V)}^2 + A\left\|\sum_{i=k+1}^{\infty}\beta_i\phi_i\right\|_{l^p(V)}^p.$$

Then, by equation (4.46),

$$\left\|\sum_{i=k+1}^{\infty}\beta_i\phi_i\right\|_{l^2(V)}^2 = \sum_{i=k+1}^{\infty}\|\beta_i\phi_i\|_{l^2(V)}^2.$$

Then we see

$$\sum_{i=k+1}^{\infty}\|\beta_i\phi_i\|_{l^2(V)}^2 = \sum_{i=k+1}^{\infty}|\beta_i|^2\|\phi_i\|_{l^2(V)}^2.$$

Given $\|\phi_i\|^2_{l^2(V)} = 1$ and inequality (4.2) we get

$$\begin{aligned}
\epsilon \left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^2_{l^2(V)} + A \left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^p_{l^p(V)} &= \epsilon \left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^2_{l^2(V)} + A \left(\left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^2_{l^p(V)} \right)^{p/2} \\
&\leq \epsilon \left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^2_{l^2(V)} + A \left(\left\| \sum_{i=k+1}^{\infty} \beta_i \phi_i \right\|^2_{l^2(V)} \right)^{p/2} \\
&= \epsilon \sum_{i=k+1}^{\infty} \|\beta_i \phi_i\|^2_{l^2(V)} + A \left(\sum_{i=k+1}^{\infty} \|\beta_i \phi_i\|^2_{l^2(V)} \right)^{p/2} \\
&= \epsilon \sum_{i=k+1}^{\infty} \beta_i^2 + A \left(\sum_{i=k+1}^{\infty} \beta_i^2 \right)^{p/2}.
\end{aligned}$$

Therefore, equation (4.51) becomes

$$\Phi(u) \geq \frac{1}{2} \sum_{i=k+1}^{\infty} (\lambda_i - \lambda) \beta_i^2 - \epsilon \sum_{i=k+1}^{\infty} \beta_i^2 - A \left(\sum_{i=k+1}^{\infty} \beta_i^2 \right)^{p/2}. \tag{4.52}$$

If $u \in N$, then

$$r^2 = \|u\|^2_E = \sum_{i=k+1}^{\infty} \lambda_i \beta_i^2 \geq \lambda_{k+1} \sum_{i=k+1}^{\infty} \beta_i^2,$$

which implies

$$\sum_{i=k+1}^{\infty} \beta_i^2 \leq \frac{r^2}{\lambda_{k+1}}.$$

Now we will do some algebraic manipulation to equation (4.51).

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{2} \sum_{i=k+1}^{\infty} (\lambda_i - \lambda) \beta_i^2 - \epsilon \sum_{i=k+1}^{\infty} \beta_i^2 - A \left(\sum_{i=k+1}^{\infty} \beta_i^2 \right)^{p/2} \\
&= \frac{1}{2} \sum_{i=k+1}^{\infty} \lambda_i \beta_i^2 - \frac{\lambda}{2} \sum_{i=k+1}^{\infty} \beta_i^2 - \epsilon \sum_{i=k+1}^{\infty} \beta_i^2 - A \left(\sum_{i=k+1}^{\infty} \beta_i^2 \right)^{p/2} \\
&= \frac{1}{2} \sum_{i=k+1}^{\infty} \lambda_i \beta_i^2 - \left(\frac{\lambda}{2} + \epsilon \right) \sum_{i=k+1}^{\infty} \beta_i^2 - A \left(\sum_{i=k+1}^{\infty} \beta_i^2 \right)^{p/2} \\
&\geq \frac{r^2}{2} - \left(\frac{\lambda}{2} + \epsilon \right) \frac{r^2}{\lambda_{k+1}} - A \left(\frac{r^2}{\lambda_{k+1}} \right)^{p/2}
\end{aligned}$$

Thus equation (4.52) becomes

$$\Phi(u) \geq \left(\frac{1}{2} - \frac{\lambda}{2\lambda_{k+1}} - \frac{\epsilon}{\lambda_{k+1}} \right) r^2 - A \left(\frac{r^p}{\lambda_{k+1}^{p/2}} \right). \tag{4.53}$$

Choose $\epsilon = \frac{\lambda_{k+1} - \lambda}{4}$. Thus

$$\frac{1}{2} - \frac{\lambda}{2\lambda_{k+1}} - \frac{\epsilon}{\lambda_{k+1}} = \frac{1}{2} - \frac{\lambda}{2\lambda_{k+1}} - \frac{\lambda_{k+1} - \lambda}{4\lambda_{k+1}}$$

$$
\begin{aligned}
&= \frac{1}{2} - \frac{\lambda}{2\lambda_{k+1}} - \frac{1}{4}\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}} \\
&= \frac{1}{2} - \frac{\lambda}{2\lambda_{k+1}} - \frac{1}{4}\left(\frac{\lambda_{k+1}}{\lambda_{k+1}} - \frac{\lambda}{\lambda_{k+1}}\right) \\
&= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \\
&= \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right).
\end{aligned}
$$

So equation (4.53) becomes

$$
\Phi(u) \geq \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^2 - A\left(\frac{r^p}{\lambda_{k+1}^{p/2}}\right). \tag{4.54}
$$

Next we create a function in terms of r of the left hand side of equation (4.54).

$$
g(r) = \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r^2 - A\left(\frac{r^p}{\lambda_{k+1}^{p/2}}\right). \tag{4.55}
$$

Taking its derivative we get

$$
\begin{aligned}
g'(r) &= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) r - \left(\frac{Apr^{p-1}}{\lambda_{k+1}^{p/2}}\right) \\
&= r\left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{Apr^{p-2}}{\lambda_{k+1}^{p/2}}\right)\right)
\end{aligned}
$$

By setting the derivative equal to zero we can find potential minimum or maximum values of $g(r)$ in order to find a lower bound for the function $\Phi(u)$.

$$
\begin{aligned}
g'(r) = 0 &\Longrightarrow r\left(\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{Apr^{p-2}}{\lambda_{k+1}^{p/2}}\right)\right) = 0 \\
&\Longrightarrow r = 0 \text{ or } r = \sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}
\end{aligned}
$$

Since we stated earlier that $r > 0$ we will not consider the case of $r = 0$ and show that the the function $g(r)$ has a local maximum at the point $r = \sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}$.

Taking the second derivative of $g(r)$ we get

$$
g''(r) = \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{Ap(p-1)r^{p-2}}{\lambda_{k+1}^{p/2}}\right).
$$

Calculating $g''\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right)$ and simplifying we get

$$
g''\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right) = \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{Ap(p-1)}{\lambda_{k+1}^{p/2}}\right)\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right)^{p-2}
$$

$$= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{Ap(p-1)}{\lambda_{k+1}^{p/2}}\right)\left(\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\right)$$
$$= \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \left(\frac{p-1}{2}\right)\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)$$
$$= \left(\frac{1}{2} - \frac{p-1}{2}\right)\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)$$
$$= \left(\frac{2-p}{2}\right)\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)$$

Since $p > 2$, we see $\frac{2-p}{2} < 0$. Also since $\lambda < \lambda_{k+1}$, we see $\frac{\lambda}{\lambda_{k+1}} < 1$ and thus $1 - \frac{\lambda}{\lambda_{k+1}} > 0$. Therefore, $g''\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right) = \left(\frac{2-p}{2}\right)\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) < 0$. So the function $g(r)$ obtains a maximum value at $r = \sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}$.

Now since $\Phi(u) \geq g(r)$ and $g(r)$ has a maximum at $r = \sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}$, then

$$\inf_{u \in N} \Phi(u) \geq g\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right).$$

Since we want to show $\inf_{u \in N} \Phi(u) > 0$ we just need to show $g\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right) > 0$.

To show $g\left(\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}\right) > 0$ we start by writing $\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)}$ in another form.

$$\sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)} = \sqrt[p-2]{\frac{\lambda_{k+1}^{p/2}}{2Ap}\left(\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}}\right)}$$
$$= \sqrt[p-2]{\frac{\lambda_{k+1} - \lambda}{2Ap}\left(\frac{\lambda_{k+1}^{p/2}}{\lambda_{k+1}}\right)}$$
$$= \left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\frac{\lambda_{k+1}^{p/2}}{\lambda_{k+1}}\right)^{\frac{1}{p-2}}$$
$$= \left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\lambda_{k+1}^{1/2}\right).$$

Plugging this into $g(r)$ we get

$$g\left(\left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\lambda_{k+1}^{1/2}\right)\right) = \frac{1}{4}\left(\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}}\right)\left(\left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\lambda_{k+1}^{1/2}\right)\right)^2$$
$$- A\left(\frac{\left(\left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\lambda_{k+1}^{1/2}\right)\right)^p}{\lambda_{k+1}^{p/2}}\right)$$
$$= \frac{1}{4}\left(\frac{\lambda_{k+1} - \lambda}{\lambda_{k+1}}\right)\left(\frac{\lambda_{k+1} - \lambda}{2Ap}\right)^{\frac{2}{p-2}}\lambda_{k+1}$$

$$
-A\left(\frac{\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{p}{p-2}}\left(\lambda_{k+1}^{p/2}\right)}{\lambda_{k+1}^{p/2}}\right)
$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\lambda_{k+1}-\lambda\right)\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}} - A\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{p}{p-2}} \\
&= \frac{1}{4}\left(\lambda_{k+1}-\lambda\right)\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}} - A\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}} \\
&= \frac{\lambda_{k+1}-\lambda}{4}\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}} - \left(\frac{\lambda_{k+1}-\lambda}{2p}\right)\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}} \\
&= \left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}}\left(\frac{\lambda_{k+1}-\lambda}{4}-\frac{\lambda_{k+1}-\lambda}{2p}\right) \\
&= \left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}}\left(\frac{(p-2)(\lambda_{k+1}-\lambda)}{4p}\right).
\end{aligned}
$$

Since $\lambda < \lambda_{k+1}$, then $\lambda_{k+1}-\lambda > 0$. Recall $p > 2$ so $p-2 > 0$. Recall $A > 0$. Thus

$$
\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{2}{p-2}}\left(\frac{(p-2)(\lambda_{k+1}-\lambda)}{4p}\right) > 0,
$$

meaning

$$
g\left(\left(\frac{\lambda_{k+1}-\lambda}{2Ap}\right)^{\frac{1}{p-2}}\left(\lambda_{k+1}^{1/2}\right)\right) > 0.
$$

Therefore,

$$
\inf_{u\in N}\Phi(u) > 0.
$$

Consider the special choice of $z = \frac{\phi_{k+1}}{\lambda_{k+1}^{1/2}}$. Then $z \in Z$ and $\|z\|_E = 1$. Let $y = \sum_{i=1}^{k} a_i\phi_i$ and

$$
u = y + \eta z \in \partial M \subset \operatorname{Span}\{\phi_1, \phi_2, \phi_3, \ldots, \phi_{k+1}\} = Y \oplus \{s\phi_{k+1} :\in \mathbb{R}\}.
$$

We will consider two cases.

Case 1: Let $\eta = 0$. Note this means $u = y$. Then, by the definition of the set ∂M, $\|u\|_E = \|y\|_E \le \rho$. Then by equation (4.46)

$$
\|u\|^2_{l^2(V)} = \|y\|^2_{l^2(V)} = \sum_{i=1}^{k}\lambda_i a_i^2 \le \rho^2.
$$

Again we will start with equation (4.49)

$$
\Phi(y) = \Phi(u) = \frac{1}{2}\left(\left(L-\lambda I\right)u, u\right) - \sum_{x\in V} F(x,u)m(x).
$$

Then since $\sum_{x\in V} F(x,u)m(x) \ge 0$ equation (4.49) becomes

$$
\Phi(u) \le \frac{1}{2}\left(\left(L-\lambda I\right)u, u\right). \tag{4.56}
$$

Since $u = y = \sum_{i=1}^{k} a_i \phi_i$, we can see

$$\begin{aligned}
\Phi(u) &\le \frac{1}{2}\left((L - \lambda I) \sum_{i=1}^{k} a_i \phi_i, \sum_{i=1}^{k} a_i \phi_i \right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} a_i \left(L\phi_i - \lambda \phi_i \right), \sum_{i=1}^{k} a_i \phi_i \right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} a_i \left(\lambda_i \phi_i - \lambda \phi_i \right), \sum_{i=1}^{k} a_i \phi_i \right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} \left(\lambda_i - \lambda \right) a_i \phi_i, \sum_{i=1}^{k} a_i \phi_i \right)
\end{aligned}$$

Then since $\|\phi_i\|^2_{l^2(V)} = 1$ equation (4.56) becomes

$$\Phi(u) \le \frac{1}{2} \sum_{i=1}^{k} \left(\lambda_i - \lambda \right) a_i^2. \tag{4.57}$$

Since $\lambda_i \le \lambda_k \le \lambda$, then $\lambda_i - \lambda \le 0$. So we see equation (4.57) becomes

$$\Phi(u) \le \frac{1}{2} \sum_{i=1}^{k} \left(\lambda_i - \lambda \right) a_i^2 \le 0. \tag{4.58}$$

Case 2: Let $\eta \ge 0$. Note this means $u = y + \eta z$. Then, by the definition of the set ∂M, $\|u\|_E = \|y + \eta z\|_E = \rho$.

Then by equation (4.46)

$$\|u\|^2_{l^2(V)} = \|y + \eta z\|^2_{l^2(V)} = \sum_{i=1}^{k} \lambda_i a_i^2 = \rho^2 - \eta^2.$$

Again we will start with equation (4.49)

$$\Phi(y + \eta z) = \Phi(u) = \frac{1}{2}\left(\left(L - \lambda I \right) u, u \right) - \sum_{x \in V} F(x, u) m(x).$$

Working out $\frac{1}{2}\left(\left(L - \lambda I \right) u, u \right)$, equation (4.49)

$$\Phi(u) = \frac{1}{2}\|u\|^2_E - \frac{\lambda}{2}\|u\|^2_{l^2(v)} - \sum_{x \in V} F(x, u) m(x). \tag{4.59}$$

For every $\epsilon > 0$, there exists $C_2 > 0$ such that for every $u \in \mathbb{R}$

$$F(x, u) \ge C_2 |u|^q - \epsilon |u|^2.$$

So equation (4.59) becomes

$$\Phi(u) \le \frac{1}{2}\|u\|^2_E - \frac{\lambda}{2}\|u\|^2_{l^2(v)} - \sum_{x \in V} \left(C_2 |u|^q - \epsilon |u|^2 \right) m(x). \tag{4.60}$$

Reworking equation (4.60) we see

$$\begin{aligned}\Phi(u) &\leq \frac{1}{2}\|u\|_E^2 - \frac{\lambda}{2}\|u\|_{l^2(v)}^2 - \sum_{x\in V}\left(C_2|u|^q - \epsilon|u|^2\right)m(x)\\ &= \frac{1}{2}\|u\|_E^2 - \frac{\lambda}{2}\|u\|_{l^2(v)}^2 - C_2\|u\|_{l^q(v)}^q + \epsilon\|u\|_{l^2(v)}^2\\ &= \frac{1}{2}\|u\|_E^2 - \left(\frac{\lambda}{2} - \epsilon\right)\|u\|_{l^2(v)}^2 - C_2\|u\|_{l^q(v)}^q\end{aligned} \tag{4.61}$$

Recall that in a finite dimensional space all norms are equivalent and $y + \eta z$ is an element of a finite dimensional space. Consequently, there is a $K > 0$ which depends on k and q such that

$$\|y + \eta z\|_{l^q(V)} \geq K\,\|y + \eta z\|_{l^2(V)}\,.$$

Hence, for $0 \leq \eta \leq \rho$ equation (4.60) becomes

$$\Phi(u) \leq \frac{1}{2}\|u\|_E^2 - \left(\frac{\lambda}{2} - \epsilon\right)\|u\|_{l^2(v)}^2 - C_2K^q\|u\|_{l^2(v)}^q. \tag{4.62}$$

Rewriting the last term we get

$$\Phi(u) \leq \frac{1}{2}\|u\|_E^2 - \left(\frac{\lambda}{2} - \epsilon\right)\|u\|_{l^2(v)}^2 - C_2K^q\left(\|u\|_{l^2(v)}^2\right)^{q/2}. \tag{4.63}$$

Using the fact that $\|u\|_E = \rho$ we get

$$\Phi(u) \leq \frac{\rho^2}{2} - \left(\frac{\lambda}{2} - \epsilon\right)\|u\|_{l^2(v)}^2 - C_2K^q\left(\|u\|_{l^2(v)}^2\right)^{q/2}. \tag{4.64}$$

Given the fact that $\|u\|_{l^2(v)}^2 = \sum_{i=1}^{k} a_i^2 + \frac{\lambda^2}{\lambda_{k+1}}$ we get

$$\Phi(u) \leq \frac{\rho^2}{2} - \left(\frac{\lambda}{2} - \epsilon\right)\left(\sum_{i=1}^{k} a_i^2 + \frac{\eta^2}{\lambda_{k+1}}\right) - C_2K^q\left(\sum_{i=1}^{k} a_i^2 + \frac{\eta^2}{\lambda_{k+1}}\right)^{q/2}. \tag{4.65}$$

Reworking this we get

$$\Phi(u) \leq \rho^2\left(\frac{1}{2} - \frac{\frac{\lambda}{2} - \epsilon}{\lambda_k}\right) + \left(\frac{\lambda}{2} - \epsilon\right)\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\eta^2 - C_2K^q\left(\frac{\rho^2}{\lambda_k} - \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\eta^2\right)^{q/2}. \tag{4.66}$$

We will call the right hand side of the function $\tilde{g_\epsilon}(\eta)$ to get the equation

$$\tilde{g_\epsilon}(\eta) = \rho^2\left(\frac{1}{2} - \frac{\frac{\lambda}{2} - \epsilon}{\lambda_k}\right) + \left(\frac{\lambda}{2} - \epsilon\right)\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\eta^2 - C_2K^q\left(\frac{\rho^2}{\lambda_k} - \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\eta^2\right)^{q/2}. \tag{4.67}$$

Taking the derivative of this we get

$$\tilde{g_\epsilon}'(\eta) = (\lambda - 2\epsilon)\,\eta\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right) + C_2K^q q\eta\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\left(\frac{\rho^2}{\lambda_k} - \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right)\eta^2\right)^{\frac{q-2}{2}}. \tag{4.68}$$

For $0 \le \eta \le \rho$ we have $\tilde{g_\epsilon}'(\eta) \ge 0$, when $0 < \epsilon \le \frac{\lambda}{2}$. Choosing $\epsilon = \frac{\lambda}{4}$, gives

$$\Phi(u) \le \max_{0 \le \lambda \le \rho} \tilde{g}_\epsilon(\eta) = \left(\frac{1}{2} - \frac{\lambda}{4\lambda_{k+1}}\right)\rho^2 - \frac{C_2 K^q}{\lambda_{k+1}^{q/2}}\rho^q = \tilde{g}_\epsilon(\rho). \tag{4.69}$$

Therefore, there is $\rho > r > 0$ such that $\tilde{g}_\epsilon(\rho) < 0$. So we have

$$\inf_{u \in N} \Phi(u) \ge g(r) > 0 > \tilde{g}_\epsilon(\rho) \ge \sup_{u \in \partial M} \Phi(u).$$

□

Theorem 4.13. [48][50] *Let E be an infinite dimensional Banach space and let $\Phi \in C^1(E, \mathbb{R})$ be even, satisfy the Palais-Smale condition, and $\Phi(0) = 0$. If $E = Y \oplus Z$ where Y is finite dimensional and Φ satisfies*

(1) there are constants $r, \alpha \ge 0$ such that $\Phi|_{\partial B_r \cap Z} \ge \alpha$, and

(2) for nested sequences $E_1 \subset E_2 \subset \cdots$ of increasing finite dimension, there exists $\rho_i = \rho(E_i) > 0$ such that $\Phi \le 0$ on $B^c_{\rho_i} = \{x \in E_i : \|x\| > \rho_i\}$, for $i = 1, 2, \ldots$,

then Φ possesses an unbounded sequence of critical values.

We need to show that $\Phi(u)$ satisfies the hypothesis of Theorem 4.13. The fact that $\Phi(u)$ satisfies part (1) of Theorem 4.13 has been verified by by the proof of Lemma 4.12. We need another lemma to show that $\Phi(u)$ satisifies part (2) of Theorem 4.13. We define the nested sequence of finite dimensional space $\{E_m\}$ in Theorem 4.13 as follows. For $\lambda < \lambda_1$ we have $E_m = \text{Span}\{\phi_1, \phi_2, \ldots, \phi_m\}$, and for $\lambda_k \le \lambda < \lambda_{k+1}$ we have $E_m = \text{Span}\{\phi_1, \phi_2, \ldots, \phi_{m+k}\}$, when $m = 1, 2, \ldots$.

Lemma 4.14. *There exist two positive constants c_1 and c_2, depending on k and m, such that for any $u \in E_m$,*

$$\Phi(u) \le c_1 \|u\|_E^2 - c_2 \|u\|_E^q \tag{4.70}$$

Proof. We break the proof into two case. The case where $m = 1$ and the case when $m > 1$.

Case 1: Let $m = 1$. This case be done by a similar calculation to the one done in Lemma 4.12. Note that of any $u \in E_1$ there exists unique $y \in Y$ and $\eta \in \mathbb{R}$ such that $u = y + \eta z$, where Y and z are defined in the proof Lemma 4.12. So by a similar calculation to how we obtained equation (4.69) we get for $u \in E_1$

$$\Phi(u) \le \left(\frac{1}{2} - \frac{\lambda}{4\lambda_{k+1}}\right)\|u\|_E^2 - \frac{C_2 K^q}{\lambda_{k+1}^{q/2}}\|u\|_E^q. \tag{4.71}$$

This means when $m = 1$ equation (4.70) is true.

Case 2: Let $m > 1$. Take $\lambda_m = \frac{\lambda_{k+m-1}+\lambda_{k+m}}{2}$. We define the functional

$$\Phi_m(u) = \frac{1}{2}((L-\lambda_m)u, u) - \sum_{x\in V} F(x,u)m(x),$$

this is simply the functional $\Phi(u)$ with a different frequency. Note that $\lambda_{k+m-1} \le \lambda_m < \lambda_{k+m}$. Thus by replacing $k+1$ with $k+m$ in equation (4.71) we get for any $u \in E_m$,

$$\Phi(u) \le \left(\frac{1}{2} - \frac{\lambda}{4\lambda_{k+m}}\right)\|u\|_E^2 - \frac{C_2K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q. \tag{4.72}$$

Express $u = \sum_{i=1}^{k+m} a_i\phi_i$. Using

$$\|u\|_E^2 = \sum_{i=1}^{k+m} \lambda_i a_i^2 \ge \lambda_1 \sum_{i=1}^{k+m} a_i^2,$$

we get for any $u \in E_m$,

$$\begin{aligned}\Phi(u) &= \Phi_m(u) + \frac{1}{2}(\lambda_m - \lambda)\sum_{i=1}^{k+m} a_i^2 \\ &\le \left(\frac{1}{2} - \frac{\lambda}{4\lambda_{k+m}}\right)\|u\|_E^2 - \frac{C_2K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q + \frac{\lambda_m-\lambda}{2\lambda_1}\|u\|_E^2 \\ &\le \left(\frac{1}{2} - \frac{\lambda}{4\lambda_{k+m}}\right)\|u\|_E^2 - \frac{C_2K^q}{\lambda_{k+m}^{q/2}}\|u\|_E^q.\end{aligned}$$

This menas when $m > 1$ equation (4.70) is true.

Therefore there exists positive constants c_1 and c_2 for any $u \in E_m$ such that equation (4.70) is true. □

Theorem 4.15. *Assume Assumptions 4.4, 4.5, 4.6, and 4.7 are satisfied and the function $\mathcal{V}$ is coercive. Let $\kappa = 1$ and $\lambda \in \mathbb{R}$. Then*

(1) there is at least one nontrivial solution $u \in E$ of equation (4.16);

(2) if in addition the nonlinearity f is odd, then there exists an unbounded sequence of critical values of the function $\Phi(u)$. Consequently, there exists infinitely many distinct pairs of nontrivial solutions to equation (4.16) in the space E.

Proof. Since $\Phi(u)$ satisfies the Palais-Smale condition by Lemma 4.10 and the linking geometry by Lemma 4.12, then the first part of Theorem 4.15 is a consequence of the Min/Max Linking Theorem 4.9. Therefore, equation (4.16) has at least one nontrivial solution $u \in E$ when $\kappa = 1$ and $\lambda \in \mathbb{R}$.

Since $\Phi(u)$ satisfies part (1) of Theorem 4.13 by Lemma 4.12 and part (2) of Theorem 4.13 by Lemma 4.14, then by Theorem 4.13 there exists an unbounded sequence of critical values of $\Phi(u)$. Therefore, equation (4.16) has infinitely many distinct pairs of nontrivial solutions in E when $\kappa = 1$ and $\lambda \in \mathbb{R}$. □

BIBLIOGRAPHY

[1] S. Ahmad, A.C. Lazer and J.L. Paul, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. **25** (1976), 933-944, DOI 10.1512/iumj.1976.25.25074, MR427825

[2] A. Ambrosetti and P.R. Rabinowitz, *Dual variational methods in critical point theory and applications*, Journal of Functional Analysis. **14** (1973), 349-381, DOI 10.1016/0022-1236(73)90051-7, MR0370183

[3] S. Aubry, *Breathers in nonlinear lattices: Existence, linear stability and quantization*, Physica D: Nonlinear Phenomena **103** (1997), 201-250, DOI 10.1016/S0167-2789(96)00261-8, MR1464249

[4] A. Banerjee, J. Jost, *Laplacian Spectrum and Protein-Protein Interaction Networks*, arXiv:0705.3373, 2007.

[5] A. Banerjee, J. Jost, *On the spectrum of the normalized graph Laplacian*, Linear Algebra and its Applications. **428** (2008), 3015-3022, DOI 10.1016/j.laa.2008.01.029, MR2416605

[6] F. Bauer, J. Jost, *Bipartite and Neighborhood graphs and the spectrum of the normalized graph Laplace operator*, Comm. Anal. Geom **21** (2013), 787-845, DOI 10.4310/CAG.2013.v21.n4.a2, MR3078942

[7] M. Benson and S-M. Lee, *On Cordialness of Regular Windmill Graphs*, Congressus Numerantium **68** (1989), 49-58, MR995854

[8] Alain Bensoussan and Jose-Luis Menaldi, *Difference Equations on Weighted Graphs*, Journal of Convex Analysis **12** (2003), 13-44, MR2135795

[9] H. Brezis, J.M. Caron and L. Nirenberg, *Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz*, Comm. Pure Appl. Math. **33** (1980), 667-684, DOI 10.1002/cpa.3160330507, MR586417

[10] I. Cahit, *Cordial Graphs: a weaker version of graceful and harmonious graphs*, Ars Combin. **23** (1987), 201-207, MR886954

[11] I. Cahit, *On Cordial and 3-equitable Graphs*, Utilitas Math. **37** (1990), 189-198, MR1068518

[12] I. Cahit, *Recent Results and Open Problems on Cordial Graphs*, Contemporary Methods in Graph Theory, Bibligraphisches Inst. Mannhiem (1990), 209-230, MR1126229

[13] N. Cairnie and K.Edwards, *The computational complexity of cordial and equitable labelling*, Discrete Math. **216** (2000), 29-34, DOI 10.1016/S0012-365X(99)00295-2, MR1750852

[14] H Chen and J Jost, *Minimum vertex covers and the spectrum of the normalized Laplacian on trees*, Linear Algebra and its Applications, **437** (2012), 1089-1101, DOI 10.1016/j.laa.2012.04.005, MR2926157

[15] Fan R. K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, vol92, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. MR1421568

[16] G. Chartrand, S-M Lee and P. Zhang, *Uniformly Cordial Graphs*, Discrete Mathematics, **306** (2006), 726-737, DOI 10.1016/j.disc.2005.11.025, MR2234980

[17] Fan R. K. Chung, A. Grigor'yan, and S.-T. Yau, *Upper Bounds For Eigenvalues of The Discrete and Continuous Laplace Operators*, Advances in Mathematics, **117**(1996), 165-178, DOI 10.1006/aima.1996.0006, MR1371647

[18] Fan R. K. Chung and Robert P. Langlands, *A Combinatorial Laplacian with Vertex Weights*, Journal of Combinatorial Theory, **A 75** (1996), 316-327, DOI 10.1006/jcta.1996.0080, MR1401006

[19] D.C. Clark, *A variant of the Lusternik-Schnirelman theory*, Ind. Univ. Math. J. **22** (1972), 65-74, DOI 10.1512/iumj.1972.22.22008, MR296777

[20] Daniel Anthony Corral, *Friendly Index Sets of Starlike Graphs*, Master's Thesis, University of Nevada, Las Vegas, 2007.

[21] David G. Costa, *An Invitation to Variational Methods in Differential Equations*, Birkhäuser Boston, Inc., Boston, MA, 2007, MR2321283

[22] J.C. Eilbeck and M. Johansson *The discrete nonlinear Schrodinger equation - 20 years on*, in Localization and energy transfer in nonlinear systems (ed. L Vasquez, R.S. MacKay, and M.P. Zorzano), (2003), 44-67

[23] Jason W. Fleischer, Tal Carmon, Mordechai Segev, Nikos K. Efremidis, and Demetrios N. Christodoulides, *Observation of Discrete Solitons in Optically Induced Real Time Waveguide Arrays*, Phys. Rev. Lett., **90** (2003), 023902, DOI 10.1103/PhysRevLett.90.023902.

[24] Jason W. Fleischer, Mordechai Segev, Nikos K. Efremidis, and Demetrios N. Christodoulides, *Observation of two-dimensional discrete solitons in optically induced nonlinear photonic lattices*, Nature , **422** (2003) 147–150, DOI 10.1038/nature01452.

[25] S. Flach and C.R. Willis, *Discrete breathers*, Physics Reports, **295** (1998), 181-264, DOI 10.1016/S0370-1573(97)00068-9, MR1607320

[26] R. Frucht and F. Harrary, *On the Corona of Two Graphs*, Aequationes Mathematicae **4** (1970), 322-325, DOI 10.1007/BF01844162, MR281659

[27] Alexander Grigor'yan, *Analysis on Graphs*, University Lecture Series, vol 71, American Mathematical Society, Providence, RI, 2009, MR3822363

[28] Alexander Grigor'yan, Young Lin, and Yunyan Yang, *Existence of positive solutions to some nonlinear equations on locally finite graphs*, Sci. China Math. **60** (2017), DOI 10.1007/s11425-016-0422-y. MR3665801

[29] Alexander Grigor'yan, Young Lin, and Yunyan Yang, *Kazdan-Warner equations on graphs*, Calc. Var. Partial Differential Equations **55** (2016), DOI 10.1007/s00526-016-1042-3. MR3523107

[30] Alexander Grigor'yan, Young Lin, and Yunyan Yang, *Yamabe type equations on finite graphs*, arXiv.1607.04521, 2016.

[31] D. Hennig and G. P. Tsironis, *Wave transmission in nonlinear lattices*, Physics Reports **307** (1999), 333-432, DOI 10.1016/S0370-1573(98)00025-8. MR1670429

[32] Sa'ar Hersonsky, *Boundary Value Problems on Planar Graphs and Flat Surfaces with Integer Cone Sigularites, I: The Dirichlet Problem*, J. Reine Angew. Math. **670** (2012), 65-92, DOI 10.1515/crelle.2011.155. MR2982692

[33] Sa'ar Hersonsky,*Boundary value problems on planar graphs and flat surfaces with integer cone singularities, II: the mixed Dirichlet-Neumann problem*, Differential Geom. Appl., **29** (2011) 329–347, DOI 10.1016/j.difgeo.2011.03.003. MR2795842

[34] Y.S. Ho, S-M. Lee, and S.C. Shee, *Cordial Labellings of the Cartesian Product and Composition of Graphs*, Ars Combin. **29** (1990), 169-180. MR1046105

[35] M. Hovay, *A-cordial Graphs*, Discrete Mathematics, **93** (1991), 183-194, DOI 10.1016/0012-365X(91)90254-Y.

[36] Y. Jabri, *The mountain pass theorem*, Encyclopedia of Mathematics and its Applications, vol. 95, Cambridge University Press, Cambridge, 2003. Variants, generalizations and some applications. MR2012778.

[37] Jurgen Jost, *Mathematical Methods in Biology and Neurobiology*, Universitext, Springer, London, 2014. MR3157168

[38] J. Kazdan, F. Warner, *Curvature functions for compact 2-manifolds*, Ann.of Math. (2) **99** (1974), 14-47, DOI 10.2307/1971012. MR343205

[39] W.W. Kirchherr, *On the cordiality of some specific graphs*, Ars Combin. **31** (1991), 127-138. MR1110227

[40] M. Keller and D. Lenz, *Unbounded Laplacians on graphs: basic spectral properties and the heat equation*, Math. Model. Nat. Phenom. **5** (2010), 198-224, DOI 10.1051/mmnp/20105409. MR2662456

[41] S-M. Lee and H.K. Ng, *On Friendly Index Sets of Bipartite Graphs*, Ars Combin. **86** (2008), 257-271, MR2379169.

[42] S-M. Lee and H.K. Ng, *On Friendly Index Sets of Spiders*, Malays. J. Math. Sci. **8** (2014), 47-68. MR3226031

[43] Y.H. Lee, H.M. Lee, and G.J. Chang, *Cordial Labelings of Graphs*, Chinese J. Math. **20** (1992), 263-273. MR1183821

[44] J. J. Manfredi, A. M. Oberman, and A. P. Sviridov, *Nonlinear Elliptic Partial Differential Equations And P-Harmonic Functions On Graphs*, Differential Integral Equations **28** (2013), 79-102. MR3299118

[45] D. McGinn, *Friendly indices and fully cordial graphs*, J. Combin. Math. Combin. Comput. **101** (2017), 83-99, MR3676192.

[46] A. Pankov *Standing waves for discrete nonlinear Schrödinger equations: sign-changing nonlinearities*, Appl. Anal. **92** (2013), 308-317, DOI 10.1080/00036811.2011.609987. MR3022859

[47] A. Pankov and G. Zhang, *Standing waves of the discrete nonlinear Schrödinger equations with growing potentials*, Commun. Math. Anal. **5** (2008), 38-49, MR2421490.

[48] A. Pankov and G. Zhang, *Standing wave solutions of the discrete non-linear Schrödinger equations with unbounded potentials, II*, Appl. Anal. **89** (2010), 1541-1557, DOI 10.1080/00036810902942234, MR2682118.

[49] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. MR845785.

[50] P.H. Rabinowitz, *Some Minimax Theorems and Applications to Nonlinear Partial Differential Equations*, Nonlinear analysis (collection of papers in honor of Erich H. Rothe) , Academic Press, New York, (1978), pp.161-177. MR0501092

[51] E. Seah, *On the construction of cordial graphs*, Ars Combin. **31** (1991), 249-254. MR1110244

[52] Ebrahim Salehi and Daniel Corral, *Fully cordial trees*, J. Combin. Math. Combin. Comput. **98** (2016), 171-183. MR3560482

[53] E. Salehi and S. De, *On a conjecture concerning the friendly index sets of trees*, Ars Combin. **90** (2009), 371-381. MR2489539

[54] E. Salehi and S-M Lee, *On friendly index sets of trees*, Congr. Numer. **178** (2006), 173-183. MR2310230

[55] Douglas West, *Introduction to graph theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 1996. MR1367739

[56] M. Willem,*Lectures on Critical Point Theory*, Trabalhos de Matematica, University of Brasilia, Brasilia, 1983.

[57] M. Willem, *Minimax Theorems*,Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996. MR1400007.

CURRICULUM VITAE

Graduate College
University of Nevada, Las Vegas

Daniel Anthony Corral

Degrees:

Bachelor of Science - Mathematics, 2006
University of Nevada, Las Vegas

Master of Science - Mathematics, 2007
University of Nevada, Las Vegas

Master of Arts - Mathematics, 2011
University of Kentucky

Special Honors and Awards:

Honorable mention NSF Graduate Research Fellowship Program (Spring 2005)
2004 Bhatnagar Award for the Top Major in Mathematical Sciences at UNLV
NSF EPSCoR Ring True Two Award, Award Number: EPS-0132556 (Summer 2004)
The National Dean's List 2003-2004 edition (2004)
Summa Cum Laude
University Honors
Department Honors

Publications:

Ebrahim Salehi and Daniel Corral, "Fully Cordial Trees", Journal of Combinatorial Mathematics and Combinatorial Computing 98 (2016), p. 171-183.

Conference and Poster Presentations:

Invited Speaker 1141st AMS Meeting at University of Delaware: "Remarks on Lagrange Multiplier Approch to Kadzan-Warner Equations on a Finite Graph", September 29-30 2018.
Presenter 28th Midwest Conference on Combinatorics and Combinatorial Computing at University of Nevada, Las Vegas:"Fully Cordial Trees", October 22-24, 2014.
Daniel Corral and Corran Webster: "On a metric of the discrete Heisenberg group" Southern Nevada Undergraduate Research Opportunities Poster Program, 2004, Funded by National Science Foundation EPSCoR.
Daniel Corral: "On the metric boundary of the discrete Heisenberg group", Fall Honors Thesis Poster Session, Fall 2004.

Dissertation Title:

Some Graph Laplacians and Variational Methods Applied to Partial Differential Equations on Graphs

Dissertation Examination Committee:

Chairperson, David Costa, Ph.D.
Committee Member, Ebrahim Salehi, Ph.D.
Committee Member, Hossein Tehrani, Ph.D.
Graduate Faculty Representative, Pushkin Kachroo, Ph.D.