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Time-Optimal Control for One Dimensional Evacuation System

Sabiha Amin Wadoo, Pushkin Kachroo

Abstract— This paper presents design of a time-optimal controller for a model representing evacuation dynamics in one dimension. The model presented here is based on the law of conservation of mass. The model is the classical one equation model for a traffic flow based on conservation of mass with a prescribed relationship between density and velocity. The equations of motion are described by nonlinear partial differential equations. We address the optimal control problem for the space discretized dynamics thus making use of nonlinear ordinary differential equations. The objective is to synthesize a nonlinear open loop controller that evacuates people in minimum time. Necessary conditions for time-optimal solution are derived. Pontryagin's minimum principle is used to arrive at a bang-bang form for optimal control.

I. INTRODUCTION

THE objective of this paper is to design a time-optimal controller for an evacuation control system in one dimension. The dynamics that are used to model an evacuation system in this paper are based on traffic flow theory [1]. The evacuation system can be modeled like a traffic flow or a fluid flow, which can be thought of as the flow of people on a building floor or a corridor. These models are based on the basic law of conservation. In case of evacuation the conservation law can be stated as "total number of people is conserved in the system". The resulting dynamics are given by nonlinear partial differential equations [1]. The system is distributed, that is both the state and control variables are distributed in time and space. The control objective is to design open loop controller to remove people from the evacuation area by generating control commands in minimum possible time.

There are two approaches to the design of controllers for distributed systems. One approach is to formulate the control problem directly as a distributed model. In [2],[3] the authors have designed feedback controllers for an evacuation system in the distributed setting. The evacuation model is presented in a partial differential equation framework and then design of controllers is also performed using distributed setting. In another approach the controllers are designed using space and time discretized models of traffic flow [4] [5]. In order to utilize the various

linear and nonlinear control techniques available for lumped parameter systems, the distributed parameter model is space discretized [5]. The controllers are then designed on the basis of resulting linear or nonlinear ordinary differential equation model using known techniques available for such systems [6], [7]. In this paper we adopt the latter approach.

There is a wide variety of models available for traffic flow [8]. In this paper we will use a one-dimensional model which describes the behavior of people using a single partial differential equation. This is a basic one-equation model based on the equation of continuity or conservation of mass. This model also has a fundamental relationship between density of people and speed of flow. According to the law of conservation of mass, total flow of people exiting from any section cannot be higher than the total flow of the people that are entering. To represent the flows, Greenshield's model [9] is used to show the dependence of speed on the density of people. We discuss the design of time-optimal control for the space discretized model based on continuity equation. We find the necessary conditions for optimal control using minimum principle of Pontryagin [10]. The resulting dynamics are a nonlinear two-point boundary value problem and the form of optimal control is Bang-Bang control law.

II. MATHEMATICAL MODEL

In this section mathematical model of a one dimensional evacuation problem is presented. We will discuss the evacuation model of a one dimensional single exit corridor of length L . The model is similar to the one dimensional traffic flow model. Both these models are similar to fluid flow and are based on the principle of conservation. The model is described by a nonlinear hyperbolic partial differential equation.

There are two main approaches to modeling. One approach is microscopic [11] where each individual is taken into consideration and his behavior is expressed by a set of rules or an equation involving adjacent individuals. The other approach is macroscopic [12]. Here the overall behavior of the flow of people is considered. The area is treated as a series of sections within each of which the density and average velocity of people can be measured for a given time. The changes in these variables may then be described using partial differential equations. The model

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presented here is macroscopic with the dynamics being represented in terms of density, flow and speed. As a result the system is distributed with all the parameters as functions of space and time. The system dynamics are modeled based on the law of conservation of mass. To study control design problem we discretise this model in space thus resulting in dynamics represented by nonlinear ODE [5].

A. Continuity Equation Based Model

This model is based on equation of conservation of mass. The conservation law of mass in case of an evacuation system means that the number of people is conserved in the system. Let us consider the case of a single exit one dimensional corridor of length L . Let $\rho(x, t)$ denote the density of people as a function of position vector x and time t , $q(x, t)$ the flow and $v(x, t)$ the velocity vector field associated with the flow. The conservation of mass equation holds and is given by

$$\frac{\partial \rho(x, t)}{\partial t} + \text{div}(q(x, t)) = 0 \quad (1)$$

with the following initial and a boundary condition

$$\rho(x, t_0) = \rho_0(x) \quad (2)$$

$$\rho(0, t) = 0 \text{ and } \rho(L, t) = 0 \quad \forall \quad t \in [0, \infty) \quad (3)$$

The flow $q(x, t)$ is obtained as a product of density and velocity as

$$q(x, t) = \rho(x, t)v(x, t) \quad (4)$$

subject to the initial conditions and boundary conditions given by (2) and (3) respectively. To describe the relationship between velocity vector field $v(x, t)$ and density $\rho(x, t)$ we need one more equation. Here we make use of Greenshields model

$$v(x, t) = v_f(x, t) \left(1 - \frac{\rho(x, t)}{\rho_{\max}} \right) \quad (5)$$

where $v_f(x, t)$ is the free flow speed and ρ_{\max} is the jam density that is the maximum number of people that could possibly fit a single cell.

B. Discretized system dynamics

For the control design purposes the distributed system (1) can be space discretized in order to utilize the various linear and nonlinear control techniques available for lumped parameter systems. This results in a lumped parameter system described by ODEs. For this purpose the corridor is subdivided into several sections. For simplicity, we are using two sections only but the same analysis is true for any finite number of sections.

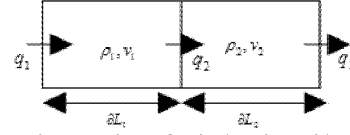


Fig.1. Sections of a single exit corridor

Space discretization is performed by dividing the length of corridor into segments as shown in Figure 1. Each segment has length ΔL_i . The dynamics for each segment of the corridor in PDE form are given by (1). For space discretization we assume for each segment the approximation $\rho_i(x, t) = \rho_i(t)$ and $q_i(x, t) = q_i(t)$. Therefore space discretization of (1) gives following continuous ODEs for two sections

$$\frac{d\rho_i(t)}{dt} = -\frac{1}{\Delta L_i} (q_{i+1}(t) - q_i(t)) \quad (6)$$

where $i = 1, 2$ is the i -th section of the corridor. The various model variables are

$\rho_i(t)$: Density in i -th section at time t (number of people/m)

$q_i(t)$: Flow of people leaving section $i-1$ section and entering section i at time t (number of people/sec)

$v_i(t)$: Velocity in i -th section at time t (m/sec)

ΔL_i : Length of section i (m)

The flow in each section is given as

$$q_i(t) = \rho_i(t)v_i(t) \quad (7)$$

Choosing flow coming out of section 2 as the output variable we have the following output equation

$$y(t) = \rho_2(t)v_2(t) \quad (8)$$

To describe the relationship between $v_i(t)$ and $\rho_i(t)$ we use (5)

$$v_i(t) = v_{if}(t) \left(1 - \frac{\rho_i(t)}{\rho_{\max}} \right) \quad (9)$$

where $v_{if}(t)$ is the free flow speed for i -th section and ρ_{\max} is the jam density.

C. Initial and boundary conditions

We assume that there is no flow going into the section 1 and we also assume that the densities in both sections are known at initial time t_0 . Thus we get the following initial and boundary conditions

$$\rho_i(t_0) = \rho_{i0} \quad (10)$$

$$\rho_1(t) = \rho(0, t) = 0 \quad \forall \quad t \in [0, \infty) \quad (11)$$

D. Control Model

To formulate the control problem based on the continuity equation model we need to choose a control variable. To do so we use Greenshields model (9), and take free flow

velocity vector field $v_{if}(t)$ as the control variable denoted by $u_i(t)$, with $u_i(t)$ being the control effort required for segment ∂L_i . If the density at a location is zero then the speed at that location will be the free flow speed. However, with the actuation system implemented, we can tell people to change the speed. Also the traffic density affects the achievable speeds, therefore we choose $v_{if}(t)$ as the control variable. We have a constraint on input

$$0 \leq v_{if}(t) \leq v_{imax} \quad (12)$$

where v_{imax} is the known upper bound on the control input $v_{if}(t)$. The space of inputs satisfying (12) is the admissible input space U .

E. State Space representation

The mathematical model given by (6) and (8) can be represented in a standard nonlinear state space form for control design purposes. The state space representation is obtained from (6) as

$$\frac{d\rho_1(t)}{dt} = -\frac{1}{\partial L_1} [q_2(t) - q_1(t)]$$

$$\frac{d\rho_2(t)}{dt} = -\frac{1}{\partial L_2} [q_3(t) - q_2(t)]$$

Using (7) and the assumption that $q_1(t) = 0$, we have

$$\frac{d\rho_1(t)}{dt} = -\frac{1}{\partial L_1} [\rho_1(t)v_1(t)]$$

$$\frac{d\rho_2(t)}{dt} = \frac{1}{\partial L_2} [\rho_1(t)v_1(t) - \rho_2(t)v_2(t)]$$

Using (9) and choosing free flow velocity $v_{if}(t)$ as the control input we get

$$\frac{d\rho_1(t)}{dt} = -\frac{\rho_1(t)}{\partial L_1} \left[1 - \frac{\rho_1(t)}{\rho_{max}}\right] u_1(t)$$

$$\frac{d\rho_2(t)}{dt} = \frac{1}{\partial L_2} \left[\rho_1(t) \left(1 - \frac{\rho_1(t)}{\rho_{max}}\right) u_1(t) - \rho_2(t) \left(1 - \frac{\rho_2(t)}{\rho_{max}}\right) u_2(t) \right] \quad (13)$$

$$y(t) = \rho_2(t) \left(1 - \frac{\rho_2(t)}{\rho_{max}}\right) u_2(t) \quad (14)$$

The dynamics in equation (13) together with output equation (14) gives the state space representation

$$\frac{d\bar{\rho}(t)}{dt} = f(\bar{\rho}(t), \bar{u}(t)) \quad (15)$$

$$y(t) = g(\bar{\rho}(t), \bar{u}(t)) \quad (16)$$

subject to initial condition $\bar{\rho}(0) = \bar{\rho}_0$. Here $\bar{\rho}(t) = [\rho_1, \rho_2]^T$, $\bar{u}(t) = [u_1, u_2]^T$ and $f = [f_1, f_2]^T$ is a vector valued function with

$$f_1 = -\frac{\rho_1(t)}{\partial L_1} \left[1 - \frac{\rho_1(t)}{\rho_{max}}\right] u_1(t)$$

$$f_2 = \frac{1}{\partial L_2} \left[\rho_1(t) \left(1 - \frac{\rho_1(t)}{\rho_{max}}\right) u_1(t) - \rho_2(t) \left(1 - \frac{\rho_2(t)}{\rho_{max}}\right) u_2(t) \right]$$

and

$$g = \rho_2(t) \left(1 - \frac{\rho_2(t)}{\rho_{max}}\right) u_2(t)$$

The model in equations (15) and (16) is the standard state space model. Since the dynamics in (15) are linear in input, these can also be written as

$$\frac{d\bar{\rho}(t)}{dt} = A(\bar{\rho}(t), t) \bar{u}(t) \quad (17)$$

where A is an array of order 2×2 given as

$$A(\bar{\rho}(t), t) = \begin{bmatrix} -\frac{\rho_1(t)}{\partial L_1} \left(1 - \frac{\rho_1(t)}{\rho_{max}}\right) & 0 \\ \frac{\rho_1(t)}{\partial L_2} \left(1 - \frac{\rho_1(t)}{\rho_{max}}\right) & \frac{\rho_2(t)}{\partial L_2} \left(1 - \frac{\rho_2(t)}{\rho_{max}}\right) \end{bmatrix} \\ = [a_1(\bar{\rho}(t), t) \quad a_2(\bar{\rho}(t), t)]$$

with $a_i(\rho(t), t)$ being the i -th column of $A(\bar{\rho}(t), t)$.

III. CONTROL DESIGN

A. Control Objective

The control objective is to design open loop controller to remove people from the evacuation area by generating control commands in minimum possible time. Mathematically speaking, we address the problem of synthesizing a controller $\bar{u}^*(t)$ that transfers system (17) from arbitrary initial condition $\bar{\rho}(t_0)$ to a specified target set in minimum time. The target set denoted by $S(t)$ is defined below

$$S(t) = \begin{cases} \rho_1(t_f) = 0 \\ \rho_2(t_f) = 0 \end{cases} \quad (18)$$

where t_f is the first instant of time when the trajectory $\rho(t)$ and $S(t)$ intersect. The target set signifies that densities in each section of the corridor at t_f should be zero.

B. Time-Optimal Control

The time-optimal control can be stated as follows: Find admissible optimal control input $\bar{u}^*(t)$ that causes (17) to follow an admissible optimal trajectory $\bar{\rho}^*(t)$, such that it

- satisfies the constraint (12)
- Forces the state $\bar{\rho}(t_0)$ of the system to go to target set $S(t)$ and
- Minimizes the performance measure $J(\bar{u})$ given by

$$J(\bar{u}) = \int_{t_0}^{t_f} dt = t_f - t_0 \quad (19)$$

Since initial time is fixed and we need to minimise $t_f - t_0$, therefore t_f is free. To find the optimal control law we use *Pontryagin's Minimum Principle* [13]. To find necessary conditions for the optimal control law define a function H as

$$H(\bar{\rho}(t), u(t), p(t), t) = 1 + p^T(t)[f(\bar{\rho}(t), u(t), t)] \quad (20)$$

Equation (20) can be expressed in its component form as

$$H(\bar{\rho}(t), u(t), p(t), t) = 1 + p_1(t)[f_1(\bar{\rho}(t), u(t), t)] + p_2(t)[f_2(\bar{\rho}(t), u(t), t)]$$

where H is called the Hamiltonian and the vector given by $p(t) = [p_1(t) \ p_2(t)]^T$ are Lagrange multipliers or costates. In terms of H the necessary conditions for $\bar{u}^*(t)$ to be optimal control law are

$$\frac{d\bar{p}^*(t)}{dt} = \frac{\partial H}{\partial p}(\bar{p}^*(t), \bar{u}^*(t), p^*(t), t) = f(\bar{\rho}^*(t), \bar{u}^*(t), \bar{p}^*(t)) \quad (21)$$

$$\begin{aligned} \frac{d\bar{p}^*(t)}{dt} &= \frac{\partial H}{\partial p}(\bar{\rho}^*(t), u^*(t), p^*(t), t) \\ &= \bar{p}^*(t) \left[\frac{\partial f}{\partial p}(\bar{\rho}^*(t), u^*(t), p^*(t)) \right] \quad \forall t \in [t_0, t_f] \quad (22) \end{aligned}$$

$$H(\bar{\rho}^*(t), \bar{u}^*(t), p^*(t), t) \leq H(\bar{\rho}^*(t), \bar{u}(t), p^*(t), t) \quad (23)$$

$\forall t \in [t_0, t_f], u(t) \in U$ with U being the admissible control space. There is one more condition that takes care of boundary conditions and is given as

$$[H(\bar{\rho}^*(t_f), \bar{u}^*(t_f), p^*(t_f), t_f)] \delta \bar{x}_f = 0 \quad (24)$$

Equation (23) indicates that an optimal control must minimize the Hamiltonian. This is called *Pontryagin's Minimum Principle* and gives an admissible optimal control space. This results in the reduced search space for control. This condition is necessary but not sufficient. Equations (21) and (22) are reduced state and costate equations respectively. The reduced dynamics (21) and (22) is a nonlinear two-point boundary-value problem which must be solved to find an optimal control. The solution of these four equations will contain four constants of integration. In order to evaluate these we make use of initial and boundary conditions, and condition (24).

C. Boundary Conditions

Let us consider the boundary conditions that occur. We assume the initial condition $\bar{\rho}^*(t_0) = \bar{\rho}_0$ is given. The final time t_f is free and final state $\rho(t_f)$ is specified by (18). Thus we have $\delta \bar{x}_f \neq 0$ in (24), therefore giving following set of boundary conditions

$$\bar{\rho}^*(t_0) = \bar{\rho}_0$$

$$\bar{\rho}^*(t_f) \in S(t)$$

$$H(\bar{\rho}^*(t_f), u^*(t_f), p^*(t_f), t_f) = 0$$

There is one additional necessary condition which is stated as: If the final time is free and Hamiltonian does not explicitly depend on time then the Hamiltonian must be identically zero when evaluated on an extremal trajectory, that is

$$H(\bar{\rho}^*(t), \bar{u}^*(t), \bar{p}^*(t)) = 0 \quad \forall t \in [t_0, t_f] \quad (25)$$

Thus for system (17) in view of equations (13) and (14) reduced necessary conditions for optimal control are

$$\begin{aligned} \frac{d\rho_1^*(t)}{dt} &= -\frac{\rho_1^*(t)}{\partial L_1} \left(1 - \frac{\rho_1^*(t)}{\rho_{\max}}\right) u_1^*(t) \\ \frac{d\rho_2^*(t)}{dt} &= \frac{\rho_1^*(t)}{\partial L_2} \left(1 - \frac{\rho_1^*(t)}{\rho_{\max}}\right) u_1^*(t) - \frac{\rho_2^*(t)}{\partial L_2} \left(1 - \frac{\rho_2^*(t)}{\rho_{\max}}\right) u_2^*(t) \\ \frac{dp_1^*(t)}{dt} &= -\frac{p_1^*(t)}{\partial L_1} \left(1 - \frac{2\rho_1^*(t)}{\rho_{\max}}\right) u_1^*(t) - \frac{\dot{p}_2^*}{\partial L_2} \left(1 - \frac{2\rho_1^*(t)}{\rho_{\max}}\right) u_1^*(t) \\ \frac{dp_2^*(t)}{dt} &= \frac{p_2^*}{\partial L_2} \left(1 - \frac{2\rho_2^*(t)}{\rho_{\max}}\right) u_2^*(t) \\ 1 + p^T(t)[f(\bar{\rho}^*(t), u^*(t), t)] &\leq 1 + p^T(t)[f(\bar{\rho}^*(t), u(t), t)] \\ &\quad \forall u(t) \in U \text{ and } \forall t \in [t_0, t_f] \end{aligned}$$

and

$$1 + p^T(t)[f(\bar{\rho}^*(t), u^*(t))] = 0, \quad \forall t \in [t_0, t_f]$$

D. Control Law

From the minimum principle we get necessary condition for optimality given by (23), which can be written in view of (20) as

$$1 + p^T(t)[f(\bar{\rho}^*(t), u^*(t), t)] \leq 1 + p^T(t)[f(\bar{\rho}^*(t), u(t), t)]$$

for all $u(t) \in U$ and $t \in [t_0, t_f]$ or equivalently,

$$p^{*T}(t)A(\bar{\rho}^*(t), t)u^*(t) \leq p^{*T}(t)[A(\bar{\rho}^*(t), t)]u(t) \quad (26)$$

Assuming the control components are independent of one another, (26) reduces to the inequality

$$p^{*T}(t)[a_i(\bar{\rho}^*(t), t)]u_i^*(t) \leq p^{*T}(t)[a_i(\bar{\rho}^*(t), t)]u_i(t) \quad (27)$$

From (27) we determine $u_i^*(t)$ in terms of $\bar{\rho}^*(t)$ and $\bar{p}^*(t)$, the optimal trajectories for densities and costates. Let us define functions $r_1^*(t)$ and $r_2^*(t)$ by the equations

$$r_i^*(t) = p^{*T}(t)[a_i(\bar{\rho}^*(t), t)], \quad i = 1, 2$$

Using the functions $r_i^*(t)$, (27) can be written as

$$r_i^* u_i^*(t) \leq r_i^* u_i(t)$$

for all $u_i(t) \in U$. Thus we need to minimize the function $r_i^* u_i(t)$. If coefficient of $u_i(t)$ is positive then

$u_i^*(t)$ must be the smallest admissible value 0. If coefficient of $u_i(t)$ is positive then $u_i^*(t)$ must be the largest admissible value $v_{i\max}$. Thus we have the following optimal control law

$$u_i^*(t) = \begin{cases} 0 & \text{if } r_i^* > 0 \\ v_{i\max} & \text{if } r_i^* < 0 \\ \text{undetermined} & \text{if } r_i^* = 0 \end{cases} \quad (28)$$

$i=1,2$ and $t \in [t_0, t_f]$. Hence for each section the optimal control law is given by (28) with functions $r_i^*(t)$ as

$$\begin{aligned} r_1^* &= (\rho_1^*(t) - \frac{\rho_1^{*2}(t)}{\rho_{\max}}) \left(\frac{p_2^*(t)}{\partial L_2} - \frac{p_1^*(t)}{\partial L_1} \right) \\ r_2^* &= -(\rho_2^*(t) - \frac{\rho_2^{*2}(t)}{\rho_{\max}}) \frac{p_2^*(t)}{\partial L_2} \end{aligned} \quad (29)$$

Equation (28) shows that the components of time-optimal control are bang- bang or piecewise constant functions of time. The control law states that the optimal control to obtain minimum time response is maximum effort throughout the interval of operation.

E. Singular Intervals

The control law $u_i^*(t)$ as given by (29) is a well defined function of $\bar{\rho}^*(t)$ and $\bar{p}^*(t)$ as long as its coefficient $p^{*T}(t)a_i(\rho^*(t), t)$ is non zero. If the coefficient is zero then switching of control is indicated. However if there is a finite time interval during which $p^{*T}(t)a_i(\rho^*(t), t) = 0$, then the coefficient of $u_i(t)$ in the Hamiltonian is zero so the necessary condition (23) provides no information about how to select $u_i^*(t)$. This interval is called singular interval and the problem is singular time-optimal. If the singular interval does not exist the problem is called normal time-optimal. In this section we verify that our problem is normal and no singular intervals can occur. To verify this consider

$$p^{*T}(t)[a_i(\bar{\rho}^*(t), t)]u_i^*(t) \leq p^{*T}(t)[a_i(\bar{\rho}^*(t), t)]u_i(t)$$

If there exists a time interval $[t_1, t_2]$ during which $r_i^* = p^{*T}(t)a_i(\rho^*(t), t) = 0$ then from (29) we have

$$\begin{aligned} (\rho_1^*(t) - \frac{\rho_1^{*2}(t)}{\rho_{\max}}) \left(\frac{p_2^*(t)}{\partial L_2} - \frac{p_1^*(t)}{\partial L_1} \right) &= 0 \\ (\rho_2^*(t) - \frac{\rho_2^{*2}(t)}{\rho_{\max}}) \frac{p_2^*(t)}{\partial L_2} &= 0 \end{aligned} \quad (30)$$

Assuming $\bar{\rho}(t) \neq \rho_{\max}$, if $r_i^*(t) = 0$ is true then from (30) we must have $p_1^*(t) = 0$ and $p_2^*(t) = 0$. Therefore the Hamiltonian $H(\bar{\rho}^*(t), \bar{u}^*(t), \bar{p}^*(t)) = 1$ for all times, which

contradicts the necessary condition (25). Therefore we conclude that there are no singular intervals for our case.

IV. SIMULATION

In this section we discuss the simulation results. From (28) we obtained the form of optimal control as the bang-bang control. Although this reduces admissible control space, we still need to solve (21) and (22) to find $\bar{\rho}^*(t)$, $\bar{p}^*(t)$ and t_f . As already mentioned this is a nonlinear two-point boundary-value problem and can not be solved analytically to get the solution. So we use an iterative numerical technique. The boundary conditions for (21) and (22) are split

$$\begin{aligned} \bar{\rho}^*(t_0) &= \bar{\rho}_0 \\ \bar{\rho}^*(t_f) &\in S(t) \end{aligned}$$

and final time is not known so we use the following numerical scheme [14]

Step1: Choose a time interval $[t_0, T]$ and an initial guess for $\bar{p}^*(t_0)$

Step 2: Using $\bar{p}^*(t_0)$ and $\bar{\rho}^*(t_0)$ evaluate $\bar{u}^*(t_0)$ and solve (21) and (22)

Step 3: At every time interval $t_k \in [t_0, T]$ we check to if see there exists a time t^* such that

$$r_i^* = p^{*T}(t^*)a_i(\rho^*(t^*), t^*) \neq 0$$

$$\bar{\rho}^*(t^*) \in S(t)$$

$$H(\rho^*(t^*), u^*(t^*), p^*(t^*), t^*) = 0$$

If any of these conditions are not satisfied we change the initial guess for $\bar{p}^*(t_0)$ and start over again. If all the conditions are met then solution generated $\bar{\rho}^*(t)$ and $\bar{u}^*(t)$, $t \in [t_0, t^*]$ is the optimal solution and t^* is the optimal time. As can be seen we can have more than one solution to the problem depending upon different values of $\bar{p}^*(t_0)$. We simulate a system for corridor subdivided into 2 sections. The length of each section is 10m. The initial densities in sections 1 and 2 are 40 and 30 people/m respectively. The exit is located at the end of section 2 and there is no flow going in through section 1. We use $\rho_{\max} = 80$ people/m, $v_{1f} = v_{2f} = 5$ m/s each. We choose the final time $T = 9$ seconds. Depending upon the initial guess of $\bar{p}^*(t_0)$ we consider the following three cases.

Case 1: Both inputs zero

The initial guess of $\bar{p}^*(t_0)$ results in both r_1^* and r_2^* as positive. Both control inputs are zero. The densities in both sections remain constant at their initial values of 40

people/m and 30 people/m as shown in Figure 2. This control law does not satisfy the conditions for optimality; therefore optimal control does not exist.

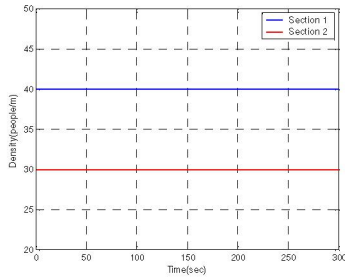


Fig. 2. No control in both sections: No Evacuation: Density plots for sections 1 and 2

Case 2: Input 1 maximum and Input 2 zero

The initial value of $\bar{p}^*(t_0)$ is chosen such that r_1^* is negative and r_2^* is positive. The control input for section 1 is maximum throughout the time interval while as control for section 2 is zero. As a result the density in section 1 is zero after $t = 6$ seconds but there is no flow in section 2 as can be seen from Figure 3. Since the exit is through section 2 the density there has reached the value of 70 people/m. Section 1 is evacuated but all the people are now in section 2. Eventually there will be a jam in section 2. Thus we see that the optimal control does not exist and there is no evacuation at all.

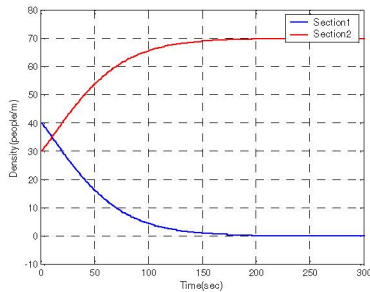


Fig. 3. No Control in section 2: Jam: Density plots for sections 1 and 2

Case 3: Both inputs maximum

For this case we choose two different sets of initial conditions $\bar{p}^*(t_0)$. Both sets result in r_1^* and r_2^* as being negative.

Case 3 (i): The control input for section 2 is maximum throughout while as control for section 1 is maximum for some time and then switches to zero. Although there is a switching but it does not affect the flow as it occurs at time at which section 1 has already been evacuated. The densities in both sections has reached zero at time $t_1^* = 7$ seconds as can be seen from Figure 4(a). We have complete evacuation in time t_1^* . This control is optimal as it satisfies all the necessary conditions. Thus we have an existence of optimal

control. To see that it is not a global minimum or a unique solution we consider another value for initial condition $\bar{p}^*(t_0)$

Case 3(ii): This case also has both the inputs maximum throughout the entire time interval. The density plots are shown in Figure 4(b). Here also we have time-optimal solution $t_2^* = 4.5$ seconds with $t_2^* < t_1^*$. Thus we see that for time-optimal control problem for certain initial values of $\bar{p}^*(t_0)$ optimal control does not exist. If it exists it is the maximum effort during the time interval of operation. Moreover there is more than one solution. Therefore, global minimum can not be found.

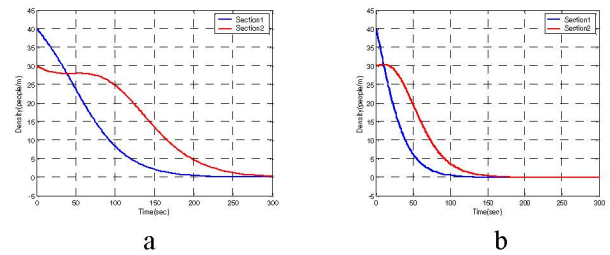


Fig. 4. Control in both sections: Time-Optimal Evacuation (a) Density plots for case (i), (b) Density plots for case (ii)

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