

AN ANALYSIS OF THE DISADVANTAGE TO PLAYERS OF MULTIPLE DECKS IN THE GAME OF 21

A Camouflaged Betting Strategy

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Summary of the Paper

- In this presentation, we will quantify the sad experiences of blackjack players who are forced by casinos to play games with multiple decks of 52-deck cards.
- The graphically displayed results imply a betting strategy which provides up to 2.6 greater expected winnings than would result from placing the same bet on every hand.

OUTLINE

- 1. The Elements of Style: Camouflage**
2. Card Counting and Countermeasures
3. Analysis: Use of Central Limit Theorem
4. Results: Golden Diagrams
5. Implied Betting Strategy
6. Tips for the Player

LES' FUNDAMENTAL RULE OF BLACKJACK

“The only famous counters are the ex-counters.”

- Most gambling systems are composed of three elements: The system itself, usually mathematical, money management, relating to your betting strategy and tolerance for loss as opposed to drive for large winnings, and camouflaging that you are playing a system. In this talk on blackjack and my later one on roulette, I will emphasize the latter, camouflaging that you are a systems player.
- The importance of camouflage is obvious. What good is it if you can count every card in a 8-deck deck pack if you are barred from playing? Our goal in camouflage is to prevent the casino from detecting you are a systems player. You would like to remain anonymous. In other words, “The only famous counters are the ex-counters.”
- My acronym “ABS” refers to the categories of camouflage: Acting, Betting, and Strategy. Here we introduce a betting camouflage technique.
- As shown in the next slide, the player has to make intelligent choices in camouflaging his or her play.

Effective Implementation of a System Requires Intelligently Designed Camouflage



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Card Counting and Casino Countermeasures (boooooo!)

- In 1962, and the 1966 second edition, mathematician Edward Oakley Thorp published *Beat the Dealer*, a ground-breaking book presenting a system to win at blackjack. This book, based on work by Thorp, in collaboration with Julian Braun and Harvey Dubner, presented computer-based winning systems for blackjack which required keeping track of the cards that have been played. This so-called "card counting" led to blackjack becoming the overwhelmingly most popular casino game.
- Casinos reacted to the advantage that a card counter gains over the house by changing some of the rules of the game and by adopting counter strategies. These included employing multiple decks rather than the single hand-held deck. Two-deck games and games employing four and six decks dealt from a so-called shoe became commonplace. Today eight-deck games are common.

The Disadvantages to Players of Multiple Decks

Players soon discovered that this use of multiple decks reduced their “expectation values,” how much they would expect to earn given they played many hands. As the number of 52-card decks in the “deck pack” increases, players discovered:

- 1) Frequency effect: The deck pack becomes favorable less frequently at all depths into the deck pack,
- 2) Magnitude effect: At any given depth, when the deck pack does become favorable the magnitude of the advantage is not as great,
- 3) Depth effect: All deck packs are favorable infrequently until a significant portion of the deck has been dealt and this occurs at greater depths the greater the number of 52-card decks in the deck pack.

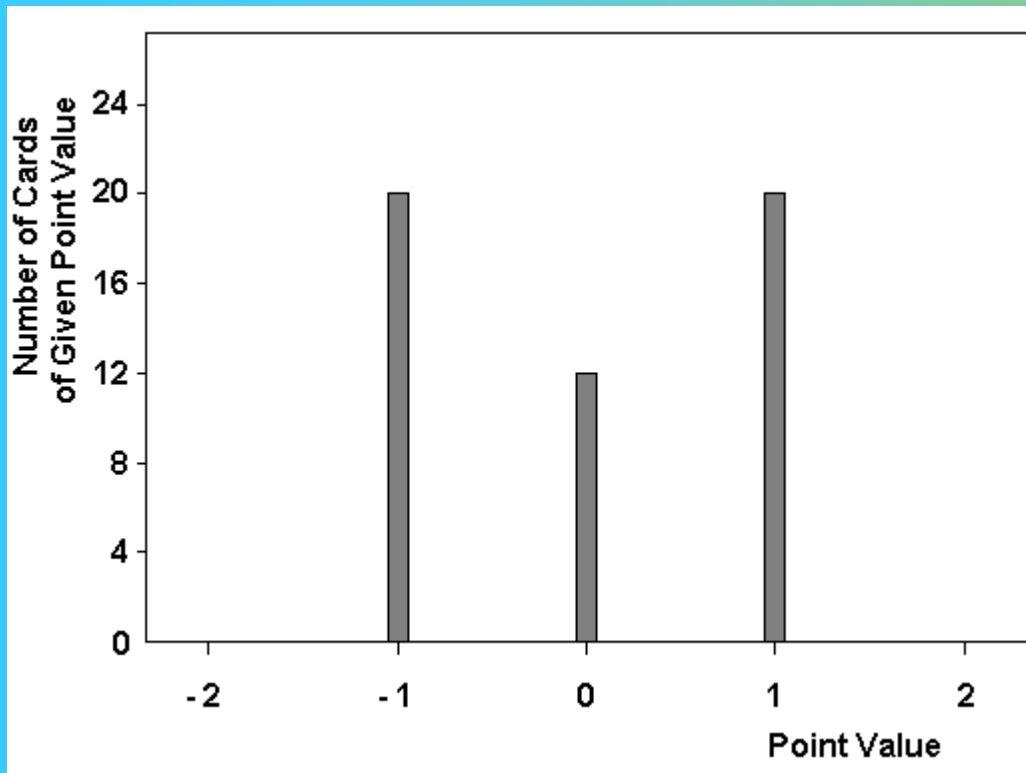
As both a player and student of the game, I was interested in determining the severity of these effects. My results, while confirming and quantifying these intuitive findings, indicated a new strategy to help players camouflage their play. That is the reason for presenting these findings today. They are clearly displayed graphically.

Thorp's Complete Point Count System

- I will throughout refer to the complete point system of Thorp, also referred to as "high-low." In it, the cards are assigned point values similar to the Goren point count in bridge.
- The computer study by Thorp and his collaborators showed that removing the 2's, 3's, 4's, 5's, and 6's from the deck pack was advantageous to the player. Removing the Aces and ten-value cards, including the face cards, was disadvantageous to the player. Removing the 7's, 8's, and 9's had little relative effect.
- To reflect these computer-based judgments, when a 2 through 6 is played the total point count is increased by +1. It's good to have those cards removed. When an Ace or ten-value card is played, the total point count is decreased by 1; they have a point value of -1.

[2's through 6's: add +1 as card is played and viewed
7's, 8's, and 9's: add 0 as card is played and viewed
10's, face cards, Aces: add -1 as card is played & viewed]

Parent Population of Card Point Values in Thorp's Complete Point Count System



The graph shows the distribution of point values in a 52-card deck. Twenty cards have a point value of -1, twenty have a point value of +1, and twelve have a point value of zero. This, from a probability analysis standpoint, is the parent distribution for a 52-card deck. For a deck pack of m such 52-card decks, the values are each multiplied by m .

Keeping the Count

- As the deck is played, the “total point count” is calculated as the arithmetic sum of the point values of the cards that have been played. If, for example, the cards played are a 2, 3, 5, Ace, Queen, and 7, the total point count would be $+3 -2 + 0 = +1$. The total point count changes as every card of non-zero point value is played from the deck pack.
- Although a total point count of, say, +5, indicates an advantage to the player, it's obvious that the advantage is greater the fewer the number of cards remaining to be played. To account for this, Thorp introduced the high-low index (next slides).
- As the player proceeds in the game, the changing value of the high-low index is the parameter guiding all player strategies -- splitting, doubling down, taking or standing, and taking insurance. A decision whether to stand or take a card holding a hand of 12 against a dealer's up-card of 3, for example, may change as play proceeds based on the changing value of the high-low index.
- For two of my favorite examples, the player splits tens against the dealer's up-card of 4 only if the high-low index exceeds +10 and splits tens against the dealer's up-card of 5 only if the high-low index exceeds +6. (In finding such results, Thorp showed that the command to “never split tens” is not correct.)
- The high-low index also provides a guide to the bet size.

Thorp's "High-Low Index"

First: Definition of Terms

- Let N_p be the number of cards in the parent population:
$$N_p = 52m,$$
where m is the number of 52-card decks in the deck pack.
- Let n be the number of cards played. This will be equal to the sample size in our subsequent analysis.
- Let f be the fraction of the deck pack that has been played, representing the "depth." Then,
$$f = n/N_p$$

The High-Low Index

Definition:

$$I = \frac{100 c}{N_p - n}$$

Equals the total point count, c , divided by the number of cards remaining in the deck pack, multiplied by 100 to provide integer values.

Since $f = n/N_p$:

$$I = \frac{100 c}{N_p (1 - f)}$$

The index is now given in terms of the fraction of the deck pack that has been played: The “depth” into the deck pack.

This yields:

$$c(I) = \frac{I N_p (1 - f)}{100}$$

Provide a value of high-low index, and this equation provides the associated value of the total point count.

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The Central Limit Theorem – 1. Math

The analysis is based on the powerful central limit theorem of probability and statistics, which can be stated as: “The distribution of the means of repeated samples of a random variable x taken from a parent population with mean \bar{x} and standard deviation σ is a normal distribution with mean \bar{x} and standard deviation σ/\sqrt{n} , where n is the size of the sample and n is ‘large.’ ” For a deck pack of m 52-card decks, the standard deviation of the parent population using the point values as assigned in the Complete Point Count system is easily shown to be

$$\sigma = \sqrt{\frac{40m}{52m-1}} \quad .$$

The mean is zero. By the CLT, the normal distribution of the means then has a mean of zero and a standard deviation given by

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \quad . \quad (\text{called “the standard error of the mean”})$$

The Central Limit Theorem – 2. Applied

- Example: The parent population for a single deck is the set of twenty -1 point value cards, 12 zero point value cards, and twenty +1 point value cards. Say we randomly select ten cards from this set and calculate the total point count. To find the mean value, we'd divide by ten. We'd do this repeatedly and then calculate the standard deviation among the means.
- We want to consider, however, not the mean value of the total point counts, but the total point count itself, the basis of the high-low index. The CLT applies equally well to such a sum, the standard deviation of this distribution being related to that of the distribution of the means simply by the sample size, the factor n , by,

$$\sigma_c = n \sigma_{\bar{x}} = \sqrt{n} \sigma .$$

The Central Limit Theorem – 3. Fudged

The CLT assumes that the sample size is much greater than unity and that the sample size is also a small fraction of the parent population, that is, $1 \ll n \ll N_p$. In the present use, both these conditions fail in some instances and the well-known “finite population correction,” α , to the calculation of the standard deviation is uniformly applied. Thus

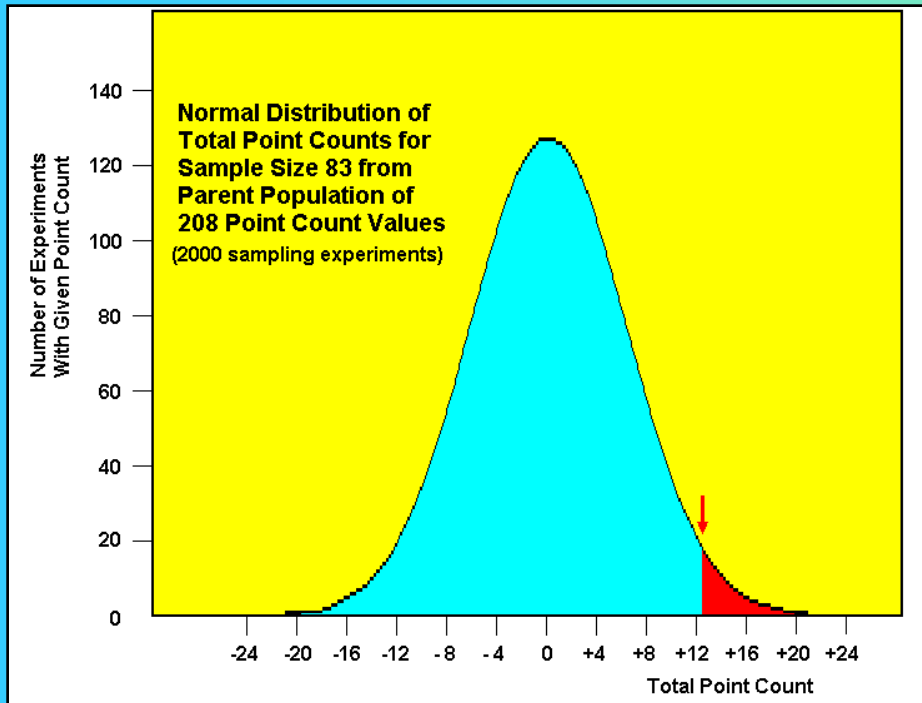
where

$$\sigma_c = \sqrt{\frac{n}{\alpha}} \sigma \quad ,$$
$$\alpha = \frac{N_p - 1}{N_p - n} \quad . \quad \text{Note that } \alpha \sim 1 \text{ for large } N_p.$$

For the mathematicians: In terms of the two parameters of the study, the number of cards in the deck pack, $N_p = 52$, and the depth into which the deck pack has been dealt, f , the expression used to calculate the value of σ can be shown to be

$$\sigma_c = N_p \left[\frac{1}{N_p - 1} \right]^{1/2} [f(1-f)]^{1/2} \sigma \quad .$$

The Normal Distribution of Total Point Counts: Example



*For a 40% depth into a deck pack of four 52-card decks, the value of the total point count at which the high-low index I becomes equal to +10 is

$$c = I(N_p - n)/100$$

$$c = (+10) \times (208 - 83)/100 = +12.5.$$

Illustrative Example

The graph displays the theoretical normal distribution, the familiar bell-shaped curve, of the total point count for a depth of 40% into a deck pack of four 52-card decks. The size of the parent population is $N_p = 4 \times 52 = 208$. The number of elements in the sample is, to the nearest integer,

$$\begin{aligned} n &= 0.4 N_p \\ &= 0.4 \times 208 = 83. \end{aligned}$$

The arrow indicates the total point count corresponding to a value of high-low index of $I = +10^*$. The shaded region to the right of the arrow has an area of 0.024 of the total area under the curve. This tells us that at this depth with these number of decks in the deck pack only a 2.4% probability exists that the high-low index will equal or exceed 10%.

The Calculations

The analysis is performed for two values of the high-low index of particular interest to card-counters, +10 and +6.

The area under the curve to the right of the total point count corresponding to a high-low index of $I = +10$ and $I = +6$ is calculated* for depths of 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80% and 90% into deck packs of 1, 2, 3, 4, 5, 6, 7, and 8 52-card decks. The resulting 9 data points for each of the 8 deck packs are connected with a smooth curve.

Because casinos detect card counters by observing varying bet sizes, those card counters wishing to escape detection often limit their maximum bet to 5 units of their minimum bet. In the complete point count system, the bet size is one-half the high-low index, if it is equal to or exceeds +2. This provides one motivation for the interest in a high-low index value of $I = +10$. A high-low index of $I \geq +10$ also resolves a potentially lucrative strategy decision, directing the player to split ten-value cards against the dealer's up-card of 4

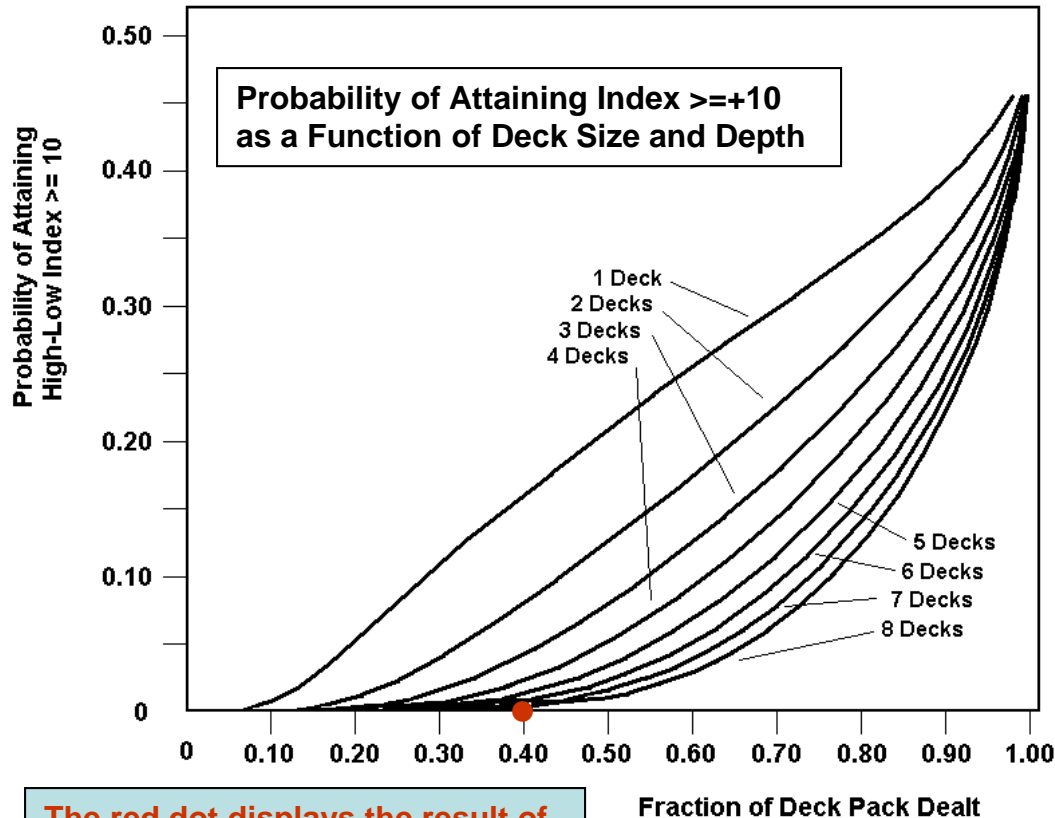
A high-low index of $I = +6$ is of interest because it resolves several difficult strategy decisions. These include standing or taking a card while holding a total of 16 against the dealer's up-card of 9, standing or taking a card while holding a total of 12 against the dealer's up-card of 3, doubling down while holding a total of 10 against the dealer's up-card of Ace, and splitting ten-value cards against the dealer's up-card of 5. The results are shown in the next slides.

***Mathematics note: The areas are calculated in terms of the "error function" by using the Winitzki approximation.**

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Results: Probability that Index $\geq +10$

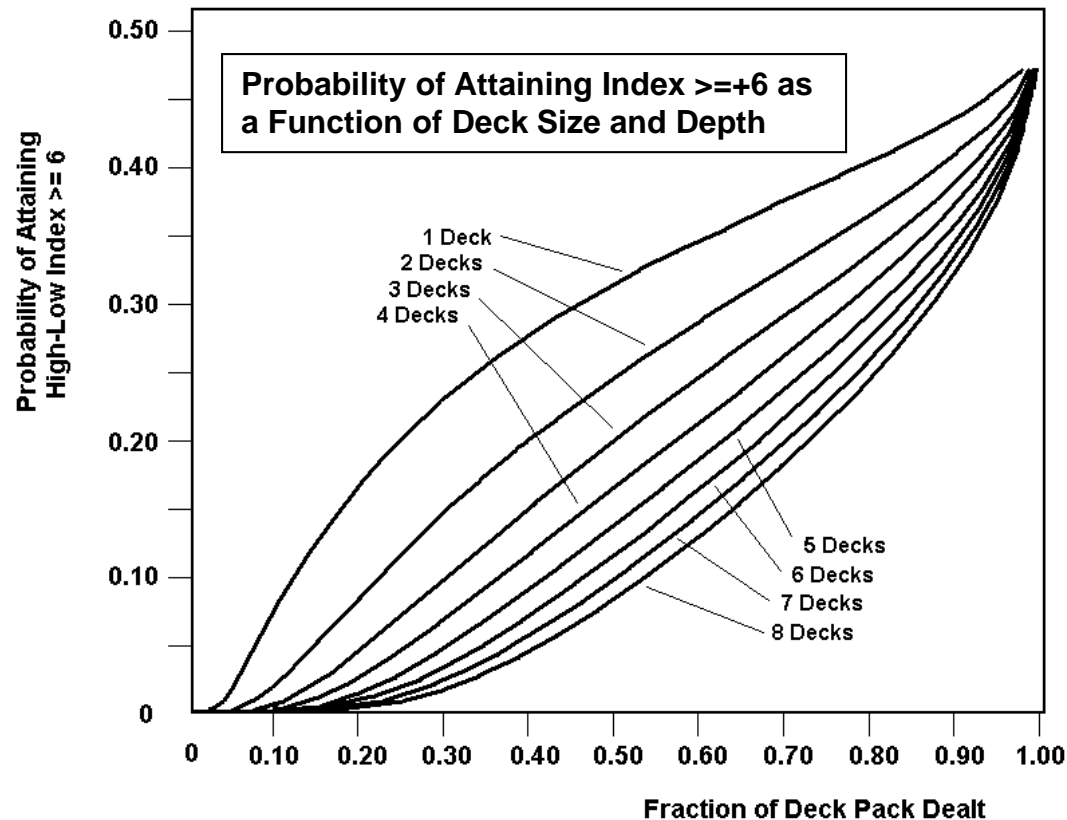


The red dot displays the result of the illustrative calculation, 2.4% probability to equal or exceed an index $\geq +10$.

We see the three effects displayed.

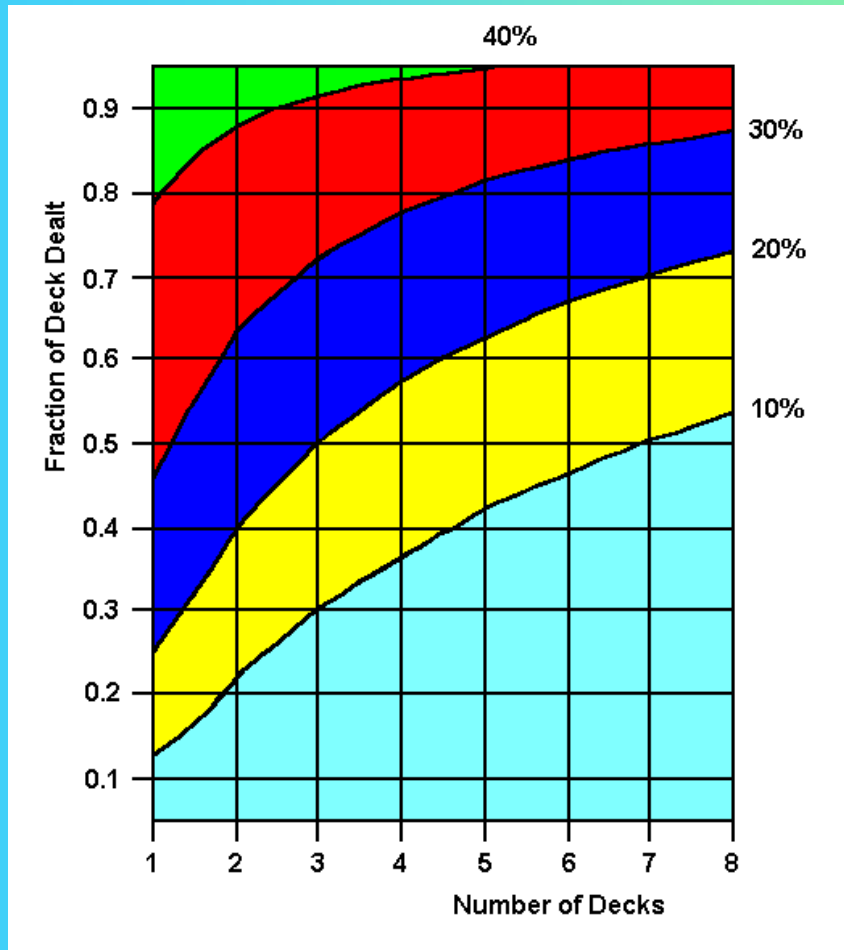
1. **Frequency effect:** a curve for a deck pack of a given number of 52-card decks lies entirely above those of larger number of decks in the deck pack. This means that larger deck packs become favorable less frequently than smaller deck packs at all depths.
2. **Magnitude effect:** At a given depth (imagine a vertical line drawn upwards at, for example, 0.50 depth), the probability is smaller the greater the number of decks in the deck pack.
3. **Depth effect:** All deck packs are favorable infrequently until a significant portion of the deck has been dealt. To achieve a given probability (imagine a horizontal line drawn to the right at, for example, the 0.20 probability level), the required depth is greater the larger the number of decks in the deck pack.

Results: Probability that Index $\geq +6$



The same effects are evident when the criterion is $I \geq +6$, a situation in which the deck pack is favorable, but not as favorable as $I \geq +10$. As expected, attaining these lower values of positive high-low index occurs more frequently; hence, these curves lie above those for $I \geq +10$. We can see this by quickly going back and forth between this and the previous slide.

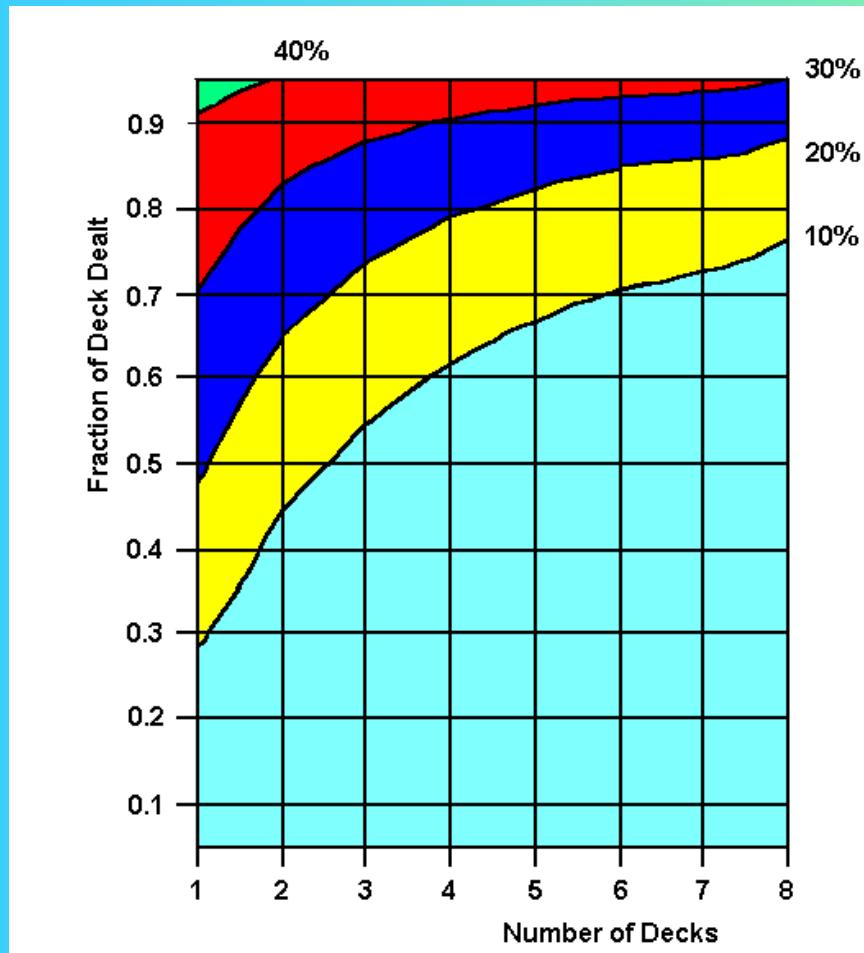
Golden Diagram for Index $I \geq +6$



TITLE: The probabilities that the high-low index will equal or exceed a value of +6 as a function of depth and number of decks in the deck pack.

The "Golden Diagram" presents the same information in a form more easily utilized by the player. The contour curves are isograms of equal levels of probability. The abscissa is the number of 52-card decks in the deck pack.

Golden Diagram for Index $I \geq +10$



TITLE: The probabilities that the high-low index will equal or exceed a value of +10 as a function of depth and number of decks in the deck pack.

The diagram can be constructed for any value of the high-low Index desired. The analysis can be applied to any counting system, which differ by the point values assigned to the cards.

Point Values in Other Systems

THE MOST POPULAR SYSTEMS, IN ORDER OF COMPLEXITY

Used in this analysis ►

Name	Acronym	2's	3's	4's	5's	6's	7's	8's	9's	10's	Aces	Developer
Complete Point Count	High-Low	1	1	1	1	1	0	0	0	-1	-1	Edward O. Thorp
Revere Plus-Minus	Revere +/-	1	1	1	1	1	0	0	-1	-1	0	Lawrence Revere
Knock-Out	K-O	1	1	1	1	1	1	0	0	-1	-1	Vancura/Fuchs
Silver Fox		1	1	1	1	1	1	0	-1	-1	-1	Ralph Stricker
Canfield Expert		0	1	1	1	1	1	0	-1	-1	0	Richard Canfield
Hi-Optimum I	Hi-Opt I	0	1	1	1	1	0	0	0	-1	0	Humble/Cooper
Advanced Plus/Minus	Uston APM	0	1	1	1	1	1	0	0	-1	-1	Ken Uston
Keep It Simple 2	KISS 2	0/1	1	1	1	1	0	0	0	-1	0	Fred Renzey
Keep It Simple 3	KISS 3	0/1	1	1	1	1	1	0	0	-1	-1	Fred Renzey
Red Seven		1	1	1	1	1	0/1	0	0	-1	-1	Arnold Snyder

The same analysis can be applied to any card counting system of choice. The relevant normal curve depends only on the mean and standard deviation of the parent population, composed of the point values of the cards in the deck pack. In some systems, the mean value of the parent population is non-zero.

Point Values in Other Systems (cont'd)

Name	Acronym	2's	3's	4's	5's	6's	7's	8's	9's	10's	Aces	Developer
Advanced Omega II		1	1	2	2	2	1	0	-1	-2	0	Bryce Carlson
Canfield Master		1	1	2	2	2	1	0	-1	-2	0	Richard Canfield
Zen Count		1	1	2	2	2	1	0	0	-2	-1	Arnold Snyder
High Optimum II	Hi-Opt II	1	1	2	2	1	1	0	0	-2	0	Humble/Cooper
Unbalanced Zen II	UPZ II	1	2	2	2	2	1	0	0	-2	-1	George C.
Mentor		1	2	2	2	2	1	0	-1	-2	-1	Fred Renzey
Revere Point Count	RPC	1	2	2	2	2	1	0	-1	-2	-2	Lawrence Revere
Advanced Point Count	Uston APC	1	2	2	3	2	2	1	-1	-3	0	Ken Uston
Strongest & Simplest	Uston SS	2	2	2	3	2	1	0	-1	-2	-2	Ken Uston
Revere 14 Count		2	2	3	4	2	1	0	-2	-3	0	Ken Uston
Advanced Point Count	RAPC	2	3	3	4	3	2	0	-1	-3	-4	Lawrence Revere
Wong Halves		0.5	1	1	1.5	1	0.5	0	-0.5	-1	-1	Stanford Wong

More complicated systems can greater discern the effects of playing given cards from the deck. The standard deviations of the parent populations in these cases are larger than for the systems whose point values are only -1, 0, and +1, but the systems can still be analyzed in terms of Golden Diagrams. The colors distinguish between systems of different maximum absolute value of the assigned point values.

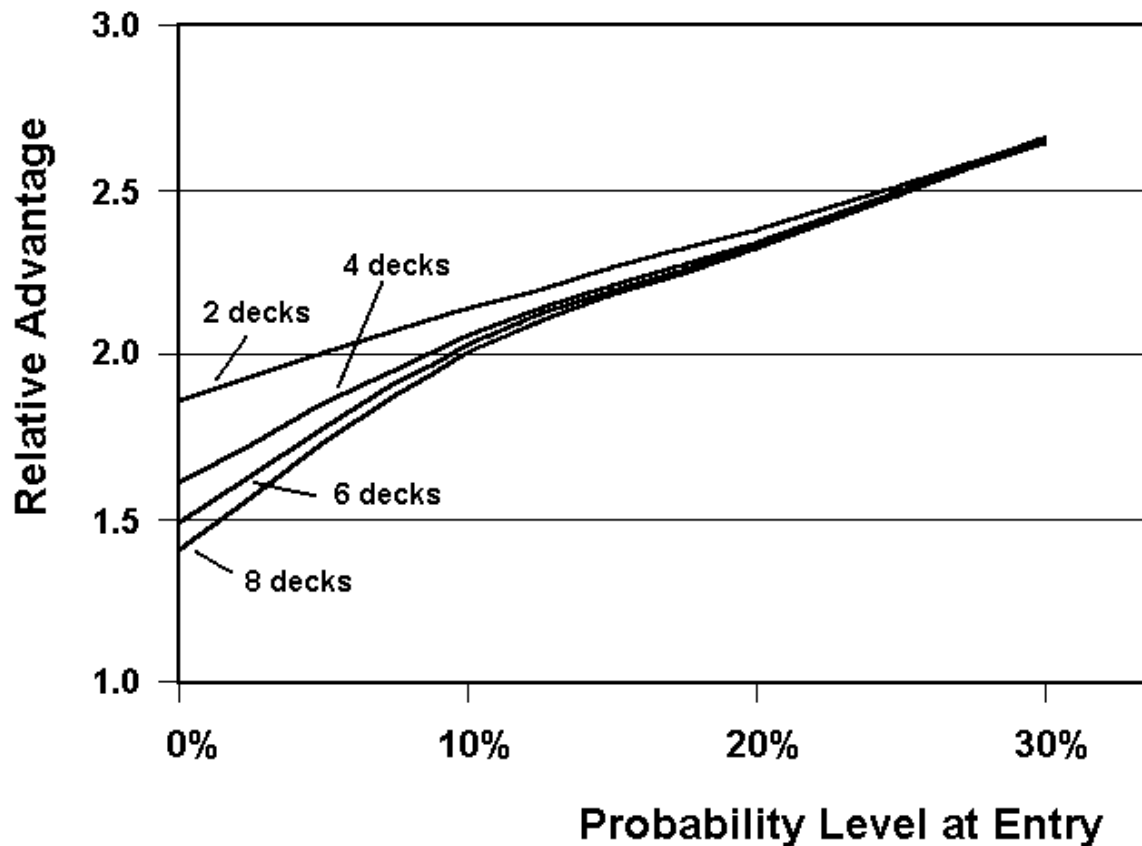
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Implications for Betting

- The results suggest a betting strategy that will increase players' advantages. Card counters increase their bet when the cards remaining in the deck pack provide an advantage to the player and decrease their bet when they favor the dealer. As noted earlier, in the complete point count system the players' bet size is one-half the high-low index, 5 units of their minimum bet for a high-low index value of +10, for example. It is 1 unit if the high-low index is less than +2.
- If the probability of obtaining a high-low index of +10 reaches 10% as play proceeds through a given deck pack, the player, in one possible betting scheme, could increase his or her minimum bet by $0.1 \times 5 \text{ units} = 0.5 \text{ units}$. Similarly, if that probability reaches 20%, 30% or 40%, the player could increase his or her minimum bet by 1.0, 1.5, and 2 units, respectively. With such a step-wise increase in the minimum bet, the expected winnings can be shown to increase markedly over that expected if the bet remains constant.

Step-Wise Betting Strategy Expectation Values



TITLE: The relative increase in expected winnings between a strategy in which the bet size is increased step-wise compared to a strategy in which the bet size remains constant as a function of the probability level at which the player begins play.

The step-wise increases in bet size can yield a factor of up to 2.6 increase in expected winnings!

The Element of Camouflage

Such a betting strategy disguises the player as playing a viable system. Casinos detect card counters by their varying and unorthodox strategy decisions (e.g. splitting tens) and, most easily, by variations in the size of their bets. The step-wise betting pattern will soon be discerned by the casino personnel, but because the player is consistent from deck to deck after shuffles he will likely be discounted as employing a home-made, non-rational, system and will avoid the scrutiny directed to card counters.

The disguise becomes even more effective if the player is a member of a blackjack team whose other members are card counters, varying their strategy decisions and bets.

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Player Tips for Using the Step-Wise Betting Strategy

Several strategies will enable the player to take greatest advantage of the step-wise betting strategy.

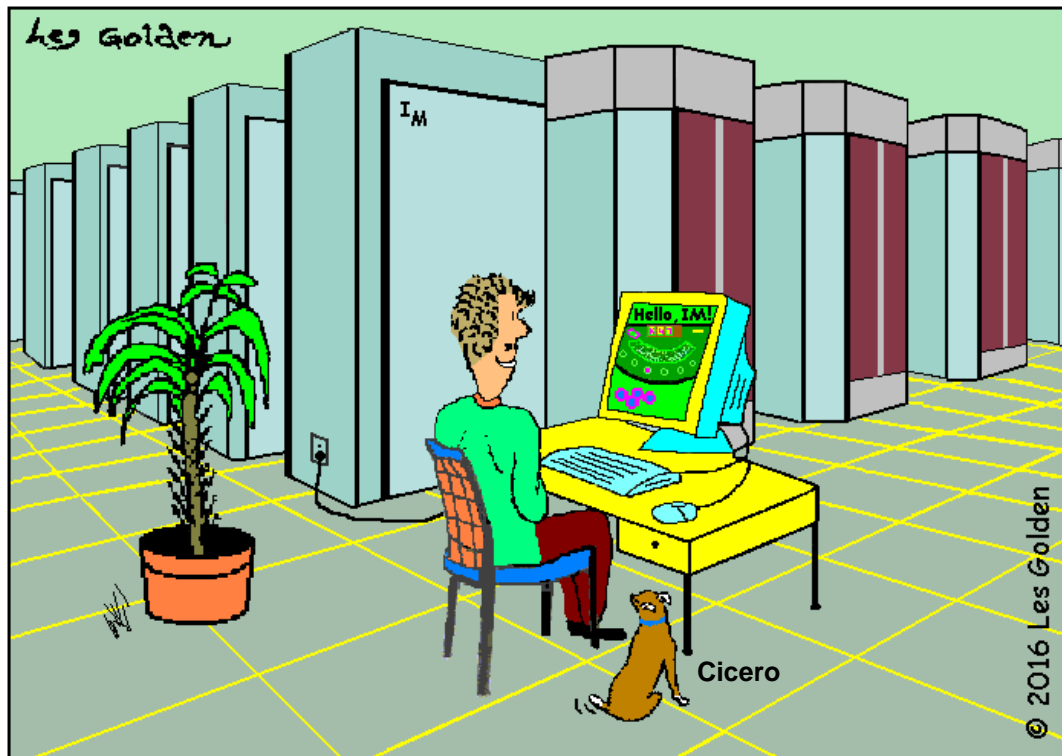
1. The player should observe dealers at various tables at various casinos to determine which deal deeply into the deck pack. The greater the depth the dealer deals before shuffling, the larger the expectation value from the step-wise betting strategy.
2. To maximize the number of hands received as the dealer deals into the deck pack, the player should sit at a table with only a few other players, preferably none.
3. Using the “striking when the deck is hot” technique will not only enable the player to place relatively larger bets but it will also lead to larger expectation values, as shown in the preceding figure.
4. Caution: Varying your playing strategy (splitting, doubling down, standing/drawing, taking insurance) with depth into the deck pack may lead to your detection, defeating the goal of the step-wise betting strategy of providing camouflage.

Our Experience in Actual Play

Although many hours of play are required to determine the relative winnings from employing the Complete Point Count system as opposed to the step-wise betting strategy as suggested by the Golden Diagrams, we have found:

- a) In the short run (hours of play) the former provides greater return. It enables winning even in the early parts of the deck. Being barred, having the deck pack shuffled prematurely, and other casino counter measures, however, are experienced.
- b) In the long run (days of play) the latter provides a greater return. The player is rarely barred and the deck is rarely shuffled early. When the deck pack becomes favorable and the player is sufficiently deep into the deck, the winnings are significant.
- c) Using both depends on your ability to camouflage your varying your strategy by, for example, sometimes purposely making a strategy decision contrary to what the Complete Point Count system indicates. The winnings using both depends highly on your skill at camouflage and will vary from player to player.

The Battle Goes On



**LEGENDARY "ILLINOIS MEDIUM" CAN COUNT
UP TO 273,084,137,314,159,265 DECKS**

Card counters face a continual battle with the casinos to counter the countering of their countered-counter measures. May the future never come when we must microminiaturize massive machines to micron-sized dimension for cerebral insertion.

Illinois Medium is happy to provide copies of slides upon request.
Please allow me to display one additional slide.

We don't own the Earth; We simply share it.



Remember Onion
June 18, 2015

Remember Tyke
August 20, 1994

Remember Harambe
May 28, 2016

Remember Stoney the bull elephant
<http://animalrights.about.com/od/saddestshow/a/StoneyDeath.htm>



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