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Variable Structure Control of Unsteady Nonlinear Aeroelastic System with Partial State Information

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This paper treats the question of control of a two-degree-of-freedom unsteady aeroelastic system with partial state information in the presence of uncertainties. The chosen model describes the plunge and pitch motion of a wing and a single trailing-edge control surface is utilized for the purpose of control. Based on the Lyapunov approach, a variable structure control law is derived. For the control law derivation, the system is treated as the interconnection of two subsystems in which the subsystem associated with the unsteady aerodynamics is input-to-state stable. The stability property of this subsystem is exploited to generate a dominating signal for feedback; thereby, the problem of state estimation of the subsystem describing the unsteady dynamics is avoided. In the closed-loop system, the pitch angle trajectory tracks reference trajectory and the state vector converges to the origin. The designed variable structure control system is simple compared to adaptive controllers and is synthesized easily using only the measured states. Moreover, the structure of the controller is independent of the dimension of subsystem associated with the unsteady aerodynamics. This is important because in literature this subsystem of varying order has been considered. Simulation results are presented which show that in the closed-loop system, regulation of the plunge and pitch trajectories are accomplished in spite of the uncertainties in the freestream velocity and elastic axis location.

Nomenclature

\( A_1, A_2, B_0, N_0, N \) = system matrices
\( A_w, A_f, B = \) system matrices
\( a = \) nondimensionalized distance from the midchord to the elastic axis
\( b = \) semichord of the wing
\( C(k) = \) Theodorsen’s function
\( C_w = \) measurement matrix
\( c_h = \) plunge degree of freedom structural damping coefficient
\( c_\alpha = \) pitch degree of freedom structural damping coefficient
\( h = \) plunge displacement coordinate
\( I_\alpha = \) mass moment of inertia about the elastic axis
\( k = \) reduced frequency \( (b\omega/u_\infty) \)
\( k_h = \) plunge degree of freedom structural spring constant
\( k_\alpha = \) pitch degree of freedom structural spring constant
\( L(t) = \) lift of the wing
\( M(t) = \) moment of the wing about the elastic axis
\( m_w = \) mass of the wing
\( m_t = \) mass of the plunge-pitch system
\( u = \) free stream velocity
\( y_f = \) output variable
\( x_w, x_f = \) state vectors

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The contributions of the paper lies in the derivation of a variable structure control law for the flutter control of an aeroelastic model with uncertain parameters which includes unsteady aerodynamics. The model has both the plunge and pitch structural nonlinearities. The model represents a prototypical aeroelastic wing section which has been traditionally used for the theoretical and experimental study of two-dimensional aeroelastic behavior. A single trailing-edge control surface is used for the control of the system. It is assumed that only the plunge displacement, pitch angle, and control surface deflection and their derivatives are measured for feedback. Based on the Lyapunov approach, a variable structure control system for the trajectory control of the pitch angle is derived. For the derivation of the VSC law, the aerelastic model is treated as the interconnection of two subsystems $S_w$ and $S_f$. The subsystem ($S_w$) describes the pitch, plunge and control surface motion and the subsystem ($S_f$) is associated with the unsteady aerodynamics. Interestingly, the $S_f$-subsystem is shown to be input-to-state stable (ISS). Since only the states associated with $S_w$ are measured and the state variables of $S_f$ cannot be measured, the ISS property of the subsystem $S_f$ is exploited to generate a dominating signal using a first-order dynamic system in the feedback path for the synthesis of the VSC law. This way, the estimation of the state variables of $S_f$, which is a difficult problem for uncertain nonlinear systems, is avoided. Interestingly, the structure of the controller is independent of the dimension of the subsystem $S_f$. This is important because unsteady dynamics are modeled with an approximation to Theodorsen’s theory yielding models for $S_f$ of different orders. Furthermore, the designed controller is attractive from the point of view of simplicity.
in implementation. In the closed-loop system, the pitch angle trajectory control is accomplished and the state vectors of \( S_w \) and \( S_f \) asymptotically converge to the origin in the state space. Simulation results are presented which show that the system responses converge to the origin for uncertainties in the freestream velocity and elastic axis locations.

The organization of the paper is as follows. Section 2 presents the aeroelastic model. Control law is designed in Section 3, and Section 4 presents simulation results.

## II. Aeroelastic Model and Control Problem

The prototypical aeroelastic wing section is shown in Fig. 1. The governing equations of motion are provided in Refs. 15-17 which are given by

\[
\begin{bmatrix}
 m_t & m_w x_\alpha b \\
 m_w x_\alpha b & I_\alpha
\end{bmatrix}
\begin{bmatrix}
 \ddot{h} \\
 \ddot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
 c_h & 0 \\
 0 & c_\alpha
\end{bmatrix}
\begin{bmatrix}
 \dot{h} \\
 \dot{\alpha}
\end{bmatrix}
+ \begin{bmatrix}
 k_h(h) & 0 \\
 0 & k_\alpha(\alpha)
\end{bmatrix}
\begin{bmatrix}
 h \\
 \alpha
\end{bmatrix}
= \begin{bmatrix}
 -L(t) \\
 M(t)
\end{bmatrix}
\]

The lift \( L(t) \) and moment \( M(t) \) represent the unsteady aerodynamics which are functions of position, velocity, acceleration, and time. The lift and moment are acting at the elastic axis of the wing. For purposes of illustration, the function \( k_h(\alpha) \) and \( k_\alpha(h) \) are considered as polynomial nonlinearities of fourth and second degree, respectively. These are given by

\[
\alpha k_\alpha(\alpha) = \alpha(k_{\alpha_0} + k_{\alpha_1} \alpha + k_{\alpha_2} \alpha^2 + k_{\alpha_3} \alpha^3 + k_{\alpha_4} \alpha^4)
\]

\[
h k_h(h) = h(k_{h_0} + k_{h_1} h^2)
\]

where \( \alpha \) is the pitch angle, \( \alpha \) is the control deflection, and \( \beta \) is the angle of attack.

Theodorsen \textsuperscript{24} derived the expressions for lift and moment, assuming harmonic motion of the airfoil, of the form \textsuperscript{15}

\[
-L(t) = -\rho b^2 s_p (u \pi \alpha + \pi \ddot{h} - \pi ba \ddot{\alpha} - u T_4 \beta - T_1 b \ddot{\beta}) - 2 \pi \rho u b s_p C(k)[u \alpha + \dot{h} + b(\frac{1}{2} - a) \dot{\alpha}]
\]

\[
+ (1/\pi)T_{10} u \beta + b(1/2\pi)T_{11} \ddot{\beta}
\]

\[
M(t) = -\rho b^2 s_p \{\pi(\frac{1}{2} - a) u b \ddot{\alpha} + \pi b^2 (\frac{1}{8} + a^2) \ddot{\alpha} + (T_4 + T_{10}) u^2 \beta + [T_1 - T_8 - (c - a) T_4 + \frac{1}{2} T_{11}] u b \ddot{\beta}
\]

\[
- T_{17}(c - a) T_1 b^2 \beta - a \pi b h \} + 2 \mu b^2 \pi s_p (\frac{1}{2} + a) C(k)[u \alpha + \dot{h} + b(\frac{1}{2} - a) \dot{\alpha} + (1/\pi) T_{10} u \beta + b(1/2\pi) T_{11} \ddot{\beta}]
\]

where \( T_i \), \( i = 1, 4, 7, 8, 10, 11 \), are described by Theodorsen depending on the elastic axis location and the control surface hinge location. The Theodorsen’s function \( C(k) \) is a complex function of the form \textsuperscript{15}

\[
C(k) = F(k) + jG(k)
\]

where \( k \) is the reduced frequency \((b \omega/u)\), and \( F(k) \) and \( G(k) \) are composed of Bessel functions. Jones developed an approximation to Theodorsen’s function for simplicity in computation which can be written as \textsuperscript{15}

\[
C(s) = 1 - \frac{0.0165 s}{s + 0.0455} - \frac{0.335 s}{s + 0.3}
\]

\[
= 0.5 \left[ 1 + \frac{a_1 s + a_0}{s^2 + p_1 s + p_0} \right]
\]

where \( s \) is the Laplace variable and \( a_1 = 0.1080075 \frac{u}{b}, a_0 = 0.006825 \frac{u^2}{b^2} \).
The complete dynamics of the unsteady aeroelastic system. Fig. 2 shows a block diagram representation where

\[ \dot{\beta} + b_{\alpha} \dot{\beta} + b_{\alpha} \beta = b_{\alpha} \beta_c \]  

where \( b_{\alpha} = 50 \), \( b_{\alpha} = 2500 \) and \( \beta_c \) is the control input to the aeroelastic model.

It will be convenient to obtain a state variable form of the complete model. The Theodorsen’s function \( C(s) \) can be treated as a second-order transfer function of a filter with input

\[ v_f(t) = [u + \dot{h} + b(0.5 - a) \dot{\alpha} + (1/\pi)T_{10}u \beta + b(1/2\pi)T_{11} \beta] \dot{\pi} \mathbf{a}_w \]  

where the vector \( \mathbf{a}_w \in \mathbb{R}^6 \) is

\[ \mathbf{a}_w = [0, u, 1/\pi T_{10} u, 1, b(5 - a), b(1/2\pi) T_{11}]^T \]  

and the partial state vector is \( x_w = (h, \alpha, \beta, \dot{h}, \dot{\alpha}, \dot{\beta})^T \in \mathbb{R}^6 \). The output of the filter is denoted as \( y_f(t) \) which is related to the input \( v_f(t) \) as

\[ \dot{y}_f(s) = C(s) \dot{v}_f(s) \]  

where \( \dot{y}_f(s) \) and \( \dot{v}_f(s) \) represent Laplace transforms of \( y_f(t) \) and \( v_f(t) \), respectively. We note that the input to the filter \( C(s) \) is a linear combination of the plunge, pitch, and control surface deflection variables.

The transfer function \( C(s) \) of the filter has a minimal realization of dimension 2. Although, one can derive a variety of realizations of \( C(s) \), we consider a representation of the filter of the form

\[ \dot{x}_f = \begin{bmatrix} \dot{h} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -p_0 & -p_1 & 1 \\ -p_1 & -p_0 & 1 \end{bmatrix} \begin{bmatrix} h \\ \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 0 \\ a_v^T \end{bmatrix} \begin{bmatrix} x_f \\ x_w \end{bmatrix} \]  

where \( x_f = (x_{f1}, x_{f2})^T \) and its output is given by

\[ y_f = 0.5 x_f + a_0 x_{f1} + a_1 x_{f2} \]

Substituting \( y_f \) from Eq. (13) in Eq. (4) and Eq. (5) and using Eqs. (1)-(5), a state variable representation for the \( x_w \)-subsystem \( (S_w) \) is obtained which is given by

\[ \frac{d}{dt} \begin{bmatrix} h \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} O_{3 \times 3} & I_{3 \times 3} \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} x_w \\ x_f \end{bmatrix} + \begin{bmatrix} O_{3 \times 2} \\ B_0 \end{bmatrix} \beta_c + \begin{bmatrix} 0_{3 \times 1} \\ N_0 \end{bmatrix} \phi(h, \alpha) \]

\[ \dot{x}_f = A_w x_w + A_f x_f + B \beta_c + N \phi(h, \alpha) \]  

where \( O \) and \( I \) denote null and identity matrices of indicated dimensions; \( A_1, A_w, A_f, B_0, B, N_0 \) and \( N \) are appropriate constant matrices; and the nonlinear function \( \phi(h, \alpha) \) is given by

\[ \phi(h, \alpha) = \begin{bmatrix} h^3 & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 \end{bmatrix}^T \]  

The \( x_f \)-subsystem \( S_f \) (Eqs. 12 and 13) and \( x_w \)-subsystem \( S_w \) (Eq. 14) together represent the complete dynamics of the unsteady aeroelastic system. Fig. 2 shows a block diagram representation in which the \( S_f \) subsystem appears in the feedback path. It is assumed here that the matrices
Suppose that \( \alpha \) makes the sliding manifold attractive. and as such differentiating Eq. (17) gives

\[
\alpha = [0, 1, 0_{p \times 4}]x_w
\]

\[
\dot{\alpha} = C_w x_w
\]

Suppose that \( \alpha_r(t) \) is a given smooth reference pitch angle trajectory converging to zero. We are interested in designing a variable structure control law such that in the closed-loop system, the pitch angle follows the reference trajectory \( \alpha_r(t) \) and the state vector \( (\dot{x}_w^T, \dot{x}_f^T)^T \) converges to the origin as well.

### III. Variable Structure Control

In this section, a variable structure control (VSC) law for the pitch angle trajectory tracking is derived. In variable structure system (VSS), the control law has discontinuity on a chosen hyper-surface (sliding surface) in the state space. The motion of the variable structure system evolves in two phases. In the first phase, the trajectories which begin from initial conditions away from the chosen sliding surface are attracted towards it in a finite time. Then in the second phase (sliding phase), the motion of the system is confined to the hypersurface and the trajectory slides along it. Interestingly, the sliding motion is insensitive to uncertainties in the system parameters.

For the purpose of design, consider a stable sliding manifold

\[
S = \dot{\alpha} + \lambda_1 \dot{\alpha} + \lambda_0 x_s
\]

\[
\dot{x}_s = \dot{\alpha}
\]

where \( \dot{\alpha} = \alpha - \alpha_r \) is the trajectory, \( \lambda_1 \) and \( \lambda_0 \) are positive real numbers. In the sliding phase, \( S(t) \equiv 0 \) and as such differentiating Eq. (17) gives

\[
\ddot{\alpha} + \lambda_1 \dot{\alpha} + \lambda_0 \alpha = 0
\]

which implies that \( (\dot{\alpha}, \ddot{\alpha}) \) tends to zero as \( t \to \infty \). Now it remains to derive a control law which makes the sliding manifold attractive.

Differentiating \( \alpha \) successively along the trajectory of Eq. (14) gives

\[
\dot{\alpha} = C_w A_w x_w
\]

\[
\ddot{\alpha} = C_w A_w (A_w x_w + A_f x_f + B \beta_c + N \phi)
\]

\[
\equiv d(a_w^* + \Delta a_w)^T x_w + (a_f^* + \Delta a_f)^T x_f + \beta_c + (n_\phi^* + \Delta n_\phi)^T \phi
\]

where \( d = C_w A_w B \), \( a_w = a_w^* + \Delta a_w = (C_w A_w^2)^T d^{-1} \), \( a_f = a_f^* + \Delta a_f = (C_w A_w A_f)^T d^{-1} \), and \( n_\phi = (n_\phi^* + \Delta n_\phi) = (C_w A_w \Delta \phi)^T d^{-1} \). Here starred vectors \( a_w^*, a_f^* \), and \( n_\phi^* \) denote nominal values and \( \Delta a_w, \Delta a_f \), and \( \Delta n_\phi \) denote uncertainties in these parameters.

Now the following assumption is made

**Assumption 1**: The scalar parameter \( d \) is unknown, but its sign is known.

Using Eq. (19) the derivative of \( S \) is given by

\[
\dot{S} = d[a_w^T x_w + a_f^T x_f + \beta_c + n_\phi^T \phi] + \lambda_1 \dot{\alpha} + \lambda_0 \alpha - \dot{\alpha}_r
\]

which is a function of the state subvector \( x_f \) associated with the unsteady aerodynamics. For stability, the control law must attenuate the effect of \( x_f \) on the error dynamics. But \( x_f \) is not measurable and cannot be used for feedback. Here instead of obtaining an estimate of \( x_f \), the stability property (to be discussed later) of the subsystem \( S_f \) is exploited to construct a signal for feedback which dominates the unaccessible signal \( x_f \).

First, we introduce the definition of input-to-state stability (ISS) from Sontag\(^{25}\).

**Definition 1 (ISS)**: The system

\[
\dot{q} = g(q, v)
\]
where \( g \) is locally Lipschitz in \( q \in R^n \) and the input \( v \in R^m \), is said to be ISS if for any \( q(0) \) and for any continuous and bounded \( v(t) \) on \([0, \infty)\), the solution exists for all \( t \geq 0 \) and satisfies

\[
||q(t)|| \leq \mu(||q(t_0)||, t - t_0) + \gamma[\sup ||v(\tau)||, t_0 \leq \tau \leq t]
\]  

(22)

for all \( t_0 \) and \( t \) such that \( 0 \leq t_0 \leq t \), where \( \mu(s, p) \) and \( \gamma(s) \) are strictly increasing functions of \( s \in R_+ \) with \( \mu(0,0) = 0, \gamma(0) = 0 \), while \( \mu \) is a decreasing function of \( p \) with \( \lim_{p \to \infty} \mu(s, p) = 0, \forall s \in R_+ \).

Here \( ||.|| \) denotes the Euclidean norm of a vector.

A. ISS Subsystem \( S_f \) and Dynamic Compensator

Indeed the subsystem \( S_f \) (Eq. 12) is ISS with respect to \( x_w \) treated as disturbance input. This can be verified as follows. First we note that \( E_f \) is a Hurwitz matrix for each value of the uncertain parameters \( \varphi_i > 0, (i = 1, 2) \). Therefore, there exists a positive definite symmetric matrix \( P_f \) (denoted as \( P_f > 0 \)) which satisfies the Lyapunov equation

\[
E_f^T P_f + P_f E_f = -I_{2 \times 2}
\]  

(23)

Of course \( P_f \) is a function of \( p_i \), i.e., \( P_f = P_f(p_0, p_1) \). We assume that \( p = (p_0, p_1) \in \Omega_f \), a closed and bounded set. Then

\[
\lambda_2||x_f||^2 \leq \lambda_{\min}(P_f)||x_f||^2 \leq x_f^T P_f x_f \leq \lambda_{\max}(P_f)||x_f||^2 \leq \lambda_1||x_f||^2
\]  

(24)

where \( \lambda_{\min} [\lambda_{\max}] \) denotes minimum [maximum] eigenvalue of \( P_f \), \( \lambda_2 = \inf_{p \in \Omega_f} \{\lambda_{\min}(P_f(p_0, p_1))\} \) and \( \lambda_1 = \sup_{p \in \Omega_f} \{\lambda_{\max}(P_f(p_0, p_1))\} \).

Now for verifying the ISS property of the subsystem \( S_f \), consider a quadratic Lyapunov function

\[
V_f = x_f^T P_f x_f
\]  

(25)

Differentiating \( V_f \) along the solution of Eq. (12) gives

\[
\dot{V}_f = x_f^T E_f^T P_f + P_f E_f x_f + 2x_f^T P_f l_f x_w
\]  

(26)

Using Young’s inequality which states that for all \( (x, y) \in R^2 \)

\[
xy \leq kx^2 + \frac{1}{4k}y^2
\]

(27)

Eq. (26) gives (choosing \( k = \frac{k_f}{2} \))

\[
|\dot{x_f}^T P l_f x_w| \leq ||x_f|| \cdot ||P_f l_f|| \cdot ||x_w|| \leq \frac{k_f}{2}||x_f||^2 + \frac{1}{2k_f}||x_w||^2 \cdot ||P_f||^2
\]  

(27)

where \( k_f \) is a positive real number. Using Eq. (27) in Eq. (26) and choosing \( 0 < k_f < 1 \), one obtain

\[
\dot{V}_f \leq -(1 - k_f)||x_f||^2 + \frac{1}{k_f}||x_w||^2 ||P_f l_f||^2 + \frac{1}{c_1}V_f + \gamma_1||x_w||^2
\]  

(28)

where \( c_1 \leq (1 - k_f)\lambda_M^{-1} \leq \frac{(1 - k_f)}{\lambda_{\max}[P_f(p_0, p_1)]} \) and \( \gamma_1 \geq k_f^{-1} ||P_f l_f||^2 \).

Solving Eq. (28), one finds that

\[
V_f(t) \leq V_f(0)e^{c_1 t} + \frac{\gamma_1}{c_1} (\sup_{\tau \in [0,t]} ||x_w(\tau)||^2)
\]  

(29)

Using Eq. (24) and the inequality \( q_1^2 + q_2^2 \leq (q_1 + q_2)^2 \) for any two positive real numbers \( q_1 \) and \( q_2 \), Eq. (29) gives

\[
||x_f(t)|| \leq (\lambda_M \lambda_{\min}^{-1})^{1/2} ||x_f(0)|| + (\lambda_{\min}^{-1} \gamma_1 c_1^{-1})^{1/2} \sup_{\tau \in [0,t]} (||x_w(\tau)||)
\]  

(30)

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According to Definition 1, it follows that the \( S_f \)-subsystem is ISS with respect to \( x_w \) as input.

The response of \( S_w \) depends on the signal \( x_f \) which is not measured. In Ref. 26, an approach is suggested which is applicable for the stabilization of systems with partial state information without an observer design for state estimation. We adopt here this approach to design the VSC law which avoids feedback of the signal \( x_f \). In view of Eq. (28), it is possible to construct a dominating signal \( \eta(t) \) which is the solution of a first-order dynamic system satisfying

\[
\dot{\eta} = -c_1 \eta + \gamma_1 ||x_w||^2
\]  

(31)

Then it follows that \( \eta(t) \geq V_f(t) \), provided that \( \eta(0) \geq \lambda M ||x_f(0)||^2 \geq V_f(0) \). For such a choice of initial condition \( \eta(0) \), one has

\[
\eta(t) \geq \lambda M ||x_f(t)||^2, t \geq 0
\]  

(32)

In the following derivation, the signal \( \eta(t) \) will be used instead of the unmeasured state \( x_f \) for feedback.

B. VSC Law

Now based on the Lyapunov method, the VSC law design will be completed. Consider a Lyapunov function

\[
V = \frac{S^2}{2|d|}
\]  

(33)

Differentiating \( V \) along the solution of Eq. (20) gives

\[
\dot{V} = sgn(d)S^T \dot{x}_w + a_T^T x + \beta_c + n_T^T \phi + (\lambda_1 \hat{\alpha} + \lambda_0 \tilde{\alpha} - \tilde{\alpha}_r)S|d|^{-1}
\]  

(34)

Using Young’s inequality and Eq. (32), one can establish the following inequalities:

\[
|sgn(d)Sa^T x| \leq |S| ||a|| ||x|| \leq |S| ||a|| (\eta \lambda m^{-1})^{1/2} \leq g_1 |S| |\eta| + \frac{||a||^2}{4g_1} \lambda m^{-1} |S|
\]

\[
(\lambda_1 \hat{\alpha} + \lambda_0 \tilde{\alpha} - \tilde{\alpha}_r)S|d|^{-1} \leq g_2 (\lambda_1 \hat{\alpha} + \lambda_0 \tilde{\alpha} - \tilde{\alpha}_r)^2 |S| + \frac{d^{-1}_i}{4g_2} |S|
\]

\[
|sgn(d)S\Delta a^T x_w| \leq g_3 |S||x_w||^2 + \frac{||\Delta a||^2}{4g_3} |S|
\]

\[
|sgn(d)S\Delta n^T \phi| \leq g_4 |S||\phi||^2 + \frac{||\Delta n||^2}{4g_4} |S|
\]  

(35)

where \( g_i (i = 1, \ldots, 4) \) are positive real numbers.

Substituting Eq. (35) and choosing the control law for eliminating all the unknown functions in Eq. (34), one obtains

\[
\beta_c = -a^T x_w - n^T \phi - G_1 sgn(Sd) - G_2 sgn(d)S - g_1 |\eta| + g_2 (\lambda_1 \hat{\alpha} + \lambda_0 \tilde{\alpha} - \tilde{\alpha}_r)^2 + g_3 |x_w|^2 + g_4 ||\phi||^2 (sgn(Sd))
\]  

(36)

Substituting the control law Eq. (36) in Eq. (34) gives

\[
\dot{V} \leq -G_1 |S| - G_2 S^2 + \frac{|S|^2}{4} ||a||^2 (g_1 \lambda m)^{-1} + (g_2 d_1^2)^{-1} + g_3^{-1} ||\Delta a||^2 + g_4^{-1} ||\Delta n||^2
\]

\[
\leq -G_1 |S| - G_2 S^2 + \mu^* |S|
\]  

(37)

where \( \mu^* \geq ||a||^2 (g_1 \lambda m)^{-1} + (g_2 d_1^2)^{-1} + g_3^{-1} ||\Delta a||^2 + g_4^{-1} ||\Delta n||^2 \).

In order to make \( \dot{V} \) negative, one sets the gain \( G_1 \) to

\[
G_1 = \mu^* + G_0
\]  

(38)

where \( G_0 > 0 \). Using Eq. (38) in Eq. (37) gives

\[
\frac{d}{dt} \left( \frac{S^2}{2} \right) \leq |d| [G_0 |S| - G_2 S^2]
\]  

(39)

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According to Eq.(39), the trajectory starting from any initial condition reaches the surface $S = 0$ in a finite time. Subsequently, the trajectory slides along $S = 0$ which according to Eq. (17) implies that $(\alpha, \dot{\alpha}) \to 0$, as $t \to \infty$. Thus pitch angle trajectory control is accomplished.

The complete closed-loop system is shown in Fig. 2. The VSC law includes a first-order dynamic compensator (Eq. 31) in the controller feedback path. Following the control law derivation of this section, it is apparent that the structure of the VSC law is independent of the order of the $S_j$ subsystem and the dominating signal $\eta$ generated by only a first-order system is sufficient for control. We note that the system has relative degree 2 since the second derivative of $\alpha$ depends explicitly on the control input and as such has zero dynamics of dimension 6. For stability in the closed-loop system, the zero dynamics must be stable. Stability of zero dynamics, when the pitch angle is zero, has been examined in several published works\textsuperscript{13,14}. Indeed computing the transfer function of the linearized system relating $\alpha$ and $\beta$, shows that the system is minimum phase for the set of values of the freestream velocities and elastic axis locations considered for simulation in the next section. Of course, if the zero dynamics have unstable equilibrium state, one can modify the output so that the new system is minimum phase and then the design can be completed following the approach of this Section 3.

IV. Simulation Results

In this section, the simulation results are presented. The model parameters are taken from Refs. 15-17 and are collected in the appendix. First the open-loop response is obtained for the chosen initial conditions $x_0(0) = [0.01, 10^6, 0.0, 0.0, 0.0]^T$ and $x_f(0) = [0.0, 0.1]^T$ for $a = 14.25$ (m/sec) and $a = -0.8424$. Fig. 3, shows that after an initial transient, the pitch angle and the plunge displacement trajectories converge to limit cycles. Apparently the uncontrolled system is unstable, and the wing undergoes periodic oscillations.

Now the simulation of the closed-loop system including the VSC law (Eq. (36)) and the dynamic controller (Eq. (31)) is performed. It is noted that the freestream velocity and the elastic axis location are the two key parameters which have significant effect on the responses of the model. Here in order to examine the robustness of the controller, $u$ and $a$ are assumed to be unknown to the designer. For the choice of the nominal values of $u = 25$ (m/sec) and $a = -0.8$, $a_w$ and $\eta^*$ are computed and used in the VSC law Eq. (36). The feedback gains are selected as $G_1 = 2.1$, $G_2 = 0.02$, $g_1 = 0.0912$, $g_2 = 0.0912$, $g_3 = 0.0912$ and $g_4 = 0.0912$. The parameters of the sliding manifold are set to $\lambda_1 = 6$, and $\lambda_0 = 9$. The parameters used in Eq. 31 for generating the dominating signal $\eta$ are selected as $c_1 = 0.0536$ and $\gamma_1 = 0.0628$. For simplicity the reference pitch angle trajectory is assumed to be zero. It is pointed out that the feedback gains satisfying the inequalities of the previous section are only sufficient for stability in the closed-loop system. Therefore, here these controller parameters have been selected after carrying out several simulations and by observing the simulated responses. Since this discontinuous control law (Eq. (36)) can cause control chattering, a smooth approximation of the signum function by a saturation function with a boundary layer thickness of $\epsilon = 0.1$ is used. In order to limit the control surface deflection $\beta$, simulation is done by clamping the magnitude of the control input $\beta$ to a maximum value of $30^\circ$. It has been observed that the VSC law designed for a higher nominal value of the freestream velocity gives improved responses; therefore, here off-nominal lower values of $u$ are considered for simulation.

A. VSC control for the off-nominal values $u = 20$ (m/sec) and $a = -0.7$

In order examine the sensitivity of the controller to parameter uncertainties, the closed-loop system is simulated for the off-nominal values of $u = 20$ (m/sec) and $a = -0.7$. It may be noted that the controller’s nominal matrices ($a_w^*$, $n_w^*$) in Eq. (36) computed for the nominal values of $u = 25$ (m/sec) and $a = -0.8$ are retained for simulation. For the selected off-nominal values of the freestream velocity and $a$, it is found that there exists limit cycle (not shown here). The open-loop system has two unstable poles (0.1393 + 13.5466i and 0.1393 - 13.5466i), but the linearized system has stable zeros (minimum phase zero dynamics). The closed-loop responses are shown in Fig. 4. The VSC law accomplishes smooth regulation of the pitch angle to zero, but the plunge motion is oscillatory and converges to zero after the initial transient. This oscillatory response of the plunge displacement is attributed to the zero dynamics. It is observed that the remaining components of the state vector
including the state $x_f$ of the sub-system $S_f$ also converge to the origin. The response time is of the order of 4 seconds. The maximum control surface deflection $\beta$ is less than 25° and the control input $\beta_c$ saturates during the segment of the transient period. It is seen that constraining the input $\beta_c$ provides flexibility in limiting the magnitude of the control surface deflection. It is noted that the designed variable structure control system is attractive from the point of simplicity in implementation compared to adaptive controllers and is synthesized easily using only the measured states.

### B. VSC control for the off-nominal values $u = 16 \text{ (m/sec)}$ and $a = -0.5$

Now simulation is done for a choice of off-nominal parameter values ($u = 16 \text{ (m/sec)}$ and $a = -0.5$). With this choice, the freestream velocity and elastic axis location have an uncertainty of about 36 percent. The initial conditions and controller parameters selected for the nominal system are retained for simulation. The responses are shown in Fig. 5. In the closed-loop system $\alpha$ smoothly converges to zero. Moreover, the state vectors $x_w$ and $x_f$ also asymptotically tend to zero since the zero dynamics are stable (minimum phase). The response time is of the order of less than 5 seconds. In the transient period, $\beta_c$ saturates and the surface deflection remains within 25°.

Extensive simulation has been performed which shows robustness of the VSC law with respect to uncertainties in $u$ and $a$. Moreover the controller has considerable flexibility and one can choose the design parameters ($G_1, G_2, g_1, \ldots g_4, \lambda_1, \lambda_0, c_1$ and $\gamma_1$) properly to obtain desirable responses in spite of uncertainties in the system using only feedback of the measured signals.

### V. Conclusions

In this paper, control of a prototypical aeroelastic wing section with pitch and plunge structural nonlinearities using a single control surface was considered. The model includes unsteady aerodynamics. For the controller synthesis, only the plunge displacement, pitch angle, control surface deflection and their derivatives were measured. Interestingly, the aeroelastic system can be represented as the interconnection of two subsystems. In this representation, the subsystem associated with the unsteady aerodynamics is input-to-state stable and appears in the feedback path. Based on such representation, a variable structure control system was derived. The states associated with the unsteady aerodynamics cannot be measured. Here instead of an observer design (which is a difficult problem for uncertain nonlinear systems), a first order dynamic system was designed for generating a dominating signal for controller synthesis. It is important to note that the structure of the controller does not depend on the dimension of the state space associated with the unsteady aerodynamics. Simulation results were presented which showed that flutter suppression can be achieved for uncertainties in the flow velocities and elastic axis locations. Moreover, the control system provides considerable flexibility in accomplishing robustness in the closed-loop system and in shaping the responses by the choice of the design parameters.

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### VI. References

Appendix

System Parameters:
The system parameters for simulation have been taken from Refs. 12 and 14.
\[ b = 0.135 \text{ m} \]
\[ m_w = 1.662 \text{ kg} \]
\[ c_h = 27.43 \text{ Ns/m} \]
\[ c_\alpha = 0.036 \text{ Ns} \]
\[ \rho = 1.225 \text{ kg/m}^3 \]
\[ I_\alpha = 0.04325 + m_w x_\alpha^2 h^2 \text{kg.m}^2 \]
\[ x_\alpha = [0.0873 - (b + ab)]/b \]
\[ T_1 = -0.0630 \]
\[ T_4 = -0.4104 \]
\[ T_7 = 0.0128 \]
\[ T_8 = 0.0964 \]
\[ T_{10} = 1.6798 \]
\[ T_{11} = 0.8551 \]
\[ k_\alpha = 2.82(1 - 22.1 \alpha + 1315.5 \alpha^2 + 8580 \alpha^3 - 17289.7 \alpha^4) \text{ N.m/rad} \]
\[ k_h = 2844.4 + 255.99 h^2 \text{ N/m} \]
Figure 2. Block diagram of the aeroelastic model using the variable structure controller

Figure 3. Open-loop response: \( u = 14.25 \text{ m/s}, \alpha = -0.8424 \)
Figure 4. Closed-loop response: \( u = 20 \text{ m/s}, \ a = -0.7 \)
Figure 5. Closed-loop response: $u = 16$ m/s, $a = -0.5$