Adaptive Local and Global Synchronization and Phase Control of Inferior Olive Neurons (Ions)

Srujan Kumar Chalike
University of Nevada, Las Vegas, srujanchalike@gmail.com

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ADAPTIVE LOCAL AND GLOBAL SYNCHRONIZATION AND PHASE
CONTROL OF INFERIOR OLIVE NEURONS (IONs)

by

Srujan Kumar Chalike

Bachelor of Technology in Electrical Engineering
Jawaharlal Nehru Technological University, Andhra Pradesh, India
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of the requirements for the

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Department of Electrical and Computer Engineering

Sahjendra N. Singh, Ph.D., Committee Chair

Henry Selvaraj, Ph.D., Committee Member

Venkatesan Muthukumar, Ph.D., Committee Member

Woosoon Yim, Ph.D., Graduate College Representative

Tom Piechota, Ph.D., Interim Vice President for Research & Dean of the Graduate College

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ABSTRACT

Adaptive Local and Global Synchronization and Phase Control of Inferior Olive Neurons (IONs)

by

Srujan Kumar Chalike

Dr. Sahjendra N. Singh, Examination Committee Chair
Professor of Electrical and Computer Engineering Department
University of Nevada, Las Vegas

Clusters of inferior olive neurons (IONs) in the olive-cerebellar system play an important role in providing synchronized motor control signals for the activation of large number of muscles. However, the dynamics of IONs are highly nonlinear and the system parameters are assumed to be unknown. The IONs evolving from arbitrary initial conditions are not synchronized. However the application of IONs for control of BAUV’s requires that IONs oscillate in unison.

The objective is to design control laws such that the controlled ION tracks the trajectories of the reference ION. The two control laws are derived based on tuning functions adaptive method for local and global synchronization. Furthermore, based on $\mathcal{L}_1$ adaptive control theory for local and global synchronization is designed. They are
completed in two steps of a back stepping design process. Using Lyapunov analysis, it is shown that in the closed-loop system, the controlled ION asymptotically tracks the trajectories of the reference ION. Phase control for the synchronization of IONs based on tuning functions and $L_1$ adaptive control method has been studied. Simulation results are presented to evaluate the performance of each control system designed.
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Figure 6.4  Adaptive synchrony of non-identical IONs for $a_1=0.01$, $a_2=0.02$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error. 

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Figure 6.6  Adaptive synchrony of identical IONs with random disturbance inputs $d_1(t)=d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.

Figure 6.7  Adaptive synchrony of identical IONs with sinusoidal disturbance inputs $d_1(t)=d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.

Figure 6.8  Adaptive synchrony of non-identical IONs with random disturbance inputs $d_1(t)=d_2(t)$ for $a_1=0.01$, $a_2=0.02$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.

Figure 6.9  Adaptive synchrony of non-identical IONs with sinusoidal disturbance inputs $d_1(t)=d_2(t)$ for $a_1=0.01$, $a_2=0.02$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.

Figure 6.10 Adaptive synchrony of identical IONs with two distinct disturbance inputs $d_1(t) \neq d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the random disturbance input, (n) shows the sinusoidal disturbance input.
Figure 6.11 Adaptive synchrony of non-identical IONs with two distinct disturbance inputs \( d_1(t) \neq d_2(t) \) for \( a_1=0.01, a_2=0.02 \) (a) \( u_1 \) and \( u_2 \), (b) \( v_1 \) and \( v_2 \), (c) \( z_1 \) and \( z_2 \), (d) \( w_1 \) and \( w_2 \), (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters \( \theta_1, \theta_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \). (j)-(k) shows the estimates of state predictors \( x_1, x_2, x_3, x_4 \), (l) shows the error, (m) shows the random disturbance input, (n) shows the sinusoidal disturbance input.

Figure 6.12 Adaptive synchrony of identical IONs with a relative phase difference between two IONs for \( a_1=a_2=0.01, d(t)=0, \mu=0 \). (a) \( u_1 \) and \( u_2 \), (b) \( v_1 \) and \( v_2 \), (c) \( z_1 \) and \( z_2 \), (d) \( w_1 \) and \( w_2 \), (f) \( u_1d \) and \( u_2 \), (g) \( v_1d \) and \( v_2 \), (h) \( z_1d \) and \( z_2 \), (i) \( w_1d \) and \( w_2 \), (j) shows the control inputs, (k) shows the error.

Figure 6.13 Adaptive synchrony of identical IONs with clamped control magnitude for \( a_1=a_2=0.01 \) (a) \( u_1 \) and \( u_2 \), (b) \( v_1 \) and \( v_2 \), (c) \( z_1 \) and \( z_2 \), (d) \( w_1 \) and \( w_2 \), (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters \( \theta_1, \theta_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \). (j)-(k) shows the estimates of state predictors \( x_1, x_2, x_3, x_4 \), (l) shows the error.
CHAPTER 1

INTRODUCTION

Inferior olive neurons (IONs) are one kind of neurons present in the body. Inferior olivary nucleus is the largest nucleus situated in the olivary body. Neuron is a nerve cell that is the basic building block of the nervous system. Neurons are specialized cells which are electrically excitable, and can use that electric excitability to receive and transmit information and are responsible for communicating information in both chemical and electrical forms [48]. Basic structure of neuron has three parts called as the dendrites, the cell body and the axon [47]. Different types of neurons are responsible for different tasks. For example, sensory neurons carry information from the sensory receptor cells located throughout the body to the brain. Motor neurons transmit information from the brain to the muscles of the body. Interneurons are responsible for communicating information between different neurons in the body. IONs exhibit two types of spatially distinct calcium currents: a) low-threshold current located in the soma and b) high-threshold current located in the dendrites [9]. In order for neurons to communicate, they need to transmit information both within the neuron and from one neuron to the next. This process utilizes both electrical signals as well as chemical messengers.

According to their electrophysiological characteristics, neurons can be classified into:
i) Tonic or regular spiking: Some neurons are typically constantly (or tonically) active.

ii) Phasic or bursting: Some neurons fire in bursts and are called phasic.

iii) Fast spiking: Some neurons are notable for their high firing rates

A neuron receives signals from other neurons at branches called dendrites and at smaller receptor sites on the cell body. It sends signals to other neurons via its axon terminals. In between is the axon, which when conditions are right, generates a signal to its terminals. Many axons also have a myelin sheath, which serves a purpose similar to that of the insulation on an electrical wire. In order to transmit information within neuron and from one neuron to other neuron synchronization plays a major part.

1.1 Literature review

Swimming and flying animals have a very seeming grace while they maneuver [18, 27], they have many control surfaces which seem to be perfectly synchronized resulting in a motion that is free of undesired forces and moments. Inferior olive (IO) neurons project to the cerebellum and contribute to motor control. They can show intriguing spatio-temporal dynamics with rhythmic and synchronized spiking. The synchronous activity of the olivo-cerebellar system in the brain plays an important role in providing motor control signals for a large number of muscles. The olivo-cerebellar system’s neuronal network is organized around clusters of inferior olive neurons [1-2]. The IONs have various dynamical behaviour including the subthreshold activity in which the membrane potential has sustained fluctuations. These subthreshold oscillators have different waveforms (sinusoidal, quasi-periodic, periodic with spikes, bursting and irregular) [6-2]
In view of the important role of the IONs in motor control, considerable effort has been made in the development of dynamical models, using laboratory test data and in the design of control systems for synchronization of IONs. Presently, several studies have been conducted on IONs such as developing biophysical models of the olivary neurons to examine their unique electrophysiological properties, mathematical models of electrically coupled neurons [9] and models of IONs using Vander pol oscillator and Fitzhugh-Nagumo (FN) dynamic systems have been obtained [10, 11]. Later authors developed ION models which provide simplicity in computation and yet are capable of approximating data obtained in laboratory tests closely [12, 13]. Lot of research is being conducted in the field of inferior olive neurons (IONs). In literature, a biology inspired rigid autonomous undersea vehicle called the biorobotic autonomous underwater vehicle (BAUV) has been developed at the Naval Undersea Warfare Center (NUWC), which has been equipped with six simultaneously rolling and pitching fins for generating large unsteady control forces for performing agile maneuvers. Application of IONs for the control of agile biorobotic autonomous undersea vehicle (BAUV) has been explored [14]. These IONs exhibit limit cycle oscillation (LCO). For the control of the BAUV, the neurons must oscillate in synchronism with specific relative phases. It has been shown in laboratory tests that a cluster of these IONs can be used for the control of fins of the BAUV. It has been found that precise control of phase angles of oscillating multiple fins attached to the BAUV is essential for an efficient propulsion and control. However, the dynamics of BAUVs are highly nonlinear and hydrodynamic coefficient are not precisely known. As such development of nonlinear control systems for BAUVs with uncertain
dynamics is of considerable importance.

Recently, phase control techniques of IONs has been considered. Researchers have considered synchronization and phase control of IONs using various approaches. In [12], a pulse type control signal has been utilized for the synchronization of IONs. Nonlinear feedback synchronizing control systems have been proposed in [14] for a cluster of IONs. In [45], Synchronization of inferior olive neurons are controlled by the hyperpolarization-activated cation current. Bifurcation of orbits and synchrony in inferior olive neurons[37] has been considered. Bifurcation analysis of two coupled calcium oscillators[38] has been conducted. Role of gap junctions in synchronized neuronal oscillations in the inferior olive neurons[39] has been investigated. Synchronization of IONs with time-delayed coupling[36] has been studied. Partial contraction analysis for coupled nonlinear oscillators[40] has been done. However for the synthesis of the control laws, it is essential to know the parameters of the IONs. In Practice, these parameters are not known precisely. To overcome this limitation, an adaptive control law for the synchrony of IONs have been designed in [15]. The design is based on the invariance and immersion approach [16].

In recent papers adaptive control laws have been designed for the synchronization of IONs using various approaches. However, from the literature it appears that an in depth study in use of synchronization of IONs for control of BAUVs in the presence of uncertainties in parameters and presence of external disturbances has to be considered. The contribution of this thesis lies in the design of adaptive control laws for the synchronization of IONs based on a) Tuning functions method with single and multiple
control inputs, and b) $\mathcal{L}_1$ adaptive method [42], with single and multiple control inputs with uncertainties in the parameters and presence of external disturbance inputs. It is assumed that parameters of the IONs are not known to the designer. The adaptive feedback is used when the system parameters are unknown. A fourth-order nonlinear model of the ION developed by Kazantsev et al. [12, 13], is considered in this study.

1.2 Thesis Outline

Studies and observations on IONs have provided a great amount of information how the IONs communicate and support potential dynamics depending upon the electrophysiological properties. It is highly desirable to use these information to derive a control law for the synchronous activity in olivo-cerebellar system.

This thesis contributes in designing an adaptive laws using various closed-loop control design methods, being motivated by the natural ability of swimming and flying animals maneuver. The implementation of control systems for synchronization of IONs in spite of uncertainties in parameters and presence of external disturbance inputs is considered. The scope of this research work covers the design of an adaptive laws for the synchronization of IONs using different approaches. The mathematical model of the IONs are presented in Chapter 2.

In chapter 3, an adaptive control law is designed based on tuning functions method for the synchronization of two IONs by using single control input. But this approach can be utilized for the synchrony of a cluster of IONs. But it is not applicable to IONs perturbed by external disturbance inputs. Using Lyapunov analysis, it is shown that
in the closed-loop system the trajectories of the controlled ION asymptotically follow the trajectories of the reference ION. The control system includes an adaption law for generating estimates of the controller parameters. Chapter 4 considers the synchronization of IONs using the tuning functions method similar to chapter 3 but in chapter 4, adaptive control law is designed by using two control inputs.

In chapter 5, an adaptive control law is designed based on $\mathcal{L}_1$ adaptive control theory for the synchronization of two IONs by using single control input. A control law is developed for the synchronization of two IONs using a smooth projection algorithm. This approach is more robust and it can be utilized for the synchrony of a cluster of IONs and also applicable to IONs perturbed by external disturbance inputs. Using Lyapunov analysis, it is shown that in the closed-loop system the trajectories of the controlled ION asymptotically follow the trajectories of the reference ION. The control system includes an adaption law for generating estimates of the unknown parameters. Chapter 6 considers the synchronization of IONs using an $\mathcal{L}_1$ adaptive control law similar to chapter 5, but here in chapter 6, adaptive control law is designed by using two control inputs. These control law is more robust and applicable to systems which have unmodelled dynamics and are perturbed by any external disturbance inputs.

Furthermore, in the closed-loop system, the relative-phase of the two IONs is controlled. Simulation results are presented which show that the adaptive law accomplished synchronization (the follower ION$_2$ tracks the reference ION$_1$), despite large uncertainties and external disturbance inputs in the ION parameters.
The mathematical model of IONs used in chapters 3, 4, 5 and 6 is presented in this chapter. Even though, the technique applied for the design of control law is different for each chapter, the model of IONs used remain the same for every chapter. These model generates oscillations by appropriate choice of parameters. The robust subthreshold oscillations of each unit emerge from Andronov-Hopf bifurcation, in the first subthreshold state. The oscillatory signal goes to second (suprathreshold) state, which hovers up and down relative to action potential threshold. When reaching threshold at the peak of a subthreshold oscillation, the unit generates a spike. The timing of the spiking is thus determined by the subthreshold oscillations. Depending on the values of the control parameters, the model qualitatively reproduces the spontaneous and stimuli-induced oscillations observed in IONs. These properties can be described by a mathematical model comprising a set of four nonlinear differential equations [12, 13]. The model essentially consists of two interconnected subsystems ($u$-$v$ and $z$-$w$ subsystems) We have two IONs in the model

1) a reference ION denoted as ION$_1$, and

2) a follower ION denoted as ION$_2$. 
The model of IONs are fourth order non-linear differential equation with 4 state variables each.

The mathematical model can be described as follows:

The reference ION (denoted as ION$_1$) is governed by

\[
\dot{u}_1 = \frac{k}{\epsilon_{Na}}[u_1^2 - u_1^3 + (u_1^2 - u_1)a_1] - \frac{k}{\epsilon_{Na}}v_1
\]

\[
\dot{v}_1 = k[u_1 - z_1 + I_{Ca} - I_{Na}]
\]

\[
\dot{z}_1 = z_1^2 - z_1^3 + (z_1^2 - z_1)a_1 - w_1
\]

\[
\dot{w}_1 = \epsilon_{Ca}(z_1 - I_{Ca} - \mu)
\]

The follower ION (denoted as ION$_2$) is governed by

\[
\dot{u}_2 = \frac{k}{\epsilon_{Na}}[(u_2^2 - u_2^3) + (u_2^2 - u_2)a_2] - \frac{k}{\epsilon_{Na}}v_2
\]

\[
\dot{v}_2 = k[u_2 - z_2 + I_{Ca} - I_{Na}] + uc_1 + d_1(t)
\]

\[
\dot{z}_2 = z_2^2 - z_2^3 + (z_2^2 - z_2)a_2 - w_2
\]

\[
\dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca} - \mu) + \epsilon_{Ca}uc_2 + d_2(t)
\]

where $x_a = (u_1, v_1, z_1, w_1)^T \in R^4$ and $x_b = (u_2, v_2, z_2, w_2)^T \in R^4$ are the state vectors associated with IONs. The ION$_2$ is controlled by the application of the control input. The ION$_1$ has no input and generates reference trajectories. The variables $z_i$ and $w_i$ are responsible for subthreshold oscillations and low-threshold ($Ca^{2+}$-dependent) spiking, and the variables $u_i$ and $v_i$ describe the higher threshold ($Na^+$-dependant) spiking. The oscillation time scales are controlled by the parameters $\epsilon_{Ca}$ and $\epsilon_{Na}$, and $I_{Ca}$ and
\( I_{Na} \) drive the depolarization levels (equilibrium points) of the system. The parameter \( k \) sets the relative time scale of the two systems. The parameter \( a_i \) (appearing in the nonlinear functions) plays an important role in shaping the trajectories of the IONs. \( uc_1, uc_2 \) are the control inputs. \( d_1(t), d_2(t) \) are the external disturbance inputs.

When there are no control inputs acting on the model \((uc_1=uc_2=0)\) then the follower ION (ION\(_2\)) will not be synchronizing with the reference ION (ION\(_1\)). This scenario is called open-loop behaviour. We simulated the two IONs with some initial conditions for the state vectors associated with the IONs. We assume that there are no control inputs and no external disturbance inputs acting on the model i.e., \( uc_1=uc_2=0, d_1(t)=d_2(t)=0 \). For a particular choice of parameters \( \epsilon_{Ca}=0.02, \epsilon_{Na}=0.001, I_{Ca}=0.01, I_{Na}=-0.59, a_1=a_2=0.01 \). The simulations results are shown in the Fig.1. From the simulation results it can be observed that the plots of \( u_1, u_2, v_1, v_2, z_1, z_2, w_1, w_2 \) starting with some arbitrary initial conditions with respect to time are not synchronized, it can be observed that after some initial transient time the trajectories of IONs are periodic and there is a phase difference between two IONs. Phase plots \( u_1 \) vs \( v_1 \), \( u_2 \) vs \( v_2 \) and \( z_1 \) vs \( w_1 \), \( z_2 \) vs \( w_2 \) have been obtained, from the figure it can observed that the plots which are periodic, overlap each other with a phase difference between them, which are known as limit cycle oscillations(LCO). Thus, one must develop a control law so that the IONs oscillate in unison. The IONs are said to be synchronized if the trajectories of the IONs satisfy \( x_a(t) = x_b(t) \) for all \( t \). That is, the trajectories of IONs overlap.

We are interested in design of an adaptive synchronizing control laws such that the controlled trajectory \( x_a(t) \) of the ION\(_1\) asymptotically converges to the solution of \( x_b(t) \).
beginning from arbitrary initial conditions, despite the uncertainties in the parameters.

In chapter 3 and 4 the control laws for adaptive synchrony are designed based on tuning functions backstepping approach [17]. In chapters 5 and 6 the control laws are designed based on the $\mathcal{L}_1$ adaptive control theory [42], with the assumption that non-linear functions of IONs are unstructured (unmodelled) and ION$_2$ is perturbed by some external disturbances ($d_1(t), d_2(t) \neq 0$).
Figure 2.1: Open-loop response of two IONs with $u_{c1} = u_{c2} = 0$
CHAPTER 3

TUNING FUNCTIONS BASED ADAPTIVE LOCAL SYNCHRONY OF
SINGLE-INPUT INFERIOR OLIVE NEURONS

The contribution of this chapter lies in the design of an adaptive control law for the synchronization of IONs using a single control input. It is assumed that parameters of the IONs are not known to the designer. A fourth-order nonlinear model of the ION developed by Kazantsev et al. [12,13], is considered in this study. The design is based on the tuning functions method. A control law is developed for the synchronization of two IONs, but this approach can be utilized for the synchrony of a cluster of IONs. Using Lyapunov analysis, it is shown that in the closed-loop system the trajectories of the controlled ION asymptotically follow the trajectories of the reference ION. The control system includes an adaption law for generating estimates of the unknown parameters. Simulation results are presented which show that the adaptive law accomplished synchronization, despite large uncertainties in the ION parameters. Furthermore, in closed-loop system, the relative phase of the two IONs is controlled.
3.1 Problem Formulation

The mathematical model of the inferior olive neurons derived by Kazantsev et al. [12-13], is a fourth-order nonlinear system. This model has been shown to reproduce the key ION electrophysiological properties. This ION model has rich dynamics and is capable of exhibiting stable and oscillatory behaviour as well as bursting and chaotic phenomena for the proper choice of its parameters and simulating pulses.

We consider two IONs, a reference ION denoted as ION$_2$ and a follower ION denoted as ION$_1$ described by

$$ION_1:$$

$$\dot{u}_1 = \frac{k}{\epsilon_{Na}}[u_1^2 - u_1^3 + (u_1^2 - u_1)a - v_1]$$

$$\dot{v}_1 = k[u_1 - z_1 + I_{Ca} - I_{Na}]$$

$$\dot{z}_1 = z_1^2 - z_1^3 + (z_1^2 - z_1)a - w_1$$

$$\dot{w}_1 = \epsilon_{Ca}(z_1 - I_{Ca} - uc_1)$$

$$ION_2:$$

$$\dot{u}_2 = \frac{k}{\epsilon_{Na}}[u_2^2 - u_2^3 + (u_2^2 - u_2)a - v_2]$$

$$\dot{v}_2 = k[u_2 - z_2 + I_{Ca} - I_{Na}]$$

$$\dot{z}_2 = z_2^2 - z_2^3 + (z_2^2 - z_2)a - w_2$$

$$\dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca})$$

The orbits of the uncontrolled ION$_1$ with $uc_1 = 0$ and ION$_2$ beginning from arbitrary initial conditions are not synchronized and there exists nonzero relative phase difference.
between the trajectories of the IONs. It is assumed here that the key parameter $a$ and the input gain of $\epsilon_{Ca}$ of the IONs are not known. Let $x_a = (u_1, v_1, z_1, w_1) \in R^4$ and $x_b = (u_2, v_2, z_2, w_2)^T \in R^4$ be the state vectors associated with IONs. We are interested in the design of an adaptive law such that $(u_2 - u_1, v_2 - v_1, z_2 - z_1, w_2 - w_1)$ asymptotically tend to zero, despite the uncertainties in the parameters $a$ and $\epsilon_{Ca}$.

3.2 Adaptive Control Law

In this section, design of a control law based on adaptive tuning method [17] is considered. For the purpose of design of a simple control law, regulation of the $(z-w)$ subsystem is considered. One could as well consider the regulation of $u_1$ to follow $u_2$ for attaining synchrony, but the resulting control system is complicated from the viewpoint of implementation.

Let $e_1(t) = z_1(t) - z_2(t)$ be the trajectory error of the IONs. A control law will be designed for regulation $e_1$ to zero. It will be seen later that if $e_1(t) = 0$, then $x_a(t) = x_b(t)$ and the IONs attain synchronism. We assume here that the state variables $z_i$ and $w_i$, $(i = 1, 2)$, are available for feedback. For the system (3.1), the relative degree of the chosen controlled variable $z_1$ is two. That is, its second derivative depends explicitly on the control input $Iext$. Therefore, adaptive design will be completed in two steps of a backstepping design procedure [17].
3.2.1 Step1:

Differentiating $e_1(t)$ along the solution of (1) and (2) gives

$$\dot{e}_1 = z_1^2 - z_2^2 - z_1^3 + z_2^3 + (z_1^2 - z_2^2 - e_1)a - w_1 + w_2 \quad (3.3)$$

Define a coordinate change as

$$e_2 = -\alpha_1 - w_1 \quad (3.4)$$

Where $\alpha_1 \in R$ is a stabilizing signal yet to be determined. Then substituting for $w_1$ from (3.4) in (3.3) gives

$$\dot{e}_1 = w_2 + f_1(z_1, z_2) + g_1(z_1, z_2)a + \alpha_1 + e_2 \quad (3.5)$$

where the nonlinear functions are

$$f_1 = z_1^2 - z_2^2 - z_1^3 + z_2^3$$

$$g_1 = z_1^2 - z_2^2 - e_1 \quad (3.6)$$

(Often the arguments of functions are suppressed for simplicity.) In view of (3.5) for regulation of $e_1$ to zero, the stabilizing signal $\alpha_1$ is selected as

$$\alpha_1(z_1, z_2, \hat{a}, w_2) = -w_2 - f_1(z_1, z_2) - g_1(z_1, z_2)\hat{a} - c_1e_1 \quad (3.7)$$

where $c_1 > 0$ and $\hat{a}$ is an estimate of the unknown parameter $a$. Note that $\alpha_1$ has been chosen to cancel the known function $f_1$ and the estimated value of $g_1a$. Substituting (3.7) in (3.5) yields

$$\dot{e}_1 = -c_1e_1 + g_1(z_1, z_2)\hat{a} + e_2 \quad (3.8)$$
where $\tilde{a} = a - \hat{a}$ is the parameter error.

For examining the asymptotic behaviour of the tracking error $e_1(t)$, consider a quadratic Lyapunov function

$$V_1(e_1, \hat{\theta}) = \frac{e_1^2}{2} + \frac{\tilde{a}^2}{2}$$  \hspace{1cm} (3.9)

where $\gamma_1 > 0$. Its derivative along the solution of (3.8) gives.

$$\dot{V}_1 = e_1(-c_1 e_1 + g_1(z_1, z_2)\tilde{a} + e_2) + \gamma_1 \tilde{a} \hat{\theta}$$

$$= -c_1 e_1^2 + e_1 e_2 + \tilde{a}[e_1 g_1(z_1, z_2) - \gamma_1 \hat{\theta}]$$  \hspace{1cm} (3.10)

Note from (3.10) that if $\tilde{a}=0$ and $e_2=0$, then $\dot{V}_1 = -c_1 e_1^2$ which implies that $e_1 \to 0$ as $t \to \infty$. However, the parameter error $\tilde{a}$ is nonzero and $e_2$ cannot be assumed to be zero because $w_1$ is not a control input. As such regulation of $e_2$ to zero using the control input $I_{ext1}$ will be considered in Step 2. It may be pointed out that although it is possible to make a choice of an adaptation law for $\hat{a}$ for cancelling $\tilde{a}$-dependent term in (3.10), it is not done in Step 1 to avoid overparameterization (dimension of dynamics of update law exceeding the dimension of unknown parameters in the system).

3.2.2 Step 2:

Now regulation of $e_2$ to zero is considered. The dynamics of $e_2$ can be obtained by differentiating $e_2$. First let us obtain the derivative of $\alpha_1$. Differentiating $\alpha_1$ given in (3.7) gives

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial z_1} \dot{z}_1 + \frac{\partial \alpha_1}{\partial z_2} \dot{z}_2 + \frac{\partial \alpha_1}{\partial w_1} \dot{w}_1 + \frac{\partial \alpha_1}{\partial \hat{a}} \dot{\hat{a}}$$  \hspace{1cm} (3.11)

substituting $\dot{z}_1$ and $\dot{w}_2$ from (3.1) in (3.11) and collecting terms gives

$$\dot{\alpha}_1 = -f_2(z_1, z_2, w_1, w_2, \hat{a}) - f_3(z_1, z_2) \hat{a}$$
\[-f_4(z_1, z_2, \hat{a})a - \epsilon_{Ca}[z_2 - I_a]\] (3.12)

where \( f_2, f_3 \) and \( f_4 \) are known functions. Using (3.12), the derivative of \( e_2 \) takes the form

\[
\dot{\epsilon}_2 = -\alpha_1 - \dot{w}_1
\]

\[
= f_2(z_1, z_2, w_1, w_2, \hat{a}) + f_3(z_1, z_2)\dot{a}
\]

\[
+f_4(z_1, z_2, \hat{a})a + \epsilon_{Ca}[-z_1 + I_{ext1} + z_2]
\] (3.13)

Adding and subtracting \( f_4\hat{a} \) in (3.13) and collecting terms gives

\[
\dot{\epsilon}_2 = \phi_o(z_1, z_2, \hat{a}, w_1, w_2) + f_3(z_1, z_2)\dot{a}
\]

\[
+f_4(z_1, z_2, \hat{a})a + \epsilon_{Ca}[e_1 + I_{ext1}]
\] (3.14)

where \( \phi_o = f_2 + f_4\hat{a} \).

The parameter \( \epsilon_{Ca} \) is not known. Let an estimate of \( \epsilon_{Ca}^{-1} = \rho \) be \( \hat{\rho} \) and the parameter error be \( \tilde{\rho} = \rho - \hat{\rho} \). Choose a control input \( I_{ext} \) as

\[
I_{ext1} = e_1 + \hat{\rho}I_a
\] (3.15)

where \( I_a \) is a new input. One easily finds that

\[
\epsilon_{Ca}(-e_1 + I_{ext1}) = \epsilon_{Ca}\hat{\rho}I_a = \epsilon_{Ca}(\rho - \hat{\rho})I_a = (1 - \epsilon_{Ca}\tilde{\rho})I_a
\] (3.16)

Substituting (3.16) in (3.14) gives

\[
\dot{\epsilon}_2 = \phi_o + f_3\dot{a} + f_4\tilde{a} + (1 - \epsilon_{Ca}\tilde{\rho})I_a
\] (3.17)

For stability analysis, consider a composite Lyapunov function

\[
V_2(e_1, e_2, \tilde{\theta}, \tilde{\rho}) = V_1(e_1) + \frac{e_2^2}{2} + \frac{\epsilon_{Ca}|\tilde{\rho}|^2\gamma_2}{2}
\]
where $\gamma_2 > 0$. Differentiating (3.18) and using (3.5) and (3.17) gives

\[ \dot{V}_2 = -c_1 e_1^2 + e_1 e_2 + \tilde{a}[e_1 g_1(z_1, z_2) - \gamma_1 \hat{a}] + e_2 \phi_o(z, w, \hat{a}) + f_3(z) \hat{a} + f_4(z, \hat{a}) \tilde{a} + I_a - \epsilon_C a \tilde{\rho} I_a] + |\epsilon_C a| \gamma_2 \tilde{\rho} \dot{\tilde{\rho}} \] (3.18)

The parameter $\epsilon_C a$ is not known. Collecting terms in (3.18) gives

\[ \dot{V}_2 = -c_1 e_1^2 + e_2 [e_1 + \phi_o(z, w, \hat{a}) + f_3(z) \hat{a} + I_a] + \tilde{a}[e_1 g_1(z) + e_2 f_4(z, \hat{a}) - \gamma_1 \hat{a}] + \tilde{\rho} \epsilon_C a [-I_a e_2 + sgn(\epsilon_C a) e_2 I_a] \] (3.19)

In view of (3.19), one chooses an adaptation law as

\[ \dot{\hat{a}} = \gamma_1^{-1} [e_1 g_1(z) + e_2 f_4(z, \hat{a})] \]

\[ \dot{\tilde{\rho}} = -\dot{\tilde{\rho}} = -\gamma_2^{-1} sgn(\epsilon_C a) e_2 I_a \] (3.20)

and the control law $I_a$ of the form

\[ I_a = -c_2 e_2 - e_1 - \phi_o(z, w, \hat{a}) - f_3(z) \hat{a} \] (3.21)

Substituting (3.20) and (3.21) in (3.19) gives

\[ \dot{V}_2 = -c_1 e_1^2 - c_2 e_2^2 \leq 0 \] (3.22)

The composite Lyapunov function $V_2(e_1, e_2, \tilde{\theta}, \tilde{\rho})$ is a positive definite function of $e_1, e_2, \tilde{\theta}, \tilde{\rho}$; and $\dot{V}_2 \leq 0$. Therefore, $V_2(t), e_1, e_2, \tilde{\theta}, \tilde{\rho} \in L_\infty[0, \infty)$, (the set of bounded functions). Integrating (3.22) yields

\[ \int_0^\infty (c_1 e_1^2 + c_2 e_2^2) dt \leq V(0) - V(\infty) < \infty \] (3.23)
which implies that $e_1, e_2 \in L_2[0, \infty)$ (the set of square integrable functions). In view of (3.8) and (3.17), it follows that $e_1, e_2 \in L_\infty[0, \infty)$. Since $e_1, e_2 \in L_\infty[0, \infty) \cap L_2[0, \infty)$ and there derivatives are bounded, according to the Barbalat’s Lemma [17], it follows that $e_1(t)$ and $e_2(t)$ asymptotically converge to zero. This implies that the $(z_1-z_2)$ and $(w_1-w_2)$ converge to zero.

Note that $(u_i-v_i)$ subsystems are forced by the periodic input $z_i$ and $z_1(t)=z_2(t)$. For the given parameters, the linearized subsystems $(u_1,v_1)$ and $(u_2,v_2)$ have asymptotically stable origin. As such $((u_1-u_2), (v_1-v_2))$ converge to zero as $t \to \infty$. But the trajectory $(u_2, v_2)$ of the reference ION is periodic for the choice of its parameters. Therefore, $ION_1$ undergoes periodic oscillations and asymptotically $x_a(t)=x_b(t)$ as $t \to \infty$.

3.3 Simulations results

This section presents results of simulation. The closed-loop system including the adaptation and control law is simulated. Various choices of the controller and adaptation gains are considered for obtaining numerical results. The reference $ION_2$ generates the trajectories for tracking by the $ION_2$. The parameter values of the neurons are taken from [15]. These are: $\epsilon_{Na} = 0.001$, $\epsilon_{Ca} = 0.02$, $k = 0.1$, $I_{Ca} = 0.018$, $I_{Na} = -0.61$, and $a = 0.01$, $k_t = 30$. The parameter $k_t$ is a time scaling parameter. The initial estimates of the parameters are $\hat{a}(0)=0.4$ and $\hat{\rho}(0)=0.5$. Note that the actual values of $a$ and $\rho$ are 0.01 and 50, respectively. These estimated values differ significantly from the actual values, but has been chosen to show robustness of the adaptive system.
Case I: Adaptive synchrony: \( c_1=20, \ c_2=10, \ \gamma_1=1, \ \gamma_2=2. \)

Distinct initial conditions of IONs are selected as \( u_1(0)=0.1, \ v_1(0)=0.2, \ z_1(0)=0.3, \ w_1(0)=0.4 \) and \( u_2(0)=0.9, \ v_2(0)=0.8, \ z_2(0)=0.7, \ w_2(0)=0.6. \) The feedback and adaptation gains are \( c_1=20, \ c_2=10, \ \gamma_1=1, \ \gamma_2=2. \) Responses are shown in Figs. (2)-(3). We observe that the trajectories of the IONs beginning from nonidentical initial conditions are not synchronized in the initial phase. But the control system quickly accomplishes synchronization following the initial transient. The Trajectory errors converge to zero (Fig. 3) and the steady-state error is very small 0.01. The control magnitude is large because the ION has fast transient responses with sharp peaking. Of course smaller values of \( c_i \) can be used for smaller control signals; but then the response time will increase.

Case II: Adaptive synchrony: effect of larger feedback gains \( c_i. \)

For examining the effect of feedback parameter, larger values \( c_1=70 \) and \( c_2=50 \) are selected. The remaining parameters and initial conditions of Case 1 are retained. Responses shown in Figs.(4)-(5) show that for the choice of larger feedback gains, error converges to zero relatively in a shorter time, but as expected larger control magnitude is needed. The steady-state error is 0.01. The estimated values \( \hat{\rho} \) and \( \hat{a} \) converge to some constant values.

Case III: Adaptive synchrony: effect of larger adaptation gains \( \gamma_i. \)

For examining the effect of adaptation gains, values of \( \gamma_1 \) and \( \gamma_2 \) are selected as 5 and 10, respectively. However remaining parameters of Case 1 are retained. Note that the adaptation gains of Case 1 are smaller than the values chosen for this case. Responses are shown in Figs. (6)-(7). The responses are somewhat similar to those of Case 1. This
shows that the effect of larger adaptation gains is minor.

**Case IV: Adaptive synchrony: effect of initial conditions of IONs.**

Now the effect of different choice of the initial conditions is examined. For this, the initial conditions of $ION_1$ are set to $u_1(0)=0.2$, $v_1(0)=0.2$, $z_1(0)=0.1$, $w_1(0)=0.1$ and for the $ION_2$ one has $u_2(0)=0.7$, $v_2(0)=0.8$, $z_2(0)=0.9$, $w_2(0)=0.8$. These initial values differ from those of Case 1. But all other parameters of Case 1 are retained for simulation. Simulated responses are shown in Figs. (8)-(9). It is observed that in spite of choice of different initial conditions, synchrony is accomplished.

**Case V: Adaptive synchrony of IONs with a relative phase difference between two IONs.** In this case, IONs are simulated with some relative phase difference between them. The IONs chosen are identical and the parameters are $a_1 = a_2=0.01$. The remaining parameters of the IONs of Case 1 are retained. Selected responses are shown in Fig.10. From the figure we can observe that there is a $90^\circ$ phase difference between two IONs, here we are trying to synchronize the IONs, by delaying the reference ION with $t_d=0.75$ and passing it through the controller. So that the follower ION, will eventually gets synchronized with the delayed reference ION and, the IONs will oscillate in unison. By this we can control the relative phase between two IONs.
Figure 3.1: Initial trajectories of the IONs: ION\(_1\) is commanded to track ION\(_2\). (a) \(u_1\) and \(u_2\), (b) \(v_1\) and \(v_2\), (c) \(z_1\) and \(z_2\), (d) \(w_1\) and \(w_2\), (e) shows the estimate of parameter \(a\), (f) shows the estimate of parameter \(\rho\), (g) shows the control input, (h) shows the trajectory error.
Figure 3.2: Trajectories of the IONs: ION\(_1\) is commanded to track ION\(_2\). (a) \(u_1\) and \(u_2\), (b) \(v_1\) and \(v_2\), (c) \(z_1\) and \(z_2\), (d) \(w_1\) and \(w_2\), (e) shows the estimate of parameter \(a\), (f) shows the estimate of parameter \(\rho\), (g) shows the control input.
Figure 3.3: Trajectory error: $u_1 - u_2$, $v_1 - v_2$, $z_1 - z_2$, and $w_1 - w_2$. 

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Figure 3.4: Trajectories of the IONs: ION₁ is commanded to track ION₂. (a) $u₁$ and $u₂$, (b) $v₁$ and $v₂$, (c) $z₁$ and $z₂$, (d) $w₁$ and $w₂$, (e) shows the estimate of parameter $a$, (f) shows the estimate of parameter $ρ$, (g) shows the control input.
Figure 3.5: Trajectory error: \( u_1 - u_2 \), \( v_1 - v_2 \), \( z_1 - z_2 \), and \( w_1 - w_2 \).
Figure 3.6: Trajectories of the IONs: ION$_1$ is commanded to track ION$_2$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the estimate of parameter $a$, (f) shows the estimate of parameter $\rho$, (g) shows the control input.
Figure 3.7: Trajectory error: $u_1 - u_2$, $v_1 - v_2$, $z_1 - z_2$, and $w_1 - w_2$.  

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Figure 3.8: Trajectories of the IONs: ION$_1$ is commanded to track ION$_2$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the estimate of parameter $a$, (f) shows the estimate of parameter $\rho$, (g) shows the control input.
Figure 3.9: Trajectory error: $u_1 - u_2$, $v_1 - v_2$, $z_1 - z_2$, and $w_1 - w_2$. 
Figure 3.10: Adaptive synchrony of identical IONs with a relative phase difference between them. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (f) $u_1$ and $u_{2d}$, (g) $v_1$ and $v_{2d}$, (h) $z_1$ and $z_{2d}$, (i) $w_1$ and $w_{2d}$, (i) shows the control input.
CHAPTER 4

TUNING FUNCTION BASED ADAPTIVE GLOBAL SYNCHRONY OF MULTI-INPUT INFERIOR OLIVE NEURONS

In the previous chapter, tuning functions based adaptive control systems for local synchrony of IONs with single control input have been designed. Controller has been designed under the assumption that there are no external disturbances present in the ION model. Contribution of this chapter lies in the design of a multi-input adaptive control system for the synchronization of two inferior olive neurons (IONs) which are not necessarily identical. The key parameters associated with the polynomial type nonlinearity of the IONs are assumed to be unknown. Based on the tuning functions adaptive backstepping design method [17], a control system is designed for the synchronization of the two IONs by unidirectional feedback. In the closed-loop system, the follower ION$_2$ asymptotically tracks the trajectory of the reference ION$_1$. Simulation results are presented which show that in the closed-loop system, the IONs synchronize despite the uncertainty in the parameters. Furthermore, in the closed-loop system, the relative phase of the two IONs is controlled.
4.1 Problem Formulation

We consider two IONs [12, 13], a reference ION denoted as ION\textsubscript{1} and a follower ION denoted as ION\textsubscript{2} described by the following equations.

ION\textsubscript{1}:

\[ \begin{align*}
\dot{u}_1 &= \frac{k}{\epsilon_{Na}} \left[ u_1^2 - u_1^3 + (u_1^2 - u_1) a_1 \right] - \frac{k}{\epsilon_{Na}} v_1 \\
\dot{v}_1 &= k [u_1 - z_1 + I_{Ca} - I_{Na}] \\
\dot{z}_1 &= z_1^2 - z_1^3 + (z_1^2 - z_1) a_1 - w_1 \\
\dot{w}_1 &= \epsilon_{Ca} (z_1 - I_{Ca} - \mu_1)
\end{align*} \] (4.1)

ION\textsubscript{2}:

\[ \begin{align*}
\dot{u}_2 &= \frac{k}{\epsilon_{Na}} \left[ (u_2^2 - u_2^3) + (u_2^2 - u_2) a_2 \right] - \frac{k}{\epsilon_{Na}} v_2 \\
\dot{v}_2 &= k [u_2 - z_2 + I_{Ca} - I_{Na}] + u c_1 \\
\dot{z}_2 &= z_2^2 - z_2^3 + (z_2^2 - z_2) a_2 - w_2 \\
\dot{w}_2 &= \epsilon_{Ca} (z_2 - I_{Ca} - \mu_1) + u c_2 + d(t)
\end{align*} \] (4.2)

where \( x_a = (u_1, v_1, z_1, w_1)^T \in R^4 \) and \( x_b = (u_2, v_2, z_2, w_2)^T \in R^4 \) are the state vectors associated with IONs and \( u c_1, u c_2 \) are the control inputs and \( d(t) \) is the disturbance input.

Define for simplicity

\[ f(u_1, a_1) = \frac{k}{\epsilon_{Na}} \left[ u_1^2 - u_1^3 + (u_1^2 - u_1) a_1 \right] \]

\[ f(z_1, a_1) = [z_1^2 - z_1^3 + (z_1^2 - z_1) a_1] \]
\[ f(u_2, a_2) = \frac{k}{\epsilon_{N_a}} [u_2^2 - u_2^3 + (u_2^2 - u_2)a_2] \] (4.3)

\[ f(z_2, a_2) = [z_2^2 - z_2^3 + (z_2^2 - z_2)a_2] \]

It is assumed that parameters \( a_1 \) and \( a_2 \) of the IONs are not known. Note that if \( a_1 \neq a_2 \) then the waveforms of the two IONs are non-identical. We are interested in the design of an adaptive law such that \( (u_2(t) - u_1(t), v_2(t) - v_1(t), z_2(t) - z_1(t), w_2(t) - w_1(t)) \) asymptotically tend to zero, despite the uncertainties in the parameters \( a_1 \) and \( a_2 \).

### 4.2 Adaptive Law

This section presents the design of an adaptive controller based on the tuning functions approach. For the purpose of design, we choose a controlled output vector

\[ y(t) = \begin{pmatrix} u_2(t) \\ z_2(t) \end{pmatrix} \] (4.4)

First a derivation of stabilizing control law is considered such that \( y(t) \) follows the reference trajectory

\[ y_r(t) = \begin{pmatrix} u_1(t) \\ z_1(t) \end{pmatrix} \] (4.5)

It will be seen later that regulation of the output vector error \( \tilde{y}(t) = y(t) - y_r(t) \) to zero will accomplish synchrony of the IONs. The design is based on the tuning functions method.
Differentiating (4.3) along the solution of (4.1) and (4.2) and using (4.3) gives

\[
\dot{\tilde{y}} = \begin{pmatrix}
\dot{u}_2 - \dot{u}_1 \\
\dot{z}_2 - \dot{z}_1
\end{pmatrix} = \begin{pmatrix}
f(u_2, a_2) + \frac{k}{\epsilon N_a} v_2 - f(u_1, a_1) + \frac{k}{\epsilon N_a} v_1 \\
f(z_2, a_2) - w_2 - f(z_1, a_1) + w_1
\end{pmatrix}
\]  

(4.6)

where

\[
f(u_2, a_2) - f(u_1, a_1) = \frac{k}{\epsilon N_a} \left[u_2^2 - u_2^3 + (u_2^2 - u_2) a_2 - u_1^2 + u_1^3 - (u_1^2 - u_1) a_1 \right]
\]

\[
\triangleq h_{01}(u_1, u_2) + \left[h_{11}(u_1), h_{12}(u_2)\right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}
\]  

(4.7)

Similarly one has

\[
f(z_2, a_2) - f(z_1, a_1) = \left[z_2^2 - z_2^3 + (z_2^2 - z_2) a_2 - z_1^2 + z_1^3 - (z_1^2 - z_1) a_1 \right]
\]

\[
\triangleq h_{02}(z_1, z_2) + \left[h_{21}(z_1), h_{22}(z_2)\right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}
\]  

(4.8)

for appropriate functions \(h_{ik}\). Define matrices \(H_0\) and \(H_1\) as

\[
H_0(u_1, u_2, z_1, z_2) = \begin{pmatrix} h_{01}(u_1, u_2) \\ h_{02}(z_1, z_2) \end{pmatrix}
\]

\[
H_1(u_1, u_2, z_1, z_2) = \begin{pmatrix} h_{11}(u_1) & h_{12}(u_2) \\ h_{21}(z_1) & h_{22}(z_2) \end{pmatrix}
\]  

(4.9)

As a result one can compactly write

\[
\begin{pmatrix}
f(u_2, a_2) - f(u_1, a_1) \\
f(z_2, a_2) - f(z_1, a_1)
\end{pmatrix} = H_0(u_1, u_2, z_1, z_2) + H_1(u_1, u_2, z_1, z_2)a
\]  

(4.10)
where $a = (a_1, a_2)^T \in \mathbb{R}^2$ is the vector of unknown parameters. Using (4.6) gives the derivative of $\tilde{y}$ as

$$
\dot{\tilde{y}} = H_0(u_1, u_2, z_1, z_2) + H_1(u_1, u_2, z_1, z_2)a - \begin{pmatrix}
    \frac{k\epsilon_{Na}^{-1}y_2}{w_2}
\end{pmatrix} = H_0(u_1, u_2, z_1, z_2) + H_1(u_1, u_2, z_1, z_2)a - \begin{pmatrix}
    \frac{k\epsilon_{Na}^{-1}y_2}{w_2}
\end{pmatrix} (4.11)
$$

4.2.1 Step 1:

Define a new vector $e = (e_1, e_2)^T$ given by

$$
\begin{pmatrix}
    e_1 \\
    e_2
\end{pmatrix} = \begin{pmatrix}
    v_2 \\
    w_2
\end{pmatrix} - \begin{pmatrix}
    \alpha_1 \\
    \alpha_2
\end{pmatrix} (4.12)
$$

where $\alpha = (\alpha_1, \alpha_2)^T \in \mathbb{R}^2$ is a stabilizing signal yet to be determined. Using (4.12) in (4.13), one obtains

$$
\dot{\tilde{y}} = H_0(u_1, u_2, z_1, z_2) + H_1(u_1, u_2, z_1, z_2)a + B[e + \alpha] (4.13)
$$

where the nonsingular matrix $B$ is

$$
B = \begin{pmatrix}
    -\frac{k}{\epsilon_{Na}} & 0 \\
    \frac{1}{\epsilon_{Na}} & 0 \\
    0 & -1
\end{pmatrix}
$$

The parameters of vector $a$ are not known. Let $\hat{a} = (\hat{a}_1, \hat{a}_2)^T \in \mathbb{R}^2$ be an estimate of the parameter vector $a$.

In Step 1, the stabilizing signal $\alpha$ is chosen for the regulation of $\tilde{y}(t)$. In view of (4.13), the stabilizing signal $\alpha(t)$ is chosen of the form

$$
\alpha(u_1, u_2, z_1, z_2, \hat{a}) = B^{-1}[-H_0(u_1, u_2, z_1, z_2) - H_1(u_1, u_2, z_1, z_2)\hat{a} - p_1\tilde{y}] (4.14)
$$
where \( p_1 > 0 \) is a design parameter. The term \( p_1 \ddot{y} \) has been included to provide damping.

Substituting (4.14) in (4.13), gives

\[
\dot{\tilde{y}} = H_1(u_1, u_2, z_1, z_2)\tilde{a} - p_1 \ddot{y} + Be \tag{4.15}
\]

where \( \tilde{a} = (a - \hat{a}) \in R^2 \) is the parameter error vector. Note that the signal \( \alpha \) has been chosen to cancel the known function \( H_0 \) and the estimated nonlinear function \( H_1a \).

For examining the stability property of the error dynamics (4.14), consider a positive definite Lyapunov function \( V \) given by

\[
V(\tilde{y}, \tilde{a}) = \frac{1}{2}(\tilde{y}^T \tilde{y} + \tilde{a}^T \Gamma^{-1} \tilde{a}) \tag{4.16}
\]

where the adaptation gain \( \Gamma \) is a positive number. Differentiating \( V \) along the solution of (4.15), gives

\[
\dot{V}(\tilde{y}, \tilde{a}) = \tilde{y}^T \dot{\tilde{y}} + \tilde{a}^T \Gamma^{-1} \dot{\tilde{a}} \tag{4.17}
\]

Substituting \( \dot{\tilde{y}} \) from (4.15) in (4.17) and collecting terms gives

\[
\dot{V}(\tilde{y}, \tilde{a}) = -p_1 ||\tilde{y}||^2 + \tilde{a}^T[H_1^T(u_1, u_2, z_1, z_2)\tilde{y} - \Gamma^{-1} \dot{\hat{a}}] + \tilde{y}^T Be \tag{4.18}
\]

Although, it is possible to choose an adaptation law based on (4.18) and cancel the unknown vector function \( H_1 \tilde{y} \), this will lead to an overparameterized update law. In the tuning functions method, the adaptation law is chosen in Step 2 for this system.
4.2.2 Step 2:

In Step 2, the control input vector \( u_c = (u_{c1}, u_{c2})^T \in \mathbb{R}^2 \) will be chosen such that the error vector \( e(t) \) tends to zero. Differentiating \( e(t) \) in (12), gives

\[
\begin{pmatrix}
\dot{e}_1 \\
\dot{e}_2 
\end{pmatrix} =
\begin{pmatrix}
\dot{v}_2 \\
\dot{w}_2 
\end{pmatrix} -
\begin{pmatrix}
\dot{\alpha}_1 \\
\dot{\alpha}_2 
\end{pmatrix} \tag{4.19}
\]

Note that \( \alpha \) is a function of \((u_1, u_2, z_1, z_2, \hat{a})\). Differentiating \( \alpha_1 \) given in (4.14) gives

\[
\dot{\alpha}(t) = \frac{\partial \alpha}{\partial u_1} \dot{u}_1 + \frac{\partial \alpha}{\partial u_2} \dot{u}_2 + \frac{\partial \alpha}{\partial z_1} \dot{z}_1 + \frac{\partial \alpha}{\partial z_2} \dot{z}_2 - B^{-1} H_1 \dot{\hat{a}} \tag{4.20}
\]

Substituting \( \dot{u}_1, \dot{u}_2, \dot{z}_1, \dot{z}_2 \) from (4.1) and (4.2), (4.20) gives

\[
\dot{\alpha}(t) = \frac{\partial \alpha}{\partial u_1} (f(u_1, a_1) - \frac{k}{\epsilon Na} v_1) + \frac{\partial \alpha}{\partial u_2} (f(u_2, a_2) - \frac{k}{\epsilon Na} v_2)
\]

\[
+ \frac{\partial \alpha}{\partial z_1} (f(z_1, a_1) - w_1) + \frac{\partial \alpha}{\partial z_2} (f(z_2, a_2) - w_2) - B^{-1} H_1 \dot{\hat{a}} \tag{4.21}
\]

Lumping the known functions and the unknown functions separately, one expresses (4.21), as

\[
\dot{\alpha}(t) = \eta_0(u_1, u_2, v_1, v_2, z_1, z_2, w_1, w_2, \hat{a}, \dot{\hat{a}}) + \eta_1(u_1, u_2, z_1, z_2, \hat{a}) \tilde{a} \tag{4.22}
\]

where \( \eta_0 \in \mathbb{R}^2 \) and \( \eta_1 \in \mathbb{R}^{2 \times 2} \) are regressor matrices. Note that \( \eta_0 \) is a function of \( \hat{a} \) and \( \dot{\hat{a}} \), and \( \eta_1 \) is a function of \( \dot{\hat{a}} \). ( The complete expressions for \( \eta_0 \) and \( \eta_1 \) are given in Appendix II). Substituting (4.20) and using (4.1) and (4.2), the derivative of \( e \) in (4.19) is given by

\[
\dot{e} =
\begin{pmatrix}
k[u_2 - z_2 + I_{Ca} - I_{Na}] + uc_1 \\
\epsilon_{Ca}(z_1 - I_{Ca} - \mu_1) + uc_2 
\end{pmatrix} - \eta_0(u_1, u_2, v_1, v_2, z_1, z_2, w_1, w_2, \hat{a}, \dot{\hat{a}}) - \eta_1(u_1, u_2, z_1, z_2, \hat{a}) \tilde{a}
\]

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\[ \dot{\eta}_2(x_1, x_2, \dot{a}, \dot{\hat{a}}) = -\eta_1(u_1, u_2, z_1, \hat{a}) + u_c \]  

(4.23)

where \( x_1 = (u_1, v_1, z_1, w_1)^T \in R^4 \), \( x_2 = (u_2, v_2, z_2, w_2)^T \in R^4 \), \( u_c = (uc_1, uc_2)^T \in R^2 \) and \( \eta_2 \) can be easily obtained from (4.23). For regulation of \( e \) to zero, consider a control input

\[ u_c = -\eta_2 - p_2 e - B^T \bar{y} \]  

(4.24)

Then substituting (4.24) in (4.23), one has

\[ \dot{e} = -B^T \bar{y} - \eta_1 \hat{a} - p_2 e \]  

(4.25)

where \( p_2 > 0 \).

Now consider a composite Lyapunov function \( V_2 \) given by

\[ V_2(x_1, x_2, \hat{a}) = V_1(\bar{y}, \hat{a}) + \frac{e^T e}{2} \]  

(4.26)

Differentiating (4.26) and using (4.18) and (4.24), gives

\[ \dot{V}_2 = \dot{V}_1 + e^T \dot{e} \]  

\[ \dot{V}_2 = -p_1 \|\bar{y}\|^2 + \hat{a}^T [H_1^T - \Gamma^{-1} \hat{a}] + \bar{y}^T B e + e^T [-B^T \bar{y} - \eta_1 \hat{a} - p_2 e] \]  

(4.27)

Collecting terms in (4.27), gives

\[ \dot{V}_2 = -p_1 \|\bar{y}\|^2 + \hat{a}^T [H_1^T - \Gamma^{-1} \hat{a} - \eta_1^T e] - p_2 e^T e \]  

(4.28)

For eliminating the unknown vector function in (4.28), the adaptation law is chosen as

\[ \dot{\hat{a}} = \Gamma^{-1} (H_1^T - \eta_1^T e) \]  

(4.29)

Substituting (4.29) in (4.28), gives

\[ \dot{V}_2 = -p_1 \|\bar{y}\|^2 - p_2 \|e\|^2 \leq 0 \]  

(4.30)
$V_2$ is a positive definite function of $\tilde{y}, \tilde{a}$ and $e$, and (4.30) implies that $\dot{V}_2 < 0$ if $\tilde{y}$ and $e$ are not zero. As such $V_2(t) < V_2(0)$ for any non-zero values of $\tilde{y}(0), \tilde{a}(0)$ and $e(0)$. Since $V_2$ is a positive definite function, $\tilde{y}$ and $e$ are bounded vector functions. Furthermore, since $\dot{V}_2 \leq 0$, $\tilde{y}(t)$ and $e(t)$ asymptotically converge to zero. Of course convergence of $\tilde{y}(t)$ to zero implies that $(u_2 - u_1)$ and $(z_2 - z_1)$ converge to zero as $t$ tends to $\infty$. Also convergence of $e$ to zero implies that $(v_2, w_2)^T$ tends to $(\alpha_1, \alpha_2)^T$. In view of (1) and (2), as $(u_1 - u_2)$ and $(z_1 - z_2)$ and its derivatives tend to zero, setting $(\dot{u}_2 - \dot{u}_1)$ and $(\dot{z}_2 - \dot{z}_1)$ to zero, one has $v_1(t) - v_2(t) = 0$ and $w_1(t) - w_2(t) = 0$, as $t \to \infty$. Therefore, in the closed-loop system ION$_1$ and ION$_2$ get synchronized.

4.3 Simulations results

This section presents the results of digital simulation. The trajectories of IONs with closed control loop are obtained. The model of the IONs given in Section 2 are used for simulation. The parameters used for the simulation are,

(1) For identical IONs $a_1 = a_2 = 0.01$

(2) Model parameters $I_{Na} = -0.59$, $I_{Ca} = 0.018$, $k = 0.1$, $\epsilon_{Ca} = 0.02$, $\epsilon_{Na} = 0.001$.

(3) The adaptation parameters are $\hat{a}_1$, $\hat{a}_2$.

(4) $uc_1$, $uc_2$ are the control inputs.

(5) $d(t)$ is the disturbance input which has been included in the equation $\dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca} - \mu_1) + uc_2 + d(t)$.

(6) $\mu$ is the bias input.

(7) $\Gamma$, $p_1$, $p_2$ are the design parameters.

The values of $p_1$ and $p_2$ are obtained by solving a Lyapunov equation $A^TP + PA = -Q$
where \( A = \begin{pmatrix} -10 & 0 \\ -0 & -2 \end{pmatrix} \) and \( Q = \begin{pmatrix} -1 & 0 \\ -0 & -1 \end{pmatrix} \).

(9) \( k_t \) is the time scaling factor.

Simulations are done for the following cases and responses were observed

**Case(i)** Simulation of two identical IONs without disturbance \((d(t)=0)\) and bias input \((\mu=0), \, (\mu=0.2)\)

**Case(ii)** Simulation of identical IONs with random and Sinusoidal Disturbances \(d(t)\).

Random disturbance which has been generated by passing a white noise through a low-pass filter of type \((\frac{b}{s+b})\), where \( b \) is a positive number \((b > 0)\) in this case we chose \( b=5 \) resulting in a pole at -5. Sinusoidal disturbance \(d(t)\) of the form \( M\sin(\omega t) \) which is passed through a low-pass filter of type \((\frac{b}{s+b})\).

**Case(iii)** Simulation of identical IONs with random and Sinusoidal Disturbances \(d(t)\) and with bias input \(\mu_1=0.5\) **Case(iv)** Simulation for the phase control of two identical IONs.

The initial conditions of the IONs used for simulation are \( x_1(0) = (0.1, 0.2, 0.3, 0.4)^T \) for ION\(_1\), and \( x_2(0) = (0.5, 0.6, 0.7, 0.8)^T \) for ION\(_2\). Note that initial conditions of the two IONs are not equal. The actual parameters of ION\(_1\) and ION\(_2\) are \( a_1 = a_2 = 0.01 \) and the initial conditions for the estimates are arbitrarily selected as \( \hat{a}(0)=(\hat{a}_1(0), \hat{a}_2(0))^T = (0.1, 0.1)^T \). Thus, the initial estimates \( \hat{a}_i(0) \) differ from the actual values of \( a_i \). It is important to note that ION\(_2\) with the adaptive law will synchronize for various choices of positive values of \( p_i, \Gamma \), but different sets of values will give different transient responses.

We have selected these values after observing the simulated responses. We may point out
that actual parameters $a_i$ are used in the ION models only for simulation, to examine the performance of the control system. But the designed adaptive control system does not require any knowledge of these actual parameters for its synthesis.

**Case I. Adaptive synchrony of identical IONs with $d(t)=0$, $\mu_1=0$, $\mu_1=0.5$**

In this case, IONs are simulated with $d(t)=0$ and $\mu_1=0$. Later IONs are simulated with $\mu_1=0.5$. The IONs chosen are identical and the parameters are $a_1 = a_2=0.01$. The initial conditions of the IONs used for simulation are $x_1(0) = (0.1,0.2,0.3,0.4)^T$ for ION$_1$, and $x_2(0) = (0.5,0.6,0.7,0.8)^T$ for ION$_2$, and the initial conditions for the estimates are arbitrarily selected as $\hat{a}(0)=(\hat{a}_1(0),\hat{a}_2(0))^T = (0.1,0.1)^T$. The chosen design parameters are $\Gamma = 10$, $p_1 = 4$, $p_2 = 4$. The trajectories of IONs and parameter estimation errors are plotted over a shorter interval in order to show the transient responses in Fig.1. It can be seen that $u_1$ and $u_2$, $v_1$ and $v_2$, $z_1$ and $z_2$, $w_1$ and $w_2$ converges quickly in a very short period of time. The trajectories $z_1$, $z_2$, $w_1$, $w_2$ are smooth trajectories when compared to $u_1$, $u_2$, $v_1$, $v_2$ it is because of the control inputs. The trajectories of the ION$_1$ and controlled ION$_2$ are shown in the Fig.2. The trajectories of the ION$_1$ and controlled ION$_2$ with a bias input $\mu_1=0.5$ are shown in the Fig.3. It is seen in Fig.3 that for chosen value of bias input $\mu_1=0.5$ the IONs are bursting. It can be observed that despite of the bursting in the reference ION$_1$, the follower ION$_2$ is able to track the trajectory of reference ION and both IONs got synchronized.

**Case II. Adaptive synchrony of Identical IONs with random and Sinusoidal Disturbances $d(t)$**

In this case, we have examined synchrony of identical IONs with random and sinu-
soidal disturbances \(d(t)\). Simulation is done using \(a_1=a_2=0.01\). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with the random disturbance of \(d(t) \in [-0.5, 0.5]\) are shown in Fig.4, and for IONs with Sinusoidal disturbance of amplitude \(M=0.1, \omega=2\) are shown in Fig.5. The trajectory error converges to zero despite the disturbance inputs.

**Case III. Adaptive synchrony of Identical IONs with a bias input \(\mu_1=0.5\), Noise and Sinusoidal Disturbances \(d(t)\)**

In this case, we have examined synchrony of identical IONs for bias value \(\mu_1=0.5\), random and sinusoidal disturbance, simulation is done using \(a_1=a_2=0.01\). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with random disturbance \(d(t) \in [-0.5, 0.5]\) are shown in Fig.6, sinusoidal disturbance of amplitude 0.1 and \(\omega=2\) are shown in Fig.7. The trajectory error converges to zero. It can be seen that for the chosen value of \(\mu_1=0.5\) and \(a\), the reference ION is bursting. As we can observe despite of the bursting in reference ION_1 the follower ION_2 is tracking the trajectory of ION_1. It is observed that synchrony is accomplished despite the presence of disturbance inputs. Simulation has been done for other sets of bifurcation parameters, and in each case synchronization and parameter identification are accomplished.

**Case IV. Adaptive synchrony of Identical IONs with a relative phase difference between two IONs.**

In this case, IONs are simulated with some relative phase difference between two IONs. The IONs chosen are identical and the parameters are \(a_1 = a_2=0.01\). The remaining parameters of the IONs of Case A are retained. Selected responses are shown
in Fig. 8. From the figure we can observe that there is a phase difference between two IONs, here we are trying to synchronize the IONs, by delaying the reference ION with $t_d=0.25$ and passing it through the controller. So that the follower ION, will eventually gets synchronized with the delayed reference ION and, the IONs will oscillate in unison. By this we can control the relative phase between two IONs.
Figure 4.1: Transient responses of identical IONs simulated for a short interval of time for $a_1 = a_2 = 0.01$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $uc_1$ and $uc_2$, (g) trajectory error $e$. (a)-(d) show the initial transient responses of trajectories of ION$_1$ and ION$_2$, (e) shows the initial transient response of estimated parameters, (f) shows the initial transient response of control inputs, (g) shows the initial transient response of the trajectory error.
Figure 4.2: Adaptive synchrony of identical IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu_1=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $uc_1$ and $uc_2$, (g) regulation error $e$. The synchronization of IONs is seen in (a)-(d). (e) shows the convergence of estimated parameters, (f) shows the control inputs (g) shows the trajectory error converges to zero.
Figure 4.3: Adaptive synchrony of identical IONs with $a_1=a_2=0.01, \mu_1=0.5, d(t)=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $u_{c1}$ and $u_{c2}$, (g) trajectory error $e$, (a)-(d) show the synchronization of IONs, (e) shows the convergence of estimated parameters. (f) shows the control inputs (g) shows the trajectory error converges to zero.
Figure 4.4: Adaptive synchrony of identical IONs with random disturbance for $a_1 = a_2 = 0.01$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $uc_1$ and $uc_2$, (g) trajectory error $e$, (h) disturbance. Despite of the random disturbance we can see the synchronization of IONs in (a)-(d). (e) shows the convergence of estimated parameters. (f) shows the control inputs (g) shows the trajectory error converges to zero, (h) shows the disturbance input.
Figure 4.5: Adaptive synchrony of identical IONs with sinusoidal disturbance for $a_1=a_2=0.01, \mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $u_{c1}$ and $u_{c2}$, (g) trajectory error $e$. The synchronization of IONs with sinusoidal disturbance is seen in (a)-(d). (e) shows the convergence of estimated parameters. (f) shows the control inputs (g) shows the trajectory error converges to zero (h) shows the sinusoidal disturbance input.
Figure 4.6: Adaptive synchrony of identical IONs with random disturbance $d(t)$ and bias value $\mu_1=0.5$, for $a_1=a_2=0.01$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $uc_1$ and $uc_2$, (g) trajectory error $e$, (h) disturbance. Despite of the random disturbance and bias value $\mu_1=0.5$, we can see the synchronization of IONs in (a)-(d). (e) shows the convergence of estimated parameters. (f) shows the control inputs (g) shows the trajectory error converges to zero, (h) shows the disturbance input.
Figure 4.7: Adaptive synchrony of identical IONs with sinusoidal disturbance $d(t)$ and bias value $\mu_1=0.5$, for $a_1=a_2=0.01$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) estimated parameters $\hat{a}_1$ and $\hat{a}_2$, (f) control inputs $u_{c1}$ and $u_{c2}$, (g) trajectory error $e$, (h) disturbance. Despite of the sinusoidal disturbance and bias value $\mu_1=0.5$, we can see the synchronization of IONs in (a)-(d). (e) shows the convergence of estimated parameters. (f) shows the control inputs (g) shows the trajectory error converges to zero, (h) shows the sinusoidal disturbance input.
Figure 4.8: Adaptive synchrony of identical IONs with a relative phase difference between two IONs for \( a_1 = a_2 = 0.01, \; d(t) = 0, \; \mu = 0 \). (a) \( u_1 \) and \( u_2 \), (b) \( v_1 \) and \( v_2 \), (c) \( z_1 \) and \( z_2 \), (d) \( w_1 \) and \( w_2 \), (e) \( u_{1d} \) and \( u_2 \), (f) \( v_{1d} \) and \( v_2 \), (g) \( z_{1d} \) and \( z_2 \), (h) \( w_{1d} \) and \( w_2 \), (i) shows the control inputs, (j) shows the error.
CHAPTER 5

\( \mathcal{L}_1 \) ADAPTIVE CONTROL SYSTEM FOR LOCAL SYNCHRONIZATION OF INFERIOR OLIVE NEURONS

In the previous chapters, tuning functions based adaptive control systems for local and global synchrony of IONs have been designed. Controllers were designed under the assumption that there are no external disturbances present in the ION model. In this chapter, the design is based on \( \mathcal{L}_1 \) Adaptive control theory [42]. We designed adaptive system applicable to IONs perturbed by disturbance input. An \( \mathcal{L}_1 \) adaptive control law includes a state predictor like indirect adaptive control systems, but it differs from the traditional adaptive control laws [51] in an important way. The control input for \( \mathcal{L}_1 \) adaptive control is generated by filtering the estimated control signal by a lowpass filter. As such high-frequency signals are not applicable to the system. Furthermore, it is assumed that the nonlinear functions of the IONs are unstructured (unmodelled).

5.1 Problem Formulation

Let us consider a reference ION (denoted as ION\(_1\)) governed by ION\(_1\):

\[
\dot{u}_1 = \frac{k}{\epsilon_{Na}}[u_1^2 - u_1^3 + (u_1^2 - u_1)a_1] - \frac{k}{\epsilon_{Na}}v_1
\]

\[
\dot{v}_1 = k[u_1 - z_1 + I_{Ca} - I_{Na}]
\]

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\[
\dot{z}_1 = z_1^2 - z_1^3 + (z_1^2 - z_1)a_1 - w_1 \quad (5.1)
\]

\[
\dot{w}_1 = \epsilon_C a (z_1 - I_C a - \mu)
\]

and a follower ION (denoted as ION\textsubscript{2}) governed by

\[
\dot{u}_2 = \frac{k}{\epsilon_{Na}} [(u_2^2 - u_3^2) + (u_2^3 - u_2)a_2] - \frac{k}{\epsilon_{Na}} v_2
\]

\[
\dot{v}_2 = k[u_2 - z_2 + I_C a - I_N a]
\]

\[
\dot{z}_2 = z_2^2 - z_2^3 + (z_2^2 - z_2)a_2 - w_2 \quad (5.2)
\]

\[
\dot{w}_2 = \epsilon_C a (z_2 - I_C a - \mu) + \epsilon_C a u_c + d(t)
\]

where \(x_a = (u_1, v_1, z_1, w_1)^T \in \mathbb{R}^4\) and \(x_b = (u_2, v_2, z_2, w_2)^T \in \mathbb{R}^4\) are the state vectors associated with IONs and \(u_c\) is the control input and \(d(t)\) is the external disturbance input.

Define for simplicity

\[
f(u_1, a_1) = \frac{k}{\epsilon_{Na}} [u_1^2 - u_1^3 + (u_1^2 - u_1)a_1]
\]

\[
f(z_1, a_1) = [z_1^2 - z_1^3 + (z_1^2 - z_1)a_1]
\]

\[
f(u_2, a_2) = \frac{k}{\epsilon_{Na}} [u_2^2 - u_2^3 + (u_2^2 - u_2)a_2]
\]

\[
f(z_2, a_2) = [z_2^2 - z_2^3 + (z_2^2 - z_2)a_2] \quad (5.3)
\]

consider the trajectory error

\[
\tilde{z} = z_2 - z_1
\]

\[
\tilde{w} = w_2 - w_1 \quad (5.4)
\]
Using (5.1) and (5.2), one obtains the differential equations for $\tilde{z}$ and $\tilde{w}$ given by

\[
\dot{\tilde{z}} = f(z_2, a_2) - f(z_1, a_1) - \tilde{w}
\]
\[
\dot{\tilde{w}} = \epsilon_{ca} \tilde{z} + \epsilon_{ca} u_c + d(t)
\]  

(5.5)

It is assume that the reference ION has bounded trajectory $(u_1, v_1, z_1, w_1)$. Substituting $z_2 = \tilde{z} + z_1$ in (4) gives

\[
\dot{\tilde{z}} = f(\tilde{z} + z_1, a_2) - f(z_1, a_1) - \tilde{w}
\]
\[
\dot{\tilde{w}} = \epsilon_{ca} \tilde{z} + \epsilon_{ca} u_c + d(t)
\]  

(5.6)

For the derivation of the control law, it is assumed that the nonlinear functions $f(\tilde{z} + z_1, a_2)$ and $f(z_1, a_1)$ are not known, and moreover these functions are unstructured (un-modeled).

Define

\[
g(\tilde{z}, t) = f(\tilde{z} + z_1, a_2) - f(z_1, a_1)
\]  

(5.7)

where $t$ denotes the dependence of the function $g(\tilde{z}, t)$ on $z_1(t)$. Then (5.5) can be compactly written as

\[
\dot{\tilde{z}} = g(\tilde{z}, t) - \tilde{w}
\]
\[
\dot{\tilde{w}} = \epsilon_{ca} \tilde{z} + \epsilon_{ca} u_c + d(t)
\]  

(5.8)

It is assumed that $\epsilon_{ca} > 0$ is an unknown parameter. We are interested in designing an adaptive control system such that IONs asymptotically attain synchrony despite the presence of disturbance input $d(t)$, unmodelled nonlinearity $g(\tilde{z}, t)$, and uncertainty in the parameter $\epsilon_{ca}$. 

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5.2 $\mathcal{L}_1$ Adaptive Control System

In this section, an adaptive synchronizing control system is designed based on the $\mathcal{L}_1$ adaptive control theory\[42\]. The derivation is based on an equivalent representation of the system (5.7). For the ION model with a bounded $d(t)$ and $u_c(t)$, the trajectory$(\tilde{z}, \tilde{w})$ of the system (5.7) is bounded over $[0, \tau]$, and it has an equivalent representation of the form

$$\dot{\tilde{z}} = \theta(t)|\tilde{z}| + \sigma_1(t) - \tilde{w} \quad (5.9)$$

$$\dot{\tilde{w}} = \epsilon_{ca}\tilde{z} + \sigma_2(t) + \epsilon_{ca}u_c$$

where $\theta(t)$, $\sigma_1(t)$, $\sigma_2(t)$ are continuous, piecewise continuously differentiable and $\sigma_2(t)=d_2(t)$. For the design of the controller it is essential to obtain certain estimates of these unknown functions and parameter $\epsilon_{ca}$. The time varying functions $\theta(t)$, $\sigma_1(t)$, $\sigma_2(t)$ and the parameter $\epsilon_{ca}$ are unknown. Let $\hat{\theta}(t)$, $\hat{\sigma}_1(t)$, $\hat{\sigma}_2(t)$ and $\hat{\epsilon}_{ca}$ be the estimates of $\theta(t)$, $\sigma_1(t)$, $\sigma_2(t)$ and $\epsilon_{ca}$.

For obtaining an estimate of the $\epsilon_{ca}$ and the unknown functions, consider a state predictor of the form

$$\dot{\hat{z}} = -\lambda_1\tilde{x}_1 + \hat{\theta}(t)|\tilde{z}| + \hat{\sigma}_1(t), \hat{z}(0) = \tilde{z}(0) \quad (5.10)$$

$$\dot{\hat{w}} = -\lambda_2\tilde{x}_2 + \hat{\epsilon}_{ca}(t)(\tilde{z} + u_c) + \hat{\sigma}_2(t), \hat{w}(0) = \tilde{w}(0)$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$, and the state prediction error is

$$\tilde{x} = \begin{pmatrix} \hat{z} - \tilde{z} \\ \hat{w} - \tilde{w} \end{pmatrix} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \quad (5.11)$$
Differentiating $\tilde{x}$ and using (5.8) and (5.9), gives the following state prediction error as

$$\dot{\tilde{x}}_1 = -\lambda_1 \tilde{x}_1 + \tilde{\theta}(t) |\tilde{z}| + \tilde{\sigma}_1(t), \tilde{x}_1(0) = 0$$

$$\dot{\tilde{x}}_2 = -\lambda_2 \tilde{x}_2 + \tilde{\epsilon}_{ca}(t) (|\tilde{z}| + u_c) + \tilde{\sigma}_2(t), \tilde{x}_2(0) = 0$$

(5.12)

where $\tilde{\theta} = \hat{\theta} - \theta, \tilde{\sigma}_1 = \hat{\sigma}_1 - \sigma_1, \tilde{\sigma}_2 = \hat{\sigma}_2 - \sigma_2$, and $\tilde{\epsilon}_{ca} = \hat{\epsilon}_{ca} - \epsilon_{ca}$ are the parameter errors.

Define

$$A = \begin{pmatrix} -\lambda_1 & 0 \\ -\lambda_2 & -\lambda_2 \end{pmatrix}$$

Then (5.11) can be written as

$$\dot{\tilde{x}} = A \tilde{x} + \begin{pmatrix} \tilde{\theta}(t) |\tilde{z}| + \tilde{\sigma}_1(t) \\ \tilde{\epsilon}_{ca}(t) (|\tilde{z}| + u_c) + \tilde{\sigma}_2(t) \end{pmatrix}$$

(5.13)

For the derivation of an adaptation law, consider a positive definite Lyapunov function $V(\tilde{x}, \tilde{\theta}, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\epsilon}_{ca})$ of the form

$$V(\tilde{x}, \tilde{\theta}, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\epsilon}_{ca}) = \tilde{x}^T P \tilde{x} + \Gamma^{-1}[\tilde{\theta}^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \tilde{\epsilon}_{ca}^2]$$

where $\Gamma > 0$ and $P$ is a $2 \times 2$ positive definite matrix (denoted as $P > 0$), which is the unique solution of the Lyapunov equation.

$$A^T P + PA = -Q$$

(5.14)

where $Q > 0$. Selecting $Q$ as diagonal matrix, it follows that the solution $P$ of (14) is a diagonal matrix of the form $P = diag(p_{11}, p_{22})$. Differentiating $V$ and using (5.11), gives

$$\dot{V} = \tilde{x}^T (A^T P + PA) \tilde{x} + 2 \tilde{x}^T P \begin{pmatrix} \tilde{\theta}(t) |\tilde{z}| + \tilde{\sigma}_1(t) \\ \tilde{\epsilon}_{ca}(t) (|\tilde{z}| + u_c) + \tilde{\sigma}_2(t) \end{pmatrix} + 2 \Gamma^{-1}[\tilde{\theta}(\hat{\theta} - \dot{\theta}) + \tilde{\sigma}_1(\hat{\sigma}_1 - \dot{\sigma}_1)]$$
\[ + \tilde{\sigma}_2 (\dot{\sigma}_2 - \dot{\sigma}_2) + \tilde{\epsilon}_{ca} \tilde{\epsilon}_{ca} \] (5.15)

Using (5.14) in (5.15) and collecting terms gives

\[
\dot{V} = -\ddot{x}^T Q \ddot{x} + 2\ddot{\theta}(\ddot{x}_1 | \tilde{z}| p_{11} + \Gamma^{-1} \dot{\theta}) + 2\ddot{\sigma}_1 (\ddot{x}_1 p_{11} + \Gamma^{-1} \dot{\sigma}_1) + 2\ddot{\sigma}_2 (\ddot{x}_2 p_{22} + \Gamma^{-1} \dot{\sigma}_2) \\
+ 2\tilde{\epsilon}_{ca} \ddot{x}_2 p_{22} (|\tilde{z}| + u_c) + 2\Gamma^{-1} (\dot{\theta} \dot{\theta} + \dot{\sigma}_1 \dot{\sigma}_1 + \dot{\sigma}_2 \dot{\sigma}_2) \] (5.16)

In view of (5.16), we select the adaptation law of the form

\[
\begin{align*}
\dot{\theta} &= \Gamma \text{Proj}(\dot{\theta}, -\ddot{x}_1 | \tilde{z}| p_{11}) \\
\dot{\sigma}_1 &= \Gamma \text{Proj}(\dot{\sigma}_1, -\ddot{x}_1 p_{11}) \\
\dot{\sigma}_2 &= \Gamma \text{Proj}(\dot{\sigma}_2, -\ddot{x}_2 p_{22}) \\
\dot{\epsilon}_{ca} &= \Gamma \text{Proj}(\dot{\epsilon}_{ca}, -\ddot{x}_2 (|\tilde{z}| + u_c) p_{22})
\end{align*}
\] (5.17)

where Proj denotes a smooth projection operator used to keep the estimated parameters within certain admissible bounding sets such that \( \theta(t), \sigma_1(t), \sigma_2(t), \epsilon_{ca} \). Then a smooth projection (termed Proj) is given by [54]

\[
\text{Proj}(\alpha, \eta) = \begin{cases} 
\eta & \text{if } \xi(\alpha) \leq 0 \\
\eta & \text{if } \xi(\alpha) \geq 0 \text{ and } \frac{\partial \xi}{\partial \alpha}(\alpha) \eta \leq 0 \\
\eta - \frac{\xi(\alpha) \frac{\partial \xi}{\partial \alpha}(\alpha)}{\left\| \frac{\partial \xi}{\partial \alpha}(\alpha) \right\|^2} \frac{\partial \xi}{\partial \alpha}(\alpha) & \text{if } \text{not}
\end{cases}
\]

where \( \frac{\partial \xi}{\partial \alpha}(\alpha) = \frac{4 \alpha - \mu}{\varepsilon \alpha_{m}} \).

Substituting (5.17) in (5.16) gives

\[
\dot{V} \leq -\ddot{x}^T Q \ddot{x} + 2\Gamma^{-1} |(\dot{\theta} \dot{\theta} + \dot{\sigma}_1 \dot{\sigma}_1 + \dot{\sigma}_2 \dot{\sigma}_2)| \] (5.18)
Because $\dot{\theta}, \ddot{\theta}, \sigma_1, \sigma_2, \dot{\sigma}_2$ are bounded, it follows from (5.17) that $\ddot{x}(t)$ is ultimately bounded. In fact, according to[42], one can show that $\ddot{x}(t)$ is uniformly bounded and $||\dddot{x}(t)||_{L_\infty}$ is proportional to $\Gamma^{1/2}$, $||\dddot{x}(t)||_{L_\infty} = max(|\dddot{x}_1(t)|, |\dddot{x}_2(t)|)$, $0 \leq t \leq \infty$ For the derivation of the control law, a backstepping design procedure is used.

5.2.1 Step 1:

Define $e_1 = \dot{w} - \alpha$. Then from (5.8) one has

$$\dot{\tilde{z}} = \theta(t)|\tilde{z}| + \sigma_1(t) - e_1 - \alpha$$

(5.19)

where $\alpha$ is a stabilizing signal yet to be determined. The stabilizing signal $\alpha$ is chosen for the regulation of $\tilde{z}$. In view of (5.19), $\alpha$ is chosen as

$$\alpha = C_1(s)[\dot{\theta}(t)|\tilde{z}| + \tilde{\sigma}_1(t)] + \lambda_1 \tilde{z}$$

(5.20)

where $C_1(s)$ is a low-pass stable filter of relative degree of at least 2 with the DC gain $C_1(0)=1$. The motivation behind the selection of $\alpha$ of the form (5.20) can be explained as follows. If $C_1(s)$ is set to 1 and the parameters errors are zero (deterministic case), then (5.20) becomes a feedback linearizing control law. The stable filter $C_1(s)$ is used to suppress the high-frequency components of the estimated signals generated by the adaptation law. Substituting the control signal $\alpha$ form (5.20) in (5.19), gives

$$\dot{\tilde{z}} = -\lambda_1 \tilde{z} - C_1(s)[(\dot{\theta} + \theta)|\tilde{z}| + (\tilde{\sigma}_1 + \sigma_1)] + \theta|\tilde{z}| + \sigma_1 - e_1$$

$$= - \lambda_1 \tilde{z} + (1 - C_1(s)(\theta|\tilde{z}| + \sigma_1) - C_1(s)[\dot{\theta}|\tilde{z}| + \tilde{\sigma}_1] - e_1$$

(5.21)

5.2.2 Step 2:

In the second step of the design procedure, one must regulate the error $e_1 = \ddot{w} - \alpha$ to zero. For this purpose, we consider the error dynamics $\dot{e}_1 = \ddot{\tilde{w}} - \alpha$. Using $\ddot{w}$ from (5.8), one
finds that

\[ \dot{\epsilon}_1 = \epsilon_{ca} \ddot{z} + \epsilon_{ca} u_c + \sigma_2(t) - \dot{\alpha} \] (5.22)

For the regulation of \( \epsilon_1 \) to zero, \( u_c \) is chosen as

\[ u_c = -k_f D(s)[\dot{\epsilon}_{ca} \ddot{z} + \dot{\epsilon}_{ca} u_c + \dot{\sigma}_2(t) + \lambda_2 (\ddot{w} - \alpha) - \dot{\alpha}] \] (5.23)

where \( k_f > 0 \). The strictly proper transfer function \( D(s) \) is chosen such that

\[ C_2(s) = \frac{\epsilon_{ca} k_f D(s)}{(1 + \epsilon_{ca} k_f D(s))} \] (5.24)

is a stable transfer function for all admissible values of \( \epsilon_{ca} \). Of course for the ION model, one has \( 0 < \epsilon_{ca} < \epsilon_{ca}^* \), where \( \epsilon_{ca}^* \) is constant. Noting that \( \epsilon_{ca} > 0 \), with a simple choice of \( D(s) = 1/s \), one finds that

\[ C_2(s) = \frac{\epsilon_{ca} k_f}{(s + \epsilon_{ca} k_f)} \] (5.25)

is stable for all \( \epsilon_{ca} > 0 \) and \( C_2(0)=1 \). For this choice of \( D(s) \), (5.22) gives the control input in a simplified form as

\[ \dot{u}_c = -k_f[\dot{\epsilon}_{ca} \ddot{z} + \dot{\epsilon}_{ca} u_c + \dot{\sigma}_2(t) + \lambda_2 (\ddot{w} - \alpha) - \dot{\alpha}] \] (5.26)

For evaluating the performance of this adaptive law, let us consider a reference model

\[ \ddot{z}_{ref} = g(\ddot{z}_{ref}, t) - \ddot{w}_{ref}, \ddot{z}_{ref}(0) = \ddot{z}(0) \]

\[ \ddot{w}_{ref} = \epsilon_{ca} \ddot{z}_{ref} + \epsilon_{ca} u_{ref} + \dot{d}, \ddot{w}_{ref}(0) = \ddot{w}(0) \] (5.27)
\[ \alpha_{\text{ref}} = C_1(s)g(\tilde{z}_{\text{ref}}, t) + \lambda_1 \tilde{z}_{\text{ref}} \]

\[ u_{\text{ref}} = -\frac{C_2(s)}{\epsilon_{ca}} \left[ \epsilon_{ca} \tilde{z}_{\text{ref}} + \sigma_2 + \lambda_2 (\tilde{w}_{\text{ref}} - \alpha_{\text{ref}}) - \dot{\alpha}_{\text{ref}} \right] \]

For stability in the closed-loop system, certain \( L_1 \)-norm condition is required to be satisfied. For a given \( \rho_0 \) such that \( \| ((\tilde{z}(0), \tilde{w}(0))^T \|_\infty < \rho_0 \), suppose that for some \( \rho_r > \rho_{in} \), the norm condition

\[ \| G(s) \|_{L_\infty} < \frac{\rho_r - \beta_0(\rho_r)}{L^*_\rho_r + \beta_1(\rho_r)} \]  

is satisfied, where \( L^*_\rho_r, \beta_0(\rho_r), \beta_1(\rho_r) \) are computable functions. Then it has been established that in the closed-loop system \( \tilde{z}, \tilde{w}, u_c, \bar{x} \) are bounded and

\[ \| \tilde{z}_{\text{ref}} - \tilde{z} \|_{L_\infty} \leq \gamma_1 \]  

\[ \| u_{\text{cref}} - u_c \|_{L_\infty} \leq \gamma_2 \]

where \( \gamma_1 \) and \( \gamma_2 \) are positive numbers and that one can achieve arbitrary desired performance bounds for the systems signals, both input and output, simultaneously by increasing the adaptation gain \( \Gamma \).

For the purpose of implementation, one may choose \( C_1(s) \) of the form

\[ C_1(s) = \frac{b_0}{(s + \mu_0)(s + \mu_1)}, b_0 > 0, \mu_0 > 0, \mu_1 > 0 \]  

(5.30)
with $\mu_0\mu_1=b_0$. Then a realization of $C_1(s)$ with input $[\hat{\theta}(t)|\tilde{z}| + \hat{\sigma}_1(t)]$ and output $\alpha_0$ can be expressed as

$$\dot{x}_{f1} = x_{f2}, x_{f1}(0) = 0$$

$$\dot{x}_{f2} = -(\mu_0 + \mu_1)x_{f2} - \mu_0\mu_1x_{f1} + b_0[\hat{\theta}(t)|\tilde{z}| + \hat{\sigma}_1(t)], x_{f2}(0) = 0$$  \hspace{1cm} (5.31)

Then $\alpha$ in (5.20) can be generated as

$$\alpha = \alpha_0 + \lambda_1 \tilde{z}$$  \hspace{1cm} (5.32)

for synthesis.

### 5.3 Simulations results

This section presents the results of digital simulation. The trajectories of IONs with closed control loop are obtained. The model of the IONs given in Section 2 are used for simulation. The parameters used for the simulation are,

(1) For identical IONs $a_1=a_2=0.01$

(2) For non-identical IONs $a_1=0.01, a_2=0.02$.

(3) Model parameters $I_{Na} = -0.59, I_{Ca} = 0.018, k = 0.1, \epsilon_{Ca} = 0.02, \epsilon_{Na} = 0.001$.

(4) The adaptation parameters are $\hat{\theta}_1, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\theta}_2, \hat{\sigma}_3, \hat{\sigma}_4, \hat{\epsilon}_{ca}$.

(5) $uc_1, uc_2$ are the control inputs.

(6) $d_1(t), d_2(t)$ are the disturbance inputs which has been included in the equations

$$\dot{v}_2 = k[u_2 - z_2 + I_{Ca} - I_{Na}] + uc_1 + d_1(t), \quad \dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca} - \mu) + u_c + d_2(t).$$

(7) $\mu$ is the bias input.
(8) $\Gamma$, $p_1$, $p_2$ are the design parameters.

The values of $p_1$ and $p_2$ are obtained by solving a Lyapunov equation $A^T P + PA = -Q$ where $A = \begin{pmatrix} -10 & 0 \\ -0 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} -1 & 0 \\ -0 & -1 \end{pmatrix}$.

(9) $k_t$ is the time scaling factor.

Simulations are done for the following cases and responses were observed

**Case(i)** Simulation of two identical IONs without disturbance ($d(t)=0$) and bias input ($\mu=0$), ($\mu=0.2$)

**Case(ii)** Simulation of two non-identical IONs without disturbance ($d(t)=0$) and bias input ($\mu=0$), ($\mu=0.2$)

**Case(iii)** Simulation of identical IONs with random and Sinusoidal Disturbances $d(t)$.

Random disturbance which has been generated by passing a white noise through a low-pass filter of type ($\frac{b}{s+b}$), where $b$ is a positive number ($b > 0$) in this case we chose $b=5$ resulting in a pole at -5. Sinusoidal disturbance $d(t)$ of the form $M \sin(\omega t)$ which is passed through a low-pass filter of type ($\frac{b}{s+b}$).

**Case(iv)** Simulation of non-identical IONs with random and Sinusoidal Disturbances $d(t)$.

**Case(v)** Simulation for the phase control of two identical IONs.

**Case(vi)** Simulation for clamped control magnitude.

The initial conditions of the IONs for simulation are $x_1(0) = (-0.1, -0.1, 0.1, 0.1)^T$ for ION$_1$, and $x_2(0) = (-0.2, 0.1, -0.1, 0.3)^T$ for ION$_2$. Note that initial conditions of the two IONs are not equal. The actual parameters of ION$_1$ and ION$_2$ are $a_1 = a_2 = 0.01$.
for identical IONs, $a_1 = 0.01$, $a_2 = 0.02$ for non-identical IONs. The initial conditions for state predictor and adaptation estimates are arbitrarily selected as $\hat{x}_1(0)=0$, $\hat{x}_2(0)=0$, $\hat{\theta}(0)=0$, $\hat{\sigma}_1(0) = 0$, $\hat{\sigma}_2(0) = 0$, $\hat{\epsilon}_{ca}(0) = 0.00002$.

It is important to note that ION$_2$ with the adaptive law will synchronize for various choices of positive values of $p_i$, $\Gamma$, but different sets of values will give different transient responses. We have selected these values after observing the simulated responses. We may point out that actual parameters $a_i$ are used in the ION models only for simulation, to examine the performance of the control system. But the designed adaptive control system does not require any knowledge of these actual parameters for its synthesis.

**Case I. $\mathcal{L}_1$ Adaptive synchrony of identical IONs with $d(t)=0$, $\mu=0$, $\mu=0.1$**

In this case, IONs are simulated with $d(t)=0$ and $\mu=0$. Later IONs are simulated with $\mu=0.1$. The IONs chosen are identical and the parameters are $a_1 = a_2=0.01$. The initial conditions used for the simulation are $x_1(0) = (-0.1, -0.1, 0.1, 0.1)^T$ for ION$_1$, and $x_2(0) = (-0.2, 0.1, -0.1, 0.3)^T$ for ION$_2$. The initial conditions for state predictor and adaptation estimates are arbitrarily selected as $\hat{x}_1(0)=0$, $\hat{x}_2(0)=0$, $\hat{\theta}(0)=0$, $\hat{\sigma}_1(0) = 0$, $\hat{\sigma}_2(0) = 0$, $\hat{\epsilon}_{ca}(0) = 0.00002$. The chosen adaptation gain is $\Gamma = 1e^5$, $p_i$ values are obtained by solving the Lyapunov equation of the form discussed in simulation results section. The trajectories of the ION$_1$ and controlled ION$_2$ without any disturbance $d(t)=0, \mu=0$ are shown in the Fig.2. The trajectories of the ION$_1$ and controlled ION$_2$ with bias value $\mu=0.1$ are shown in Fig.3.

**Case II. $\mathcal{L}_3$ Adaptive synchrony of non-identical IONs $d(t)=0$, $\mu=0$, $\mu=0.1$**

In this case, IONs are simulated with $d(t)=0$ and $\mu=0$. Later IONs are simulated with
\( \mu = 0.1 \). The IONs chosen are non-identical and the parameters are \( a_1 = 0.01, \ a_2 = 0.02 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs without \( \mu \) are shown in Fig.4 and for IONs with bias value \( \mu \) are shown in Fig.5.

**Case III.** \( \mathcal{L}_1 \) Adaptive synchrony of Identical IONs with some random and sinusoidal disturbance \( d(t), \mu = 0 \).

In this case, IONs are simulated with some random and sinusoidal disturbance, \( \mu = 0 \). The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with random disturbance are shown in Fig.6 and for IONs with sinusoidal disturbance are shown in Fig.7.

**Case IV.** \( \mathcal{L}_1 \) Adaptive synchrony of Identical IONs with random and sinusoidal disturbance \( d(t), \mu = 0.1 \).

In this case, IONs are simulated with some random and sinusoidal disturbance, bias value \( \mu = 0.1 \). The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with random disturbance, \( \mu = 0.1 \) are shown in Fig.8 and for IONs with sinusoidal disturbance, \( \mu = 0.1 \) are shown in Fig.9.

**Case V.** \( \mathcal{L}_1 \) Adaptive synchrony of Identical IONs with a relative phase difference between two IONs.

In this case, IONs are simulated with some relative phase difference between two IONs. The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses are shown
in Fig.10. From the figure we can observe that there is a phase difference between two IONs, here we are trying to synchronize the IONs, by delaying the reference ION with $t_d=0.25$ and passing it through the controller. So that the follower ION, will eventually gets synchronized with the delayed reference ION and, the IONs will oscillate in unison. By this we can control the relative phase between two IONs.

**Case VI. $L_1$ Adaptive synchrony of Identical IONs with clamped control magnitude.**

In this case, IONs are simulated with clamped control magnitude. The IONs chosen are identical and the parameters are $a_1 = a_2=0.01$. The remaining parameters of the IONs of Case A are retained. Selected responses are shown in Fig.11. From the figure we can observe that by restricting the control input magnitude to a certain value, the IONs are taking a little longer time for synchronization. By this we can say that if there is no restriction on control input magnitude the IONs will synchronize quickly.
Figure 5.1: Initial trajectories of IONs (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$. 
Figure 5.2: Adaptive synchrony of identical IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$. 
Figure 5.3: Adaptive synchrony of identical IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu=0.1$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$. 
Figure 5.4: Adaptive synchrony of non-identical IONs for $a_1=0.01$, $a_2=0.02$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$. 
Figure 5.5: Adaptive synchrony of non-identical IONs for $a_1=0.01$, $a_2=0.02$, $d(t)=0$, $\mu=0.1$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$. 
Figure 5.6: Adaptive synchrony of identical IONs with random disturbance for $a_1=a_2=0.01$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$, (j) shows the disturbance input.
Figure 5.7: Adaptive synchrony of identical IONs with sinusoidal disturbance for $a_1=a_2=0.01$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$, (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$, (j) shows the sinusoidal disturbance input.
Figure 5.8: Adaptive synchrony of identical IONs with random disturbance and bias value for $a_1=a_2=0.01$, $\mu=0.1$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$, (j) shows the disturbance input.
Figure 5.9: Adaptive synchrony of identical IONs with sinusoidal disturbance for $a_1=a_2=0.01$, $\mu=0.1$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, (f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$. (h)-(i) shows the estimates of state predictors $x_1$ and $x_2$, (j) shows the sinusoidal disturbance input.
Figure 5.10: Adaptive synchrony of identical IONs with a relative phase difference between two IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) $u_{1d}$ and $u_2$, (f) $v_{1d}$ and $v_2$, (g) $z_{1d}$ and $z_2$, (h) $w_{1d}$ and $w_2$, (i) shows the control inputs, (j) shows the error.
Figure 5.11: Adaptive synchrony of identical IONs with clamped control magnitude for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control input, f)-(g) shows estimates of Projection parameters $\theta$, $\epsilon_{ca}$, $\sigma_1$, $\sigma_2$ (h) shows the estimates of state predictors $x_1$, $x_2$, (i) shows the error.
CHAPTER 6

$\mathcal{L}_1$ ADAPTIVE CONTROL SYSTEM FOR GLOBAL SYNCHRONIZATION OF INFERIOR OLIVE NEURONS

In the previous chapters, tuning functions based adaptive control systems for local and global synchrony of IONs and $\mathcal{L}_1$ adaptive control system for local synchronization of IONs have been designed. In tuning functions method controllers were designed under the assumption that there are no external disturbances present in the ION model. In $\mathcal{L}_1$ adaptive local synchronization, controller is been designed for single input and IONs perturbed by single disturbance input. Contribution of this chapter lies in the design of an multi-input adaptive control system for the synchronization of two inferior olive neurons(IONs). In this chapter, an attempt is made to design an adaptive system applicable to IONs perturbed by two disturbance inputs. Furthermore, it is assumed that the nonlinear functions of the IONs are unstructured(unmodelled). The design is based on $\mathcal{L}_1$ Adaptive control theory [42].

6.1 Problem Formulation

Let us consider a reference ION (denoted as ION$_1$) governed by
ION1:

\[
\dot{u}_1 = \frac{k}{\epsilon_{Na}} [u_1^2 - u_1^3 + (u_1^2 - u_1)a_1] - \frac{k}{\epsilon_{Na}} v_1 \\
\dot{v}_1 = k[u_1 - z_1 + I_{Ca} - I_{Na}] \\
\dot{z}_1 = z_1^2 - z_1^3 + (z_1^2 - z_1)a_1 - w_1 \\
\dot{w}_1 = \epsilon_{Ca}(z_1 - I_{Ca} - \mu) \tag{6.1}
\]

and a follower ION (denoted as ION2) governed by

ION2:

\[
\dot{u}_2 = \frac{k}{\epsilon_{Na}} [(u_2^2 - u_2^3) + (u_2^2 - u_2)a_2] - \frac{k}{\epsilon_{Na}} v_2 \\
\dot{v}_2 = k[u_2 - z_2 + I_{Ca} - I_{Na}] + u_1 + d_1(t) \\
\dot{z}_2 = z_2^2 - z_2^3 + (z_2^2 - z_2)a_2 - w_2 \\
\dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca} - \mu) + \epsilon_{Ca}u_2 + d_2(t) \tag{6.2}
\]

where \(x_a = (u_1, v_1, z_1, w_1)^T \in R^4\) and \(x_b = (u_2, v_2, z_2, w_2)^T \in R^4\) are the state vectors associated with IONs and \(u_1, u_2, \epsilon_{Ca}\) are the control inputs and \(d_1(t), d_2(t)\) are the external disturbance inputs.

Define for simplicity

\[
f(u_1, a_1) = \frac{k}{\epsilon_{Na}} [u_1^2 - u_1^3 + (u_1^2 - u_1)a_1] \\
f(z_1, a_1) = [z_1^2 - z_1^3 + (z_1^2 - z_1)a_1] \\
f(u_2, a_2) = \frac{k}{\epsilon_{Na}} [u_2^2 - u_2^3 + (u_2^2 - u_2)a_2] \tag{6.3}
\]
\[ f(z_2, a_2) = [z_2^2 - z_2^3 + (z_2^2 - z_2)a_2] \]

Consider the trajectory error

\[
\begin{align*}
\tilde{u} &= u_2 - u_1 \\
\tilde{v} &= v_2 - v_1 \\
\tilde{z} &= z_2 - z_1 \\
\tilde{w} &= w_2 - w_1
\end{align*}
\]

Using (6.1) and (6.2), one obtains the differential equations for \( \tilde{u}, \tilde{v}, \tilde{z} \) and \( \tilde{w} \) given by

\[
\begin{align*}
\dot{\tilde{u}} &= f(u_2, a_2) - f(u_1, a_1) - k\epsilon_{na}^{-1} \tilde{v} \\
\dot{\tilde{v}} &= k(\tilde{u} - \tilde{z}) + uc_1 + d_1(t) \\
\dot{\tilde{z}} &= f(z_2, a_2) - f(z_1, a_1) - \tilde{w} \\
\dot{\tilde{w}} &= \epsilon_{ca} \tilde{z} + \epsilon_{ca} uc_2 + d_2(t)
\end{align*}
\]

It is assume that the reference ION has bounded trajectory(\( u_1, v_1, z_1, w_1 \)). Substituting \( u_2=\tilde{u}+u_1, z_2=\tilde{z}+z_1 \) in (5.4) gives

\[
\begin{align*}
\dot{\tilde{u}} &= f(\tilde{u} + u_1, a_2) - f(u_1, a_1) - k\epsilon_{na}^{-1} \tilde{v} \\
\dot{\tilde{v}} &= k(\tilde{u} - \tilde{z}) + uc_1 + d_1(t) \\
\dot{\tilde{z}} &= f(\tilde{z} + z_1, a_2) - f(z_1, a_1) - \tilde{w} \\
\dot{\tilde{w}} &= \epsilon_{ca} \tilde{z} + \epsilon_{ca} uc_2 + d_2(t)
\end{align*}
\]
For the derivation of the control law, it is assumed that the nonlinear functions \( f(\bar{u} + u_1, a_2), f(u_1, a_1), f(\bar{z} + z_1, a_2) \) and \( f(z_1, a_1) \) are not known, and moreover these functions are unstructured (unmodelled).

Define

\[
\begin{align*}
    g_1(\bar{u}, t) &= f(\bar{u} + u_1, a_2) - f(u_1, a_1) \\
    g_2(\bar{z}, t) &= f(\bar{z} + z_1, a_2) - f(z_1, a_1)
\end{align*}
\]

(6.7)

where \( t \) denotes the dependence of the function \( g_1(\bar{u}, t) \) on \( u_1(t) \) and \( g_2(\bar{z}, t) \) on \( z_1(t) \).

Then (5) can be compactly written as

\[
\begin{align*}
    \dot{\bar{u}} &= g_1(\bar{u}, t) - k\epsilon^{-1}_{ca} \bar{v} \\
    \dot{\bar{v}} &= k(\bar{u} - \bar{z}) + uc_1 + d_1(t) \\
    \dot{\bar{z}} &= g(\bar{z}, t) - \bar{w} \\
    \dot{\bar{w}} &= \epsilon_{ca} \bar{z} + \epsilon_{ca} uc_2 + d_2(t)
\end{align*}
\]

(6.8)

It is assumed that \( \epsilon_{ca} > 0 \) is an unknown parameter. We are interested in designing an adaptive control system such that IONs asymptotically attain synchrony despite the presence of disturbance inputs \( d_1(t) \) and \( d_2(t) \), unmodelled nonlinearities \( g_1(\bar{u}, t), g_2(\bar{z}, t) \), and uncertainty in the parameter \( \epsilon_{ca} \).

6.2 \( \mathcal{L}_1 \) Adaptive Control System

In this section, an adaptive synchronizing control system is designed based on the \( \mathcal{L}_1 \) adaptive control theory [42]. The derivation is based on an equivalent representation of the system (6.8). Note that the \((\bar{z}, \bar{w})\) system is developed for \((\bar{u}, \bar{v})\) system. Because
system will be controlled by $L_1$ adaptive feedback, $\tilde{z}$ will be a bounded function (It will be showed later that indeed $\tilde{z}(t)$ is bounded.) For the ION model with a bounded $d_1(t)$, $d_2(t)$ and $uc_1(t)$, $uc_2(t)$ the trajectory($\hat{u}$, $\hat{v}$, $\tilde{z}$, $\tilde{w}$) of the system (6.8) is bounded over $[0,\tau]$, and it has an equivalent representation of the form

$$\dot{\hat{u}} = \theta_1(t)|\hat{u}| + \sigma_1(t) - k\epsilon_{ca}^{-1}\tilde{v}$$

$$\dot{\hat{v}} = k(\hat{u}) + uc_1 + \sigma_2(t)$$

$$\dot{\tilde{z}} = \theta_2(t)|\tilde{z}| + \sigma_3(t) - \tilde{w}$$

$$\dot{\tilde{w}} = \epsilon_{ca}\tilde{z} + \sigma_4(t) + \epsilon_{ca}uc_2$$

where $\theta_1(t)$, $\theta_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$, $\sigma_4(t)$ are continuous, piecewise continuously differentiable. The equivalent form of $\tilde{v}$ is based on boundedness of $\tilde{z}$, which will be controlled by $uc_2$ For the design of the controllers it is essential to obtain certain estimates of these unknown functions and parameter $\epsilon_{ca}$. The time varying functions $\theta_1(t)$, $\theta_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$, $\sigma_4(t)$ and the parameter $\epsilon_{ca}$ are unknown. Let $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{\sigma}_1(t)$, $\hat{\sigma}_2(t)$, $\hat{\sigma}_3(t)$, $\hat{\sigma}_4(t)$ and $\hat{\epsilon}_{ca}$ be the estimates of $\theta_1(t)$, $\theta_2(t)$, $\sigma_1(t)$, $\sigma_2(t)$, $\sigma_3(t)$, $\sigma_4(t)$ and $\epsilon_{ca}$.

For obtaining an estimate of the $\epsilon_{ca}$ and the unknown functions, consider a state predictor of the form

$$\dot{\hat{u}} = -\lambda_1\tilde{x}_1 + \hat{\theta}_1(t)|\hat{u}| + \hat{\sigma}_1(t) - k\epsilon_{ca}^{-1}\tilde{v}, \hat{u}(0) = \tilde{u}(0)$$

$$\dot{\hat{v}} = -\lambda_2\tilde{x}_2 + k\tilde{u} + \hat{\sigma}_2(t) + uc_1, \hat{v}(0) = \tilde{v}(0)$$

$$\dot{\tilde{z}} = -\lambda_1\tilde{x}_3 + \hat{\theta}_2(t)|\tilde{z}| + \hat{\sigma}_3(t), \tilde{z}(0) = \tilde{z}(0)$$

$$\dot{\tilde{w}} = -\lambda_2\tilde{x}_4 + \hat{\epsilon}_{ca}(t)(\tilde{z} + uc_2) + \hat{\sigma}_4(t), \tilde{w}(0) = \tilde{w}(0)$$

(6.10)
where $\lambda_1 > 0$ and $\lambda_2 > 0$, and the state prediction error is

$$\ddot{x} = \begin{pmatrix} \ddot{u} - \ddot{u} \\ \ddot{v} - \ddot{v} \end{pmatrix} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}$$ (6.11)

$$\ddot{y} = \begin{pmatrix} \ddot{z} - \ddot{z} \\ \ddot{w} - \ddot{w} \end{pmatrix} = \begin{pmatrix} \ddot{x}_3 \\ \ddot{x}_4 \end{pmatrix}$$ (6.12)

Differentiating $\ddot{x}$, $\ddot{y}$ and using (6.9) and (6.10), gives the following state prediction error as

$$\dot{\ddot{x}}_1 = -\lambda_1 \ddot{x}_1 + \tilde{\theta}_1(t)|\dddot{u}| + \tilde{\sigma}_1(t), \ddot{x}_1(0) = 0$$

$$\dot{\ddot{x}}_2 = -\lambda_2 \ddot{x}_2 + k(\dddot{u}) + uc_1 + \tilde{\sigma}_2(t), \ddot{x}_2(0) = 0$$ (6.13)

$$\dot{\ddot{x}}_3 = -\lambda_1 \ddot{x}_3 + \tilde{\theta}_2(t)|\dddot{z}| + \tilde{\sigma}_3(t), \ddot{x}_3(0) = 0$$

$$\dot{\ddot{x}}_4 = -\lambda_2 \ddot{x}_4 + \tilde{\epsilon}_{ca}(t)(|\dddot{z}| + uc_2) + \tilde{\sigma}_4(t), \ddot{x}_4(0) = 0$$ (6.14)

where $\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1$, $\tilde{\theta}_2 = \hat{\theta}_2 - \theta_2$, $\tilde{\sigma}_1 = \hat{\sigma}_1 - \sigma_1$, $\tilde{\sigma}_2 = \hat{\sigma}_2 - \sigma_2$, $\tilde{\sigma}_3 = \hat{\sigma}_3 - \sigma_3$, $\tilde{\sigma}_4 = \hat{\sigma}_4 - \sigma_4$, and $\tilde{\epsilon}_{ca} = \hat{\epsilon}_{ca} - \epsilon_{ca}$ are the parameter errors. Define

$$A = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$$

Then (6.13) and (6.14) can be written as

$$\dot{\ddot{x}} = A\ddot{x} + \begin{pmatrix} \tilde{\theta}_1(t)|\dddot{u}| + \tilde{\sigma}_1(t) \\ k\dddot{u} + uc_1 + \tilde{\sigma}_2(t) \end{pmatrix}$$ (6.15)

$$\dot{\ddot{y}} = A\ddot{y} + \begin{pmatrix} \tilde{\theta}_2(t)|\dddot{z}| + \tilde{\sigma}_3(t) \\ \tilde{\epsilon}_{ca}(t)(|\dddot{z}| + uc_2) + \tilde{\sigma}_4(t) \end{pmatrix}$$ (6.16)

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For the derivation of an adaptation law, consider two positive definite Lyapunov functions \( V_1(\tilde{x}, \tilde{\theta}_1, \tilde{\sigma}_1, \tilde{\sigma}_2) \), \( V_2(\tilde{y}, \tilde{\theta}_2, \tilde{\sigma}_3, \tilde{\sigma}_4, \tilde{\epsilon}_{ca}) \) of the form

\[
V_1(\tilde{x}, \tilde{\theta}_1, \tilde{\sigma}_1, \tilde{\sigma}_2) = \tilde{x}^T P \tilde{x} + \Gamma^{-1}[\tilde{\theta}_1^2 + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2]
\]

\[
V_2(\tilde{y}, \tilde{\theta}_2, \tilde{\sigma}_3, \tilde{\sigma}_4, \tilde{\epsilon}_{ca}) = \tilde{y}^T P \tilde{y} + \Gamma^{-1}[\tilde{\theta}_2^2 + \tilde{\sigma}_3^2 + \tilde{\sigma}_4^2 + \tilde{\epsilon}_{ca}^2]
\]

where \( \Gamma > 0 \) and \( P \) is a 2×2 positive definite matrix (denoted as \( P > 0 \)), which is the unique solution of the Lyapunov equation.

\[
A^T P + PA = -Q \quad (6.17)
\]

where \( Q > 0 \). Selecting \( Q \) as diagonal matrix, it follows that the solution \( P \) of (6.17) is a diagonal matrix of the form \( P = \text{diag}(p_{11}, p_{22}) \). Differentiating \( V_1, V_2 \) and using (6.11), gives

\[
\dot{V}_1 = \tilde{x}^T (A^T P + PA) \tilde{x} + 2\tilde{x}^T P \left( \begin{array}{c} \tilde{\theta}_1(t)|\tilde{u}| + \tilde{\sigma}_1(t) \\ \tilde{\sigma}_2(t) \end{array} \right) + 2\Gamma^{-1}[\tilde{\theta}_1(\dot{\tilde{\theta}}_1 - \dot{\tilde{\theta}}_1) + \tilde{\sigma}_1(\dot{\tilde{\sigma}}_1 - \dot{\tilde{\sigma}}_1)]
\]

\[
+ \tilde{\sigma}_2(\dot{\tilde{\sigma}}_2 - \dot{\tilde{\sigma}}_2) \quad (6.18)
\]

\[
\dot{V}_2 = \tilde{y}^T (A^T P + PA) \tilde{y} + 2\tilde{y}^T P \left( \begin{array}{c} \tilde{\theta}_2(t)|\tilde{z}| + \tilde{\sigma}_3(t) \\ \tilde{\epsilon}_{ca}(t)(|\tilde{z}| + uc_2) + \tilde{\sigma}_4(t) \end{array} \right) + 2\Gamma^{-1}[\tilde{\theta}_2(\dot{\tilde{\theta}}_2 - \dot{\tilde{\theta}}_2) + \tilde{\sigma}_3(\dot{\tilde{\sigma}}_3 - \dot{\tilde{\sigma}}_3)]
\]

\[
+ \tilde{\sigma}_4(\dot{\tilde{\sigma}}_4 - \dot{\tilde{\sigma}}_4) + \tilde{\epsilon}_{ca}(\dot{\tilde{\epsilon}}_{ca}) \quad (6.19)
\]

Using (6.13) in (6.15) and (6.14) in (6.16) and collecting terms gives

\[
\dot{V}_1 = -\tilde{x}^T Q \tilde{x} + 2\tilde{\theta}_1(\tilde{x}_1|\tilde{u}|p_{11} + \Gamma^{-1}\dot{\tilde{\theta}}_1) + 2\tilde{\sigma}_1(\tilde{x}_1p_{11} + \Gamma^{-1}\dot{\tilde{\sigma}}_1) + 2\tilde{\sigma}_2(\tilde{x}_2p_{22} + \Gamma^{-1}\dot{\tilde{\sigma}}_2)
\]

\[
+ 2\tilde{x}_2p_{22}(|\tilde{u}| + uc_1) - 2\Gamma^{-1}(\tilde{\theta}_1\dot{\tilde{\theta}}_1 + \tilde{\sigma}_1\dot{\tilde{\sigma}}_1 + \tilde{\sigma}_2\dot{\tilde{\sigma}}_2) \quad (6.20)
\]
\[ \dot{V}_2 = -\hat{y}^T Q \hat{y} + 2\hat{\theta}_2(\ddot{x}_3|\ddot{z}|p_{11} + \Gamma^{-1}\dot{\hat{\theta}}_2) + 2\hat{\sigma}_3(\ddot{x}_3 p_{11} + \Gamma^{-1}\dot{\hat{\sigma}}_3) + 2\hat{\sigma}_4(\ddot{x}_4 p_{22} + \Gamma^{-1}\dot{\hat{\sigma}}_4) \]

\[ + 2\hat{\epsilon}_c a \dot{x}_4 p_{22}(|\ddot{z}| + uc_2 + \Gamma^{-1}\ddot{\hat{\epsilon}}_c) - 2\Gamma^{-1}(\ddot{\theta}_2 \dot{\hat{\theta}}_2 + \ddot{\sigma}_3 \dot{\hat{\sigma}}_3 + \ddot{\sigma}_4 \dot{\hat{\sigma}}_4) \]  

(6.21)

In view of (6.20) and (6.21), we select the adaptation law of the form

\[ \dot{\hat{\theta}}_1 = \Gamma \text{Proj}(\hat{\theta}_1, -\ddot{x}_1|u |p_{11}) \]

\[ \dot{\hat{\sigma}}_1 = \Gamma \text{Proj}(\hat{\sigma}_1, -\ddot{x}_1 p_{11}) \]

\[ \dot{\hat{\sigma}}_2 = \Gamma \text{Proj}(\hat{\sigma}_2, -\ddot{x}_2 p_{22}) \]

\[ \dot{\hat{\theta}}_2 = \Gamma \text{Proj}(\hat{\theta}_2, -\ddot{x}_3|\ddot{z}|p_{11}) \]

\[ \dot{\hat{\sigma}}_3 = \Gamma \text{Proj}(\hat{\sigma}_3, -\ddot{x}_3 p_{11}) \]

\[ \dot{\hat{\sigma}}_4 = \Gamma \text{Proj}(\hat{\sigma}_4, -\ddot{x}_4 p_{22}) \]  

(6.22)

\[ \dot{\hat{\epsilon}}_c a = \Gamma \text{Proj}(\hat{\epsilon}_c a, -\ddot{x}_4(|\ddot{z} + uc_2|) p_{22}) \]

where \( \text{Proj} \) denotes a smooth projection operator used to keep the estimated parameters within certain admissible bounding sets such that \( \theta(t), \sigma_1(t), \sigma_2(t), \sigma_3(t), \sigma_4(t), \epsilon_c a \) are restricted to that admissible range. Then a smooth projection (termed \( \text{Proj} \)) is given by [54]

\[ \text{Proj}(\alpha, \eta) = \begin{cases} 
\eta & \text{if } \xi(\alpha) \leq 0 \\
\eta & \text{if } \xi(\alpha) \geq 0 \text{ and } \frac{\partial \xi}{\partial \alpha}(\alpha) \eta \leq 0 \\
\eta - \frac{\xi(\alpha) \frac{\partial \xi}{\partial \alpha}(\alpha) \eta}{\| \frac{\partial \xi}{\partial \alpha}(\alpha) \|^2} & \text{if not}
\end{cases} \]
where \( \frac{\partial \kappa}{\partial \alpha}(\alpha) = \frac{4a-\mu}{\epsilon \alpha_m} \).

Substituting (6.17) in (6.16) gives

\[
\dot{V}_1 \leq -\tilde{x}^T Q \tilde{x} + 2\Gamma^{-1}|(\theta_1 \dot{\theta}_1 + \sigma_1 \dot{\sigma}_1 + \sigma_2 \dot{\sigma}_2)| \tag{6.23}
\]

\[
\dot{V}_2 \leq -\tilde{y}^T Q \tilde{y} + 2\Gamma^{-1}|(\phi_2 \dot{\phi}_2 + \sigma_3 \dot{\sigma}_3 + \sigma_4 \dot{\sigma}_4)| \tag{6.24}
\]

Because \( \tilde{x}(t), \dot{\tilde{x}}(t), \tilde{y}(t), \dot{\tilde{y}}(t) \) are ultimately bounded. In fact, according to[42], one can show that \( \tilde{x}(t), \dot{\tilde{x}}(t), \tilde{y}(t), \dot{\tilde{y}}(t) \) are uniformly bounded and \( ||\tilde{x}(t)||_{L\infty}, ||\tilde{y}(t)||_{L\infty} \) are proportional to \( \Gamma^{1/2}, \)

\[
||\tilde{x}(t)||_{L\infty} = \max(|\tilde{x}_1(t)|, |\tilde{x}_2(t)|), \quad 0 \leq t \leq \infty \quad \text{and} \quad ||\tilde{y}(t)||_{L\infty} = \max(|\tilde{y}_3(t)|, |\tilde{y}_4(t)|), \quad 0 \leq t \leq \infty \]

For the derivation of the control laws, a backstepping design procedure is used.

6.2.1 Step 1:

Define \( e_1 = \tilde{v} - \alpha_1 \) and \( e_2 = \tilde{w} - \alpha_2 \) then one compactly write as

\[
e = \begin{pmatrix} \tilde{v} - \alpha_1 \\ \tilde{w} - \alpha_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}
\]

Then from (6.9) one has

\[
\dot{\tilde{u}} = \theta_1(t) |\tilde{u}| + \sigma_1(t) - k\epsilon^{-1}_{na}(e_1 + \alpha_1) \tag{6.25}
\]

\[
\dot{\tilde{z}} = \phi_2(t) |\tilde{z}| + \sigma_3(t) - e_2 - \alpha_2 \tag{6.26}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are the stabilizing signals yet to be determined. The stabilizing signals \( \alpha_1 \) and \( \alpha_2 \) are chosen for the regulation of \( \tilde{u}, \tilde{z} \). In view of (6.19), \( \alpha_1 \) is chosen as

\[
k\epsilon^{-1}_{na}\alpha_1 = C_1(s)|\tilde{\theta}_1(t)||\tilde{u}| + \tilde{\sigma}_1(t)| + \lambda_1 \tilde{u} \tag{6.27}
\]
α_2 is chosen as
\[ \alpha_2 = C_1(s)[\hat{\theta}_2(t)|\tilde{z}| + \hat{\sigma}_3(t)] + \lambda_1 \tilde{z} \]  
\hspace{1cm} (6.28)

where \( C_1(s) \) is a low-pass stable filter of relative degree of at least 2 with the DC gain \( C_1(0)=1 \). The motivation behind the selection of \( \alpha_1 \) and \( \alpha_2 \) of the form (6.20) can be explained as follows. If \( C_1(s) \) is set to 1 and the parameters errors are zero (deterministic case), then (6.20) becomes a feedback linearizing control law. The stable filter \( C_1(s) \) is used to suppress the high-frequency components of the estimated signals generated by the adaptation law. Substituting the stabilizing signals \( \alpha_1 \) from (6.27) and \( \alpha_2 \) from (6.28) in (6.25) and (6.26), gives

\[ \dot{\hat{u}} = -\lambda_1 \hat{u} - C_1(s)[(\hat{\theta}_1 + \theta_1)|\tilde{z}| + (\hat{\sigma}_1 + \sigma_1)] + \theta_1|\tilde{u}| + \sigma_1 - e_1 \]

\hspace{1cm} (6.29)

\[ \dot{\hat{z}} = -\lambda_1 \hat{z} - C_1(s)[(\hat{\theta}_2 + \theta_2)|\tilde{z}| + (\hat{\sigma}_3 + \sigma_3)] + \theta|\tilde{z}| + \sigma_3 - e_2 \]

\hspace{1cm} (6.30)

6.2.2 Step 2:

In the second step of the design procedure, one must regulate the error \( e \) to zero. For this purpose, we consider the error dynamics \( \dot{e}_1 = \hat{v} - \hat{\alpha}_1, \dot{e}_2 = \hat{w} - \hat{\alpha}_2 \). Using \( \hat{v} \) and \( \hat{w} \) from (6.8), one finds that

\[ \dot{e}_1 = k\hat{u} + uc_1 + \sigma_2(t) - \hat{\alpha}_1 \]  
\hspace{1cm} (6.31)

\[ \dot{e}_2 = \epsilon_{ca}\hat{z} + \epsilon_{ca}uc_2 + \sigma_4(t) - \hat{\alpha}_2 \]  
\hspace{1cm} (6.32)

For the regulation of \( e_1 \) and \( e_2 \) to zero, \( uc_1 \) and \( uc_2 \) are chosen as

\[ uc_1 = -k_fD(s)[k\hat{u} + uc_1 + \hat{\sigma}_2(t) + \lambda_2(\hat{v} - \alpha_1) - \hat{\alpha}_1] \]  
\hspace{1cm} (6.33)
\[ uc_2 = -k_f D(s)[\dot{\epsilon}_{ca} \dot{z} + \epsilon_{ca} uc_2 + \dot{\sigma}_4(t) + \lambda_2(\dot{w} - \alpha_2) - \dot{\alpha}_2] \]  

(6.34)

where \( k_f > 0 \). The strictly proper transfer function \( D(s) \) is chosen such that

\[ C_2(s) = \frac{\epsilon_{ca} k_f D(s)}{(1 + \epsilon_{ca} k_f D(s))} \]  

(6.35)

is a stable transfer function for all admissible values of \( \epsilon_{ca} \). Of course for the ION model, one has \( 0 < \epsilon_{ca} < \epsilon_{ca}^* \), where \( \epsilon_{ca}^* \) is constant. Noting that \( \epsilon_{ca} > 0 \), with a simple choice of \( D(s) = 1/s \), one finds that

\[ C_2(s) = \frac{\epsilon_{ca} k_f}{s + \epsilon_{ca} k_f} \]  

(6.36)

is stable for all \( \epsilon_{ca} > 0 \) and \( C_2(0)=1 \). For this choice of \( D(s) \), (6.33) and (6.34) gives the control inputs in a simplified form as

\[ \dot{uc}_1 = -k_f[k\ddot{u} + uc_1 + \dot{\sigma}_2(t) + \lambda_2(\ddot{v} - \alpha_1) - \dot{\alpha}_1] \]  

(6.37)

\[ \dot{uc}_2 = -k_f[\epsilon_{ca} \dot{z} + \epsilon_{ca} uc_2 + \dot{\sigma}_4(t) + \lambda_2(\dot{w} - \alpha_2) - \dot{\alpha}_2] \]  

(6.38)

For evaluating the performance of this adaptive law, let us consider a reference model

\[ \dot{u}_{ref} = g(\ddot{u}_{ref}, t) - k_{na}^{-1} \ddot{\theta}_{ref}, \bar{u}_{ref}(0) = \bar{u}(0) \]

\[ \dot{v}_{ref} = \bar{u}_{ref} + uc_{ref1} + d_2, \bar{v}_{ref}(0) = \bar{v}(0) \]  

(6.39)

\[ \dot{z}_{ref} = g(\ddot{z}_{ref}, t) - \ddot{\theta}_{ref}, \bar{z}_{ref}(0) = \bar{z}(0) \]

\[ \dot{w}_{ref} = \epsilon_{ca} \dot{z}_{ref} + \epsilon_{ca} uc_{ref2} + d_4, \bar{w}_{ref}(0) = \bar{w}(0) \]  

(6.40)

\[ \alpha_{1ref} = C_1(s)g(\ddot{u}_{ref}, t) + \lambda_1 \ddot{\theta}_{ref} \]

\[ \alpha_{2ref} = C_1(s)g(\ddot{z}_{ref}, t) + \lambda_1 \ddot{\theta}_{ref} \]
\[ uc_{1ref} = -C_2(s)[\bar{u}_{ref} + \sigma_2 + \lambda_2(\bar{v}_{ref} - \alpha_{1ref}) - \dot{\alpha}_{1ref}] \]

\[ uc_{2ref} = -\frac{C_2(s)}{\epsilon_{ca}} [\epsilon_{ca}\bar{z}_{ref} + \sigma_4 + \lambda_2(\bar{w}_{ref} - \alpha_{2ref}) - \dot{\alpha}_{2ref}] \]

For stability in the closed-loop system, certain \( L_1 \)-norm condition is required to be satisfied. For a given \( \rho_0 \) such that \( ||(\bar{u}(0), \bar{v}(0), \bar{z}(0), \bar{w}(0))^T||_\infty < \rho_0 \), suppose that for some \( \rho_r > \rho_{in} \), the norm condition

\[ ||G(s)||_{L_\infty} < \frac{\rho_r - \beta_0(\rho_r)}{\mathcal{L}^*_\rho r \rho_r + \beta_1(\rho_r)} \quad (6.41) \]

is satisfied, where \( \mathcal{L}^*_\rho r \), \( \beta_0(\rho_r) \), \( \beta_1(\rho_r) \) are computable functions. Then it has been established in [42] that in the closed-loop system \( \bar{u}, \bar{v}, uc_1, \bar{x}, \bar{z}, \bar{w}, uc_2, \bar{y} \) are bounded and

\[ ||\bar{u}_{ref} - \bar{u}||_{L_\infty} \leq \gamma_1 \quad (6.42) \]

\[ ||uc_{ref1} - uc_1||_{L_\infty} \leq \gamma_2 \]

\[ ||\bar{z}_{ref} - \bar{z}||_{L_\infty} \leq \gamma_1 \quad (6.43) \]

\[ ||uc_{ref2} - uc_2||_{L_\infty} \leq \gamma_2 \]

where \( \gamma_1 \) and \( \gamma_2 \) are positive numbers and that one can achieve arbitrary desired performance bounds for the systems signals, both input and output, simultaneously by increasing the adaptation gain \( \Gamma \).

For the purpose of implementation, one may choose \( C_1(s) \) of the form

\[ C_1(s) = \frac{b_0}{(s + \mu_0)(s + \mu_1)}, b_0 > 0, \mu_0 > 0, \mu_1 > 0 \quad (6.44) \]
with \( \mu_0 \mu_1 = b_0 \). Then a realization of \( C_1(s) \) with input \([\hat{\theta}_1(t)|\bar{u}| + \bar{\sigma}_1(t)]\), and with input \([\hat{\theta}_2(t)|\bar{z}| + \bar{\sigma}_3(t)]\). The outputs \( \alpha_{01} \) and \( \alpha_{02} \) can be expressed as

\[
\dot{x}_f = x_f, x_f(0) = 0
\]

\[
\dot{x}_f = -(\mu_0 + \mu_1)x_f - \mu_0 \mu_1 x_f + b_0[\hat{\theta}_1(t)|\bar{u}| + \bar{\sigma}_1(t)], x_f(0) = 0 \quad (6.45)
\]

\[\alpha_{01} = x_f \]

\[
\dot{x}_f = x_f, x_f(0) = 0
\]

\[
\dot{x}_f = -(\mu_0 + \mu_1)x_f - \mu_0 \mu_1 x_f + b_0[\hat{\theta}_2(t)|\bar{z}| + \bar{\sigma}_3(t)], x_f(0) = 0 \quad (6.46)
\]

\[\alpha_{02} = x_f \]

Then \( \alpha_1 \) and \( \alpha_2 \) in (20) can be generated as

\[
\alpha_1 = \alpha_{01} + \lambda_1 \bar{u} \quad (6.47)
\]

\[
\alpha_2 = \alpha_{02} + \lambda_1 \bar{z} \quad (6.48)
\]

for synthesis.

### 6.3 Simulations results

This section presents the results of digital simulation. The trajectories of IONs with closed control loop are obtained. The model of the IONs given in Section 2 are used for simulation. The parameters used for the simulation are,

(1) For identical IONs \( a_1 = a_2 = 0.01 \)

(2) For non-identical IONs \( a_1 = 0.01, a_2 = 0.02 \).

(3) Model parameters \( I_{Na} = -0.59, I_{Ca} = 0.018, k = 0.1, \epsilon_{Ca} = 0.02, \epsilon_{Na} = 0.001 \).
(4) The adaptation parameters are $\hat{\theta}_1, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\theta}_3, \hat{\sigma}_4, \hat{\epsilon}_{ca}$.

(5) $uc_1, uc_2$ are the control inputs.

(6) $d_1(t), d_2(t)$ are the disturbance inputs which has been included in the equations
\[
\dot{v}_2 = k[u_2 - z_2 + I_{Ca} - I_{Na}] + uc_1 + d_1(t), \quad \dot{w}_2 = \epsilon_{Ca}(z_2 - I_{Ca} - \mu) + u_c + d_2(t).
\]

(7) $\mu$ is the bias input.

(8) $\Gamma, p_1, p_2$ are the design parameters.

The values of $p_1$ and $p_2$ are obtained by solving a Lyapunov equation $A^T P + PA = -Q$
where $A = \begin{pmatrix} -10 & 0 \\ -0 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

(9) $k_t$ is the time scaling factor.

Simulations are done for the following cases and responses were observed

**Case(i)** Simulation of two identical IONs without disturbance ($d(t) = 0$) and bias input ($\mu = 0$), ($\mu = 0.2$)

**Case(ii)** Simulation of two non-identical IONs without disturbance ($d(t) = 0$) and bias input ($\mu = 0$), ($\mu = 0.2$)

**Case(iii)** Simulation of identical IONs with random and Sinusoidal Disturbances $d_1(t), d_2(t)$ which has been generated by passing it through a low-pass filter of type $\left( \frac{b}{s+b} \right)$ where $b$ is a positive number ($b > 0$) in this case we chose $b = 5$ that has a pole at -5.

**Case(iv)** Simulation of non-identical IONs with random and Sinusoidal Disturbances $d_1(t), d_2(t)$.

**Case(v)** Simulation of identical IONs with distinct disturbances $d_1(t), d_2(t)$.

**Case(vi)** Simulation of non-identical IONs with distinct disturbances $d_1(t), d_2(t)$. 

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Case (vii) Simulation for the phase control of two identical IONs.

Case (viii) Simulation for clamped control magnitude.

The initial conditions of the IONs for simulation are \( x_1(0) = (-0.1, -0.1, 0.1, 0.1)^T \) for ION_1, and \( x_2(0) = (-0.2, 0.1, -0.1, 0.3)^T \) for ION_2. Note that initial conditions of the two IONs are not equal. The initial conditions for state predictor and adaptation estimates are arbitrarily selected as \( \hat{x}_1(0) = 0, \hat{x}_2(0) = 0, \hat{\theta}_1(0) = 0, \hat{\sigma}_1(0) = 0, \hat{\sigma}_2(0) = 0, \hat{x}_3(0) = 0, \hat{x}_4(0) = 0, \hat{\theta}_2(0) = 0, \hat{\sigma}_3(0) = 0, \hat{\sigma}_4(0) = 0, \hat{\epsilon}_{ca}(0) = 0.0002. \)

It is important to note that ION_2 with the adaptive law will synchronize for various choices of positive values of \( p_i, \Gamma \), but different sets of values will give different transient responses. We have selected these values after observing the simulated responses. We may point out that actual parameters \( a_i \) are used in the ION models only for simulation, to examine the performance of the control system. But the designed adaptive control system does not require any knowledge of these actual parameters for its synthesis.

**Case 1.** \( L_1 \) Adaptive synchrony of identical IONs with \( d_1(t) = d_2(t) = 0, \mu = 0, \mu = 0.2 \)

In this case, IONs are simulated with \( d(t) = 0 \) and \( \mu = 0 \). Later IONs are simulated with \( \mu = 0.2 \). The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The initial conditions used for the simulation are \( x_1(0) = (-0.1, -0.1, 0.1, 0.1)^T \) for ION_1, and \( x_2(0) = (-0.2, 0.1, -0.1, 0.3)^T \) for ION_2. The initial conditions for state predictor and adaptation estimates are arbitrarily selected as \( \hat{x}_1(0) = 0, \hat{x}_2(0) = 0, \hat{\theta}_1(0) = 0, \hat{\sigma}_1(0) = 0, \hat{\sigma}_2(0) = 0, \hat{x}_3(0) = 0, \hat{x}_4(0) = 0, \hat{\theta}_2(0) = 0, \hat{\sigma}_3(0) = 0, \hat{\sigma}_4(0) = 0, \hat{\epsilon}_{ca}(0) = 0.0002. \) The chosen adaptation gain is \( \Gamma = 1e^5 \), \( p_i \) values are obtained by solving the Lyapunov equation.
of the form discussed in simulation results section. The trajectories of the ION_1 and controlled ION_2 without any disturbance \( d(t)=0, \mu=0 \) are shown in the Fig.1. The trajectories of the ION_1 and controlled ION_2 with bias value \( \mu=0.1 \) are shown in Fig.2. It is seen in Fig.2 that for chosen value of bias input \( \mu_1=0.2 \) the IONs are bursting. It can be observed that despite of the bursting in the reference ION_1, the follower ION_2 is able to track the trajectory of reference ION and both IONs got synchronized.

**Case II.** \( L_1 \) Adaptive synchrony of non-identical IONs \( d_1(t)=d_2(t)=0, \mu=0, \mu=0.2 \)

In this case, IONs are simulated with \( d(t)=0 \) and \( \mu=0 \). Later IONs are simulated with \( \mu=0.2 \). The IONs chosen are non-identical and the parameters are \( a_1=0.01, a_2=0.02 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs without \( \mu \) are shown in Fig.3 and for IONs with bias value \( \mu \) are shown in Fig.4.

**Case III.** \( L_1 \) Adaptive synchrony of Identical IONs with random and Sinusoidal Disturbances \( d_1(t), d_2(t) \).

In this case, IONs are simulated with some random disturbance inputs \( d_1(t)=d_2(t) \) and sinusoidal disturbance inputs \( d_1(t)=d_2(t) \). The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with random disturbances are shown in Fig.5 and for IONs with sinusoidal disturbances are shown in Fig.6.

**Case IV.** \( L_1 \) Adaptive synchrony of non-identical IONs with random and Sinusoidal Disturbances \( d_1(t), d_2(t) \).

In this case, IONs are simulated with some random disturbance inputs \( d_1(t)=d_2(t) \)
and sinusoidal disturbance inputs \( d_1(t) = d_2(t) \). The IONs chosen are non-identical and the parameters are \( a_1 = 0.01, a_2 = 0.02 \). The remaining parameters of the IONs of Case A are retained. Selected responses for IONs with random disturbances are shown in Fig.7 and for IONs with sinusoidal disturbances are shown in Fig.8.

**Case V.** \( \mathcal{L}_1 \) Adaptive synchrony of Identical IONs with distinct disturbances \( d_1(t) \) and \( d_2(t) \)

In this case, IONs are simulated with some random disturbance \( d_1(t) \) and sinusoidal disturbance \( d_2(t) \). The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses are shown in Fig.9.

**Case VI.** \( \mathcal{L}_1 \) Adaptive synchrony of non-Identical IONs with distinct disturbances \( d_1(t) \) and \( d_2(t) \)

In this case, IONs are simulated with some random disturbance \( d_1(t) \) and sinusoidal disturbance \( d_2(t) \). The IONs chosen are non-identical and the parameters are \( a_1 = 0.01, a_2 = 0.02 \). The remaining parameters of the IONs of Case A are retained. Selected responses are shown in Fig.10.

**Case VII.** \( \mathcal{L}_1 \) Adaptive synchrony of Identical IONs with a relative phase difference between two IONs.

In this case, IONs are simulated with some relative phase difference between two IONs. The IONs chosen are identical and the parameters are \( a_1 = a_2 = 0.01 \). The remaining parameters of the IONs of Case A are retained. Selected responses are shown in Fig.11. From the figure we can observe that there is a phase difference between two
IONs, here we are trying to synchronize the IONs, by delaying the reference ION with $t_d=0.25$ and passing it through the controller. So that the follower ION, will eventually gets synchronized with the delayed reference ION and, the IONs will oscillate in unison. By this we can control the relative phase between two IONs.

*Case VIII.* $L_1$ Adaptive synchrony of Identical IONs with clamped control magnitude.

In this case, IONs are simulated with clamped control magnitude. The IONs chosen are identical and the parameters are $a_1 = a_2=0.01$. The remaining parameters of the IONs of Case A are retained. Selected responses are shown in Fig.12. From the figure we can observe that by restricting the control input magnitude to a certain value, the IONs are taking a little longer time for synchronization. By this we can say that if there is no restriction on control input magnitude the IONs will synchronize quickly.
Figure 6.1: Initial trajectories of IONs (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
Figure 6.2: Adaptive synchrony of identical IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
Figure 6.3: Adaptive synchrony of identical IONs for $a_1 = a_2 = 0.01$, $d(t) = 0$, $\mu = 0.2$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
Figure 6.4: Adaptive synchrony of non-identical IONs for $a_1 = 0.01$, $a_2 = 0.02$, $d(t) = 0$, $\mu = 0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
Figure 6.5: Adaptive synchrony of non-identical IONs for $a_1=0.01$, $a_2=0.02$, $d(t)=0$, $\mu=0.2$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
Figure 6.6: Adaptive synchrony of identical IONs with random disturbance inputs $d_1(t)=d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$ .(j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.
Figure 6.7: Adaptive synchrony of identical IONs with sinusoidal disturbance inputs $d_1(t)=d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.
Figure 6.8: Adaptive synchrony of non-identical IONs with random disturbance inputs $d_1(t)=d_2(t)$ for $a_1=0.01, a_2=0.02$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the disturbance input.
Figure 6.9: Adaptive synchrony of non-identical IONs with sinusoidal disturbance inputs $d_1(t) = d_2(t)$ for $a_1 = 0.01$, $a_2 = 0.02$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$. (l) shows the error, (m) shows the disturbance input.
Figure 6.10: Adaptive synchrony of identical IONs with two distinct disturbance inputs $d_1(t) \neq d_2(t)$ for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error, (m) shows the random disturbance input, (n) shows the sinusoidal disturbance input.
Figure 6.11: Adaptive synchrony of non-identical IONs with two distinct disturbance inputs \( d_1(t) \neq d_2(t) \) for \( a_1=0.01, \ a_2=0.02 \) (a) \( u_1 \) and \( u_2 \), (b) \( v_1 \) and \( v_2 \), (c) \( z_1 \) and \( z_2 \), (d) \( w_1 \) and \( w_2 \), (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters \( \theta_1, \theta_2, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \). (j)-(k) shows the estimates of state predictors \( x_1, x_2, x_3, x_4 \), (l) shows the error, (m) shows the random disturbance input, (n) shows the sinusoidal disturbance input.
Figure 6.12: Adaptive synchrony of identical IONs with a relative phase difference between two IONs for $a_1=a_2=0.01$, $d(t)=0$, $\mu=0$. (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (f) $u_{1d}$ and $u_2$, (g) $v_{1d}$ and $v_2$, (h) $z_{1d}$ and $z_2$, (i) $w_{1d}$ and $w_2$, (j) shows the control inputs, (k) shows the error.
Figure 6.13: Adaptive synchrony of identical IONs with clamped control magnitude for $a_1=a_2=0.01$ (a) $u_1$ and $u_2$, (b) $v_1$ and $v_2$, (c) $z_1$ and $z_2$, (d) $w_1$ and $w_2$, (e) shows the control inputs, (f)-(i) shows estimates of Projection parameters $\theta_1$, $\theta_2$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$. (j)-(k) shows the estimates of state predictors $x_1$, $x_2$, $x_3$, $x_4$, (l) shows the error.
CHAPTER 7

CONCLUSION

The main objective of this research work is to design an adaptive control laws for the synchronization of two inferior olive neurons (IONs) [12, 13] which are not necessarily identical. The key parameters associated with the polynomial type nonlinearity of the IONs are not known. Furthermore, in the closed-loop the relative phase of the two IONs is controlled. In chapter 3, the adaptive control law is designed based on tuning functions backstepping approach [17] with single control input. The signal $z$ of the $(z - w)$ system was treated as an output. Using Lyapunov analysis the derivation of the adaptive control law was completed in two steps of a backstepping design procedure. It is assumed that the parameters are not known to the designer and there are no external disturbances present in the model. Simulation results were presented which showed that the follower ION$_2$ will track the reference ION$_1$ starting from arbitrary initial conditions, and the IONs attain synchronism despite of the uncertainties in the parameters [16]. However, this design of control law attains synchronism, but the control input magnitude which is used for attaining synchronism is very large. The IONs may not synchronize when there is large difference between the two arbitrary initial conditions of IONs and, if there are any external disturbances present in the model.
In chapter 4, the design of the adaptive control law is based on tuning functions back-stepping approach [17] with two control inputs. The system parameters were assumed to be unknown. In the closed-loop system, it was shown that IONs attain synchronism and oscillate in unison after some initial transient time. In this design of adaptive control system it is assumed that there is some bias input ($\mu$) and a disturbance input $d(t)$ acting on the model. It is observed from the simulation results that despite the presence of bias and disturbance input the IONs attain synchronism. This control law is limited for only certain range of bias input and very low disturbance input.

In chapter 5, the design of the adaptive control law is based on $L_1$ adaptive control theory [42]. The control law is designed for nonlinear strict-feedback systems in the presence of uncertain system input gain and unknown time-varying nonlinearities. It is assumed that the parameters of IONs are not known. Furthermore, it is assumed that IONs are perturbed by a disturbance input and the nonlinear functions of the IONs are unstructured(unmodelled). This design of adaptive control law is more robust, it can applied to the systems, in which nonlinearities of system are not known. In simulation results it has been observed that despite of the nonlinearities and perturbed by disturbance input the IONs attain synchronism. This control law can be applicable to the nonlinear systems which are unstructured. Furthermore, in chapter 6, the design of an adaptive control law is based on $L_1$ adaptive control law similar to chapter 5, but in chapter 6, adaptive law is designed with two control inputs. It is assumed that the parameters of IONs are not known. These control law is more robust and applicable to systems which have unmodelled dynamics and are perturbed by any external disturbance.
inputs. The control magnitude required for attaining synchronism is less and the two IONs can start at any arbitrary initial conditions. Even though, there are large difference between the initial conditions the IONs still attain synchronism as these adaptation scheme is very robust.

In chapters 3, 4, 5 and 6, in the closed-loop system, the relative-phase of the two IONs is controlled. Simulation results are presented which show that the adaptive law accomplished synchronization (the follower ION2 tracks the reference ION1), despite large uncertainties and external disturbance inputs in the ION parameters. This thesis provides various methods of designing an adaptive control systems for the synchronization of two IONs. However, this type of control law can be applied to any number of follower IONs to track the reference ION assuming that the nonlinearities of the system are unstructured and parameters of the system are not known.
APPENDIX I

MODEL PARAMETERS

The model parameters of the inferior olive neurons (IONs) given by:

Identical IONs: \( a_1 = a_2 = 0.01 \)

Non-identical IONs: \( a_1 = 0.01, a_2 = 0.02 \)

\[ I_{Na} = -0.59. \]

\[ I_{Ca} = 0.018. \]

\[ k = 0.1. \]

\[ \epsilon_{Ca} = 0.02. \]

\[ \epsilon_{Na} = 0.001. \]
\[ q = \frac{k}{\epsilon_{Na}} \]
\[ r = \frac{\epsilon_{Na}}{k} \]
\[ \alpha_1 = -(u_1^2 - u_1^3 - u_2^2 + u_2^3 - v_1) - (r\phi_1\dot{a}_1) + (r\psi_1\dot{a}_2) - (P_1r(u_1 - u_2)) \]
\[ \alpha_2 = -(z_1^2 - z_1^3 - z_2^2 + z_2^3 - w_1) - (\phi_2\dot{a}_1) + (\psi_2\dot{a}_2) - P_1(z_1 - z_2) \]
\[ G_{11} = (3qu_1^5) + (u_1^4(-5q - 3\dot{a}_1)) + u_1^3(5\dot{a}_1 + 2q) + (u_1^2(3v_1q - 2\dot{a}_1)) - 2u_1v_1q \]
\[ G_{12} = -(3qu_2^5) + (u_2^4(5q + 3\dot{a}_2)) + u_2^3(-5\dot{a}_2 - 2q) + (u_2^2(-3v_2q + 2\dot{a}_2)) + 2u_2v_2q \]
\[ G_{13} = -k(u_1 - z_1 + I_{Ca} - I_{Na}) \]
\[ G = G_{11} + G_{12} + G_{13} \]
\[ H_{11} = (3z_1^5) + (z_1^4(-5 - 3\dot{a}_1)) + (z_1^3(5\dot{a}_1 + 2)) + (z_1^2(3w_1 - 2\dot{a}_1)) - 2z_1w_1 \]
\[ H_{12} = -(3z_2^5) + (z_2^4(5 + 3\dot{a}_2)) + (z_2^3(-5\dot{a}_2 - 2)) + (z_2^2(-3w_2 + 2\dot{a}_2)) + 2z_2w_2 \]
\[ H_{13} = -\epsilon_{Ca}(z_1 - I_{Ca}) \]
\[ H = H_{11} + H_{12} + H_{13} \]
\[ G_1 = -(2qu_1^4) + (u_1^3(3q + 2\dot{a}_1)) - (u_1^2(q + 3\dot{a}_1)) + (u_1(\dot{a}_1 - 2v_1q)) + v_1q \]
\[ G_2 = (2qu_2^4) - (u_2^3(3q + 2\dot{a}_2)) + (u_2^2(q + 3\dot{a}_2)) - (u_2(\dot{a}_2 - 2v_2q)) - v_2q \]
\[ H_{14} = -(2z_1^4) + (z_1^3(3 + 2\dot{a}_1)) + (z_1^2(-1 - 3\dot{a}_1)) + (z_1(\dot{a}_1 - 2w_1)) + w_1 \]
\[ H_2 = (2z_2^4) - (z_2^3(3 + 2\dot{a}_2)) + (z_2^2(1 + 3\dot{a}_2)) - (z_2(\dot{a}_2 - 2w_2)) - w_2 \]
\[ I_5 = (3u_1^4) - (5u_1^3) + (2u_1^2) - (2u_1^3r\dot{a}_1) + (3u_1^2r\dot{a}_1) - ((ru_1\dot{a}_1)) - r\phi_1 \]
\[ I_6 = -(3u_2^4) + (5u_2^3) - (2u_2^2) + (2u_2^3r\dot{a}_2) - (3u_2^2r\dot{a}_2) + (u_2r\dot{a}_2) + r\psi_1 \]
\[ I_7 = (3z_1^4) - (5z_1^3) + (2z_1^2) - (2z_1^3\dot{a}_1) + (3z_1^2\dot{a}_1) - (z_1\dot{a}_1) - \phi_2 \]

\[ I_8 = -(3z_2^4) + (5z_2^3) - (2z_2^2) + (2z_2^3\dot{a}_2) - (3z_2^2\dot{a}_2) + (z_2\dot{a}_2) + \psi_2 \]

\[ \eta_1 = \begin{pmatrix} I_5 & I_6 \\ I_7 & I_8 \end{pmatrix} \]

\[ N_{01} = -(r_q G) - (r G_1 \dot{a}_1) - (r G_2 \dot{a}_2) - (r \phi_1 \dot{a}_1) + (r \psi_1 \dot{a}_2) - P_1 v_2 + P_1 \alpha_1 + (P_1^2 r (u_1 - u_2)) \]

\[ N_{02} = -H - (\dot{a}_1 H_{14}) - (\dot{a}_2 H_2) - (\phi_2 \dot{a}_1) + (\psi_2 \dot{a}_2) - P_1 w_2 + P_1 \alpha_2 + P_1^2 \]

\[ \eta_0 = \begin{pmatrix} N_{01} \\ N_{02} \end{pmatrix} \]
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VITA
Graduate College
University of Nevada, Las Vegas

Srujan Kumar Chalike

Local Address:
4248 Cottage Circle, Apt 4
Las Vegas, Nevada 89119

Home Address:
H.no 12-14-110, Lalapet
Vinobha Nagar, Secunderabad
Andhra Pradesh
India 500017

Degrees:
Bachelors of Technology, Electrical Engineering, 2008
Bharat Institute of Engineering and Technology
Jawaharlal Nehru Technological University, Hyderabad, India

Publications:
Lee, K.W, Chalike, S.K, S.N.Singh, Tuning Functions Based Adaptive Synchrony in Inferior Olive Neurons

Thesis Title:
Adaptive Control Systems for Synchronization of Inferior Olive Neurons (IONs)

Thesis Examination Committee:

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Chairperson, Dr. Sahjendra Singh, Ph.D.
Committee Member, Dr. Venkatesan Muthukumar, Ph.D.
Committee Member Dr. Henry Selvaraj, Ph.D.
Graduate Faculty Representative, Dr. Woosoon Yim, Ph.D.